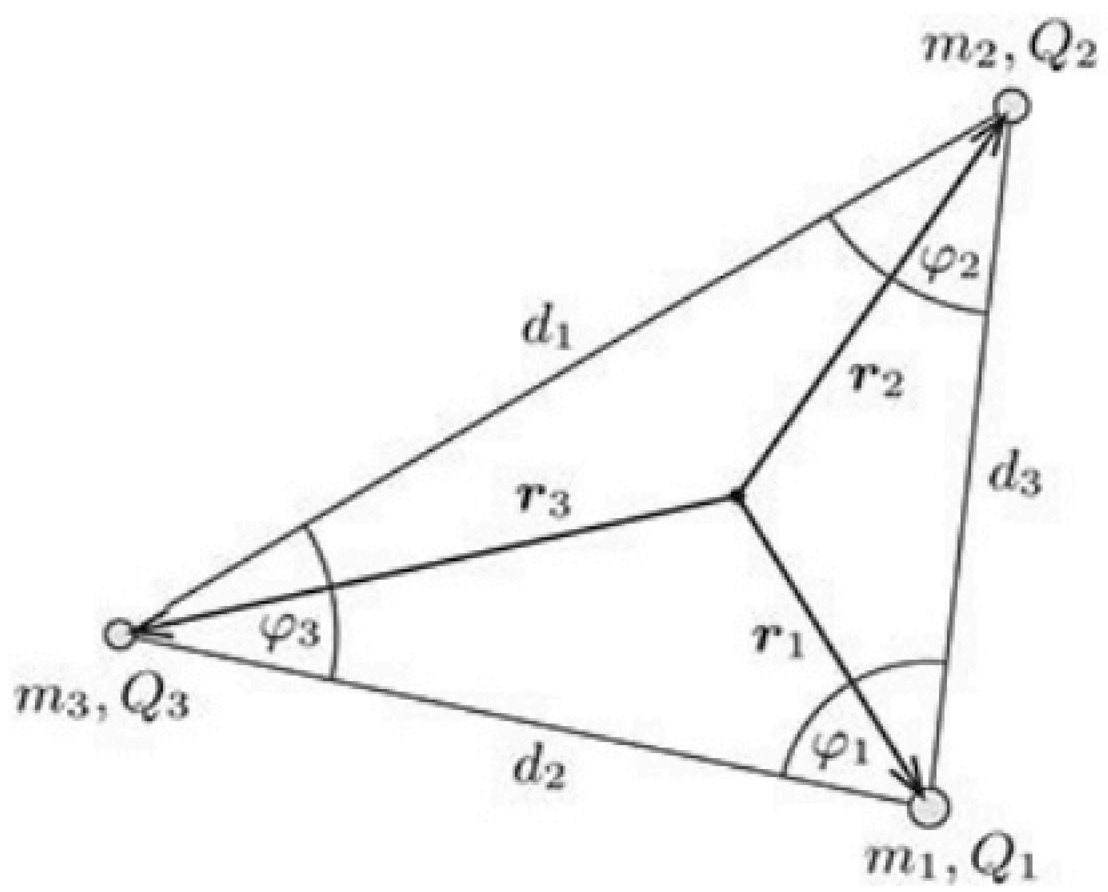
S130 In the absence of external forces, the centre of mass (CM) of the three-pearl system remains at rest. It is therefore convenient to choose it as

the origin of a vector coordinate system in which at any given time the position vectors of the pearls are $\boldsymbol{r}_i$, and their distances from each other are d_i ($i = 1, 2, 3$) (*see* figure).
The net force acting on pearl 1 is the vector sum of the electrostatic forces exerted on it by the other two pearls:

$$\boldsymbol{F}_1 = m_1\boldsymbol{a}_1 = k_e\frac{Q_1Q_2}{d_3^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2) + k_e\frac{Q_1Q_3}{d_2^3}(\boldsymbol{r}_1 - \boldsymbol{r}_3). \tag{1}$$

Because the vector origin has been chosen to be at the position of the CM:

$m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 = \mathbf{0}.$



Substituting for $\boldsymbol{r}_3$ from this into equation (1) gives the equation of motion for pearl 1 as

$$m_1\boldsymbol{a}_1 = k_eQ_1\left(\frac{Q_2}{d_3^3} + \frac{Q_3}{d_2^3}\left[\frac{m_1}{m_3} + 1\right]\right)\boldsymbol{r}_1 + k_eQ_1\left(\frac{Q_3}{d_2^3}\frac{m_2}{m_3} - \frac{Q_2}{d_3^3}\right)\boldsymbol{r}_2. \tag{2}$$

Now, the condition for a straight-line trajectory for pearl 1 is that the direction of its acceleration should be the same as that of its instantaneous position vector. Consequently, the factor multiplying $\boldsymbol{r}_2$ on the right-hand side of equation (2) must be zero. This happens if $Q_2d_2^3/m_2 = Q_3d_3^3/m_3$.
Similar conditions can be written for the other two pearls. So we get the requirement that, at any instant,

$$\frac{Q_1}{m_1}d_1^3 = \frac{Q_2}{m_2}d_2^3 = \frac{Q_3}{m_3}d_3^3 = \lambda, \tag{3}$$

where λ has the same value for all three bodies. The common value λ is not a constant of the motion, but changes with time, as the distances between the charges increase.

