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Perspective

Independent domination in graphs: A survey and recent results

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ABSTRACT

A set S of vertices in a graph G is an independent dominating set of G if S is an independent set and every vertex not in S is adjacent to a vertex in S. In this paper, we offer a survey of selected recent results on independent domination in graphs.

1. Introduction

An independent dominating set in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. Independent dominating sets have been studied extensively in the literature. In this paper, we survey selected results on independent domination in graphs.

Dominating and independent dominating sets. A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum size of a dominating set.

A set is *independent* (or *stable*) if no two vertices in it are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G. The *independent domination number* of G, denoted by $i(G)$, is the minimum size of an independent dominating set. The *independence number* of G, denoted $\alpha(G)$, is the maximum size of an independent set in G. It follows immediately that $\gamma(G) \le i(G) \le \alpha(G)$.

A dominating set of G of size $\gamma(G)$ is called a γ-*set*, while an independent dominating set of G of size $i(G)$ is called an i-*set*.

History. The idea of an independent dominating set arose in chessboard problems. In 1862, de Jaenisch [30] posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. A graph G may be formed from an 8×8 chessboard by taking the squares as the vertices with two vertices adjacent if a queen situated on one square attacks the other square. The graph G is known as the *queens graph*. The minimum number of mutually non-attacking queens that attack all the squares of a chessboard is the independent domination number $i(G)$. For the queens graph G, we note that $\alpha(G) = 8$, $i(G) = 7$, and $\gamma(G) = 5$.

The theory of independent domination was formalized by Berge [6] and Ore [91] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [21,22].

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Notation. A graph G has vertex set $V(G)$, order $|V(G)|$, and edge set $E(G)$. For vertex v, the *open neighborhood* of v, denoted $N(v)$, is $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood of* v, denoted $N[v]$, is $\{v\} \cup N(v)$. For a set $S \subseteq V(G)$, its *open neighborhood*, denoted $N(S)$, is $\bigcup_{v \in S} N(v)$.

We denote the degree of v in G by $d(v)$. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). A vertex of degree 1 is a *leaf*, and a vertex of degree 0 is an *isolated vertex*.

We denote by C_n the cycle on n vertices, P_n the path on n vertices, K_n the complete graph on n vertices, and $K_{r,s}$ the complete bipartite graph with partite sets of size r and s.

For a set $S \subseteq V(G)$, *the subgraph induced by* S is denoted by $G[S]$. If G does not contain a graph F as an induced subgraph, then we say that G is F*-free*. In particular, we say that a graph is *claw-free* if it is $K_{1,3}$-free.

1.1. Common graphs and exact values

The independent domination number of some common graphs is given in Proposition 1.1:

Proposition 1.1. (a) *For the path and cycle,* $i(P_n) = i(C_n) = \lceil n/3 \rceil$.
(b) *For the complete bipartite graph,* $i(K_{r,s}) = \min(r, s)$.

Two graph operations occur frequently in the construction of extremal graphs. For r a positive integer, the *expansion* $\exp(G, r)$ of a graph G is the graph obtained from G by replacing each vertex v of G with an independent set I_v of size r and replacing each edge vw by a complete bipartite graph with partite sets I_v and I_w. The *corona* $\mathrm{cor}(G)$ (sometimes denoted $G \circ K_1$) is the graph obtained from G by adding a pendant edge at each vertex of G. More generally, the *generalized corona* $\mathrm{cor}(G, r)$ is the graph obtained from G by adding r pendant edges to each vertex of G. A tree is a *double-star* if it has diameter 3; in particular, let $S_{r,r} = \mathrm{cor}(P_2, r)$.

Proposition 1.2. (a) $i(\exp(G, r)) = r \cdot i(G)$.
(b) $i(\mathrm{cor}(G, r)) = r|V(G)| - (r - 1)\alpha(G)$.

Proof. (a) Consider any two nonadjacent vertices x and y in a graph; if $N(x) = N(y)$, then any independent dominating set contains either both x and y or neither of them. It follows that if D is an independent dominating set of $\exp(G, r)$, then for every vertex of G, D either contains all of I_v or none of I_v. Furthermore, $\{v : I_v \subseteq D\}$ is an independent dominating set of G.

(b) Let D be an independent dominating set of $\mathrm{cor}(G, r)$. For every vertex v of G, D contains either v or all r leaves adjacent to v. It follows that for D to be as small as possible, D must contain as many vertices of G as possible, namely a maximum independent set. □

In Section 2 we investigate fundamental bounds on $i(G)$. In Section 3 we explore the graphs where $i(G)$ equals $\gamma(G)$ or $\alpha(G)$. The value of $i(G)$ in regular graphs is the focus of Section 4. Thereafter we look at further bounds and graph families (Section 5), other results (Section 6), the complexity of the parameter (Section 7), and some generalizations and extensions (Section 8). (Some results not mentioned here can be found in the books by Haynes, Hedetniemi, and Slater [67,68].)

2. Bounds on the independent domination number

2.1. General bounds

The first result establishes a simple relationship between the independent domination number and the maximum degree of a graph, and was given by Berge [7].

Proposition 2.1 ([7])**.** *For a graph G with n vertices and maximum degree Δ,*

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \leq i(G) \leq n - \Delta.$$

The upper bound was improved by Blidia et al. [10]. Earlier, Bollobás and Cockayne [11] observed the following useful property of minimum dominating sets.

Observation 2.2 ([11])**.** *If G is an isolate-free graph, then there exists a γ-set D such that for every $v \in D$, there exists a vertex $u \in V(G) \setminus D$ such that $N[u] \cap D = \{v\}$ (called an external private neighbor).*

Using this observation, Bollobás and Cockayne [11] proved the following upper bound on the independent domination number.

Theorem 2.3 ([11])**.** *If G is an isolate-free graph on n, then* $i(G) \leq n + 2 - \gamma(G) - \lceil n/\gamma(G) \rceil$.

Proof. By Observation 2.2, there exists a γ-set D such that every vertex $v \in D$ has an external private neighbor. For each vertex $v \in D$, choose an external private neighbor v'. By the Pigeonhole Principle, there is a vertex $y \in D$ that is adjacent to at least $(n - |D|)/|D|$ vertices of $V(G) \setminus D$. Let D' be a maximal independent set containing y. Since $D' \cap N(y) = \emptyset$ and D' can contain at most one of x and x' for every vertex $x \in D \setminus \{y\}$, it follows that $|D'| \le n - (\gamma(G) - 1) - \lceil (n - \gamma(G))/\gamma(G) \rceil$. Since $i(G) \le |D'|$, the result follows. $\square$

Since the upper bound in Theorem 2.3 is maximized at $\gamma(G) = \sqrt{n}$, one immediately obtains the following bound, first noted by Favaron [39] (and also proved in [50]):

Theorem 2.4 ([39]). *If G is an isolate-free graph on n vertices, then $i(G) \le n + 2 - 2\sqrt{n}$.*

For examples of equality in the above theorem, take $G = \mathrm{cor}(K_m, m-1)$. Note that G has order m^2 and $i(G) = (m-1)^2 + 1 = n + 2 - 2\sqrt{n}$ (by Proposition 1.2). Brigham et al. [14] investigated the graphs that attain (the floor of) the bound in Theorem 2.4. In particular, they showed that if n is a square, then the generalized coronas given above are the only extremal graphs.

