CHAPTER 2

(© Eyebyte/Alamy)

Noticeable deformation occurred in this chain link just before excessive stress caused it to fracture.

STRAIN

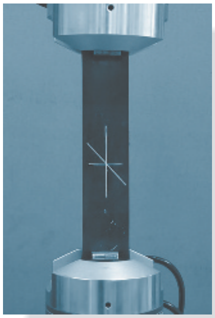
CHAPTER OBJECTIVES

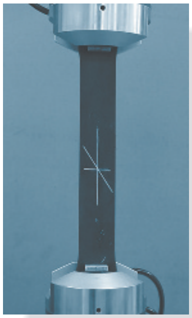
- In engineering the deformation of a body is specified using the concepts of normal and shear strain. In this chapter we will define these quantities and show how they can be determined for various types of problems.

2.1 DEFORMATION

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation***, and they may be highly visible or practically unnoticeable. For example, a rubber band will undergo a very large deformation when stretched, whereas only slight deformations of structural members occur when a building is occupied. Deformation of a body can also occur when the temperature of the body is changed. A typical example is the thermal expansion or contraction of a roof caused by the weather.

In a general sense, the deformation will not be uniform throughout the body, and so the change in geometry of any line segment within the body may vary substantially along its length. Hence, to study deformation, we will consider line segments that are very short and located in the neighborhood of a point. Realize, however, that the deformation will also depend on the orientation of the line segment at the point. For example, as shown in the adjacent photos, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.





Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.

2.2 STRAIN

In order to describe the deformation of a body by changes in the lengths of line segments and changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiment, and once the strain is obtained, it will be shown in the next chapter how it can be related to the stress acting within the body.

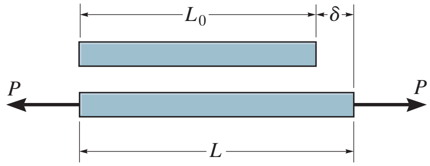


Fig. 2–1

Normal Strain. If an axial load P is applied to the bar in Fig. 2–1, it will change the bar's length L_0 to a length L. We will define the ***average normal strain*** ϵ (epsilon) of the bar as the change in its length δ (delta) $= L - L_0$ divided by its original length, that is

$$\boxed{\epsilon_{\text{avg}} = \frac{L - L_0}{L_0}} \tag{2–1}$$

The normal strain at a point in a body of arbitrary shape is defined in a similar manner. For example, consider the very small line segment Δs located at the point, Fig. 2–2. After deformation it becomes $\Delta s'$, and the change in its length is therefore $\Delta s' - \Delta s$. As $\Delta s \rightarrow 0$, in the limit the normal strain at the point is therefore

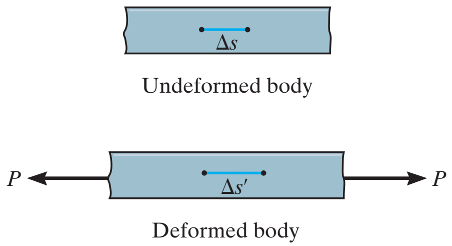


Fig. 2–2

$$\boxed{\epsilon = \lim_{\Delta s \rightarrow 0} \frac{\Delta s' - \Delta s}{\Delta s}} \tag{2–2}$$

In both cases ϵ (or ϵ_{avg}) is a change in length per unit length, and it is positive when the initial line elongates, and negative when the line contracts.

Units. As shown, normal strain is a *dimensionless quantity*, since it is a ratio of two lengths. However, it is sometimes stated in terms of a ratio of length units. If the SI system is used, where the basic unit for length is the meter (m), then since ϵ is generally very small, for most engineering applications, measurements of strain will be in micrometers per meter (μm/m), where $1\ \mu\text{m} = 10^{-6}$ m. For experimental work, strain is sometimes expressed as a percent. For example, a normal strain of $480(10^{-6})$ can be reported as 480 μm/m, or 0.0480%. Or one can state the strain as simply 480 μ (480 "micros").

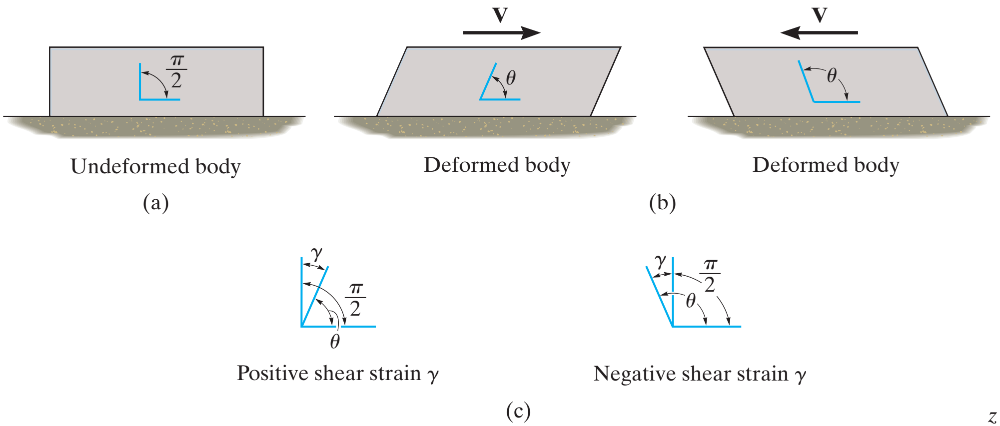


Fig. 2–3

Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the *change in angle* that occurs between them is referred to as ***shear strain***. This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the two perpendicular line segments at a point in the block shown in Fig. 2–3*a*. If an applied loading causes the block to deform as shown in Fig. 2–3*b*, so that the angle between the line segments becomes θ, then the shear strain at the point becomes

$$\boxed{\gamma = \frac{\pi}{2} - \theta} \qquad (2\text{–}3)$$

Notice that if θ is smaller than $\pi/2$, Fig. 2–3*c*, then the shear strain is *positive*, whereas if θ is larger than $\pi/2$, then the shear strain is *negative*.

