Heterogeneous Agents and General Equilibrium in Financial Markets

Tyler Abbot

Abstract

This paper investigates the effects of heterogeneous preferences on risk premia, on asset price dynamics, and on distributions of wealth and consumption. I consider a continuous time economy populated by an arbitrarily large number of agents whose CRRA preferences differ in their risk aversion parameter. I derive closed form expressions for the interest rate, the market price of risk, and drift and diffusion of consumption weights. I find that agents dynamically self select into one of three groups depending on their preferences: leveraged investors, diversified investors, and saving divestors. The thresholds for selection are driven by a wedge between the market price of risk and the risk free rate. I then simulate the economy using a parallelized/GPU implementation of path monte-carlo methods to estimate asset prices and compare the qualitative and quantitative features to real world observations. The model does well in replicating a falling dividend to price ratio over time, low risk free rate, and rising inequality. However, changes in the initial condition have non-trivial effects on the short run outcomes of the model, which points towards the need for a more coherent theory of the distribution of risk preference.

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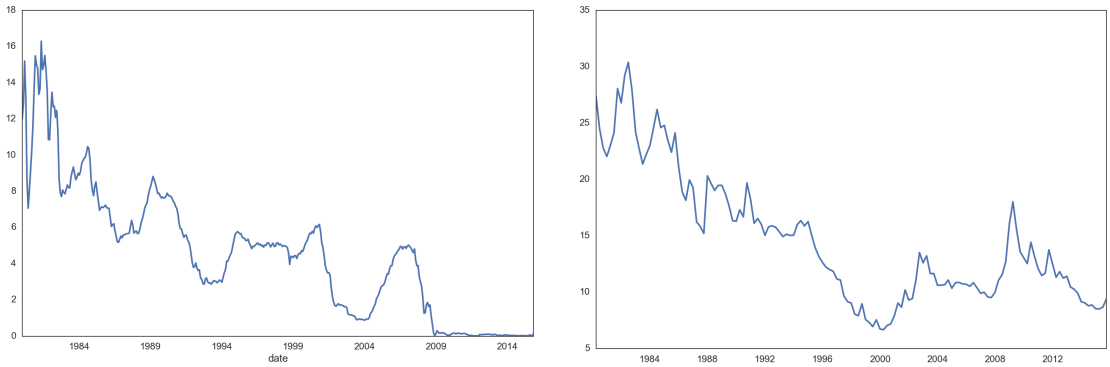
Introduction

Each day, trillions of dollars worth of financial assets change hands. Being simply a piece of paper, a financial security gives its bearer the right to a stream of future dividends and capital gains for the infinite future. The price of this abstract object is so difficult to determine that if you ask two analysts for an exact price they will generally disagree. This fact has been well documented in studies such as Andrade, Crump, Eusepi, and Moench (2014) or Carlin, Longstaff, and Matoba (2014). These papers consider disagreement to be driven by imperfect information, but we should expect financial analysts to be well informed. The Grossman-Stiglitz Paradox tells us that if prices do not contain all of the information about an asset, then there is room for competition on information gathering, so banks and financial institutions have an incentive to gather information. Thus analysts spend much of their time studying the market and its constituents. Because of this, it seems unlikely that this disagreement is driven entirely by informational frictions.

If we abstract away from imperfect information, we can try to think of other reasons for disagreement about asset prices. The fact that two people in the same situation with the same information might price a risky asset differently could also be explained by differences in their preference towards risk. This line of thinking is in direct contrast to that of a representative agent framework and it is this contrast that makes heterogeneous preference models such a fruitful pursuit. Take for instance the aspect of trade in financial assets previously mentioned. With a single agent there can be no exchange because there is no counter party. We look for a set of prices to make the representative agent indifferent to consuming everything, holding the entire capital stock, etc. In order to have exchange in an economic model we must introduce two or more agents who differ in some way.

When returns are endogenous and we allow there to exist inherent differences in agents' risk preference, the problem ceases to be time consistent and we end up with constantly shifting distributions of wealth,, consumption, and financial variables. Although this time inconsistency makes the use of dynamic programming impossible, we can employ the martingale method pioneered by Harrison and Pliska (1981) and further refined by Karatzas and Shreve (1998). In fact, this time-inconsistent nature of the problem may be the very characteristics that brings it closer to the real world. I think few people would claim that interest rates and dividend yields are stationary processes (see Figure 1; sources: FRED, Yahoo! Finance), but that they have exibited clear downward trends since the 1980's. This paper finds that these trends are consistent with an economy populated by agents with heterogeneous risk preferences.

Since Dumas (1989), there has been both theoretical and empirical work on heterogeneous risk preferences. The field lay relatively dormant until Coen-Pirani (2004) studied the introduction of Epstein-Zin preferences into an economy of two agents and found the counter-intuitive result that, under certain parametrizations, the most risk averse agent



(a) 3 month treasury rate

(b) Dividend yield of US economy

Fig. 1. The evolution of both interest rates and dividend yields since 1980 show how financial variables are clearly non-stationary. Note: "Dividend yield" is here calculated as the ratio of GDP to the level of the S&P 500.

could dominate. Since then, the empirical work has focused on trying to estimate the distribution of risk aversion that might prevail in the economy, as in Kimball, Sahm, and Shapiro (2008). Theoretical work has tried to identify how and why different agents with different preferences participate in different ways. Guvenen (2006) uses participation constraints to study the effects of allowing only low risk aversion agents to participate in financial markets and are able to replicate well the real world wealth distribution. This approach assumes the participation choice, while in a more complete markets vain, Bhamra and Uppal (2014), Chabakauri (2013), Gârleanu and Panageas (2015), and Cozzi (2011) have all worked on studying the interactions of two heterogeneous agents, where the participation (or portfolio) decision is made endogenous. This paper extends that work by making the choice endogenous among N agents.

The majority of the theoretical work on heterogeneous risk preferences focuses on agent survival and asymptotic properties. Authors attempt to derive closed form solutions for asset prices when there are two agents, assuming that the results generalize to many agents. These techniques often miss the dynamics of the market and the complexity one would find using an economy populated by many heterogeneous agents. I try to add to the literature by deriving expressions for the dynamics of agents' consumption, interest rates, the market price of risk, and an estimable expression of asset prices when the number of agents is arbitrarily large. Using these I simulate an economy, studying its properties over the short run. My work most closely matches that of Cvitanić, Jouini, Malamud, and Napp (2011), who study an economy populated by agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. However, they focus on issues of long run survival and price. I build on their results by studying how changes in the distribution of preferences effect the short run dynamics and speed of convergence of the model, while focusing on a single aspect of heterogeneity: risk aversion.

If we think of individual agents as each having a supply and demand function for risky assets and risk free bonds, it is possible to think of a model of heterogeneous risk preferences as one of market break down. Each agent populates a single, theoretical market, but only one market can clear. The market which clears is the one corresponding to the agent who is indifferent between buying or selling their shares or bonds. In fact, the formulas for the risk free rate and the market price of risk derived in this paper resemble greatly those in Basak and Cuoco (1998). In that paper, two agents participate in the economy, but one is restriceted from participating in the financial market. However in this paper, contrary to the limited participation literature, the clearing markets for stocks and bonds do not have to correspond to the same agent, nor does the corresponding agent even need to exist in the economy. We will see in section 2.2 that two moments of the distribution of shares determine the market clearing agents. Additionally, these values will vary over time and will be endogenously determined.

In studying the dynamics of the model, I simulate several different economies and compare the results. In particular I focus on increasing the number of agents in the economy over a given spread for the risk aversion parameter. This has two key effects on the initial conditions and parameters of the model. First, increasing the number of agents has the effect of reducing their share of the endowment in the financial asset. Second, it changes the weighted average of the elasticity of intertemporal substitution (EIS), which is a key determinant in the model's outcomes[1]. These changes imply a change in the distribution of financial wealth and a change in the distribution of preferences. Because of these effects, simulations of this model are highly influenced by the choice of the underlying distribution's support. This points towards a possible future direction of research in studying the limiting where $N \to \infty$, removing this degree of freedom in the parameterization.

In the formulas and simulations I find that preference heterogeneity drives a wedge between the interest rate and the market price of risk, providing a partial explanation for the equity risk premium puzzle of Mehra and Prescott (1985). Additionally, the dividend-price ratio is trend-stationary and indicates predictability of stock returns up to and around this trend. This result is consistent with those of Campbell and Shiller (1988b) and Campbell and Shiller (1988a), who find that the returns on stocks can be predicted as a function of dividend yield. This could explain the result of Mankiw (1981), who rejects the permanent income hypothesis on the basis that asset price comovements with the stochastic discount facter are forecastable.

In Section 1, I construct a continuous time model of financial markets populated by a finite number of agents who differ in their preferences towards risk. Section 2 solves the model up to an estimable equation for asset prices, giving closed form solutions for the interest rate, the market price of risk, and dynamics, as well as discussing market

[1] A fruitful direction for future research would be to exchange the preferences in this model with Epstien-Zin preferences, to differentiate between risk aversion and EIS.

segmentation. In Section 3, I give simulation results for changing the number of agents over a given support. Finally, Section 4 concludes and gives some ideas for future research and applications. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

In this section, I will describe the general setting of the model. The key components are the definition of agent heterogeneity, the economic uncertainty, agent optimization, portfolio admissibility, and equilibrium conditions. The solution method will be discussed in the following section.

1.1. Agent Heterogeneity

I consider a continuous time economy populated by a finite number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. Each agent has constant relative risk aversion (CRRA) preferences such that their risk aversion parameter, γ_i, is drawn from a distribution $f(\gamma)$ whose support is open and bounded [2] such that $\gamma_i \in (\underline{\gamma}, \overline{\gamma})$:

$$U_i(c(t)) = \frac{c(t)^{1-\gamma_i} - 1}{1 - \gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

$$\gamma_i \sim f(\gamma)$$

In this paper, I will consider $\gamma \in (1, \overline{\gamma})$ for ease of exposition. This model encompasses a representative agent, a two agent frame-work, and up to a countable, finite number of agents.

1.2. Fincancial Markets

Agents have available to them one risky asset, representing claims to a Lucas tree whose dividend process follows a geometric Brownian motion, and risk free borrowing and lending at an interest rate $r(t)$ in zero net supply. All uncertainty in the model is driven by a standard Wiener process, $W(t)$, defined on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F})$. Thus the evolution of dividends in the economy is given by

$$\frac{dD(t)}{D(t)} = \mu_D dt + \sigma_D dW(t) \tag{1}$$

[2]This ensures that no agent is sure that they are the most or least risk averse. If the support were closed, an agent whose preferences were on an extremum could know their position in the distribution relative to others. This would create problems for the assumption of a nash equilibrium and change the best response of these extreme agents.

where μ_D and σ_D are constants. Agents can continuously trade in claims to the dividend process whose price, $S(t)$, also follows a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dW(t) \tag{2}$$

whose individual shares are denoted $\omega^i(t)$[3]. Here $\mu_S(t)$ and $\sigma_S(t)$ are time varying and determined in general equilibrium. Additionally, agents can borrow and lend at a time varying interest rate $r(t)$ using a risk free bond, whose individual share is denoted $b^i(t)$. The price of the risk free bond, denoted $S^0(t)$, thus follows a deterministic[4] process whose dynamics are given by

$$\frac{dS^0(t)}{S^0(t)} = r(t)dt \tag{3}$$

1.3. Budget Constraints and Individual Optimization

All agents are initially endowed with a share, $\omega_i(0)$, in the tree. Assume also that agents are initially endowed with zero savings or borrowing. Define individual wealth as $X^i(t) = \omega^i(t)S(t) + b^i(t)S^0(t)$. At any time t an agents dynamic budget constraint can be written as

$$dX^i(t) = \left[r(t)X^i(t) + \omega^i(t)S(t)\left(\mu_S(t) + \frac{D(t)}{S(t)} - r(t)\right) - c^i(t)\right]dt + \omega^i(t)\sigma_s(t)S(t)dW(t) \tag{4}$$

where the set of variables $\{c^i(t), \omega^i(t), X^i(t), D(t), S(t), r(t), W(t)\}$ represent an agent's consumption, asset holdings, and wealth , as well as the dividend, market clearing asset price, market clearing risk free interest rate, and Wiener process governing the Brownian motion. The budget constraint is standard (see for example Karatzas, Lehoczky, and Shreve (1987)) and can be interpreted as the instantaneous change in wealth being equal to the return on savings plus the net returns on holding the risky asset plus a mean zero random term governed by the risk process. This stochastic differential equation admits

[3]A simplifying assumption is that agents do not have access to a storage technology for dividends nor a market for trade in consumption and thus must consume their dividend flow. This implies that $c^i(t) = \omega^i(t)D(t)$.

