We can see the relevant current distribution as the superposition of three current loops on the left, front and bottom faces of the cube, as shown in Fig. 7.23. Each current loop produces a magnetic field of magnitude B at the center, parallel to the area vector associated with the current loop (whose direction is determined by the right-hand-rule). Therefore, the relevant magnetic field is

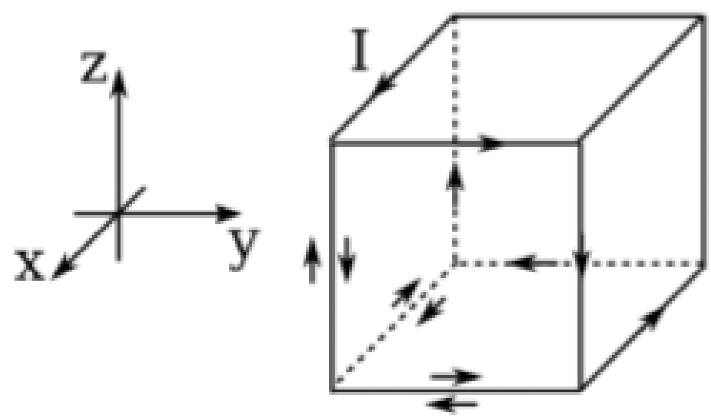

Figure 7.23: Superposition of currents

$$\boldsymbol{B}' = -B\hat{\boldsymbol{\imath}} + B\hat{\boldsymbol{\jmath}} + B\hat{\boldsymbol{k}},$$

which is of magnitude $\sqrt{3}B$ and directed from the center towards the top-right vertex on the back face in Fig. 7.23.

Define the origin at the bottom end of the cylinder (smaller z-coordinate) and consider the solenoid in cylindrical coordinates. Slice the solenoid into rings of thickness dh. Suppose that we wish to calculate the magnetic field at a certain point P with coordinates $(0, 0, z)$. We have previously derived in an example problem that a circular ring of radius R that carries an anti-clockwise current I produces a vertical magnetic field

$$B_z = \frac{\mu_0 I R^2}{2(z^2 + R^2)^{\frac{3}{2}}}$$

at a point that is at a perpendicular height z from the ring, along the symmetrical axis. Applying this result to this problem, the contribution to the magnetic field at the origin due to an infinitesimal ring between z-coordinates h and $h + dh$ is

$$dB_z = \frac{\mu_0 \eta I dh R^2}{2((h - z)^2 + R^2)^{\frac{3}{2}}},$$

as the particular infinitesimal ring carries current $\eta I dh$ and because point P is a distance $z-h$ away from the ring at z-coordinate h. Then, the total magnetic field at P is obtained by integrating the above over all rings from $h=0$ to $h=l$.

$$B_z = \int_0^l \frac{\mu_0 \eta I R^2}{2((h-z)^2+R^2)^{\frac{3}{2}}} dh.$$

Making the substitutions $h-z = R\tan\theta$ and $dh = R\sec^2\theta d\theta$,

$$B_z = \int_{\tan^{-1}\frac{-z}{R}}^{\tan^{-1}\frac{l-z}{R}} \frac{\mu_0 \eta I}{2} \cos\theta d\theta = \left[\frac{\mu_0 \eta I}{2}\sin\theta\right]_{\tan^{-1}\frac{-z}{R}}^{\tan^{-1}\frac{l-z}{R}}$$

$$= \frac{\mu_0 \eta I}{2}\left(\frac{l-z}{\sqrt{(l-z)^2+R^2}} + \frac{z}{\sqrt{z^2+R^2}}\right).$$

