1 The Ising model

> This model was suggested to Ising by his thesis adviser, Lenz. Ising solved the one-dimensional model, ..., and on the basis of the fact that the one-dimensional model had no phase transition, he asserted that there was no phase transition in any dimension. As we shall see, this is false. It is ironic that on the basis of an elementary calculation and erroneous conclusion, Ising's name has become among the most commonly mentioned in the theoretical physics literature. But history has had its revenge. Ising's name, which is correctly pronounced "E-zing," is almost universally mispronounced "I-zing."
>
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1.1 Definitions

The Ising model is easy to define, but its behavior is wonderfully rich. To begin with we need a lattice. For example we could take Z^d, the set of points in R^d all of whose coordinates are integers. In two dimensions this is usually called the square lattice, in three the cubic lattice and in one dimension it is often refered to as a chain. (There are lots of other interesting lattices. For example, in two dimensions one can consider the triangular lattice or the hexagonal lattice.) I will use letters like i to denote a site in the lattice. For each site i we have a variable σ_i which only takes on the values $+1$ and -1. Ultimately we want to work with the full infinite lattice. One approach to doing this is to first work with finite lattices and then try to take an "infinite volume limit" in which the finite lattice approaches the full infinite lattice in some sense. (A different approach will be described latter.) In the case of the lattice Z^d we could start with the subset

$$\Lambda_L = \{(i_1, i_2, \cdots, i_d) : |i_j| \leq L, j = 1, 2, \cdots, d\}$$

and then let $L \to \infty$.

The subset Λ_L is finite, so the number of possible spin configurations $\{\sigma_i\}_{i \in \Lambda_L}$ is finite. We will use σ as shorthand for one of these spins configurations, i.e., an assignment of $+1$ or -1 to each site. We are going to define a probability measure μ_L on this set of configurations. It will depend on an "energy function" or "Hamiltonian" H. The simplest choice for H is

$$H(\sigma) = - \sum_{i,j \in \Lambda_L : |i-j|=1} \sigma_i \sigma_j$$

The sum is over all pairs of sites in our finite volume which are nearest neighbors in the sense that the Euclidean distance between them is 1. The probability measure is given by

$$\mu_L(\sigma) = \exp(-\beta H(\sigma))/Z$$

where Z is a constant chosen to make this a probability measure. Explicitly,

$$Z = \sum_\sigma \exp(-\beta H(\sigma))$$

where the sum is over all the spin configurations on our finite volume. In our definition of H we have only included the nearest neighbor pairs such that both sites are in the finite volume. This is usually called "free" boundary conditions.

The possibility of imposing other boundary conditions will be important in the discussion of phase transitions. We could add to the Hamiltonian nearest neighbor pairs of sites with one site in Λ_L and the other site outside of Λ. We then specify some fixed choice for each spin just outside of Λ. For example we might take all these outside spins to be $+1$. We will refer to this as $+$ boundary conditions. $-$ boundary conditions are defined in the obvious way. Boundary conditions that use a mixture of $+$ and $-$ for the fixed boundary spins are of interest, e.g., in the study of domain walls or interfaces. We will encounter them later.

Having defined a probability measure we can compute expectations with respect to it. Of course, since the probability space is finite these expectations are nothing more than finite sums. Physicists denote this expectation or sum by $< \ >$. For example,

$$< \sigma_0 >= \frac{1}{Z} \sum_\sigma \sigma_0 \exp(-\beta H(\sigma))$$

gives the average value of the spin at the origin. When we need to make explicit which boundary conditions we are using we will write $< \ >^+$, $< \ >^-$ or $< \ >^{free}$. If we use free boundary conditions then the model has what is called a "global spin flip" symmetry. This simply means that if we replace every spins σ_i by $-\sigma_i$, then $H(\sigma)$ does not change. Since σ_0 changes sign under this transformation, we see that $< \sigma_0 >_{free} = 0$. This argument does not apply in the case of $+$ or $-$ boundary conditions. A very important question is how much $< \sigma_0 >$ depends on the choice of the boundary conditions. We will address this question in the next section.

There is nothing special about the spin at the origin. We can compute the expectation of any function that only depends on the spins in the volume Λ. Two that are of particular interest in physics are the magnetization

$$M = < \sum_{i \in \Lambda} \sigma_i >$$

and the energy

$$E = < H >$$

Both of these quantities will grow like $|\Lambda|$ as the volume goes to infinity. To obtain something which has a chance of having an infinite volume limit we need to divide them by the number of sites in the volume. The result of doing this for the energy is usually called the energy per site. For the total magnetization the result is often called simply the magnetization.

As we said at the start we eventually want to look at the infinite lattice. So we would like to take the limit as $L \to \infty$ of quantities like the magnetization, expectations like $< \sigma_0 >_L$ or more generally expectations $< f(\sigma) >_L$ where $f(\sigma)$ is any function that only depends on finitely many spins. This infinite volume limit is often called the "thermodynamic" limit. The question of its existence is a monumental one, but we will not have anything to say about it. However, we do want to introduce a slightly more mathematical point of view of the infinite volume limit which we will pursue further in section 1.4. The reason for the restriction to finite volumes was to have only a finite number of terms in the Hamiltonian. The infinite volume Hamiltonian is only a formal creature, although it is often written down with the understanding that some sort of limiting process is needed. However, it does make sense to talk about probability measures on the infinite lattice. The space of configurations on the lattice Z^d is $\{-1, +1\}^{Z^d}$ and this space has a natural topology - the product topology that comes from using the discrete topology on $\{-1, +1\}$. Hence the Borel sets provide a σ-algebra and we can talk about Borel probability measures on the space of spin configurations. What one would like to show is that μ_L^+ converges to such a measure μ^+ in the sense that

$$\lim_{L\to\infty} < f(\sigma) >^+_L = \int f(\sigma) d\mu^+$$

for functions $f(\sigma)$ which only depend on finitely many spins. Integrals with respect to such an infinite volume limit are often denoted $< \ >$, and we will follow this convention. In this case it is natural to denote it by $< \ >^+$ to indicate the possible dependence on the $+$ boundary conditions used to define the finite volume measure.