If the equalities (3) are satisfied, then the accelerations of the pearls can, after a bit of careful algebra, be shown to be given by the following formula:

$$\boldsymbol{a}_i = k_e \frac{Q_1 Q_2 Q_3 (m_1 + m_2 + m_3)}{\lambda m_1 m_2 m_3} \boldsymbol{r}_i \qquad \text{for } i = 1, 2, 3.$$

Accordingly, the ratio of the accelerations of the pearls is the same as the ratio of the lengths of their position vectors, and the proportionality factor is the same for all three bodies at any given time; consequently, the ratio of the distances from the origin of the three pearls remains *constant in time*. These ratios can be found from condition (3) and the given proportion for the mass-to-charge ratios of the three pearls:

$$d_1 : d_2 : d_3 = \sqrt[3]{\frac{m_1}{Q_1}} : \sqrt[3]{\frac{m_2}{Q_2}} : \sqrt[3]{\frac{m_3}{Q_3}} = 1 : \frac{1}{\sqrt[3]{2}} : \frac{1}{\sqrt[3]{3}}.$$

With such distance and charge-to-mass ratios, the three pearls move in such a way that at any moment the triangle they form is similar to their initial triangle, and (using the cosine law) the angles of that triangle are

$$\varphi_1 = \arccos\left(\frac{d_2^2 + d_3^2 - d_1^2}{2 d_2 d_3}\right) = 84.2°, \qquad \varphi_2 = 52.2°, \qquad \varphi_3 = 43.6°.$$

Note. Because of the similar forms of Coulomb's law and Newton's law of universal gravitational, the above solution can also be applied to three point masses moving in each other's gravitational fields. Their trajectories can be straight lines if the connection between their distances and their 'mass-to-gravitational-charge' ratios are in accord with equalities (3). As the 'mass-to-gravitational-charge' ratio is actually the inertial-mass-to-gravitational-mass ratio, which is the same for every body (for the sake of simplicity and practical common sense, this ratio is unity), rectilinear motion of the bodies, in the gravitational field of each other, can occur only if $d_1 = d_2 = d_3$, that is, the bodies are at the vertices of an equilateral triangle. Of course, the gravitational force is always attractive, and so the distance between the bodies decreases after they are released from rest.

We will consider the disk in polar coordinates along the plane of disk. Define the origin at the center of the disk and the z-axis to be normal to the disk. Consider an infinitesimal area element on the disk at a radial distance r and polar angle θ with sides $rd\theta$ and dr. The distance between this element and P is r' given by

$$r'^2 = h^2 + r^2.$$

Therefore, the magnitude of the electric field at P due to this element is

$$\frac{dq}{4\pi\varepsilon_0 r'^2} = \frac{\sigma r dr d\theta}{4\pi\varepsilon_0 (h^2 + r^2)}.$$

Due to the symmetry of the disk, the electric field at P can only be in the z-direction. Therefore, we simply need to integrate the z-component of the electric field at P due to all infinitesimal elements over the entire disk. Thus, the electric field at P is

$$\begin{aligned} E_z &= \int_0^R \int_0^{2\pi} \frac{\sigma r d\theta dr}{4\pi\varepsilon_0 (h^2 + r^2)} \cdot \frac{h}{\sqrt{h^2 + r^2}} \\ &= \int_0^R \frac{\sigma h r dr}{2\varepsilon_0 \left(h^2 + r^2\right)^{\frac{3}{2}}} \\ &= \left[-\frac{\sigma h}{2\varepsilon_0 \sqrt{h^2 + r^2}} \right]_0^R \\ &= \frac{\sigma}{2\varepsilon_0} - \frac{\sigma h}{2\sqrt{h^2 + R^2}}. \end{aligned}$$

In the next problem, slice the truncated cone into disks of infinitesimal thickness in the direction perpendicular to the axis of the cone. Now, we argue that these disks must make the same contribution to the electric field at the vertex of the original cone. Let the perpendicular distances between two of such disks to the vertex be h_1 and h_2. Observe that the electric field at the vertex due to one disk must be proportional to its charge and inversely proportional to the squared distance between it and the vertex. As compared to the first disk, the distance between the second disk and the vertex is $\frac{h_2}{h_1}$ times that of the first disk. However, its charge is also larger by a factor of $\frac{h_2^2}{h_1^2}$ due to the differences in area. The two effects cancel out — causing the contributions by the two disks to be equal. Therefore, the electric field

due to the entire truncated cone is equivalent to the disk at the top, with a surface charge density of ρl. The electric field at the vertex is then

$$E_z = \frac{\rho l}{2\varepsilon_0} - \frac{\rho l h}{2\sqrt{h^2 + R^2}}.$$

The image charge configuration in this case is that in Fig. 6.6, with an additional image charge $q_3 - q_2$ at the center of the shell where q_2 is the image charge in Fig. 6.6. This will ensure that the shell is an equipotential surface with total charge q_3. Then, the electric field in the region outside the shell due to this image charge configuration must be the correct solution, as guaranteed by the second uniqueness theorem. The force on q_1 at this instance is then

$$F = \frac{q_1(q_3 - q_2)}{4\pi\varepsilon_0 r^2} + \frac{q_1 q_2}{4\pi\varepsilon_0 (r-a)^2}.$$

