Modelling and Simulation
Practical Assignment 2: Percolation

Rick van Veen (s1883933)* Laura Baakman (s1869140)*

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1. Experiments

This section presents our exploration of the parameter space of the percolation model. In section 1.1 we discuss the influence of the probability parameter p. In this section we often use the size of the cluster to compare the influence of the different parameters. We have defined the cluster size as the number of points that are occupied.

Section 1.2 explores the effect of the size of the system on the generated clusters. In section 1.3 we attempt to determine the fractal dimension of the finite clusters as a function of p. Finally, section 1.4 presents a short analysis of the impact of the used connectivity.

1.1. Probability

One important property of clusters is their size, and how that size depends on the parameter p. Since one can only determine the size of a finite cluster, we only consider size to be defined for non-percolating clusters. Therefore in the following discussion on the size of clusters we do not consider percolation clusters. Kenzel et al. [2] describe this relation as follows: for small values of p we get a large number of small clusters. As p increases we find positive correlation between p and the average cluster size until p reaches some threshold value p_c. For $p > p_c$ we get either a small finite cluster or a percolating cluster.

*These authors contributed equally to this work.

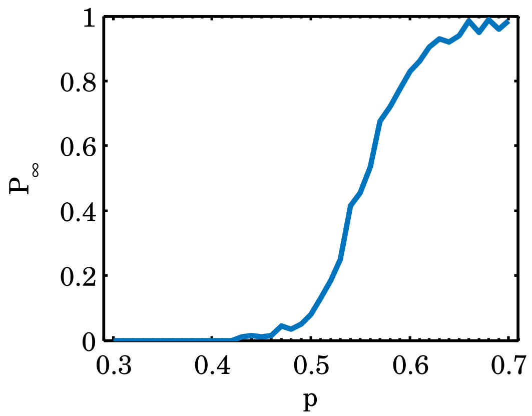


Figure 1: *Ratio of percolating clusters, P_∞, as a function of p. Ratios are calculated over $r_{max} = 200$ runs on a 41×41 grid.*

As $p > p_c$ increases the probability of ending with a finite cluster decreases, until we always get the percolating cluster for $p = 1$. Note that although in theory this cluster should cover the full grid, this is not necessarily the case in our model, since it stops growing as soon as one border site is occupied.

To find an indication of the value of p_c with our model we have let it generate a cluster on a 41×41 grid for $p = 0.31, 0.32, \ldots, 0.7$. For each value of p we grow $r_{max} = 200$ clusters.

Figure 2 presents the mean and standard deviation of the size of the finite clusters as a function of p. In this figure we observe the effect of p on the mean cluster size described by Kenzel et al. Furthermore, based on these data

