To compute the capacitance of a set-up, we inject $q$ amount of charge to the conductor and find its resultant potential $\phi$ relative to infinity. Afterwards, we can calculate the capacitance $C$ as $\frac{q}{\phi}$. In this case, when $q$ amount of charge is added to the metal sphere, it is distributed evenly over its surface due to symmetry. Since the electric field of a spherical shell of charge $q$ is identical to that of a point charge $q$ placed at its center for regions outside of the shell, the potential at the surface of the metal sphere is the potential of a point located a distance $R$ away from a point charge $q$.

$$\phi = \frac{q}{4\pi\varepsilon_0 R}.$$

The capacitance of the metal sphere is thus

$$C = \frac{q}{\phi} = 4\pi\varepsilon_0 R$$

We define the origin to be at the left edge where the distance between the plates is the smallest.

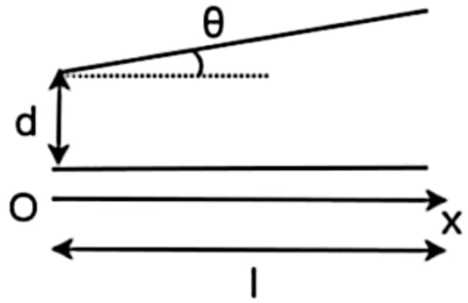


Figure 6.25: Tilted plate

Define the x-axis to be positive rightwards in Fig. 6.25. The separation between the plates at a coordinate $x$ is $d + x\sin\theta$. The set-up can be taken to be many capacitors of infinitesimal thickness, obtained from making myriad vertical cuts, connected in parallel since the potential differences across

them are identical due to the equipotential nature of both plates. Each of these infinitesimal plates has length $dx$ and hence area $wdx$. The plate separation of an infinitesimal plate at coordinate $x$ is again $d + x\sin\theta$. Since $\theta$ is small, the electric field between the plates is approximately vertical so the capacitance of an infinitesimal plate is analogous to that of a parallel-plate capacitor, $\varepsilon_0 \frac{wdx}{d+x\sin\theta}$. The total capacitance of the system is obtained by determining the equivalent capacitance of these infinitesimal elements. Since the infinitesimal capacitors are connected in parallel, the equivalent capacitance is simply the sum (integral) of the individual capacitances.

$$
\begin{aligned}
C_{eq} &= \int_0^l \frac{\varepsilon_0 w}{d + x\sin\theta} dx \\
&= \left[\frac{\varepsilon_0 w}{\sin\theta} \ln|d + x\sin\theta|\right]_0^l \\
&= \frac{\varepsilon_0 w}{\sin\theta} \ln\left(1 + \frac{l\sin\theta}{d}\right).
\end{aligned}
$$

Let the cylinder possess charge $q$ which is distributed uniformly across its curved surface due to its axial symmetry and infinite nature. Then, we can draw a concentric cylindrical Gaussian surface of an arbitrary length $l$ and an arbitrary radius $r$, $r_1 \leq r \leq r_2$, around the cylinder. Let the linear charge density of the cylinder be $\lambda = \frac{q}{L}$. By Gauss' law, the electric field at a radius $r$ from the central axis of the system is

$$E \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\implies E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

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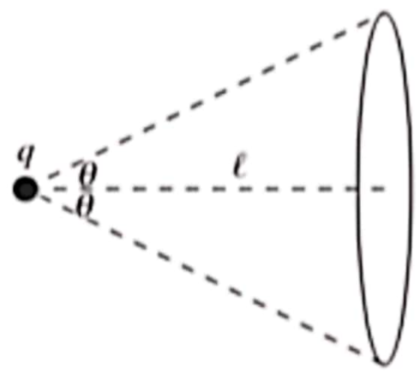
The potential difference between the shell and the cylinder is then

$$\Delta V = -\int_{r_2}^{r_1} \frac{\lambda}{2\pi\varepsilon_0 r} dr = \left[-\frac{\lambda}{2\pi\varepsilon_0} \ln r\right]_{r_2}^{r_1}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1}.$$

The capacitance per unit length of this system is thus

$$\frac{C}{L} = \left|\frac{\lambda}{\Delta V}\right| = \frac{2\pi\varepsilon_0}{\ln \frac{r_2}{r_1}}.$$




**Solution.** Let $\ell = R\cos\theta$, and deform the disk into a spherical cap with radius $R$. Then the answer is then just $kq/\epsilon_0$, where $k$ is the ratio of the area of the cap to the total area of the sphere. In spherical coordinates,

$$k = \frac{1}{4\pi}\int_0^\theta 2\pi \sin\theta \, d\theta = \frac{1-\cos\theta}{2}$$

so the answer is

$$\frac{1-\cos\theta}{2}\frac{q}{\epsilon_0}.$$

You can also show this using the original flat Gaussian surface, though that takes more work.

