HETEROGENEOUS RISK PREFERENCES IN FINANCIAL MARKETS

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Abstract. This paper builds a continuous time model of N heterogeneous agents whose CRRA preferences differ in their level of risk aversion and considers the Mean Field Game (MFG) in the limit as N becomes large. The model represents a natural extension of other work on heterogeneous risk preferences (e.g. Cvitanic, et. al., (2011) "Financial Markets Equilibrium with Heterogeneous Agents". Review of Finance, 16, 285-321) to a continuum of types. I add to the previous literature by characterizing the limit in N and by studying the short run dynamics of the distribution of asset holdings. I find that agents dynamically self select into one of three groups depending on their preferences: leveraged investors, diversified investors, and saving divestors, driven by a wedge between the market price of risk and the risk free rate. The solution is characterized by dependence on individual holdings of the risky asset, which in the limit converge to a stochastic flow of measures. In this way, the mean field is not dependent on the state, but on the control, making the model unique in the literature on MFG and providing a convenient approach for simulation. I simulate both the finite types and continuous types economies and find that both models match qualitative features of real world financial markets. However, the continuous types economy is more robust to the definition of the support of the distribution of preferences and computationally less costly than the finite types economy.

Introduction

Each day, trillions of dollars worth of financial assets change hands. Being simply a piece of paper, a financial security gives its bearer the right to a stream of future dividends and capital gains for the infinite future. The price of this abstract object is so difficult to determine that if you ask two analysts for an exact price they will generally disagree. This fact has been well documented in studies such as Andrade et al. (2014) or Carlin et al. (2014). These observations are in direct contrast to

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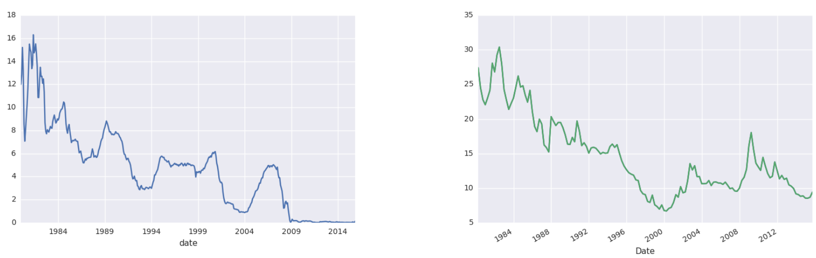
a representative agent model of financial markets. Take for instance the aspect of trade in financial assets previously mentioned. With a single agent there can be no exchange because there is no counter party. We look for a set of prices to make the representative agent indifferent to consuming everything, holding the entire capital stock, etc. In order to have exchange in an economic model we must introduce two or more agents who are heterogeneous in some way.

In this paper I build a continuous time model of heterogeneous risk preferences and study the limit as $N \to \infty$. The majority of the theoretical work on heterogeneous risk preferences focuses on two agents (Dumas (1989); Coen-Pirani (2004); Guvenen (2006); Bhamra and Uppal (2014); Chabakauri (2013); Gârleanu and Panageas (2015); Cozzi (2011)). My work most closely resembles that of Cvitanić et al. (2011), who study an economy populated by N agents who differ in their risk aversion parameter, their rate of time preference, and their beliefs. However, they focus on issues of long run survival and price. I build on their results by studying how changes in the distribution of preferences effect the short run dynamics of the model, while focusing on a single aspect of heterogeneity: risk aversion. Additionally, I take their work to the limit as $N \to \infty$. This formulation results in very similar results in terms of economic intuition, but simulation is more robust and one can explicitly model the distribution of preferences.

Models of a continuum of agents are not necessarily new, but the study of such models in continuous time stochastic settings has recently garnered a large amount of attention thanks to a series of papers by Jean-Michel Lasry and Pierre-Louis Lions (Lasry and Lions (2006a), Lasry and Lions (2006b), Lasry and Lions (2007)). These authors studied the limit of N-player stochastic differential games as $N \to \infty$ and agents' risk is idiosyncratic, dubbing the system of equations governing the limit a "Mean Field Game" (MFG). Their work has then been applied to macroeconomics in works such as Moll (2014) and Achdou et al. (2014). However, these papers focus on idiosyncratic risk and do not study the problem of aggregate shocks. Recent work, such as Carmona et al. (2014), Carmona and Delarue (2013), Chassagneux et al. (2014), and Cardaliaguet et al. (2015) to name but a few, has focused on the issue of analyzing equilibria in MFG models with common noise. The approach is often to use a stochastic Pontryagin maximum principle to derive a system of forward-backward stochastic differential equations governing the solution. In this paper I take a different approach, solving a MFG model with common noise using Girsanov theory and the martingale method, a tool ubiquitous in finance. The solution is characterized by mean field dependence through the control, as opposed to the state, and the equilibrium is a stochastic field described by an Ito diffusion process. This result is reminiscent of infinite dimensional models of the term structure (Carmona and Tehranchi (2007)), but where the cylindrical Brownian motion is homogeneous in the state. This points towards a new way to consider control in the mean field setting, where one begins from an infinite dimensional space and models the idiosyncratic risk and aggregate risk as a correlated cylindrical Brownian motion.

The qualitative features of the economy which are matched by models of heterogeneous preferences are characterized by non-stationary paths. When returns are endogenous and we allow there to exist inherent differences in agents' risk preferences, the problem ceases to be time consistent and we end up with constantly shifting distributions of wealth and consumption. Additionally, financial variables cease to take on steady state values. The dynamics of financial variables become themselves stochastic and the number of state variables explodes. Although this time inconsistency and growth in the number of state variables makes the use of dynamic programming impossible[1], we can employ the martingale method pioneered by Harrison and Pliska (1981) and further refined by Karatzas et al. (1987). In fact, the time-inconsistent nature of the problem may be the very characteristic that brings it closer to the real world. I think few people would claim that interest rates and dividend yields are stationary processes (see Figure 1; sources: FRED, Yahoo! Finance), but that they have exhibited clear downward trends since the 1980s This paper finds that these trends are consistent with an economy populated by agents with heterogeneous risk preferences.



(a) 3 month treasury rate

(b) Dividend yield of US economy

FIGURE 1. The evolution of both interest rates and dividend yields since 1980 show how financial variables are clearly non-stationary. Note: "Dividend yield" is here calculated as the ratio of GDP to the level of the S&P 500 in order to match the model presented below.

In the simulations presented in Section 4, we will see that the dividend-price ratio exhibits a similar downward trend. This indicates predictability of stock returns up to some deterministic drift. This result is consistent with those of Campbell and Shiller (1988b) and Campbell and Shiller (1988a), who find that the returns on stocks can be predicted as a function of dividend yield. This could also explain the result of Mankiw (1981), who rejects the permanent income hypothesis on the basis that asset price co-movements with the stochastic discount factor are forecast-able When agents exhibit heterogeneous preferences, the stochastic discount factor does not correspond to a specific agent in every period, but to a time varying level of risk aversion. This level is falling through time and correlated to asset prices. Asset prices are rising faster than dividends at the same time and move more than one to one with dividends. This excess volatility and time variation in the stochastic discount factor produce a slightly predictable dividend yield.

If we think of individual agents as each having a supply and demand function for risky assets and risk free bonds, it is possible to think of a model of heterogeneous risk preferences as one of market break down. Each agent populates a single theoretical market, but only one market can clear, as in models of the leverage

[1]Or at least impractical for mere mortals.

cycle, e.g. Geanakoplos (2010)[2]. The market which clears is the one corresponding to the agent who is indifferent between buying or selling their shares or bonds. In fact, the formulas for the risk free rate and the market price of risk derived in this paper resemble greatly those in Basak and Cuoco (1998). In that paper, two agents participate in the economy, but one is restricted from participating in the financial market while the participating agent determines the value of financial assets. However in this paper, contrary to the limited participation literature, the clearing markets for stocks and bonds do not have to correspond to the same agent, nor does the corresponding agent even need to exist in the economy. We will see in Section 2.2 that two moments of the distribution of consumption shares determine the market clearing agents. Additionally, these values will vary over time and will be endogenously determined.

To compare the finite types case to the MFG formulation I simulate several economies and study the results. I take the naive approach of evenly discretizing the support for the preference parameter and find that the results converge very slowly in the number of bins. Conversely, I find that, when approximating integrals using Gaussian quadrature, that the MFG simulation converges very quickly in the number of nodes. Additionally, I find that one could not match the MFG model using the finite types simulation without a prohibitively large number of simulated agents. These results imply that despite the mathematics of the MFG economy being slightly more complex, the simulation is greatly simplified, more robust, and more versatile.

The paper is organized as follows: in Section 1, I construct a continuous time model of financial markets populated by a finite number of agents who differ in their preferences towards risk. Section 2 solves the model up to an estimable equation for asset prices, giving closed form solutions for the interest rate, the market price of risk, and dynamics, as well as discussing market segmentation. In Section 4, I give simulation results for changing the number of agents over a given support. Finally, Section 5 concludes and gives some ideas for future research and applications. The more technical analysis and proofs have been relegated to the appendix.

1. The Model

In this section, I will describe the general setting of the model. The key components are the definition of agent heterogeneity, the economic uncertainty, agent optimization, portfolio admissibility, and equilibrium conditions. The solution method will be discussed in the following section.

1.1. **Agent Heterogeneity.**

I consider a continuous time economy populated by a number, N, of heterogeneous agents indexed by $i \in \{1, 2, ..., N\}$. Each agent has constant relative risk aversion (CRRA) preferences with relative risk aversion γ_i:

$$U(c(t), \gamma_i) = \frac{c(t)^{1-\gamma_i} - 1}{1 - \gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

Additionally, agents will begin with a possibly heterogeneous initial wealth, $X^i(0) = x_i$. Agents' initial conditions will be distributed according to a density $(\gamma_i, x_i) \sim f(\gamma, x)$. In this paper, I will consider $\gamma \in (1, \overline{\gamma})$ for ease of exposition and for simulations I will take $f(\gamma, x) = f(\gamma)\delta_{x^*}$, where δ_x is the Dirac delta function, such

[2] I also should thank John Geanakoplos, since his work inspired this paper.

that agents begin with homogeneous wealth. These assumptions can be relaxed, but can induce numerical difficulties.

Assuming this type of random initial condition and preference level provides the independence necessary to study asymptotics. In the sense of Lasry and Lions (2007), this is similar to the independent noise on individual state variables. Independence in this noise allows for the propagation of chaos, while here independence in the initial condition and preference parameter do the same. These assumptions can be thought of economically as a sample. The economy as a whole is too large to measure, so an econometrician must study a random sample of individuals.

