CH4

2023年3月23日　　上午 10:14

4.2 Elastic Deformation of an Axially Loaded Member

軸向負荷

Here we wish to find the ***relative displacement*** (delta) of one end of the bar with respect to the other end as caused by this loading.

Using the method of sections, a differential element (or wafer) of length dx and cross-sectional area $A(x)$ is isolated from the bar at the arbitrary position x. The stress and strain in the element are

$$\sigma = \frac{P(x)}{A(x)} \quad \text{and} \quad \epsilon = \frac{d\delta}{dx}$$

Provided the stress does not exceed the proportional limit, we can apply Hooke's law;

$$\sigma = E(x)\epsilon$$

$$\frac{P(x)}{A(x)} = E(x)\left(\frac{d\delta}{dx}\right)$$

$$d\delta = \frac{P(x)dx}{A(x)E(x)}$$

變型量

原長

$$\boxed{\delta = \int_0^L \frac{P(x)dx}{A(x)E(x)}} \qquad (4\text{–}1)$$

截面 * E值

where

δ = displacement of one point on the bar relative to the other point
L = original length of bar
$P(x)$ = internal axial force at the section, located a distance x from one end
$A(x)$ = cross-sectional area of the bar expressed as a function of x
$E(x)$ = modulus of elasticity for the material expressed as a function of x.

P(x)　　A(X)

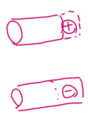
Constant Load and Cross-Sectional Area. In many cases the bar will have a constant cross-sectional area A; and the material will be homogeneous, so E is constant. the internal force P throughout the length of the bar is also constant.

$$\int_0^L dx = L \Leftrightarrow \boxed{\delta = \frac{PL}{AE}} \qquad (4\text{–}2)$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each *segment of the bar* where these quantities remain *constant*.

$$\boxed{\delta = \sum \frac{PL}{AE}} \qquad (4\text{–}3)$$

Sign Convention. We will consider both the force and displacement to be *positive* if they cause *tension* and *elongation*, respectively, Fig. 4–4; whereas a *negative* force and displacement will cause *compression* and *contraction*, respectively.



EXAMPLE 4.1

軸向負荷变形量

鋼棒　鋁管截面

$\boxed{\delta = \frac{AE}{PL}}$

The assembly shown in Fig. 4–6*a* consists of an aluminum tube *AB* having a cross-sectional area of 400 mm². A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. Take E_{st} = 200 GPa, E_{al} = 70 GPa.

求

⇨ 80 kN→ 時的 C 变形量

Fig. 4-6

鋼棒伸長　$\delta_{steel} = \delta_{BC}$

鋁棒压縮　$\delta_{al.} = \delta_B$

SOLUTION

Internal Force. The free-body diagram of the tube and rod segments in Fig. 4–6*b*, shows that the rod is subjected to a tension of 80 kN and the tube is subjected to a compression of 80 kN.

EXAMPLE 4.1 CONTINUED

Displacement. We will first determine the displacement of end *C* with respect to end *B*. Working in units of newtons and meters, we have

$$\delta_{st} = \delta_{C/B} = \frac{PL}{AE} = \frac{[+80(10^3)\text{N}]\,(0.6\text{ m})}{\pi\,(0.005\text{ m})^2[200\,(10^9)\,\text{N/m}^2]} = +0.003056\text{ m}\rightarrow$$

600 mm　5　200×10^3 N/mm²　$= 3.056$ mm ※

The positive sign indicates that end *C* moves *to the right* relative to end *B*, since the bar elongates.

The displacement of end *B* with respect to the *fixed* end *A* is

$$\delta_{al} = \delta_B = \frac{PL}{AE} = \frac{[-80(10^3)\text{ N}](0.4\text{ m})}{[400\text{ mm}^2(10^{-6})\text{ m}^2/\text{mm}^2][70(10^9)\text{ N/m}^2]}$$

$$= -0.001143\text{ m} = 0.001143\text{ m}\rightarrow$$

400 mm　70×10^3　1.14 mm

Here the negative sign indicates that the tube shortens, and so *B* moves to the *right* relative to *A*.

