EXERCISE 2

$(y_1, \dots, y_5)$ sample from $Y_i \sim \text{Pois}(e^{\beta_1})$ indep. $i = 1, \dots, 5$

$(y_6, \dots, y_{10})$ sample from $Y_i \sim \text{Pois}(e^{\beta_1 + \beta_2})$ indep. $i = 6, \dots, 10$

a) • $Y_i \sim \text{Poisson}(\mu_i)$ $i = 1, \dots, 10$ indep.

• $\eta_i = \beta_1 + \beta_2 x_i$ with $x_i = \begin{cases} 0 & i = 1, \dots, 5 \\ 1 & i = 6, \dots, 10 \end{cases}$

• $\log \mu_i = \eta_i$

b) $f(y_i ; \mu_i) = e^{-\mu_i} \mu_i^{y_i} \cdot \frac{1}{y_i!}$

$f(y_1, \dots, y_{10} ; \underline{\mu}) = \prod_{i=1}^{10} \left(e^{-\mu_i} \mu_i^{y_i} \frac{1}{y_i!} \right)$

$L(\underline{\mu}) \propto \prod_{i=1}^{10} e^{-\mu_i} \mu_i^{y_i}$ likelihood

$\ell(\underline{\mu}) = \sum_{i=1}^{10} \{ -\mu_i + y_i \log \mu_i \}$ loglikelihood. Since the parameters of interest are (β_1, β_2) and $\mu_i = e^{\beta_1 + \beta_2 x_i}$

$$\ell(\beta_1, \beta_2) = \sum_{i=1}^{10} \left\{ -e^{\beta_1 + \beta_2 x_i} + y_i (\beta_1 + \beta_2 x_i) \right\} \quad \leftarrow \text{loglikelihood of } (\beta_1, \beta_2)$$

$$= -\sum_{i=1}^{10} e^{\beta_1 + \beta_2 x_i} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=1}^{10} x_i y_i$$

$$= -\sum_{i=1}^{5} e^{\beta_1} - \sum_{i=6}^{10} e^{\beta_1 + \beta_2} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=6}^{10} y_i$$

$$= -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \beta_1 \sum_{i=1}^{10} y_i + \beta_2 \sum_{i=6}^{10} y_i$$

$$\ell_*(\beta_1, \beta_2) = \begin{cases} \dfrac{\partial \ell(\underline{\beta})}{\partial \beta_1} = -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \sum_{i=1}^{10} y_i \\ \dfrac{\partial \ell(\underline{\beta})}{\partial \beta_2} = -5e^{\beta_1 + \beta_2} + \sum_{i=6}^{10} y_i \end{cases} \quad \text{score function}$$

$$\ell_{**}(\beta_1, \beta_2) = \begin{bmatrix} -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} & -5e^{\beta_1 + \beta_2} \\ -5e^{\beta_1 + \beta_2} & -5e^{\beta_1 + \beta_2} \end{bmatrix} = -5 \begin{bmatrix} e^{\beta_1} + e^{\beta_1 + \beta_2} & e^{\beta_1 + \beta_2} \\ e^{\beta_1 + \beta_2} & e^{\beta_1 + \beta_2} \end{bmatrix}$$

observed info is $j(\underline{\beta}) = -\ell_{**}(\underline{\beta})$

The MLE can be found by noticing that

for sample 1 $(y_1, \dots, y_5) = \underline{y}_1$ the expected value is $\mathbb{E}[Y_i] = \mu_1 = e^{\beta_1}$

for sample 2 $(y_6, \dots, y_{10}) = \underline{y}_2$ the expected value is $\mathbb{E}[Y_i] = \mu_2 = e^{\beta_1 + \beta_2}$

The function from (μ_1, μ_2) to (β_1, β_2) is bijective: $(\beta_1, \beta_2) = f(\mu_1, \mu_2)$