1.2. **Financial Markets.** Agents have available to them a forest of N Lucas trees with perfectly correlated dividends, $D(t)$, which follow a geometric Brownian motion, and risk free borrowing and lending at an interest rate $r(t)$ in zero net supply. Each Lucas tree represents an average dividend process or per-capita production

$$\frac{dD(t)}{D(t)} = \mu_D dt + \sigma_D dW(t) \tag{1.1}$$

where μ_D and σ_D are constants. Agents can continuously trade in claims to the dividend process whose price, $S(t)$, also follows a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dW(t) \tag{1.2}$$

whose share will be denoted $\pi(t, \gamma_i, x_i) = \pi^i(t)$. Throughout the paper the notation is suppressed where possible for readability, but one should remember that the index i implies dependence both on the initial condition in x and the preference parameter γ. Here $\mu_S(t)$ and $\sigma_S(t)$ are time varying and determined in general equilibrium. Additionally, agents can borrow and lend at a time varying interest rate $r(t)$ using a risk free bond, whose individual share is denoted $b^i(t)$. The price of the risk free bond, denoted $S^0(t)$, thus follows a deterministic[3] process whose dynamics are given by

$$\frac{dS^0(t)}{S^0(t)} = r(t)dt \tag{1.3}$$

1.3. **Budget Constraints and Individual Optimization.** All agents are initially endowed with a share in the average tree, as well as an initial bond position. If we denote by $X_1^i(t)$ and $X_2^i(t)$ an individuals financial wealth held in risky assets and risk free assets, respectively, then total wealth can be written $X^i(t) = X_1^i(t) + X_2^i(t) = \pi^i(t)S(t) + b^i(t)S^0(t)$, where $\pi^i(t)$ and $b^i(t)$ represent individual shares in the risky and risk free asset, respectively. At any time t an agents dynamic budget constraint can be written as

$$\begin{aligned} dX^i(t) = &\left[X^i(t)r(t) \; + \pi^i(t)S(t)\left(\mu_S(t) + \frac{D(t)}{S(t)} - r(t)\right) - c^i(t)\right]dt \\ &+ \pi^i(t)\sigma_s(t)S(t)dW(t) \end{aligned} \tag{1.4}$$

where the set of variables $\{c^i(t), \pi^i(t), X^i(t), D(t), S(t), r(t), W(t)\}$ represent an agent's consumption, risky asset holdings, and wealth, as well as the dividend, market clearing asset price, market clearing risk free interest rate, and Wiener process governing the Brownian motion, respectively. This stochastic differential

[3]It is not necessarily the case that $r(t)$ is deterministic.

equation admits a unique strong form solution (see Yong and Zhou (1999), Theorem 6.14) given by

$$
\begin{aligned}
X^i(t) = \exp\left\{\int_0^t r(s)ds\right\}\Bigg[x_i \\
+ \int_0^t \exp\left\{-\int_0^s r(u)du\right\}\left(\pi^i(s)S(s)\left(\mu_S(t) + \frac{D(t)}{S(t)} - r(t)\right) - c^i(t)\right)ds \\
\int_0^t \exp\left\{-\int_0^s r(u)du\right\}\pi^i(s)\sigma_S(s)S(s)dW(s)\Bigg]
\end{aligned}
$$

Using these facts an individual agent's constrained maximization subject to instantaneous changes in wealth can be written as:

$$
\begin{array}{ll}
\max\limits_{\{c^i(u), pi^i(s), b^i(u)\}_{u=t}^{\infty}} & \mathbb{E}\int_t^{\infty} e^{-\rho(u-t)}\frac{c^i(u)^{1-\gamma_i} - 1}{1-\gamma_i}du \\
\text{s.t.} & \textit{Equation } (1.4)
\end{array}
$$

1.4. **Admissibility.** If an agent's optimal policy implies that they could hold more debt than equity, driving their wealth into negative territory, they could borrow infinitely. This case is ruled out in the real world and we should thus limit our attention to a smaller set of admissible policies. This issue was first treated in the continuous time setting by Karatzas et al. (1986) and I follow their assumptions here. Assume shares $\pi^i(t)$ and consumption $c^i(t)$ measurable, adapted, real valued processes such that

$$
\begin{aligned}
\int_0^{\infty} \pi^i(t)^2 dt < \infty \quad \text{a.s.} \\
\int_0^{\infty} c^i(t)dt < \infty \quad \text{a.s.}
\end{aligned}
$$

Then we can define the set of admissible policies by the following:

Definition 1. *A pair of policies $(\pi^i(t), c^i(t))$ is said to be admissible for the initial endowment $x_i \geq 0$ for agent i's optimization problem if the wealth process $X(t)$ satisfies*

$$X^i(t) \geq 0, \quad \forall\, t \in [0, \infty) \quad a.s.$$

Denote by $\mathcal{A}(x_i)$ the set of all such admissible pairs.

1.5. **Equilibrium.** Each agent will be considered to be a price taker. This implies an Arrow-Debreu type equilibrium concept.

Definition 2. *An equilibrium in this economy is defined by a set of processes $\{r(t), S(t), \{c^i(t), X^i(t), \pi^i(t)\}_{i=1}^N\}\ \forall\ t$, given preferences and initial endowments, such that $\{c^i(t), \pi^i(t), X^i(t)\}$ solve the agents' individual optimization problems and* *the following set of market clearing conditions is satisfied:*

$$\begin{aligned}\frac{1}{N}\sum_i c^i(t) &= D(t)\\ \frac{1}{N}\sum_i \pi^i(t) &= 1 \qquad (1.5)\\ \frac{1}{N}\sum_i X^i(t) &= S(t)\end{aligned}$$

2. Equilibrium Characterization

This section will derive a solution to each agent's maximization problem and give results on the characteristics of equilibrium. I briefly describe the martingale method and then formulas for financial market variables as functions of consumption weights, which can be describes as Itô diffusion processes.

2.1. **The Static Problem.**
Following Karatzas and Shreve (1998) we can define the stochastic discount factor as

$$H_0(t) = exp\left(-\int_0^t r(u)du - \int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta(u)^2 du\right) \tag{2.1}$$

where

$$\theta(t) = \frac{\mu_s(t) + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)} \tag{2.2}$$

represents the market price of risk. This implies that the stochastic discount factor also follows a diffusion of the form

$$\frac{dH_0(t)}{H_0(t)} = -r(t)dt - \theta(t)dW(t) \tag{2.3}$$

It is important to keep in mind that agents are not discounting at their own rate, but at a market rate. This is because each agent knows that their only choice is to buy or sell assets at the market rate, assuming that there is no arbitrage. If it were possible for there to be many markets, each one clearing at an individual agent's price, one could buy risky assets in a market with a risk averse agent and sell them in a market with a risk neutral agent at a higher price, making a positive profit.

The process defined by $H_0(t)exp\left\{\int_0^t r(s)ds\right\}$ is a martingale under the measure $\mathbb{P}$. To make use of Girsanov theory we can define a new measure

$$\mathbb{Q}(A) = \mathbb{E}\left[H_0(t)exp\left\{\int_0^t r(s)ds\right\}\mathbb{1}_A\right], \quad A \in \mathcal{F}_t$$

Then we can rewrite the wealth process in terms of a new process $\tilde{W}(t)$ defined by

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s)ds$$

which is a Brownian motion under $\mathbb{Q}$. Thus we have

$$\begin{aligned}X_t^i \exp\left\{\int_0^t -r(s)ds\right\} + \int_0^t \exp\left\{-\int_0^s r(u)du\right\}c^i(s)ds =\\ x_i + \int_0^t \exp\left\{-\int_0^s r(u)du\right\}\pi^i(s)\sigma_S(s)S(s)d\tilde{W}_s\end{aligned} \tag{2.4}$$

By the definition of $\tilde{W}(t)$, the right hand side of Equation (2.4) is a local martingale under $\mathbb{Q}$. This implies that the left hand side is then a super-martingale under $\mathbb{Q}$, and we have

$$\mathbb{E}^{\mathbb{Q}}\left[X_t \exp\left\{\int_0^t -r(s)ds\right\} + \int_0^t \exp\left\{-\int_0^s r(u)du\right\} c^i(s)ds\right] \le x_i$$

Following Proposition 2.6 from Karatzas et al. (1987), given an admissible pair $(\pi^i(t), c^i(t))$ we can rewrite each agent's dynamic problem as a static one beginning at time $t = 0$

$$\begin{aligned} \max_{\{c^i(u)\}_{u=0}^{\infty}} \quad & \mathbb{E}\int_0^{\infty} e^{-\rho u}\frac{c^i(u)^{1-\gamma_i}-1}{1-\gamma_i}du \\ \text{s.t.} \quad & \mathbb{E}\int_0^{\infty} H_0(u)c^i(u)du \le x_i \end{aligned}$$

If we denote by Λ_i the Lagrange multiplier in individual i's problem, then the first order conditions can be rewritten as

$$c^i(u) = \left(e^{\rho u}\Lambda_i H_0(u)\right)^{\frac{-1}{\gamma_i}} \tag{2.5}$$

which holds for every agent in every period. It is important to point out that the Lagrange multiplier is constant in time and a function only of the preference parameter and initial condition: $\Lambda_i = \Lambda(\gamma_i, x_i)$. This will be a key fact in deriving the convergence in N.

2.2. **Consumption Weights.** Given each agent's first order conditions, we can derive an expression for consumption as a fraction of per-capita dividends.

Proposition 1. *One can define the consumption of individual, i, at any time, t, as a share $\omega^i(t)$ of the per-capita dividend, $D(t)$, such that*

$$c^i(t) = \omega^i(t)D(t) \tag{2.6}$$

$$\textit{where } \omega^i(t) = \frac{N\left(\Lambda_i e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^N \left(\Lambda_j e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_j}}} \tag{2.7}$$

This expression recalls the results in Basak and Cuoco (1998) or Cuoco and He (1994), where $\omega(t)$ acts like a time-varying Pareto-Negishi weight. In those works, however, participation is driven by an imperfection in the information structure or some exogenous constraint. Here the choice of participation is driven by preferences towards risk. The value of the stochastic discount factor is equal across agents, but differs in its weight for each agent as they differ in risk aversion. This leads one to think that perhaps it would be better to think of this as an incomplete market. If markets were fully complete, there would be a risky asset for each agent, but here agents are forced to bargain over a single asset.

To derive an expression for the risk free rate and the market price of risk, we will need the following lemma about the drift and diffusion of agents' consumption processes:

Lemma 1. *If we model an agent's consumption as a geometric Brownian motion with time varying drift and diffusion coefficients $\mu_{c^i}(t)$ and $\sigma_{c^i}(t)$, then we have the* *following relationship between $\mu_{c^i}(t)$, $\sigma_{c^i}(t)$, $r(t)$ and $\theta(t)$, and for all $i \in \{1, ..., N\}$*

$$r(t) = \rho + \mu_{c_i}(t)\gamma_i - (1+\gamma_i)\gamma_i \frac{\sigma_{c_i}(t)^2}{2}$$
$$\theta(t) = \sigma_{c_i}(t)\gamma_i$$

These formulas are very similar to those one would find in a standard representative agent model. However, these expressions hold simultaneously for all agents, meaning that the growth rate and volatility of consumption for each agent must adjust, while for a representative agent they would be replaced by the drift and diffusion of the dividend process. In order to better understand how these values adjust, rewrite Lemma 1 in terms of $\mu_{c^i}(t)$ and $\sigma_{c^i}(t)$ and differentiate to get

$$\frac{\partial \mu_{c_i}(t)}{\partial \theta(t)} = \frac{1+\gamma_i}{\gamma_i^2}\theta(t) \tag{2.8}$$

$$\frac{\partial \mu_{c_i}(t)}{\partial r(t)} = \frac{1}{\gamma_i} \tag{2.9}$$

$$\frac{\partial \sigma_{c_i}(t)}{\partial \theta(t)} = \frac{1}{\gamma_i} \tag{2.10}$$

$$\frac{\partial \sigma_{c_i}(t)}{\partial r(t)} = 0 \tag{2.11}$$

Equations (2.8) and (2.9) imply that the growth rate of every individual's consumption is increasing in both the market price of risk and in the interest rate. All things being equal, holding portfolios and preferences constant, a higher market price of risk implies greater returns. Thus, any given agent will earn more on their portfolio and can expect a higher (or less negative) growth rate in consumption. However, the magnitude of this effect depends both on the prevailing market price of risk and the agent's preferences. First, consider Equation (2.8). When $\gamma_i = 1$, the coefficient is 2 and as γ_i increases the coefficient falls asymptotically towards zero. For more risk averse agents, the change in the expected growth rate of consumption in response to changes in $\theta(t)$ is smaller, going to zero as gamma goes to infinity. This is driven by a consumption smoothing motive. More risk averse agents dislike fluctuations in their consumption and are thus less sensitive to changes in the market.

Second, consider Equation (2.9). Every agent's expected growth rate in consumption is increasing in the interest rate. This makes sense for agents who are net lenders, as they see greater returns on their savings, but this is counter-intuitive for agents who are net borrowers. It implies that, despite having to pay a higher interest rate on their borrowing they prefer to grow their consumption more quickly. This is driven by a wealth effect. An increase in the interest rate lowers the stochastic discount factor, reducing the price of consumption today and in the future. A higher interest rate implies a lower present value of lifetime consumption, whether an agent is a lender or borrower. This makes the budget constraint less binding for both. Because markets are complete, agents borrow solely to finance their consumption choices. So the loosening of the budget constraint will cause an increase in consumption growth rates for all agents despite their financial position.

Finally, Equations (2.10) and (2.11) imply that diffusion in consumption is increasing in the market price of risk, but this effect is decreasing in γ_i, while changes in the interest rate have no effect on consumption volatility. First, Equation (2.11) states that a change in the risk free rate has no effect on the volatility of any agent's consumption, except for its indirect effect on the market price of risk. Agents need to be compensated for volatility in their consumption stream and that compensation comes only from risky assets. Second, Equation (2.10) is decreasing in γ_i as a more risk averse agent will respond less to changes in the market; more risk averse agents desire a smoother consumption path. However, why consumption co-moves positively with the market price of risk is unclear. In order to understand this effect, we need to understand the determinants of the market price of risk.