Since both displacements are to the right, the displacement of *C* relative to the fixed end *A* is therefore

$$(\overset{+}{\rightarrow})\qquad \delta_C = \delta_B + \delta_{C/B} = 0.001143\text{ m} + 0.003056\text{ m} = 0.00420\text{ m} = 4.20\text{ mm}\rightarrow \text{ ※} \qquad \textit{Ans.}$$

$\delta_c = \delta_{st} + \delta_{al}$
$= 3.056 + 1.142$
$= 4.198$ mm ※

4.3 Principle of Superposition

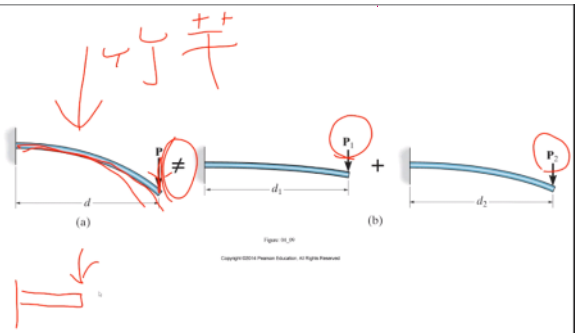
疊加定理

The following two conditions must be satisfied if the principle of superposition is to be applied.

1. ***The loading must be linearly related to the stress or displacement that is to be determined.*** For example, the equations $\sigma = P/A$ and $\delta = PL/AE$ involve a linear relationship between P and σ or δ.
2. ***The loading must not significantly change the original geometry or configuration of the member.*** If significant changes do occur, the direction and location of the applied forces and their moment arms will change.

$P_2 \rightarrow \delta_2$　⇒ $P_3 = P_1 + P_2$，$\delta_3 = \delta_1 + \delta_2$ #
$P_1 \rightarrow \delta_1$

☆前題 < 線性 / 不能有太大形變



静不定（靜力学 can't solve），需結合材力

☆4.4 Statically Indeterminate Axially Loaded Member

Consider the bar shown in Fig. 4–10*a* which is fixed supported at both of its ends. From the free-body diagram, Fig. 4–10*b*, equilibrium requires

$$+\uparrow \Sigma F = 0;\qquad F_B + F_A - P = 0$$

This type of problem is called ***statically indeterminate***, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic condition***. Hence, the compatibility condition becomes

$$\delta_{A/B} = 0$$

This equation can be expressed in terms of the applied loads by

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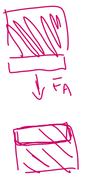
$$\delta_{A/B} = 0$$

This equation can be expressed in terms of the applied loads by using a *load–displacement relationship*, which depends on the material behavior.

☆ $$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$



$\frac{PL}{AE}$ δ_A

$\downarrow F_A$

$\frac{PL}{AE}$ δ_B

☆ $\delta_A = \delta_B$

compatibility condition

假令成2段，分別有不同位移

EXAMPLE 4.4

The steel rod shown in Fig. 4–11*a* has a diameter of 10 mm. It is fixed to the wall at *A*, and before it is loaded, there is a gap of 0.2 mm between the wall at *B*' and the rod. Determine the reactions at *A* and *B* if the rod is subjected to an axial force of $P = 20$ kN as shown. Neglect the size of the collar at *C*. Take $E_{st} = 200$ GPa.

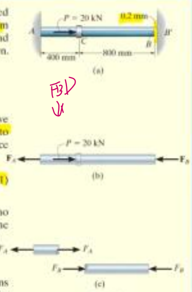


Fig. 4-11

FBD

SOLUTION

假設P大到能使B→B'

Equilibrium. As shown on the free-body diagram, Fig. 4–11*b*, we will *assume* that force *P* is large enough to cause the rod's end *B* to contact the wall at *B*'. The problem is statically indeterminate since there are two unknowns and only one equation of equilibrium.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -F_A - F_B + 20(10^3)\text{ N} = 0 \qquad (1)$$

$F_A + F_B = 20000$ N —①

Compatibility. The force *P* causes point *B* to move to *B*', with no further displacement. Therefore the compatibility condition for the rod is

$$\delta_{B/A} = 0.0002 \text{ m}$$

Load-Displacement. This displacement can be expressed in terms of the unknown reactions using the load–displacement relationship, Eq. 4–2, applied to segments *AC* and *CB*, Fig. 4–11*c*. Working in units of newtons and meters, we have