Consider now all graphs with n vertices and minimum degree at least δ. Favaron [39] proved an upper bound on $i(G)$ for $\delta \ge 2$, and she conjectured the extremal value as a function of n and δ. This conjecture was proved for $\delta = 2$ by Glebov and Kostochka [51] and in general by Sun and Wang [106]:

Theorem 2.5 ([106]). *If graph G of order n has minimum degree at least δ, then $i(G) \le n + 2\delta - 2\sqrt{\delta n}$.*

Earlier, Favaron [39] showed that for every positive integer δ, the bound in Theorem 2.5 is attained for infinitely many graphs. Haviland [62] improved the bound of Favaron when δ is large relative to the order and showed:

Theorem 2.6 ([62]). *Let G be a graph of order n with minimum degree at least δ. If $n/4 \le \delta \le 2n/5$, then $i(G) \le 2(n-\delta)/3$; if $2n/5 \le \delta \le n/2$, then $i(G) \le \delta$.*

2.2. Domination and independence

In his 1962 book, Berge [6] observed that an independent set is maximal independent if and only if it is dominating. Thus $i(G)$ equals the minimum size of a maximal independent set in G. He also observed that every maximal independent set in a graph G is a minimal dominating set of G. The *upper domination number* of G, denoted by $\Gamma(G)$, is the maximum size of a minimal dominating set. Hence we have the following inequalities:

Theorem 2.7 ([6]). *For every graph G, $\gamma(G) \le i(G) \le \alpha(G) \le \Gamma(G)$.*

Indeed, this is part of the canonical domination chain that was first observed by Cockayne et al. [24] in 1978. A detailed discussion on the domination chain can be found in Chapter 3 in [68]. That the bounds $\gamma(G) \le i(G) \le \alpha(G)$ are sharp may be seen by taking G to be the corona $\mathrm{cor}(H)$ of any graph H, which satisfies $\gamma(G) = i(G) = \alpha(G) = |V(H)|$. However, the difference between any two of these parameters can be make arbitrarily large. For example, for $r \ge 1$, if G is the double-star $S_{r,r}$, then $\gamma(G) = 2$, $i(G) = r + 1$ and $\alpha(G) = 2r$. These bounds are discussed further in Section 3.

The domination chain stated in Theorem 2.7 suggests the following question. Given integers s_1, s_2, s_3, s_4, does there exist a graph G for which $\gamma(G) = s_1$, $i(G) = s_2$, $\alpha(G) = s_3$, and $\Gamma(G) = s_4$? If such a graph G exists, then we call the sequence (s_1, s_2, s_3, s_4) a *domination sequence*. These were characterized by Cockayne and Mynhardt [25]:

Theorem 2.8 ([25]). *A sequence (s_1, s_2, s_3, s_4) of integers is a domination sequence if and only if the following three conditions hold:* (a) $1 \le s_1 \le s_2 \le s_3 \le s_4$; (b) $s_1 = 1$ *implies that* $s_2 = 1$; *and* (c) $s_3 = 1$ *implies that* $s_4 = 1$.

2.3. $K_{1,k}$-free graphs

Allan and Laskar [1] proved that the independence number is equal to the domination number for all claw-free graphs:

Theorem 2.9 ([1]). *If G is a claw-free graph, then $\gamma(G) = i(G)$.*

Proof. Let G be a claw-free graph. Among all γ-sets, choose D such that the graph $G[D]$ has the fewest edges. Suppose D is not independent. Then there exist vertices u and v in D that are adjacent. Let $P_v = \{w \in V(G) : N[w] \cap D = \{v\}\}$ be the private neighbors of v. By the minimality of D, the set P_v is nonempty. Since G is claw-free, the set P_v is a clique. Therefore for any $v' \in P_v$, the set $D' = (D \setminus \{v\}) \cup \{v'\}$ is a γ-set such that $G[D']$ has fewer edges that $G[D]$, a contradiction. $\square$

This result was extended by Bollobás and Cockayne [11]:

Theorem 2.10 ([11]). *For $k \ge 3$, if G is $K_{1,k}$-free, then $i(G) \le (k-2)\gamma(G) - (k-3)$.*

Zverovich and Zverovich [120] proved that the inequality in Theorem 2.10 is actually true for a wider class of graphs.

Theorem 2.11 ([120]). *For $k \ge 3$, if G does not contain two induced subgraphs isomorphic to $K_{1,k}$ having different centers and exactly one edge in common, then $i(G) \le (k-2)\gamma(G) - (k-3)$.*

2.4. Bipartite graphs

Since every bipartite graph is the union of two independent sets, each of which dominates the other, we have the following well-known bound on the independent domination number of a bipartite graph:

Proposition 2.12. *If G is a bipartite graph without isolated vertices on n vertices, then $i(G) \leq n/2$.*

That the bound in Proposition 2.12 is sharp may be seen by taking $K_{n/2,n/2}$. Actually, one can obtain extremal graphs of arbitrarily large minimum degree and diameter. For example, by Proposition 1.2, any expansion of a generalized corona of an even cycle has independent domination number equal to half its order.

Ma and Cheng [85] gave the following characterization of the connected bipartite graphs achieving equality in the bound of Proposition 2.12.

Theorem 2.13 ([85])**.** *If G is a connected bipartite graph on n vertices with partite sets X and Y, then $i(G) = n/2$ if and only if $|\{v \in X : N(v) \subseteq N(S)\}| \geq |N(S)|$ for every subset $S \subseteq X$.*

Proof. Let D be any independent dominating set, and let $T = Y \setminus D$. Then $D \cap X = |\{v \in X : N(v) \subseteq T\}|$ and $T = N(D \cap X)$. Thus, writing S for $D \cap X$, D has size less than $n/2$ if and only if $|\{v \in X : N(v) \subseteq N(S)\}| < |N(S)|$. $\square$

2.5. Trees

As a special case of Proposition 2.12, for every tree T we have $i(T) \leq n/2$, and this bound is sharp. Favaron [40] proved the following bound for trees, which was originally conjectured by McFall and Nowakowski [87].

Theorem 2.14 ([40])**.** *If T is a tree with n vertices and ℓ leaves, then $i(G) \leq (n + \ell)/3$.*

The bound in Theorem 2.14 is achieved, for example, by the path P_{3k+1} when $k \geq 1$. Other families achieving equality include the corona $T = \mathrm{cor}(P_{2k})$ of a path of order $2k$. Such a corona T has $n = 4k$ and $\ell = 2k$ with $i(T) = 2k = (n + \ell)/3$. The full list of extremal graphs is provided in [40]; indeed, the proof of Theorem 2.14 is an inductive proof that includes the characterization of the extremal graphs.

3. Graphs with $i(G) = \gamma(G)$ or $i(G) = \alpha(G)$

3.1. Graphs with $i(G) = \gamma(G)$

It remains an open problem to characterize the graphs G such that $i(G) = \gamma(G)$. A necessary and sufficient forbidden-subgraph list characterizing such graphs is impossible, since the addition of a new vertex adjacent to all vertices of a graph G produces a graph G' containing G as an induced subgraph with $\gamma(G') = i(G') = 1$.