[4]It is not necessarily the case that $r(t)$ is deterministic.

a unique strong form solution (see Yong and Zhou (1999), Theorem 6.14) given by

$$
\begin{aligned}
X_t^i = \exp\left\{\int_0^t r(s)ds\right\}\Bigg[X^i(0) \\
+ \int_0^t \exp\left\{-\int_0^s r(u)du\right\}\left(\omega^i(t)S(t)\left(\mu_S(t)+\frac{D(t)}{S(t)}-r(t)\right)-c^i(t)\right)ds \\
\int_0^t \exp\left\{-\int_0^s r(u)du\right\}\omega^i(s)\sigma_S(s)S(s)dW_s\Bigg]
\end{aligned}
$$

Using these facts an individual agent's constrained maximization subject to instantaneous changes in wealth can be written as:

$$
\begin{array}{ll}
\max_{\{c^i(u),a^i(s),b^i(u)\}_{u=t}^{\infty}} & \mathbb{E}\int_t^{\infty} e^{-\rho(u-t)}\frac{c^i(u)^{1-\gamma_i}-1}{1-\gamma_i}du \\
\text{s.t.} & eq.\ (4)
\end{array}
$$

1.4. Admissibility

In order to ensure that an agent's optimal policy satisfies our real world assumptions about feasibility, it is necessary to impose some restrictions on the admissible set of policies. By "real world" assumptions I am referring to the non-negativity of consumption and wealth. If an agent's optimal policy implies that they could hold more debt than equity, driving their wealth into negative territory, they could borrow infinitely. This case is ruled out in the real world and we should thus limit our attention to a smaller set of admissible policies. This issue was first treated in Karatzas, Lehoczky, Sethi, and Shreve (1986) and I follow their assumptions here.

Define the portfolio process $\omega^i(t)$ and consumption process $c^i(t)$ measurable, adapted, real valued processes such that

$$
\int_0^{\infty} \omega^i(t)^2 dt < \infty \quad \text{a.s.}
$$
$$
\int_0^{\infty} c^i(t)dt < \infty \quad \text{a.s.}
$$

Then we can define the set of admissibile policies by the following:

Definition 1. *A pair of policies $(\omega^i(t), c^i(t))$ is said to be admissible for the initial endowment $X^i(0) \geq 0$ for agent i's optimization problem if the wealth process $X(t)$ satisfies*

$$
X^i(t) \geq 0, \quad \forall\, t \in [0,\infty) \quad a.s.
$$

Denote by $\mathcal{A}(X^i(0))$ the set of all such admissible pairs.

This definition of the admissible set will guarantee that agents' wealth processes remain positive and that they satisfy a supermartingale property under the maringale measure defined by the stochastic discount factor. This fact will allow us to solve a static problem as opposed to the dynamic one.

1.5. Equilibrium

Each agent will be considered to be a price taker. This implies an Arrow-Debreu type equilibrium concept.

Definition 2. *An equilibrium in this economy is defined by a set of processes* $\{r(t), S(t), \{c^i(t), X^i(t), \omega^i(t))\}_{i=1}^N\}\ \forall\ t$, *given preferences and initial endowments, such that* $\{c^i(t), \omega^i(t), X^i(t)\}$ *solve the agents' individual optimization problems and the following set of market clearing conditions is satisfied:*

$$
\begin{aligned}
\sum_i c^i(t) &= D(t) \\
\sum_i \omega^i(t) &= 1 \\
\sum_i X^i(t) &= S(t)
\end{aligned}
\tag{5}
$$

The definition of equilibrium can be interpreted as agents choosing their consumption and portfolio choices to maximize their individual utilities. These choices in turn must satisfy the market clearing conditions for consumption, asset holdings, and wealth. In order to do so, interest rates and asset prices adjust.

2. Equilibrium Characterization

This section will derive a solution to each agent's maximization problem and give results on the characteristics of equilibrium. In deriving the equilibrium, it is interesting to note that the choices of agents can be seen as determining first and foremost their consumption, which in turn determines their portfolio choice. These choices give rise to a market clearing price of risk, a risk free interest rate, and a stock price. All of these outcomes can be compared (see appendix A) to an economy populated by a single, representative agent.

2.1. The Static Problem

Given that there is only one market and one risky asset, there is only one stochastic discount factor for the market. Following Karatzas and Shreve (1998) we can define the stochastic discount factor as

$$H_0(t) = exp\left(-\int_0^t r(u)du - \int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta(u)^2 du\right) \tag{6}$$

where

$$\theta(t) = \frac{\mu_s(t) + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)} \tag{7}$$

represents the market price of risk. This implies that the stochastic discount factor also follows a diffusion of the form

$$\frac{dH_0(t)}{H_0(t)} = -r(t)dt - \theta(t)dW(t) \tag{8}$$

It is important to keep in mind that agents are not discounting at their own rate, but at a market rate. This is because each agent knows that their only choice is to buy or sell assets at the market rate, assuming that there is no arbitrage. If it were possible for there to be many markets, each one clearing at an individual agent's price, one could buy risky assets in a market with a risk averse agent and sell them in a market with a risk neutral agent at a higher price, making a positive profit.

The process defined by $H_0(t)exp\left\{\int_0^t r(s)ds\right\}$ is a martingale under the measure $\mathbb{P}$. To make use of Girsanov theory we can define a new measure

$$\mathbb{Q}(A) = \mathbb{E}\left[H_0(t)exp\left\{\int_0^t r(s)ds\right\}\mathbb{1}_A\right], \quad A \in \mathcal{F}_t$$

Then we can rewrite the wealth process in terms of a new process $\tilde{W}(t)$ defined by

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s)ds$$

which is a Brownian motion under $\mathbb{Q}$. Thus we have

$$\begin{aligned} X_t^i \exp\left\{\int_0^t -r(s)ds\right\} + \int_0^t \exp\left\{-\int_0^s r(u)du\right\} c^i(s)ds = \\ X^i(0) + \int_0^t \exp\left\{-\int_0^s r(u)du\right\} \omega^i(s)\sigma_S(s)S(s)d\tilde{W}_s \end{aligned} \tag{9}$$

By the definition of $\tilde{W}(t)$, the right hand side of eq. (9) is a local martingale under $\mathbb{Q}$. This implies that the left hand side is then a supermartingale under $\mathbb{Q}$, and we have

$$\mathbb{E}^{\mathbb{Q}}\left[X_t \exp\left\{\int_0^t -r(s)ds\right\} + \int_0^t \exp\left\{-\int_0^s r(u)du\right\} c^i(s)ds\right] \leq X^i(0)$$

Following Propositions 2.6 from Karatzas et al. (1987), given an admissible pair $(a^i(t), c^i(t))$ we can rewrite each agent's dynamic problem as a static one

$$\begin{aligned} \max_{\{c^i(u)\}_{u=0}^{\infty}} \quad & \mathbb{E}\int_0^{\infty} e^{-\rho u}\frac{c^i(u)^{1-\gamma_i}-1}{1-\gamma_i}du \\ \text{s.t.} \quad & \mathbb{E}\int_0^{\infty} H_0(u)c^i(u)du \leq X_0^i \end{aligned}$$

If we denote by Λ_i the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c^i(u) = \left(e^{\rho u}\Lambda_i H_0(u)\right)^{\frac{-1}{\gamma_i}} \tag{10}$$

which holds for every agent in every period.

2.2. Consumption Weights

Given each agent's first order conditions, we can derive an expression for consumption as a fraction of total dividends.

Proposition 1. *One can define the consumption of individual, i, at any time, t, as a share $\omega^i(t)$ of the total dividend, $D(t)$, such that*

$$c^i(t) = \omega^i(t)D(t) \tag{11}$$

$$\textit{where } \omega^i(t) = \frac{\left(\Lambda_i e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda_j e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_j}}} \tag{12}$$

This expression recalls the results in Basak and Cuoco (1998) or Cuoco and He (1994), where $\omega(t)$ acts like a time-varying pareto-negishi weight. In those works, however, participation is driven by an imperfection in the information structure or some exogenous constraint. Here the choice of participation is driven by preferences towards risk. The value of the stochastic discount factor is equal across agents, but differs in its weight for each agent as they differ in risk aversion. This leads one to think that perhaps it would be better to think of this as an incomplete market. If markets were fully complete, there would be a risky asset for each agent, but here agents are forced to bargain over a single asset.

To derive an expression for the risk free rate and the market price of risk, we will need the following lemma about the drift and diffusion of agents' consumption processes:

Lemma 1. *If we model an agent's consumption as a geometric Brownian motion with time varying drift and diffusion coefficients $\mu_{c^i}(t)$ and $\sigma_{c^i}(t)$, then we have the following* *relationship between $\mu_{c^i}(t)$, $\sigma_{c^i}(t)$, $r(t)$ and $\theta(t)$, and for all $i \in \{1, ..., N\}$*

$$
\begin{aligned}
r(t) &= \rho + \mu_{c_i}\gamma_i - (1+\gamma_i)\gamma_i \frac{\sigma_{c_i}^2}{2} \\
\theta(t) &= \sigma_{c_i}\gamma_i
\end{aligned}
$$

These formulas are identical to those one would find in a standard representative agent model. However, these expressions hold here simultaneously for all agents, meaning that the growth rate and volatility of consumption for each agent must adjust, while in a representative agent they would be replaced by the drift and diffusion of the dividend process. In order to better understand how these values adjust, rewrite Lemma 1 in terms of $\mu_{c^i}(t)$ and $\sigma_{c^i}(t)$ and differentiate to get

$$
\begin{aligned}
\frac{\partial \mu_{c_i}(t)}{\partial \theta(t)} &= \frac{1+\gamma_i}{\gamma_i^2}\theta(t) \\
\frac{\partial \mu_{c_i}(t)}{\partial r(t)} &= \frac{1}{\gamma_i} \\
\frac{\partial \sigma_{c_i}(t)}{\partial \theta(t)} &= \frac{1}{\gamma_i} \\
\frac{\partial \sigma_{c_i}(t)}{\partial r(t)} &= 0
\end{aligned}
$$

These partial derivatives imply that the growth rate of every individual's consumption is increasing in both the market price of risk and in the interest rate. All things being equal, holding portfolios and preferences constant, a higher market price of risk implies greater returns. Thus, any given agent will earn more on their portfolio and can expect a higher (or less negative) growth rate in consumption. However, the magnitude of this effect depends both on the prevailing market price of risk and the agent's preferences.

First, consider $\frac{\partial \mu_{c_i}(t)}{\partial \theta(t)}$. When $\gamma_i = 1$, the coefficient is 2 and as γ_i increases the coefficient falls asymptotically towards zero. For more risk averse agents, the change in the expected growth rate of consumption in response to changes in $\theta(t)$ is smaller, going to zero as gamma goes to infinity. This is driven by a consumption smoothing motive. More risk averse agents dislike fluctuations in their consumption and are thus less sensitive to changes in the market. That being said, they will need to dynamically reallocate their portfolio in order to offset the effect of shocks on their consumption. This effect will manifest itself in the sign of the diffusion of their portfolio share, given in Proposition 3 below.