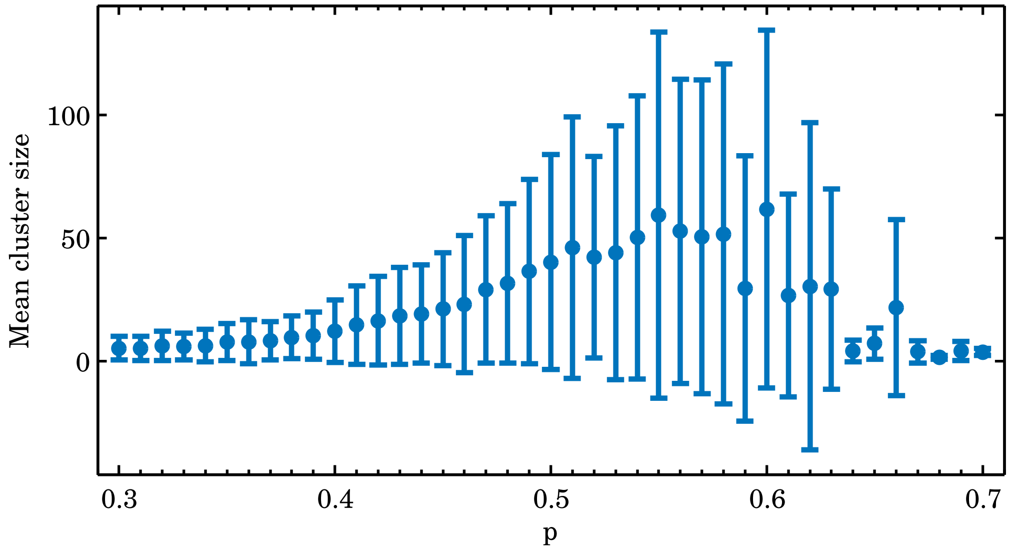


Figure 2: *Mean cluster sizes, represented as points, and standard deviations, indicated by the vertical error bars, as a function of p, with a step size of 0.1. The mean and standard deviation were calculated over 200 runs on a 41×41 grid.*

one would estimate p_c to be approximately 0.55.

Kenzel et al. also predicted that the number of percolating clusters relative to the number of finite clusters would grow for $p > p_c$ until $p = 1$, where the only possibility would be a percolating cluster. To observe this effect figure 1 shows P_∞, which is the ratio of the number of percolating clusters to the number of finite clusters. This graph is based on the same data as figure 2. Based on this graph we would say that $p_c \approx 0.4$. This number is lower than the value for p_c based on figure 2. This is probably caused by the relatively small grid sizes, which causes us to classify some clusters as percolating, that are actually finite.

Stauffer [4] has found p_c to be approximately 0.5928 for a square lattice. As stated earlier our lower estimation of p_c is quite likely caused by our small grid.

1.2. System Size

In section 1.1 we postulated that our relatively small grid influenced the found value of p_c. This section qualitatively discusses the relation between the size of the grid and the cluster. We repeat the experiment discussed in section 1.1 but varied $N = 2, 6, \ldots, 60$ instead of p. Since we are mostly interested in the size of finite clusters we choose $p = 0.5 < p_c$. Instead of the size of grid we now measure Q the ratio of the size of finite clusters to the number of sites in the grid.

Figure 3 shows the mean and standard deviation of Q as a function of N. In this graph we observe that average Q decreases as N increases, i.e. the size of the clusters does not grow as hard as the number of sites in the grid.

Figure 4 shows that P_∞ reaches zero for $N > 40$, which indicates that for this value of N there are no more percolating clusters. This fits with the theory discussed in section 1.1, which states that we get only finite clusters for $p < p_c$. We have percolating clusters for $p = 0.5 < p_c$ since the grid is not large enough to hold the finite clusters.

Physics by computer heeft een hoop theorie hiervoer die ik niet helemaal begrijp... of waarvoor we in ieder geval een ander soort plotjes