The first result involving forbidden subgraphs that implies equality of the parameters γ and i was Theorem 2.9, which showed that every claw-free graph G, and hence every line graph, satisfies $i(G) = \gamma(G)$. Later, Topp and Volkmann [108] found 16 graphs F such that being F-free implies that $i(G) = \gamma(G)$. Several other properties of a graph also imply this condition. One such property is that the vertices of degree at least 3 form an independent set:

Proposition 3.1. *If G is a graph in which the vertices of degree at least 3 form an independent set, then $i(G) = \gamma(G)$.*

The proof of the proposition is similar to that of Theorem 2.9.

A special case of such graphs are graphs that have an efficient dominating set. An *efficient dominating set* (or *independent perfect dominating set*) in a graph is a set S such that $\{N[s] : s \in S\}$ is a partition of $V(G)$. Such graphs have been studied in [3,82] and elsewhere.

The class of trees T with $i(T) = \gamma(T)$ was first described by Harary and Livingston [58], but this description is rather complex. Cockayne et al. [19] characterized such trees in terms of the sets $\mathcal{A}(T)$ and $\mathcal{A}_i(T)$ of vertices of the tree T that are contained in all its γ-sets and i-sets, respectively. These sets were characterized by Mynhardt [88] using a tree-pruning procedure.

Another characterization of trees T with $i(T) = \gamma(T)$ was given by us in [32]. This approach uses graphs where every vertex has a label, and there is a set of operations that allow one to extend a graph. A similar constructive characterization of the trees such that every γ-set is an i-set was provided in [70]. Also, Ma and Cheng [85] gave a constructive characterization of trees with independent domination number one-half their order.

Fricke et al. [46] defined a graph to be *i-excellent* if every vertex belongs to some i-set. They observed that the set of *γ-excellent* trees (trees where every vertex is in some minimum dominating set) is properly contained in the set of i-excellent trees. For an example of an i-excellent tree that is not γ-excellent, take the double-star $S_{r,r}$ for $r \geq 2$. A constructive characterization of i-excellent trees is given in [69].

3.2. Domination-perfect graphs

Motivated by the concept of perfect graphs in the chromatic sense, Sumner and Moore [105] defined a graph G to be *domination perfect* if $\gamma(H) = i(H)$ for every induced subgraph H of G. Theorem 2.9 yields the following result.

Corollary 3.2. *Claw-free graphs are domination perfect.*

For example, line graphs are domination perfect. Sumner and Moore [105] established that it is not necessary to check every induced subgraph of a graph in order to determine if it is domination perfect.

Theorem 3.3 ([105]). *A graph is domination perfect if and only if $\gamma(H) = i(H)$ for every induced subgraph H of G with $\gamma(H) = 2$.*

Zverovich and Zverovich [120] offered a finite forbidden induced-subgraph characterization of domination-perfect graphs. Fulman [47] showed that this characterization is not correct, and he presented a sufficient condition for a graph to be domination perfect in terms of eight forbidden induced subgraphs. Topp and Volkmann [108] found thirteen graphs F such that being F-free implies that a graph is domination perfect. Subsequently, Zverovich and Zverovich [121] provided a forbidden induced-subgraph characterization of domination-perfect graphs in terms of seventeen forbidden induced subgraphs. Each of these seventeen graphs G is *minimally domination imperfect* in that it is not domination perfect, but $\gamma(H) = i(H)$ for every proper induced subgraph H of G.

Theorem 3.4 ([121]). *A graph is domination perfect if and only if it contains none of the seventeen graphs $G_1, \ldots, G_{17}$ shown in Fig. 1 as an induced subgraph.*

For a survey on domination perfect graphs, we refer the reader to [103].

3.3. Well-covered graphs

A graph is *well-covered* if the independent domination number is equal to the independence number. Equivalently, every maximal independent set is a maximum independent set of the graph. For example, the balanced complete bipartite graphs are well-covered. The concept of well-covered graphs was introduced by Plummer [95].
Ravindra [97] characterized the well-covered bipartite graphs.

Theorem 3.5 ([97]). *A connected bipartite graph G is well-covered if and only if it contains a perfect matching M such that for every edge $uv \in M$, $G[N[u] \cup N[v]]$ is a complete bipartite graph.*

As an immediate consequence of Theorem 3.5, we have a characterization of well-covered trees.

Corollary 3.6. *A tree is well-covered if and only if it is K_1 or the corona of a tree.*

Corollary 3.6 was extended by Finbow and Hartnell [41], who showed that a graph of girth at least 8 is well-covered if and only if it is the corona of a graph of girth at least 8. Later, Finbow et al. [42] characterized well-covered graphs of girth at least 5. A number of other classes of well-covered graphs have been completely described, including well-covered block graphs and unicyclic graphs [109], well-covered cubic graphs [16], well-covered graphs that contain neither 4- nor 5-cycles [43], and 4-connected claw-free well-covered graphs [60]. For a survey on well-covered graphs we refer the reader to Plummer [96].

4. Regular graphs

We saw earlier that, for any fixed minimum degree, there are graphs with $i(G) = n - o(n)$. However, if we require the graph to be regular, this is not the case. By double counting the edges joining an independent set and its complement, one obtains:

Observation 4.1 ([99]). *If G is a regular graph on n vertices with no isolated vertex, then $i(G) \le \alpha(G) \le n/2$.*

It is not hard to show that $i(G) = n/2$ only for the balanced complete bipartite graphs. Of course, $i(G) \ge n/(r+1)$ if G is r-regular (Proposition 2.1).

4.1. Cubic graphs

Upper bounds. The question of best possible bounds for cubic graphs remains unresolved. Lam et al. [80] gave a proof of the following:

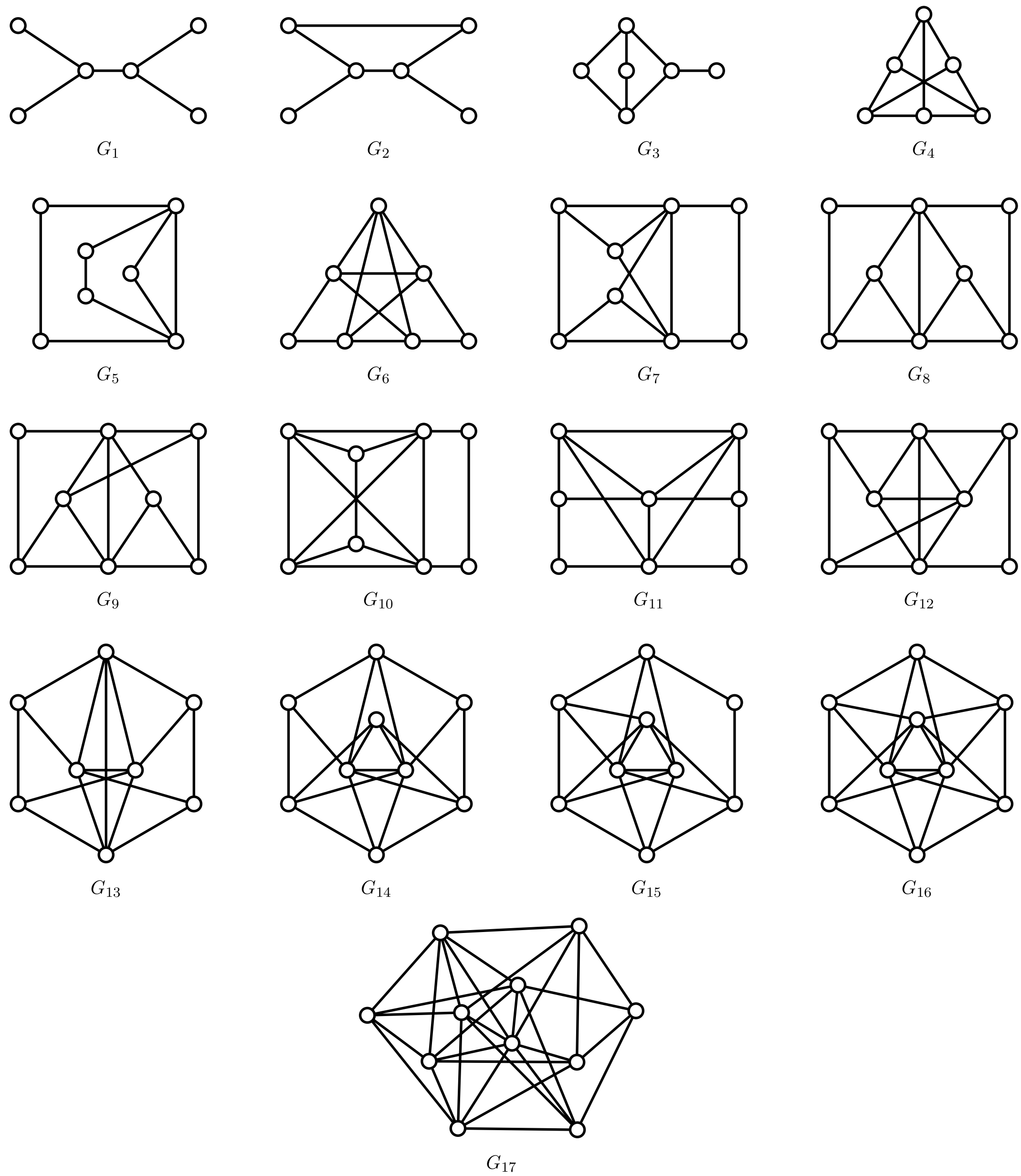

Fig. 1. Minimal domination imperfect graphs $G_1, \ldots, G_{17}$.

Theorem 4.2 ([80]). *If G is a connected cubic graph G on n vertices other than $K_{3,3}$, then $i(G) \leq 2n/5$.*

The proof is an intricate strong induction argument. Equality in Theorem 4.2 holds for the prism $C_5 \,\square\, K_2$, shown in Fig. 2. (Recall that for graphs G and H, the *Cartesian product* $G \,\square\, H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.)

We believe that this is the only such graph, and indeed we conjecture the following:

Conjecture 4.3. *If G is a connected cubic graph on n vertices other than $K_{3,3}$ and $C_5 \,\square\, K_2$, then $i(G) \leq 3n/8$.*

Two infinite families $\mathcal{G}_{\text{cubic}}$ and $\mathcal{H}_{\text{cubic}}$ of connected cubic graphs G with $i(G) = 3|V(G)|/8$ can be constructed as follows.

For $k \geq 1$, consider two copies of the path P_{4k} with respective vertex sequences $a_1 b_1 c_1 d_1 \cdots a_k b_k c_k d_k$ and $w_1 x_1 y_1 z_1 \cdots w_k x_k y_k z_k$. For each $1 \leq i \leq k$, join a_i to w_i, b_i to x_i, c_i to z_i, and d_i to y_i. To complete the construction of graphs in $\mathcal{G}_{\text{cubic}}$, join a_1 to d_k and w_1 to z_k.

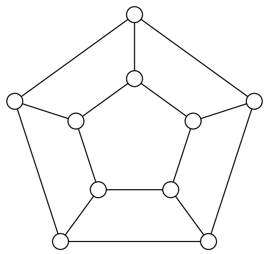

Fig. 2. The prism $C_5 \square K_2$.

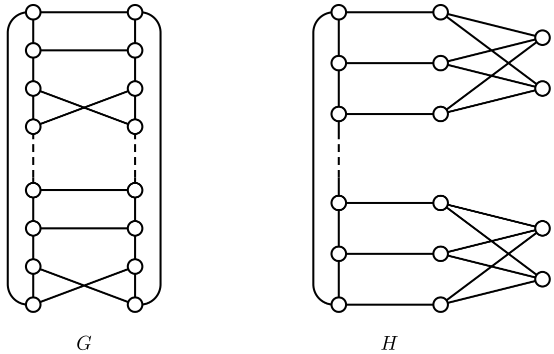

Fig. 3. Graphs $G \in \mathcal{G}_{\rm cubic}$ and $H \in \mathcal{H}_{\rm cubic}$ of order n with $i(G) = i(H) = 3n/8$.

For $\ell \geq 1$, consider a copy of the cycle $C_{3\ell}$ with vertex sequence $a_1b_1c_1 \cdots a_\ell b_\ell c_\ell a_1$. For each $1 \leq i \leq \ell$, add the vertices $\{w_i, x_i, y_i, z_i^1, z_i^2\}$, and join a_i to w_i, b_i to x_i, and c_i to y_i. To complete the construction of graphs in $\mathcal{H}_{\rm cubic}$, for each $1 \leq i \leq \ell$ and $j \in \{1, 2\}$, join z_i^j to each of the vertices w_i, x_i, and y_i.

Graphs in the families $\mathcal{G}_{\rm cubic}$ and $\mathcal{H}_{\rm cubic}$ are illustrated in Fig. 3.

Proposition 4.4 ([54]). *If $G \in \mathcal{G}_{\rm cubic} \cup \mathcal{H}_{\rm cubic}$ has order n, then $i(G) = 3n/8$.*

Perhaps even more than Conjecture 4.3 is true, in that the only extremal graphs are those in $\mathcal{G}_{\rm cubic} \cup \mathcal{H}_{\rm cubic}$. We have confirmed by computer search that this is true when $n \leq 20$.

Recall that a property $\mathcal{B}$ of a random graph holds *asymptotically almost surely* (a.a.s.) if the probability that $\mathcal{B}$ holds tends to 1 as n tends to infinity. Duckworth and Wormald [33] proved the following result.

Theorem 4.5 ([33]). *For a random cubic graph G on n vertices, $i(G)$ a.a.s. satisfies $0.2641n \leq i(G) \leq 0.27942n$.*

The upper bound in Theorem 4.5 was achieved by using differential equations to analyze the performance of a randomized greedy algorithm that is based on repeatedly choosing vertices of current minimum degree and deleting edges. The lower bound was calculated by means of a direct expectation argument.

γ versus i. We consider next the relationship between the independent and ordinary domination numbers in cubic graphs. In 1991, Barefoot et al. [4] gave a class of 2-connected cubic graphs for which the difference between i and γ is unbounded. However, they conjectured that for any 3-connected cubic graph the difference is at most 1. Their conjecture was disproved in multiple papers, including [26,78,100,119,122], which collectively showed that there are cubic graphs that are 3-connected with γ and i arbitrarily far apart.

Our family $\mathcal{G}_{\rm cubic}$ provides a simple example of a family of 3-connected cubic graphs with γ and i arbitrarily far apart, since it is not hard to show the following:

Proposition 4.6 ([54]). *If $G \in \mathcal{G}_{\rm cubic}$ has order n, then $\gamma(G) = \lceil 5n/16 \rceil$.*

As an immediate consequence of Propositions 4.4 and 4.6, if $G \in \mathcal{G}_{\rm cubic}$ has order n, then $i(G) - \gamma(G) = \lfloor n/16 \rfloor$. This suggests the following question:

Question 4.7. *Is it true that $i(G) - \gamma(G) \leq \lfloor n/16 \rfloor$ for any 3-connected cubic graph G of order at least 12?*

The ratio of the independence number to the domination number in a cubic graph cannot be too large, as is evident from the following result.