Cartesian Strain Components. We can generalize our definitions of normal and shear strain and consider the undeformed element at a point in a body, Fig. 2–4*a*. Since the element's dimensions are very small, its deformed shape will become a parallelepiped, Fig. 2–4*b*. Here the *normal strains* change the sides of the element to

$$(1 + \epsilon_x)\Delta x \qquad (1 + \epsilon_y)\Delta y \qquad (1 + \epsilon_z)\Delta z$$

which produces a *change in the volume of the element*. And the *shear strain* changes the angles between the sides of the element to

$$\frac{\pi}{2} - \gamma_{xy} \qquad \frac{\pi}{2} - \gamma_{yz} \qquad \frac{\pi}{2} - \gamma_{xz}$$

which produces a *change in the shape of the element.*

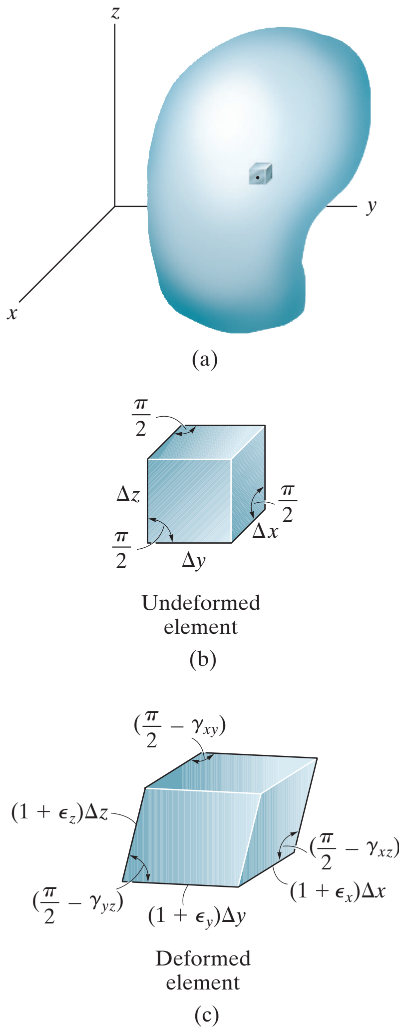


Fig. 2–4

2

Small Strain Analysis. Most engineering design involves applications for which only *small deformations* are allowed. In this text, therefore, we will assume that the deformations that take place within a body are almost infinitesimal. For example, the *normal strains* occurring within the material are *very small* compared to 1, so that $\epsilon \ll 1$. This assumption has wide practical application in engineering, and it is often referred to as a *small strain analysis*. It can also be used when a change in angle, $\Delta\theta$, is small, so that $\sin \Delta\theta \approx \Delta\theta$, $\cos \Delta\theta \approx 1$, and $\tan \Delta\theta \approx \Delta\theta$.

The rubber bearing support under this concrete bridge girder is subjected to both normal and shear strain. The normal strain is caused by the weight and bridge loads on the girder, and the shear strain is caused by the horizontal movement of the girder due to temperature changes.

IMPORTANT POINTS

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components all depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about its size.

EXAMPLE 2.1

Determine the average normal strains in the two wires in Fig. 2–5 if the ring at A moves to A'.

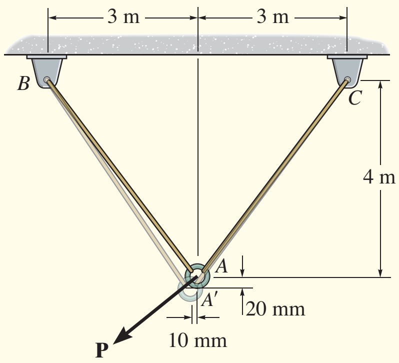


Fig. 2–5

SOLUTION

Geometry. The original length of each wire is

$$L_{AB} = L_{AC} = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}$$

The final lengths are

$$L_{A'B} = \sqrt{(3\text{ m} - 0.01\text{ m})^2 + (4\text{ m} + 0.02\text{ m})^2} = 5.01004\text{ m}$$

$$L_{A'C} = \sqrt{(3\text{ m} + 0.01\text{ m})^2 + (4\text{ m} + 0.02\text{ m})^2} = 5.02200\text{ m}$$

Average Normal Strain.

$$\epsilon_{AB} = \frac{L_{A'B} - L_{AB}}{L_{AB}} = \frac{5.01004\text{ m} - 5\text{ m}}{5\text{ m}} = 2.01(10^{-3})\text{ m/m} \qquad \textit{Ans.}$$

$$\epsilon_{AC} = \frac{L_{A'C} - L_{AC}}{L_{AC}} = \frac{5.02200\text{ m} - 5\text{ m}}{5\text{ m}} = 4.40(10^{-3})\text{ m/m} \qquad \textit{Ans.}$$

EXAMPLE 2.2

When force **P** is applied to the rigid lever arm *ABC* in Fig. 2–6*a*, the arm rotates counterclockwise about pin *A* through an angle of 0.05°. Determine the normal strain in wire *BD*.