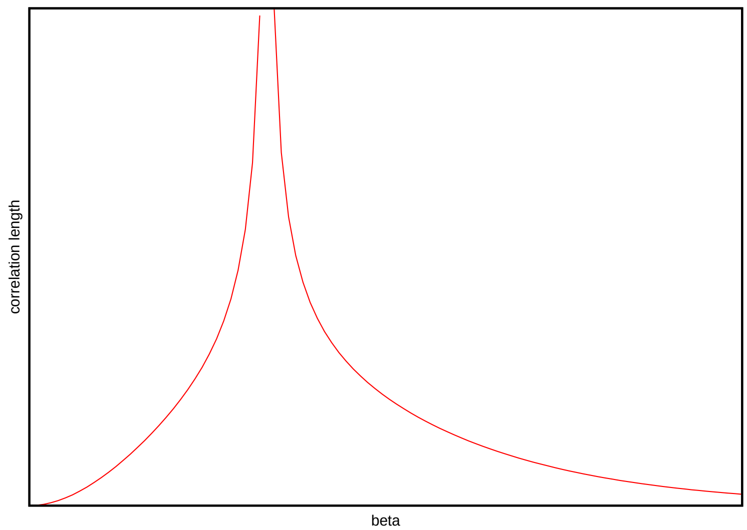

Figure 1: Qualitative behavior of the correlation length as a function of inverse temperature

Another quantity of physical interest which will play a major role in the renormalization group is the correlation length. A natural question is how much the spins σ_i and σ_j at two sites are correlated, especially if the spins are far apart. The simplest measurement of this is

$$< \sigma_i \sigma_j > - < \sigma_i >< \sigma_j >$$

If the spins are independent then this quantity would be zero. This quantity is called a "truncated" correlation function and denoted by $< \sigma_i ; \sigma_j >$. What typically happens is that this quantity decays exponentially as $|i-j| \to \infty$ for all values of β but one. So

$$< \sigma_i ; \sigma_j > \sim \exp(-|i-j|/\xi)$$

as $|i-j| \to \infty$ where ξ is a length scale which is called the "correlation length". The behavior of this correlation length as a function of β is typically that shown in figure ??. The correlation length is to some extent a measure of how interesting the system is. A short correlation length means that distant spins are very weakly correlated.

We end this section by defining a few other quantities of physical interest. Before we do this we first introduce a slightly more general Hamiltonian. If there is an external magnetic field on the system then the energy is given by

$$H = -\sum_{<i,j>} \sigma_i \sigma_j - h \sum_i \sigma_i$$

where h, the magnetic field is an external parameter. Of course for this to make sense we must restrict to a finite volume and specify how we treat the boundary. The "free energy" F is defined by

$$\exp(-\beta F) = Z = \sum_\sigma \exp(-\beta H)$$

F will be a function of the two parameters β and h and the choice of the finite volume. It will usually grow with the number of sites in the volume. The free energy per site, f, is simply F divided by the number of sites. It ought to have an infinite volume limit. If we differentiate the free energy per site with respect to the magnetic field we get minus the magnetization,

$$m = -\frac{\partial f}{\partial h}$$

When we defined the magnetization before there was no magnetic field term in H. To obtain this magnetization we should set $h = 0$ after taking the derivative. However, we can talk about the magnetization of the system with an external field present, $h \neq 0$. The magnetization is then a function of β and h. The derivative of the magnetization with respect to h is called the magnetic susceptibility and denoted by χ,

$$\chi = \frac{\partial m}{\partial h} = -\frac{\partial^2 f}{\partial h^2}$$

By taking the derivative of βf with respect to β one obtains the energy per site. The second derivative of f with respect to β is known as the specific heat and denoted

$$C = \frac{\partial^2 f}{\partial \beta^2}$$

In the above definitions we would like to work with the infinite volume limit of the free energy per site f. This limit is known to exist under very general conditions. However, there is no reason that all the derivatives we have been happily writing down need exist. At some values of the parameters they will not, and this lack of analyticity is another way of detecting phase trasitions.

Exercises:

1.1.1 (easy) By explicity computing $\frac{\partial^2 f}{\partial h^2}$, show that the susceptibility can be written as a sum of truncated correlation functions:

$$\chi = \beta \sum_{i,j\in\Lambda} < \sigma_i ; \sigma_j >$$

1.1.2 (long, but important) The partition function for the Ising chain $-L, -L+1, \cdots, L-1, L$ with various boundary conditions is

$$\begin{aligned} Z^{free} &= \sum_{\sigma_{-L},\cdots,\sigma_L} \exp(\beta \sum_{i=-L}^{L-1} \sigma_i \sigma_{i+1}) \\ Z^{+} &= \sum_{\sigma_{-L},\cdots,\sigma_L} \exp(\beta \sum_{i=-L}^{L-1} \sigma_i \sigma_{i+1} + \beta\sigma_{-L} + \beta\sigma_L) \\ Z^{-} &= \sum_{\sigma_{-L},\cdots,\sigma_L} \exp(\beta \sum_{i=-L}^{L-1} \sigma_i \sigma_{i+1} - \beta\sigma_{-L} - \beta\sigma_L) \end{aligned}$$

Z^{free} is easy to compute. Note that

$$\sum_{\sigma_L} \exp(\beta\sigma_{L-1}\sigma_L) = 2\cosh(\beta)$$

regardless of what the value of σ_{L-1} is. Use this to show that $Z^{free} = [2\cosh(\beta)]^{2L+1}$.

To compute correlation functions it is useful to introduce something called the transfer matrix. Let T be the two by two matrix

$$T = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

Normally one would write the entries of such a matrix as T_{ij} with i, j running from 1 to 2. Here we will denote the entries by $T(\sigma, \sigma')$ where σ and σ' take on the values $+1$ and -1 in that order. So $T(+1,+1)$ is the upper left entry of the matrix. The important thing to note is that we now have $T(\sigma_i, \sigma_{i+1}) = \exp(\beta \sigma_i \sigma_{i+1})$. Define a vector

$$\phi_+ = \begin{pmatrix} e^{\beta} \\ e^{-\beta} \end{pmatrix}$$

Show that

$$Z^{\pm} = (\phi_{\pm}, T^{2L} \phi_{\pm})$$

Compute the eigenvalues and eigenvectors of T. Use this to show

$$Z^+ = Z^- = [2\cosh(\beta)]^{2L+2} + [2\sinh(\beta)]^{2L+2}$$

Show that the free energy per site has an infinite volume limit and it is the same for all three choices of boundary conditions.

Let D be the matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that

$$< \sigma_0 >^+ = \frac{(\phi_+, T^L D T^L \phi_+)}{(\phi_+, T^{2L} \phi_+)}$$

Use this and your diagonalization of T to show that $< \sigma_0 >^+ \to 0$ as $L \to \infty$. Show that

$$< \sigma_0 \sigma_l >^+ = \frac{(\phi_+, T^L D T^l D T^{L-l} \phi_+)}{(\phi_+, T^{2L} \phi_+)}$$

Use this to show that the infinite length limit of this correlation function exists and is given by

$$\lim_{L \to \infty} < \sigma_0 \sigma_l >^+ = (\tanh(\beta))^l$$

and so the correlation length is given by

$$\xi = \frac{-1}{\log(\tanh(\beta))}$$

Sketch a graph of this correlation length as a function of β.