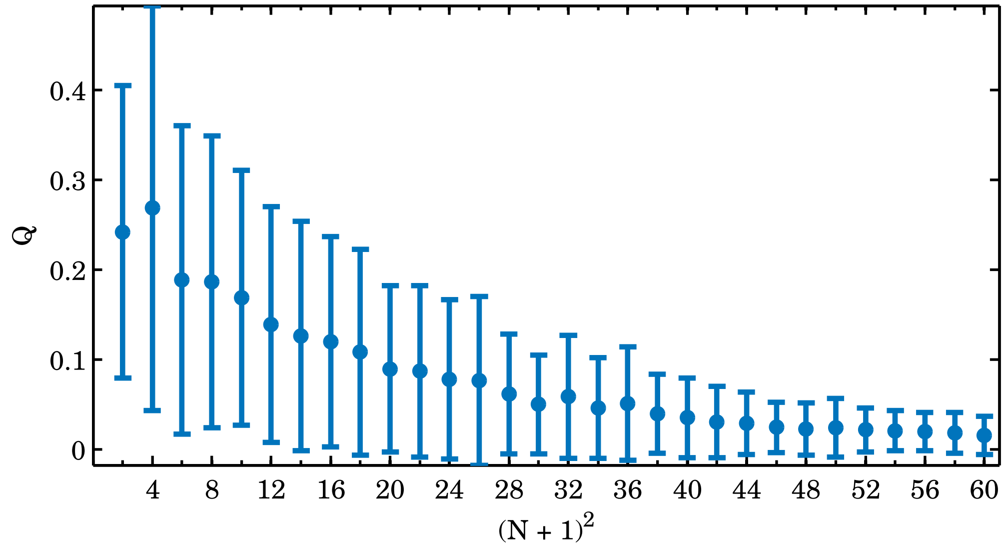


Figure 3: *The mean, represented as points, and standard deviations, indicated by the error bars, of the ratio of the cluster size to number of sites in the grid, i.e.* $(N+1)^2$*.The mean and standard deviation were calculated over 200 runs with* $p = 0.5$*.*

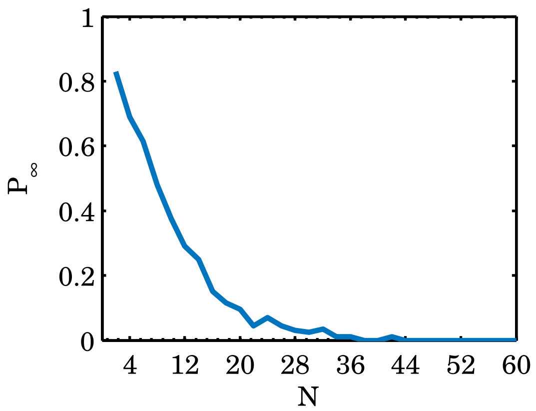


Figure 4: *Ratio of percolating clusters to the number of finite clusters,* P_∞*, as a function of* N*. Ratios are calculated over* $r_{max} = 200$ *runs with* $p = 0.5$*.*

If the lattice size is infinite we are no longer limited by the size of the lattice, in essence we remove one of our stop conditions. In this situation the theory presented in section 1.1 holds, i.e. as long as $p < p_c$ we only have finite clusters. As $p > p_c$ the number of percolating clusters relative to the number of finite clusters increases until we always get a percolating cluster for $p = 1$.

1.3. Fractal Dimension

Hoe verandert de fractal dimension ρ als een functie van p? Ik het geprobeerd, zie branch fractalDimension, maar ik krijg dimensions eruit die allemaal lager zijn dan wat het hoort te zijn en het plotten gaat volledig stuk. In boxcount staat onderaan een functie de de fractal dimension berekent, op het moment gebruikt die de mean van de gradient, maar volgens Biehl zou je lineare regressie moeten gebruiken. Allebei geven ze crap resulaten.

Falconer [1] describes the fractal dimension as some number ρ such that

$$(1) \quad M_\varepsilon(\rho) \sim c\varepsilon^{-s}$$

where c and s are constants and $M_\varepsilon(\rho)$ are measurements at different scales ε for $\varepsilon \to 0$. Falconer then shows that the fractal dimension can be estimated "as minus the gradient of a log-log graph plotted over a suitable range of ε"[1].

One way to get the measurements M_ε is to use box-counting. When one uses this algorithm the different scales mentioned in Falconer's definition are the sizes of the boxes.

We have used the function `box-count` by Moisy [3] to determine the fractal dimension of one percolation cluster. This implementation of the box-counting algorithm uses box sizes that are a power of two, consequently $\varepsilon = 1, 2, 4, \ldots 2^q$ where q is the smallest integer such that $q \leq (2N + 1)$.

Figure 5a shows the cluster of which we have determined the fractal dimension using box-counting. It was generated with $N = 80$, $p = 0.7$. Figure 5b presents the number of boxes as a function of the size of the boxes, figure 5c presents the minus of the gradient of figure 5b. From this graph we can infer the box-counting dimension by finding the local dimension for which the gradient is approximately stable. Using this method we find the fractal dimension to be 1.879, which neatly approximates the dimension 1.896 mentioned by Stauffer [4]. The small difference between these numbers can be explained by the relatively small size of our cluster and the fact that we present the fractal dimension of only cluster instead of the average over multiple clusters.