Second, notice that every agent's expected growth rate in consumption is increasing in the interest rate. This makes sense for agents who are net lenders, as they see greater returns on their savings, but this is counter-intuitive for agents who are net borrowers. It implies that, despite having to pay a higher interest rate on their borrowing they prefer to grow their consumption more quickly. This is driven by a wealth effect. An increase in the interest rate lowers the stochastic discount factor, reducing the price of consumption today and in the future. This makes the budget constraint less binding for both the lender and the borrower because they both face a higher economy-wide risk free rate. They use this economy-wide variable to calculate the present value of their future consumption and equate it to their wealth. Thus, ceterus paribus, a higher interest rate implies a lower present value of lifetime consumption, whether an agent is a lender or borrower. Because markets are complete, agents borrow solely to finance their consumption choices. So the loosening of the budget constraint will cause an increase in consumption growth rates for all agents despite their financial position.

Finally, notice that diffusion in consumption is increasing in the market price of risk, but this effect is decreasing in γ_i. First, it makes sense that this value is decreasing in γ_i, as a more risk averse agent will respond less to changes in the market; more risk averse agents desire a smoother consumption path. However, why it co-moves positively with the market price of risk is unclear. In order to understand this effect, we need to understand the determinants of the market price of risk.

2.3. The Risk-Free Rate and Market Price of Risk

Given Lemma 1, we can derive expressions for the market price of risk and the risk free rate:

Proposition 2. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ *and* $\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$ *such that*

$$r(t) = \rho + \frac{\mu_D}{\xi(t)} - \frac{1}{2}\frac{\xi(t) + \phi(t)}{\xi(t)^3}\sigma_D^2 \tag{13}$$

$$\theta(t) = \frac{\sigma_D}{\xi(t)} \tag{14}$$

by Lemma 1 and eq. (5)*.*

Proposition 2 is in terms of only certain moments of the joint distribution of asset holdings and risk aversion: $\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$. These represent the weighted averages of elasticity of intertemporal substitution and its square, where the weights correspond to consumption weights. In other words, an agent's preferences only effect the market clearing interest rate and market price of risk up to their amount of participation in the market[5].

[5]This is highly reminiscent of a mean field game, in which agents consider the effect of the distribution of agents on their own optimal control.

In (14), we can see that the market price of risk in the heterogeneous economy is equal to the market price of risk that would prevail in a representative agent economy populated by an agent whose elasticity of inter-temporal substitution is equal to the consumption weighted average in our economy. This is because the market price of risk is determined by agents choosing the diffusion of their consumption. Each agent will increase or decrease their asset holdings such that the diffusion of their consumption is equal the market price of risk scaled down by their risk aversion (Lemma 1). Simply by market clearing, we know that the weighted average of their diffusions must equal the diffusion of the dividend process. In other words the sum of the variation in individuals' consumption can be no larger than the variation in the whole economy. It follows that the market price of risk must be the diffusion in the dividend process scaled by the weighted average of the EIS.

Looking at (13), the first two terms are very reminiscent of the interest rate in a representative agent economy populated by the same agent that would determine the market price of risk. That is if we were to use a representative agent model where the agent's CRRA parameter satisfied $\frac{1}{\gamma} = \xi(t)$ we would find the same market price of risk and nearly the same interest rate. However, the last term of (13) is slightly different. We can rewrite eq. (13) as the interest rate that would prevail in our hypothetical economy where $\frac{1}{\gamma} = \xi(t)$, plus an extra term:

$$r(t) = \rho + \frac{\mu_D}{\xi(t)} - \frac{1}{2}\frac{\xi(t)+1}{\xi(t)^2}\sigma_D^2 - \frac{1}{2}\frac{1}{\xi(t)}\left(\frac{\phi(t)}{\xi(t)^2} - 1\right)\sigma_D^2$$

If it were the case that $\phi(t) = \xi(t)^2$, then this additional term would be zero and the interest rate and market price of risk in this model could be exactly matched by those in an economy populated by a representative agent with time varying risk aversion, similar to the model of habit formation by Campbell and Cochrane (1999). However, given that the weights $\omega^i(t)$ sum to one we can apply the discrete version of Jensen's inequality to show[6] that $\phi(t) > \xi(t)^2, \quad \forall t < \infty$. This causes the additional term to be strictly negative. The risk free rate is then lower than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge", which I'll define as $\frac{\phi(t)}{\xi(t)^2} > 1$, between the price of risk and the price for risk free borrowing. As the difference between $\xi(t)^2$ and $\phi(t)$ grows this wedge becomes greater. The driving force behind the heterogeneity wedge and its effect on the equity risk premium is the market segmentation

[6] $\phi(t) = \sum_i \frac{\omega^i(t)}{\gamma_i^2} = \frac{\omega^1(t)}{\gamma_1^2} + \frac{\omega^2(t)}{\gamma_2^2} + \frac{\omega^3(t)}{\gamma_3^2} + \cdots = (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_1^2} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_2^2}\right) + \frac{\omega^3(t)}{\gamma_3^2} + \cdots > (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_1} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_2}\right)^2 + \frac{\omega^3(t)}{\gamma_3^2} + \cdots > (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_1} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{\gamma_2} + \frac{\omega^3(t)}{\gamma_3} + \ldots\right)^2 = \left(\sum_i \frac{\omega^i(t)}{\gamma_i}\right)^2 = (\xi(t))^2$, by the strict concavity of the quadratic and induction.

that occurs when agents differ in their preferences towards risk.

2.4. Market Segmentation

When this economy is populated by two or more agents who have different values of γ, the markets for risky and risk free assets will never clear at the same level and will generate a market segmentation involving three distinct groups. Define $\{\gamma_r(t), \gamma_\theta(t)\}$ to be the RRA parameters in a representative agent economy that would produce the same interest rate and market price of risk, respectively:

$$
\begin{aligned}
r(t) &= \rho + \gamma_r(t)\mu_D - \gamma_r(t)(1+\gamma_r(t))\frac{\sigma_D^2}{2} \\
\theta(t) &= \gamma_\theta(t)\sigma_D
\end{aligned}
$$

Equating these expressions to those in Proposition 2 we can solve for these preference levels, such that

$$
\begin{aligned}
\gamma_r(t) &= \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\left(\frac{\mu_D}{\sigma_D^2}\right)^2 - \frac{\mu_D}{\sigma_D^2}\left(1 + \frac{2}{\xi(t)}\right) + \frac{\xi(t)+\phi(t)}{\xi(t)^3} + \frac{1}{4}} \\
\gamma_\theta(t) &= \frac{1}{\xi(t)}
\end{aligned}
$$

Finally, with a bit of algebra, it can be shown that $\gamma_r(t) < \gamma_\theta(t)$, $\forall t < \infty$. This implies that the markets for risky and risk-free assets do not coincide in finite t. Additionally, it shows that the two markets overlap (see Figure 2). This implies a sort of market segmentation with three groups: leveraged investors, diversifying investors, and saving divestors.

The three market segments in this economy represent buyers and sellers of risky and risk-free assets. Agents who have low risk aversion will sell bonds in order to buy a larger share in the risky asset. Agents with a middling level of risk aversion will be purchasing both bonds and shares in the risky asset. They do this by capitalizing their gains in the risky asset. As we'll see in section 3, as the low risk aversion agents dominate the market, they drive up asset prices, producing high returns. The diversifying investors capitalize these gains in both the risky and the risk-free assets. Finally, agents with high risk aversion will be purchasing bonds and shrinking their share in the risky asset. In general, these agents are simply exchanging with the low risk aversion agents their risky shares for bonds. This causes the risky asset to be concentrated amongst the low risk aversion investors as time progresses and pushes up asset prices, again as we'll see in section 3.

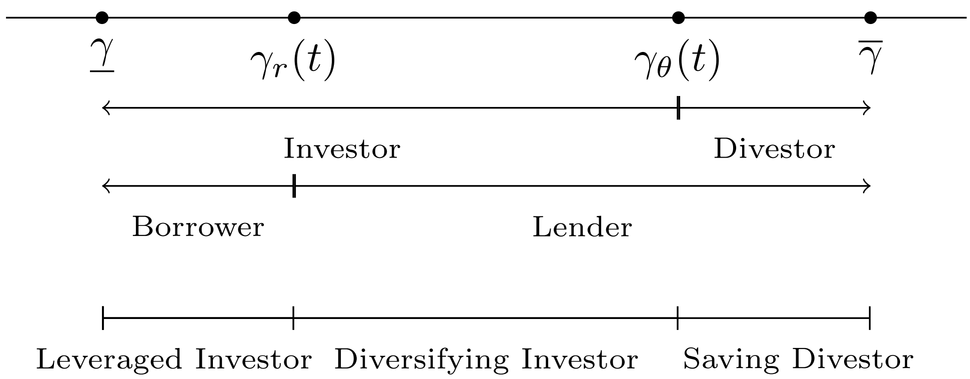


Fig. 2. The market is segmented depending on an agent's preferences relative to the representative agent solution. The markets for risky and risk-free assets do not coincide and there are three segments. Agents with low risk aversion are simultaneously borrowing and investing. Agents with middling risk aversion are lending and investing. Agents with high risk aversion are saving and divesting.

2.5. Consumption Weight Dynamics

We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Proposition 1.

Proposition 3. *Assuming consumption weights also follow a geometric Brownian motion such that*

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t)$$

an application of Itô's lemma to (12) *gives expressions for* $\mu_{\omega^i}(t)$ *and* $\sigma_{\omega^i}(t)$*:*

$$\mu_{\omega^i}(t) = (r(t) - \rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \frac{\theta(t)^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right] \tag{15}$$

$$\sigma_{\omega^i}(t) = \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) \tag{16}$$

The expressions in Proposition 3 describe how an agent's position relative to the consumption weighted average of elasticity of inter-temporal substitution and its square determines the growth rate in their relative share in consumption and the covariance of that share with the risk process. In turn, $\omega^i(t)$ determines, by a one-to-one correspondence, an agent's portfolio choice. For this analysis, we'll use γ_θ from the previous section (dropping the notation (t) for simplicity), such that $\xi(t) = \frac{1}{\gamma_\theta}$.

Consider first the case where an agent's preferences coincide with the weighted average, ie $\gamma_i = \gamma_\theta$. In (16), which describes how an agent's consumption weight co-varies with the risk process, $\sigma_{\omega^i} = 0$. If an agent has the same EIS as the market then they will not desire to vary their asset holdings in the face of shocks. As in the analysis of the previous sub-section, this is because the agent is perfectly in agreement with the market. However, notice that in this case $\mu_{\omega^i} = \theta(t)^2 \left(\frac{1}{\gamma_\theta^2} - \phi(t) \right) = \sigma_D^2 \left(1 - \frac{\phi(t)}{\xi(t)^2} \right)$, by Proposition 2. This is an indicator of the speed with which the economy is moving through this equilibrium. Although the agent is instantaneously satisfied with the current market price of risk, they are deterministically moving out of this position. The speed with which this is occuring is driven by the heterogeneity wedge, $\frac{\phi(t)}{\xi(t)^2}$. When this wedge is high, the rate at which the marginal agent moves out of the marginal position is greater.