EXAMPLE 4.4 CONTINUED

longer shorter

$$\delta_{B/A} = 0.0002 \text{ m} = \frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE}$$

$$0.0002 \text{ m} = \frac{F_A(0.4 \text{ m})}{\pi(0.005 \text{ m})^2[200(10^9)\text{ N/m}^2]} - \frac{F_B(0.8 \text{ m})}{\pi(0.005 \text{ m})^2[200(10^9)\text{ N/m}^2]}$$

$0.2 = \frac{400F_A - 800F_B}{5^2\pi * 200000}$

$\Rightarrow 400F_A - 800F_B = 3141592.165$ —②

or

$$F_A(0.4 \text{ m}) - F_B(0.8 \text{ m}) = 3141.59 \text{ N·m} \qquad (2)$$

①,②联立解出

Solving Eqs. 1 and 2 yields

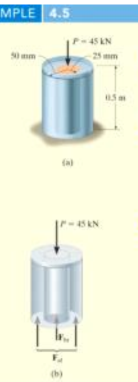
$$F_A = 16.0 \text{ kN} \quad F_B = 4.05 \text{ kN} \qquad \textit{Ans.}$$

Since the answer for F_B is *positive*, indeed end *B* contacts the wall at *B*' as originally assumed.

NOTE: If F_B were a negative quantity, the problem would be statically determinate, so that $F_B = 0$ and $F_A = 20$ kN.

㊀ 静定 ～

EXAMPLE 4.5



The aluminum post shown in Fig. 4–12*a* is reinforced with a brass core. If this assembly supports an axial compressive load of $P = 45$ kN, applied to the rigid cap, determine the average normal stress in the aluminum and the brass. Take $E_{al} = 70(10^3)$ MPa and $E_{br} = 105(10^3)$ MPa.

SOLUTION

Equilibrium. The free-body diagram of the post is shown in Fig. 4–12*b*. Here the resultant axial force at the base is represented by the unknown components carried by the aluminum, $\mathbf{F}_{al}$, and brass, $\mathbf{F}_{br}$. The problem is statically indeterminate. Why?

Vertical force equilibrium requires

$F_{al} + F_{br} = 45$ kN

$$+\uparrow \Sigma F_y = 0; \qquad -45 \text{ kN} + F_{al} + F_{br} = 0 \qquad (1)$$

Compatibility. The rigid cap at the top of the post causes both the aluminum and brass to displace the same amount. Therefore,

$$\delta_{al} = \delta_{br}$$

Using the load–displacement relationships,

$$\frac{F_{al}\cancel{L}}{A_{al}E_{al}} = \frac{F_{br}\cancel{L}}{A_{br}E_{br}}$$

$$F_{al} = F_{br}\left(\frac{A_{al}}{A_{br}}\right)\left(\frac{E_{al}}{E_{br}}\right)$$

$F_{al} = \frac{F_{br}}{A_{br}E_{br}} * A_{al}E_{al}$

$= \frac{(45 - F_a)}{25^2\pi * 105000} * (50^2 - 25^2)\pi \cdot 70000$

$\Rightarrow F_{al} = 30$ kN, $F_{br} = 15$ kN

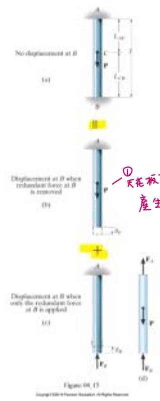
* $\sigma_{al} = \frac{F}{A} = \frac{30000}{(50^2 - 25^2)\pi} = 5.092$ MPa #

* $\sigma_{st} = \frac{F}{A} = \frac{15000}{25^2\pi} = 7.639$ MPa # Ans.

4.5 The Force Method of Analysis for Axially Loaded Members

It is also possible to solve statically indeterminate problems by writing the compatibility equation using the principle of superposition. This method of solution is often referred to as the ***flexibility*** or ***force method*** *of analysis*.