1.2 Phase transitions - the role of the boundary conditions

There are a variety of ways to look at phase transitions. We will start by asking whether or not the boundary conditions make a difference in the infinite volume limit. Let's start with a very specific instance of this question. In a finite volume we might guess that + boundary conditions would make the spin at the origin a little more likely to be $+1$ than -1. This is indeed the case; in fact we will prove that $< \sigma_0 >_L^+ > 0$ in a moment. The important question is what happens when we take the infinite volume limit, $L \to \infty$. Does $< \sigma_0 >_L^+$ converge to zero or to some nonzero value? (There is of course the third possibility that it does not coverge at all.) The answer depends on the parameter β.

Theorem 1.2.1: If the number of dimensions is at least two, then there is a positive number β_c such that for $\beta < \beta_c$ the limit $\lim_{L\to\infty} < \sigma_0 >_L^+$ exists and is zero while for $\beta > \beta_c$ the limit exists and is strictly greater than zero.

We will only prove a weaker (but still very interesting) version of this theorem. What we will show is that there are two positive constants β_1 and β_2 such that if $\beta < \beta_1$ then the limit is zero, while if $\beta > \beta_2$ the limit is not zero. β_1 will be much smaller than β_2, so our proofs won't say anything about what happens in the rather large interval $[\beta_1, \beta_2]$.

We start with the case of small β. This is the same as the temperature being large, and this regime is usually called the "high temperature" regime. Of course if $\beta = 0$ then the spins are independent and the model is completely trivial. The intuition is that if β is small, then the spins should only be weakly correlated. The influence of the boundary condition will decay rapidly as we move in from the boundary.

When β is small, a natural approach is to attempt an expansion around the trivial case of $\beta = 0$. This can be done, but it is not a trivial affair. Some explanation of why it is not trivial will help to motivate the following proof. Consider the average value of the spin at the origin:

$$< \sigma_0 >= \frac{\sum_\sigma \sigma_0 \exp(-\beta H(\sigma))}{\sum_\sigma \exp(-\beta H(\sigma))}$$

For a finite volume the numerator and denominator are obviously entire functions of β. To conclude that their quotient is analytic somewhere we need to know that the denominator does not vanish there. At $\beta = 0$ the denominator is not zero, and so it does not vanish in a neighborhood of zero. However, the region in which it does not vanish will depend on the finite volume we are considering and may well shrink to just the origin as we take the infinite volume limit. Note that if we expand either the numerator or denominator in a power series about $\beta = 0$, the coeffecients will have a strong volume dependence. (The coeffecient of β^n will grow like the number of sites raised to the n.) By contrast we would hope that $< \sigma_0 >$ is analytic in a neighborhood of $\beta = 0$ with coeffecients that actually have a limit in the infinite volume limit. (This is true although we will not prove it.) The crucial observation is that there must be a lot of cancellation going on between the numerator and denominator. This need to "cancel the numerator and denominator" will appear in our proofs.

Theorem 1.2.2: In any number of dimensions there is a positive number β_1 such that for $\beta < \beta_1$ the limits $\lim_{L\to\infty} < \sigma_0 >_L^+$ and $\lim_{L\to\infty} < \sigma_0 >_L^-$ are both zero. (β_1 will depend on the number of dimensions.)

Proof: The quantity $\sigma_i\sigma_j$ only takes on the values $+1$ and -1. This and the fact that sinh and cosh are odd and even functions respectively, yields the following identity.

$$\begin{aligned}\exp(\beta\sigma_i\sigma_j) &= \cosh(\sigma_i\sigma_j\beta) + \sinh(\sigma_i\sigma_j\beta) \\ &= \cosh\beta + \sigma_i\sigma_j\sinh\beta = \cosh\beta[1 + \sigma_i\sigma_j\tanh\beta]\end{aligned}$$

Using this identity we can rewrite the partition function as

$$Z = \sum_\sigma \exp(\sum_{<i,j>} \sigma_i\sigma_j) = (\cosh\beta)^N \sum_\sigma \prod_{<i,j>} [1 + \sigma_i\sigma_j\tanh\beta]$$

N is the number of bonds in the sum in the original Hamiltonian. We now expand out the product over the N bonds $< i,j >$. A term in the resulting mess is specified by choosing either 1 or $\sigma_i\sigma_j\tanh\beta$ for each bond $< i,j >$. Some of the bonds $< i,j >$ have one site outside the volume. For these bonds σ_i is fixed to be $+1$ for the site outside the volume and σ_i is not summed over in the sum on σ. We let B denote the set of bonds for which we take $\sigma_i\sigma_j\tanh\beta$. Then

$$Z = (\cosh\beta)^N \sum_\sigma \sum_B \prod_{<i,j>\in B} \sigma_i\sigma_j\tanh\beta$$

where B is summed over all subsets of the set of nearest neighbor bonds such that at least one of the endpoints is in the finite volume. Letting $|B|$ denote the number of bonds in B we can rewrite the above as

$$Z = (\cosh \beta)^N \sum_B (\tanh \beta)^{|B|} \sum_\sigma \prod_{<i,j> \in B} \sigma_i \sigma_j$$

A wonderful thing is about to happen. Since each σ_i can only be $+1$ or -1, σ_i raised to an even power is just 1, and σ_i raised to an odd power is just σ_i. So $\prod_{<i,j> \in B} \sigma_i \sigma_j$ will just be a product over sites i of either 1 or σ_i. The sum over σ is just a product over sites i of a sum over σ_i. However, $\sum_{\sigma_i} \sigma_i = 0$, and so the sum over σ will yield zero unless $\prod_{<i,j> \in B} \sigma_i \sigma_j$ contains an even number of σ_i's for every site i in the finite volume. This happens if and only if for every site i the number of bonds in B which hit i is even. When the product $\prod_{<i,j> \in B} \sigma_i \sigma_j$ is equal to 1, the sum over σ produces a factor of $2^{|\Lambda|}$.

Let ∂B denote the sites $i \in \Lambda$ such that the number of bonds hitting i is odd. ∂B only consists of sites inside the finite volume Λ. Bonds in B can hit sites outside of Λ. For these sites there is no constraint that the number of bonds hitting the site must be even. In fact, these sites can be hit by at most one bond in B. Our result can now be rewritten

$$Z = 2^{|\Lambda|} (\cosh \beta)^N \sum_{B : \partial B = \emptyset} (\tanh \beta)^{|B|}$$

We now repeat the above for the numerator

$$\sum_\sigma \sigma_0 \exp(-\beta H(\sigma))$$

We now need to ask when the quantity $= \sum_\sigma \sigma_0 \prod_{<i,j> \in B} \sigma_i \sigma_j$ is not zero. Clearly the answer is that it is not zero if and only if $\partial B = \{0\}$. Thus we have

$$\sum_\sigma \sigma_0 \exp(-\beta H(\sigma)) = 2^{|\Lambda|} (\cosh \beta)^N \sum_{B : \partial B = \{0\}} (\tanh \beta)^{|B|}$$

and so

$$< \sigma_0 >= \frac{\sum_{B : \partial B = \{0\}} (\tanh \beta)^{|B|}}{\sum_{B : \partial B = \emptyset} (\tanh \beta)^{|B|}}$$

We just cancelled factors of $2^{|\Lambda|} (\cosh \beta)^N$ between the numerator and denominator. The numerator and denominator in the above both still grow exponentially with the volume, so more cancellation between the numerator and denominator is yet to come. All of the terms in the above sums are positive, so we now see that $<\sigma_0>^+> 0$.