2.3. **The Risk-Free Rate and Market Price of Risk.** Given Lemma 1, we can derive expressions for the market price of risk and the risk free rate:

Proposition 2. *The interest rate and market price of risk are fully determined by the sufficient statistics* $\xi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ *and* $\phi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$ *such that*

$$r(t) = \rho + \frac{\mu_D}{\xi(t)} - \frac{1}{2}\frac{\xi(t) + \phi(t)}{\xi(t)^3}\sigma_D^2 \tag{2.12}$$

$$\theta(t) = \frac{\sigma_D}{\xi(t)} \tag{2.13}$$

by Lemma 1 and Equation (1.5)*.*

Proposition 2 is in terms of only certain moments of the empirical joint distribution of consumption shares and risk aversion: $\xi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$. These represent the first and second moment of the distribution of elasticity of intertemporal substitution (EIS) with respect to consumption shares. In other words, an agent's preferences only effect the market clearing interest rate and market price of risk up to their amount of participation in the market for consumption.

In (2.13), we can see that the market price of risk in the heterogeneous economy is equal to the market price of risk that would prevail in a representative agent economy populated by an agent whose elasticity of inter-temporal substitution is equal to the consumption weighted average in our economy. This is because the market price of risk is determined by agents choosing the diffusion of their consumption. In the face of shocks each agent will increase or decrease their consumption such that the diffusion of their consumption is equal the market price of risk scaled down by their risk aversion (see Lemma 1).

Looking at (2.12), the first two terms are very reminiscent of the interest rate in a representative agent economy populated by the same agent that would determine the market price of risk. That is if we were to use a representative agent model where the agent's CRRA parameter satisfied $\frac{1}{\gamma} = \xi(t)$ we would find the same market price of risk and nearly the same interest rate. We can rewrite Equation (2.12) as the interest rate that would prevail in our hypothetical economy where plus an extra term:

$$r(t) = \rho + \frac{\mu_D}{\xi(t)} - \frac{1}{2}\frac{\xi(t) + 1}{\xi(t)^2}\sigma_D^2 - \frac{1}{2}\frac{1}{\xi(t)}\left(\frac{\phi(t)}{\xi(t)^2} - 1\right)\sigma_D^2$$

If it were the case that $\phi(t) = \xi(t)^2$, then this additional term would be zero and the interest rate and market price of risk in this model could be exactly matched by those in an economy populated by a representative agent with time varying risk aversion, similar to the model of habit formation by Campbell and Cochrane (1999). However, we can apply the discrete version of Jensen's inequality to show[4] that $\phi(t) > \xi(t)^2, \quad \forall\, t < \infty$. This causes the additional term to be strictly negative. The risk free rate is then lower than it would be in an economy populated by a representative agent. This introduces a sort of "heterogeneity wedge", which I'll define as $\frac{\phi(t)}{\xi(t)^2} > 1$, between the price of risk and the price for risk free borrowing. the larger the difference between $\xi(t)^2$ and $\phi(t)$ the greater the wedge. This wedge is also one plus the coefficient of variation squared in the effective distribution of EIS. The more diverse the consumption shares of individual agents with respect to the elasticity of intertemporal substitution, the greater the wedge. The driving force behind the heterogeneity wedge is the market segmentation that occurs when agents differ in their preferences towards risk.

2.4. **Market Segmentation.** When this economy is populated by two or more agents who have different values of γ, the markets for risky and risk free assets will never clear at the same level and will generate a market segmentation involving three distinct groups. Define $\{\gamma_r(t), \gamma_\theta(t)\}$ to be the RRA parameters in a representative agent economy that would produce the same interest rate and market price of risk, respectively:

$$
\begin{aligned}
r(t) &= \rho + \gamma_r(t)\mu_D - \gamma_r(t)(1+\gamma_r(t))\frac{\sigma_D^2}{2} \\
\theta(t) &= \gamma_\theta(t)\sigma_D
\end{aligned}
$$

Equating these expressions to those in Proposition 2 we can solve for these preference levels, such that

$$
\begin{aligned}
\gamma_r(t) &= \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\left(\frac{\mu_D}{\sigma_D^2}\right)^2 - \frac{\mu_D}{\sigma_D^2}\left(1 + \frac{2}{\xi(t)}\right) + \frac{\xi(t)+\phi(t)}{\xi(t)^3} + \frac{1}{4}} \\
\gamma_\theta(t) &= \frac{1}{\xi(t)}
\end{aligned}
$$

Finally, with a bit of algebra, it can be shown that $\gamma_r(t) < \gamma_\theta(t)$, $\forall t < \infty$. This implies that the markets for risky and risk-free assets do not coincide in finite t. Additionally, it shows that the two markets overlap (see Figure 2). This implies a sort of market segmentation with three groups: leveraged investors, diversifying investors, and saving divestors.

The three market segments in this economy represent buyers and sellers of risky and risk-free assets. Agents who have low risk aversion will sell bonds in order to buy a larger share in the risky asset. Agents with a middling level of risk aversion will be purchasing both bonds and shares in the risky asset. They do this by capitalizing their gains in the risky asset. As we'll see in Section 4, as the low risk

---

[4] $\phi(t) = \frac{1}{N}\sum_i \frac{\omega^i(t)}{\gamma_i^2} = \frac{1}{N}\left(\frac{\omega^1(t)}{\gamma_1^2} + \frac{\omega^2(t)}{\gamma_2^2} + \frac{\omega^3(t)}{\gamma_3^2} + \dots\right) = (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_1^2} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_2^2}\right) + \frac{\omega^3(t)}{N\gamma_3^2} + \cdots > (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_1} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_2}\right)^2 + \frac{\omega^3(t)}{N\gamma_3^2} + \cdots > (\omega^1(t) + \omega^2(t))\left(\frac{\omega^1(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_1} + \frac{\omega^2(t)}{\omega^1(t)+\omega^2(t)}\frac{1}{N\gamma_2} + \frac{\omega^3(t)}{N\gamma_3} + \dots\right)^2 = \left(\frac{1}{N}\sum_i \frac{\omega^i(t)}{\gamma_i}\right)^2 = (\xi(t))^2$, by the strict concavity of the quadratic and induction.

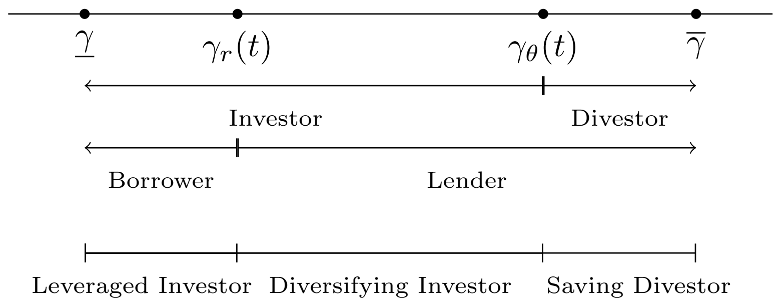


FIGURE 2. The market is segmented depending on an agent's preferences relative to the representative agent solution. The markets for risky and risk-free assets do not coincide and there are three segments. Agents with low risk aversion are simultaneously borrowing and investing. Agents with middling risk aversion are lending and investing. Agents with high risk aversion are saving and divesting.

aversion agents dominate the market, they drive up asset prices, producing high returns. The diversifying investors capitalize these gains in both the risky and the risk-free assets. Finally, agents with high risk aversion will be purchasing bonds and shrinking their share in the risky asset. In general, these agents are simply exchanging with the low risk aversion agents their risky shares for bonds. This causes the risky asset to be concentrated amongst the low risk aversion investors as time progresses and pushes up asset prices, again as we'll see in Section 4.

2.5. **Consumption Weight Dynamics.** We can study the dynamics of an agent's consumption weight by applying Itô's lemma to the expression given in Proposition 1.

Proposition 3. *Assuming consumption weights also follow a geometric Brownian motion such that*

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t)$$

an application of Itô's lemma to (2.7) *gives expressions for* $\mu_{\omega^i}(t)$ *and* $\sigma_{\omega^i}(t)$*:*

$$\begin{aligned} \mu_{\omega^i}(t) &= (r(t)-\rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) \\ &\quad + \frac{\theta(t)^2}{2}\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right] \end{aligned} \tag{2.14}$$

$$\sigma_{\omega^i}(t) = \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) \tag{2.15}$$

Consider first the case where an agent's preferences coincide with the weighted average, ie $\gamma_i = \gamma_\theta = \frac{1}{\xi(t)}$ (as in Section 2.4). In (2.15), which describes how an agent's consumption weight co-varies with the risk process, $\sigma_{\omega^i} = 0$. If an agent has the same EIS as the market then they will not desire to vary their consumption weight in the face of shocks. As in the analysis of the previous sub-section, this is because the agent is perfectly in agreement with the market. However, notice that in this case $\mu_{\omega^i} = \theta(t)^2 \left(\frac{1}{\gamma_\theta^2} - \phi(t)\right) = \sigma_D^2 \left(1 - \frac{\phi(t)}{\xi(t)^2}\right)$, by Proposition 2. This is an indicator of the speed with which the economy is moving through this equilibrium. Although the agent is instantaneously satisfied with the current market price of risk, they are deterministically moving out of this position. The speed with which this is occurring is driven by the heterogeneity wedge, $\frac{\phi(t)}{\xi(t)^2}$. When this wedge is high, the rate at which the marginal agent moves out of the marginal position is greater.

Next consider the case where an agent is more patient than the weighted average, that is $\gamma^i > \gamma_\theta$. Then $\sigma_{\omega^i} < 0$ and agent i's weight is negatively correlated to the market. This implies that if an agent is more patient than the average, or alternatively more risk averse, then their consumption share will increase when there are negative shocks and decrease when there are positive shocks. This is a prudence motive and these agents can be thought of as playing a "buy low, sell high" strategy. They do not want to grow their consumption faster than the economy, but to pad their position against future shocks. They smooth consumption over time, providing a very stable consumption path. For this reason, their decisions are driven not by a desire to increase their consumption today, but to insure themselves against shocks in the distant future and, in turn, increase their wealth. These agents will have a consumption stream which is less volatile than the economy.

Conversely, if an agent is less risk averse than the average, ie $\gamma^i < \gamma_\theta$, their consumption shares covary positively with the market. These agents are essentially buying high and selling low, a strategy that will cause their wealth to be highly volatile. An agent with a lower risk aversion has a higher elasticity of intertemporal substitution and, thus, can be thought of as less patient. Given a shock to the dividend process, the expected growth rate remains constant, but the level shifts permanently because of the martingale property of the Brownian motion. Since less patient agents see the current output of the dividend as more important than its long-run behavior, present shocks have a greater effect on their personal price. Thus, a negative shock causes them to reduce their price and in turn their consumption shares, while a positive shock causes them to increase their price and consumption share. These are the day-traders, riding booms and busts to try to make a quick buck while not losing their shirts. Although they may benefit in the short fun, their consumption will be more volatile than the economy.

The analysis of (2.14) is quite difficult for the case of $\gamma_i \neq \gamma_\theta$. The first term is the product of two separate terms: one involving the interest rate and rate of time preference, the other the agent's position in the distribution. If the interest rate is above the rate of time preference, the first term is positive. If the interest rate differs from the rate of time preference then the agent should desire to shift consumption across time periods, either from today to tomorrow or vice versa. However, the direction will be determined by their preference. If $\gamma_i > \gamma_\theta$ then the product will be negative and this first term will contribute negatively to their growth rate $\mu_{\omega^i}(t)$. The opposite is true when $\gamma_i < \gamma_\theta$. The combined effect of these two terms is to say that if an agent is less patient than the average and the interest rate is greater than their rate of time preference, they will want to grow their consumption faster than the rate of growth in the economy, while if they are more patient than the average then they will tend to grow their consumption more slowly than the rate of growth in the economy. This effect is only partial, however, and it is necessary to take into consideration the second term.

The second term is quite a bit more complex. The term in brackets is a sort of quadratic in deviations from the weighted average of risk aversion. Whether this term is positive or negative depends in a complicated way on $\xi(t)$ and $\phi(t)$ [5]. It is sufficient to note that, when the distribution is not too skewed, there exists a level of risk aversion such that if an agent is above this the second term in (2.14) is negative and that this level of risk aversion is not equal to γ_θ or γ_r. This is related to the deterministic nature of the shifting distribution of asset holdings. Although these two preference levels represent the instantaneous market clearing levels, they do not reflect how the distribution is evolving over time.

