CS 4487 Assignment 4

Exercise 1

[2 points]. Prove that the eigenvalues of a real symmetric matrix $A \in \mathbb{R}^{N\times N}$, where $A = A^T$, are real-valued (*i.e.*, not complex-valued). Prove also that the eigenvectors of A corresponding to different eigenvalues are orthogonal to each other.

First part:

Let $\lambda \in C$be the eigenvalue of A => $Av = \lambda v$, v is the eigenvector,

$$
\begin{aligned}
(Av)^T &= (\lambda v)^T \\
v^T A^T &= \bar{\lambda} v^T \\
v^T A &= \bar{\lambda} v^T \\
v^T A v &= \bar{\lambda} v^T v \\
v^T (\lambda v) &= \bar{\lambda} v^T v \\
\lambda v^T v &= \bar{\lambda} v^T v \\
\lambda &= \bar{\lambda}
\end{aligned}
$$

As the conjugate of λ is same as λ, the eigenvalues are real valued.

Second part:

Let $\lambda$and μbe different eigenvalues of A and v and w are the corresponding eigenvectors

$$
\begin{aligned}
\lambda < v, w > \; &= \; < \lambda v,\ w > \\
&= \; < Av,\ w > \\
&= \; < v,\ A^T w > \\
&= \; < v,\ Aw > \\
&= \; < v,\ \mu w > \\
&= \mu < v, w >
\end{aligned}
$$

$$=> \ (\lambda - \mu) < v, w > \; = \; 0$$

as $\lambda$and $\mu$are different eigenvalues, two eigenvectors v and w are orthogonal.

Exercise 2

[5 points]. Derive the K-th largest direction of variance in principal component analysis (PCA).

Exercise 3

[5 points]. 1) Suppose that a discrete-time linear system has outputs $y[n]$ for the given inputs $x[n]$, as shown in Fig. 1. Determine the response $y_4[n]$ when the input is as shown in Fig. 2.

a) [1 point]. Express $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

b) [1 point]. Using the fact that the system is linear, determine $y_4[n]$, the response to $x_4[n]$.

c) [1 point]. From the input-output pairs in Fig. 1, determine whether the system is time-invariant.

2) Determine the discrete-time convolution of $x[n]$ and $h[n]$ for the following two cases.

a) [1 point]. As shown in Fig. 3.

b) [1 point]. As shown in Fig. 4.

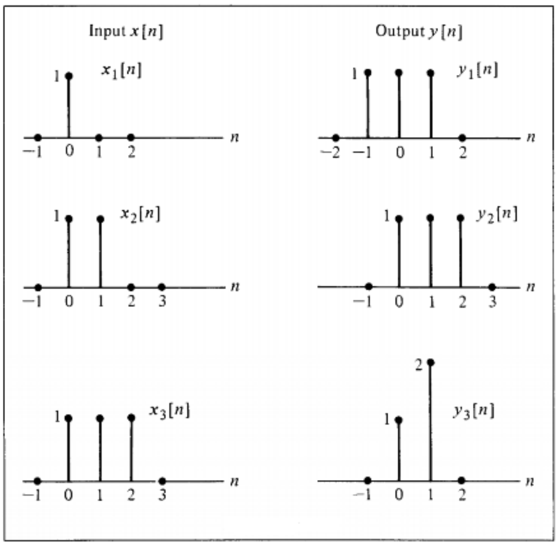

Figure 1:

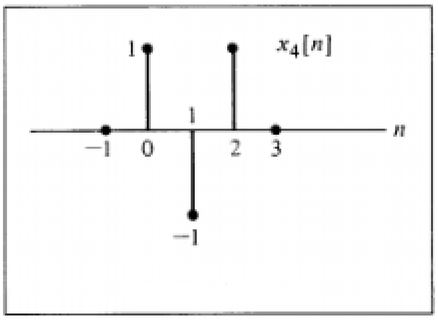

Figure 2:

a) $x_4(n) = 2x_1(n) - 2x_2(n) + x_3(n)$

b) As the system is linear , $y_4(n) = y_1(n) - 2y_2(n) + y_3(n)$

$y_4(n) = 2\{0\,1\,1\,1\,0\,0\} - 2\{0\,0\,1\,1\,1\,0\} + \{0\,0\,1\,2\,0\,0\} = \{0\,2\,1\,2\,-2\,0\}$

c) The system is not time invariant, as we get $y_2[n] = x_2[n] + \delta(n+2)$, if we input n as delayed input n_0, the output will not be the same.