Let B be a subset of bonds with $\partial B = \{0\}$. We are going to show that there is a path of bonds ω in B which starts at 0 and ends at some site just outside of Λ. We construct it as follows. Since $\partial B = \{0\}$, B must contain at least one bond which has 0 as an endpoint. Pick one such bond and let i_1 be the other endpoint of this bond. The number of bonds hitting i_1 must be even and there is at least one, so there must be another one. Let i_2 be its other endpoint. The number of bonds hitting i_2 must also be even so we can find a bond in B which we have not picked yet which also hits i_2. We then let i_3 be its other endpoint. We continue this process to find a sequence of bonds in B of the form $< 0, i_1 >, < i_1, i_2 >, < i_2, i_3 > \cdots < i_{n-1}, i_n >$. It is possible that we return to a site that we have already visited, i.e., some i_k may equal some i_j with $j < k$. Even when this happens it is still true that the number of bonds chosen so far which hit the site in question odd and so there is still at least one unchosen bond in B which hits the site. Eventually we must run out of bonds, but the only way the construction can end is for the site i_n to be outside of Λ. For these sites the number of bonds in B hitting the site need not be even. (In fact there is at most one bond in B hitting each of these sites.) The path of bonds in B we have constructed from 0 to the boundary need not be unique. For each set B we make some choice of this path ω, and we define $C = B \setminus \omega$. Note that $\partial C = \emptyset$.

Consider the map $B \to (\omega, C)$ from the set of B's with $\partial B = \{0\}$ into the set of pairs (ω, C) where ω is a walk from 0 to the boundary and C is a set of bonds with $\partial C = \emptyset$. The crucial point here is that the map is one to one. Thus we have the following upper bound:

$$\sum_{B:\partial B=\{0\}} (\tanh \beta)^{|B|} \leq \sum_{\omega} \sum_{C:\partial C=\emptyset} (\tanh \beta)^{|\omega|+|C|}$$

We could impose the constraint that $C \cap \omega = \emptyset$ on the sum over C, but we are free to drop it and get a weaker bound. The sum over C in the above repoduces the denominator, and so the above inequality is the same as

$$<\sigma_0>\leq \sum_{\omega} (\tanh \beta)^{|\omega|}$$

(The cancellation between the numerator and denominator was just achieved by finding an upper bound on the numberator which contained the denominator as a factor.)

The rest is easy. Consider all the walks ω of exactly n steps. Forgetting about the fact that the walk must end at the boundary, we can bound the number of such walks that start at 0 by $(2d-1)^n$. ($2d-1$ is the number of choices you have for a direction at each step.) The shortest walk which goes from 0 to the boundary has $L+1$ bonds, so our upper bond becomes

$$< \sigma_0 >^+ \leq \sum_{n=L+1}^{\infty} (\tanh \beta)^n (2d-1)^n$$

Let β_1 be the solution of $(\tanh \beta_1)(2d-1) = 1$ so that $\beta < \beta_1$ implies $(\tanh \beta)(2d-1) < 1$. This implies that the upper bond goes to zero as $L \to \infty$. $\square$

Theorem 1.2.3: In two or more dimensions there is a positive number β_2 such that for $\beta > \beta_2$ the limit $\liminf_{L \to \infty} < \sigma_0 >_L^+$ is strictly greater than zero for + boundary conditions. (β_2 will depend on the number of dimensions.)

Remark: We have used the lim inf instead of lim in the statement of the theorem since we do not know a priori that this limit exits. It is known to exist.

Proof:

When β is large we expect that the measure will be supported mainly on configurations for which $\sigma_i = \sigma_j$ for most of the bonds $< i, j >$. We are going to develop a geometric representation ("contours") of the spin configurations that is motivated by this. We first consider the case of two dimensions. With each spin configuration we associate a "contour" Γ. Γ will be a subset of bonds in the "dual" lattice. To construct the dual lattice we put a site at the center of each square in the original lattice. So the dual lattice is the set of points in R^d such that each coordinate is of the form $\frac{1}{2}$ plus an integer. For each bond $< i, j >$ in the original lattice there is a unique bond in the dual lattice which bisects $< i, j >$. We include this bond in the dual lattice if and only if $\sigma_i \neq \sigma_j$. Recall that we are imposing + boundary conditions. The bonds in the dual lattice that separate a site just outside of Λ from a site i in Λ will be in Γ if $\sigma_i = -1$. We leave it as an exercise for the reader to show that the number of bonds in the dual lattice hitting a particular site in the dual lattice must be even, i.e., $0, 2$ or 4. Thus we can think the contour Γ as being made up of a collection of loops. More precisely, $\partial \Gamma = \emptyset$ where $\partial \Gamma$ is the set of sites in the dual lattice which are hit by an odd number of bonds in Γ.

Suppose we are given the contour Γ. Then we know for every bond $< i, j >$ whether $\sigma_i = \sigma_j$ or $\sigma_i \neq \sigma_j$. Since we know the spins are $+1$ outside of Λ we can determine the configuration everywhere. Thus the contour completely determines the spin configuration. As we noted above, not every contour arises from some spin configuration. The contour must satisfy $\partial\Gamma = \emptyset$. We claim that if Γ does satisfy this condition then there is a spin configuration σ such that $\Gamma(\sigma) = \Gamma$. We define σ as follows. Pick a site i in Λ. Consider all the paths of bonds in the lattice which go from i to a site outside of Λ. The condition $\partial\Gamma = \emptyset$ implies that either every path crosses Γ an odd number of times or an even number of times. In the odd case we let $\sigma_i = -1$ and in the even case we let $\sigma_i = +1$. It is then easy to see that $\Gamma(\sigma) = \Gamma$. Thus there is a one to one correspondence between spin configurations σ and contours Γ with $\partial\Gamma = \emptyset$. Note that with free boundary conditions or periodic boundary conditions the correspondence would be two to one, i.e., each contour configuration would correspond to two spin configurations related by a global spin flip.

If β is large then we expect that σ_i is usually equal to σ_j and so the bond in the dual lattice which separates i and j is rarely in Γ. Thus Γ typically consists of small pieces that are well separated from each other,a "dilute sea of contours". We will say that Γ "encloses the origin" if every path of bonds from outside Λ to the origin must cross Γ an odd number of times. Γ encloses the origin if and only if the spin at the origin is not equal to the fixed boundary spins, i.e., $\sigma_0 = -1$. Thus

$$\mu(\{\sigma : \sigma_0 = -1\}) = \mu(\{\sigma : \Gamma(\sigma)\, encloses\, 0\})$$

Throughout this proof μ denotes μ_L^+. We have made the dependence of the contour $\Gamma(\sigma)$ on the spin configuration σ explicit since we are about to introduce contours that do not depend on σ.

