Advanced Statistical Mechanics

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Week 1: 1D Ising model and the transfer matrix

1　1D Ising Model

1.1　Partition Function

The 1D Ising model is model of a 1D magnet. We start by having N spins pointing up or down. Its partition function is

$$Z = \sum_{\sigma_k = \pm 1} e^{-\frac{E}{T}} \tag{1.1}$$

where

$$E = -J \sum_{k=0}^{N-1} \sigma_k \sigma_{k+1} - J \sigma_N \sigma_1 \tag{1.2}$$

We denote

$$K = \frac{J}{T} \tag{1.3}$$

from now on. N is the length of the chain. Then

$$Z = \sum_{\sigma_1 = \pm 1, \sigma_2 = \pm 1 \dots, \sigma_N = \pm 1} \exp \left(K \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1} + K \sigma_N \sigma_1 \right). \tag{1.4}$$

To compute its partition function, we introduce the transfer matrix

$$T_{\sigma_1, \sigma_2} = e^{K \sigma_1 \sigma_2} = \begin{pmatrix} T_{1,1} & T_{1,-1} \\ T_{-1,1} & T_{-1,-1} \end{pmatrix} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \tag{1.5}$$

With its help, the partition function can be rewritten as

$$Z = \text{Tr } T^N. \tag{1.6}$$

The matrix T has to eigenvalues,

$$\lambda_1 = 2 \cosh K, \ \ \lambda_2 = 2 \sinh K. \tag{1.7}$$

Then the partition function is given by

$$Z = \text{Tr } T^N = \lambda_1^N + \lambda_2^N = (2 \cosh K)^N + (2 \sinh K)^N. \tag{1.8}$$

Notice that if $N \gg 1$, then the smaller eigenvalue does not have to be kept here:

$$Z = (2 \cosh K)^N \left(1 + (\tanh K)^N \right) \approx (2 \cosh K)^N. \tag{1.9}$$

This is because $\tanh K < 1$. More generally,

$$Z \approx \lambda_1^N, \tag{1.10}$$

where λ_1 is the largest eigenvalue of the transfer matrix. This approximate equality becomes more and more precise as N gets larger.

1.2   Correlation Function

It is also instructive to compute the correlation function, which tells whether spins in different positions know about each other. It is given by

$$\langle \sigma_n \sigma_m \rangle = \frac{1}{Z} \sum_{\sigma_l = \pm 1} e^{K \sum_{k=1}^{N-1} \sigma_k \sigma_{k+1} + K \sigma_N \sigma_1} \sigma_n \sigma_m. \tag{1.11}$$

Let us assume, without loss of generality, that $n < m$. If this is not so, we can relabel what is n and what is m. We can convince ourselves that with the help of the transverse matrix, we can rewrite the correlation function as

$$\langle \sigma_n \sigma_m \rangle = \frac{1}{Z} \text{Tr} \left[T^{n-1} \tau_3 T^{m-n} \tau_3 T^{N-m+1} \right]. \tag{1.12}$$

Here τ_3 is the third Pauli matrix, given by

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.13}$$

To compute this trace, we employ notations and techniques borrowed from Quantum Mechanics. Introduce notations for the two eigenvectors of the transfer matrix T, $|1\rangle$ and $|2\rangle$, such that

$$T |1\rangle = \lambda_1 |1\rangle , \ T |2\rangle = \lambda_2 |2\rangle . \tag{1.14}$$

Then we write

$$\text{Tr} \left[T^{n-1} \tau_3 T^{m-n} \tau_3 T^{N-m+1} \right] = \sum_{a=1,2} \langle a| T^{n-1} \tau_3 T^{m-n} \tau_3 T^{N-m+1} |a\rangle . \tag{1.15}$$

The next step parallels what is called in quantum mechanics and quantum field theory as "Lehmann decomposition". We insert the "complete set of states"

$$\sum_{a=1,2; b=1,2; c=1,2; \ d=1,2; f=1,2} \langle a| T^{n-1} |b\rangle \langle b| \tau_3 |c\rangle \langle c| T^{m-n} |d\rangle \langle d| \tau_3 |f\rangle \langle f| T^{N-m+1} |a\rangle . \tag{1.16}$$