Next consider the case where an agent is more patient than the weighted average, that is $\gamma^i > \gamma_\theta$. Then $\sigma_{\omega^i} < 0$ and agent i's weight is negatively correlated to the market. This implies that if an agent is more patient, or alternatively more risk averse, than the average then their asset holding will increase when there are negative shocks and decrease when there are positive shocks. This is a prudence motive and these agents can be thought of as playing a "buy low, sell high" strategy. They do not want to grow their consumption faster than the economy, but to pad their position against future shocks. For this reason, their portfolio decisions are driven not by a desire to increase their consumption today, but to insure themselves against shocks in the distant future and, in turn, increase their wealth. These are the Warren Buffets of the world, living in the same home for 30 years while becoming the richest person on earth.

Conversely, if an agent is less risk averse than the average, ie $\gamma^i < \gamma_\theta$, their asset holdings covary positively with the market. These agents are essentially buying high and selling low, a strategy that in the long run will leave them under water in terms of wealth (see section 3). An agent with a lower risk aversion has a higher elasticity of inter-temporal substitution and, thus, can be thought of as less patient. Given a shock to the dividend process, the expected growth rate remains constant, but the level shifts permanently because of the martingale property of the Brownian motion. Since less patient agents see the current output of the dividend as more important than its long-run behavior present shocks have a greater effect on their personal price. Thus, a negative shock causes them to reduce their price and in turn their asset holdings, while a positive shock causes them to increase their price and asset holdings. These are the day-traders, riding booms and busts to try to make a quick buck, while not losing their shirts.

The analysis of (15) is quite difficult for the case of $\gamma_i \neq \gamma_\theta$. The first term is the product of two separate terms: one involving the interest rate and rate of time preference, the other the agent's position in the distribution. If the interest rate is above the rate of time preference, the first term is positive. If the interest rate differs from the rate of time preference then the agent should desire to shift consumption across time periods, either from today to tomorrow or vice versa. However, the direction will be determined by their preference. If $\gamma_i > \gamma_\theta$ then the product will be negative and this first term will contribute negatively to their growth rate $\mu_{\omega^i}(t)$. The opposite is true when $\gamma_i < \gamma_\theta$. The combined effect of these two terms is to say that if an agent is less patient than the average and the interest rate is greater than their rate of time preference, they will want to grow their consumption faster than the rate of growth in the economy, while if they are more patient than the average then they will tend grow their consumption more slowly than the rate of growth in the economy. This effect is only partial, however, and it is necessary to take into consideration the second term.

The second term is quite a bit more complex. The term in brackets is a sort of quadratic in deviations from the weighted average of risk aversion. Whether this term is positive or negative depends in a complicated way on $\xi(t)$ and $\phi(t)$ [7]. It is sufficient to note that, when the distribution is not too skewed, there exists a level of risk aversion such that if an agent is above this the second term in (15) is negative and that this level of risk aversion is not equal to γ_θ or γ_r. This is related to the deterministic nature of the shifting distribution of asset holdings. Although these two preference levels represent the instantaneous market clearing levels, they do not reflect how distribuiton the is evolving over time.

2.6. Portfolios

Tightly linked to risky shares are portfolio weights. Given the knowledge of an agent's consumption weight dynamics it is possible to derive the evolution of their portfolio weights, $\pi^i(t) = \frac{\omega^i(t)S(t)}{X(t)}$.

Proposition 4. *Assuming portfolio weights $\pi^i(t) = \frac{\omega^i(t)S(t)}{X(t)}$ follow a geometric Brownian motion such that*

$$\frac{d\pi^i(t)}{\pi^i(t)} = \mu_{\pi^i}(t)dt + \sigma_{\pi^i}(t)dW(t)$$

an application of Itô's lemma gives expressions for $\mu_{\pi^i}(t)$ and $\sigma_{\pi^i}(t)$:

$$\mu_{\pi^i}(t) = \mu_{\omega^i}(t) + (1-\pi^i(t))\left(\mu_S(t) - r(t) + \sigma_S(t)(\sigma_{\omega^i}(t) + \pi^i(t)\sigma_S(t))\right) \tag{17}$$

$$\sigma_{\pi^i}(t) = \sigma_{\omega^i}(t) + (1-\pi^i(t))\sigma_S(t) \tag{18}$$

Notice that this portfolio weight can be greater than one, but cannot be negative. An agent's portfolio choice is driven by their consumption weight and by whether or not they are a net borrower or lender. If $1-\pi^i(t) > 0$, the agent has positive savings, otherwise

[7] It can be shown that the roots of this quadratic are $\frac{1}{\gamma_i} = \xi(t) - \frac{1}{2} \pm \sqrt{\frac{1}{4} + \phi(t) - \xi(t)^2}$. Because $\phi(t) > \xi(t)^2$, there will always be two real roots. However, whether these are positive or negative depends on the values of $\xi(t)$ and $\phi(t)$.

they are a net borrower.

2.7. Asset Prices

Now, given expressions to describe the evolution of consumption choices over time, one can give a formula describing asset prices. Bear in mind that it is not trivial to solve for each individual agent's asset price, as it depends in a non-linear way on their consumption weight. However, given consumption will evolve one can use market clearing to derive the expression in Proposition 5.

Proposition 5. *Under a transversality condition on wealth, that is if we assume* $\mathbb{E}_t\left[\lim_{s\to\infty} H_0(s)X^i(s)\right] = 0$*, then it can be shown that asset prices satisfy the following:*

$$S(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u)du \tag{19}$$

Proposition 5 matches classic asset pricing formulas and defines asset prices today in terms of expectations of the future outcome of the dividend process, discounted at the market rate. Consider (19), which we can rewrite by substituting for $H_0(t)$ using (10) as

$$S(t) = \mathbb{E}_t \int_t^\infty e^{-\rho(u-t)} \left(\frac{c^i(u)}{c^i(t)}\right)^{-\gamma_i} D(u)du$$

for every i, which is exactly equal to the asset pricing formula derived from the Euler equation in a representative agent economy. The key difference, however, is that the dynamics of the consumption process in this economy are not equal to the dynamics of the dividend process. It may be possible to construct an agent whose consumption share remains fixed over a short period of time and thus whose price in a representative agent economy would equal the price in the heterogeneous economy, but it is not necessarily true that that agent exists in the model being solved here. For instance, in the case of two agents, the price will always be somewhere between the price that would prevail in the two individuals' autarkic economies[8].

Asset price dynamics being necessary for numerical simulation, it is possible to derive an estimable expression for the volatility of asset prices. The following proposition is identical to one in Cvitanić et al. (2011)[9] and is thus provided without proof:

Proposition 6. *The volatility of the stock price is given by*

$$\sigma_S(t) = \sigma_D + \frac{\mathbb{E}_t \int_t^\infty (\theta(t) - \theta(u)) H_0(u) D(u)du}{\mathbb{E}_t \int_t^\infty H_0(u) D(u)du} \tag{20}$$

[8] This could be addressed in a model of continuous types, but that is for future research.

[9] I am grateful to the authors of that paper for helping me to understand the Malliavan Calculus and the Clark-Okone theorem.

This proposition essentially states that, if $\theta(s) < \theta(t)\ \forall\ s > t$, then there will be excess volatility. This will be true in the present model and I refer the interested reader to the previously mentioned paper for a thorough treatment of the asymptotic results. From the statement in Proposition 6, one can find $\mu_S(t)$ using eq. (14), or by similarly matching coefficients in the Clark-Ocone derivation.

The last proposition involves leverage. We are interested in the evolution of wealth, which is made up of risky assets and bond holdings. So, we should study how an individual's bond holdings and economy-wide borrowing evolve over time. First, we need a lemma about individual drift and diffusion coefficients.

Lemma 2. *Individual savings, $b^i(t)$, have the following dynamics:*

$$db^i(t) = \mu_{b^i}(t)dt + \sigma_{b^i}(t)dW(t)$$

where

$$\mu_{b^i}(t) = \frac{-(\mu_{\omega^i}(t) + \sigma_{\omega^i}(t)\sigma_S(t))\omega^i(t)S(t)}{S^0(t)} \tag{21}$$

$$\sigma_{b^i}(t) = \frac{-\sigma_{\omega^i}(t)\omega^i(t)S(t)}{S^0(t)} \tag{22}$$

First, consider (22) and recall our analysis about $\sigma_{\omega^i}(t)$. The sign of $\sigma_{b^i}(t)$ is opposite that of $\sigma_{\omega^i}(t)$. This is fairly logical, given the closed aspect of the economy, as agents can only finance short run changes in their risky asset holdings through debt. Following a positive shock, agents with low risk aversion desire to increase their holdings of the risky asset to profit from the higher dividends being paid. To do so, they will leverage their position and will thus borrow more. In this way, agents with low risk aversion are financing their portfolio growth in the face of positive shocks by borrowing on the bond market. We will see in section 3 that as the low risk aversion agents dominate, their price dominates the market as well, causing prices to rise. So, their borrowing is indirectly causing rising asset prices.

Next, consider eq. (21). An agent's savings decision depends on both the growth rate of their weight and the covariance of that weight with asset prices. By Proposition 6 we know that $\sigma_S(t) > 0$, so the sign of the product $\sigma_{\omega^i}(t)\sigma_S(t)$ is determined by the sign of $\sigma_{\omega^i}(t)$. If an agent is more patient than the average, $\sigma_{\omega^i}(t) < 0$. But if the same agent is still growing their portfolio, they could still be reducing their savings (or increasing their borrowing). Agents who are "diversifying investors" must thus have a positive growth rate in their shares and a positive growth rate in their bonds. This implies that $\mu_{\omega^i}(t) > 0$, $\sigma_{\omega^i}(t) < 0$, and $-\frac{\mu_{\omega^i}(t)}{\sigma_{\omega^i}(t)} > \sigma_S(t)\omega^i(t)S(t)$. This implies that diversified investors are dynamically adjusting their portfolio position to keep their mean-variance profile above their actual exposure to asset prices. They are hedging their bets by buying low and selling high, but simultaneously reducing their overall exposure by shifting some of their capital gains to risk free assets.

Now, we can use Lemma 2 to study the evolution of total borrowing and total financial leverage in the economy.

Proposition 7. *If we define total value of borrowing in the economy as* $\Gamma(t) = \frac{1}{2}S^0(t)\sum_{i=1}^{N}|b^i(t)|$ *and total financial leverage as* $L(t) = \frac{\Gamma(t)}{S(t)}$, *then it can be shown that these processes have the following dynamics:*

$$
\begin{aligned}
d\Gamma(t) &= \mu_\Gamma(t)dt + \sigma_\Gamma(t)dW(t)\\
\frac{dL(t)}{L(t)} &= \mu_L(t)dt + \sigma_L(t)dW(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_\Gamma(t) &= r(t)\Gamma(t) + \frac{1}{2}S^0(t)\sum_{i=1}^{N}\left[sgn(b^i(t))\left(\mu_{b^i}(t) - \frac{1}{2}\sigma_{b^i}(t)^2\right) + \frac{1}{2}|b^i(t)|\sigma_{b^i}(t)\right]\\
\sigma_\Gamma(t) &= \frac{1}{2}S^0(t)\sum_{i=1}^{N}sgn(b^i(t))\sigma_{b^i}(t)\\
\mu_L(t) &= \frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{\Gamma(t)} - \mu_S(t) + \sigma_S(t)^2\\
\sigma_L(t) &= \frac{\sigma_\Gamma(t)}{\Gamma(t)} - \sigma_S(t)
\end{aligned}
$$

Although these expressions are quite complicated, there are a few interesting features to note. First, notice that total debt, $\Gamma(t)$, cannot be written as a geometric Brownian motion, but that it will be positive by definition. Also, notice that the drift in leverage is determined by several terms in $\Gamma(t)$ and drift and diffusion of stock prices. First, you have a term involving $\Gamma(t)$, which is determined by the covariance of stock prices and total debt. Only if debt, its drift, and its diffusion grow proportionally, will the drift in leverage be more or less constant. Finally, notice that diffusion in leverage is decreasing in the diffusion of stock prices. In fact, if $\Gamma(t)$ were to grow faster than $\sigma_\Gamma(t)$, then leverage would move counter-cyclically This will indeed be the case.