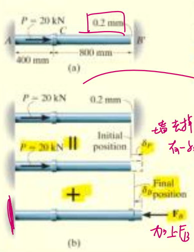
☆ 令 wall 不存在，以另一力替代



①假設B地不存在 產生一δ形变

Figure 04_15

EXAMPLE 4.8 Using Forced Method



牆去掉 有一δ_P

加上F_B

直徑10

The A-36 steel rod shown in Fig. 4–16*a* has a diameter of 10 mm. It is fixed to the wall at *A* and before it is loaded there is a gap between the wall at *B'* and the rod of 0.2 mm. Determine the reactions at *A* and *B'*. Neglect the size of the collar at *C*. Take $E_{st} = 200$ GPa.

SOLUTION

Compatibility. Here we will consider the support at *B'* as redundant. 多餘 Using the principle of superposition, Fig. 4–16*b*, we have

$$(\xrightarrow{+}) \qquad \cancel{0.0002 \text{ m}}\ 0.2\text{mm} = \delta_P - \delta_B \qquad (1)$$

Load-Displacement. The deflections δ_P and δ_B are determined from Eq. 4–2.

$$\delta_P = \frac{PL_{AC}}{AE} = \frac{[20(10^3)\text{ N}](\cancel{0.4\text{ m}}\ 400)}{\pi(\cancel{0.005\text{ m}}\ 5)^2[200(10^{\cancel{9}\,3})\text{ N/m}^2]} = \cancel{0.5093(10^{-3})\text{ m}}\ 0.5093 \text{ mm}$$

→ 400 + 800 = 1200

$$\delta_B = \frac{F_B L_{AB}}{AE} = \frac{F_B(1.20\text{ m})}{\pi(\cancel{0.005\text{ m}}\ 5^2)^2[200(10^{\cancel{9}\,3})\text{ N/m}^2]} = 76.3944(10^{-9})F_B$$

1.64×10^{-5} mm F_B

$\underset{P造成}{\delta_P} - \underset{牆反制的}{\delta_B} \doteq 0.2$ mm

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EXAMPLE 4.8 CONTINUED

Fig. 4–16

$0.2 = 0.5093 - 7.64 \times 10^{-5} F_B$

$F_B = 4048.42$ N ※

Substituting into Eq. 1, we get

$$0.0002 \text{ m} = 0.5093(10^{-3}) \text{ m} - 76.3944(10^{-9})F_B$$

$$F_B = 4.05(10^3) \text{ N} = 4.05 \text{ kN} \qquad \text{Ans.}$$

Equilibrium. From the free-body diagram, Fig. 4–16*c*,

$$\xrightarrow{+} \Sigma F_x = 0; \quad -F_A + 20 \text{ kN} - 4.05 \text{ kN} = 0 \quad F_A = 16.0 \text{ kN} \qquad \text{Ans.}$$

$F_A \leftarrow \boxed{\rightarrow}_{20k} \leftarrow F_B$ (4.05k)

$F_A = 20 - 4.05 = 16$ kN ※

2种 S.I.
- compatability ⇒ 分2段考慮 — 相減 = 0 < $F_A + F_B = P$, $\delta_A + \delta_B = \delta_{all}$; 吃掉 Gap
- Forced Method ⇒ 拿掉 wall — 讓產生变形

4.6 Thermal Stress — 熱應力 ε

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. Ordinarily this expansion or contraction is *linearly* related to the temperature increase or decrease that occurs. If this is the case, and the material is homogeneous and isotropic, it has been found from experiment that the displacement of a member having a length *L* can be calculated using the formula

$\delta \rightarrow \alpha\,\Delta T$

$$\boxed{\delta_T = \alpha\,\Delta T L} \qquad (4\text{–}4)$$

總变形量

where

熱膨脹係數 < CTE >

α = a property of the material, referred to as the ***linear coefficient of thermal expansion***. The units measure strain per degree of temperature. They are 1/°C (Celsius) or 1/K (Kelvin) in the SI system. Typical values are given on the inside back cover