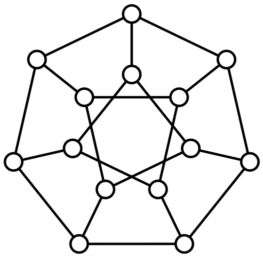

Fig. 4. The generalized Petersen graph G_{14}.

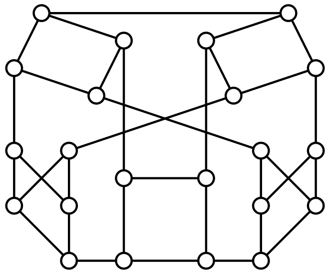

Fig. 5. The bipartite cubic graph G_{22} with $i(G_{22}) = 4n/11$.

Theorem 4.8 (*[54]*). *If G is a connected cubic graph, then $i(G)/\gamma(G) \leq 3/2$, with equality if and only if $G = K_{3,3}$.*

This has recently been improved in [102]:

Theorem 4.9 (*[102]*). *If G is a connected cubic graph other than $K_{3,3}$, then $i(G)/\gamma(G) \leq 4/3$.*

This bound is sharp because of $C_5 \,\square\, K_2$, but we [102] conjecture that this graph is the only graph where equality holds.

Girth constraints. Finally in this section, we consider some conjectures and open questions about cubic graphs of higher girth. The first conjecture is due to Verstraete [114].

Conjecture 4.10 (*[114]*). *If G is a connected cubic graph on n vertices with girth at least 6, then $i(G) \leq n/3$.*

We remark that the girth requirement in the above conjecture is essential, since the generalized Petersen graph G_{14} shown in Fig. 4 of order 14 has independent domination number 5.

Perhaps the graph G_{14} is the only exception when relaxing the girth condition in Conjecture 4.10 from 6 to 5. Indeed, we pose the following conjecture.

Conjecture 4.11. *If G is a connected bipartite cubic graph on n vertices other than $K_{3,3}$, then $i(G) \leq 4n/11$.*

We have confirmed by computer search that Conjecture 4.11 is true when $n \leq 26$ (see [54]). If Conjecture 4.11 is true, then the bound is achieved by the bipartite cubic graph G_{22} of order $n = 22$ with $i(G_{22}) = 8$ shown in Fig. 5.

4.2. Regular graphs of fixed regularity

Let c_r denote the supremum of $i(G)/n$ taken over all connected r-regular graphs G of order n except $K_{r,r}$. By Observation 4.1, it follows that $c_r \leq 1/2$. It is easy to see that $c_2 = 3/7$. We saw above that $c_3 = 2/5$. In general, the value c_r is somewhat nondecreasing as we now observe:

Lemma 4.12. *For all positive integers r and s, $c_{rs} \geq c_r$.*

Proof. The expansion of a graph has the same ratio of $i(G)/n$. $\square$

But it is unclear what happens in general. We pose the following questions.

Question 4.13. *Is it true that c_r tends to $1/2$ as $r \to \infty$?*

Question 4.14. (a) *Is it true that $c_4 = 3/7$?*
(b) *Is is true that if G is a connected 4-regular graph other than $K_{4,4}$, then $i(G)/\gamma(G) \leq 3/2$?*

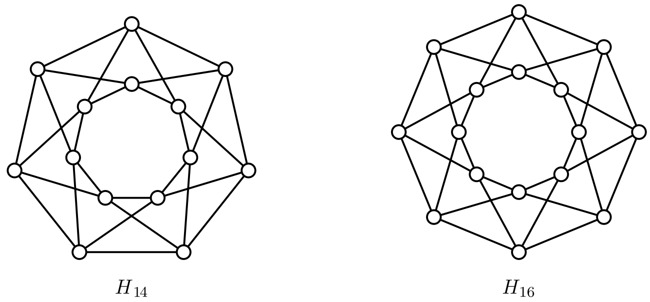

Fig. 6. The expansions H_{14} and H_{16}.

If part (a) is true, then the bound is achieved by the 4-regular graph H_{14} shown in Fig. 6. If part (b) is true, then the bound is achieved, for example, by the 4-regular graphs H_{14} and H_{16} shown in Fig. 6. We remark that both graphs have domination number 4 and independent domination number 6.

Duckworth and Wormald [34] determined lower and upper bounds on the size of a maximum independent set in random δ-regular graphs for each fixed $\delta \geq 3$. Harutyunyan et al. [61] studied independent dominating sets in graphs of girth at least 5 and proved the following result.

Theorem 4.15 (*[61]*). *There is a constant $c > 0$ such that for every d-regular graph G on n vertices of girth at least 5, it holds that $i(G) \leq n(\log d + c)/d$.*

Since the graph consisting of $n/(2d)$ disjoint copies of the complete bipartite graph $K_{d,d}$ has no independent dominating set of size less than $n/2$, the girth condition in Theorem 4.15 cannot be relaxed.

4.3. Regular graphs of large degree

Favaron [39] was the first to improve the upper bound of Observation 4.1 for large δ.

Theorem 4.16 (*[39]*). *If G is a δ-regular graph on n vertices with $\delta \geq n/2$, then $i(G) \leq n - \delta$, with equality only for complete multipartite graphs with vertex classes all of the same order.*

Haviland [62,63] improved the upper bound of Observation 4.1 for values of δ with $n/4 \leq \delta \leq n/2$. We remark that $(3 - \sqrt{5})/2 \approx 0.3820$.

Theorem 4.17 (*[62,63]*). *If G is a δ-regular graph on n vertices with $\delta \leq n/2$, then*

$$i(G) \leq \begin{cases} n - \sqrt{n\delta}, & \text{if } n/4 \leq \delta \leq (3-\sqrt{5})n/2, \\ \delta, & \text{if } (3-\sqrt{5})n/2 \leq \delta \leq n/2. \end{cases}$$

This bound was improved for $\delta \geq 2n/5$ in [54].

5. Further bounds and graph families

5.1. Chromatic number

MacGillivray and Seyffarth [84] established a sharp upper bound on the independence number of a graph in terms of the order of the graph and the chromatic number. We present here a short proof of their result using probabilistic methods.

Theorem 5.1 (*[84]*). *If G is a connected graph on n vertices with $\chi(G) = k \geq 3$, then $i(G) \leq (k-1)n/k - (k-2)$, and this bound is sharp.*

Proof. Consider any k-coloring of the graph G using colors $1, \ldots, k$. Construct a maximal independent set S in the following way. Choose a color at random, take all vertices of that color, and extend the color class to a maximal independent set. For a vertex v to be in S, it is necessary that none of its neighbors has the chosen color. The probability that v is chosen is therefore at most $(k - D(v))/k$, where $D(v)$ is the number of different colors in $N(v)$; that is,

$$\Pr(v \in S) \leq (k - D(v))/k.$$

Since the graph G is not $(k-1)$-colorable, we cannot eliminate any color. It follows that for each color c, there is a vertex v_c of color c that has neighbors of every other color. For each color c, select one such vertex v_c and let $X = \{v_1, v_2, \ldots, v_k\}$.