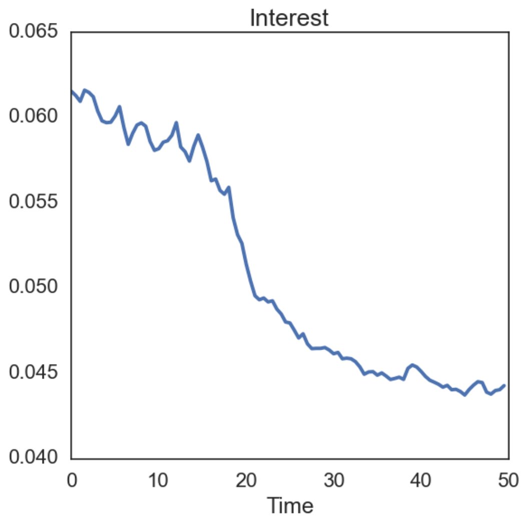
Many of these propositions are complex and difficult to grasp in the abstract. They describe the evolution of either an individual agent or the entire economy and involve relative terms and sums. They also reference each other and co-move throughout time. It is because of this complexity that it is easiest to understand how this economy behaves by using a numerical simulation. Armed with Propostions 1 to 3, 5 and 7, we can now simulate an economy and study its dynamics over time.

3. Simulation Results and Analysis

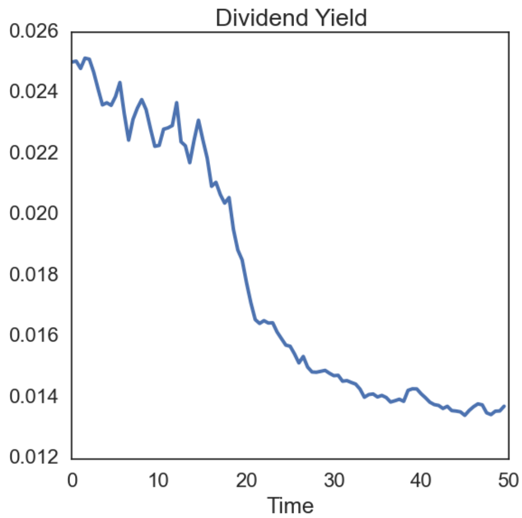
In this section, I review some simulation results and compare them. For all of the simulations, I will hold the following group of parameters fixed at the given values: $\mu_D = 0.03$, $\sigma_D = 0.06$, and $\rho = 0.01$. These settings correspond to a yearly parameterization. Additionally, for simulating asset prices by Monte Carlo, I need to specify a truncation level $T = 1000$, as well as the number of path iterations $M = 5120$ [10]. Finally, I simulate forward 50 periods at a $\Delta t = 0.5$ discretization. I'll show first simulation results for a shocked five agent model [11]. Then I'll look at two versus five agents, showing how this simple change drastically changes the level and the dynamics of all variables, while leaving unchanged the asymptotic value of aggregate variables. The conclusion to be drawn here is that the discretization inherent in the calibration has a non-trivial effect on the short run characteristics of the model.

3.1. Five Agents: Shocked

Let's begin with five agents with CRRA parameters $[\gamma_i] = [1, 2, 3, 4, 5]$. For this first simulation I'll allow the shock process, $dW(t)$, to realize away from its expectation.



(a) Interest Rate



(b) Dividend Yield

Fig. 3. Dividend yield defined as $\frac{D(t)}{S(t)}$ and interest rate, $r(t)$, for five agents under a shocked process.

The first thing to note is how strikingly similar[12] the true realization of dividend yields in Figure 1 are to those in Figure 3. Here we see clear negative trends in both interest

[10] A multiple of 512 for technical reasons. For a full description of the numerical method, see appendix C. The programs are available on request.

[11] Results for a single agent match the theoretical solution given in appendix A.

[12] This is mostly random, but is very satisfying. It is, however, not random that the trends in both interest and dividend yield are negeative.

and dividend yield. First, as the most impatient agent begins to dominate the market for risky assets, the arket interest rate begins to converge towards their preferred rate. That is, in the long run, the prevailing interest rate will correspond to that which one would find in an economy populated by a single agent with the lowest value of γ_i. Similarly, asset prices are converging to a similar long run value. This causes $S(t)$ to grow faster than $D(t)$, pushing down the dividend yield.

The downward trend in dividend yields points towards a possible explanation for the predictability in asset prices described in Campbell and Shiller (1988b) and Campbell and Shiller (1988a). Here dividend yield is trend stationary, meaning that it moves around a negative trend. Because of this, we can expect periods of high yield relative to this trend to predict subsequent periods of low yield. In this way, asset prices have a predictable component[13] The conclusion I draw from this is that the portion of asset price movements that are explainable are driven by the time variation in the discount factor, moving across market clearing values of γ^θ and γ^r.

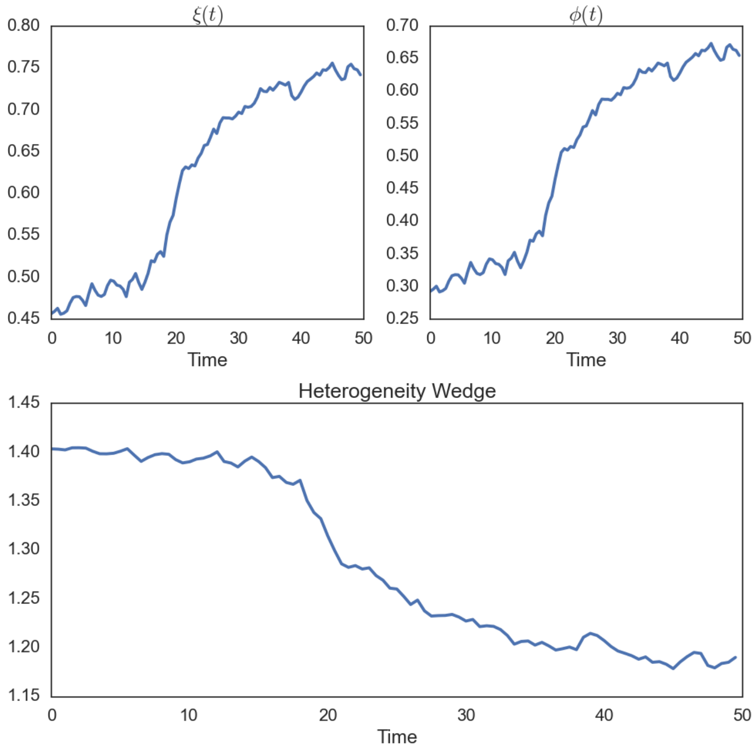


Fig. 4. Sufficient statistics for the distribution of the risk aversion, where $\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$, and the heterogeneity wedge is defined as $\frac{\phi(t)}{\xi(t)^2}$.

The distribution of asset holdings can be described by the evolution of $\xi(t)$ and $\phi(t)$, as well as the heterogeneity wedge $\frac{\phi(t)}{\xi(t)^2}$. These values are displayed in Figure 4. You'll notice that the market clearing level of risk aversion, or from the other direction the

[13]Indeed, a quick regression of dividend yield on past values shows that in a simple AR(1) model previous values of the dividend yield have a strongly significant coefficient.

average EIS, is converging towards one. Additionally, the wedge introduced by agent heterogeneity is falling over time. It is these facts that are driving the falling interest rate and market price of risk. The evolution of $\xi(t)$ and $\phi(t)$ is driven by agents consumption and wealth choices. In Figure 5 you can see that the least risk averse agent has a very volatile wealth process. This agent is highly leveraged and is the most exposed to swings in asset prices. Additionally, as mentioned in section 2, they play a buy high sell low strategy that causes their wealth to collapse in the long run. The dynamics of these variables are easier to see around a non-stochastic trend as in the next subsection.

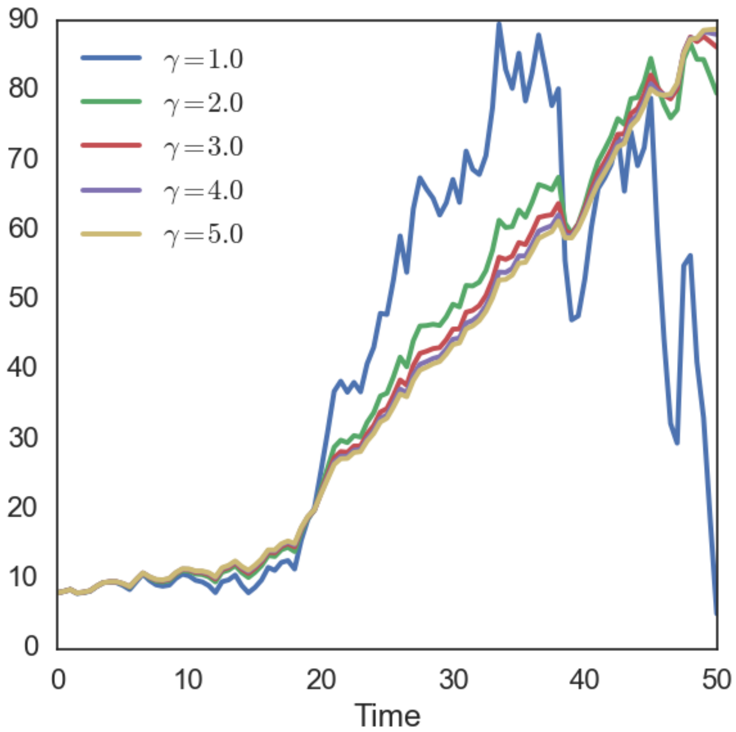
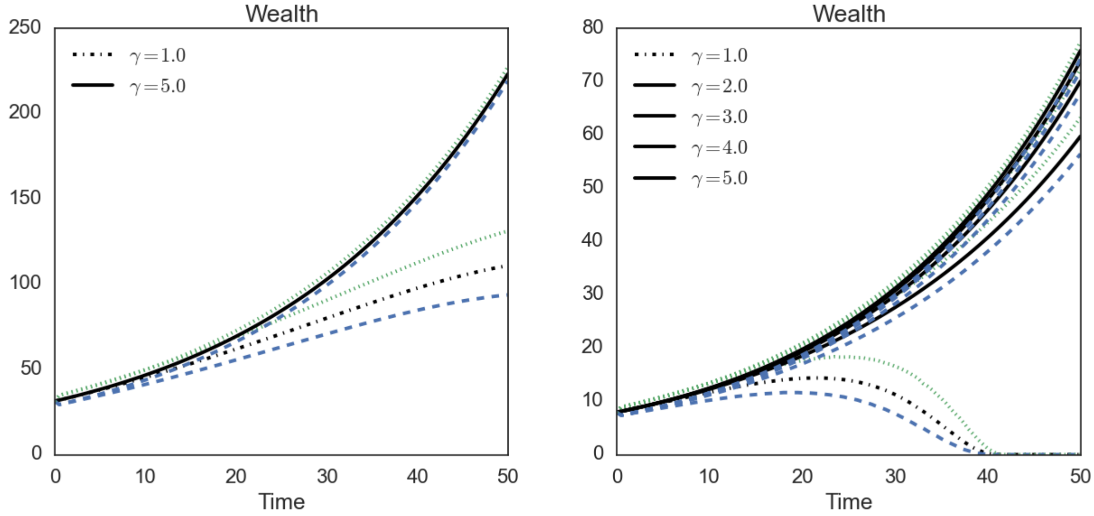


Fig. 5. Agent wealth in a five agent economy subject to shocks. The most volatile line corresponds to the least risk averse agent.

