An Empirical Stochastic Model for the Geometry of Two-Dimensional Crack Growth in Soil (with discussion)

ARTICLE *in* GEODERMA · JULY 2000

Impact Factor: 2.77 · DOI: 10.1016/S0016-7061(00)00015-X

An empirical stochastic model for the geometry of two-dimensional crack growth in soil.

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Abstract

A model is presented for the fragmentation of drying soil, based on the geometry of two-dimensional crack growth. The model restricts itself to the formation of two-dimensional cracks. The pertinent parameters used to generate cracking patterns include those used to characterise crack growth development as a random walk, fragmentation of peds above a certain size threshold and attraction of cracks within defined distances. A sensitivity analysis is carried out to examine the effect of varying these parameters on final cracking patterns. The variation of curvature and threshold to aggregate splitting are shown to have significant effects. Simulations based on this model are illustrated, and compared with cracks in real soil.

Keywords: soil cracks; aggregate size; geometry; random walk.

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1. Introduction

The ability of a soil to crack during drying dictates many of the transport processes that occur in the soil profile, as well as the stability of the soil. The size and tortuosity of cracks determine to a large extent the rate at which solutes and micro-organisms are transported in the profile, and the distribution and the connectivity of these cracks determine flow pathways, and thus control dispersal of substances in the soil profile. Much work has been done in relation to the quantification of crack patterns (O'Callaghan & Loveday, 1973; Wells & Theng, 1988; Preston *et al.*, 1997) and attempts have been made to examine and model the mechanisms involved in soil cracking (Morris *et al.*, 1992; Hull, 1994) and in other materials (Ehart *et al.*, 1996).

Modelling the formation of cracks in drying soil can be attempted in different ways. Many authors have concentrated on the conditions under which soil cracking occurs, and how these conditions affect measurable properties of the cracks (Andersland & Almoussawi, 1987; Lima & Grismer, 1992; Morris *et al.*, 1992; Yassoglou *et al.*, 1994). Another approach is to consider models based on the failure under stress of bonds between adjacent particles (Meakin 1987; Skjeltorp & Meakin, 1988). Some work has provided statistical summaries of crack geometry (Scott *et al.*, 1986; Noever, 1991). However, modelling the full cracking patterns does not seem to have been attempted, with the exception of the model of Gray *et al.* (1976), which although elegant, is too simple to model most types of crack growth. The spatial models of Dai & Ozawa (1997) allow a large set of texture-like structures to be simulated, some of which have a resemblance to cracks. However, we have sought a model that is based on parameters which can be related to crack growth in real soil structures. Such a model will help understanding of how soil properties affect the patterns of crack formation, and will be useful in simulating the transport of water, gases and micro-organisms through fragmented soil.

2. The model

2.1 Initial conditions and assumptions

We consider only two-dimensional cracks, such as will form in a thin layer of soil (Corte & Higashi, 1960; Preston *et al.*, 1997). The model incorporates crack growth as a random walk, allows fragmentation of peds above a certain size threshold, and has a facility for cracks to be attracted to one another, within a defined distance. These three relatively simple processes define the geometry of cracks.

2.2 Crack growth

The `tip' of a growing crack extends randomly in a direction that is allowed to change by small random amounts. In our model these amounts (call them $\delta\theta$) are uniformly and independently distributed in the range $[-\alpha,\alpha]$ and changes occur at regular frequent intervals (δt), too closely spaced to be seen individually. Between these intervals, the crack grows by an amount δl. Use of a different shape of distribution for $\delta\theta$ (provided the mean is zero), or irregular intervals, should not substantially change the resulting crack pattern, since the aggregation of many small independent changes is, by the Central Limit Theorem (Kitchens, 1987), independent of the distribution of the changes. However, the spread of $\delta\theta$ ($\alpha/\sqrt{3}$ for the uniform distribution we use) is important and determines the average curvature of the evolving cracks. It should be noted that the distribution of the radius of curvature is scale dependent, and so our use of the term does not correspond exactly to the usual mathematical definition. For the growth model, the mean curvature *at any given scale* is proportional to the standard deviation of $\delta\theta$. The value of δt affects only the speed of crack growth and not the final result. he growth model is illustrated in Fig 1.