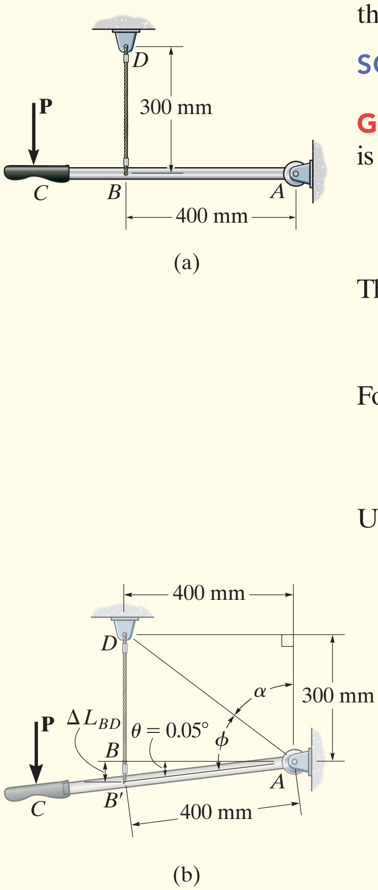


Fig. 2–6

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point *A* is shown in Fig. 2–6*b*. From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400\text{ mm}}{300\text{ mm}}\right) = 53.1301°$$

Then

$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle *ABD* the Pythagorean theorem gives

$$L_{AD} = \sqrt{(300\text{ mm})^2 + (400\text{ mm})^2} = 500\text{ mm}$$

Using this result and applying the law of cosines to triangle $AB'D$,

$$\begin{aligned} L_{B'D} &= \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})\,(L_{AB'})\cos\phi} \\ &= \sqrt{(500\text{ mm})^2 + (400\text{ mm})^2 - 2(500\text{ mm})\,(400\text{ mm})\cos 36.92°} \\ &= 300.3491\text{ mm} \end{aligned}$$

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}}$$

$$= \frac{300.3491\text{ mm} - 300\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

SOLUTION II

Since the strain is small, this same result can be obtained by approximating the elongation of wire *BD* as ΔL_{BD}, shown in Fig. 2–6*b*. Here,

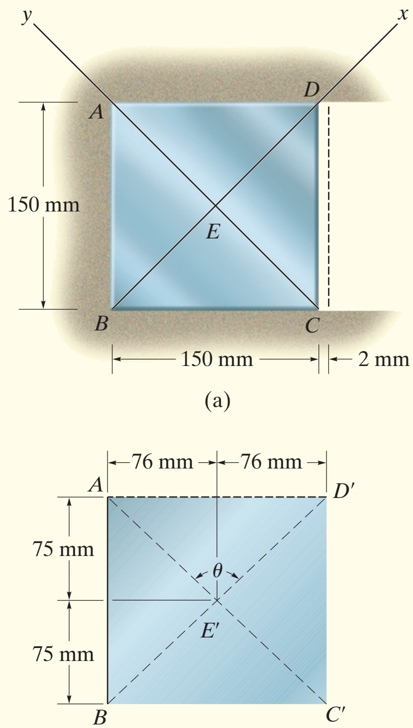
$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi\text{ rad})\right](400\text{ mm}) = 0.3491\text{ mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \qquad \textit{Ans.}$$

EXAMPLE 2.3

The plate shown in Fig. 2–7*a* is fixed connected along AB and held in the horizontal guides at its top and bottom, AD and BC. If its right side CD is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal AC, and (b) the shear strain at E relative to the x, y axes.



(a)

(b)

Fig. 2–7

SOLUTION

Part (a). When the plate is deformed, the diagonal AC becomes AC', Fig. 2–7*b*. The lengths of diagonals AC and AC' can be found from the Pythagorean theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along AC is

$$(\epsilon_{AC})_{\text{avg}} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at E relative to the x and y axes, which are 90° apart, it is necessary to find the change in the angle at E. After deformation, Fig. 2–7*b*,

$$\tan\left(\frac{\theta}{2}\right) = \frac{76\text{ mm}}{75\text{ mm}}$$

$$\theta = 90.759° = \left(\frac{\pi}{180°}\right)(90.759°) = 1.58404\text{ rad}$$

Applying Eq. 2–3, the shear strain at E is therefore the change in the angle AED,

$$\gamma_{xy} = \frac{\pi}{2} - 1.58404\text{ rad} = -0.0132\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the once 90° angle becomes larger.

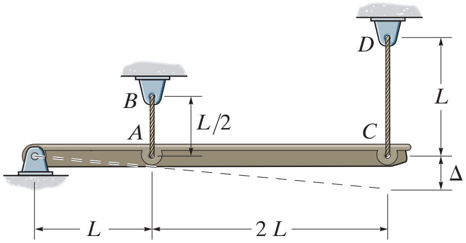
NOTE: If the x and y axes were horizontal and vertical at point E, then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point E.

