Geometric Algorithms
Assignment 1

Laura Baakman (s1869140)

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A

The Euclidean distance between the n-dimensional points $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as:

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}. \tag{1}$$

My implementation of this formula is provided in Listing 1.
The length of the polygonal line created by drawing line segments between consecutive points from the list $[\boldsymbol{p_0}, \ldots, \boldsymbol{p_n}]$ is:

$$\sum_{i=0}^{n} d(\boldsymbol{p_i}, \boldsymbol{p_{i+1}}). \tag{2}$$

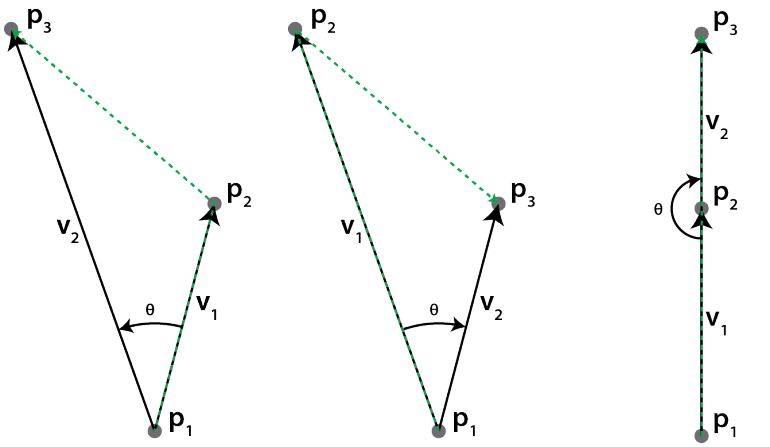
The first step of my implementation computes the list `pairs`, which contains all pairs of points between which the distance should be computed. Actually computing this distance, using the function `euclidean_distance` and summing the results gives the length of the requested line, see Listing 2.

Listing 1: Function that computes the Euclidean distance between two points.

```
def euclidean_distance(a, b):
    """Compute the euclidean distance between the n-dimensional points
    a and b."""
    return(
        sqrt(
            sum(
                [(a_i - b_i)**2 for a_i, b_i in zip(a, b)]
            )
        )
    )
```

Listing 2: Function that computes the length of the polygonal path between consecutive points.

```
def length_of_connecting_path(points):
    """Compute the length of the polygonal line that connects
    consecutive points."""
    pairs = zip(points, points[1:])
    return(
        sum(
            [euclidean_distance(a, b) for (a, b) in pairs]
        )
    )
```



(a) left turn **(b)** right turn **(c)** no turn

Figure 1: A path, the green dashed line, through the points P_1, P_2 and P_3 making (a) a left, (b) a right turn and (c) no turn.

Listing 3: Mathematica code used to compute the value of q.

```
p1 = {p1x, p1y, 0};
p2 = {p2x, p2y, 0};
p3 = {p3x, p3y, 0};

v1 = p2 - p1;
v2 = p3 - p2;

Cross[v1, v2]
```

B

Figure 1 presents the three possible types of paths that form a straight line through the points $\boldsymbol{P_1}$, $\boldsymbol{P_2}$ and $\boldsymbol{P_3}$. Converting these three 2D points to three dimensional ones, by adding a z-coordinate that is zero, allows us to use the cross product to determine the angle θ, using the definition of the cross product:

$$\boldsymbol{v_1} \times \boldsymbol{v_2} = \|\boldsymbol{v_1}\| \cdot \|\boldsymbol{v_2}\| \cdot \boldsymbol{n} \cdot \sin\theta. \tag{3}$$

Where the $\boldsymbol{v_1} = \boldsymbol{P_2} - \boldsymbol{P_1}$, $\boldsymbol{v_2} = \boldsymbol{P_3} - \boldsymbol{P_1}$ and $\boldsymbol{n} = [0, 0, 1]$ represents the normal of the plane that contains the points.