Hence for each $v \in X$, we have $D(v) = k - 1$ and $\Pr(v \in S) = 1/k$. For each $v \in V \setminus X$, we have $D(v) \geq 1$ and $\Pr(v \in S) \leq (k - D(v))/k \leq (k-1)/k$. We can therefore bound the expected size of S by

$$\begin{aligned}\mathbb{E}(|S|) &= \sum_{v \in X} \Pr(v \in S) + \sum_{v \in V \setminus X} \Pr(v \in S) \\ &\leq k\left(\frac{1}{k}\right) + (n-k)\left(\frac{k-1}{k}\right) \\ &= (k-1)n/k - (k-2).\end{aligned}$$

Thus there is a maximal independent set of at most this quantity.

That this bound is sharp may be seen for $k \geq 3$ by taking the generalized coronas $G = \mathrm{cor}(K_k, r)$ with $r \geq 1$. □

5.2. Planar graphs

Combining Theorem 5.1 with the Four Color Theorem, we have the following upper bound on the independent domination number of a planar graph in terms of its order.

Theorem 5.2 (*[84]*)**.** *If G is a planar graph on n vertices, then* $i(G) \leq 3n/4 - 2$.

This is best possible because of the graphs $\mathrm{cor}(K_4, r)$, where $r \geq 1$. Similar results can be obtained for graphs embedded on other surfaces.

MacGillivray and Seyffarth [84] showed that if we restrict attention to planar graphs of diameter 2, then the upper bound can be improved:

Theorem 5.3 (*[84]*)**.** *If G is a planar graph on n vertices with diameter* 2*, then* $i(G) \leq \lceil n/3 \rceil$.

The graphs achieving equality in the upper bound in Theorem 5.3 are also characterized in [84].

5.3. Triangle-free graphs

Haviland [64] considered triangle-free graphs. This was later extended by Goddard and Lyle [55] in the following theorem (where part (c) was also established in [101]):

Theorem 5.4 (*[55]*)**.** *Let G be a triangle-free graph on n vertices.*

(a) *There exist G with* $i(G) = n - o(n)$.
(b) *If* $\delta(G) \geq 3n/20$*, then* $i(G) \leq n/2$*, and this is sharp for* $\delta(G) \leq n/4$.
(c) *If* $\delta(G) \geq n/4$*, then* $i \leq \max(n - 2\delta(G), \delta(G))$*, and this is sharp.*

Equality in part (c) of Theorem 5.4 is obtained for graphs such as the following: take a path P_4 with vertex set v_1, v_2, v_3, v_4 and replace each v_i with an independent set A_i whose vertices have the same open neighborhood, where $|A_1| = |A_4| = n/2 - \delta$ and $|A_2| = |A_3| = \delta$.

In [55] we constructed triangle-free graphs G with $i(G) > n/2$ for all n and k such that n is a multiple of 5 and $0 < k = \delta(G) < n/10$ as follows. For a positive integer δ, let G_δ be obtained from the corona $\mathrm{cor}(C_5)$ of a 5-cycle by replacing each leaf by an independent set of size $n/5 - \delta$ and replacing each vertex of the 5-cycle by an independent set of size δ. We posed the following question.

Question 5.5 (*[55]*)**.** *Is it true that every triangle-free graph G on n vertices with* $\delta(G) \geq n/10$ *satisfies* $i(G) \leq n/2$*?*

5.4. Graphs of diameter 2

Recall that the maximum independent domination number over all isolate-free graphs with n vertices is $n + 2 - 2\sqrt{n}$ (Theorem 2.4). However, the extremal graphs, namely the generalized coronas $\mathrm{cor}(K_m, m-1)$ for $m \geq 2$, have diameter 3. The bound can be improved slightly if we consider only graphs of diameter 2.

Theorem 5.6. *If G is a graph of order n and diameter* 2*, then* $i(G) \leq n - 3 \cdot 2^{-2/3} n^{2/3} + o(n^{2/3})$*, and this is sharp.*

Proof. Let I be a maximal independent set of G. We are done if $|I| < n - 3n^{2/3}$, so assume otherwise.

Choose a subset X of $V(G) \setminus I$ in the following way. Let X_I be the set of vertices $x \in V(G) \setminus I$ such that there is no $y \in V(G) \setminus I$ with $N(x) \cap I \subsetneq N(y) \cap I$. Let $X \subseteq X_I$ be a largest subset of X_I such that no two vertices in X have identical neighborhoods in I; that is, for every pair of vertices x and x' in X_I such that $N(x) \cap I = N(x') \cap I$, keep only one representative for X.

Since the graph G has diameter 2, every two vertices in I have a common neighbor; by the construction of X, they must have a common neighbor in X. If we let $t = |X|$ and let d denote the maximum number of edges between a vertex of X and the set I, then

$$t\binom{d}{2} \geq \binom{|I|}{2} \geq \binom{n-3n^{2/3}}{2}. \tag{1}$$

At the same time, if we let x be a vertex in X with $|N(x) \cap I| = d$ and extend the independent set $\{x\} \cup (I \setminus N(x))$ to a maximal independent set I_x, then, by the construction of X, no vertex of $X \setminus \{x\}$ belongs to the set I_x. Hence, $i(G) \leq |I_x| \leq n - |N(x) \cap I| - |X \setminus \{x\}| = n - d - t + 1$.

If we minimize $d + t$ subject to constraint (1), we get $d \approx 2t \approx 2^{1/3}n^{2/3}$, and the desired upper bound follows.

To show that this bound is sharp, let $k \geq 2$ be an integer and consider a decomposition of K_{k^2} into $k(k+1)$ edge-disjoint copies of K_k. Such copies of K_k correspond to the lines of an affine plane of order k, which is known to exist when k is a power of a prime [113]. Then construct a graph H as follows: start with K_{k^2}, and for each copy F of K_k in the decomposition, add $2k$ vertices adjacent only to the vertices of F to form the graph H. Note that H has order $n = k^2 + 2k \cdot k(k+1) = 2k^3 + 3k^2$. Each vertex of the clique K_{k^2} is in $k+1$ copies of K_k, and so has $2k(k+1)$ neighbors outside the clique. It follows that $i(H) = n - 3k^2 - 2k + 1$. $\square$

5.5. Chessboard graphs

Yaglom and Yaglom [116] determined the number of a given chess piece needed to attack the whole board; in another words, the domination number of the graph constructed by taking each square as a vertex and joining two vertices if a piece sitting on one square attacks the other. Their solutions for the case of rooks, bishops and kings have the pieces nonattacking. Since the rooks, bishops and kings graphs are claw-free, it follows that their domination numbers and independent domination numbers are equal.

Theorem 5.7 (*[116]*).

(a) *$i(R_n) = n$ for the rooks graph R_n on the $n \times n$ board.*
(b) *$i(B_n) = n$ for the bishops graph B_n on the $n \times n$ board.*
(c) *$i(K_n) = (\lceil n/3 \rceil)^2$ for the kings graph K_n on the $n \times n$ board.*

Several papers have provided upper bounds for the queens graph. The current best asymptotic upper bound is due to Östergård and Weakley [92]. The best lower bound is the same as the one for domination, given by Finozhenok and Weakley [45].

