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Induced-Paired Dominating Sets in Lexicographic Product of Graphs

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V(G)$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A dominating set $S \subseteq V(G)$ is an induced-paired dominating set if every component of the induced subgraph $G[S]$ is a K_2. The minimum cardinality of an induced-paired dominating set of G is the induced-paired domination number $\gamma^{ip}(G)$. We show that if two graphs G and H do not admit any induced-paired dominating set, then their lexicographic product, $G \circ H$, does not admit either. However, if any of the graphs G or H admits any induced-paired dominating set, then $G \circ H$ admits. In addition, we show bounds and characterization for $\gamma^{ip}(G \circ H)$.

1 Introduction

Consider the situation where a graph G models a facility or a multiprocessor network with limited-range detection devices (sensing for example, movement, heat or size) placed at chosen vertices of G. The purpose of

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these devices is to detect and precisely identify the location of an intruder such as a thief, saboteur, vandal or fire that may suddenly be present at some vertex.

As it is costly to install and maintain detection devices, it is logical to determine the locations of the minimum number of devices that can, among them, precisely determine an intruder at any location. Sometimes such a device can determine if an intruder is in its neighborhood but cannot detect if the intruder is at its own location. In this case, it is required to find an *open-dominating* set. In addition, if it is necessary to minimize redundancy, each device is required to have at most one other device in its neighborhood. In this case, it is required to find an *open-independent* set. For this definitions, see notation and terminology in the end of this section.

We consider finite, simple, and undirected graphs. For a graph G, the vertex set is denoted $V(G)$ and the edge set $E(G)$. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. A dominating set S is a *paired dominating set* if the subgraph induced by S has a perfect matching. A dominating set $S \subseteq V(G)$ is an *induced-paired dominating set* if every component of the subgraph induced by S is a complete graph with 2 vertices. Paired dominating sets were introduced by Haynes and Slater [3, 4].

Every graph without isolated vertices has a paired dominating set, but not all these graphs have an induced-paired dominating set. The cycle graph on five vertices, C_5, has a paired dominating set, formed by four vertices that induce a P_4, but has no induced-paired dominating set. If an induced-paired dominating set exists in a given graph G, it is often of interest to identify the set of minimum size among such sets in G, which is denoted by $\gamma^{ip}(G)$.

The decision problem associated with induced-paired domination is NP-complete even when restricted to bipartite graphs and, for general graphs, there are bounds on $\gamma^{ip}(G)$ [2]. Also in [2] a characterization of those triples (a, b, c) of positive integers $a \leq b \leq c$ for which a graph has domination number a, paired-domination number b, and induced-paired domination number c is given and the authors characterize the cycles and trees that have induced-paired dominating sets.

In this paper we analyze the existence of induced-paired domination sets in the lexicographic products of two graphs. Consider G and H two graphs with vertex sets $V(G) = \{g_1, g_2, ..., g_n\}$ and $V(H) = \{h_1, h_2, ..., h_m\}$. The *lexicographic product* of G and H, denoted by $G \circ H$, is the graph with vertex set $V(G \circ H) = V(G) \times V(H)$ and edge set $E(G \circ H) = \{(g_i, h_j)(g_k, h_l) \mid g_i g_k \in E(G), \text{ or } g_i = g_k \text{ and } h_j h_l \in E(H)\}$. See Figure 1 for an example. There are many results about dominating sets and their variations in lexicographic product of two graphs [1, 5, 6, 7].

Our contributions. It appears that the decision problem associated with induced-paired domination is difficult as we mentioned that it remains $\mathcal{NP}$-complete even for just bipartite graphs. However, we study some graph classes for which the induced-paired dominating number is bounded and also present characterizations of $\gamma^{ip}(G \circ H)$ considering cases when G or H or both of them admit induced-paired domination sets.

