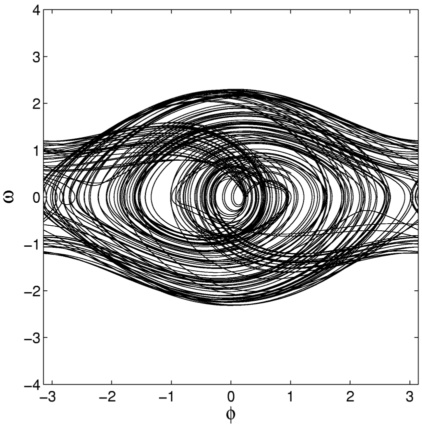
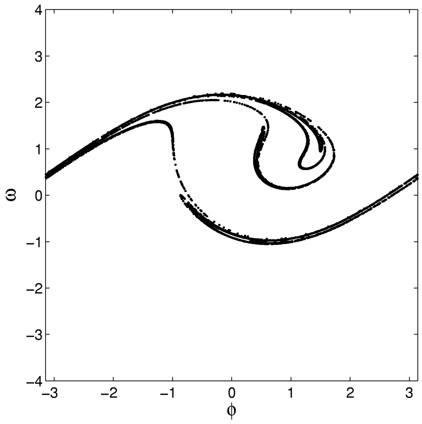
3. The driven pendulum

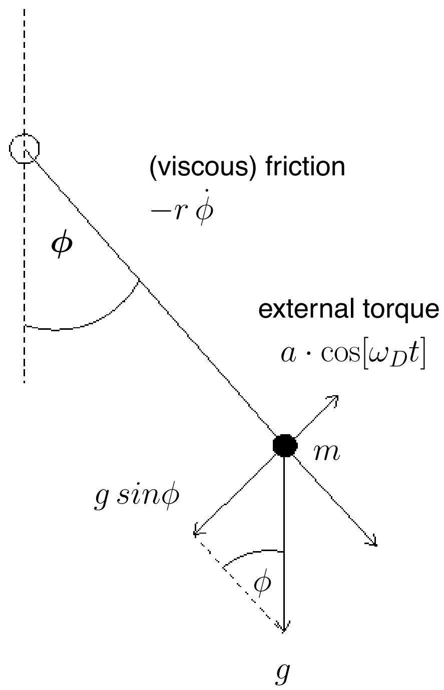




(phase space plot and Poincaré section, as explained below)

The discussion follows to a large extent the chapter *The Chaotic Pendulum* in W. Kinzel and G. Reents, *Physics by Computer* (Springer, 1998).

Further information and related Mathematica and C programs as well as Java applets are available at `http://theorie.physik.uni-wuerzburg.de/TP3/physbc.html`



The driven, damped pendulum:

(we set $m = g = \ell = 1$ for convenience)

$$\ddot\phi = -\sin\phi - r\,\dot\phi + a\,\cos[\omega_D\,t]$$

ω_D : frequency of the driving torque
a : amplitude of the driving torque
r : friction coefficient

rewritten as a system of first order O.D.E. in three dimensions $\{\phi, \omega, \theta\}$

$$\dot\phi = \omega$$
$$\dot\omega = -\sin\phi - r\,\omega + a\,\cos\theta$$
$$\dot\theta = \omega_D$$

The ideal pendulum

as a test case for numerical integration

$$\dot{\phi} = \omega$$
$$\dot{\omega} = -\sin\phi$$

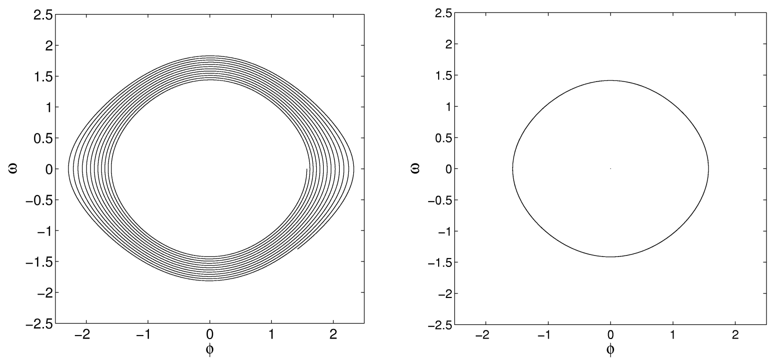


Fig. 3: *Phase space plot, i.e.* $\omega = \dot{\phi}$ *vs.* ϕ *as obtained for initial conditions* $\phi(0) = \pi/2$ *and* $\omega(0) = 0$.