[Fig 1 near here]

For any sample of drying soil, the cracking will eventually stop, and it will be seen that aggregates smaller than some area are stable and will not split further. What this maximum stable aggregate size is will depend on many things including the thickness and composition of the soil. Our model assumes that cracking takes place in three stages:

(i) When aggregates are many times bigger than their stable size, cracks begin at a number, N_0, of independent random positions in the soil , and simultaneously grow at random as described above. When they come close to an existing crack, less than a distance *b*, they are attracted towards it, and terminate when they meet it. During attraction, the difference between the crack growth direction and a direction perpendicular to the attracting crack decreases by a proportion *a* at each time step: If the crack is growing in a direction θ, and ϕ denotes a direction perpendicular to the attracting crack, then at the next time step $\theta \rightarrow \phi + a(\theta - \phi) + \delta\theta$. Alternatively, cracks terminate on meeting the edge of the soil area.

(ii) After all cracks formed in stage (i) have terminated, cracks begin at random positions on existing cracks, and grow and terminate as in stage (i). The initial growth direction is perpendicular to the existing crack. Each crack terminates before the next one starts. This stage continues until the area of the largest aggregate is less than A_1.

(iii) When, after crack formation, aggregates come to be several times their stable size, they exhibit a tendency to split in two. At each step, the largest aggregate splits, unless some aggregates have an elongation (defined below) greater than 3, in which case the aggregate with the largest product of area and elongation product splits. For each aggregate, the crack starts on the edge, at the point nearest the centroid, but subject to a little random perturbation. It then grows as in stage (ii). These steps continue until the largest aggregate size is less than A_2, at which point the cracking terminates.

[Fig 2 near here]

The three stages of crack growth are illustrated in Figure 2. The model is fairly complex in that it has many parameters. These are listed in Table 1. The final geometry of the cracking patterns is defined by these simple rules. Analysis of the patterns is provided by relatively simple quantifiers, a description of which is provided below.

Two simulations of the model are shown in Fig 3, with two examples of real soil cracks for comparison. Details on the generation of the real crack images are given by

Preston *et. al.*, (1997). The parameters for the simulated cracks were $N_o = 3$, $A_1 = 30$ cm^2, $A_2 = 2.4$ cm^2, $\delta l = 0.1$ mm, $\alpha = 0.1$ radians, $b = 2.5$ mm, and $a = 0.1$.
It can be seen that the simulated cracks capture many of the features of the real soil images. However, there are a few ways in which they differ: some real cracks finish without reaching another crack, and the shape of real aggregates seems more uniform.

It is possible to add further refinements to the model in order to produce even more realistic cracking patterns. The width of the cracks can be varied, becoming thinner later in the simulation. Some cracks, perhaps the thinnest ones, can be made to terminate at random without having reached another crack. The cracking process can also be made inhomogeneous. It would be fairly straightforward to allow the process parameters to vary systematically or stochastically with position in the image. We have not used such add-ons to the model, as they do not affect the geometrical properties of the cracks studied, and may distract from them. We could add a parameter to the model to specify crack termination, for example, but this parameter would act independently of the others and could easily be adapted to match that of any given set of real cracks.

[Fig 3 near here]

3. Analysis of cracking patterns

Changing any of the parameters in the crack growth model will have some effect on the resulting realisations. These may be very apparent, for example changes resulting from varying the stable aggregate size. They may be more subtle: much more detailed examination will be needed to see the effect of the crack attraction parameters *a* and *b*. We have chosen five summaries of the crack patterns which are easily measured and characterize the geometry of the aggregates formed. Note that measures, such as mean aggregate size and crack curvature, are not included since they are closely related to explicit parameters of the model. For all measures, aggregates bordering the image edge are excluded from the analysis.

Aggregate size coefficient of variation.
This is the ratio of the standard deviation of aggregate area to its mean. This is a scale-free measure of size variability, and assesses the uniformity of the aggregates.

Mean aggregate compactness.
For any aggregate, this is defined as

$$\text{compactness} = 4\pi\,(\text{area}/(\text{perimeter})^2)$$

This has a value of 1 for a disc, and decreases as the shape becomes more irregular.