I can obtain the MLE $(\hat{\mu}_1, \hat{\mu}_2)$ and obtain $(\hat{\beta}_1, \hat{\beta}_2) = f(\hat{\mu}_1, \hat{\mu}_2)$

the MLE of μ_1 and μ_2 are

$\hat{\mu}_1 = \bar{y}_1 = \frac{1}{5} \sum_{i=1}^{5} y_i$ sample mean of $\underline{y}_1$

$\hat{\mu}_2 = \bar{y}_2 = \frac{1}{5} \sum_{i=6}^{10} y_i$ sample mean of $\underline{y}_2$

$$\Rightarrow \begin{cases} \hat{\mu}_1 = e^{\hat{\beta}_1} \\ \hat{\mu}_2 = e^{\hat{\beta}_1 + \hat{\beta}_2} \end{cases} \Rightarrow \quad \hat{\beta}_1 = \log \hat{\mu}_1 = \log \bar{y}_1$$

$$\hat{\beta}_1 + \hat{\beta}_2 = \log \hat{\mu}_2 \Rightarrow \hat{\beta}_2 = \log \hat{\mu}_2 - \hat{\beta}_1 = \log \hat{\mu}_2 - \log \hat{\mu}_1$$

hence $\hat{\beta}_1 = \log \bar{y}_1$

$\hat{\beta}_2 = \log \bar{y}_2 - \log \bar{y}_1 = \log \frac{\bar{y}_2}{\bar{y}_1}$ $\quad \left(\begin{matrix} e^{\hat{\beta}_1} = \bar{y}_1 \\ e^{\hat{\beta}_2} = \frac{\bar{y}_2}{\bar{y}_1} \end{matrix} \right)$

Alternatively, one can solve the likelihood equations

$$\begin{cases} -5e^{\beta_1} - 5e^{\beta_1 + \beta_2} + \sum_{i=1}^{10} y_i = 0 & \rightarrow 5e^{\beta_1} = \sum_{i=1}^{10} y_i - \sum_{i=6}^{10} y_i \rightarrow e^{\beta_1} = \frac{1}{5} \left(\sum_{i=1}^{10} y_i - \sum_{i=6}^{10} y_i \right) = \bar{y}_1 \\ -5e^{\beta_1 + \beta_2} + \sum_{i=6}^{10} y_i = 0 & \rightarrow e^{\beta_1 + \beta_2} = \dfrac{\sum_{i=6}^{10} y_i}{5} = \bar{y}_2 \end{cases}$$

c) $\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \dot{\sim} N_2 \left(\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, j(\hat{\underline{\beta}})^{-1} \right)$

the marginal $\hat{\beta}_1 \dot{\sim} N(\beta_1, [j(\hat{\underline{\beta}})^{-1}]_{(1,1)})$

↳ element in position (1, 1)

d) we already noticed that

for sample 1 $\mathbb{E}[Y_i] = \mu_1 = e^{\beta_1}$

for sample 2 $\mathbb{E}[Y_i] = \mu_2 = e^{\beta_1 + \beta_2}$

$$\frac{\mu_2}{\mu_1} = \frac{e^{\cancel{\beta_1} + \beta_2}}{\cancel{e^{\beta_1}}} = e^{\beta_2} \Rightarrow \beta_2 = \log \frac{\mu_2}{\mu_1} = \log \frac{\mathbb{E}[Y_i \mid \text{sample 2}]}{\mathbb{E}[Y_i \mid \text{sample 1}]}$$

Hence β_2 represent the log of the ratio between the means of the two samples.

Or, noting that $\mu_2 = e^{\beta_2} \cdot \mu_1$

the mean of sample 2 is obtained by multiplying the mean of sample 1 of a coefficient e^{β_2}

e) The saturated model is a model with n parameters $\mu_1, \dots, \mu_n$.

In this case, I have one parameter for each observation and the estimates are

$\hat{\mu}_i = y_i \quad \forall i$

Hence the max of the loglikelihood is

$$\tilde{\ell}(\text{saturated}) = \sum_{i=1}^{10} \{ -\hat{\mu}_i + y_i \log \hat{\mu}_i \} = \sum_{i=1}^{10} \{ -y_i + y_i \log y_i \}$$