Every $\Gamma(\sigma)$ can be written as the union of its connected components. Note that $\partial\Gamma(\sigma) = \emptyset$ implies that the boundary of each connected component is empty. If $\Gamma(\sigma)$ encloses the origin then at least one of these connected components encloses the origin. (A sum of even numbers cannot be odd.) For a connected contour γ that encloses the origin we let

$$E_\gamma = \{\sigma : \gamma \subset \Gamma(\sigma)\}$$

Then the event that $\Gamma(\sigma)$ encloses the origin is contained in $\cup_\gamma E_\gamma$ where the union is over connected contours γ which enclose the origin and satisfy

$\partial\gamma = \emptyset$. (We have inclusion here rather than equality since whenever there are an even number of connnected components that enclose the origin the spin at the origin will be +1.) Hence

$$\mu(\{\sigma : \sigma_0 = -1\}) \leq \mu(\cup_\gamma E_\gamma) \leq \sum_\gamma \mu(E_\gamma)$$

The following two lemmas will finish the proof for us.

Lemma 1.2.4: For any contour γ

$$\mu(E_\gamma) \leq \exp(-2\beta|\gamma|)$$

where $|\gamma|$ is the number of bonds in γ.

Lemma 1.2.5: In two or more dimensions there is a constant c which depends only on the number of dimensions such that the number of connected contours γ enclosing the origin with exactly n bonds is bounded by c^n.

We will prove the two lemmas after we complete the proof of the theorem. The two lemmas and our bound above yield

$$\mu(\{\sigma : \sigma_0 = -1\}) \leq \sum_n \exp(-2\beta n)c^n$$

where n is summed over the possible number of bonds in connected contours. In two dimensions the smallest contour that encloses the origin has 4 bonds in it, so the sum over n starts at 4. If β is large enough then the above series converges to something less than 1/2. Since

$$< \sigma_0 >= \mu(\{\sigma : s_0 = +1\}) - \mu(\{\sigma : s_0 = -1\})$$

this completes the proof. $\square$

Proof of Lemma 1.2.4: It is convenient to redefine the Hamiltonian to be

$$H = -\sum_{<i,j>} (\sigma_i\sigma_j - 1)$$

(This only shifts H by a constant and so does not change $\mu(E_\gamma)$.) Any bond with $\sigma_i = -\sigma_j$ contributes 2 to H, and any bond with $\sigma_i = \sigma_j$ contributes 0.

Thus $H(\sigma) = 2|\Gamma|$. Since there is a one to one correspondence between spin configurations σ and contours Γ with $\partial\Gamma = \emptyset$, we have

$$Z = \sum_{\Gamma:\partial\Gamma=\emptyset} e^{-2\beta|\Gamma|}$$

The numerator in the definition of $\mu(E_\gamma)$ is equal to

$$Z\mu(E_\gamma) = \sum_{\Gamma:\partial\Gamma=\emptyset,\gamma\subset\Gamma} e^{-2\beta|\Gamma|}$$

For each Γ with $\partial\Gamma = \emptyset$ and $\gamma \subset \Gamma$, we let $\Gamma' = \Gamma \setminus \gamma$. Note that $\partial\Gamma' = \emptyset$ and there is a one to one correspondence between Γ which contain γ and have empty boundary and Γ' which are disjoint from γ and have empty boundary.

$$\begin{aligned} Z\mu(E_\gamma) &= \sum_{\Gamma':\partial\Gamma'=\emptyset,\Gamma'\cap\gamma=\emptyset} e^{-2\beta|\Gamma\cup\gamma|} \\ &= e^{-2\beta|\gamma|} \sum_{\Gamma':\partial\Gamma'=\emptyset,\Gamma'\cap\gamma=\emptyset} e^{-2\beta|\Gamma|} \\ &\leq e^{-2\beta|\gamma|} \sum_{\Gamma':\partial\Gamma'=\emptyset,} e^{-2\beta|\Gamma|} \\ &= e^{-2\beta|\gamma|} Z \end{aligned}$$

Dividing by Z, this proves the lemma. □

The proof of lemma 1.2.5 is different in three and two dimensions, so it is time to explain how the proof of the theorem works in three dimensions. The dual lattice is defined in the same way, i.e., it is the sites in R^3 each of whose coordinates equals an integer plus 1/2. Rather than being made up of bonds in the dual lattice, the contours will now be made up of unit squares whose vertices are sites in the dual lattice. (For some reason these squares are usually called plaquettes.) Each bond $< i, j >$ in the lattice has a unique plaquette in the dual lattice that bisects the bond. We include the plaquette in $\Gamma(\sigma)$ if $\sigma_i \neq \sigma_j$. The boundary of a contour is now defined to be the set of bonds in the dual lattice that are hit by an odd number of plaquettes in the contour. The contours that arise from spins configurations have empty boundary, and there is a one to one correspondence between spin configuration and contours with empty boundary. All the previous proofs go through.

Proof of Lemma 1.2.5: In two dimensions one can show that given a connected γ with $\partial\gamma = \emptyset$ it is possible to find a path of bonds which visits every bond in γ exactly once. This implies that the number of γ which contain a fixed bond and have a total of n bonds is at most $(2d-1)^{n-1}$. Consider the n vertical bonds which lie bewteen the origin and the site $(n, 0)$. Any γ with n bonds that encloses the origin must contain at least one of these bonds. Thus the total number of γ with n bonds may be bounded by $n(2d-1)^n$ which is trivally bounded by c^n. Unfortunately this argument does not work in three dimensions, so we will present a cruder bound that works in any number of dimensions greater than one.

We give the proof in the language of three dimensions. Let γ be a connected contour ($\partial\gamma = \emptyset$) with exactly n plaquettes. (Actually, we will not need the fact that it is a contour, only that it is connected.) We claim that we can find a sequence of plaquettes $p_1, p_2, \cdots, p_m$ such that every plaquette in γ appears in the list at least once, each pair of consecutive plaquettes, p_i and p_{i+1}, are connected, i.e., have at least a vertex in common, and such that $m \leq 2n$.

The number of plaquettes connected to a fixed plaquette in the dual lattice is some dimension dependent constant which we denote by c_1. The connectivity implies that the number of possible p_{i+1} for a fixed p_i is at most c_1. Thus the number of sequences with p_1 fixed and m plaquettes is bounded by c_1^{m-1}. Since m is at most $2n$, the number of sequences with p_1 fixed is bounded by c^n for some c. The sequence completely determines γ, so the number of γ with n plaquettes which contain a fixed plaquette is bounded by c^n. Any γ that encloses the origin must contain one of the n plaquettes which are perpendicular to the line from the origin to $(n, 0, 0)$. So we get a bound of the form nc^n.