Elongation.
This is related to compactness, and is defined as the square root of the ratio of the second moments along the major and minor axes of the shape. (A moment is the mean of the product of powers of deviations of x and y values about their mean. They can be used to summarise the spread in x and y values**.** The general theory of moments in image analysis is presented by Glasbey & Horgan, 1995). If s_{xx}, s_{yy}, s_{xy} denote

respectively the variances of the X and Y co-ordinates of the shape, and their covariance, then the direction, ϕ of the major axis can be obtained from

$$\phi = (1/2)\tan^{-1}(2s_{xy}/(s_{yy} - s_{xx})), \quad \text{if } s_{yy} > s_{xx}$$

and is otherwise this expression plus $\pi/2$ where $\tan^{-1}$ produces output over the range $-\pi/2$ to $\pi/2$. The second moments along the major and minor axes, termed λ_1 and λ_2, are then obtained from

$$\lambda_1 = s_{xx} \sin 2\phi + s_{yy} \cos^2\phi + 2\, s_{xy} \sin\phi\cos\phi.$$
$$\lambda_2 = s_{xx} \cos^2\phi + s_{yy} \sin^2\phi - 2\, s_{xy} \sin\phi\cos\phi.$$

and the elongation is defined as

$$\text{Elongation} = \sqrt{(\lambda_1/\lambda_2)}$$

It is a global measure of shape which would, for example, be large for the silhouette of a banana and low (near 1) for that of an orange. Unlike compactness, it is not sensitive to how irregular the boundary is.

Distribution of number of edges and number of neighbours.
Each aggregate boundary will be produced by three or more distinct growing cracks. We record the distribution of this number. The cracks forming the aggregate boundaries will separate them from other neighbouring aggregates. The number of these is related to the number of edges, but may be more. Again, we record the distribution.

4. Effect of parameter variation

Although a single realisation of the crack growth model will generate many aggregates, their shape and other properties are not independent. To achieve true replication, the model was run 12 times for each set of parameter values, and standard errors of the summary statistics are based on between-realisation variation. The parameter space of the model is of high dimensions, and would be cumbersome to explore in full. We will examine each parameter in turn, exhibit some realisations, and investigate the effect of the parameters on the crack pattern measures defined above. To give a scale to the simulations, we assume that the square shown in Fig 3(a) & 3(b) has sides of length 10 cm. We then use as the `base' model one where $N_o = 3$, $A_1 = 30$ cm^2, $A_2 = 2.4$ cm^2, $\delta l = 0.1$ mm, $\alpha = 0.1$ radians, $b = 2.5$ mm, and $a = 0.1$

4.1 Varying stable aggregate size

We assume that varying the stable aggregate size will have the most noticeable effect on the crack pattern. If A_1, the size at which splitting starts, and the other parameters, are varied in proportion, the only effect will be to change the scale at which we are looking at the model. Thus, we will not examine this parameter - its effect can be simulated by looking at the other realisations (e.g. in Fig 3) from a closer or further distance. We examine instead the effect of the other parameters, holding constant the stable aggregate size.

4.2 Crack curvature

This is an explicit part of the model through the effect of the parameter α. To see its effect, simulations were run for values of α = 0.05, 0.1 & 0.2 radians. Examples of the resulting crack patterns are shown in Fig 4. Table 2 shows means for the pattern summaries. We see that the curvature significantly affects all the crack pattern measures used.

[Fig 4 near here]

4.3 Splitting threshold

This is another explicit parameter, A_1, in the model. Fig 5 illustrates the effect of setting it to values of 30, 14 and 6 cm^2. There is little difference between Fig 5(a) and 5(b). However, Fig 5(c) looks quite different. Its resemblance to real cracks is questionable - it looks more like some irregular field pattern. This demonstrates that the two-stage model (section 2.2) is necessary - cracks growing independently at random are not sufficient. Table 3 shows summary statistics for the 12 simulations with each value of A_1. This also demonstrates the differences between A_1 = 6 cm^2 compared with 14 cm^2 or 30 cm^2. These latter two do not differ significantly at $p < 0.05$ in any of the measures, but we note the monotonic trend from 6 cm^2 to 30 cm^2 in all of the measures (i.e. the value for 30 cm^2 extrapolates the difference between 6 cm^2 and 14 cm^2).

[Fig 5 near here]

4.4 Initial crack number

Crack growth starts with a number of cracks appearing in the soil and growing in a random direction as described above. Later cracks start from existing cracks. It is plausible that the number of initial cracks, N_0, may have some influence on the resulting patterns. To investigate this, we ran simulations with N_0 = 1, 8 and 16. Sample patterns are shown in Fig 6. The differences are subtle, but more regularity in the pattern is evident when N_0 is smaller. The results are summarised in Table 4. The effect of N_0 is most apparent in the area coefficient of variation ($p<0.001$), and there is also a significant effect on elongation ($p<0.05$). There is no clear evidence of the effect on the other measures.