2

2

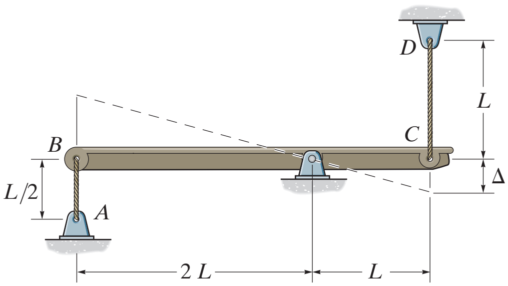
PRELIMINARY PROBLEMS

P2–1. A loading causes the member to deform into the dashed shape. Explain how to determine the normal strains ϵ_{CD} and ϵ_{AB}. The displacement Δ and the lettered dimensions are known.



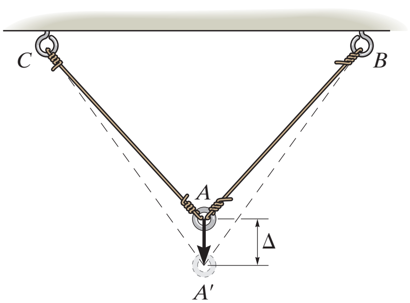
Prob. P2–1

P2–2. A loading causes the member to deform into the dashed shape. Explain how to determine the normal strains ϵ_{CD} and ϵ_{AB}. The displacement Δ and the lettered dimensions are known.



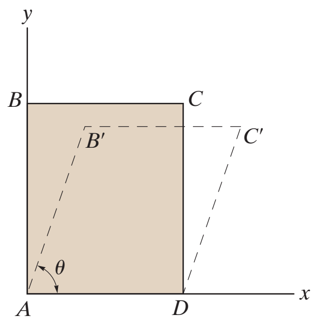
Prob. P2–2

P2–3. A loading causes the wires to elongate into the dashed shape. Explain how to determine the normal strain ϵ_{AB} in wire AB. The displacement Δ and the distances between all lettered points are known.



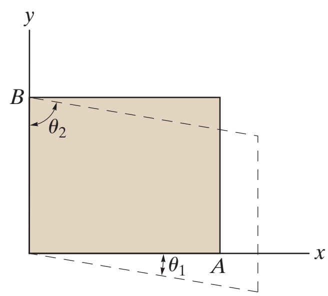
Prob. P2–3

P2–4. A loading causes the block to deform into the dashed shape. Explain how to determine the strains ϵ_{AB}, ϵ_{AC}, ϵ_{BC}, $(\gamma_A)_{xy}$. The angles and distances between all lettered points are known.



Prob. P2–4

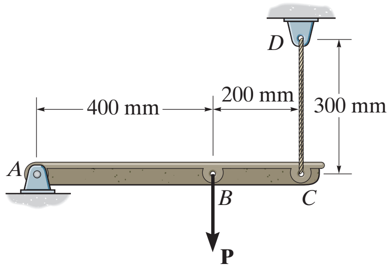
P2–5. A loading causes the block to deform into the dashed shape. Explain how to determine the strains $(\gamma_A)_{xy}$, $(\gamma_B)_{xy}$. The angles and distances between all lettered points are known.



Prob. P2–5

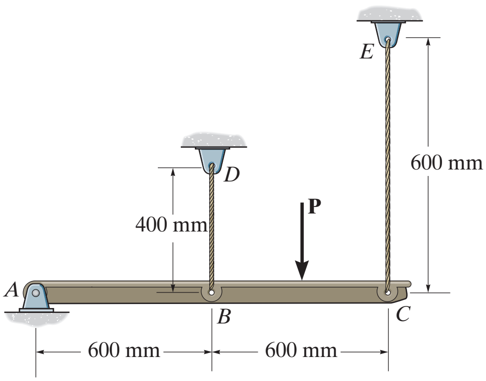
FUNDAMENTAL PROBLEMS

F2–1. When force **P** is applied to the rigid arm *ABC*, point *B* displaces vertically downward through a distance of 0.2 mm. Determine the normal strain in wire *CD*.



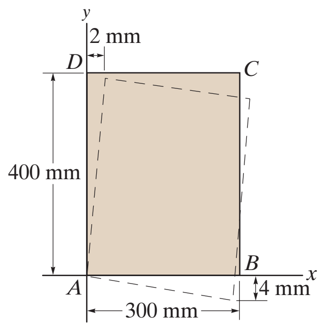
Prob. F2–1

F2–2. If the force **P** causes the rigid arm *ABC* to rotate clockwise about pin *A* through an angle of 0.02°, determine the normal strain in wires *BD* and *CE*.



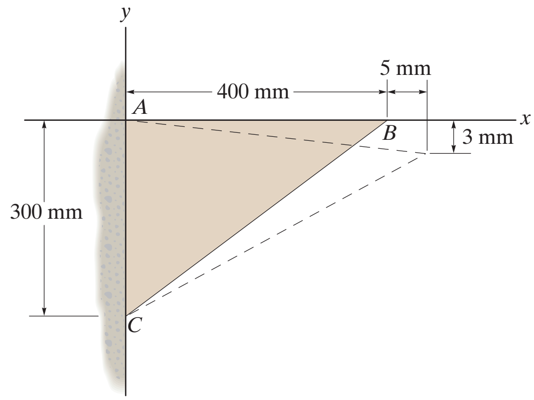
Prob. F2–2

F2–3. The rectangular plate is deformed into the shape of a parallelogram shown by the dashed line. Determine the average shear strain at corner *A* with respect to the *x* and *y* axes.