3.2. Two or Five Agents: No Shock

In order to look at distributional outcomes for different numbers of agents, it is easiest to study the non-stochastic path (that where $dW(t) = 0$) and a 95% confidence interval around this path. Here I study the outcome for an economy with two agents where $[\gamma_i] = [1, 5]$ and the outcome for another economy with five agents where $[\gamma_i] = [1, 2, 3, 4, 5]$. The major differences between the results are that when there are more agents, credit constraints become binding in shorter time and rates of convergence are slower. However, the change in the number of agents has a significant change in the level of all variables, implying that mispecification of the support of the distribution of risk preferences has a non-trivial effect on the model's short run predictions.



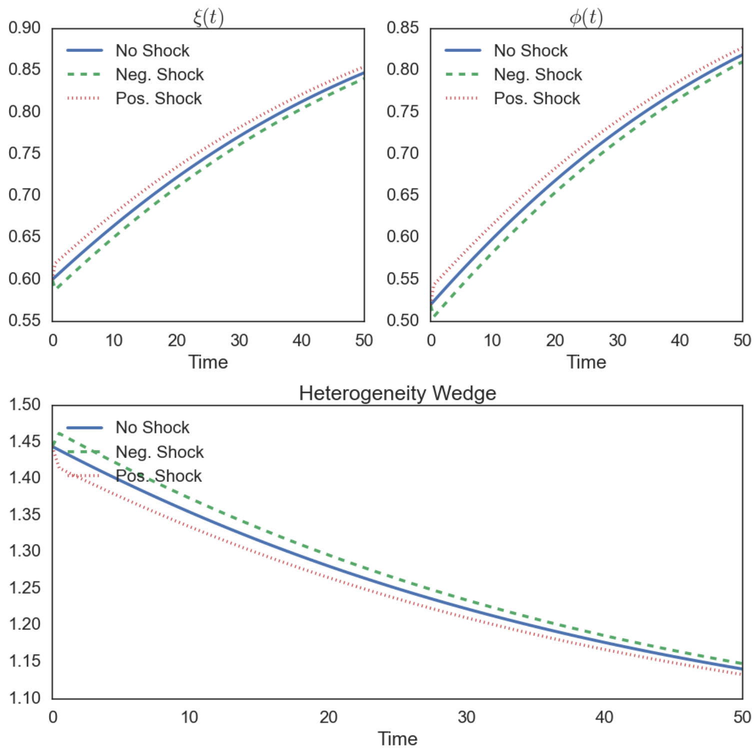
(a) Two Agents (b) Five Agents

Fig. 6. Wealth around the non-stochastic trend and a 95% confidence interval (dots is positive, dash is negative shocks).

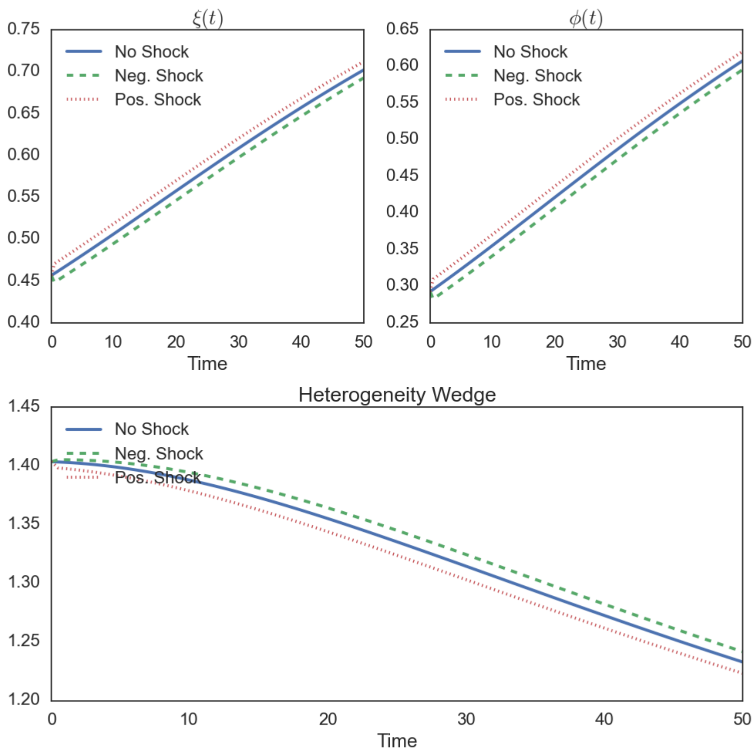
Observe in Figure 6 individual wealth processes over time. The most striking feature of these two plots is downward drift in the trend for the wealth of the least patient (least risk averse) agent. In fact, as you can see in Figure 6(b), this agent's wealth converges to zero in a finite amount of time, as they borrow to finance the growth in their consumption shares. This agent eventually fully leverages their position. In this way they are dominating the market for consumption, but living hand to mouth in the sense that they have no financial wealth. Additionally, in the periods before they reach the constraint, their wealth is highly volatile and its sensitivity to shocks one sided. By this I mean that a positive shock has a greater effect than a negative one. This is driven by their dynamic portfolio choice, where in the face of a negative shock they reduce their asset holdings and in the face of a positive shock they increase their holdings (see Proposition 3). Once they arrive at the constraint, however, they no longer have collateral to use to buffer their exposure and their state is absorbed at zero.

The distribution of consumption shares across individuals can be summed up by the sufficient statistics $\xi(t)$ and $\phi(t)$, as shown in Figure 7. There you can see that the heterogeneity wedge, defined as $\frac{\phi(t)}{\xi(t)^2}$, although initially higher for two agents, converges more slowly for five agents. This is driven partly by the reduction in the initial shares of individual agents and partly by the dynamics of these variables, which are determined by higher moments of the distribution of consumption weights. Additionally, both $\xi(t)$ and $\phi(t)$ are higher lower for five agents than for two, causing a higher interest rate and market price of risk.

Although these facts are interesting, the comparison between aggregate variables across simulations points towards a breakdown in the superposition principle mentioned



(a) Two Agents



(b) Five Agents

Fig. 7. Sufficient statistics for the distribution of the risk aversion, where $\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$, and the heterogeneity wedge is defined as $\frac{\phi(t)}{\xi(t)^2}$.

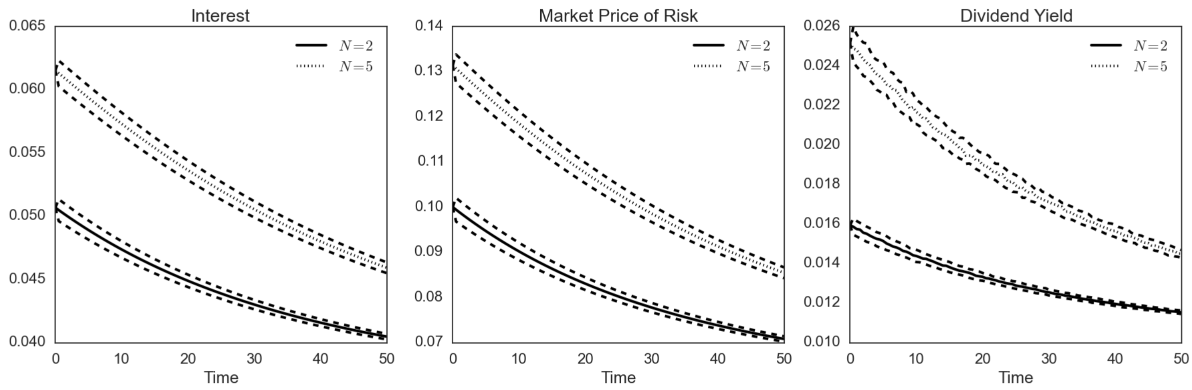


Fig. 8. Comparison across simulations for aggregate variables.

in the introduction. In Figure 8 you can see that the underlying assumptions of the model imply that changing the support of the distribution of risk aversion causes a non-trivial shift in levels, volatility, and rate of convergence of the interest rate, market price of risk, and the dividend yield. If one believes that individuals fall into a finite number of risk aversion types, then indeed changing the number of agents over a fixed support has no effect on the aggregate variables in the model. However, if one thinks there is a continuum of types, then the way that one discretizes or bins this distribution into a finite support has a substantial effect on the model's outcome.

4. Conclusion

The distribution of preferences has a large effect on the outcomes of the model. These effects are driven mainly by consumption weighted averages of the elasticity of intertemporal substitution. In this model, the consumption weights correspond to shares of the total stock market. The implication is that, logically, the amount of participation by individuals determines to what degree their preferences effect price. In fact, the evolution of individual shares is determined by each agent's relative position to the weighted averages of EIS. Given the heterogeneity in preferences, markets for risk free bonds and risky assets clear at different levels, implying three groups. Leveraged investors have low risk aversion and borrow in order to grow their share in the risky asset. Saving divestors are highly risk averse and lend in order to shrink their share in the risky asset. Somewhere in the middle we have the diversifying investor, who is growing their share in the stock market and simultaneously lending by buying bonds. Whether an agent belongs to one of these groups, determined by their preferences, drives their borrowing behavior and in turn the evolution of their financial wealth.

Outcomes of this model are driven by a heterogeneity wedge[14] which describes how different are the market clearing risk free rate and market price of risk. When this wedge is high, corresponding to two very different marginal investors and a diverse group of investors, asset prices are low, interest rates are high, and dividend yields are high. Conversely, when this wedge is low, corresponding to a concentration of risky assets towards a single agent, asset prices are high, interest rates are low, and divedend yield is low. However, these statements are history contingent, for example a representative agent economy will always have a low wedge. However, conditional on there being multiple agents, these statements hold for the heterogeneity wedge in relative terms.

Additionally, dividend yield in this model evolves around a deterministic trend. This implies a predictable component in stock market returns. A negative shock to this economy implies a shift of the distribution of asset holdings towards more risk averse agents. This reduces asset prices and predicts a faster growth rate in the dividend in the future. We know from these simmulations that economies with a lower weighted average of EIS will have a higher rate of return on risky assets. Papers such as Campbell and Shiller (1988a), Campbell and Shiller (1988b), Mankiw (1981), and Hall (1979) drew differing conclusions about the standard model of asset prices. However, broadly speaking, they all deduced that there was some portion of asset prices that was slightly predictable as a funciton of the growth rate in aggregate consumption. In the model presented here, we can take a step towards explaining this predictability as the dividend yield being not stationary, but trend stationary.

Finally, I've shown through simulation that the addition of more agents to a model with heterogeneous preferences has a substantial effect on model predictions. Changes to the underlying support of preferences, that is to the number of types of agents, has a non-trivial effect on the model's outcome. Because of this, it is difficult to draw conclusions about this model if one believes in a continuum of types. Because of this, I believe a fruitful direction for future study would be to look at the introduciton of a continuum of agents. This points towards work in the mean field game litereature on common shocks (see Guéant, Lasry, and Lions (2011) for an introduction to mean field games and Carmona, Delarue, and Lacker (2014) for a discussion of the case of common noise).

This paper represents only a first step towards, hopefully, very enlightening aspects of heterogeneity in finance. Here, we've seen that simply increasing the number of agents over the same support of a distribution of relative risk aversion can have effects on the amount and volatility of leverage in an economy, the rate of change in the dividend price ratio, and can cause a glut of lending. Further research on this topic could include a perpetual youth framework to attempt to achieve stationarity, financial frictions that limit borrowing, or allowing for default in borrowing.

---

[14] Defined as $\frac{\phi(t)}{\xi(t)^2} = \frac{\sum \frac{\omega^i(t)}{\gamma_i^2}}{\left[\sum \frac{\omega^i(t)}{\gamma_i}\right]^2}$.