[Fig 6 near here]

Experimentation with other parameters in the model showed little effect. The growth rate δl only affects the speed of the algorithm, provided is it less than the resolution of the image displayed. If it is substantially more, cracks would appear to be composed of short straight segments. Changing the crack attraction distance and rate parameters had little noticeable effect, except near crack junctions, where increasing a made the join more strictly perpendicular, and reducing b introduced apparent sudden changes of direction in crack growth.

5. Comparison with real soil cracks

The images shown in Fig 3(c) and 3(d) were obtained from a study of crack patterns by Preston *et al.*, (1997), who give details about the soil preparation and image formation. To compare these patterns with those in our model, we obtained summary statistics for all images from two of the sites studied, Cruden Bay (5 images) and Insch (4 images) in Grampian Region, Scotland, of which Fig 3(c) and 3(d), respectively are examples. Because of difficulties and ambiguities in accurately tracing all of the individual cracks, the number of edges was not obtained. The results are shown in Table 5.

There is a good match in compactness and number of neighbours, where the results are similar to those in the various simulations in the previous section. The comparison is less satisfactory with size coefficient of variation and elongation. Aggregates at the two sites are more variable in size and less elongated than those generated by the model. The mean elongation in our model can be reduced a little by more strongly selecting elongated aggregates for splitting. However, when an aggregate is to be split, the crack will have the start position and growth direction that would approximately split it in two, but will thereafter change direction at random. It may be that some refinement of this would be needed in order to produce slightly less elongated aggregates. The size coefficient of variation may be increased by this, or some heterogeneity of the process parameters over the soil region might be needed.

The next stage in the model is to investigate the practical utility of estimating the parameters used in the model from soil samples, and to functionally link these estimations to fundamental properties of the soil. Our main model parameters include assuming cracks wandering randomly and, within defined distances, being attracted to other cracks. Additionally, we allow individual peds to form above a certain size threshold. In the many attempts at observing crack generation in soil, as far as we know, none have been able to adequately observe for the spatio-temporal patterns of cracking. However, with digital time-lapse facilities now becoming common place there are clear methodologies becoming available for such observations to take place. One of us (IMY) has already carried out preliminary studies on the cracking behaviour in pure clay and simple soil systems, using time-lapse facilities. Preston *et al*. (1997) have made strong links between cracking properties (including the heterogeneity and connectivity of cracking patterns, as measured by fractal analysis) to particle size analysis, stability of peds and particles. Unpublished work (personal communication Young & Crawford) has linked the circularity of generated cracks to the cohesiveness of individual volumes in soil. This latter work and that of Preston *et al.* (1997), both implicate the clay fraction of the soil as crucial to the final observed cracking patterns.

So there are strong potential links between the type of patterns generated in our model and fundamental properties of soils. However, the real test will be to initially relate what we are seeing from our relatively simple test assays to what we really observe in the field, and then to link the micro/meso-crack features of our soil patterns to a larger scale.

6. Discussion

We have presented a model for the growth of cracks in a medium such as soil. We believe this model gives realisations which reasonably resemble those in real soil, and which capture some of the essential features of how cracks grow. The model may thus be of help in understanding crack growth, and in interpreting the patterns seen in soil.

Study of the model we have developed leads to a number of conclusions about soil cracking.

- At a macroscopic level, crack patterns may be explained by a reasonably simple process based on random growth, aggregate splitting and crack attraction.

- Cracks growing randomly and independently do not lead to realistic patterns. The tendency for aggregates to split according to their size is the most essential part of the model.
- Crack curvature has a significant effect on the resulting crack patterns.
- The number of initial crack starting positions affects the regularity of the aggregates formed, with greater regularity resulting from fewer start positions.
- Any study of soil cracks should allow for heterogeneity of the parameters affecting the crack growth. It is likely that this heterogeneity exists at many area scales..

The model has not presented some of the characteristics seen in real soil: crack width is invariant in the model and not necessarily so in real soil; cracks always join other cracks in the model and do not in real soil; the crack process is more homogeneous than is observed in real soil, where soil properties (thickness of the cracking layer, moisture content etc.) will vary over many area scales. Crack depth varies and is important: it could be an additional aspect of crack growth. Interactions might occur, with crack width changing with depth. However, these considerations are independent of the geometrical process constructed, and could readily be incorporated in a more general model, if this should prove useful.