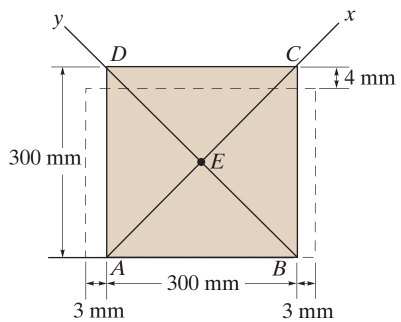
Prob. F2–3

F2–4. The triangular plate is deformed into the shape shown by the dashed line. Determine the normal strain along edge *BC* and the average shear strain at corner *A* with respect to the *x* and *y* axes.



Prob. F2–4

F2–5. The square plate is deformed into the shape shown by the dashed line. Determine the average normal strain along diagonal *AC* and the shear strain at point *E* with respect to the *x* and *y* axes.



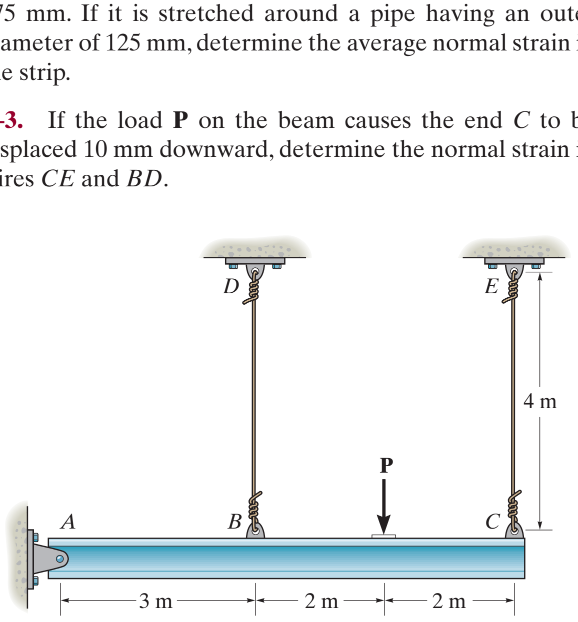
Prob. F2–5

PROBLEMS

2–1. An air-filled rubber ball has a diameter of 150 mm. If the air pressure within it is increased until the ball's diameter becomes 175 mm, determine the average normal strain in the rubber.

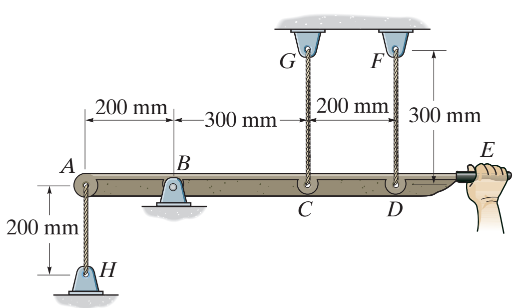
2–2. A thin strip of rubber has an unstretched length of 375 mm. If it is stretched around a pipe having an outer diameter of 125 mm, determine the average normal strain in the strip.

2–3. If the load **P** on the beam causes the end C to be displaced 10 mm downward, determine the normal strain in wires CE and BD.



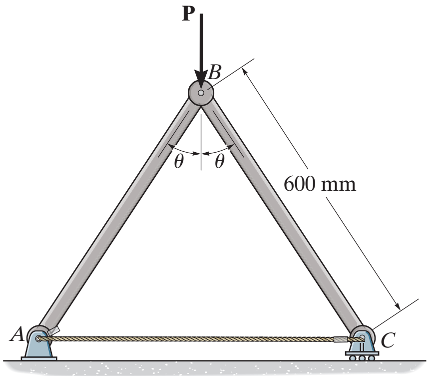
Prob. 2–3

*__2–4.__ The force applied at the handle of the rigid lever causes the lever to rotate clockwise about the pin B through an angle of 2°. Determine the average normal strain in each wire. The wires are unstretched when the lever is in the horizontal position.



Prob. 2–4

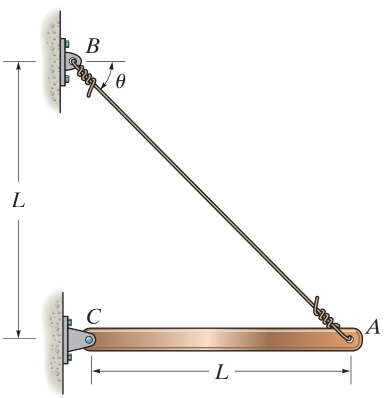
2–5. The pin-connected rigid rods AB and BC are inclined at $\theta = 30°$ when they are unloaded. When the force **P** is applied θ becomes 30.2°. Determine the average normal strain in wire AC.



Prob. 2–5

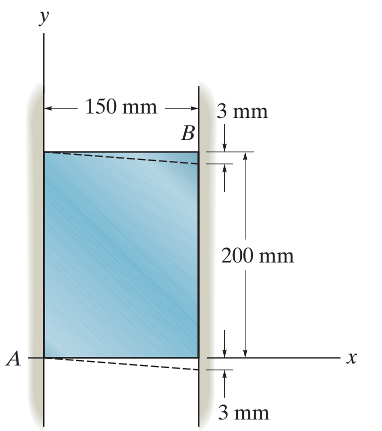
2–6. The wire AB is unstretched when $\theta = 45°$. If a load is applied to the bar AC, which causes θ to become 47°, determine the normal strain in the wire.

2–7. If a horizontal load applied to the bar AC causes point A to be displaced to the right by an amount ΔL, determine the normal strain in the wire AB. Originally, $\theta = 45°$.