Since all values but the last of the vector $\boldsymbol{n}$ are zero the result of (3) will be of the form $[0, 0, q]$, where q is influenced by $\sin\theta$. The norms of the vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are greater than or equal to zero by definition. And does not influence the value of q. From this we can conclude that the sign, or the lack thereof of q is completely dependent upon $\sin\theta$. As we are only interested in the direction of the angle, not in its size we can use this to determine the type of path.

If the points are collinear the angle θ is π and which results in $q = 0$. If q is negative the path makes a left turn, if q is positive the path makes a right turn.

Using Mathematica, see Listing 3, we get an expression for the value of q:

$$q = -\boldsymbol{P_{1}}_y\boldsymbol{P_{2}}_x + \boldsymbol{P_{1}}_x\boldsymbol{P_{2}}_y + \boldsymbol{P_{1}}_y\boldsymbol{P_{3}}_x - \boldsymbol{P_{2}}_y\boldsymbol{P_{3}}_x - \boldsymbol{P_{1}}_x\boldsymbol{P_{3}}_y + \boldsymbol{P_{2}}_x\boldsymbol{P_{3}}_y, \tag{4}$$

where $\boldsymbol{P_{1}}_y$ represents the y-coordinate of the point $\boldsymbol{P_1}$.

C

Convex Hull

To find the convex hull of the generated set we use the ConvexHull algorithm given in the exercise. Since the code necessary to compute `L_upper` and `L_lower` is comparative it has been extracted to a method called `half_convex_hull` provided in Listing 4. The function `half_convex_hull` uses the method `make_right )turn` that determines if the path defined by the last three points in the set `L` makes a right turn. This method, uses the value of q defined in (4), see Listing 5.

Listing 4: Method that computes the upper (or lower) half of the convex hull.

```
def half_convex_hull(L, for_range, points_sorted):
    """
    Compute the upper or lower part of the convex hull.

    Args:
        L: The initial set to be used for the this half of the convex
    hull.
        for_range: The range of points_sorted to be considered.
        points_sorted: The sorted list of points of which the half
    convex hull is computed.
    """
    for i in for_range:
        L.append(points_sorted[i])
        while (
            len(L) > 2 and
            not make_right_turn(L[-3], L[-2], L[-1])
        ):
            L.pop(-2)
    return L
```

Listing 5: Method that computes the type of a path defined by `p1`, `p2` and `p3`.

```
def make_right_turn(p1, p2, p3):
    """Return true if the line drawn through p1, p2 and p3 makes a
    right turn."""
    q = -(p1[1]*p2[0]) + p1[0]*p2[1] + p1[1]*p3[0] - p2[1]*p3[0] - p1
    [0]*p3[1] + p2[0]*p3[1]
    return (q > 0)
```

This method is called twice to compute the convex hull, see Listing 6.

See Figure 2a for a visualization of a convex hull of 1000 points, generated with the provided `generate_points()` method. Figure 2b shows the convex hull of 100 points and all possible straight lines between points in the set in green, showing that the convex hull is indeed convex.

Area

The area A of a irregular planar non-intersecting polygon with vertices (x_1, y_1), $\ldots, (x_n, y_n)$ is defined as:

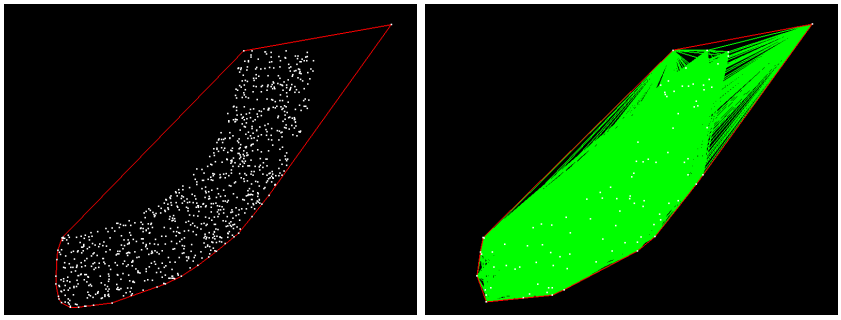
$$\begin{aligned} A &= \frac{1}{2} \cdot \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} \\ &= \frac{1}{2} \cdot (x_1 \cdot y_2 - x_2 \cdot y_1 + x_2 \cdot y_3 - x_3 \cdot y_2 \\ &\qquad + \ldots + \\ &\qquad x_{n-1} \cdot y_n - x_n \cdot y_{n-1} + x_n \cdot y_1 - x_1 \cdot y_n). \end{aligned} \tag{5}$$