Notation and terminology. For a graph G, the *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. A subset $R \subseteq V(G)$ is *independent* if no two vertices in R are adjacent, i.e, for every $v \in R$, $|N[v] \cap R| \leqslant 1$. A subset $S \subseteq V(G)$ is *open-independent* if for every $v \in S$, $|N(v) \cap S| \leqslant 1$. A subset $S \subseteq V(G)$ is *open-dominating* if for every $v \in V(G)$, $|N(v) \cap S| \geqslant 1$. A induced-paired dominating set is a set that is both an open-independent set and an open-dominating set, then, a set $S \subseteq V(G)$ is an induced-paired dominating set if for every $v \in S$, $|N(v) \cap S| \leqslant 1$ and for every $v \in V(G)$, $|N(v) \cap S| \geqslant 1$.

2 Results

Our first result collect some basics results on the induced-paired dominating number in some particular classes of graphs.

Proposition 2.1. *Let G be a graph. If G has an edge uv such that $N[u] \cup N[v] = V(G)$, then $\gamma^{ip}(G) = 2$.*

Proof. First, set $S = \{u, v\}$ such that uv is an edge of G and $N[u] \cup N[v] = V(G)$. Note that $|N(u) \cap S| \leqslant 1$, $|N(v) \cap S| \leqslant 1$ and for every $w \in V(G)$, $|N(w) \cap S| \geqslant 1$. So, S is an induced-paired dominating set of G. ■

Observe that complete graphs and complete bipartite graphs are examples of graphs which satisfy the statement of Proposition 2.1 and so the induced-paired domination number for these classes is 2.

For the next results we consider the set $X_i = \{(g_i, h_j) \in V(G \circ H) : 1 \leqslant j \leqslant m\}$ and the component G_i, defined as the subgraph of $G \circ H$ induced by X_i, i.e $G_i = (G \circ H)[X_i]$. Figure 1 shows a graph $G \circ H$ and its representation with the components $X_i's$. The dashed edges represent that there is an edge between all the vertices of the respective components.

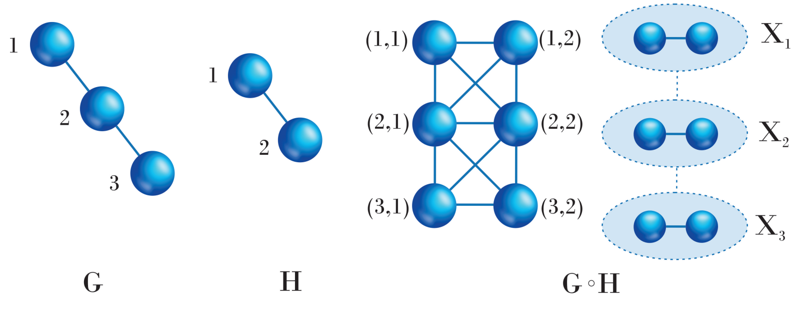

Figure 1: Example of graphs G and H, the resulting lexicographic product and the representation of the components $X_i's$.

In the next two lemmas, we check some properties of induced-paired dominating sets in $G \circ H$.

Lemma 2.1. *Consider D an induced-paired dominating set of $G \circ H$ such that $|D \cap X_i| \leqslant 1$ for every $1 \leqslant i \leqslant n$. Then, G admits an induced-paired dominating set D' such that $|D'| = |D|$.*

Proof. Let D be an induced-paired dominating set in $G \circ H$ as described. Consider the set $D' = \{g_i \mid (g_i, h_j) \in D\}$. Note that $|D'| = |D|$. We will show that D' is an induced-paired dominating set in G. Since that D is an open dominating set, we have that $|N((g_k, h_l)) \cap D| \geqslant 1$, for every $(g_k, h_l) \in V(G \circ H)$. Then, by construction of D', $|N(g_k) \cap D'| \geqslant 1$ for every $g_k \in V(G)$, implying that D' is an open-dominating set of G. By hypothesis, D is an open-independent set. Then $|N((g_k, h_l)) \cap D| \leqslant 1$, for every $(g_k, h_l) \in D$. Again, by construction of D', $|N(g_k) \cap D'| \leqslant 1$ for every $g_k \in D'$ and D' is an open-independent set in G. Therefore, G admits an induced-paired dominating set D' such that $|D'| = |D|$. ■