The model we have developed attempts to capture the geometry of soil cracks. As such, we have not made explicit reference to the physical mechanisms occuring in the soil. We can however envisage what these are. The forces produced in shrinking drying soil will tend to keep cracks growing in the same direction, if the soil were homogeneous. It is not, of course, and the local microscopic variations in the binding forces between individual soil particles will cause the crack directions to change. We speculate that the greater the variation, the more will the cracks curve. It may be possible to express the crack growth model in these terms. The forces leading to cracking become less as the soil splits into smaller aggregates. Eventually, they will become less than the binding forces in the soil, and cracking will stop. This leads to there being a maximum stable aggregate size. The phenomenon of crack attraction can be understood in terms of the cracks seeking to reduce the shrinking force by splitting the soil into aggregates. (For a discussion of the mechanics, see Raats, 1984). Dividing aggregates approximately in two will split them with minimum crack length. It should be possible to re-parametrise our model in terms of these processes. We have opted for simplicity, by describing the process geometrically. We hope that this will make our model more adaptable to a wider range of soil types and conditions.

An alternative approach to modelling the cracks would be to explicitly construct a network of bonds between soil particles, and simulate the application of drying forces. This would involve a different, more microscopic view of the soil.

Our geometrical model provides a link between local small-scale features, and the crack patterns observed at a macroscopic level. The model can, using a random process with a modest number of parameters, capture how cracks develop. Relating the parameters used in the model to the properties of real soil is the next challenge. We also expect to be able to use the model to study the influence of cracks on soil transport properties. For example, gases can diffuse and water flow more readily through cracks than through the soil aggregates. Diffusion and flow simulations should allow the importance of cracks and the effect of different cracking patterns to be studied.

Acknowledgements

This work was supported by the Scottish Office Agriculture, Environment and Fisheries Department.

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Table 1. Parameters used in the crack growth model

Name	Role
N_o	Number of cracks starting in soil area
A_1	Aggregate area after which splitting starts (cm^2)
A_2	Maximum area of stable aggregate (cm^2)
δl	Crack growth length in time (mm^{-1})
α	Maximum change in direction (radians)
b	Distance at which crack attraction starts (mm)
a	Rate of crack attraction (mm^{-1})

Table 2. Mean crack pattern summaries for varying curvature (a)

		Curvature		
Property	a=0.05	a=0.1	a=0.2	SED
Size c.v	0.29	0.34	0.46	0.028***
Elongation	1.76	1.67	1.87	0.07*
Compactness	0.744	0.725	0.694	0.016***
Edges	3.93	3.89	3.79	0.034***
Neighbours	6.11	6.04	6.02	0.042*

SED = standard error of difference

Table 3. Mean crack pattern summaries for varying splitting threshold (A_1)

		Threshold		
Property	A_1=30 cm^2	A_1 = 14 cm^2	A_1 =6 cm^2	SED
Size c.v.	0.341	0.358	1.032	0.067***
Elongation	1.67	1.75	2.38	0.08***
Compactness	0.725	0.716	0.657	0.012***
Edges	3.89	3.88	3.58	0.052***
Neighbours	6.037	6.074	6.128	0.108***

Table 4. Mean crack pattern summaries for varying initial crack numbers.(N_o)

Property	N_o = 1	N_o = 8	N_o = 16	SED
Size c.v.	0.31	0.36	0.43	0.03**
Elongation	1.7	1.79	1.9	0.08*
Compactness	0.729	0.707	0.708	0.014
Edges	3.870	3.850	3.845	0.035
Neighbours	6.09	6.07	6.18	0.07

Table 5. Crack pattern summaries for soil from Cruden Bay and Insch

Property	Cruden Bay	Insch	SED
Size c.v.	0.62	0.51	0.05
Elongation	1.43	1.42	0.01
Compactness	0.807	0.740	0.015***
Neighbours	5.97	6.06	0.07

Figure Captions

Fig. 1. Crack growth. At regular intervals, the growth direction changes by a small random amount

Fig. 2. Stages of crack formation

Fig. 3. (a) & (b) Two examples of crack simulations. (c) & (d) Real soil cracks.

Fig. 4. Effect of crack curvature. (a) α=0.05, (b) α=0.1, (c) α=0.2.

Fig. 5. Effect of splitting threshold. (a) A_1= 30 cm^2, (b) A_1= 14 cm^2, (c) A_1= 6 cm^2

Fig. 6. Effect of initial crack number. (a) N_o = 1, (b) N_o = 8, (c) N_o = 16