~ 20 ppm (20×10^{-6}) = 20 Mε

ΔT = the algebraic change in temperature of the member → 可下對人的物件 產生的 總变形量

L = the original length of the member

δ_T = the algebraic change in the length of the member

$\varepsilon_T = \alpha\,\Delta T$

EXAMPLE 4.9 $\Delta T \rightarrow \sigma, F$

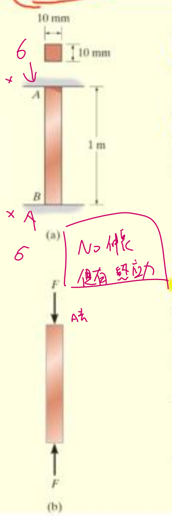
用 force Method

The A-36 steel bar shown in Fig. 4–17*a* is constrained to just fit between two fixed supports when $T_1 = 30°$C. If the temperature is raised to $T_2 = 60°$C, determine the average normal thermal stress developed

用force Metho

EXAMPLE 4.9

$\delta T \rightarrow \sigma, F$



The A-36 steel bar shown in Fig. 4–17*a* is constrained to just fit between two fixed supports when $T_1 = 30°C$. If the temperature is raised to $T_2 = 60°C$, determine the average normal thermal stress developed in the bar.

SOLUTION

Equilibrium. The free-body diagram of the bar is shown in Fig. 4–17*b*. Since there is no external load, the force at *A* is equal but opposite to the force at *B*; that is,

$$+\uparrow \Sigma F_y = 0; \qquad F_A = F_B = F$$

$F = F_A = F_B$

The problem is statically indeterminate since this force cannot be determined from equilibrium.

Compatibility. Since $\delta_{A/B} = 0$, the thermal displacement δ_T at *A* that occurs, Fig. 4–17*c*, is counteracted by the force **F** that is required to push the bar δ_F back to its original position. The compatibility condition at *A* becomes

$$(+\uparrow) \qquad \delta_{A/B} = 0 = \delta_T - \delta_F$$

溫度導致伸長

Applying the thermal and load–displacement relationships, we have

actually loaded

$$0 = \underset{伸}{\alpha \Delta T L} - \underset{縮}{\frac{FL}{AE}}$$

牆壁導致伸長

Thus, from the data on the inside back cover,

$$F = \alpha \Delta T A E$$

EXAMPLE 4.9 CONTINUED

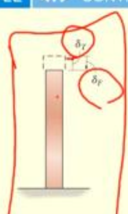


Fig. 4–17

$$F = [\underset{\alpha}{12(10^{-6})/°C}](\underset{\delta T}{60°C - 30°C})(\underset{A}{0.010\ m})^2\,[\underset{E}{200(10^6)\ kN/m^2}]$$

$$= 7.2\ kN$$

Since **F** also represents the internal axial force within the bar, the average normal compressive stress is thus

$$\sigma = \frac{F}{A} = \frac{7.2\ kN}{(0.010\ m)^2} = 72\ 000\ kPa = 72\ MPa \qquad \textit{Ans.}$$

$\frac{7200}{10^2} = 72\ MPa$ #

NOTE: From the magnitude of **F**, it should be apparent that changes in temperature can cause large reaction forces in statically indeterminate members.

$F = \alpha\ \delta T\ A\ E$
(N) mm² → MPa #

4.7 Stress Concentrations 應力集中

In 不連續區: $\sigma = \frac{F}{A} * K$

因孔何不連續造成的狀態

Figure: 04_PH06

This saw blade has grooves cut into it in order to relieve both the dynamic stress that develops within it as it rotates and the thermal stress that develops as it heats up. Note the small circles at the end of each groove. These serve to reduce the stress concentrations that develop at the end of each groove.

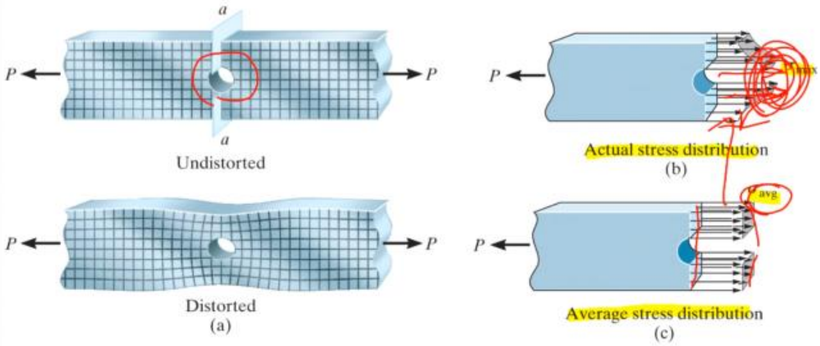


Figure: 04_20

In both of these cases, *force equilibrium* requires the magnitude of the *resultant force* developed by the stress distribution to be equal to P. In other words,