It remains to prove the claim. Consider sequences of plaquettes $p_1, p_2, \cdots, p_m$ with the following properties:

(i) each p_i is in γ

(ii) p_i and p_{i+1} are connected for $i = 1, 2, \cdots, m-1$.

(iii) m is at most twice the number of distinct plaquettes in the sequence.

Pick a sequence with the maximal number of distinct plaquettes. We would like to show this number is n. If it is less than n, then we can find a plaquette p in γ which is not in the sequence but which is connected to some plaquette in the sequence, say p_j. Now consider the sequence $p_1, p_2, \cdots, p_{j-1}, p_j, p, p_j, p_{j+1}, \cdots, p_m$. The length of this sequence is $m + 2$ and it has one more distinct plaquette than the original sequence. So this longer sequence still satisfies (iii). So this contradicts the maximality of $p_1, \cdots, p_m$. $\square$

What we have proved about $< \sigma_0 >$ has the following trivial but important consequence.

Corollary Assume that $\lim_{L\to\infty} < \sigma_0 >_L^+$ exists for all β. Call it $m(\beta)$. Then $m(\beta)$ is not an analytic function in a neighborhood of the positive real axis.

Proof: We have proved that $m(\beta)$ is zero on $[0, \beta_1)$. If it were analytic in a neighborhood of the positive real axis this would imply that it was zero in this neighborhood. This would contradict the fact that it is not zero on (β_2, ∞). $\square$

Exercises:

1.2.1 (very easy, but note that this gives yet another proof that the one dimensional model does not have a phase transition.) The proof of theorem 1.2.2 was valid in any number of dimensions. Show that in one dimension this argument can be used to show that $< \sigma_0 >$ converges to zero in the infinite volume limit for all β. Hint: in one dimension the possible sets B with $\partial B = \emptyset$ or $\partial B = \{0\}$ are quite limited. Also note that $\tanh \beta < 1$ for all β.

1.2.2 (easy, but an important point) Where does the proof of theorem 1.2.3 break down in one dimension?

1.2.3 (not too hard, not too exciting) Use the proof of theorem 1.2.2 to show that if $\beta < \beta_1$ then with $+$ or $-$ boundary conditions the magnetization per site converges to zero in the infinite volume limit. Hint: the total magnetization includes sites near the boundary. For these sites the effect of the boundary conditions is not neglible. You will need to make use of the fact that the fraction of such sites (the number of them divided by the total number of sites in the volume) goes to zero in the infinite volume limit.

1.2.4 (takes some thought, but the conclusion is worth it) Assume that the infinite volume limit of $< \sigma_i \sigma_j >$ exists. Adapt the proof of theorem 1.2.2 to show that for $\beta < \beta_1$ this correlation function decays exponentially as $|i - j| \to \infty$. In other words, show that the correlation length is finite. Then show that the correlation length goes to zero as $\beta \to 0$.

1.2.5 (takes some serious thought, but if you can do this one then you have a solid understanding of the Peiels argument) Adapt the proof of theorem

1.2.3 to show that in two dimensions, if β is large enough then there is a constant $\delta > 0$ such that for any two sites i and j,

$$< \sigma_i \sigma_j > \geq \delta$$

(Note: this is true for any boundary conditions, so we may wish to use free boundary conditions.) Hint: If $\sigma_i \neq \sigma_j$ what can you say about the contour Γ.

1.2.6 (purely a graph theory problem) Show that the dual of the triangular lattice is the hexagonal lattice and vice versa.

1.3 Phase transitions and critical phenomena

Crudely speaking a phase transition is when an abrupt change occurs in the infinite volume system at some values of the parameters like temperature and magnetic field. The values of the parameters where this happens are called a *critical point*. The goal of this section is to make this statement more meaningful and to describe what is believed to happen in the vicinity of a phase transition - so called critical phenomena. Most of the statements in this section about what happens near the critical point are not rigorous statements. Our goal here is to sketch a picture of what is widely believed to be true.

In the previous section we have already seen two ways of thinking about a phase transition. The first is that there is an inverse temperature β_c such that for $\beta < \beta_c$ the boundary conditions do not matter and for $\beta > \beta_c$ they do. The second is that some quantity is not analytic at β_c.

Now we consider what happens as we let β approach β_c. Recall that β is the reciprocal of the temperature, $T = 1/\beta$. We will use the temperature T in the following since that is the convention. As the temperature T approaches the critical temperaure some physically relevant quantities will diverge while others will converge to zero. The way in which they do so is extremely interesting. We begin with the correlation length. In one of the exercises we proved that the correlation length goes to zero as the temperature goes to infinity. It can also be proved that it goes to zero as the temperature goes to zero. As $T \to T_c$ the correlation length ξ diverges to ∞. This is a third way to think about a phase transition - the correlation length diverges at the critical point. (Actually this only happens for the class of phase transitions known as second order transitions. More on this point later.)

Define

$$t = (T - T_c)/T_c$$

so t measure the deviation from the critical temperature in a dimensionless way. The correlation length diverges as a negative power of t:

$$\xi \sim |t|^{-\nu},\ as\, T \to T_c$$

The number ν is called a "critical exponent". (It is conceivable that the correlation length would diverge with different critical exponents when we let $T \to T_c$ from the left and from the right. Thus we could define a high temperature and low temperature ν. This is sometimes done, but it is widely believed that they are the same.)

As $T \to T_c$ the specific heat and the magnetic susceptibility also diverge to ∞. (Recall that these quantities are just the second derivative of the free energy with respect to the magnetic field h and with respect to β.) They also diverge as t to some negative power:

$$\begin{aligned}\chi &= \frac{\partial m}{\partial h} = -\frac{\partial^2 f}{\partial h^2} \sim |t|^{-\gamma} \\ C &= \frac{\partial^2 f}{\partial \beta^2} \sim |t|^{-\alpha}\end{aligned}$$

In all both these formulae we set h to 0 after taking the partial derivatives.

If $T < T_c$ and we impose + boundary conditions on the system then the magnetization per site, m, will be strictly positive in the infinite volume limit. As $T \to T_c^-$ the magnetization m will converge to zero:

$$m \sim (-t)^\beta$$

Of course this β has nothing to do with the inverse temperature β. Despite the plethora of Greek letters, it is convention to use β for both this critical exponent and the inverse temperature.

The above four critical exponents all describe behavior as $T \to T_c$. There are two more critical exponents which are defined at $T = T_c$. When $T = T_c$ the magnetization per site is zero. If we impose a magnetic field h it will be nonzero. The behavior of this magnetization as we let the field go to zero defines yet another critical exponent:

$$|m| \sim |h|^{1/\delta}$$

At the critical point the correlation length is infinite. However, the correlation functions still decay, but only as a power of the distance. This power is another critical exponent. The convention is to define it as follows:

$$< \sigma_0 \sigma_x > \sim \frac{1}{|x|^{d-2+\eta}}$$

One way to motivate the $d-2$ in the power is to recall that the integral kernel of $(-\Delta)^{-1}$ is $1/|x-y|^{d-2}$ if $d > 2$. This remark will make a little more sense in chapter 3.