Left: numerical integration with the naive Euler method, time step $dt = 0.01$.
Right: num. integration with the (Runge Kutta based) `Matlab` procedure `ode45`.

For $a = 0, r = 0$ the total energy is perfectly conserved $\rightarrow$
initial conditions must be perfectly restored after one period ($T = 2\pi\sqrt{\ell/g}$ for small ϕ)
Euler integration yields (here) an artificial increase of the energy!
(see the discussion in the supplement *Ordinary Differential Equations*)

The damped pendulum

without external driving ($a = 0$) and weak damping ($r < 2$)

$$\dot{\phi} = \omega$$
$$\dot{\omega} = -\sin\phi - r\,\omega$$

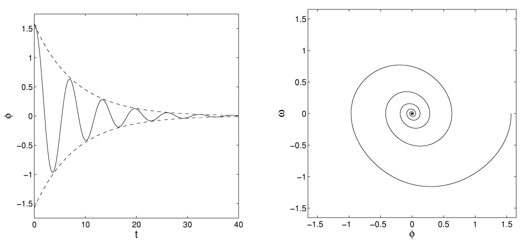


Fig. 4: Left: $\phi(t)$ *for* $r = 0.25$ *and initial conditions* $\phi(0) = \pi/2$ *and* $\omega(0) = 0$. *The envelopes are* $\pm\phi(0)\cdot\exp[-rt/2]$ *and the frequency is* $\omega_r = (1 - r^2/4)^{1/2}$ *for small* ϕ.
Right: *the corresponding phase space plot in the* $(\phi,\,\omega)$*–plane*

From *almost all* initial conditions, the system approaches the stable fixed point $\phi = 0, \omega = 0$. The pendulum comes to rest, all energy is dissipated due to friction.

Exception: $\phi = \pi, \omega = 0$ is an *unstable* fixed point.

Note: Trajectories in the phase space plot cannot cross, because the pair $(\phi, \dot{\phi})$ uniquely determines the further temporal evolution of the system

Closely related statement (yet not obvious):
There is no chaotic motion in two-dimensional (continuous) phase space

```
  function di = dgl(t,y)
% function diff(phi,omega)
% diff. eqs. for the damped, driven pendulum
% phi' = omega, omega' = - r omega - sin phi
% uses GLOBAL variables amp,omd,r
  global amp omd r
  phi= y(1); omega=y(2); di = zeros(2,1);
  di(1) = omega;
  di(2) = -r*omega - sin(phi) + amp*cos(omd*t);
```

```
  function pend(ampin,tmax,all)
% pend(amp,tmax,all)
% fixed: omd=2/3, r=1/4
% all=1: 0<t<tmax, only phase space plot
% all=0: 300 pi<t<tmax, addtl. Poincare section
  if (nargin==2) all =0; end
% default accuracy of ode45 is not sufficient!
  op = odeset('RelTol',1.e-8);

  clear global amp omd r
  global amp omd  r
  amp=ampin; omd=2/3; r=0.25; pd=2*pi/omd;
% initial conditions (horizontal pendulum)
  phi0 = pi/2.; om0=0;

  if all == 0
%   initial phase not plotted
%   t is multiple of 2 Pi/omd as to guarantee
%   correct initial value of omd*t later
    [T,Y] = ode45(@dgl,[0 100*pd],[phi0 om0],op);
    lst = length(Y(:,1));
    phi0 =   Y(lst,1); om0 = Y(lst,2);
  end
 close all; figure(1); hold on; box on;
 axis square; axis ([-pi pi -4 4]);

% integration and phase space plot up to tmax, plot
% phi-intervals [-pi pi],[pi 3pi],... on top of e.a.
 [T,Y]=ode45(@dgl,[0 tmax],[phi0 om0],op);
 for i=ceil(min(Y(:,1))/2./pi):ceil(max(Y(:,1)/2./pi))
   plot( Y(:,1) - i*2*pi, Y(:,2));
 end
 hold off;

% the Poincare section after initial integration
  if all ==0
   figure(2);  hold on; box on;
   axis([-pi pi -4 4]); axis square;

%  first period and dot plotted
   [T,Y] = ode45(@dgl,[0  pd],[phi0 om0],op);
   lst = length(Y(:,1));
   plot(mod(Y(lst,1)+pi,2*pi)-pi, Y(lst,2),'k.');

% further periods (pd) up to tmax
   for i=2: floor( tmax/pd )
    [T,Y]=ode45(@dgl,[(i-1)*pd i*pd],[Y(lst,1) Y(lst,2)],op);
    lst = length(Y(:,1));
    plot( mod( Y(lst,1)+pi,2*pi) - pi, Y(lst,2),'k.');
   end
   hold off;
  end
```