Theorem 5.8 (*[45,92]*). *For the queens graph Q_n on the $n \times n$ board:*

(a) *$i(Q_n) \geq n/2$, except for $n \in \{3, 11\}$ when $i(Q_n) = (n-1)/2$.*
(b) *$i(Q_n) \leq 61n/111 + O(1)$.*

For $n \leq 10$, the exact value of $i(N_n)$ for the knights graph N_n on the $n \times n$ board was found by computer search in [72]: $i(N_2) = i(N_3) = i(N_4) = 4$, $i(N_5) = 5$, $i(N_6) = 8$, $i(N_7) = 13$, $i(N_8) = i(N_9) = 14$, and $i(N_{10}) = 16$. However, there is very little known about independent dominating sets in the knights graph in general. In particular, no good upper bounds are known for $i(N_n)$ for large n.

5.6. Product graphs

The following conjecture was made by Vizing in 1968, after being posed by him as a problem in [115].

Vizing's Conjecture. *For every pair of graphs G and H, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.*

Vizing's Conjecture remains one of the major open problems in domination theory. A recent survey and results on Vizing's Conjecture can be found in [13]. Here we mention a similar problem for independent domination.

The analogous conjecture for independent domination number is false since there are nontrivial graphs G and H such that $i(G \square H) < i(G)i(H)$. Further, there are graphs with $\gamma(G \square H) < i(G)\gamma(H)$ and $i(G \square H) < i(G)\gamma(H)$. Nevertheless, there is a related conjecture given in [13].

Conjecture 5.9. *For all graphs G and H, $\gamma(G \square H) \geq \min\{i(G)\gamma(H), i(H)\gamma(G)\}$.*

The truth of Conjecture 5.9 would imply Vizing's conjecture. On the other hand, the conjecture that $i(G \square H) \geq \gamma(G)\gamma(H)$ for all graphs G and H is a consequence of Vizing's Conjecture. Perhaps this could be proven without first proving Vizing's Conjecture.

For specific cartesian products, Harary and Livingston [59] conjectured that $i(Q_n) = \gamma(Q_n)$ for all cubes except Q_5. This is known to be true for Q_n in the special case when $n = 2^k - 1$, since such cubes have efficient dominating sets [57].

Other graph products have been considered. For example, some are investigated by Nowakowski and Rall [90].

6. Other results

Nordhaus–Gaddum bounds. Nordhaus–Gaddum type bounds on the sum of the independent domination numbers of a graph and its complement are easy to establish.

Proposition 6.1. *If G is a graph of order $n \geq 2$, then $3 \leq i(G) + i(\overline{G}) \leq n + 1$.*

The lower bound follows immediately from the observation that if $i(G) = 1$ or $i(\overline{G}) = 1$, then $i(\overline{G}) \geq 2$ or $i(G) \geq 2$, respectively. That this lower bound is sharp may be seen by considering the graph $G = K_{1,n-1}$ with $i(G) = 1$ and $i(\overline{G}) = 2$. Applying the upper bound of $i(G) \leq n - \Delta(G)$, we have $i(G) + i(\overline{G}) \leq 2n - (\Delta(G) + \Delta(\overline{G})) \leq 2n - (\Delta(G) + \delta(\overline{G})) = 2n - (n-1) = n + 1$. That this bound is sharp may be seen by taking $G = K_n$ or $\overline{G} = K_n$.

If one does not allow isolates in either G or $\overline{G}$, then the upper bound in Proposition 6.1 can be improved as shown by us in [53].

Theorem 6.2 *([53]). If G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ has an isolated vertex, then $i(G) + i(\overline{G}) \leq n + 4 - \lfloor 2\sqrt{n} \rfloor$, and this is sharp.*

Trivially, if G is a graph of order $n \geq 2$, then $2 \leq i(G)i(\overline{G})$. Finding sharp upper bounds on the product of the independent domination numbers of a graph and its complement proved to be more challenging. This problem was studied by several authors, including [18,20], culminating in our result from [53]:

Theorem 6.3 *([53]). Let $b(n) = \lfloor (n+4)/4 \rfloor \, \lfloor (n+6)/4 \rfloor$. Then, for all graphs G of order n,*

$$i(G)i(\overline{G}) \leq \begin{cases} n & \text{if } n \leq 7, \\ b(n) + 1 & \text{if } n = x^2 \text{ for } x \text{ odd, or } n = x^2 - 1 \text{ for } x \text{ even}, \\ b(n) & \text{otherwise}, \end{cases}$$

and this is best possible for all n.

The proof is based on the result by Entringer et al. [37] that gives the minimum order of a graph in which every vertex is in both a clique and an independent set of specified size.

Haviland [65] asked about the maximum value of the product of $i(G)$ and $i(\overline{G})$ for regular graphs, and wondered whether this product is always $o(n^2)$. Recently, we [54] showed that it is possible to construct regular graphs such that the product is $\Theta(n^2)$. The best bound we have is:

Proposition 6.4 *([54]). There exists a family of regular graphs G of order n such that $i(G) \cdot i(\overline{G}) \geq n^2/169 - O(n)$.*

As regards the sum of the parameters, we conjecture:

Conjecture 6.5 *([54]). For a regular graph G on n vertices that is neither complete nor empty,*

$$i(G) + i(\overline{G}) \leq n/2 + 2.$$

i-critical graphs. A graph G is *i-critical* if $i(G) < i(G - v)$ for every vertex v. These graphs were first considered by Ao (see [2]). Recently, Edwards and MacGillivray [36] calculated the maximum diameter of an i-critical graph:

Theorem 6.6 *([36]). The diameter of an i-critical graph G is at most $2(i(G) - 1)$, and this is sharp.*

Domination edge-critical graphs. A noncomplete graph G is *k-domination-edge-critical* if $\gamma(G) = k$ and $\gamma(G + e) < k$ for every edge $e \in E(\overline{G})$. In 1983, Sumner and Blitch [104] conjectured that if G is a k-domination-edge-critical graph for $k \geq 3$, then $\gamma(G) = i(G)$. This conjecture became a major outstanding conjecture in domination theory for a while, as a great deal of heuristic and computer-generated data supported it. However, Ao et al. [2] gave counterexamples for all $k \geq 4$. Moreover, in 1999 van der Merwe [111,112] provided an elegant construction that gives for each s such that $s \geq 3$ a connected 3-domination-edge-critical graph G with $i(G) = s$.

Domination bicritical graphs. A graph G is *domination bicritical* if the removal of any pair of vertices decreases the domination number. In 2005, Brigham et al. [15] posed the following question: Is it true that if G is a connected domination bicritical graph, then $\gamma(G) = i(G)$? This question has yet to be settled.

Idomatic number and fall colorings. The *idomatic number* of a graph is the maximum number of disjoint independent dominating sets in the graph. The terminology is due to Zelinka [118], but the parameter was introduced by Cockayne and Hedetniemi [23]. Payan [93] showed that it is not true that every regular graph has idomatic number more than 1, thus refuting a conjecture of Berge.

A *fall coloring* of a graph, also called an *idomatic partition*, is a partition of the vertices into independent dominating sets. That is, it is a proper coloring such that every vertex has every color in its open neighborhood. As Cockayne and Hedetniemi [22] observed, not every graph has a fall coloring (for example, C_5 does not).