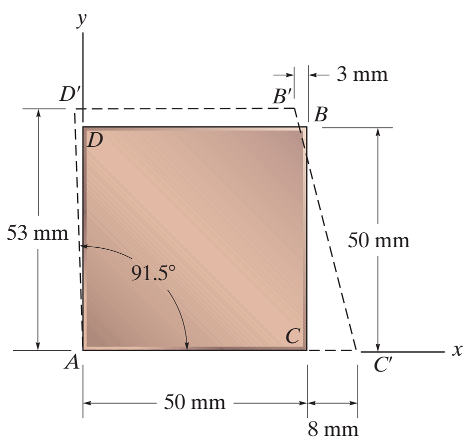
Probs. 2–6/7

*2–8. The rectangular plate is subjected to the deformation shown by the dashed line. Determine the average shear strain γ_{xy} in the plate.



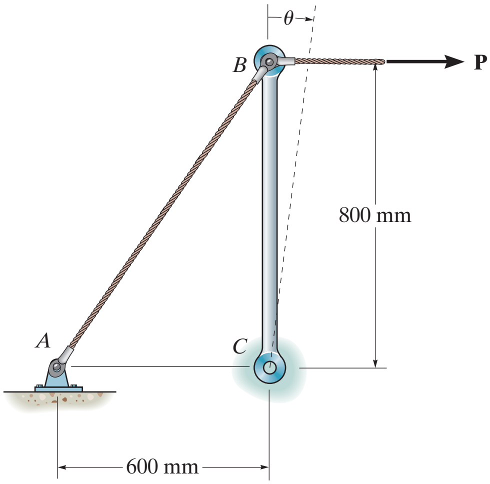
Prob. 2–8

2–9. The square deforms into the position shown by the dashed lines. Determine the shear strain at each of its corners, A, B, C, and D, relative to the x, y axes. Side $D'B'$ remains horizontal.



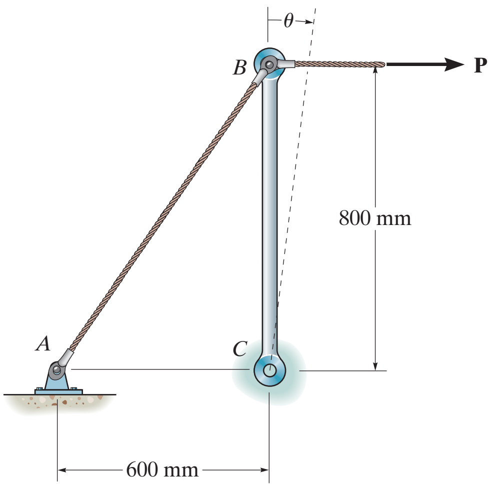
Prob. 2–9

2–10. Part of a control linkage for an airplane consists of a rigid member CB and a flexible cable AB. If a force is applied to the end B of the member and causes it to rotate by $\theta = 0.5°$, determine the normal strain in the cable. Originally the cable is unstretched.



Prob. 2–10

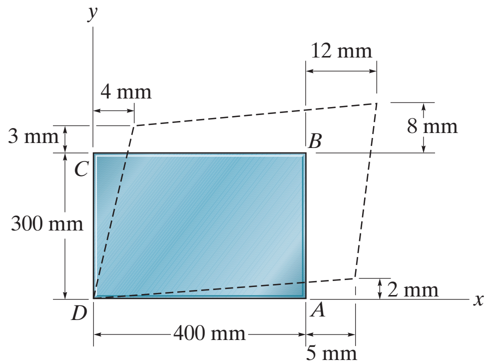
2–11. Part of a control linkage for an airplane consists of a rigid member CB and a flexible cable AB. If a force is applied to the end B of the member and causes a normal strain in the cable of 0.004 mm/mm, determine the displacement of point B. Originally the cable is unstretched.



Prob. 2–11

***2–12.** Determine the shear strain γ_{xy} at corners A and B if the plastic distorts as shown by the dashed lines.

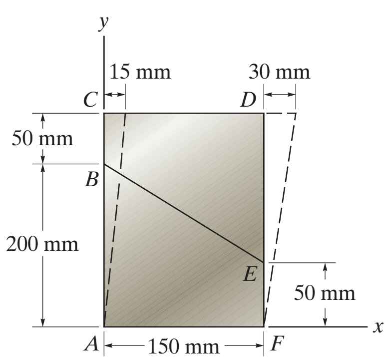
2–13. Determine the shear strain γ_{xy} at corners D and C if the plastic distorts as shown by the dashed lines.



Probs. 2–12/13

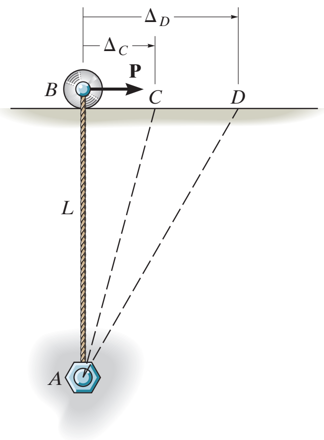
2–14. The material distorts into the dashed position shown. Determine the average normal strains ϵ_x, ϵ_y and the shear strain γ_{xy} at A, and the average normal strain along line BE.

2–15. The material distorts into the dashed position shown. Determine the average normal strains along the diagonals AD and CF.