2.6. **Asset Prices and Portfolios.** Now, given expressions to describe the evolution of consumption choices over time, one can give a formula describing asset prices. Bear in mind that it is not trivial to solve for each individual agent's asset price, as it depends in a non-linear way on their consumption weight. However, given how the discount factor will evolve one can use market clearing to derive the expression in Proposition 4.

Proposition 4. *Under a transversality condition on wealth, that is if we assume* $\mathbb{E}_t\left[\lim_{s\to\infty} H_0(s)X^i(s)\right] = 0$, *then it can be shown that asset prices satisfy the following:*

$$S(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)} D(u)du \tag{2.16}$$

Proposition 4 matches classic asset pricing formulas and defines asset prices today in terms of expectations of the future outcome of the dividend process, discounted at the market rate. Consider (2.16), which we can rewrite by substituting for $H_0(t)$ using (2.5) as

$$S(t) = \mathbb{E}_t \int_t^\infty e^{-\rho(u-t)} \left(\frac{c^i(u)}{c^i(t)}\right)^{-\gamma_i} D(u)du$$

for every i, which is exactly equal to the asset pricing formula derived from the Euler equation in a representative agent economy. The key difference, however, is that the dynamics of the consumption process in this economy are not equal to the dynamics of the dividend process. It may be possible to construct an agent whose consumption share remains fixed over a short period of time and thus whose price in a representative agent economy would equal the price in the heterogeneous economy, but it is not necessarily true that that agent exists in the model being solved here. For instance, in the case of two agents, the price will always be somewhere between the price that would prevail in the two individuals' autarkic economies.

Asset price dynamics and portfolios being necessary for numerical simulation, it is possible to derive estimable expressions for the volatility of asset prices and for

[5]It can be shown that the roots of this quadratic are $\frac{1}{\gamma_i} = \xi(t) - \frac{1}{2} \pm \sqrt{\frac{1}{4} + \phi(t) - \xi(t)^2}$. Because $\phi(t) > \xi(t)^2$, there will always be two real roots. However, whether these are positive or negative depends on the values of $\xi(t)$ and $\phi(t)$.

the optimal portfolio weight of agents. The following proposition is identical to two given in Cvitanić et al. (2011)[6] and is thus provided without proof:

Proposition 5. *The volatility of the stock price is given by*

$$\sigma_S(t) = \sigma_D + \frac{\mathbb{E}_t \int_t^\infty (\theta(t) - \theta(u)) H_0(u) D(u) du}{\mathbb{E}_t \int_t^\infty H_0(u) D(u) du} \tag{2.17}$$

and optimal portfolios by

$$\pi^i(t)\sigma_S(t)S(t)H_0(t) = \theta(t)\mathbb{E}_t \int_t^\infty H_0(u)D(u)\omega^i(u)du \tag{2.18}$$

$$+ \frac{1-\gamma_i}{\gamma_i}\mathbb{E}_t \int_t^\infty H_0(u)D(u)\theta(u)\omega^i(u)du \tag{2.19}$$

This proposition essentially states that, if $\theta(s) < \theta(t)$, then there will be excess volatility, and that risk averse agents will hold a smaller share of their wealth in the risky asset. I refer the interested reader to the previously mentioned paper for a thorough treatment of the asymptotic results. From the statement in Proposition 5, one can find $\mu_S(t)$ using Equation (2.13), or by similarly matching coefficients in the Clark-Ocone derivation.

3. Extension to Infinite Types

Consider now the limiting case as $N \to \infty$. This corresponds to a special type of mean field game with common noise, where the idiosyncratic volatility is degenerate. That is, although there are two degrees of randomness in the model corresponding to the random initial condition and the Brownian motion, there is no idiosyncratic risk process. Agents' states evolve idiosyncratically because of their heterogeneous preferences, but are subject only to a common noise. This implies for a given level of wealth and a given preference level, γ, all agents will have the same control. This fact is similar to symmetry in permutations of the state in Lasry and Lions (2007) and other papers on mean field games, but one can think of the preference parameter as being a degenerate state variable, i.e. $d\gamma = 0$. Additionally, because the constraint is determined by initial wealth, one can consider heterogeneity in the initial condition as being the key driver of the mean field. This characteristic makes the model dependent on the initial condition and the realization of the Brownian motion. Because of this the mean field will be with respect to the control and the determinant distribution will be over the initial condition.

If we take the control, $\omega^i(t) = \omega(t, \gamma_i, x_i)$, we have a function of an empirical mean:

$$\omega(t, \gamma_i, x_i) = \frac{N\left(\Lambda(\gamma_i, x_i)e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^N \left(\Lambda(\gamma_j, x_j)e^{\rho t}H_0(t)\right)^{\frac{-1}{\gamma_j}}}$$

By the strong law of large numbers (assuming the variance in consumption across agent types is bounded), the empirical average converges to the mean with respect

[6] I am grateful to the authors of that paper for helping me to understand the derivation using the Malliavan Calculus and the Clark-Ocone theorem.

to the distribution of the initial condition:

$$\omega(t, \gamma_i, x_i) \underset{N \to \infty}{\to} = \omega(t, \gamma, x) = \frac{\left(\Lambda(\gamma, x) e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma}}}{\int \left(\Lambda(\gamma, x) e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma}} dF(\gamma, x)}$$

Since $\omega(t, \gamma, x)$ acts as the only state variable determining all individual and aggregate outcomes in the model, the same is true for the rest of the propositions, where one simply replaces the market clearing conditions and the variables $\xi(t)$ and $\phi(t)$:

$$\begin{aligned}
D(t) &= \int c(t, \gamma, x) dF(\gamma, x) \\
1 &= \int \omega(t, \gamma, x) dF(\gamma, x) \\
S(t) &= \int X(t, \gamma, x) dF(\gamma, x) \\
\xi(t) &= \int \frac{\omega(t, \gamma, x)}{\gamma} dF(\gamma, x) \\
\phi(t) &= \int \frac{\omega(t, \gamma, x)}{\gamma^2} dF(\gamma, x) \\
\theta(t) &= \frac{\sigma_D}{\xi(t)} \\
r(t) &= \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2} \frac{\xi(t) + \phi(t)}{\xi(t)^3} \sigma_D^2
\end{aligned}$$

The market clearing condition for consumption weights implies something intriguing about their relationship to the initial distribution. If we think of $\omega(t, \gamma, x)$ as a ratio of probability measures, we can think of $\omega(t, \gamma, x)$ as the Radon-Nikodym derivative of a stochastic measure with respect to the distribution of the initial condition. That is, define $\omega(t, \gamma, x) = \frac{dG(t, \gamma, x)}{dF(\gamma, x)}$. Then we have

$$\begin{aligned}
\int \omega(t, \gamma, x) dF(\gamma, x) &= 1 \\
\int \frac{dG(t, \gamma, x)}{dF(\gamma, x)} dF(\gamma, x) &= \\
\int dG(t, \gamma, x) &=
\end{aligned}$$

The evolution of this distribution would be difficult to describe directly, but the expressions in Proposition 4 give the dynamics of this stochastic distribution. So $\omega(t, \gamma, x)$ allows one to calculate exactly the evolution of this stochastic distribution by use of a change of measure. Alternatively, one can think of $\omega(t, \gamma, x)$ as a sort of importance weight, where as the share of risky assets is concentrated towards one area in the support, the weight of this area grows in the determination of asset prices.

Additionally, the Radon-Nikodym interpretation allows one to compare the continuous types to finite types. Say for instance we would like to discretize the above expression for the market clearing condition on $\omega(t, \gamma, x)$ using a Riemann sum with an evenly space partition (e.g. a midpoint rule):

$$\int \omega(t,\gamma,x)dF(\gamma,x) \approx \frac{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}{JK}\sum_{k=1}^{K}\sum_{j=1}^{J}\omega(t,\gamma_k,x_j)f(\gamma_k,x_j)$$

This looks quite similar to the market clearing conditions in the finite type model (Equation (1.5)). So, could we construct a finite types sample that matches this approximation, at least initially? Make the identification $N = JK$ and notice that since $\omega(t,\gamma,x)$ is a geometric Brownian motion, such that $\omega(t,\gamma,x) = \omega(0,\gamma,x)\hat{\omega}(t,\gamma,x)$ where $\hat{\omega}(t,\gamma,x)$ is a stochastic process. If we define the initial condition on omega as $\omega(0,\gamma,x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})f(\gamma,x)}$, then the above market clearing condition becomes

$$\int \omega(t,\gamma,x)dF(\gamma,x) \approx \frac{1}{N}\sum_{k=1}^{K}\sum_{j=1}^{J}\hat{\omega}(t,\gamma_k,x_j)$$

This market clearing condition looks exactly like the condition in Equation (1.5). However, this has particular implications about the Radon-Nikodym derivative. From the definition of the Radon-Nikodym derivative we can write

$$\begin{aligned} G(t,A) &= \int_A \omega(t,\gamma,x)dF(\gamma,x) \\ &= \int_A \omega(t,\gamma,x)f(\gamma,x)d\gamma dx \end{aligned}$$

Substituting the imposed definition of $\omega(t,\gamma,x)$ we have

$$G(t,A) = \int_A \frac{\hat{\omega}(t,\gamma,x)}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}d\gamma dx$$

Now, since $\hat{\omega}(0,\gamma,x) = 1$, the above implies

$$G(0,A) = \int_A \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}d\gamma dx$$

Thus the initial condition of the stochastic measure $G(0,A)$ is a uniform distribution.

All of this to say that if one attempts to approximate the continuous model by a finite model not taking into account the initial distribution $f(\gamma,x)$, one can only generate a certain initial condition. That is, the product distribution $\omega(0,\gamma,x)f(\gamma,x) = \frac{1}{(\overline{\gamma}-\underline{\gamma})(\overline{x}-\underline{x})}$ in a simulation of finite types and any Riemann sum approximation to the integral. One could also attempt to use a monte-carlo scheme, sampling many agents from the initial distribution. However, the variance will be large for any value of N which one can compute on a desktop computer.

The continuous types model encompasses the discrete types model completely, in that if the true distribution $f(\gamma,x)$ were a discrete distribution one could get identical results. On the other hand, the continuous types model seems more complex at first glance. Although this seeming addition of complexity provides little in the way of new economic insight, it does provide several nice explicit modeling tools. First, the joint distribution of initial wealth and risk aversion is explicitly modeled. In a model of finite types one can only model a product distribution such that $\omega(0,\gamma,x)f(\gamma,x)$ is uniform. Second, but closely related, is the computational simplification provided by the continuum. For finite types one must simulate many agents in order to match some distribution of preferences. Because the main drivers of financial variables in this model are moments of the distribution of risk preferences, one can simulate quadrature points to approximate a continuous distribution, whereas to do the same for the finite types model would require many simulated agents. This fact will become quite apparent in simulations.

4. Simulation Results and Analysis

In this section, I review the simulation strategy as well as some simulation results and compare them[7]. The underlying assumption in all of these simulations is that I am attempting to approximate a continuous distribution of types (for a recent survey on estimating risk preferences see Barseghyan et al. (2015) and for evidence on heterogeneity see Chiappori et al. (2012)). One could argue that this is the goal of any model of heterogeneous risk preferences with finite types, as in Dumas (1989) or Chabakauri (2013), but that one assumes finite types for tractability and with the hope that the results generalize. This section will try to convince you that, quantitatively, the results for the continuous types model are not the same as those for the finite types model and that the continuous types model is in fact computationally less costly than an equally accurate finite types model.

These simulations can be separated into two steps: simulating the processes for consumption shares and the associated market clearing interest rate and market price of risk, and estimating portfolios and asset prices by Monte Carlo. First, the Ito processes are discretized using a Euler-Murayama scheme. Then, individual nodes or agents are simulated by taking an initial distribution for $\omega(0, \gamma, x)$. This initial distribution implies a certain initial asset price and portfolio weights, both of which are calculated by Monte Carlo using a Rhomberg extrapolation (see Glasserman (2003) or Guyon and Henry-Labordère (2013)). Using this, one can generate an initial wealth and initial bond holdings for individual agents. This initial distribution also implies values for $\xi(0)$ and $\phi(0)$, which imply values for $r(0)$ and $\theta(0)$. Then, using the above mentioned scheme I simulate forward the set of consumption weights, $\omega^i(t)$ or $\omega(t, \gamma^i, x^i)$, estimating asset prices and portfolio weights each period in the same fashion.