Lemma 2.2. *Let G and H be graphs such that both G and $G \circ H$ admit induced-paired dominating sets. Then, $G \circ H$ has an induced-paired dominating set D such that $\gamma^{ip}(G \circ H) = |D|$ and $|D \cap X_i| \leqslant 1$ for every $1 \leqslant i \leqslant n$.*

Proof. Consider G and $G \circ H$ as described and let D be an induced-paired dominating set of $G \circ H$ such that $\gamma^{ip}(G \circ H) = |D|$. Consider the graph G' with vertex set $V(G') = \{G_1, G_2, \ldots, G_n\}$ and edge set $E(G') = \{(G_i, G_j) | ((g_i, h_k)(g_j, h_k)) \in E(G \circ H)\}$ and set $D' = \{G_i | X_i \cap D = \emptyset\}$. Suppose, by contradiction, that every induced-paired dominating set D of $G \circ H$, such that $\gamma^{ip}(G \circ H) = |D|$, has at least one component X_i such that $|X_i \cap D| > 1$. Then, for every neighbour g_k of g_i in G, we must have $D \cap X_k = \emptyset$, otherwise D would not be an open-independent set. Therefore, $G'[D']$ has at least a isolated vertex, implying that G' does not admit any induced-paired dominating set. This implies that G does not admit any induced-paired dominating set, since that G is isomorphic to G'. A contradiction. ■

The lexicographic product $C_5 \circ C_5$ does not admit any induced-paired dominating set. We show that when G and H do not admit induced-paired dominating sets then the lexicographic product $G \circ H$ also do not admit any induced-paired dominating set.

Theorem 2.1. *Let G and H be graphs such that both do not admit any induced-paired dominating set. Then $G \circ H$ also does not admit any induced-paired dominating set.*

Proof. Let G be a graph with n vertices and H any graph that does not admit any induced-paired dominating set. Suppose, by contradiction, that $G \circ H$ admits an induced-paired dominating set D. If $|D \cap X_i| \leqslant 1$, for $1 \leqslant i \leqslant n$ then, by Lemma 2.1, G admits an induced-paired dominating set D' such that $|D'| = |D|$, a contradiction.

Consider that $|D \cap X_i| > 1$ for some $i \in \{1, 2, \cdots, n\}$. In this case, consider $D' = \{h_j \mid (g_i, h_j) \in D\}$ and $D(X_i) = D \cap X_i$. Note that $D' \subseteq V(H)$. If there is a vertex $h_k \in V(H)$ such that $|N(h_k) \cap D'| > 1$, then, since the subgraph induced by X_j is isomorphic to H, we have that $|N(g_i, h_k) \cap D(X_i)| > 1$. Thus, D' is an open-independent set in H. Moreover, since that $D \cap X_l = \emptyset$, for every l such that $g_l \in N_G(g_i)$, the vertices in X_i are open-dominated only by vertices in $D(X_i)$. Implying that D' is an open-dominating set in H. So, H admits an induced-paired dominating set, a contradiction.

Thus, we can conclude that $G \circ H$ does not admit any induced-paired dominating set. ■

Now, we analyze the case where one of the graphs, G and H, has an induced-paired dominating set. First, we will consider that the graph G admits and we will present a characterization showing that $\gamma^{ip}(G \circ H) = \gamma^{ip}(G)$.

Theorem 2.2. *Let $G \circ H$ be a lexicographic product of two graphs. If G admits some induced-paired dominating set, then $\gamma^{ip}(G \circ H) = \gamma^{ip}(G)$.*

Proof. Let G be a graph with n vertices and D an induced-paired dominating set of G such that $|D| = \gamma^{ip}(G)$.

