Transfer matrix solution to the 1D Ising model

The most popular approach to solving the 2D Ising model is via the so called transfer matrix method. We can get some idea of how this method works by using it to solve the 1D model. In particular we can use this technique to solve the 1D Ising model in the presence of an external magnetic field, something not possible with our previous direct summation approach. This solution will allow us to rigorously demonstrate that the 1D model does not exhibit a phase transition (i.e., no spontaneous magnetization).

Consider an N-site 1D Ising model with nearest neighbor ferromagnetic coupling J and periodic boundary conditions (i.e., $i+N=i$) in an external magnetic field B. The Hamiltonian for this model can be written as follows:

$$H = -J\sum_{i=1}^{N} s_i s_{i+1} - B\sum_{i=1}^{N} s_i = -\sum_{i=1}^{N}\left\{Js_i s_{i+1} + B(s_i + s_{i+1})/2\right\}.$$

The partition function is:

$$\begin{aligned}
Z(N,T) &= \sum_{s_1}\cdots\sum_{s_N} e^{-\beta H} \\
&= \sum_{s_1}\cdots\sum_{s_N} \exp\left(\beta\sum_{i=1}^{N}\left\{Js_i s_{i+1} + B(s_i + s_{i+1})/2\right\}\right) \\
&= \sum_{s_1}\cdots\sum_{s_N} \prod_{i=1}^{N} \exp\left(\beta\left\{Js_i s_{i+1} + B(s_i + s_{i+1})/2\right\}\right) \\
&= \sum_{s_1}\cdots\sum_{s_N} \prod_{i=1}^{N} P_{s_i s_{i+1}}
\end{aligned}$$

where the four possible Boltzmann factors P_{mn} are simply given by:

$$P_{11} = e^{\beta(J+B)}, \quad P_{1-1} = P_{-11} = e^{-\beta J}, \text{ and } P_{-1-1} = e^{\beta(J-B)}.$$

Treating the P_{mn} as matrix elements, we define the "transfer matrix" $\mathbf{P} = \begin{bmatrix} P_{11} & P_{1-1} \\ P_{-11} & P_{-1-1} \end{bmatrix}$.

Before going on, convince yourself that the partition function really can be written in terms of these "matrix elements" as shown above.

Now recall from linear algebra that the square of a matrix $\mathbf{P}$ can be defined in terms of matrix elements as follows: $(\mathbf{P}^2)_{ij} = \sum_k P_{ik} P_{kj}$. Similarly, the cube of $\mathbf{P}$ is defined by matrix elements: $(\mathbf{P}^3)_{ij} = \sum_k \sum_l P_{ik} P_{kl} P_{lj}$. With this in mind, we rewrite the above partition function as

$$Z(N,T) = \sum_{s_1}\cdots\sum_{s_N} P_{s_1 s_2} P_{s_2 s_3} P_{s_3 s_4} \cdots P_{s_{N-1} s_N} P_{s_N s_1} = \sum_{s_1} (\mathbf{P}^N)_{s_1 s_1} = Tr(\mathbf{P}^N)$$

where $Tr(\,)$ denotes the trace. Notice that $\mathbf{P}^N$ is just a 2×2 matrix. Okay, make sure you agree with the above result since it's the key to our solution.

From here we will let two theorems of linear algebra do all the work for us. The first theorem is that the trace of a matrix is independent of the representation of the matrix and is equal to the sum of the matrix eigenvalues (which, of course, give the diagonal representation of the matrix). The second theorem we need, which may not be quite as familiar, is that if λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{P}$, then $\lambda_1{}^N$ and $\lambda_2{}^N$ are the eigenvalues of the matrix $\mathbf{P}^N$. The implications of these two theorems are that all we need to do is find the eigenvalues of the transfer matrix $\mathbf{P}$ and we're done(!) since the partition function is now simply:

$$Z(N,T) = \lambda_1^N + \lambda_2^N \ .$$

The eigenvalues are given, as usual, by the vanishing of the secular determinant:

$$|\mathbf{P} - \lambda \mathbf{I}| = 0.$$

Leaving the algebra as an exercise for the reader we arrive at the result:

$$\boxed{\lambda_\pm = e^{\beta J}\left\{\cosh(\beta B) \pm \left(\sinh^2(\beta B) + e^{-4\beta J}\right)^{1/2}\right\}}$$

We can achieve further simplification by considering the free energy, which is given by

$$\begin{aligned}\beta F &= -\ln Z(N,T) \\ &= -\ln(\lambda_+^N + \lambda_-^N) \\ &= -N\ln(\lambda_+) - \ln\left[1 + (\lambda_- / \lambda_+)^N\right].\end{aligned}$$

Since $\lambda_- < \lambda_+$, the second term in the above expression becomes negligible with respect to the first for large N. Thus we are completely justified in writing the partition function simply in terms of the larger of the two eigenvalues as:

$$\boxed{Z(N,T) = \lambda_+^N = e^{N\beta J}\left\{\cosh(\beta B) + \left(\sinh^2(\beta B) + e^{-4\beta J}\right)^{1/2}\right\}^N}$$

You should verify for yourself that in the zero-field case (B=0) this expression reduces to our previously derived result. Well, that's about it. As I said above, this result can be used to prove that the 1D Ising model does NOT undergo a phase transition. To demonstrate this fact we must compute the magnetization given by $m = (1/N) \sum_i \langle s_i \rangle$. Referring back to our first expressions for the Hamiltonian and partition function, notice that m can cleverly be written as a derivative of $\ln Z(N,T)$ with respect to βB:

$$m = \frac{1}{N}\frac{\partial \ln Z(N,T)}{\partial(\beta B)} \ .$$

In order to demonstrate the absence of a phase transition we must show that $m \neq 0$ for B=0, T>0 is not possible ... but I think I'll leave this as another exercise for the reader (or perhaps as a final exam question).