EXERCISE 3

Note: the "residual standard error" of a model is $\sqrt{\frac{SSE}{(n-p)}}$
with n sample size and p number of covariates

a) Denoting with $Y_i = mpg_i$ $\quad i = 1, \dots, 32$ (response variable)
and with x_{ij} = value of the j-th covariate on the i-th car
$j = 1, \dots, 9 \quad i = 1, \dots, 32$
(with $x_{i1} = 1$ for all i, since the model includes the intercept)

The model can be written as:

$$Y_i = \beta_1 + \beta_2 \underbrace{x_{i2}}_{wt_i} + \beta_3 \underbrace{x_{i3}}_{am_i} + \dots + \beta_9 \underbrace{x_{i9}}_{vs_i} \qquad \text{" model A"}$$

with $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$, $\quad i = 1, \dots, 32$.

b) Sample space $\mathcal{Y} = \mathbb{R}^{32}$
Parameter space $\Theta = \mathbb{R}^9 \times (0, +\infty)$

c) 1. t-value of "hp"
this value corresponds to the observed test statistic for testing the hypothesis

$$\begin{cases} H_0: \beta_6 = 0 \\ H_1: \beta_6 \neq 0 \end{cases}$$

$$t^{obs} = \frac{\hat{\beta}_6 - 0}{\sqrt{\hat{var}(\hat{\beta}_6)}} = \frac{\hat{\beta}_6}{\hat{se}(\hat{\beta}_6)} = \frac{-0.0214}{0.0162} = -1.321$$

2. estimate of "qsec"
this value is $\hat{\beta}_{qsec} = \hat{\beta}_8$ (max. lik. estimate)
we can derive it inverting the formula used in point 1.

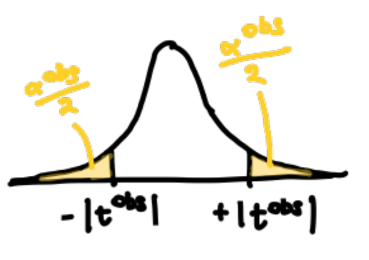
$$t^{obs} = \frac{\hat{\beta}_8}{\hat{se}(\hat{\beta}_8)} \Rightarrow \hat{\beta}_8 = t^{obs} \cdot \hat{se}(\hat{\beta}_8)$$
$$= 1.229 \cdot 0.6587 = 0.8095$$

3. Pr(>|t|) of "vs"
this is the pvalue of the test of significance of $\beta_{vs} = \beta_9$

$$\begin{cases} H_0: \beta_9 = 0 \\ H_1: \beta_9 \neq 0 \end{cases}$$

the test statistic

$$T = \frac{\hat{\beta}_9}{\sqrt{\hat{var}(\hat{\beta}_9)}} \overset{H_0}{\sim} t_{n-p} = t_{32-9} = t_{23}$$



the pvalue is
$$\alpha^{obs} = \mathbb{P}_{H_0}(|T| > |t^{obs}|)$$
$$= 2\, \mathbb{P}_{H_0}(T > |t^{obs}|)$$
$$= 2\, \mathbb{P}_{H_0}(T > 0.183)$$
where $T \sim t_{23}$
Student's t with 23 dof.

from the table we know that:
$\mathbb{P}_{H_0}(T \leq 0.183) < 0.90$
$\Rightarrow \mathbb{P}_{H_0}(T > 0.183) = 1 - \mathbb{P}_{H_0}(T \leq 0.183) > 0.10$
$\Rightarrow \alpha^{obs} = 2[1 - \mathbb{P}_{H_0}(T \leq 0.183)] > 0.20$
I do not reject H_0 for all usual significance levels.

d) We want to test

$$\begin{cases} H_0: \beta_2 = \beta_3 = \dots = \beta_8 = \beta_9 = 0 \\ H_1: \text{at least one of } \beta_j \text{ is } \neq 0 \quad (j = 2, \dots, 9) \end{cases}$$