Now $|a\rangle$ are the eigenstates for T, so we should immediately be able to see that $a = b$, $c = d$, $f = a$. Moreover, it is clear that

$$\langle a| T^n |a\rangle = \lambda_a^n \tag{1.17}$$

for any n and a. So we find

$$\sum_{a,b} \lambda_a^{n-1} \langle a| \tau_3 |b\rangle \lambda_b^{m-n} \langle b| \tau_3 |a\rangle \lambda_a^{N-m+1} = \sum_{a,b} \lambda_a^{N+n-m} \lambda_b^{m-n} \langle a| \tau_3 |b\rangle \langle b| \tau_3 |a\rangle . \tag{1.18}$$

This in principle is good enough, this is the mathematical answer to the quantity we are trying to calculate. However, for large chain length N, which is the limit we are interested in, we can further simplify this expression. We know that $\lambda_1 > \lambda_2$. So since N is large, a should be chosen to be 1 only, as $a = 2$ results in a much smaller contribution to the sum $(\lambda_1^N \gg \lambda_2^N)$. Then we find

$$\sum_{b=1,2} \lambda_1^{N+n-m} \lambda_b^{m-n} \langle 1| \tau_3 |b\rangle \langle b| \tau_3 |1\rangle = \lambda_1^N \left(|\langle 1| \tau_3 |1\rangle|^2 + \frac{\lambda_2^{m-n}}{\lambda_1^{m-n}} |\langle 1| \tau_3 |2\rangle|^2 \right). \tag{1.19}$$

Finally, we can now write down the answer for $N \gg 1$, by observing that to get the correlation function we need to divide this by $Z = \lambda_1^N$

$$\langle \sigma_n \sigma_m \rangle = |\langle 1| \tau_3 |1\rangle|^2 + \frac{\lambda_2^{m-n}}{\lambda_1^{m-n}} |\langle 1| \tau_3 |2\rangle|^2. \tag{1.20}$$

This is the answer for the correlation function. We see it consists of two parts. One is completely independent of n and m. So as far as the separation between the spins in the correlation function is concerned, it is a separation independent constant. The other depends on $m - n$. Let us elucidate this dependence further:

$$\frac{\lambda_2^{m-n}}{\lambda_1^{m-n}} |\langle 1| \tau_3 |2\rangle|^2 = |\langle 1| \tau_3 |2\rangle|^2 \, e^{-(m-n) \ln[\lambda_1/\lambda_2]}. \tag{1.21}$$

We know that $\lambda_1 > \lambda_2$, so the logarithm of their ratio is positive.

It is customary to introduce the correlation length ξ, such that

$$\xi = \frac{1}{\ln [\lambda_1/\lambda_2]}. \tag{1.22}$$

Then we have

$$\langle \sigma_n \sigma_m \rangle = |\langle 1| \tau_3 |1\rangle|^2 + |\langle 1| \tau_3 |2\rangle|^2 \, e^{-\frac{|m-n|}{\xi}}. \tag{1.23}$$

(we agreed before that $m > n$, but for generality we put the absolute value sign in the exponential). We see that the correlation function between the two spins consists of two parts: a distance independent constant, and the second term which decays with the distance as soon as $m - n \gg \xi$. This explains why ξ is called the correlation length. ξ controls the distance at which spins stop noticing each other.

Finally, we observe that the distance independent constant is nothing but the product of the two correlations

$$\langle \sigma_n \rangle = \langle 1| \tau_3 |1\rangle = \langle \sigma_m \rangle. \tag{1.24}$$

We can see this by repeating the arguments which led to (1.20), but this time for this simpler correlation function. Then we can say

$$\langle \sigma_n \sigma_m \rangle - \langle \sigma_n \rangle \langle \sigma_m \rangle = |\langle 1| \tau_3 |2\rangle|^2 \, e^{-\frac{|m-n|}{\xi}}. \tag{1.25}$$

The difference on the left hand side is called the connected correlation function, it always goes to zero at large separation. This elucidates the meaning of ξ further.

Finally, we note that all this is not particularly necessary because in the 1D Ising model

$$\langle \sigma_n \rangle = \frac{1}{Z} \sum_{\sigma} \sigma_n e^{K \sum_{k=1}^{N} \sigma_k \sigma_{k+1} + K \sigma_N \sigma_1} = 0. \tag{1.26}$$

This is because for each term in the sum with some set of σ, there is another term where all σ take opposite values. These tersm have identical energies, and the sum vanishes.