For all of the simulations, I will hold the following group of parameters fixed at the given values: $\mu_D = 0.03$, $\sigma_D = 0.06$, and $\rho = 0.01$. These settings correspond to a yearly parameterization. Additionally, for simulating asset prices by Monte Carlo, I need to specify a truncation level $T = 300$, as well as the number of path iterations $M = 6.24 \times 10^5$. Finally, I simulate forward 25 periods at a $\Delta t = 0.5$ discretization. These simulations were run on a set of Amazon Web Services c4.8xlarge compute optimized cluster nodes, each with two Intel Xeon E5-2666 v3 (Haswell) processors sporting 36 virtual cores and 60Gb of memory.

I'll show first simulation results for a shocked five agent model. Then I'll look at two, five, and ten agents around a non-stochastic sample path, showing how the change in number of agents drastically changes the level and the dynamics of all variables, while leaving unchanged the asymptotic value of aggregate variables. Finally, I'll compare this to simulating a continuous type economy with two, five, or ten quadrature points. We will see that the effect of changing the number of quadrature points converges quickly to zero when the initial distribution is uniform.

[7]In the present draft, the simulations are only preliminary. Given the high computing cost, I was unable to complete simulations with enough paths to reduce variance in the ratio estimators of volatility and portfolios, but these are currently running and will be available as soon as possible.

Thus, in terms of robustness to the discretization choice, the model of continuous types dominates the model of finite types.

4.1. **Finite Types versus Continuous Types.** If one believes there to be a continuous distribution of risk preferences, one could choose to use either a finite types model or an approximation of a model of continuous types. However, the computational complexity and accuracy will not be equal across the two approximations. For finite types, we can consider different numbers of types, approximating the continuous distribution of types by a histogram, and see if the assumptions about the discretization alter the results. For the continuous types model we must approximate the integrals in some way, here choosing to use a quadrature rule, and consider how these approximations change the results.

As an example, consider the definition of $\xi(t)$ and its associated quadrature approximation:

$$\xi(t) = \int \frac{\omega(t,\gamma,x)}{\gamma} dF(\gamma,x) \approx \sum_{m=1}^{M} \sum_{k=1}^{K} \psi_m \psi_k \frac{\omega(t,\gamma_m,x_k) f(\gamma_m,x_k)}{\gamma_m}$$

where (ψ_m, ψ_k) are the appropriate quadrature weights and (γ_m, x_k) the associated quadrature points. The useful feature here is that the points are fixed in time, that is if one would like to simulate this model forward, it is only necessary to fix a set of points (γ_m, x_k) and simulate forward the associated consumption weights $\omega(t,\gamma_m,x_k)$. In this way, we can compare the accuracy and robustness of the two types of simulations, finite types or continuous types, for a given number of points simulated. One should expect (and we will see this is the case) that the results are not the same, as the simulated models have drastically different assumptions about the distribution $f(\gamma,x)$, as discussed in Section 3.

For finite types, changing the number of simulated points changes the distribution $f(\gamma,x)$ in the model, while for the continuous types simulation, changing the number of quadrature points does not change the assumptions about $f(\gamma,x)$, but only affects the accuracy of the quadrature approximation. This will be the key feature that differentiates the two simulations. Although the qualitative features will be similar, the robustness of the continuous types simulation will be far superior to the finite types simulation. However, for longer time periods the continuous types approximation will break down. This is driven by the fact that the agent with the lowest risk aversion will dominate the economy in the long run (see Cvitanić et al. (2011)). Mass will eventually build up on the lower area of the support and, when one fixes the quadrature points, this area will be below the lowest quadrature point. Because of this, a quadrature approximation is most likely not the most accurate method for longer simulations, but it is sufficient for comparison purposes here.

4.2. **Five Agents: Shocked.** Let's begin with five agents with CRRA parameters $[\gamma_i] = [1, 3.25, 5.5, 7.75, 10]$. For this first simulation I'll allow the shock process, $dW(t)$, to realize away from its expectation.

Figure 3 shows the interest rate and dividend yield in this first economy. Here we see clear negative trends in both processes. First, as the most impatient agent begins to dominate the market for risky assets, the market interest rate begins to converge towards their preferred rate. That is, in the long run, the prevailing interest rate will correspond to that which one would find in an economy populated by a single agent with the lowest value of γ_i. Similarly, asset prices are converging

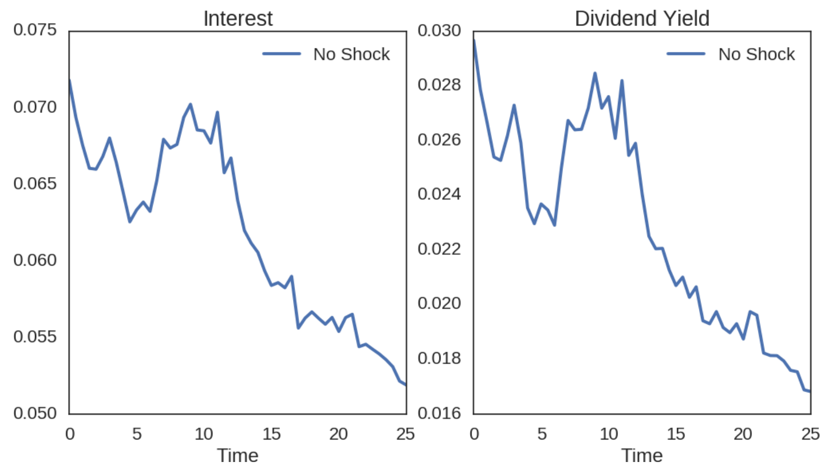


FIGURE 3. Dividend yield defined as $\frac{D(t)}{S(t)}$ and interest rate, $r(t)$, for five agents under a shocked process.

to a similar long run value. This causes $S(t)$ to grow faster than $D(t)$, pushing down the dividend yield.

The downward trend in dividend yields points towards a possible explanation for the predictability in asset prices described in Campbell and Shiller (1988b) and Campbell and Shiller (1988a). Here dividend yield is trending downward as asset prices are growing faster than the dividend. Because of this, we can expect periods of high yield relative to this trend to predict subsequent periods of low yield. In this way, asset prices have a predictable component[8] The conclusion I draw from this is that the portion of asset price movements that are explainable are driven by the time variation in the discount factor, moving across market clearing values of γ^{θ} and γ^{r}.

In the face of a changing interest rate and market price of risk, agents are dynamically optimizing their consumption shares and portfolio weights, depicted in Figure 4. In relative terms, the most risk neutral agent stands apart from the rest of the economy in the level and volatility of their choices. Additionally, we can observe the differences in agents' choices in the face of shocks. During periods 5 through around 8 the economy undergoes a small recession. The most risk neutral agent reduces their portfolio shares and their consumption weights. However, the opposite is true of the rest of the agents, and in fact the most risk averse agents increase their shares and weights by the largest amount. Their consumption choices will change the effective distribution of preferences and their portfolio choices will determine the volatility in their wealth.

The distribution of consumption shares can be described by the evolution of $\xi(t)$ and $\phi(t)$, as well as the heterogeneity wedge $\frac{\phi(t)}{\xi(t)^2}$. These values are displayed in Figure 5. You'll notice that the market clearing level of risk aversion, or conversely

[8]Indeed, a quick regression of dividend yield on past values shows that in a simple AR(1) model previous values of the dividend yield have a strongly significant coefficient.

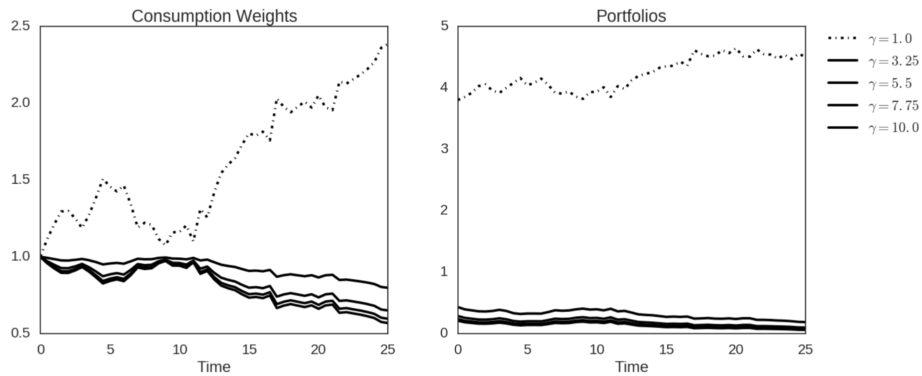


FIGURE 4. Agent portfolio shares in a five agent economy subject to shocks.

FIGURE 5. Sufficient statistics for the distribution of risk aversion, where $\xi(t) = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i^2}$, and the heterogeneity wedge is defined as $\frac{\phi(t)}{\xi(t)^2}$.

the average EIS, is converging towards one. Additionally, the wedge introduced by agent heterogeneity is falling over time. It is these facts that are driving the falling interest rate and market price of risk. Additionally, portfolio and consumption decisions affect the evolution of agents' wealth. In Figure 6 you can see that the least risk averse agent has a very volatile wealth process and a large amount of borrowing. This agent is highly leveraged and is the most exposed to swings in asset prices, so their wealth process moves more than one to one with the dividend process. These results are difficult to interpret, as one can't necessarily separate the effects of shocks from the drift. To that end, we can look at simulations around a sort of non-stochastic trend.

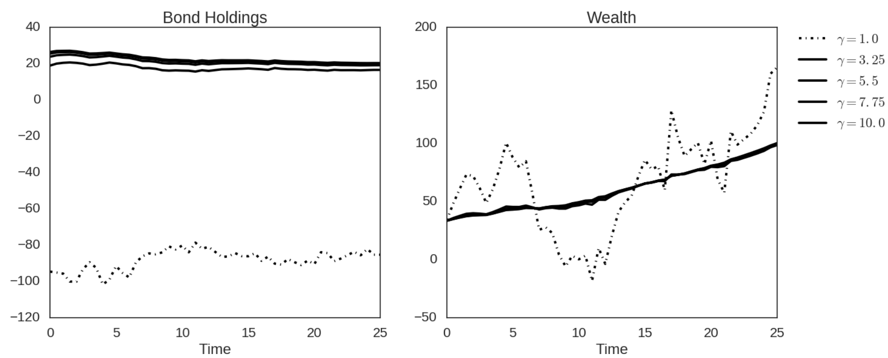


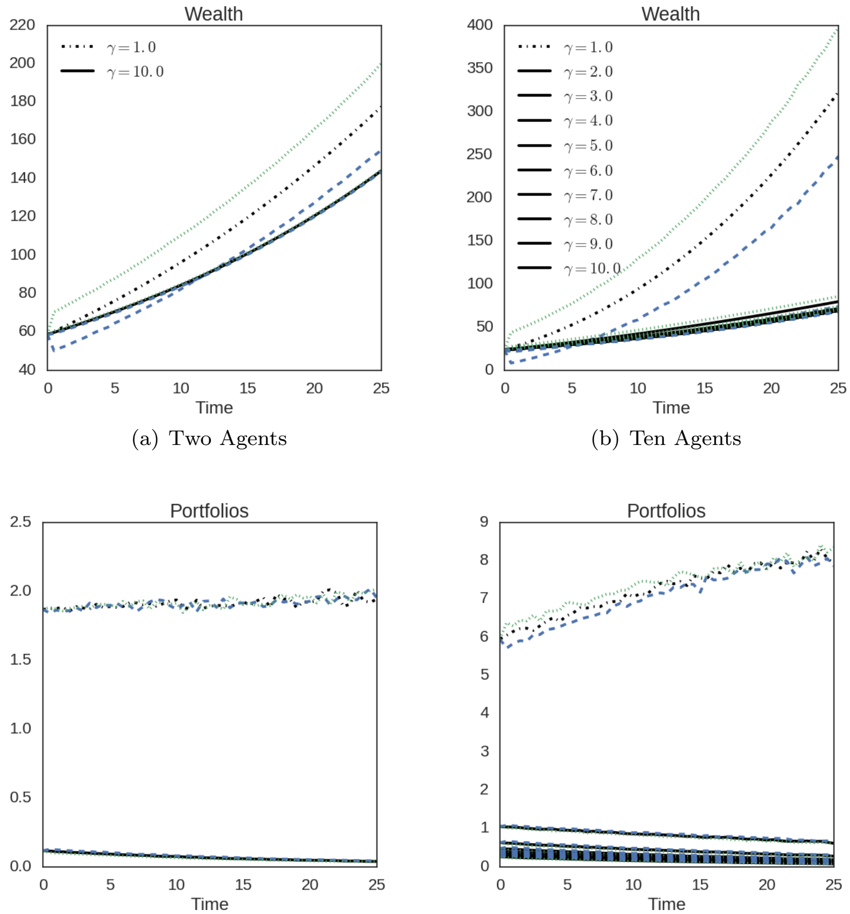
FIGURE 6. Agent bond holdings and wealth in a five agent economy subject to shocks. The most volatile line corresponds to the least risk averse agent.