Appendix A. The Representative Agent Solution

This appendix contains (without proof) the solution to the representative agent economy, i.e. the case where $N = 1$.

$$
\begin{aligned}
r(t) &= \rho + \gamma\mu_D - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2 \\
\theta(t) &= \sigma_D\gamma \\
\frac{S(t)}{D(t)} &= \frac{1}{\rho + \mu_D(\gamma-1) - \frac{1}{2}\sigma^2\gamma(\gamma-1)} \\
\mu_S(t) &= \mu_D \\
\sigma_S(t) &= \sigma_D
\end{aligned} \tag{23}
$$

Appendix B. Proofs

Proof of Proposition 1. Taking ratios of consumption first order conditions for two arbitrary agents, i and j we find

$$\frac{c^i(t)}{c^j(t)} = \Lambda_j^{\frac{1}{\gamma_j}} \Lambda_i^{\frac{-1}{\gamma_i}} \left(H_0(t)e^{\rho t}\right)^{\frac{1}{\gamma_j} - \frac{1}{\gamma_i}}$$

To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption

$$
\begin{aligned}
& & \sum_{j=1}^{N} c^j(t) &= D(t) \\
\Leftrightarrow & & \frac{\sum_{j=1}^{N} c^j(t)}{c^i(t)} &= \frac{D(t)}{c^i(t)} \\
\Leftrightarrow & & c^i(t) &= \frac{c^i(t)}{\sum_{j=1}^{N} c^j(t)} D(t) \\
\Leftrightarrow & & c^i(t) &= \left(\frac{\left(e^{\rho t}\Lambda_i(t)H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(e^{\rho t}\Lambda_j(t)H_0(t)\right)^{\frac{-1}{\gamma_j}}} \right) D(t) \\
\Leftrightarrow & & c^i(t) &= \omega^i(t)D(t)
\end{aligned}
$$

□

Proof of Lemma 1. Modeling consumption as a geometric Brownian motion implies that for every agent i the consumption process can be described by the stochastic differential equation

$$\frac{dc^i(t)}{c^i(t)} = \mu_{c^i}(t)dt + \sigma_{c^i}(t)dW(t) \tag{24}$$

Armed with this knowledge, take the first order condition for an arbitrary agent i's maximization problem, solve for $H_0(s)$, and apply Itô's lemma:

$$\begin{aligned}
H_0(t) &= \frac{1}{\Lambda_i} c^i(t)^{-\gamma_i} e^{-\rho t} \\
\Rightarrow \frac{dH_0(t)}{H_0(t)} &= \left(-\rho - \gamma\mu_{c^i}(t) + \gamma(1+\gamma)\frac{\sigma_{c^i}(t)^2}{2}\right) dt - \left(\gamma_i \sigma_{c^i}(t)\right) dW(t)
\end{aligned}$$

Now, match coefficients to those in (8) to find

$$\begin{aligned}
r(t) &= \rho + \gamma_i \mu_{c^i}(t) - \gamma_i(1+\gamma_i)\frac{\sigma_{c^i}(t)^2}{2} \\
\theta(t) &= \gamma_i \sigma_{c^i}(t)
\end{aligned}$$

Solving for μ_{c^i} and σ_{c^i} gives

$$\begin{aligned}
\mu_{c^i}(t) &= \frac{r(t) - \rho}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2}\frac{\theta(t)^2}{2} \\
\sigma_{c^i}(t) &= \frac{\theta(t)}{\gamma_i}
\end{aligned}$$

□

Proof of Proposition 2. Recall the definition of consumption dynamics in (24) and the market clearing condition for consumption in (5). Apply Itô's lemma to the market clearing condition:

$$\begin{aligned}
&\sum_{i=1}^{N} c^i(t) = D(t) \Rightarrow \sum_{i=1}^{N} dc^i(t) = dD(t) \\
\Leftrightarrow &\sum_{i=1}^{N} \left(c^i(t)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right) = D(t)\mu_D dt + D(t)\sigma_D dW(t) \\
\Leftrightarrow &\frac{\sum_{i=1}^{N} \left(c^i(s)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right)}{\sum_{i=1}^{N} c^i(t)} = \mu_D dt + \sigma_D dW(t) \\
\Leftrightarrow &\sum_{i=1}^{N} \omega^i(t)\mu_{c^i}(t)dt + \sum_{i=1}^{N} \omega^i(t)\sigma_{c^i}(t)dW(t) = \mu_D dt + \sigma_D dW(t)
\end{aligned}$$

By matching coefficients we find

$$\mu_D = \sum_{i=1}^{N} \omega^i(t)\mu_{c^i}(t)$$
$$\sigma_D = \sum_{i=1}^{N} \omega^i(t)\sigma_{c^i}(t)$$

Now use Lemma 1 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk to find

$$\theta(t) = \frac{\sigma_D}{\xi(t)}$$
$$r(t) = \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t) + \phi(t)}{\xi(t)^3}\sigma_D^2$$

where

$$\xi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$$
$$\phi(t) = \sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$$

□

Proof of Proposition 3. Assume that consumption weights follow a geometric Brownian motion given by

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t) \tag{25}$$

Recall the definition of consumption weights in (12) and gather terms:

$$\omega^i(t) = \frac{\left(\Lambda^i e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda_j e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_j}}}$$
$$\Leftrightarrow \omega^i(t) = \left[\sum_{j=1}^{N} \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_0(t)\right)^{\frac{1}{\gamma_i} - \frac{1}{\gamma_j}}\right]^{-1} \tag{26}$$

Recall the definition of Itô's lemma, where $\omega^i(t)$ is a function of $H_0(t)$ and t:

$$d\omega_i(t) = \frac{\partial \omega^i(t)}{\partial t}dt + \frac{\partial \omega^i(t)}{\partial H_0(t)}dH_0(t) + \frac{1}{2}\frac{\partial^2 \omega^i(t)}{\partial H_0(t)^2}(dH_0(t))^2$$

Substituting for $dH_0(t)$ by (8) and using the Itô box calculus to see that $(dH_0(t))^2 =$

$H_0(t)^2\theta(t)^2 dt$, we see that

$$\begin{aligned}\frac{d\omega^i(t)}{\omega^i(t)} &= \frac{1}{\omega^i(t)}\left(\frac{\partial\omega^i(t)}{\partial t} - r(t)H_0(t)\frac{\partial\omega^i(t)}{\partial H_0(t)} + H_0(t)^2\theta(t)^2\frac{1}{2}\frac{\partial^2\omega^i(t)}{\partial H_0(t)^2}\right)dt \\ &\quad - \theta(t)\frac{1}{\omega^i(t)}\frac{\partial\omega^i(t)}{\partial H_0(t)}dW(t)\end{aligned}$$

Matching coefficients with those in (25) it is clear that

$$\begin{aligned}\mu_{\omega^i} &= \frac{1}{\omega^i(t)}\left(\frac{\partial\omega^i(t)}{\partial t} - r(t)H_0(t)\frac{\partial\omega^i(t)}{\partial H_0(t)} + H_0(t)^2\theta(t)^2\frac{1}{2}\frac{\partial^2\omega^i(t)}{\partial H_0(t)^2}\right) \\ \sigma_{\omega^i} &= -\theta(t)\frac{1}{\omega^i(t)}\frac{\partial\omega^i(t)}{\partial H_0(t)}\end{aligned}$$

Differentiating the expression in (26), carrying out some painful algebra, and simplifying gives

$$\begin{aligned}\mu_{\omega^i}(t) &= (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \frac{\theta(t)^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right] \\ \sigma_{\omega^i}(t) &= \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right)\end{aligned}$$

□

Proof of Proposition 4. The dynamics of an agent's portfolio weights follow directly from an application of Ito's lemma and the Ito product rule:

$$\begin{aligned}d\pi^i(t) &= d\left[\left(\omega^i(t)S(t)\right)\left(\frac{1}{X(t)}\right)\right] \\ &= d(\omega^i(t)S(t))\frac{1}{X(t)} + \omega^i(t)S(t)d\frac{1}{X(t)} + d(\omega^i(t)S(t))d\frac{1}{X(t)} \\ &= \pi^i(t)\left[\mu_{\omega^i}(t) + (1-\pi^i(t))\left(\mu_S(t) - r(t) + \sigma_S(t)(\sigma_{\omega^i}(t) + \pi^i(t)\sigma_S(t)\right)\right]dt \\ &\quad + \pi^i(t)\left[\sigma_{\omega^i}(t) + (1-\pi^i(t))\sigma_S(t)\right]dW(t)\end{aligned}$$

where I've ommitted the algebra for simplicity. □

Proof of Proposition 5. Following a trick in Gârleanu and Panageas (2015), we can arrive at an expression for asset prices. Take a straight forward application of Itô's lemma to the time t present value of time u wealth:

$$\begin{aligned}d(H_0(u)X^i(u)) &= X^i(u)dH_0(u) + H_0(u)dX^i(u) + dH_0(u)dX^i(u) \\ &= X^i(u)\left(-r(u)H_0(u) - \theta(u)H_0(u)dW(u)\right) \\ &\quad + H_0(u)\left[\left(r(u)X^i(u) + a^i(u)S(u)\left(\mu_S(u) + \frac{D(u)}{S(u)} - r(u)\right) - c^i(u)\right)du \right. \\ &\quad \left. a^i(u)S(u)\sigma_S(u)dW(u)\right] - \theta(u)H_0(u)\omega^i(u)\sigma_S(u)S(u)du\end{aligned}$$

Now, notice that in this economy agents asset holdings are simply their consumption weight (ie $c^i(s) = \omega^i(s)D(s)$ and $a^i(s) = \omega^i(s)$) and that $\sigma_S(t)\theta(t) = \mu_S(t) + \frac{D(t)}{S(t)} - r(t)$ by (7). This implies that the above expression simplifies to

$$d(H_0(s)X^i(s)) = -H_0(u)\omega^i(u)D(u)du + H_0(u)[\omega^i(u)\sigma_S(u)S(u) - X^i(u)\theta(u)]dW(u)$$

By the definition of the Itô differential this is equivalent to

$$\begin{aligned}\lim_{u\to\infty} H_0(u)X^i(u) - H_0(t)X^i(t) &= -\int_t^\infty H_0(u)\omega^i(u)D(u)du \\ &\quad + \int_t^\infty H_0(u)[\omega^i(u)\sigma_S(u)S(u) - X^i(u)\theta(u)]dW(u)\end{aligned}$$

If we take expectations, then the first term on the left hand side is zero by a transversality condition on the present value of wealth. Also, notice that the Brownian integral on the right hand side is zero in expectation by the martingale property (Oksendal (1992)). So we can write

$$-H_0(t)X^i(t) = -\mathbb{E}_t \int_t^\infty H_0(u)\omega^i(u)D(u)du$$

Finally, we arrive at an expression for wealth today

$$X^i(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\omega^i(u)D(u)du$$

Now take the market clearing condition for wealth and substitute this new formula

$$\begin{aligned}
S(t) &= \sum_{i=1}^{N} X^i(t) \\
&= \sum_{i=1}^{N} \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} \omega^i(u) D(u) du \\
&= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} \left(\sum_{i=1}^{N} \omega^i(u) \right) D(u) du \\
&= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u) du
\end{aligned}$$

□

Proof of Lemma 2. Individual savings is determined by wealth and asset holdings, so we can use Itô's product rule ($d(XY) = YdX + XdY + dXdY$), along with (4), (2), and Proposition 2 to find $dX(t)$ in terms of the dynamics of $b^i(t)$, then match coefficients.