The potential differences across the two capacitors must be identical as they are connected by conducting wires in parallel. The system is then equal to two capacitors connected in parallel with a total charge $q$ and equivalent capacitance $C_1 + C_2$. The energy loss is then

$$\Delta U = \frac{q^2}{2(C_1 + C_2)} - \frac{q^2}{2C_1} = -\frac{C_2 q^2}{2C_1(C_1 + C_2)}.$$

The electric field at a radial distance $r$ from the center of the spherical shell, $r_1 \leq r \leq r_2$, is given by Gauss' law as

$$E = \frac{q}{4\pi\varepsilon_0 r^2}.$$

The potential difference between the inner and outer shells is given by

$$\Delta V = -\int_{r_2}^{r_1} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Since the potential energy stored in a capacitor is $U = \frac{1}{2}q\Delta V$, the current potential energy stored in the spherical shell is given by

$$U = \frac{1}{2}q\Delta V = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Integrating the energy density of the electric field would also yield the same result. In the next problem, observe that $q$ amount of charge will be uniformly induced on the inner surface of the shell so that the electric field within the shell is zero. This leaves $-q$ amount of charge uniformly distributed on the outer surface. We can determine the external work done by determining the change in potential energy of the system. There are several methods to go about this. Firstly, we can sum the potential energy of the charge due to the two shells of charges and the potential energy between the shells of charges (which was the previous result). To compute the former, simply observe that the potential at the center of a spherical shell of charge $Q$ and radius $R$ is simply

$$\frac{Q}{4\pi\varepsilon_0 R},$$

as all charges are equidistant from the center. The potential at the center due to the two spherical shells is then

$$V_{center} = \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2}.$$



The potential energy of the charge due to the shells is then

$$U_{charge} = -qV_{center} = \frac{q^2}{4\pi\varepsilon_0 r_2} - \frac{q^2}{4\pi\varepsilon_0 r_1}.$$

The total potential energy of the initial set-up is then

$$U = U_{charge} + U_{shells} = \frac{q^2}{4\pi\varepsilon_0 r_2} - \frac{q^2}{4\pi\varepsilon_0 r_1} + \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{q^2}{8\pi\varepsilon_0 r_2} - \frac{q^2}{8\pi\varepsilon_0 r_1}.$$

The potential energy of the set-up after the charge has been extracted is zero. Therefore, the external work done is

$$W = \Delta U = 0 - U = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

There is another perspective to the change in potential energy involving the electric fields. The electric field of the initial set-up is that of a point charge, excluding the portion in the spherical shell of inner radius $r_1$ and outer radius $r_2$ while the electric field of the final set-up is simply that of a point charge. Therefore, the increase in potential energy is that carried by the field in the spherical shell — this is essentially the previous result. Thus,

$$W = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

The electric field strength at the location of one infinitely large plate, due to the other plate, is $\frac{\sigma}{2\varepsilon_0}$ by Gauss' law. Therefore, the electrostatic force on the movable plate is

$$F_{elec} = -\sigma A \cdot \frac{\sigma}{2\varepsilon_0} = -\frac{\sigma^2 A}{2\varepsilon_0},$$

where the negative sign indicates that the electrostatic force acts along the direction that tends to reduce the distance between the two plates. The external force applied in moving the plate must then be negative of this to not result any change in kinetic energy of the plate. The work done by the external force in increasing the plate separation by $x$ is then

$$W = F_{ext}x = \frac{\sigma^2 Ax}{2\varepsilon_0}.$$

Now, consider the energy density of the field. The field between the two plates has a uniform magnitude $\frac{\sigma}{\varepsilon_0}$ and direction. The field outside the gap



is zero. The energy density between the plates is then

$$\frac{1}{2}\varepsilon_0 E^2 = \frac{\sigma^2}{2\varepsilon_0}.$$

The change in the potential energy associated with the electric field in increasing the plate separation by $x$ is the change in volume, $Ax$, multiplied by the energy density above. This must also be equal to the work done by the external force.

$$W = \Delta U = \frac{\sigma^2 Ax}{2\varepsilon_0}.$$

The change in potential energy can also be determined via the potential energy stored in a capacitor $U = \frac{q^2}{2C}$. Then,

$$W = \Delta U = \frac{q^2}{2} \cdot \left(\frac{1}{C'} - \frac{1}{C}\right) = \frac{\sigma^2 A^2}{2}\left(\frac{d+x}{\varepsilon_0 A} - \frac{d}{\varepsilon_0 A}\right) = \frac{\sigma^2 Ax}{2\varepsilon_0}.$$

The electric field strength at the location of one infinitely large plate, due to the other plate, is $\frac{\sigma}{2\varepsilon_0}$ by Gauss' law. Therefore, the electrostatic force on the movable plate is

$$F_{elec} = -\sigma A \cdot \frac{\sigma}{2\varepsilon_0} = -\frac{\sigma^2 A}{2\varepsilon_0},$$

where the negative sign indicates that the electrostatic force acts along the direction that tends to reduce the distance between the two plates. The external force applied in moving the plate must then be negative of this to not result any change in kinetic energy of the plate. The work done by the external force in increasing the plate separation by $x$ is then

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$$W = \Delta U = \frac{\sigma^2 Ax}{2\varepsilon_0}.$$

The change in potential energy can also be determined via the potential energy stored in a capacitor $U = \frac{q^2}{2C}$. Then,

$$W = \Delta U = \frac{q^2}{2} \cdot \left(\frac{1}{C'} - \frac{1}{C}\right) = \frac{\sigma^2 A^2}{2}\left(\frac{d+x}{\varepsilon_0 A} - \frac{d}{\varepsilon_0 A}\right) = \frac{\sigma^2 Ax}{2\varepsilon_0}.$$