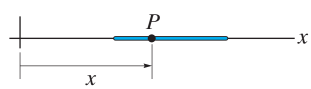
Probs. 2–14/15

***2–16.** The nylon cord has an original length L and is tied to a bolt at A and a roller at B. If a force **P** is applied to the roller, determine the normal strain in the cord when the roller is at C, and at D. If the cord is originally unstrained when it is at C, determine the normal strain ϵ'_D when the roller moves to D. Show that if the displacements Δ_C and Δ_D are small, then $\epsilon'_D = \epsilon_D - \epsilon_C$.



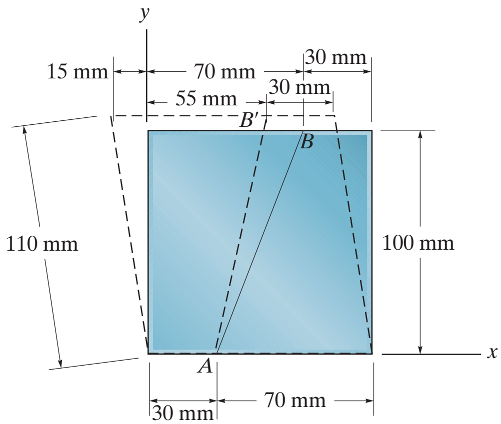
Prob. 2–16

2–17. A thin wire, lying along the x axis, is strained such that each point on the wire is displaced $\Delta x = kx^2$ along the x axis. If k is constant, what is the normal strain at any point P along the wire?



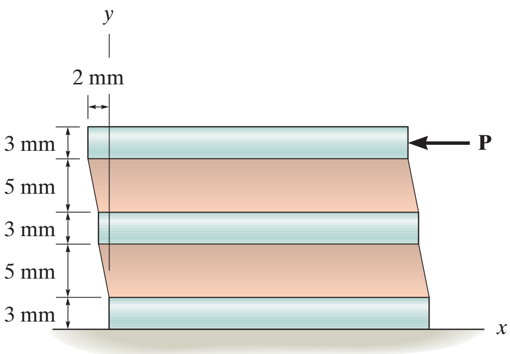
Prob. 2–17

2–18. The block is deformed into the position shown by the dashed lines. Determine the average normal strain along line AB.



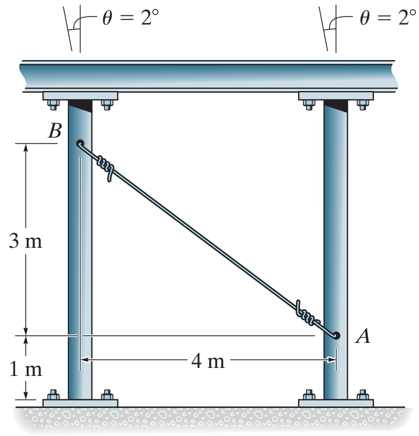
Prob. 2–18

2–19. Nylon strips are fused to glass plates. When moderately heated the nylon will become soft while the glass stays approximately rigid. Determine the average shear strain in the nylon due to the load **P** when the assembly deforms as indicated.



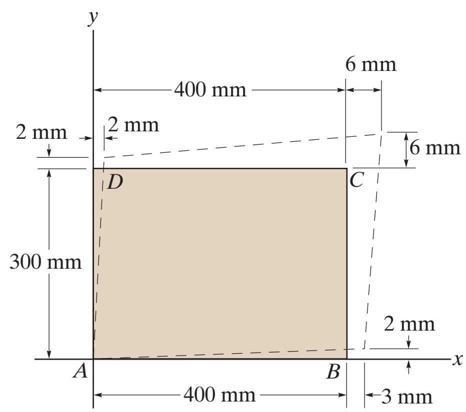
Prob. 2–19

***2–20.** The guy wire AB of a building frame is originally unstretched. Due to an earthquake, the two columns of the frame tilt $\theta = 2°$. Determine the approximate normal strain in the wire when the frame is in this position. Assume the columns are rigid and rotate about their lower supports.



Prob. 2–20

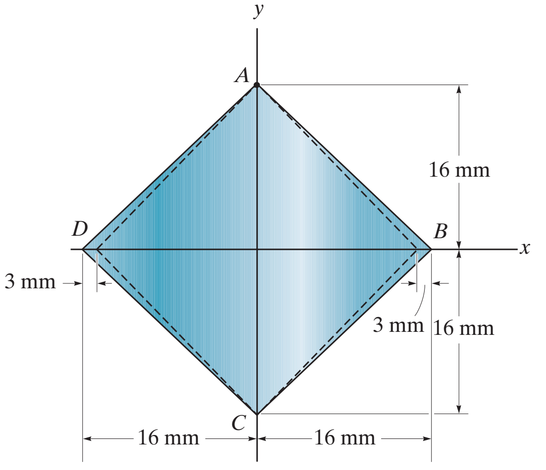
2–21. The rectangular plate is deformed into the shape shown by the dashed lines. Determine the average normal strain along diagonal AC, and the average shear strain at corner A relative to the x, y axes.



Prob. 2–21

2

2–22. The corners B and D of the square plate are given the displacements indicated. Determine the shear strains at A and B.

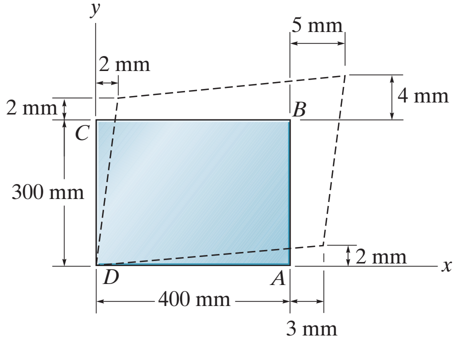


Prob. 2–22

2–23. Determine the shear strain γ_{xy} at corners A and B if the plate distorts as shown by the dashed lines.