1.4. Connectivity

We consider two different connectivities, namely four- and eight-connectivity, which are illustrated in figure 7. In this section we discuss the influence of the 8-connectivity on the size of the cluster.

Figure 6 shows two clusters which have been grown using the same probabilities but different connectivities.

Observaties.

The results of performing the same experiment as discussed in section 1.1 with the eight-connectivity mask, are presented in figure 8 and 9. We have changed the range of p to $p = 0.2, 0, 21, \ldots, 0.99$, since using the range used for four-connectivity seemed too small.

It should be noted that for some values of p, especially larger values there are no mean cluster sizes, since no finite clusters were found. This indicates that one is much more likely to encounter a percolating cluster with eight-connectivity than with four-connectivity.

Explain why this makes sense

Hoe zorgt dit ervoor dat de mean size van een cluster kleiner is?

This is confirmed by figure 9, where we find that only for very low values of p P_∞ is zero, and that P_∞ quickly approaches 1.

These findings suggest that p_c is much lower when eight-connectivity is used. Based on this, admittedly small experiment, one would guess p_c to be approximately 0.2 when eight-connectivity is used. More research is needed to find the actual value of p_c when eight-connectivity is used.

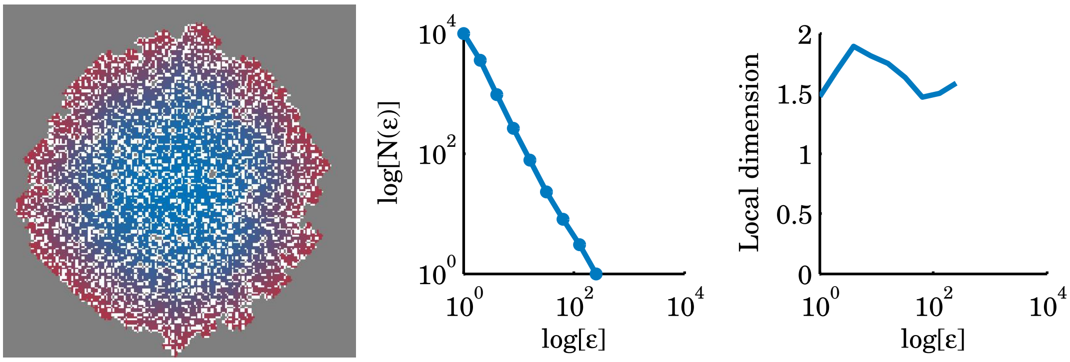
Influence of size

Influence on fractal dimension

References

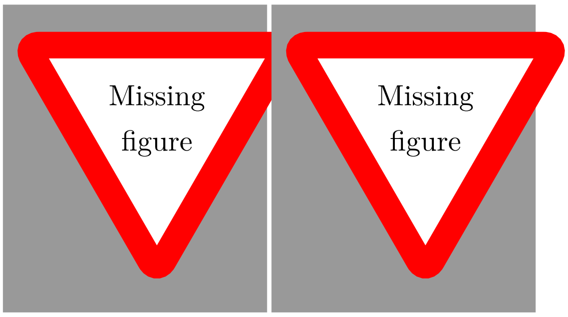
[1] Kenneth Falconer. *Fractal geometry: mathematical foundations and applications*. John Wiley & Sons, 2004.

[2] Wolfgang Kenzel et al. *Physics by computer*. Springer-Verlag New York, Inc., 1997.



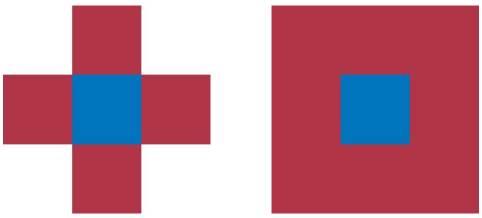
(a) *The cluster.* **(b)** *The number of boxes as function of the box size.* **(c)** *The minus gradient of figure 5b.*

Figure 5: *(a) The cluster ($N = 80, p = 0.7$) used to compute the box-counting dimension. (b) The number of boxes used to cover that cluster as a function of the box size. (c) The gradient of the function plotted in (b).*