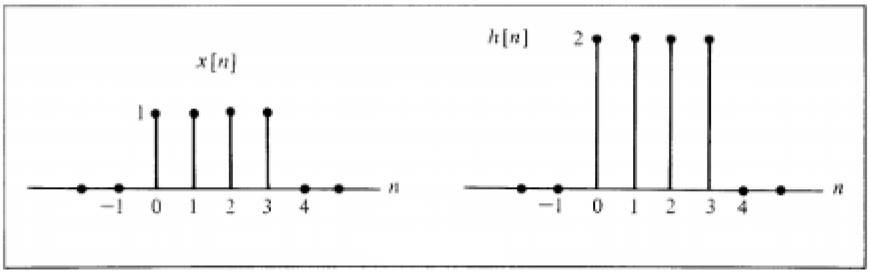

Figure 3:

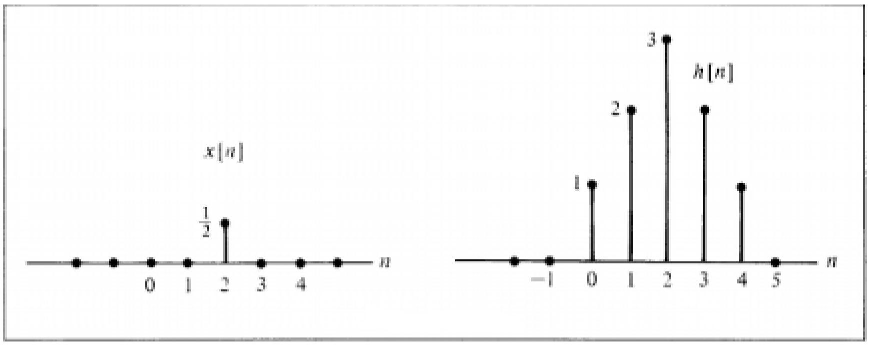

Figure 4:

2)

a)

h\x	0	1	1	1	1	0
0	0	0	0	0	0	0
2	0	2	2	2	2	0
2	0	2	2	2	2	0
2	0	2	2	2	2	0
2	0	2	2	2	2	0
0	0	0	0	0	0	0

Summing up diagonally, we get y[n] = {0 0 2 4 6 8 6 4 2 0 0} b)

h\x	0	0	0.5	0	0
1	0	0	0.5	0	0
2	0	0	1	0	0
3	0	0	1.5	0	0
2	0	0	1	0	0
1	0	0	0.5	0	0

Summing up diagonally, we get y[n] = {0 0 0.5 1 1.5 1 0.5 0 0}

Exercise 4

[3 points]. From the lecture slides, we know that the convolution of discrete-time signals $x[n]$ and $y[n]$, for $n \in [-\infty, +\infty]$, is defined as

$$z[n] = (x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m] = \sum_{m=-\infty}^{\infty} x[n-m]y[m]. \tag{1}$$

Here we introduce the Discrete-time Fourier Transform (DTFT) of a discrete-time signal $x[n]$:

$$X(\omega) = \mathrm{DTFT}(x) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{2}$$

Try to prove the Convolution Theorem:

$$Z(\omega) = \mathrm{DTFT}(x * y) = \mathrm{DTFT}(x) \cdot \mathrm{DTFT}(y) = X(\omega) \cdot Y(\omega). \tag{3}$$

$$Z(w) = DTFT(x \cdot y)$$

$$= \sum_{n=-\infty}^{\infty} (x \cdot y)[n]e^{j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[n-k]e^{j\Omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]e^{j\Omega n}$$

Put n - k = m , n = m + k

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m+k=-\infty}^{\infty} y[m]e^{-j\Omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} y[m]e^{-j\Omega m}e^{-j\Omega k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k} \sum_{m=-\infty}^{\infty} y[m]e^{-j\Omega m}$$

$$= X(w) \cdot Y(w)$$