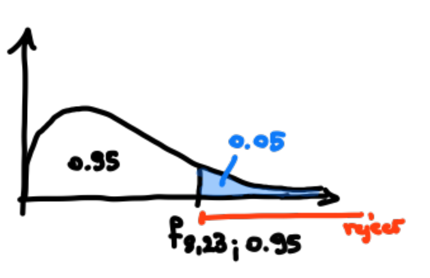
we use the statistic

$$F = \frac{\tilde{\Sigma}^2 - \hat{\Sigma}^2}{\hat{\Sigma}^2} \cdot \frac{n-p}{p-1} \overset{H_0}{\sim} F_{p-1, n-p} = F_{8,23}$$

with $\tilde{\Sigma}^2$ estimator of σ^2 under the null model $\tilde{\Sigma}^2 = \frac{SST}{n}$
$\hat{\Sigma}^2$ estimator under the full model "A" $\hat{\Sigma}^2 = \frac{SSE}{n}$

$$F = \frac{SST - SSE}{SSE} \cdot \frac{23}{8} = \frac{SSR/SST}{SSE/SST} \cdot \frac{23}{8} = \frac{R^2}{1-R^2} \cdot \frac{23}{8}$$

with the data I get $f^{obs} = \frac{0.8678}{1 - 0.8678} \cdot \frac{23}{8} = 18.87$



at a significance level of 5%: I reject H_0 if $f^{obs} > f_{8,23;0.95}$ ($= 2.3748$)

hence I reject H_0.

e) model B
$$Y_i = \beta_1 + \beta_2 \underbrace{x_{i2}}_{wt_i} + \beta_3 \underbrace{x_{i3}}_{am_i} + \varepsilon_i$$

We can use a test for comparing nested models, since model B can be obtained as a restriction of model A. Specifically, we need to test

$$\begin{cases} H_0: \beta_4 = \beta_5 = \dots = \beta_8 = \beta_9 = 0 \\ H_1: \text{at least one of } \beta_j \text{ is } \neq 0 \quad (j = 4, \dots, 9) \end{cases}$$

the test statistic is

$$F = \frac{\tilde{\Sigma}^2 - \hat{\Sigma}^2}{\hat{\Sigma}^2} \cdot \frac{n-p}{p-p_0} \overset{H_0}{\sim} F_{p-p_0, n-p} = F_{6,23}$$

where
$\tilde{\Sigma}^2$ estimator of σ^2 under H_0 (model B) $\quad \tilde{\Sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i^B)^2 = \frac{SSE_B}{n}$
$\hat{\Sigma}^2$ estimator under the full model A $\quad \hat{\Sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i^A)^2 = \frac{SSE_A}{n}$

$n = 32$
$p = 9 \quad p_0 = 3$

$SSE_A = (n-p) \cdot (\text{residual SE})^2 = 23 \cdot 2.544^2 = 148.85$
$SSE_B = (n-p_0) \cdot (\text{residual SE})^2 = 29 \cdot 3.098^2 = 278.33$

$$f^{obs} = \frac{278.33 - 148.85}{148.85} \cdot \frac{23}{6} = 3.334$$

- using a 5% significance level, I reject H_0 if $f^{obs} > f_{6,23;0.95}$ ($= 2.527$)
  Hence, I reject H_0: I prefer model A.

- using a 1% significance level, I reject H_0 if $f^{obs} > f_{6,23;0.99}$ ($= 3.71$)
  Here, I do not reject H_0: I prefer model B.

f) introducing the interaction we obtain the model

$$Y_i = \beta_1 + \beta_2 x_{i2} + \beta_3 x_{i3} + \delta z_i + \varepsilon_i$$

where $x_{i2} = wt_i$ (weight of car i)

$$x_{i3} = am_i = \begin{cases} 1 & \text{if car } i \text{ has manual transmission} \\ 0 & \text{if car } i \text{ has automatic transmission} \end{cases}$$

$$z_i = x_{i2} \cdot x_{i3} = \begin{cases} x_{i2} = wt_i & \text{if } am_i = 1 \quad \text{(weight of car } i \text{ if manual)} \\ 0 & \text{if } am_i = 0 \end{cases}$$

interpretation of parameters
- consider a car with automatic transmission ($x_{i3} = 0$, $z_i = 0$)
  the mean consumption is
  $\mu_i = \beta_1 + \beta_2 x_{i2}$
- for a car with manual transmission ($x_{i3} = 1$, $z_i = x_{i2}$)
  $\mu_i = \beta_1 + \beta_2 x_{i2} + \beta_3 + \delta x_{i2}$
  $= (\beta_1 + \beta_3) + (\beta_2 + \delta) x_{i2}$

β_1 is the intercept for cars with an automatic transmission
$\beta_1 + \beta_3$ is the intercept for cars with a manual transmission
β_2 is the effect on the mean consumption of increasing the weight of the car of 1 unit, for cars with an automatic transmission
$\beta_2 + \delta$ is the effect on the mean consumption of increasing the weight of the car of 1 unit, for cars with a manual transmission