Consider $V(G) = \{g_1, g_2, \cdots, g_n\}$ and $V(H) = \{h_1, h_2, \cdots, h_m\}$. Construct the set $D' = \{(g_i, h_1) \in V(G_i) | g_i \in D\}$. We will show that D' is an induced-paired dominating set in $G \circ H$. Since that D is an induced-paired dominating set of G, we have that $|N(g_i) \cap D| \geqslant 1$, for every $g_i \in D$ and $|N(g_k) \cap D| \leqslant 1$, for every $g_k \in V(G)$. Then, by construction of D', $|N((g_i, h_1)) \cap D'| \geqslant 1$, for every $(g_i, h_1) \in D'$, and $|N((g_k, h_l)) \cap D'| \leqslant 1$, for every $(g_k, h_l) \in V(G \circ H)$. Therefore, D' is an induced-paired dominating set of $G \circ H$. Since that $|D' \cap X_i| \leqslant 1$, for all $1 \leqslant i \leqslant n$, we have that $|D'| = |D|$. Therefore $\gamma^{ip}(G \circ H) \leqslant \gamma^{ip}(G)$.

Now, suppose let D' be a minimum induced-paired dominating set in $G \circ H$. By contradiction, suppose that $|D'| < \gamma^{ip}(G)$. If $|X_i \cap D'| \leqslant 1$, for every $1 \leqslant i \leqslant n$, by Lemma 2.1, the graph G admits an induced-paired dominating set D such that $|D| < \gamma^{ip}(G)$, a contradiction. If there is a component X_i of $G \circ H$, such that $|D' \cap X_i| > 1$ for some $1 \leqslant i \leqslant n$, by Lemma 2.2, the graph $G \circ H$ admits a induced-paired dominating set D, such that $|D| = |D'|$ and $|X_i \cap D| \leqslant 1$. Again, by Lemma 2.1, the graph G admits an induced-paired dominating set D such that $|D| < \gamma^{ip}(G)$, a contradiction. Therefore, $\gamma^{ip}(G \circ H) \geqslant \gamma^{ip}(G)$. ■

Consider $S \subseteq V(G)$ a set such that $E(G[S])$ is an independent set of edges and the number of vertices not dominated by S in G is as small as possible. We will call this set an *almost induced-paired dominating* set, an IPD' set for short. The vertices not dominated by the IPD' set will be called *problem vertices* and the *problem vertex number* is the number of such vertices. In the last result, we show an upper bound for $\gamma^{ip}(G \circ H)$ when the graph G does not admit any induced-paired dominating set but H admits such a set.

Theorem 2.3. *Let G be a graph that does not admit any induced-paired dominating set, H a graph that admits such a set and A' an almost induced-paired dominating set of G, then $\gamma^{ip}(G \circ H) \leqslant |A'| + k \cdot \gamma^{ip}(H)$, where k is the problem vertex number of G with respect to A'.*

Proof. Consider A' a IPD' set of G and B' an induced-paired dominating set in H such that $|B'| = \gamma^{ip}(H)$. Construct the sets A and B such that $A = \{(g_i, h_1) \mid g_i \in A'\}$ and $B = \{(g_l, h_j) \mid h_j \in B'$ and g_l is a problem vertex of G with respect to $A'\}$. Note that $|B| = k \cdot \gamma^{ip}(H)$. We will show that $D = A \cup B$ is an induced-paired dominating set in $G \circ H$.

For the definition of problem vertex, we have that $N(g_i) \cap A' = \emptyset$, where g_i is a problem vertex. For the construction of D, we have that $N((g_i, h_j)) \cap A = \emptyset$, for all $(g_i, h_j) \in B$. Then, the set D is an open-independent set of $G \circ H$. By construction of A, the set D dominates all the vertices of $G \circ H$, with the exception of the vertices in the components X_i such that g_i is a problem vertex in G. Since the vertices in B form a open-dominating set in $G \circ H\, [X_i]$, we obtain that D is a open-dominating set in $G \circ H$. Therefore D is a induced-paired dominating set in $G \circ H$ and $\gamma^{ip}(G \circ H) \leqslant |A'| + k \cdot \gamma^{ip}(H)$. ■

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