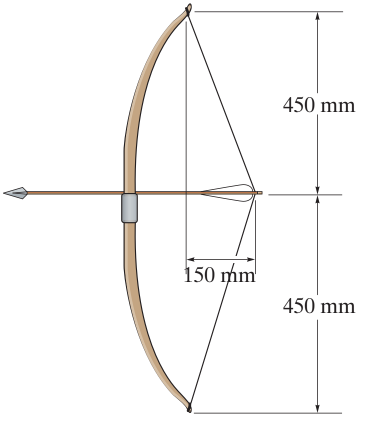
***2–24.** Determine the shear strain γ_{xy} at corners D and C if the plate distorts as shown by the dashed lines.

2–25. Determine the average normal strain that occurs along the diagonals AC and DB.



Probs. 2–23/24/25

2–26. If the unstretched length of the bowstring is 887.5 mm, determine the average normal strain in the string when it is stretched to the position shown.

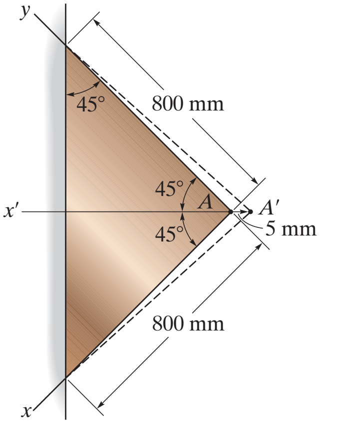


Prob. 2–26

2–27. The triangular plate is fixed at its base, and its apex A is given a horizontal displacement of 5 mm. Determine the shear strain, γ_{xy}, at A.

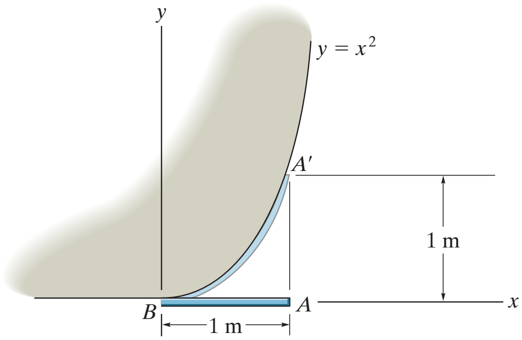
***2–28.** The triangular plate is fixed at its base, and its apex A is given a horizontal displacement of 5 mm. Determine the average normal strain ϵ_x along the x axis.

2–29. The triangular plate is fixed at its base, and its apex A is given a horizontal displacement of 5 mm. Determine the average normal strain $\epsilon_{x'}$ along the x' axis.



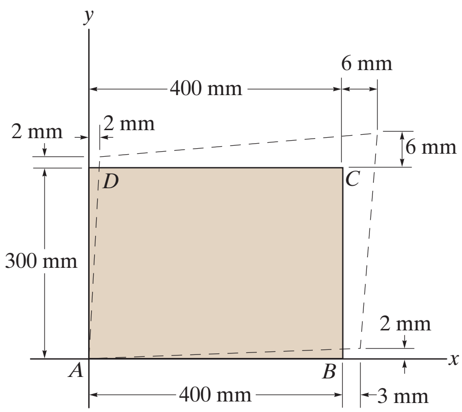
Probs. 2–27/28/29

2–30. The rubber band AB has an unstretched length of 1 m. If it is fixed at B and attached to the surface at point A', determine the average normal strain in the band. The surface is defined by the function $y = (x^2)$ m, where x is in meters.



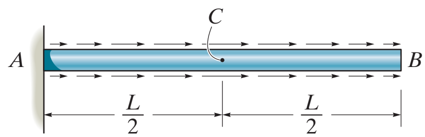
Prob. 2–30

2–31. The rectangular plate is deformed into the shape shown by the dashed lines. Determine the average normal strain along diagonal BD, and the average shear strain at corner B relative to the x, y axes.



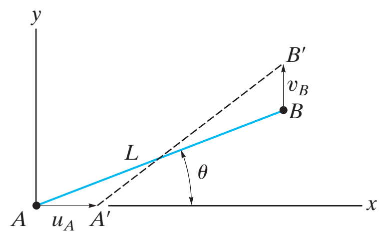
Prob. 2–31

***2–32.** The nonuniform loading causes a normal strain in the shaft that can be expressed as $\epsilon_x = k \sin\left(\frac{\pi}{L}x\right)$, where k is a constant. Determine the displacement of the center C and the average normal strain in the entire rod.



Prob. 2–32

2–33. The fiber AB has a length L and orientation θ. If its ends A and B undergo very small displacements u_A and v_B respectively, determine the normal strain in the fiber when it is in position $A'B'$.



Prob. 2–33

2–34. If the normal strain is defined in reference to the final length $\Delta s'$, that is,

$$\epsilon' = \lim_{\Delta s' \to 0}\left(\frac{\Delta s' - \Delta s}{\Delta s'}\right)$$

instead of in reference to the original length, Eq. 2–2, show that the difference in these strains is represented as a second-order term, namely, $\epsilon - \epsilon' = \epsilon\epsilon'$.