Alternatively, one can convince oneself that (by figuring out the eigenvectors of the transfer matrix T)

$$|1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \langle 2| = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.27}$$

Then

$$\langle 1| \tau_3 |1\rangle = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0. \tag{1.28}$$

So in our particular example,

$$\langle \sigma_n \sigma_m \rangle - \langle \sigma_n \rangle \langle \sigma_m \rangle = \langle \sigma_n \sigma_m \rangle = |\langle 1| \tau_3 |2\rangle|^2 \, e^{-\frac{|m-n|}{\xi}}. \tag{1.29}$$

So we learned that the spins are correlated exponentially, with the correlation length

$$\xi = \frac{1}{\ln[\lambda_1/\lambda_2]} = \frac{1}{\ln[1/\tanh K]} = \frac{1}{\ln[1/\tanh(J/T)]}. \tag{1.30}$$

This correlation length is always finite at any temperature. At very small temperature, it becomes larger and eventually, as $T \to 0$, it diverges to infinity. It is sraightforward to check that as $T \ll J$,

$$\xi \approx \frac{1}{2} e^{\frac{2J}{T}}. \tag{1.31}$$

So we learned that the partition function is given by the largest eigenvalue of the transfer matrix, and that the correlation length is the inverse logarithm of the ratio of the highest eigenvalue to the next highest eigenvalue of the transfer matrix.

1.3 Connection with quantum mechanics

In quantum mechanics we have states $|\psi\rangle$ satisfying the Schödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{1.32}$$

Then

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle. \tag{1.33}$$

We can see a connection between the evolution operator for some time t, e^{-iHt} and the transfer matrix T. To make the equivalence more precise, we assume that t is imaginary and introduce the imaginary time

$$\tau = it. \tag{1.34}$$

The imaginary time is actually real, which is sometimes confusing (confusing name). Then

$$\frac{\partial}{\partial \tau}|\psi\rangle = -H|\psi\rangle. \tag{1.35}$$

The tranfer matrix can be identified with the evolution operator

$$T = e^{-H\bar{\tau}} \tag{1.36}$$

for some arbitrarily chosen $\bar{\tau}$, representing a "unit" of time.

For example, for our transfer matrix (1.5) we find after some clever manipulation of the Pauli matrices

$$T = Ce^{\gamma\tau_1}, \tag{1.37}$$

where τ_1 is the first Pauli matrix

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.38}$$

and

$$\sinh\gamma = \frac{\exp(-K)}{\sqrt{2\sinh(2K)}}, \quad C = \sqrt{2\sinh(2K)}. \tag{1.39}$$

It follows from here that

$$H\bar{\tau} = -\gamma\tau_1 - \ln C. \tag{1.40}$$

The term $\ln C$ is thought of as multiplied by a unit matrix; H itself is a two by two matrix and $\bar{\tau}$ is a number. Also keep in mind that $\bar{\tau}$ is some arbitrary interval of imaginary time, while τ_1 is a completely different thing, a matrix. We find a remarkable correspondence: the 1D Ising model is equivalent to a quantum mechanics of 1 spin-1/2 particle which

moves in a magnetic field leading to the Pauli Hamiltonian (1.40), basically representing magnetic field pointing in the x-direction, represented by τ_1.

This is part of a general and mysterious statistical physics - quantum mechanics correspondence.

The fact that only the highest eigenvalue of T plays a role corresponds to the well known fact that only the ground state of a quantum system is important in its evolution for a long time. We find

$$Z = e^{-N\bar{\tau}E_0}, \tag{1.41}$$

where E_0 is the ground state energy (lowest eigenstate of the Hamiltonian H),

$$\lambda_1 = e^{-\bar{\tau}E_0}. \tag{1.42}$$

Correspondingly, the correlation length is inversely proportional to the excitation gap in the spectrum

$$\xi = \frac{1}{\bar{\tau}(E_1 - E_0)}. \tag{1.43}$$

That's why often people say "system with a large gap", having in mind short correlation length.

Or "gapless system", having in mind a system with $\xi = \infty$.