Substituting $q_2 = -\frac{R}{r}q_1$ and $a = \frac{R^2}{r}$ that were derived previously,

$$F = \frac{q_1(rq_3 + Rq_1)}{4\pi\varepsilon_0 r^3} - \frac{q_1^2 Rr}{4\pi\varepsilon_0 (r^2 - R^2)^2}$$

in the direction away from the shell.

The potential differences across the two capacitors must be identical as they are connected by conducting wires in parallel. The system is then equal to two capacitors connected in parallel with a total charge q and equivalent capacitance $C_1 + C_2$. The energy loss is then

$$\Delta U = \frac{q^2}{2(C_1 + C_2)} - \frac{q^2}{2C_1} = -\frac{C_2 q^2}{2C_1(C_1 + C_2)}.$$

Suppose that the inner conductor carries a linear charge density λ while the outer conductor carries $-\lambda$. The electric field at a perpendicular distance r from the axis of the capacitor, $a \leq r \leq b$, is given by Gauss' law as

$$E = \frac{\lambda}{2\pi\varepsilon_0 r},$$

directed radially outwards from the inner cylinder. As λ is increased, the breakdown electric field will occur at $r = a$.

$$E_b = \frac{\lambda}{2\pi\varepsilon_0 a}.$$

The potential difference between the two conductors is

$$V = -\int_b^a E dr = \frac{\lambda}{2\pi\varepsilon_0} \ln\frac{b}{a} = E_b a \ln\frac{b}{a}.$$

The capacitance per unit length is

$$c = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}}.$$

The stored energy per unit length is thus

$$u = \frac{1}{2}cV^2 = \frac{1}{2} \cdot \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}} \cdot \left(E_b a \ln\frac{b}{a}\right)^2 = \pi\varepsilon_0 E_b^2 a^2 \ln\frac{b}{a}.$$

To maximize the energy per unit length,

$$\frac{du}{da} = 0$$

$$2a \ln\frac{b}{a} + a^2 \cdot -\frac{1}{a} = 0$$

$$\frac{b}{a} = \sqrt{e}$$

as $a \neq 0$. To maximize the potential difference between the two conductors,

$$\frac{dV}{da} = 0$$

$$\ln\frac{b}{a} + a \cdot -\frac{1}{a} = 0$$

$$\frac{b}{a} = e.$$

The electric field at a radial distance r from the center of the spherical shell, $r_1 \leq r \leq r_2$, is given by Gauss' law as

$$E = \frac{q}{4\pi\varepsilon_0 r^2}.$$

The potential difference between the inner and outer shells is given by

$$\Delta V = -\int_{r_2}^{r_1} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Since the potential energy stored in a capacitor is $U = \frac{1}{2}q\Delta V$, the current potential energy stored in the spherical shell is given by

$$U = \frac{1}{2}q\Delta V = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Integrating the energy density of the electric field would also yield the same result. In the next problem, observe that q amount of charge will be uniformly induced on the inner surface of the shell so that the electric field within the shell is zero. This leaves $-q$ amount of charge uniformly distributed on the outer surface. We can determine the external work done by determining the change in potential energy of the system. There are several methods to go about this. Firstly, we can sum the potential energy of the charge due to the two shells of charges and the potential energy between the shells of charges (which was the previous result). To compute the former, simply observe that the potential at the center of a spherical shell of charge Q and radius R is simply

$$\frac{Q}{4\pi\varepsilon_0 R},$$

as all charges are equidistant from the center. The potential at the center due to the two spherical shells is then

$$V_{center} = \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2}.$$

The potential energy of the charge due to the shells is then

$$U_{charge} = -qV_{center} = \frac{q^2}{4\pi\varepsilon_0 r_2} - \frac{q^2}{4\pi\varepsilon_0 r_1}.$$

The total potential energy of the initial set-up is then

$$\begin{aligned} U = U_{charge} + U_{shells} &= \frac{q^2}{4\pi\varepsilon_0 r_2} - \frac{q^2}{4\pi\varepsilon_0 r_1} + \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \\ &= \frac{q^2}{8\pi\varepsilon_0 r_2} - \frac{q^2}{8\pi\varepsilon_0 r_1}. \end{aligned}$$

The potential energy of the set-up after the charge has been extracted is zero. Therefore, the external work done is

$$W = \Delta U = 0 - U = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

There is another perspective to the change in potential energy involving the electric fields. The electric field of the initial set-up is that of a point charge, excluding the portion in the spherical shell of inner radius r_1 and outer radius r_2 while the electric field of the final set-up is simply that of a point charge. Therefore, the increase in potential energy is that carried by the field in the spherical shell — this is essentially the previous result. Thus,

$$W = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$