The driven, damped pendulum

three-dim. phase space, or explicit t-dep.:

$$\dot{\phi} = \omega$$
$$\dot{\omega} = -\sin\phi - r\,\omega + a\,\cos[\omega_D t]$$

consider in the following: $r = 0.25,\ \omega_D = 2/3,\ \phi(0) = \pi/2,\ \omega(0) = 0$

generic behavior for small driving force:
the driving can compensate the friction; after a more or less complex initial phase, the system approaches an *attractor*, i.e. a periodic state

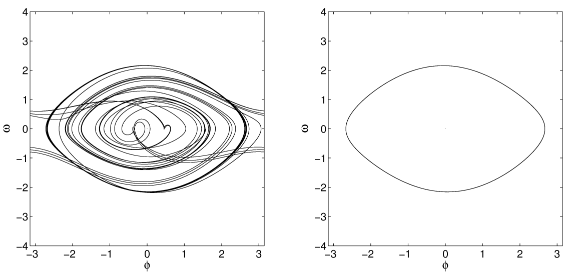


Fig. 5: *Phase space plot* ω *vs.* ϕ *for driving force amplitude* $a = 0.6$ *in the time interval* $[0, 1000]$ *including the initial phase (left) and for the stationary oscillation only (right).*

The Poincaré section

In order to visualize the behavior in three-dimensional phase space $(\phi, \omega = \dot{\phi}, \theta = \omega_D t)$, we make *stroboscopic* snapshots of the position in (ϕ, ω) after the initial phase $[0, t_{init}]$

at times $t_{init} + \frac{2\pi j}{\omega_D}$, $j = 0, 1, 2, \ldots$ (multiples of the period $T_D = 2\pi/\omega_D$),
i.e. for $\omega_D = 2/3$: $t_{init} + 0, 3\pi, 6\pi, \ldots$

Poincaré sections display the intersection of the three-dim. trajectories with equidistant planes of constant $\theta = \omega_D t$

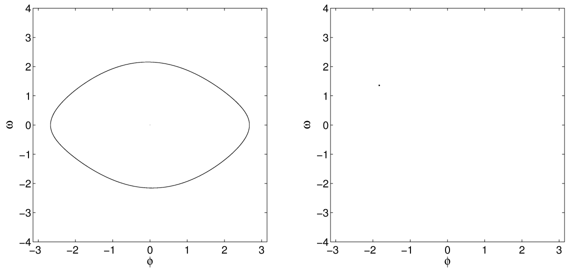


Fig. 6: *Stationary attractor in (ϕ, ω) for driving force amplitude $a = 0.6$ (left, same as above) and the corresponding Poincaré section (right), where the motion is represented by a single dot (exact position depends on t_{init}).*

For small amplitude a of the driving torque, we observe a simple periodic movement with frequency ω_D as imposed by the external force