4.3. **Increasing Agents: No Shock.** In order to look at distributional outcomes for different numbers of agents, it is easiest to study the non-stochastic path (that where $dW(t) = 0$) and a 95% confidence interval around this path. Here I study the outcome for several economies with two, five, and ten agents. Each simulation takes the vector of preference parameters to be evenly spaced over $[1, 10]$[9]. The motivation behind these simulations is to consider the convergence in the number of agents and to compare to the simulations of continuous types in Section 4.4. The change in the number of agents has a significant change in the level of all variables, implying that misspecification of the support of the distribution of risk preferences has a non-trivial effect on the model's short run predictions about market variables.

Observe in Figure 7 individual wealth processes over time for two and ten agents, where I've highlighted the lowest risk aversion level for simplicity. The most striking feature of these two plots is the growing variance of the wealth of the least patient (least risk averse) agent and the difference in the growth rate of wealth across the two simulations. The least risk averse agent is maintaining a fairly constant or slightly growing portfolio share throughout time, but the value of that share is growing. In Figure 8 you'll see that dividend yield is falling through time, implying that asset prices are growing faster than the economy. Recall that the volatility in any agent's wealth is given by $\pi^i(t)\sigma_s(t)S(t)$, so as $S(t)$ grows this volatility becomes larger. Because the most risk neutral agent has such a large share in the stock market, their exposure to this volatility is greater.

The distribution of consumption shares across individuals can be summed up by the sufficient statistics $\xi(t)$ and $\phi(t)$, as shown in Figure 9 for all three simulations. There you can see that the heterogeneity wedge, defined as $\frac{\phi(t)}{\xi(t)^2}$, converges more slowly for five agents. This is driven by the dynamics of these variables, which are

[9]For two agents $[\gamma_i] = [1, 10]$, for five agents $[\gamma_i] = [1, 3.25, 5.5, 7.75, 10]$, and for ten agents $[\gamma_i] = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]$



(a) Two Agents

(b) Ten Agents

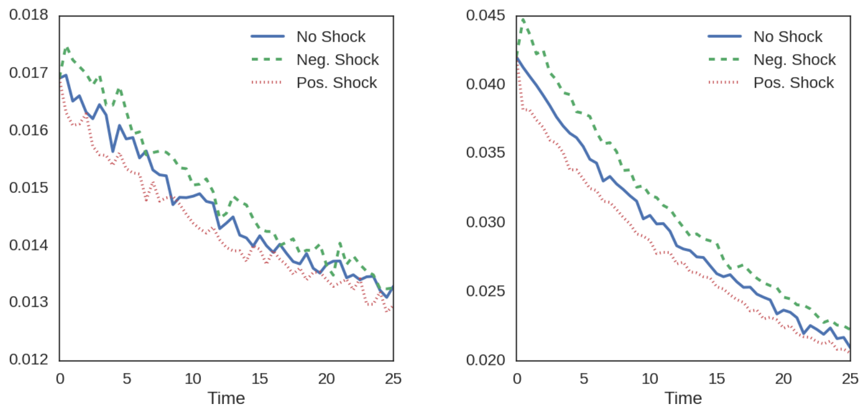
(c) Two Agents

(d) Ten Agents

FIGURE 7. Wealth and portfolios around the non-stochastic trend and a 95% confidence interval (dots is positive, dash is negative shocks).

determined by higher moments of the distribution of consumption weights. Because both $\xi(t)$ and $\phi(t)$ are non linear transformations of γ, changing the number of agents will change the level of both variables. Beyond this, the more agents there are, the more total mass there is in the economy. Although the averages are what is important for levels, changing the mass of agents over the support changes the rate of convergence, as in order for the most risk neutral agent to dominate they must accumulate a greater consumption share to bring $\xi(t)$ and $\phi(t)$ to the same long run level. Additionally, both $\xi(t)$ and $\phi(t)$ are lower for ten agents than for five, and for five agents than for two, and the simulations are converging very slowly in the number of agents.

These facts combine to cause the interest rate and market price of risk to be higher for more agents. In Figure 10 you can see that the underlying assumptions



(a) Two Agents (b) Ten Agents

FIGURE 8. Dividend yield, $\frac{D(t)}{S(t)}$ around the non-stochastic trend and a 95% confidence interval (dots is positive, dash is negative shocks).

of the model imply that changing the support of the distribution of risk aversion causes a shift in levels, volatility, and rate of convergence of the interest rate and market price of risk. If one believes there is a continuum of types, then the way that one discretizes or bins this distribution into a finite support has a substantial effect on the model's outcome. This is the driving interest in a continuous types simulation, as explicitly modeling the continuous distribution will provide stability to simulations with a comparable computational cost.

4.4. **Continuous Types: No Shock.** Consider a similar exercise, attempting to model a continuous distribution by some finite set of points, except instead use the model of continuous types. Instead of changing the number of agents one changes the number of quadrature points used for approximation. Here I'll use two, five, and ten quadrature points to match the above simulations. Additionally, since the above simulations began with an equal number of agents consuming equal shares in the consumption good and holding equal wealth, the initial marginal distribution of wealth will be a point mass and the distribution of preferences will be uniform (ie $f(\gamma, x) = \frac{\delta_{x*}}{\overline{\gamma}-\underline{\gamma}} = \frac{\delta_{x*}}{9}$) [10]. The key point is that the change in the number of quadrature points changes the points in the support in which one is interested, but this change does not necessarily affect the underlying simulation. I present plots for sufficient statistics of the consumption shares and the aggregate variables. The results for shares and wealth follow a similar trend, but are more difficult to compare visually given they are distributions instead of finite dimensional vectors. You will see that the qualitative characteristics are the same as before, but that the simulations converge much more quickly in the number of quadrature points than the finite type simulations do in the number of agent types.

[10]This also reduces the required number of quadrature points, since the initial distribution is essentially one dimensional.

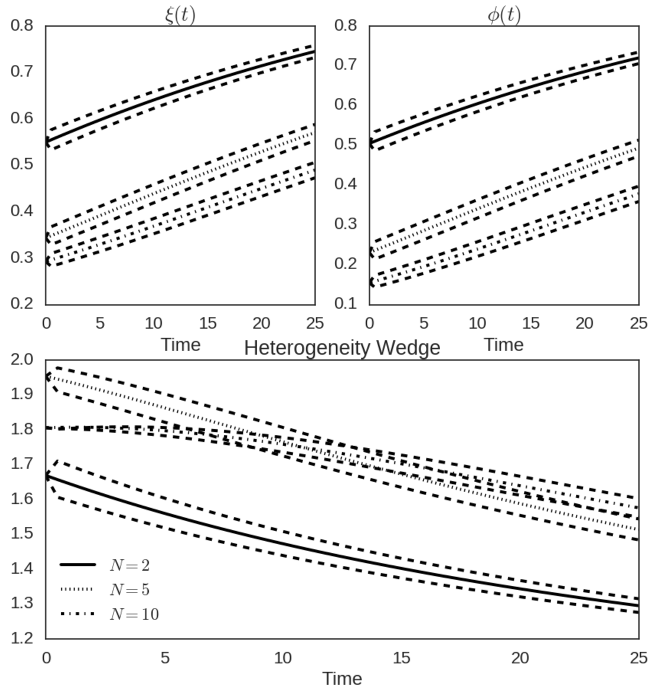


FIGURE 9. Sufficient statistics for the distribution of the risk aversion with finite types, where $\xi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i}$ and $\phi(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\omega^i(t)}{\gamma_i^2}$, and the heterogeneity wedge is defined as $\frac{\phi(t)}{\xi(t)^2}$. N corresponds to the number of types.

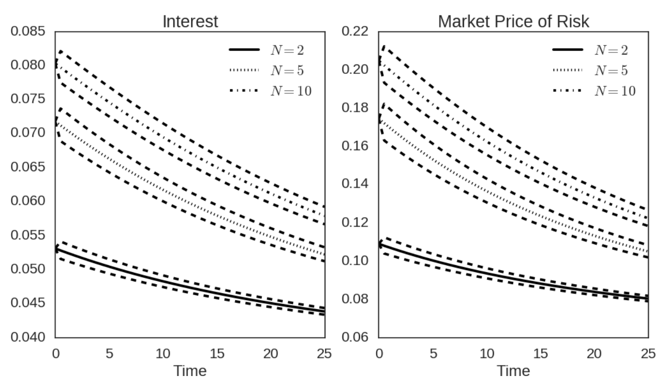


FIGURE 10. Comparison across simulations for aggregate variables with finite types. N corresponds to the number of types.

The heterogeneity wedge $\frac{\phi(t)}{\xi(t)^2}$ converges quickly to one particular initial condition and evolves along a very similar trend[11] which is smooth and hump-shaped.

[11]The lines begin to diverge towards the end of the simulation. This is related to an earlier point about the quadrature approximation. As the most risk neutral agent accumulates all the wealth

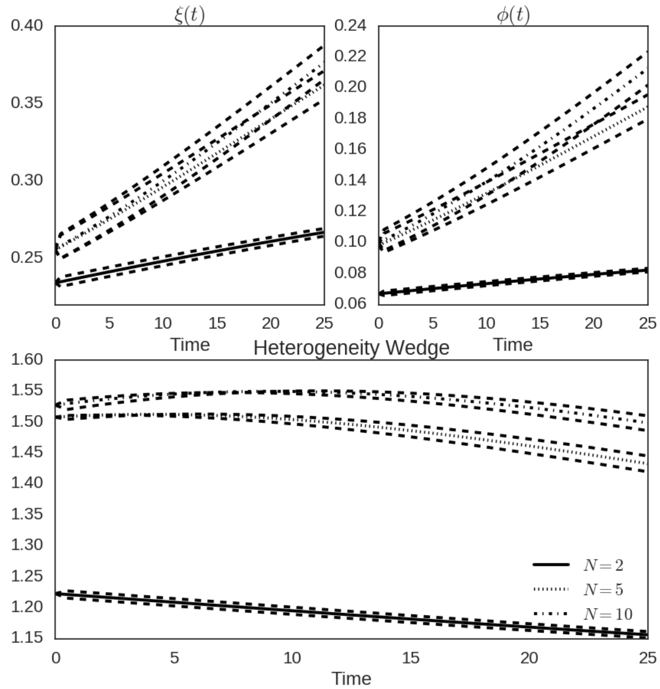


FIGURE 11. Sufficient statistics for the distribution of the risk aversion with continuous types, where $\xi(t) = \int \frac{\omega(\gamma,x,t)}{\gamma} dF(\gamma,x)$ and $\phi(t) = \int \frac{\omega(\gamma,x,t)}{\gamma^2} dF(\gamma,x)$, and the heterogeneity wedge is defined as $\frac{\phi(t)}{\xi(t)^2}$. N corresponds to the number of quadrature points.

As you can see in Figure 11, the lines for five and ten quadrature points lie on top of each other. Additionally, the finite types simulation seems to be converging to the same path, albeit much more slowly. In fact, in order to match the continuous types model by using the finite types simulations one needs many agent types, a prohibitive number. This implies that, if one is attempting to match a continuous distribution of types, a low number (e.g. two, three, etc.) is insufficient to match the level of the heterogeneity wedge that would be present in the true model. Beyond the number required, the definition of the continuous distribution frees the modeler from specifying preference levels and instead can simply specify the shape of the distribution. This removes a degree of freedom, but provides another by separating the distributions as described in Section 3.