Here comes the amazing fact about critical exponents. Until now we have been concentrating on the nearest neighbor Ising model on the lattice $\mathbb{Z}^d$. Now consider the nearest neighbor Ising model on another lattice in the same number of dimensions. For example, instead of the square lattice $\mathbb{Z}^2$ we could use the hexagonal or triangular lattices. In general the critical β for the model will depend on the lattice. However, the critical exponents do not. They are believed to be *exactly* the same for all lattices in the same number of dimensions. This phenomenon is called " universality". But wait, there is more. We can also change the Hamiltonian. For example, for $\mathbb{Z}^2$ we can include terms $-\sigma_i \sigma_j$ in the Hamiltonian when the distance between i and j is either 1 or $\sqrt{2}$. The new terms in the Hamiltonian favor the spins lining up even more than in the original Ising model, so we might expect that the critical temperature of this model is higher than that of the original Ising model. However, the critical exponents of the two models are again believed to be *exactly* the same. To spell out a few more examples, we could include all terms $-\sigma_i \sigma_j$ with $|i-j| \leq l$, and we should get the same critical exponents regardless of the choice of the cutoff distance l. We could include interactions between four spins at a time. In general a wide class of models will all have the same critical exponents. These critical exponents will depend on the number of dimensions, but not on the details of the microscopic interaction or the particular lattice. (Some caution is in order here. Not all Ising-type models in the same number of dimensions will have the same critical exponents. A sure way to get different exponents is to include "power law" interactions. For example,

$$H = -\sum_{i,j} \frac{\sigma_i \sigma_j}{|i-j|^p}$$

with the sum over all pairs of sites i, j is expected to have critical exponents that depend on p. There are more subtle ways to get different critical exponents which go under the name of "multicritical points.")

In addition to the universality of the critical exponents, there are relations between the various exponents. They come in two flavors: "scaling laws" and "hyperscaling laws." The scaling laws are

$$\alpha + 2\beta + \gamma = 2$$

and

$$\beta\delta = \beta + \gamma$$

(There is a third scaling law which involves a critical exponent Δ that we did not define.) The hyperscaling laws are

$$d\nu = 2 - \alpha$$

and

$$2 - \eta = d\frac{(\delta - 1)}{(\delta + 1)}$$

As always, d is the number of dimensions. It is sometimes said that the difference between scaling laws and hyperscaling laws is that the latter contain the number of dimensions, d, explicitly while the former do not. This is not quite accurate. Obviously we could use the the two hyperscaling laws to derive an equation which did not depend on d, but this equation should still be considered a hyperscaling law.

The scaling laws are believed to hold in all number of dimensions. The hyperscaling laws are only expected to hold in low enough dimensions. For Ising type models they should hold in 2 and 3 dimensions. These scaling laws and the universality of the critical exponents are among the big questions that the renormalization group sheds some light on.

In two dimensions the Ising model with nearest neighbor interaction only and no magnetic field has been solved exactly. The solution yields the following exponents.

$$\alpha = log, \quad \beta = 1/8, \quad \gamma = 7/4, \quad \delta = 15, \quad \nu = 1, \quad \eta = 1/4$$

The statement $\alpha = log$ means that the specific heat diverges as $-log(|t|)$. It is easy to check that these values satisfy all the scaling and hyperscaling relations. (The value of α should be taken to be zero.) The exact solution is only in the case of zero external field. The values for those critical exponents that require an external field for their definition do not follow from the exact solution but require additional nonrigorous argument. The two dimensional

Ising model may also be solved when the coupling β for the bonds in the horizontal direction is different from the coupling for the bonds in the vertical direction. One finds that the critical exponents are still the same.

In addition to the exact values of the critical exponents from the solution of the two dimensional Ising model there is another set of values known as "mean field" values. Mean field refers to an approximation method in physics which we will eventually look at in chapter 3. The amazing thing about this approximation is that it is believed to be exact if the number of dimensions is sufficiently large. For the Ising type models we have been studying, $d = 5$ is sufficiently large, i.e, in 5 and more dimensions the critical exponents exactly equal the values predicted by the mean field approximation. (In 4 dimensions they are believed to agree up to logarithmic corrections.) This "mean field behavior" has been proved for some of the exponents for the nearest neighbor Ising model.

Our fourth way of thinking about phase transitions is in terms of symmetry breaking. Many physical phase transitions are accompanied by symmetry breaking, but it is certainly possible to have mathematical models in which there is a phase transition but no natural symmetry breaking. In the Ising model the symmetry group is just the group of order 2, $\mathbb{Z}/2$. The symmetry is given by sending all σ_i to $-\sigma_i$. Of course this is not a symmetry if we impose $+$ or $-$ boundary conditions. But for $\beta < \beta_c$, we get the same infinite volume limit with both boundary conditions and the infinite volume probably measure does have this symmetry. For $\beta > \beta_c$ we get two different probability measures in the infinite volume limit and we say the symmetry is broken. Physicists usually say the symmetry is "spontaneously broken."

Here is a fifth way to think about a phase transition. Until now we have been thinking of keeping the spacing between lattice sites fixed (usually to 1) and letting the finite volume go to infinity. We can instead fix a finite volume (e.g. a square in two dimensions) and then use a lattice with spacing a, e.g., $a\mathbb{Z}^d$. Then we let $a \to 0$ instead of letting the volume grow. If the system has a finite correlation length then this correlation length will go to zero as we let $a \to 0$. So the microspopic randomness in the system will not be seen at macroscopic length scales. Figure 2 shows a typical domain wall for $\beta > \beta_c$ (the low temperature phase). In the scaling limit the domain wall will just be a straight line and so the same for almost all spin configurations.

Figure 3 shows the domain wall when $\beta = \beta_c$. Even in the scaling limit the curve will be random. The fifth characterization of a critical point is that the microscopic randomness produces random structures on a macroscopic

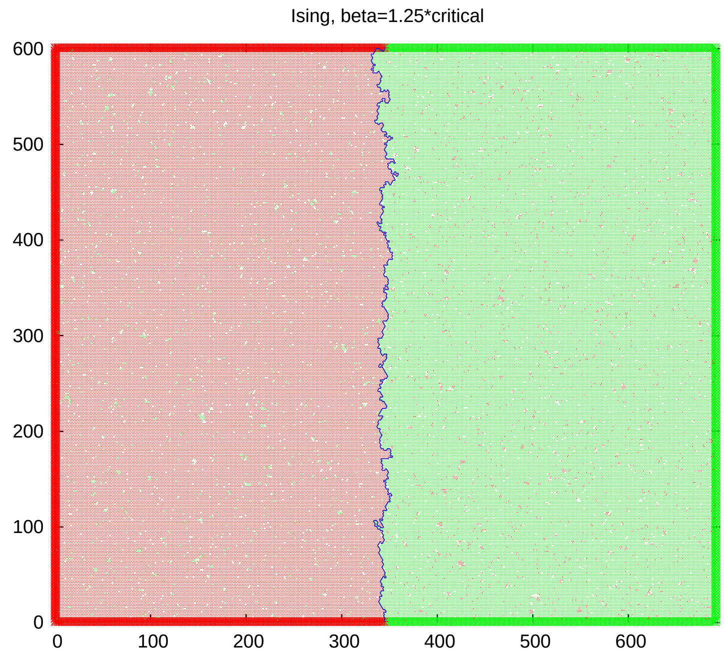

Figure 2: Domain wall forced by mixed boundary conditions at $\beta = 1.25\beta_c$.

scale.