$$\boxed{P = \int_A \sigma \, dA} \qquad (4\text{–}5)$$

This integral *graphically* represents the total *volume* under each of the stress-distribution diagrams shown in Fig. 4–20*b* or Fig. 4–21*b*. The resultant **P** must act through the *centroid* of each *volume*. The results of these investigations are usually reported in graphical form using a ***stress-concentration factor*** K. We define K as a ratio of the Maximum stress to the average normal stress acting at the cross section; i.e.,

$\sigma_{max} = K \sigma_{avg}$

$$\boxed{K = \frac{\sigma_{max}}{\sigma_{avg}}} \qquad (4\text{–}6)$$

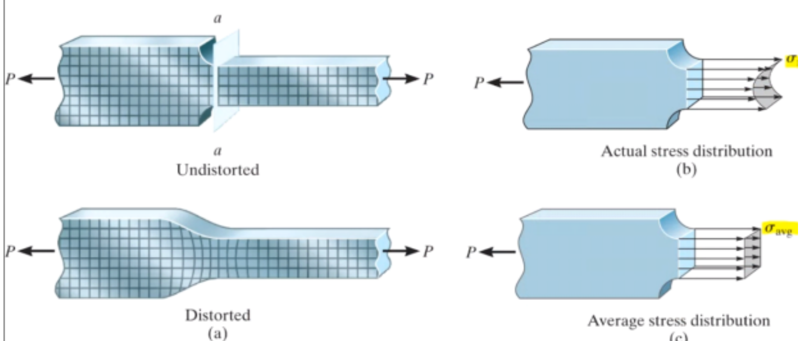


Figure: 04_21

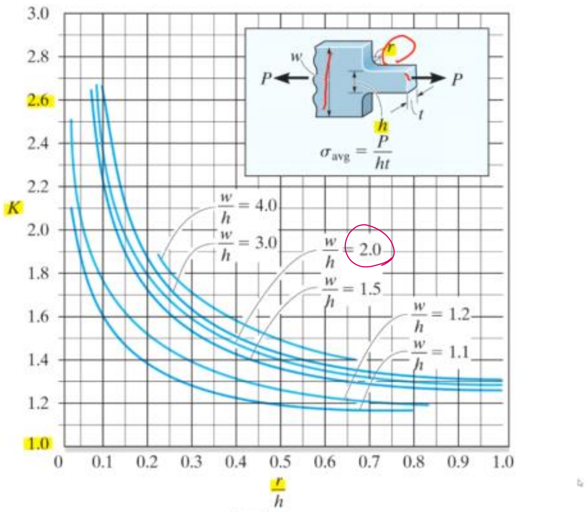


Figure: 04_23

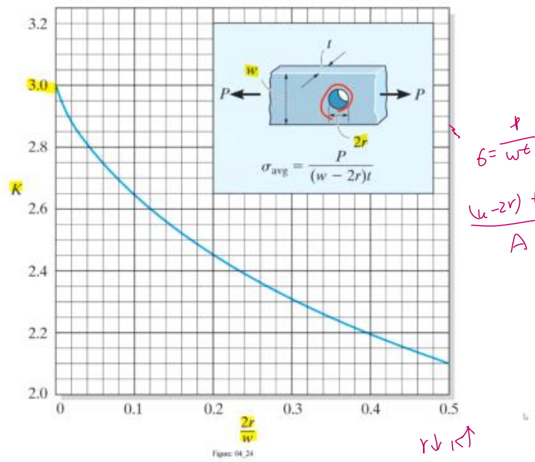


Figure: 04_24

$\sigma = \frac{P}{wt}$

$\frac{(w-2r)t}{A} * k = \sigma$

$r\downarrow\ k\uparrow$

HW (Due next week)

- 4-5, 8, 15, 23, 35, 37, 59, 70, 89