To solve the next problem, slice the solenoid into thin cylindrical shells with radial distances between r and $r+dr$. Observe that the current per unit cross sectional area of the solenoid is $J = \frac{\eta I}{r_1 - r_0}$. Therefore, the current per unit length of this cylindrical shell is Jdr. With the same origin at the bottom of the solenoid, the contribution to the magnetic field at a point P along the symmetrical axis by this cylindrical shell is

$$dB_z = \frac{\mu_0 J dr}{2}\left(\frac{l-z}{\sqrt{(l-z)^2+r^2}} + \frac{z}{\sqrt{z^2+r^2}}\right),$$

where we have replaced the previous current per unit length ηI with Jdr, and R with the variable r. The total magnetic field at P is then

$$B_z = \int_{r_0}^{r_1} \frac{\mu_0 J}{2}\left(\frac{l-z}{\sqrt{(l-z)^2+r^2}} + \frac{z}{\sqrt{z^2+r^2}}\right) dr.$$

To evaluate the integral $\int \frac{a}{\sqrt{a^2+x^2}} dx$, make the substitutions $x = a\tan\theta$ and $dx = a\sec^2\theta$. Then,

$$\int_{x_0}^{x_1} \frac{a}{\sqrt{a^2+x^2}} dx = \int_{\tan^{-1}\frac{x_0}{a}}^{\tan^{-1}\frac{x_1}{a}} a\sec\theta d\theta$$

$$= [a\ln|\sec\theta + \tan\theta|]_{\tan^{-1}\frac{x_0}{a}}^{\tan^{-1}\frac{x_1}{a}} = a\ln\left|\frac{\sqrt{x_1^2+a^2}+x_1}{\sqrt{x_0^2+a^2}+x_0}\right|.$$

Therefore,

$$B_z = \frac{\mu_0 \eta I (l-z)}{2(r_1 - r_0)} \ln \left| \frac{\sqrt{r_1^2 + (l-z)^2} + r_1}{\sqrt{r_0^2 + (l-z)^2} + r_0} \right| + \frac{\mu_0 \eta I z}{2(r_1 - r_0)} \ln \left| \frac{\sqrt{r_1^2 + z^2} + r_1}{\sqrt{r_0^2 + z^2} + r_0} \right|.$$

Solution. (a) First, we can get the intuition using a single wire. In this case,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}$$

in cylindrical coordinates. Upon a 90° rotation, $\hat{\boldsymbol{\theta}}$ turns into $\hat{\mathbf{r}}$, giving

$$\mathbf{E} = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{r}}$$

which is a valid electrostatic field, as it's simply the electric field of an charged wire with linear charge density $\lambda = \mu_0\epsilon_0 I$. So by superposition, rotating the $\mathbf{B}$ field of the two wires would also give a valid electrostatic field. (Of course, this isn't really physically meaningful, since electric and magnetic fields don't even have the same units. It's just a mathematical trick.)

We can also prove the correspondence more generally. The key criterion for a valid magnetostatic field is $\nabla \cdot \mathbf{B} = 0$, which for such two-dimensional setups is $\partial_x B_x + \partial_y B_y = 0$. Now, when we rotate by 90°, we define an electric field by $E_y = B_x$ and $E_x = -B_y$, which implies $\partial_x E_y - \partial_y E_x = 0$. But in such a two-dimensional setup, this is equivalent to $\nabla \times \mathbf{E} = 0$, which is the condition to have a valid electrostatic field.

(b) The field lines of $\mathbf{B}$ are always parallel to $\mathbf{B}$. Now, this artificial $\mathbf{E}$ is always perpendicular to $\mathbf{B}$, and equipotentials are always perpendicular to $\mathbf{E}$, so the equipotentials follow the magnetic field lines.

On the other hand, we know precisely what the potential is in this problem. By integrating the $1/r$ field, the potential is proportional to $\log r$, so

$$V(r) \propto \log(r_+) - \log(r_-) = \log(r_+/r_-)$$

where r_+ and r_- are the distances to the two wires. So the equipotentials have constant r_+/r_-. We've already found, when investigating the method of images for spheres , that this implies the equipotentials are *circles*, specifically circles of Appolonius. So the magnetic field lines are circles!

i. What are the dimensions of the quantity q_m?