(a) *4-connectivity* **(b)** *8-connectivity*

Figure 6: *The results of growing a cluster with the same grid of probabilites with (a) four-connectivity and (b) eight-connectivity.*



(a) *4-connectivity* **(b)** *8-connectivity*

Figure 7: *Connectivity masks for (a) four-connectivity and (b) eight-connectivity. The red squares are considered neighbors of the blue center square.*

[3] F. Moisy. *Computing a fractal dimension with Matlab: 1D, 2D and 3D Box-counting.* 2008. URL: `http://www.fast.u-psud.fr/~moisy/ml/boxcount/html/demo.html`.

[4] Dietrich Stauffer. *Introduction to percolation theory.* Taylor & Francis, 1985. Chap. 2, pp. 15–58.

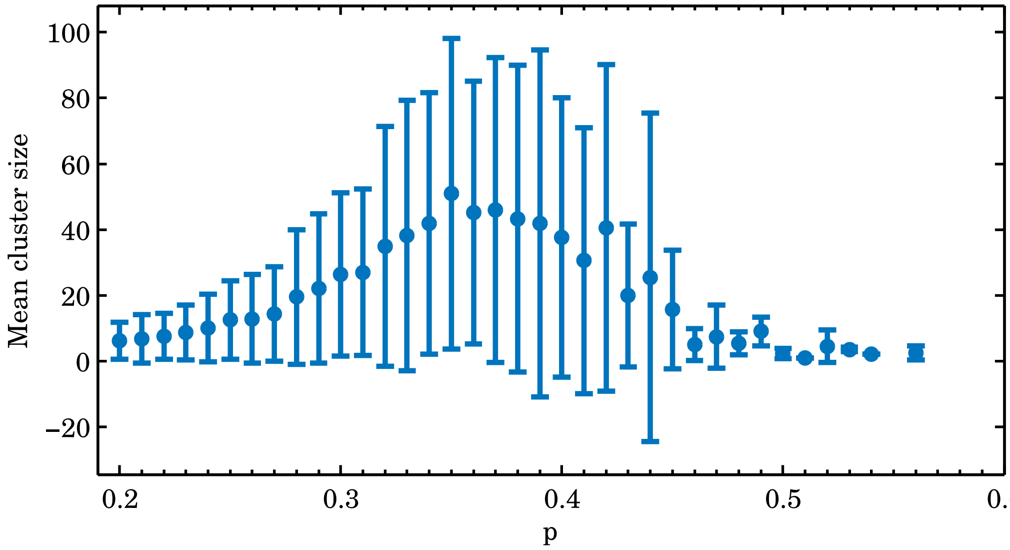


Figure 8: *Mean cluster sizes, represented as points, and standard deviations, indicated by the vertical error bars, as a function of $p = 0.2, 0, 21, \ldots, 0.99$ when eight-connectivity is used. The mean and standard deviation were calculated over 200 runs on a 41×41 grid.*

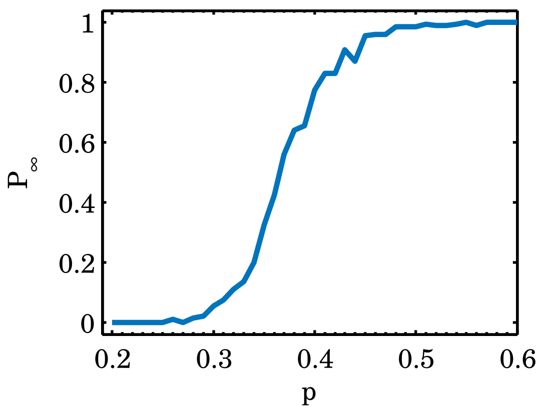


Figure 9: *Ratio of percolating clusters, P_∞, as a function of $p = 0.2, 0, 21, \ldots, 0.99$ when eight-connectivity is used. Ratios are calculated over $r_{max} = 200$ runs on a 41×41 grid.*