$$\begin{aligned}
dX^i(t) &= d(\omega^i(t)S(t) + b^i(t)S^0(t)) \\
&= (d\omega^i(t))S(t) + \omega^i(t)(dS(t)) + (d\omega^i(t))(dS(t)) + (db^i(t))S^0(t) + b^i(t)(dS^0(t)) \\
&= \left[\mu_{\omega^i}(t) + \mu_S(t) + \sigma_{\omega^i}(t)\sigma_S(t)\right] \omega^i(t)S(t)dt + \left[\mu_{b^i}(t) + r(t)b^i(t)\right] S^0(t)dt \\
&+ \left[\sigma_{\omega^i}(t) + \sigma_S(t)\right] \omega^i(t)S(t)dW(t) + \sigma_{b^i(t)}S^0(t)dW(t)
\end{aligned}$$

Now, matching coefficients, we have the following two relationships:

$$\begin{aligned}
\left[\mu_{\omega^i}(t) + \mu_S(t) + \sigma_{\omega^i}(t)\sigma_S(t)\right] \omega^i(t)S(t) + \left[\mu_{b^i}(t) + r(t)b^i(t)\right] S^0(t) & \\
= \left[r(t)X^i(t) + \omega^i(t)S(t)(\mu_S(t) - r(t))\right] & \\
\left[\sigma_{\omega^i}(t) + \sigma_S(t)\right] \omega^i(t)S(t) + \sigma_{b^i(t)}S^0(t) = \omega^i(t)\sigma_S(t)S(t) &
\end{aligned}$$

which simplifies to give

$$\begin{aligned}
\mu_{b^i}(t) &= \frac{-(\mu_{\omega^i}(t) + \sigma_{\omega^i}(t)\sigma_S(t))\omega^i(t)S(t)}{S^0(t)} \\
\sigma_{b^i}(t) &= -\frac{\sigma_{\omega^i}(t)\omega^i(t)S(t)}{S^0(t)}
\end{aligned}$$

where

$$db^i(t) = \mu_{b^i}(t)dt + \sigma_{b^i}(t)dW(t)$$

□

Proof of Proposition 7. For this proof we need an intermediate result about Itô's lemma and absolute values. If we think of $|X(t)| = (X(t)^2)^{\frac{1}{2}} = f(X(t))$ where $dX(t) = \mu_X(t)dt + \sigma_X(t)dW(t)$, then we can use Itô's lemma to show that

$$d|X(t)| = \left(\frac{X(t)}{|X(t)|}\mu_X(t) + \frac{1}{2}\frac{|X(t)|^2 - X(t)}{|X(t)|}\sigma_X(t)^2\right)dt + \sigma_X(t)\frac{X(t)}{|X(t)|}dW(t) \tag{27}$$

Now, we can use (27), along with Lemma 2 to find the dynamics of total debt in the economy:

$$\begin{aligned}
d\Gamma(t) &= d\left(\frac{1}{2}S^0(t)\sum_{i=1}^{N}|b^i(t)|\right) \\
&= dS^0(t)\left(\frac{1}{2}\sum_{i=1}^{N}|b^i(t)|\right) + \frac{1}{2}S^0(t)\sum_{i=1}^{N}d|b^i(t)| + \frac{1}{2}dS^0(t)\sum_{i=1}^{N}d|b^i(t)| \\
&= \left(r(t)S^0(t)\frac{1}{2}\sum_{i=1}^{N}|b^i(t)|\right)dt + \\
&\frac{1}{2}S^0(t)\sum_{i=1}^{N}\left[\left(\frac{b^i(t)}{|b^i(t)|}\mu_b^i(t) + \frac{1}{2}\frac{|b^i(t)|^2 - b^i(t)}{|b^i(t)|}\sigma_b^i(t)^2\right)dt + \sigma_b^i(t)\frac{b^i(t)}{|b^i(t)|}dW(t)\right]
\end{aligned}$$

Notice that $\frac{b^i(t)}{|b^i(t)|} = sgn(b^i(t))$...

$$\begin{aligned}
d\Gamma(t) = &\left(r(t)\Gamma(t) + \frac{1}{2}S^0(t)\sum_{i=1}^{N}\left[sgn(b^i(t))\mu_b^i(t) + \frac{1}{2}(|b^i(t)| - sgn(b^i(t)))\sigma_{b^i}(t)^2\right]\right)dt \\
&+ \frac{1}{2}S^0(t)\sum_{i=1}^{N}sgn(b^i(t))\sigma_{b^i}(t)dW(t)
\end{aligned}$$

Which implies that total debt follows the following stochastic process

$$d\Gamma(t) = \mu_\Gamma(t)dt + \sigma_\Gamma(t)dW(t)$$

where

$$\begin{aligned}
\mu_\Gamma(t) &= r(t)\Gamma(t) + \frac{1}{2}S^0(t)\sum_{i=1}^{N}\left[sgn(b^i(t))\left(\mu_{b^i}(t) - \frac{1}{2}\sigma_{b^i}(t)^2\right) + \frac{1}{2}|b^i(t)|\sigma_{b^i}(t)\right] \\
\sigma_\Gamma(t) &= \frac{1}{2}S^0(t)\sum_{i=1}^{N}sgn(b^i(t))\sigma_{b^i}(t)
\end{aligned}$$

Next, define total financial leverage to be $L(t) = \frac{\Gamma(t)}{S(t)}$ and recall Itô's quotient rule: $d\frac{X}{Y} = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2\right)$. So, we can define the dynamics of total financial leverage:

$$
\begin{aligned}
dL(t) = d\frac{\Gamma(t)}{S(t)} &= \frac{\Gamma(t)}{S(t)}\left(\frac{d\Gamma(t)}{\Gamma(t)} - \frac{dS(t)}{S(t)} - \frac{d\Gamma(t)}{\Gamma(t)}\frac{dS(t)}{S(t)} + \left(\frac{dS(t)}{S(t)}\right)^2\right) \\
&= \frac{\Gamma(t)}{S(t)}\left[\left(\frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{\Gamma(t)} - \mu_S(t) + \sigma_S(t)^2\right)dt + \left(\frac{\sigma_\Gamma(t)}{\Gamma(t)} - \sigma_S(t)\right)dW(t)\right]
\end{aligned}
$$

Thus, total financial leverage follows a diffusion process such that

$$
dL(t) = \mu_L(t)dt + \sigma_L(t)dW(t)
$$

where

$$
\begin{aligned}
\mu_L(t) &= \frac{\mu_\Gamma(t) - \sigma_\Gamma(t)\sigma_S(t)}{S(t)} - L(t)\left(\mu_S(t) - \sigma_S(t)^2\right) \\
\sigma_L(t) &= \frac{\sigma_\Gamma(t)}{S(t)} - L(t)\sigma_S(t)
\end{aligned}
$$

$\square$

Appendix C. Numerical Simulation Method

Gathering all of the stochastic processes we have the following definitions to describe the evolution of the economy:

$$
\begin{aligned}
\frac{dD(t)}{D(t)} &= \mu_D dt + \sigma_D dW(t)\\
\frac{d\omega^i(s)}{\omega^i(s)} &= \mu_{\omega^i}(s)dt + \sigma_{\omega^i}(s)dW(t)\\
\frac{dH_0(t)}{H_0(t)} &= -r(t)dt - \theta(t)dW(t) \qquad (28)\\
\theta(t) &= \frac{\sigma_D}{\xi(t)}\\
r(t) &= \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t)+\phi(t)}{\xi(t)^3}\sigma_D^2
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{\omega^i}(t) &= (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \frac{\theta(t)^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right]\\
\sigma_{\omega^i}(t) &= \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right)\\
\xi(t) &= \sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i}\\
\phi(t) &= \sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i^2}
\end{aligned}
$$

given a set of initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$. All of the above variables can be determined as a function of the realization of the risk process $W(t)$. If we combine those values with an estimation of asset prices and the following formulas

$$
\begin{aligned}
\theta(t) &= \frac{\mu_s + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)}\\
\frac{dS(t)}{S(t)} &= \mu_s(t)dt + \sigma_s(t)dW(t)
\end{aligned}
$$

we can back out the coefficients $\mu_S(t)$ and $\sigma_S(t)$ and study the dynamics of the economy, as well as characteristics of asset prices.

Note that it can be shown (Oksendal (1992)) that if a stochastic process $Z(t)$ follows a geometric Brownian motion with drift and diffusion $\mu_Z(t)$ and $\sigma_Z(t)$, then

$$
Z(t+\Delta t) = Z(t)e^{(\mu_Z - \frac{1}{2}\sigma_Z^2)\Delta t + \sigma_Z(W(t+\Delta t)-W(t))} \qquad (29)
$$

The numerical scheme follows the following steps:

1. Specify a time discretization such that $t \in \{0, 1, ..., T\}$ and a time step Δt. Note that the specification of parameters and this time step will determine the discretization as being yearly, quarterly, monthly, etc.
2. Specify a set of agents indexed by $i \in \{1, ..., N\}$ for some number N and each agent's risk aversion parameter γ_i.
3. Specify initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$.
4. Simulate a process $\{dW(t)\}_{t=0}^T$ where $dW(t) \sim \mathcal{N}(0, \Delta t)$.
5. Using (28), (29), and the simulated Wiener process, for each period $t \in \{0, 1, ..., T\}$ calculate $\{D(t), \{\omega^i(t)\}_{i=1}^N, r(t), \theta(t), \xi(t), \phi(t)\}$.
6. Using the monte-carlo approach described in appendix C.1, for each period $t \in \{0, 1, ..., T\}$ calculate $\hat{S}(t)$, $\hat{\sigma_S}(t)$, and $\hat{\mu_S}(t)$.
7. Given the process for $\hat{S}$, calculate wealth $X^i(t)$ for each period using the definitions $X^i(0) = \omega^i(0)S(0) + b^i(0) = \omega^i(0)S(0)$ (where $b^i(t)$ is risk free bond holdings and we assume agents enter the model with no savings/debt) and (4).
8. Calculate any measures you might find enlightening!

C.1. Estimating Asset Prices

The expression we wish to estimate is given by

$$S(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u) du$$

In order for the integral to be defined, it must be that the integrand converges towards zero as $u \to \infty$. If this is the case, then we could estimate the integral by truncating the upper bound at some level, $t + T$. In this way we would look to approximate the true asset price in the economy by another:

$$S(t) \approx S^*(t) = \mathbb{E}_t \int_t^{t+T} \frac{H_0(u)}{H_0(t)} D(u) du$$

This expression can easily be estimated by monte-carlo. To estimate the integral, I'll use the trapezoid rule, but note that in the numerical simulation this is exact, given that time is discretized. Define the discretization by partitioning the interval $(t, t + T)$ into H evenly spaced intervals such that Δ_t is the distance between points in the partition. To estimate the expectation, I will use monte-carlo (see Casella and Robert (2013)) by sampling M paths for the process $W(t)$ and simulating the economy along these paths to extract processes H_0 and D. Indicating draws by a super-script m the estimator is given by:

$$\hat{S}^*(t) = \frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{2}\left(D(t) + \frac{H_0^m(T)}{H_0(t)}D^m(T)\right)\right.$$
$$\left. + \sum_{i=1}^{M-1}\frac{H_0^m(t+\Delta_t i)}{H_0(t)}D^m(t+\Delta_t i)\right] \tag{30}$$

Given the computational simplicity of this expression, it can be calculated quite easily using parallel methods on a graphics processing unit (GPU). I use an element-wise product and pairwise summation to calculate the expectation, resulting in a 300 times speed up (code available upon request).

To estimate, the steps are as follows for a given distribution of $\{\gamma_i\}_{i=1}^N$ and an initial condition for the distribution of wealth $\{\omega^i(t)\}_{i=1}^N$:

1. Simulate M sample paths for $dW(t)$ of length T, where M is an integers, using the knowledge that $dW(t) \sim \mathcal{N}(0, t)$.
2. Using the M sample paths, simulate the evolution of M different economies populated by the same agents under the same initial condition. Extract the values for $(H_0(t), D(t))$.
3. Calculate asset prices at period t as the Monte carlo approximation given in (30).

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