Solution

By the second expression, q_m must be measured in Newtons per Tesla. But since Tesla are also Newtons per Ampere per meter, then q_m is also measured in Ampere meters.

ii. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > d$. Write your answer in terms of q_m, d, z, and any necessary fundamental constants.

Solution

Adding the two terms,

$$B(z) = -\frac{\mu_0}{4\pi}\frac{q_m}{z^2} + \frac{\mu_0}{4\pi}\frac{q_m}{(z-d)^2} = \frac{\mu_0 q_m}{4\pi}\left(\frac{1}{(z-d)^2} - \frac{1}{z^2}\right).$$

iii. Evaluate this expression in the limit as $d \to 0$, assuming that the product $q_m d = p_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of p_m, z, and any necessary fundamental constants.

Solution

By combining the fractions,

$$B(z) = \frac{\mu_0 q_m}{4\pi}\frac{z^2 - (z-d)^2}{z^2(z-d)^2} = \frac{\mu_0 q_m}{4\pi}\frac{2zd - d^2}{z^2(z-d)^2}.$$

In the limit $d \to 0$, the d^2 term in the numerator can be neglected, and the denominator can be approximated as z^4, giving

$$B(z) = \frac{\mu_0 q_m d}{2\pi z^3} = \frac{\mu_0 p_m}{2\pi z^3}.$$

i. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > 0$. Write your answer in terms of I, r, z, and any necessary fundamental constants.

Solution

Applying the Biot-Savart law, with $\mathbf{s}$ the vector from the point on the loop to the point on the z axis,

$$B(z) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l} \times \mathbf{s}}{s^3} = \frac{\mu_0 I}{4\pi} \frac{2\pi r}{r^2 + z^2} \sin\theta$$

where θ is the angle between the point on the loop and the center of the loop as measured by the point on the z axis, so

$$\sin\theta = \frac{r}{\sqrt{r^2 + z^2}}.$$

Then we have

$$B(z) = \frac{\mu_0 I}{4\pi} \frac{2\pi r^2}{(r^2 + z^2)^{3/2}}.$$

ii. Let kIr^γ have dimensions equal to that of the quantity p_m defined above in Part aiii, where k and γ are dimensionless constants. Determine the value of γ.

Solution

We know p_m must have dimensions of Amperes times meters squared, so $\gamma = 2$.

iii. Evaluate the expression in Part bi in the limit as $r \to 0$, assuming that the product $kIr^\gamma = p'_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of k, p'_m, z, and any necessary fundamental constants.

Solution

Using our previous result,

$$B(z) = \frac{\mu_0 I}{4\pi} \frac{2\pi r^2}{(r^2 + z^2)^{3/2}} \approx \frac{\mu_0 I}{2\pi} \frac{\pi r^2}{z^3} = \frac{\mu_0}{2\pi} \frac{\pi}{k} \frac{p'_m}{z^3}$$

iv. Assuming that the two approaches are equivalent, $p_m = p'_m$. Determine the constant k in Part bii.

Solution

By inspection, $k = \pi$.

i. Assume that $R \gg L$ and only Gilbert type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

Solution

The monopoles that make up the dipoles cancel out except on the flat surfaces. Then the cylinder acts like a parallel plate capacitor.
If the size of a dipole is d, then the surface density of monopole charge is

$$\sigma_m = q_m/d^2.$$

Using the analogy with a parallel place capacitor, the magnitude of B is

$$B = \mu_0 \sigma_m = \mu_0 \frac{p_m}{d^3}$$

and the direction is to the left.

ii. Assume that $R \ll L$ and only Ampère type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

Solution

The currents that make up the dipoles all cancel out except on the cylindrical surfaces. Then the cylinder acts like a solenoid, with

$$B = \frac{\mu_0 I}{d}$$

where I/d is the surface current density. The magnitude of B is

$$B = \frac{\mu_0 I}{d} = \mu_0 \frac{p_m}{d^3}$$

and the direction is to the right.