In terms of financial variables, the fact that $\xi(t)$ and $\phi(t)$ are converging implies directly that $r(t)$, $\theta(t)$, and $S(t)$ will be converging quickly in the number of

in the long run, and because the quadrature points do not include the end point, the simulations are converging to different long run solutions. This problem would be alleviated with a different integral approximation which does include the endpoints, but as mentioned before this is left for later study.

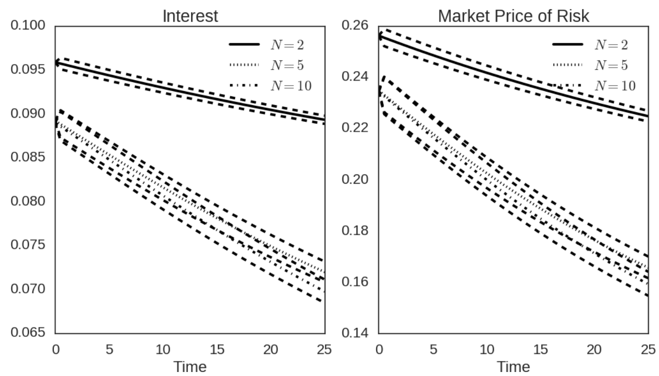


FIGURE 12. Comparison across simulations for aggregate variables with continuous types. N corresponds to the number of quadrature points.

quadrature points. In Figure 12 one can see that indeed the solutions for five and ten quadrature points are almost identical. However, they are diverging towards the end of the simulation. In fact, because they are converging to different values, they will cause the Monte Carlo estimators to differ, as these rely on long path simulations of the stochastic discount factor. Indeed, in Figure 13 you can see that the dividend yield is diverging and that the level of the stock price volatility is different. Although this is somewhat disappointing, the results are still superior to those for finite types and may be improved with a different integral approximation.

Clearly, if one wishes to match aggregate financial variables in a simulation with heterogeneous preferences, the continuous types model is computationally less costly, requiring fewer points to be simulated to get convergence. Additionally, one does not need to choose bins to be simulated, simply a preferred method to simulate the integral in $\xi(t)$ and $\phi(t)$. However, as already discussed, the simulations are beginning to diverge. This is caused by the fact that mass is accumulating at the lower end of the risk aversion distribution. Because the Gaussian quadrature rule used to calculate the integrals does not cover this area well, the integral become poorly approximated. In the long run, any numerical integral approximation will be poor as mass accumulates to a singularity at the lower bound of the distribution, but one which takes this area into account, or at least places more points in the lower section of the support, would be preferable.

5. CONCLUSION

In this paper I have studied how the distribution of risk preferences affects financial variables, consumption shares, and portfolio distributions. The distribution of risk preferences has a large effect on financial variables driven mainly by consumption weighted averages of the EIS. The implication is that the amount of participation by individuals in the market for consumption, not the market for risky assets, determines to what degree their preferences affect price. In fact, the evolution of individual shares is determined by each agent's relative position to two weighted averages of the EIS and its square. Given the heterogeneity in preferences, markets for risk free bonds and risky assets clear at different levels, implying three groups.

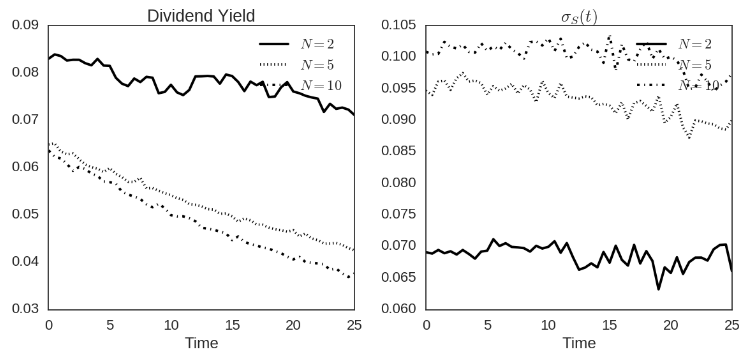


FIGURE 13. Comparison across simulations for aggregate variables with continuous types. N corresponds to the number of quadrature points.

Leveraged investors have low risk aversion and borrow in order to grow their share in the risky asset. Saving divestors are highly risk averse and lend in order to shrink their share in the risky asset. Somewhere in the middle we have the diversifying investor, who is growing their share in the stock market and simultaneously lending by buying bonds.

Outcomes are driven by a heterogeneity wedge[12] which describes how different are the market clearing risk free rate and market price of risk. This value can also be thought of as the squared coefficient of variation plus one in the weighted distribution of EIS. When this wedge is high, corresponding to two very different marginal investors and/or a diverse group of investors, asset prices are low, interest rates are high, and dividend yields are high. Conversely, when this wedge is low, corresponding to a concentration of consumption shares towards a single agent, asset prices are high, interest rates are low, and dividend yield is low. Naturally, when one agent dominates the variation in preferences is low. However, these statements are history contingent, for example a representative agent economy will always have a low wedge. Conditional on there being multiple agents, these statements hold for the heterogeneity wedge in relative terms.

Additionally, dividend yield in this model is falling over time and co-moves negatively with the growth rate in dividends. This implies a predictable component in stock market returns. A negative shock to this economy implies a shift of the distribution of consumption shares towards more risk averse agents. This reduces asset prices and predicts a faster growth rate in the dividend in the future. We know from these simulations that economies with a lower weighted average of EIS will have a higher rate of return on risky assets. Papers such as Campbell and Shiller (1988a), Campbell and Shiller (1988b), Mankiw (1981), and Hall (1979) drew differing conclusions about the standard model of asset prices, but, broadly speaking, they all deduced that there was some portion of asset prices that was slightly predictable as a function of the growth rate in aggregate consumption. In

[12]Defined as $\frac{\phi(t)}{\xi(t)^2} = \frac{\sum \frac{\omega^i(t)}{\gamma_i^2}}{\left[\sum \frac{\omega^i(t)}{\gamma_i}\right]^2}$ or $\frac{\int \frac{\omega(t,\gamma,x)}{\gamma^2} dF(\gamma,x)}{\left[\int \frac{\omega(t,\gamma,x)}{\gamma} dF(\gamma,x)\right]^2}$ in the continuous types case.

the model presented here, we can take a step towards explaining this predictability as the dividend yield co-moves with the average EIS. This is similar to a model of time varying preferences, but where individuals preferences remain constant and aggregate features vary over time.

Finally, I've shown how to extend the finite types model to one of a continuum of types. The results are reminiscent of theoretical work on Mean Field Games (MFG) with common noise, such as Carmona et al. (2014). However, this model takes a novel approach to solving such a MFG model by applying the Martingale Method, a typical tool in mathematical finance. The feature which makes this particular model so tractable is the dependence on the initial condition. This, in turn, is driven by market completeness. Agents seek to grow their consumption at some rate relative to the growth rate in the economy and do so by accumulating financial assets. They can accumulate assets by borrowing essentially without limit[13]. An interesting direction for future research would be to carry this approach over to incomplete markets, as in Chabakauri (2015), to study how borrowing constraints would affect the accumulation of assets and market dynamics.

[13]CRRA preferences rule out the chance of default and agents are always able to borrow.

Appendix A. Proofs

Proof of Proposition 1. Taking ratios of consumption first order conditions for two arbitrary agents, i and j we find

$$\frac{c^i(t)}{c^j(t)} = \Lambda_j^{\frac{1}{\gamma_j}} \Lambda_i^{\frac{-1}{\gamma_i}} \left(H_0(t)e^{\rho t}\right)^{\frac{1}{\gamma_j}-\frac{1}{\gamma_i}}$$

To solve for the consumption weight of an individual i, take the market clearing condition in consumption and divide through by agent i's consumption

$$\begin{aligned}
& \frac{1}{N}\sum_{j=1}^{N} c^j(t) = D(t) \\
\Leftrightarrow \quad & \frac{\frac{1}{N}\sum_{j=1}^{N} c^j(t)}{c^i(t)} = \frac{D(t)}{c^i(t)} \\
\Leftrightarrow \quad & c^i(t) = \frac{c^i(t)}{\frac{1}{N}\sum_{j=1}^{N} c^j(t)} D(t) \\
\Leftrightarrow \quad & c^i(t) = \left(\frac{N\left(e^{\rho t}\Lambda_i(t)H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(e^{\rho t}\Lambda_j(t)H_0(t)\right)^{\frac{-1}{\gamma_j}}}\right) D(t) \\
\Leftrightarrow \quad & c^i(t) = \omega^i(t) D(t)
\end{aligned}$$

□

Proof of Lemma 1. Modeling consumption as a geometric Brownian motion implies that for every agent i the consumption process can be described by the stochastic differential equation

$$\frac{dc^i(t)}{c^i(t)} = \mu_{c^i}(t)dt + \sigma_{c^i}(t)dW(t) \tag{A.1}$$

Armed with this knowledge, take the first order condition for an arbitrary agent i's maximization problem, solve for $H_0(s)$, and apply Itô's lemma:

$$\begin{aligned}
& H_0(t) = \frac{1}{\Lambda_i} c^i(t)^{-\gamma_i} e^{-\rho t} \\
\Rightarrow & \frac{dH_0(t)}{H_0(t)} = \left(-\rho - \gamma_i\mu_{c^i}(t) + \gamma_i(1+\gamma_i)\frac{\sigma_{c^i}(t)^2}{2}\right) dt - \left(\gamma_i\sigma_{c^i}(t)\right) dW(t)
\end{aligned}$$

Now, match coefficients to those in (2.3) to find

$$\begin{aligned}
r(t) &= \rho + \gamma_i\mu_{c^i}(t) - \gamma_i(1+\gamma_i)\frac{\sigma_{c^i}(t)^2}{2} \\
\theta(t) &= \gamma_i\sigma_{c^i}(t)
\end{aligned}$$

Solving for μ_{c^i} and σ_{c^i} gives

$$\begin{aligned}
\mu_{c^i}(t) &= \frac{r(t)-\rho}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2}\frac{\theta(t)^2}{2} \\
\sigma_{c^i}(t) &= \frac{\theta(t)}{\gamma_i}
\end{aligned}$$

□

Proof of Proposition 2. Recall the definition of consumption dynamics in (A.1) and the market clearing condition for consumption in (1.5). Apply Itô's lemma to the market clearing condition:

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} c^i(t) = D(t) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} dc^i(t) = dD(t) \\
\Leftrightarrow &\frac{1}{N}\sum_{i=1}^{N} \left(c^i(t)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right) = D(t)\mu_D dt + D(t)\sigma_D dW(t) \\
\Leftrightarrow &\frac{\frac{1}{N}\sum_{i=1}^{N}\left(c^i(s)\mu_{c^i}(t)dt + c^i(t)\sigma_{c^i}(t)dW(t)\right)}{D(t)} = \mu_D dt + \sigma_D dW(t) \\
\Leftrightarrow &\frac{1}{N}\sum_{i=1}^{N}\omega^i(t)\mu_{c^i}(t)dt + \frac{1}{N}\sum_{i=1}^{N}\omega^i(t)\sigma_{c^i}(t)dW(t) = \mu_D dt + \sigma_D dW(t)
\end{aligned}
$$

By matching coefficients we find

$$
\begin{aligned}
\mu_D &= \frac{1}{N}\sum_{i=1}^{N}\omega^i(t)\mu_{c^i}(t) \\
\sigma_D &= \frac{1}{N}\sum_{i=1}^{N}\omega^i(t)\sigma_{c^i}(t)
\end{aligned}
$$

Now use Lemma 1 to substitute the values for consumption drift and diffusion, then solve for the interest rate and the market price of risk to find

$$
\begin{aligned}
\theta(t) &= \frac{\sigma_D}{\xi(t)} \\
r(t) &= \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t)+\phi(t)}{\xi(t)^3}\sigma_D^2
\end{aligned}
$$

where

$$
\begin{aligned}
\xi(t) &= \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i} \\
\phi(t) &= \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i^2}
\end{aligned}
$$

□

Proof of Proposition 3. Assume that consumption weights follow a geometric Brownian motion given by