More complex behavior

- periodic with $2\pi/\omega_D$, **pendulum turns over** once per cycle

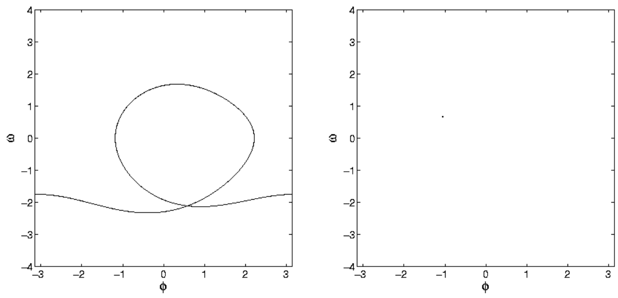


Fig. 7:
Attractor and Poincaré section for $a = 0.97$

- **periods** $\boldsymbol{n \cdot 2\pi/\omega_D}$, e.g. $n = 2$

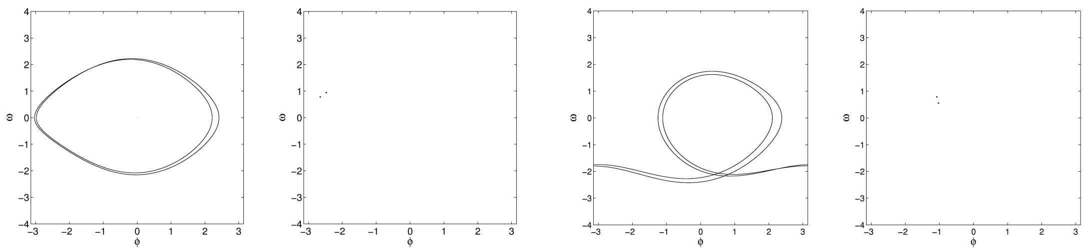


Fig. 8: *Attractor and Poincaré section for $a = 0.665$ (left) and $a = 0.98$ (right)*

- **multiple periods**, e.g. $n = 4$ and $n = 7$

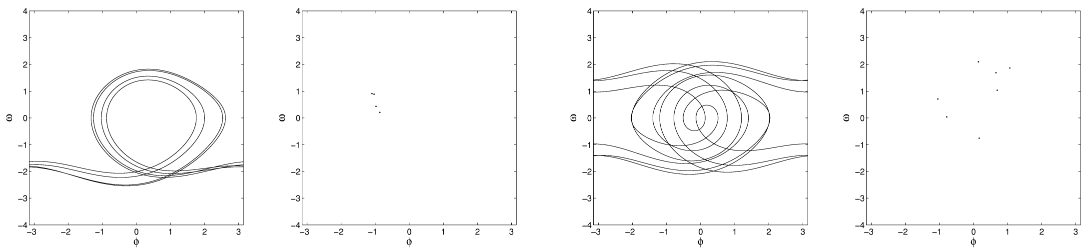


Fig. 9: *Attractor and Poincaré section for* $a = 1.0$ *(left) and* $a = 0.85$ *(right)*

Note: we can observe *period doubling*, e.g. for $a = 0.97, 0.98, 1.0$

- **Chaotic behavior**

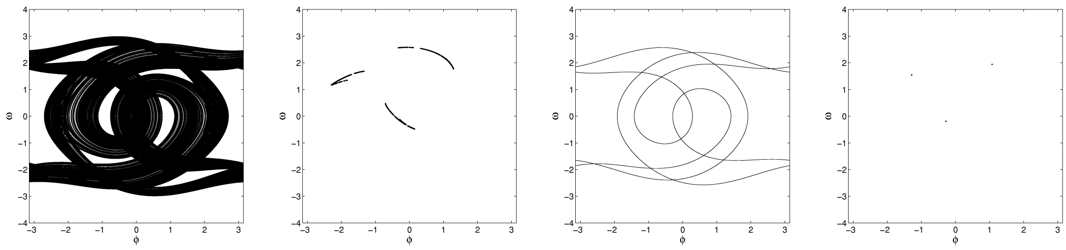


Fig. 10: *Chaotic behavior at* $a = 1.3$ *(left) displays three bands in the Poincaré section. It is reminiscent of the* $n = 3$ *periodic state for* $a = 1.2$ *(right)*

- **Strange attractors**

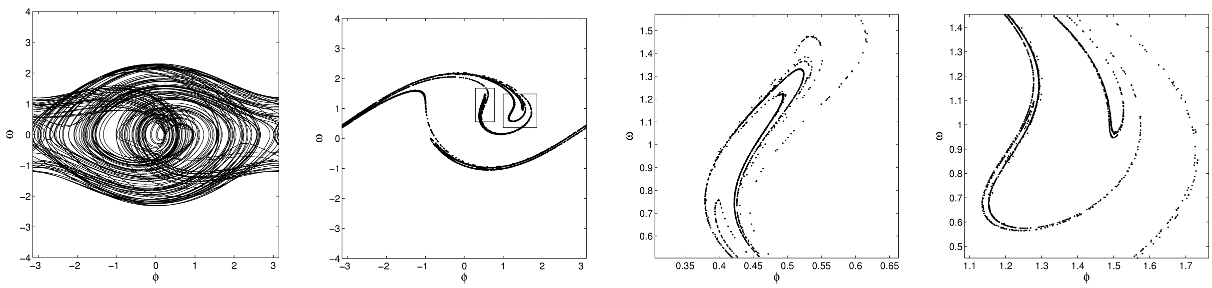


Fig. 11:
Left: *phase space plot for* $a = 0.7$*, here the integration was up to* $t_{max} = 1000$*, with increasing* t_{max} *the outer envelope curve is completely filled.*
Right: *corresponding Poincaré section and two different close-ups as marked by rectangles in the second plot. The resolution is limited by the integration time (here* $t_{max} = 100000$*).*