Listing 6: Method that computes the convex hull of the inputted set of points.

```
def convex_hull(cv_points):
    """Compute the convex hull of the passed points using the passed
    points."""
    cv_points.sort()

    L_upper = half_convex_hull(cv_points[0:2], range(2, len(cv_points))
    , cv_points)
    L_lower = half_convex_hull(cv_points[-2:], reversed(range(0, len(
    cv_points) - 2)), cv_points)

    return(L_upper + L_lower[1:len(cv_points) - 1])
```



(a) Convex hull of 1000 points. **(b)** Convex hull of 100.

Figure 2: Visualizations of the convex hull, shown in red, of a set of points, with in (b) not only the convex hull but also all possible straight lines between points in the set, shown in green.

Listing 7: Method that computes the area of an irregular polygon.

```
def area_irregular_polygon(points):
    """
    Compute the area  of an irregular polygon.

    The polygon is defined by the passed points, which are the polygons
     vertices.
    """
    points.append(points[0])
    first_sum = sum([x * y for ([x, _], [_, y]) in zip(points, points
    [1:])])
    second_sum = sum([x * y for ([x, _], [_, y]) in zip(points[1:],
    points)])
    return ((first_sum - second_sum) / 2)
```

Using this formula, the method to compute the area of a irregular polygon is implemented as in Listing 7.

D

For the purpose of this exercise I have defined a class `Line` that contains all methods pertaining to the lines. Listing 8 presents the class definition and its constructors and factory methods. The factory method `randomWithFloat` generates a random line, with the coefficients `a` and `b` as floats, `randomWithFraction` does the same but uses `Fraction` objects.

Intersection

The x-coordinate intersection of two lines $f(x) = a_1 \cdot x + b_1$ and $g(x) = a_2 \cdot x + b_2$ can be found by solving $f(x) = g(x)$. Using Mathematica, see Listing 9, we get the solution of this equation:

$$x = \frac{-b_1 + b_2}{a_1 - a_2} \tag{6}$$

Point on Line

Test

Listing 8: The definition of the line class and its constructors.

```
class Line(object):

    """Class to represent a line of the form y = ax + b using two
    coefficients a, b."""

    def __init__(self, a, b):
        """Construct a Line object."""
        super(Line, self).__init__()
        self.a = a
        self.b = b

    @classmethod
    def randomWithFloat(cls, bounds=(-100, 100)):
        """
        Return a Line object, that is initialized with randomly chosen
    floats.

        By default the random chosen floats are between -100 and 100,
    passing a
        tuple (min, max) as the bounds parameter changes these values.
        """
        (minimum, maximum) = bounds
        return cls(
            uniform(minimum, maximum),
            uniform(minimum, maximum)
        )

    @classmethod
    def randomWithFraction(cls, bounds=(-100, 100)):
        """
        Return a Line object, that is initialized with randomly chosen
    fractions.

        By default the random chosen floats are between -100 and 100,
    passing
        a tuple (min, max)as the bounds parameter changes these values.
        """
        (minimum, maximum) = bounds
        return cls(
            Fraction.from_float(uniform(minimum, maximum)),
            Fraction.from_float(uniform(minimum, maximum))
        )
```

Listing 9: Mathematica code used to derive an expression for the x-coordinate of two lines.

```
Solve[a1 * x + b1 == a2 * x + b2, x]
```