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t) \tag{A.2}$$

Recall the definition of consumption weights in (2.7) and gather terms:

$$\omega^i(t) = \frac{\left(\Lambda^i e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^N \left(\Lambda_j e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_j}}}$$

$$\Leftrightarrow \omega^i(t) = \left[\sum_{j=1}^N \Lambda_j^{\frac{-1}{\gamma_j}} \Lambda_i^{\frac{1}{\gamma_i}} \left(e^{\rho t} H_0(t)\right)^{\frac{1}{\gamma_i} - \frac{1}{\gamma_j}}\right]^{-1} \tag{A.3}$$

Recall the definition of Itô's lemma, where $\omega^i(t)$ is a function of $H_0(t)$ and t:

$$d\omega_i(t) = \frac{\partial \omega^i(t)}{\partial t} dt + \frac{\partial \omega^i(t)}{\partial H_0(t)} dH_0(t) + \frac{1}{2} \frac{\partial^2 \omega^i(t)}{\partial H_0(t)^2} (dH_0(t))^2$$

Substituting for $dH_0(t)$ by (2.3) and using the Itô box calculus to see that $(dH_0(t))^2 = H_0(t)^2 \theta(t)^2 dt$, we see that

$$\begin{aligned}\frac{d\omega^i(t)}{\omega^i(t)} &= \frac{1}{\omega^i(t)} \left(\frac{\partial \omega^i(t)}{\partial t} - r(t) H_0(t) \frac{\partial \omega^i(t)}{\partial H_0(t)} + H_0(t)^2 \theta(t)^2 \frac{1}{2} \frac{\partial^2 \omega^i(t)}{\partial H_0(t)^2} \right) dt \\ &\quad - \theta(t) \frac{1}{\omega^i(t)} \frac{\partial \omega^i(t)}{\partial H_0(t)} dW(t)\end{aligned}$$

Matching coefficients with those in (A.2) it is clear that

$$\begin{aligned}\mu_{\omega^i}(t) &= \frac{1}{\omega^i(t)} \left(\frac{\partial \omega^i(t)}{\partial t} - r(t) H_0(t) \frac{\partial \omega^i(t)}{\partial H_0(t)} + H_0(t)^2 \theta(t)^2 \frac{1}{2} \frac{\partial^2 \omega^i(t)}{\partial H_0(t)^2} \right) \\ \sigma_{\omega^i}(t) &= -\theta(t) \frac{1}{\omega^i(t)} \frac{\partial \omega^i(t)}{\partial H_0(t)}\end{aligned}$$

Differentiating the expression in (A.3), carrying out some painful algebra, and simplifying gives

$$\begin{aligned}\mu_{\omega^i}(t) &= (r(t) - \rho) \left(\frac{1}{\gamma_i} - \xi(t) \right) + \frac{\theta(t)^2}{2} \left[\left(\frac{1}{\gamma_i^2} - \phi(t) \right) - 2\xi(t) \left(\frac{1}{\gamma_i} - \xi(t) \right) + \left(\frac{1}{\gamma_i} - \xi(t) \right) \right] \\ \sigma_{\omega^i}(t) &= \theta(t) \left(\frac{1}{\gamma_i} - \xi(t) \right)\end{aligned}$$

□

Proof of Proposition 5. Following a trick in Gârleanu and Panageas (2015), we can arrive at an expression for asset prices. Take a straight forward application of Itô's lemma to the time t present value of time u wealth:

$$\begin{aligned}d(H_0(u) X^i(u)) &= X^i(u) dH_0(u) + H_0(u) dX^i(u) + dH_0(u) dX^i(u) \\ &= X^i(u) \left(-r(u) H_0(u) - \theta(u) H_0(u) dW(u) \right) \\ &\quad + H_0(u) \left[\left(r(u) X^i(u) + \pi^i(u) S(u) \left(\mu_S(u) + \frac{D(u)}{S(u)} - r(u) \right) - c^i(u) \right) du \right. \\ &\quad \left. + \pi^i(u) S(u) \sigma_S(u) dW(u) \right] - \theta(u) H_0(u) \pi^i(u) \sigma_S(u) S(u) du\end{aligned}$$

Now, notice that $c^i(s) = \omega^i(s) D(s)$ and that $\sigma_S(t)\theta(t) = \mu_S(t) + \frac{D(t)}{S(t)} - r(t)$ by (2.2). This implies that the above expression simplifies to

$$d(H_0(s) X^i(s)) = -H_0(u) \omega^i(u) D(u) du + H_0(u) [\pi^i(u) \sigma_S(u) S(u) - X^i(u) \theta(u)] dW(u)$$

By the definition of the Itô differential this is equivalent to

$$\begin{aligned}\lim_{u\to\infty} H_0(u)X^i(u) - H_0(t)X^i(t) = &-\int_t^\infty H_0(u)\omega^i(u)D(u)du \\ &+ \int_t^\infty H_0(u)[\omega^i(u)\sigma_S(u)S(u) - X^i(u)\theta(u)]dW(u)\end{aligned}$$

If we take expectations, then the first term on the left hand side is zero by a transversality condition on the present value of wealth. Also, notice that the Brownian integral on the right hand side is zero in expectation by the martingale property (Oksendal (1992)). So we can write

$$-H_0(t)X^i(t) = -\mathbb{E}_t \int_t^\infty H_0(u)\omega^i(u)D(u)du$$

Finally, we arrive at an expression for wealth today

$$X^i(t) = \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\omega^i(u)D(u)du$$

Now take the market clearing condition for wealth and substitute this new formula

$$\begin{aligned}S(t) &= \frac{1}{N}\sum_{i=1}^N X^i(t) \\ &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\omega^i(u)D(u)du \\ &= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}\left(\frac{1}{N}\sum_{i=1}^N \omega^i(u)\right)D(u)du \\ &= \mathbb{E}_t \int_t^\infty \frac{H_0(u)}{H_0(t)}D(u)du\end{aligned}$$

□

Appendix B. Numerical Simulation Method

Gathering all of the stochastic processes we have the following definitions to describe the evolution of the economy:

$$\begin{aligned}\frac{dD(t)}{D(t)} &= \mu_D dt + \sigma_D dW(t) \\ \frac{d\omega^i(s)}{\omega^i(s)} &= \mu_{\omega^i}(s)dt + \sigma_{\omega^i}(s)dW(t) \\ \frac{dH_0(t)}{H_0(t)} &= -r(t)dt - \theta(t)dW(t) \\ \theta(t) &= \frac{\sigma_D}{\xi(t)} \\ r(t) &= \frac{\mu_D}{\xi(t)} + \rho - \frac{1}{2}\frac{\xi(t)+\phi(t)}{\xi(t)^3}\sigma_D^2\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
\mu_{\omega^i}(t) &= (r(t)-\rho)\left(\frac{1}{\gamma_i}-\xi(t)\right)+\frac{\theta(t)^2}{2}\left[\left(\frac{1}{\gamma_i^2}-\phi(t)\right)-2\xi(t)\left(\frac{1}{\gamma_i}-\xi(t)\right)+\left(\frac{1}{\gamma_i}-\xi(t)\right)\right]\\
\sigma_{\omega^i}(t) &= \theta(t)\left(\frac{1}{\gamma_i}-\xi(t)\right)\\
\xi(t) &= \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i}\\
\phi(t) &= \frac{1}{N}\sum_{i=1}^{N}\frac{\omega^i(t)}{\gamma_i^2}
\end{aligned}$$

given a set of initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$. All of the above variables can be determined as a function of the realization of the risk process $W(t)$. If we combine those values with an estimation of asset prices, volatilities, portfolios, and the following formulas

$$\begin{aligned}
\theta(t) &= \frac{\mu_s+\frac{D(t)}{S(t)}-r(t)}{\sigma_s(t)}\\
\frac{dS(t)}{S(t)} &= \mu_s(t)dt+\sigma_s(t)dW(t)
\end{aligned}$$

we can back out the coefficients $\mu_S(t)$ and $\sigma_S(t)$ and study the dynamics of the economy, as well as characteristics of asset prices.

The numerical scheme follows the following steps:

(1) Specify a time discretization such that $t \in \{0, 1, ..., T\}$ and a time step Δt. Note that the specification of parameters and this time step will determine the discretization as being yearly, quarterly, monthly, etc.
(2) Specify a set of agents indexed by $i \in \{1, ..., N\}$ for some number N, each agent's risk aversion parameter γ_i, and each agents' initial wealth $X^i(0)$.
(3) Specify initial conditions $\{\omega^i(0)\}_{i=1}^N$ and $D(0)$.
(4) Simulate a process $\{dW(t)\}_{t=0}^T$ where $dW(t) \sim \mathcal{N}(0, \Delta t)$.
(5) Using (B.1) and the simulated Wiener process, for each period $t \in \{0, 1, ..., T\}$ calculate $\{D(t), \{\omega^i(t)\}_{i=1}^N, r(t), \theta(t), \xi(t), \phi(t)\}$.
(6) Using the monte-carlo approach described in Appendix B.1, for each period $t \in \{0, 1, ..., T\}$ calculate $\hat{S}(t)$, $\hat{\pi}^i$, $\hat{\sigma_S}(t)$, and $\hat{\mu_S}(t)$.
(7) Given the processes for $\hat{S}$ and $\hat{\pi}$, calculate wealth $X^i(t)$ and bond holdings $b^i(t)$ for each period using the definitions $X^i(0) = \pi^i(0)S(0) + b^i(0)$ and (1.4).
(8) Calculate any measures you might find enlightening!

B.1. **Monte Carlo Method.** The expressions we wish to estimate (Equations (2.16) to (2.18)) are of the form

$$Y(t) = \mathbb{E}_t \int_t^{\infty} f\left((Z(u)\right) du$$

for some function f and some multidimensional stochastic process $Z(t)$.

In order for the integral to be defined, it must be that the integrand converges towards zero as $u \to \infty$. If this is the case, then we could estimate the integral by truncating the upper bound at some level, $t + T$. In this way we would look to approximate the true value in the economy by another:

$$Y(t) \approx Y^*(t) = \mathbb{E}_t \int_t^{t+T} f(Z(t))du$$

This expression can easily be estimated by monte-carlo. Given that one must discretize the stochastic process in numerical simulations, replacing $Z(t)$ by its Euler-Murayama approximation $\hat{Z}(t)$ implies that a trapezoid rule exactly approximates the time integral. Define the discretization by partitioning the interval $(t, t + T)$ into H evenly spaced intervals such that Δ_t is the distance between points in the partition. Sample M paths for the process $W(t)$ and simulate the economy along these paths to extract processes in $\hat{Z}(t)$. Indicating draws by a super-script m the estimator is given by:

$$\begin{aligned} \hat{Y^*}(t) = \frac{1}{M} \sum_{m=1}^{M} \Bigg[\frac{1}{2} \left(f\left(\hat{Z}^m(t)\right) + f\left(\hat{Z}^m(t+T)\right)\right) \\ + \sum_{i=1}^{T/\Delta_t - 1} f\left(\hat{Z}^m(t+\Delta_t i)\right)\Bigg] \end{aligned} \tag{B.2}$$

Given the computational simplicity of this expression, it can be calculated quite easily. However, the dimension of the random variables involved is very large. For example, if $Delta_t = 0.5$ and $T = 100$, the driving noise is 600 dimensional. Add to that the fact that the state variable has the same number of dimensions, but multiplied by the number of agents or nodes. This implies that for 10 agents the random variables involved are 6000 dimensional. This means that variance will be high for even seemingly large numbers of sample paths. In the set of simulations currently included in the paper, 6.24×10^5 simulation paths are used, but the estimators still exhibit noisy behavior, in particular for estimating portfolios and volatility, which are ratio estimators.

To calculate these values, the present set of simulations was run on a cluster of 6 AWS c4.8xlarge compute optimized nodes, each with 36 vCPU's. The simulations ran in roughly 18 hours.

To estimate, the steps are as follows for a given distribution of $\{\gamma_i\}_{i=1}^N$ and an initial condition for the distribution of wealth $\{\omega^i(t)\}_{i=1}^N$:

(1) Simulate M sample paths for $dW(t)$ of length T, where M is an integers, using the knowledge that $dW(t) \sim \mathcal{N}(0, t)$.
(2) Using the M sample paths, simulate the evolution of M different economies populated by the same agents under the same initial condition. Extract the values for $\left(H_0(t), D(t), \{\omega^i(t)\}_{i=1}^N, \theta(t)\right)$.
(3) Approximate all of the expectations in Equations (2.16) to (2.18) using Equation (B.2).
(4) Repeat the above process for every period.

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