For graphs G and H, the *direct product* $G \times H$ (which also goes by other names) is the graph with vertex set $V(G) \times V(H)$, where two vertices (x, y) and (v, w) are adjacent if and only if $xv \in E(G)$ and $yw \in E(H)$. Dunbar et al. [35] considered the direct product of two complete graphs and observed that this has a fall coloring only when the number of colors is equal to the order of one of the complete graphs. The question for the direct product of three of more complete graphs was considered by Valencia-Pabon [110] and Klavžar and Mekiš [77].

Laskar and Lyle [81] considered fall colorings of cubes. For example, they showed that no hypercube has a fall 3-coloring, but for all $k \geq 4$ (and of course $k = 2$) all sufficiently large hypercubes have a fall k-coloring. Earlier, Lyle et al. [83] showed that:

Proposition 6.7 ([83]). *A strongly chordal graph G has a fall coloring if and only if $\omega(G) = \delta(G) + 1$, where $\omega(G)$ is the clique number of G.*

7. Complexity questions

In this section we consider the complexity of determining the independent domination number of a graph. In general, the parameter is NP-complete, as shown in the book by Garey and Johnson [48]. The problem remains NP-complete when restricted to some common families of graphs:

Theorem 7.1 ([27,117,76,17,86]). *The problem of determining whether $i(G) \leq k$ for input G and k is NP-complete even when G is restricted to bipartite graphs, to line graphs, to circle graphs, to unit disk graphs, or to planar cubic graphs.*

Irving [75] showed that unless $P = \mathrm{NP}$, there is no polynomial-time algorithm to approximate the independent domination number within a constant factor. Heggernes and Telle [73] showed that it is NP-hard to determine whether a graph can be partitioned into k independent dominating sets (that is, has a fall coloring with k colors) for any fixed $k \geq 3$. Henning et al. [74] showed that it is NP-complete to decide whether a given graph has two disjoint independent dominating sets.

It is straightforward to calculate the independent domination number of a tree in linear time, first observed in [8]. This was slightly generalized in [94], and a polynomial-time algorithm for graphs of bounded treewidth was given by Telle and Proskurowski [107]. Farber [38] showed that there is a linear-time algorithm to determine the independent domination number of chordal graphs. Kratsch and Stewart [79] gave a polynomial-time algorithm for cocomparability graphs.

8. Generalizations

There have been several generalizations of the concept of independent domination. We mention just a few here.

Borowiecki et al. [12] considered a graph property $\mathcal{P}$ and defined $i_{\mathcal{P}}(G)$ of a graph G to be the minimum size of a maximal subset of $V(G)$ with property $\mathcal{P}$. The original i corresponds to the property $\mathcal{P}$ of having no edges. This parameter was considered by Hedetniemi et al. [71] for $\mathcal{P}$ the property of being acyclic, and by Haynes et al. [66] for the property of being H-free for some graph H.

In particular, we discuss the case when $\mathcal{P}$ is the property of having maximum degree at most k, as considered by Fink and Jacobson [44]. They defined a subset S of $V(G)$ to be *k-independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. A k-independent set S of G is *maximal* if for every vertex $v \in V(G) \setminus S$, the set $S \cup \{v\}$ is not k-independent. The *lower k-independence number* $i_k(G)$ is the minimum size of a maximal k-independent set in G. Hence, $i_1(G) = i(G)$. Blidia et al. [9] established the following relationships between two lower independence parameters.

Theorem 8.1 ([9]). *For every graph G and integers j and k with $1 \leq j \leq k$, $i_{k+1}(G) \leq (k - j + 2)i_j(G)$. Equality can occur only when $j = 1$ or $j = k$.*

As a consequence of Theorem 8.1, we have $i_{k+1}(G) \leq (k + 1)i(G)$.

There have also been generalizations based on the alternative formulation that $i(G)$ is the minimum size of a dominating set with property $\mathcal{P}$. For example, Goddard et al. [52] showed that Theorem 2.4 generalizes for a large class of properties.

There have been generalizations where the conditions are strengthened. For example, an *independent 2-dominating set* is a set S such that S is independent and every vertex not in S is adjacent to at least two vertices in S (see Haynes et al. [66]). Such a set does not always exists; consider for example the cycle C_5. Indeed, Croitoru and Suditu [28] (who called this a *perfect stable*) showed that it is NP-hard to determine whether a graph has such a set. In a different direction, Mynhardt [89] considered the case where one looks for the minimum size of a *k-maximal independent set*, where k-maximal means one cannot obtain a larger independent set by removing less than k vertices and then adding k vertices.

There have also been generalizatons based on distance. For $s \geq r \geq 1$, Beineke and Henning [5] defined a set S of vertices to be an *(r, s)-set* if no two vertices of S are within distance r of each other and every vertex in G is within distance s from some vertex of S. They defined $i_{r,s}(G)$ to be the minimum size of such a set. The parameter i corresponds to $i_{1,1}$. They showed that $i_{1,s}(G) \leq n/s$ provided $n \geq s$. They conjectured that $i_{1,s}(T) \leq n/(s + 1)$ when T is a tree, and proved this for $s \leq 3$. Later, Gimbel and Henning [49] extended the result of Theorem 2.4 to distance independent domination:

Theorem 8.2 ([49]). *For $k \geq 1$, if G is a connected graph of on $n \geq k+1$ vertices, then $i_{1,k}(G) \leq (n+k+1-2\sqrt{n})/k$, and this bound is sharp.*

There is also a fractional version of independent domination. A *fractional independent set* is an assignment f of nonnegative reals to the vertices such that for each edge uv the sum $f(u)+f(v)$ is at most 1, a *fractional dominating set* is an assignment f such that for each vertex v the sum $\sum_{w\in N[v]} f(w)$ is at least 1, and a *maximal fractional independent set* is a fractional independent set that is also a fractional dominating set. Fractional domination in graphs was studied, for example, by Domke et al. [31] and by Grinstead and Slater [56], while fractional independent sets were studied by Kumar et al. [98].

Dahme et al. [29] considered an extension based on real numbers. For real number c with $0 < c < 1$, they defined $i_c(G)$ to be the minimum size of a set S such that $|N(v) \cap S| \geq c \cdot d(v)$ for all $v \in V(G) \setminus D$ and $|N(v) \cap S| \leq c \cdot d(v)$ for all $v \in D$. They showed that such a set always exists.

9. Conclusion

In this paper we surveyed selected results on independent dominating sets in graphs. These results establish key relationships between the independent domination number and other parameters, including the domination number, the independence number, and the chromatic number. Further, these results establish optimal upper bounds on the independent domination number in terms of the order itself, the order and the maximum degree, and the order and the minimum degree. Structural results on domination-perfect graphs were presented, as were results on the independent domination number in various families of graphs, including planar graphs, triangle-free graphs, and graphs with restricted diameter. The complexity questions associated with the independent domination number were also discussed.

We recall here several interesting open problems and conjectures on the independent domination number. Several questions on regular graphs are attractive and worth investigating: in particular, the conjecture that $i(G) \leq 3n/8$ for every connected cubic graph of order more than 10, the upper bounds for connected 4-regular graphs, the general behavior of the maximum ratio of the independent domination number to the domination number, and the conjecture that $i(G) \leq n/3$ for every connected cubic graph with girth at least 6. There are also several intriguing open questions such as whether every triangle-free graph with $\delta(G) \geq n/10$ satisfies $i(G) \leq n/2$, which if answered would shed more light on the complexity of the independent domination number.

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