Strange attractors
– resemble a *puff pastry dough* with self–similar structures
– are **fractal** objects with dimension $1 < d_f < 2$ in the Poincaré section
more than a line, less than an area, see chapter 2 for more about fractals

The Poincaré section defines a discrete sequence of the form

$$\begin{pmatrix} \phi(t_j) \\ \omega(t_j) \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \phi(t_{j+1}) \\ \omega(t_{j+1}) \end{pmatrix} \; = \; \begin{pmatrix} \phi(t_j + 2\pi/\omega_D) \\ \omega(t_j + 2\pi/\omega_D) \end{pmatrix}$$

similar to the *Logistic Map* or the *Chirikov Map*. The iteration is, of course, more complicated and amounts to the numerical integration of the O.D.E.

The bifurcation diagram . . . is obtained by plotting the values of ϕ or ω that occur in the Poincaré section (after the initial phase) vs. the control parameter, e.g. for varying a with fixed r. It displays period doubling acc. to the Feigenbaum scenario as well as windows of periodic behavior and different routes to chaos.

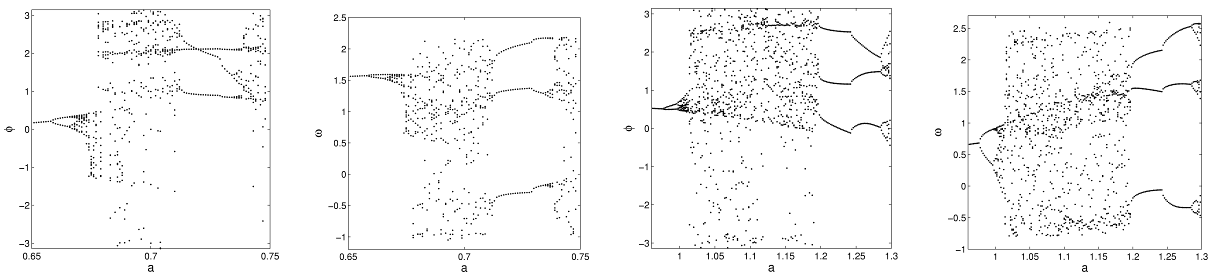


Fig. 12: *Portions of the bifurcation diagrams of ϕ (first, third) and ω (second, fourth) vs. a for $r = 0.25$. The quality of the plots is limited by the resolution in a (0.002) and the number of recorded values (100) after the initial phase.*

Suggestions and remarks

- As in the Frenkel-Kontorova model (see assignment 1), chaootic motion emerges from the competition of different time or length scales. Here:
  ω_o: frequency of the undisturbed oscillator
  ω_D: frequency of the external torque
  $1/r$: time scale introduced by the damping
- suggestion: study the cases with $\omega_D = \omega_o$ or ω_D/ω_o irrational
- other examples of chaotic systems:
  non-linear mechanical systems: double pendulum, *billiards*, . . .
  less simple systems: weather, stock market, ecological systems . . .
- Chaotic behavior is also relevant in quantum mechanical systems
- In principle, deterministic chaos is known since Poincaré. Computer simulations (models!) have made it feasible to study chaotic motion in detail and display it.
- Chaos (the *death of Laplace's demon*) is one of the foundations of Statistical Physics
- Additional literature (see also assignment 1)
  G.L. Baker, J.P. Gallub, *Chaotic Dynamics* , Cambridge Univ. Press (1996)
  H.-J. Jodl, H.J. Korsch, *Chaos: A Program Collection for PC*, Springer (1994)
  E. Ott, *Chaos in Dynamical Systems*, Cambridge University Press (1993)
  H.G. Schuster, *Deterministic Chaos: An Introduction*, VCH (1995)