We end this crash course in critical phenomena with a brief discussion of a different type of phase transition from those we have been considering. At the critical point $T = T_c$ the second derivates of the free energy f with respect to β and with respect to the magnetic field h both diverge, so f is not analytic at the critical point. The first derivatives of f with respect to β and h are both continuous functions. The phase transition at $T = T_c$ is called a "second order" transition since second derivatives of the free energy are the lowest order derivatives which fail to be continuous. There are also first order transitions in which first derivatives of the free energy fail to be continuous.

We take the Ising model at very low temperature and consider the effect of a small magnetic field. We have already seen that the effect of the boundary conditions is quite dramatic - they produce two very different infinite volume limits. If we take $h > 0$ then we will get an infinite volume limit in which $< \sigma_0 >> 0$. What is more subtle is that if we take $h > 0$, take the infinite volume limit and then let $h \to 0^+$ we will obtain a state $< \quad >^{h=0+}$ in which the expected value of σ_0 is positive. In fact $< \quad >^{h=0+}$ is the same as $< \quad >^+$ the infinite volume state we get from + boundary conditions. Likewise we can obtain $< \quad >^-$ by imposing a negative h and taking $h \to 0^-$ after we have

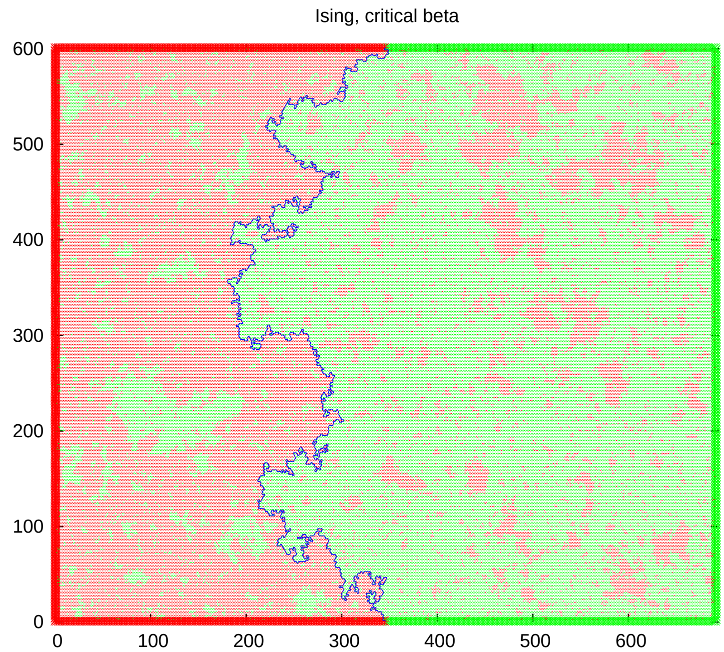

Figure 3: Domain wall forced by mixed boundary conditions at $\beta = \beta_c$.

taken the infinite volume limit. It is crucial that we let $h \to 0$ only after we have taken the infinite volume limit. Now consider the magnetization m as a function of h. (We are keeping the temperature fixed at some value below T_c.) As $h \to 0^+$, m will converge to $< \sigma_0 >^+$ while it converges to $< \sigma_0 >^-$ when $h \to 0^-$. Since $< \sigma_0 >^+$ and $< \sigma_0 >^-$ are not equal, m has a jump discontinuity at $h = 0$. This is a first order transition since a first derivative of the free energy, namely the magnetization, is discontinuous.

First order transitions are quite different from second order ones. One of the most important differences is that the correlation length does not diverge as we approach the first order transition. Note also that there is actually an entire interval of first order transitions at $h = 0$ and $0 \leq T \leq T_c$. The power law divergences that we saw at second order transitions are not present at first order transitions, so there are no critical exponents associated with the first order transitions. First order transitions are simpler than second order ones and at very low temperatures there is a wonderful rigorous treatment of first order transitions due to Pirogov and Sinai. There is a nice rigorous renormalization group approach to this "Pirogov-Sinai" theory by Gawedski, Kotecky and Kupiainen.

Exercises:

1.3.1 (really easy) Show that the scaling laws hold for both the $d = 2$ Ising critical exponents and for the mean field exponents, but the hyperscaling laws only hold for the former.

Critical exponents

$$t = (T - T_c)/T_c$$

ν, correlation length :

$$\xi \sim t^{-\nu} \quad as \quad t \to 0, \quad \nu_{2d} = 1, \quad \nu_{MF} = 1/2$$

γ, magnetic susceptibility :

$$\chi = \frac{\delta m}{\delta h} = -\frac{\delta^2 f}{\delta h^2} \sim |t|^{-\gamma} \quad as \quad t \to 0, \quad \gamma_{2d} = 7/4, \quad \gamma_{MF} = 1$$

α, specific heat :

$$C = \frac{\delta^2 f}{\delta \beta^2} \sim |t|^{-\alpha} \quad as \quad t \to 0, \quad \alpha_{2d} = log, \quad \alpha_{MF} = 0(disc.)$$

β, spontaneous magnetization :

$$m = \frac{\delta f}{\delta h} \sim (-t)^{\beta} \quad as \quad t \to 0^-, \quad \beta_{2d} = 1/8, \quad \beta_{MF} = 1/2$$

δ, response to magnetic field at $T = T_c$:

$$|m| \sim h^{1/\delta} \quad as \quad h \to 0, \quad \delta_{2d} = 15, \quad \delta_{MF} = 3$$

η, correlation function at T_c :

$$< \sigma_0 \sigma_x > \sim \frac{1}{|x|^{d-2+\eta}} \quad as \quad |x| \to \infty, \quad \eta_{2d} = 1/4, \quad \eta_{MF} = 0$$

Scaling laws:

$$\alpha + 2\beta + \gamma = 2$$

$$\beta\delta = \beta + \gamma$$

Hyperscaling laws:

$$d\nu = 2 - \alpha$$

$$2 - \eta = d\frac{(\delta - 1)}{(\delta + 1)}$$

See table on p. 111 of Goldenfeld for 3d values and experimental values.