Robotic Kinematics & Motion Planning Survey

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Abstract

A survey of sections 6.1-6.3 in [CLO07]. The ideas of modelling robotic arm movement algebraically are introduced, including the problems of finding the subset of $\mathbb{R}^2$ that the series of arms can reach (The Forward Kinematic problem) as well as given a point in $\mathbb{R}^2$, what are the possible configurations of the arms so that the last joint is at a specific point and orientation (The Inverse Kinematic problem). Grobner bases are the main way we solve these problems, while also looking into specialization, singularities, and motion planning.

1 Introduction

Consider a robot arm composed of joints and rigid segments connected in series with a fixed point at one end and a "hand" at the other. Let

$$\mathcal{J} := S^1 \times \cdots S^1 \times I_1 \times \cdots I_p$$

be the joint space of the robot where S^1 defines the angles of the revolute joints and I_j defines the extended lengths of the prismatic joints. Then the possible positions of the hand can be represented by the points (a, b) of a region $U \subset \mathbb{R}^2$ and the possible orientations of the hand are given by the vectors $\mathbf{u}$ in $V = S^1$. The configuration space of the robot's hand is defined as $C = U \times V$ and the mapping $f : \mathcal{J} \to C$ which determines the hand configuration from the different joint settings. The forward kinematic problem involves explicitly describing f in terms of the joint settings and the dimensions of the rigid segments of the robot's arm. The inverse kinematic problem deals with how to determine one or all of the $j \in \mathcal{J}$ given $c \in C$ such that $f(i) = c$.

Many different applications of kinematic motion have been looked at symbolically throughout the years starting as early as the Industrial Revolution in textbooks and is included in The Kinematics of Machinery [Reu76], but unfortunately the mathematical developments for solving such robotic arm questions were not advanced enough for the required techniques. The more modern approach and with it the modern notation was first brought forward in 1955 by Denavit, J. Hartenberg, R.S where matrix notation is established and the use of computers to solve these problems is now feasible. Grobner bases were introduced by Buchberger [Buc06], and first applied to this problem in a SIAM conference held in July 1987 in New York also by Buchberger.

These problems and methods play an important role in computer science and have many useful

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applications such as robotics, computer games, artificial intelligence, and animation which implements inverse kinematics for movement.

The subject area of kinematic motion planning is still active, with numerical approximations to solutions also being important, as sometimes a precise solution can not be found. Solving analytically using a Grobner basis becomes computationally expensive, but with Taylor approximations 'good enough' is possible. Typically no more than 6 or 7 DOF (degrees of freedom) are used, as this is sufficient for any kind of desired movement in 2D or 3D space.

The complexity of this problem comes down to efficient calculations of Grobner bases, of which there is an upper bound on degrees of $2(\frac{d^2}{2}+d)^{2^{n-1}}$ where in our case $d = n$ (shown later), but this is usually solved much more efficiently (as there are only products of variables of 1 or 2 degrees in IK). We could not find any strict bounds on the types of calculations done here.

High-level ideas of the chapter:

1. Model and formalize the problem of robotic arm movement.
2. Obtain system of equations to parameterize the final hand position.
3. Solve system of polynomial equations via Grobner basis.
4. Specialize the result to exact values (as opposed to their symbolic counterparts), and show when this maintains the Grobner properties.
5. Use the linear approximation of the parameterization to find kinematic singularities (bad solutions when viewed geometrically).
6. Go over a worked example to see it in action.

We will also be proving a couple textbook questions to gain a deeper understanding of the subject.

2 Preliminaries

In this section, we establish the notation which will be used throughout the paper and some important background which we shall need to prove our claims in the next sections.

We will be restricting ourselves to a plane (i.e in $\mathbb{R}^2$) so as all joints lay on the same plane. An arm A has a series of segments labelled $(S_1, ..., S_n)$ of lengths $(l_1, ..., l_n)$, some of which are variable length (if there is a prismatic joint) or are otherwise connected by a revolute joint (with no restrictions on rotation). We label each of the revolute joint rotations as $\{\theta_1, ..., \theta_n\}$. We will also assume that the joint settings are independent of each other. Note that in solving the polynomial system, we do not care about the specifics of l_i, so prismatic joints are not an issue until specialization and solving for specific cases.

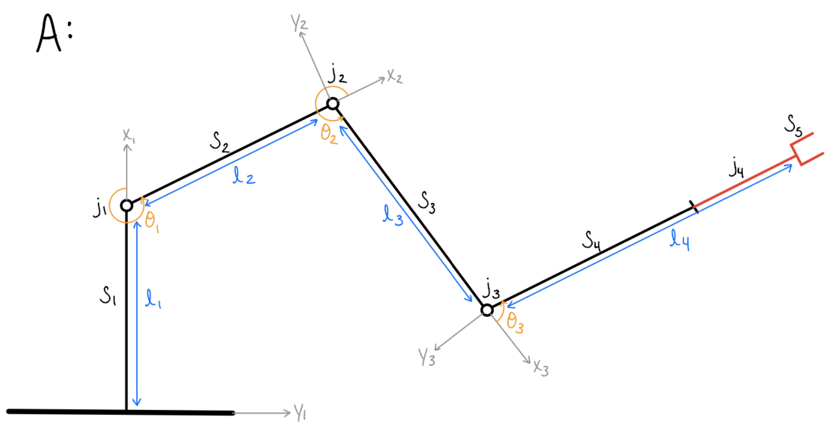
Next we define the space that we will be working over. If there are n revolute joints and p prismatic joints our input space can be parameterized as

$$\mathcal{J} := S^1 \times \cdots S^1 \times I_1 \times \cdots I_p$$

Where S^1 is the circle of radius one, or by an angle $\theta \in [0, 2\pi]$ and I_j is the settings of the j-th prismatic joint, and represents how far out a prismatic joint is extended, as well as any other static segments. We define $V \subset \mathbb{R}^2$ as the possible regions such that the last joint (also called a hand) may move to.

We may also be interested in modelling the hand of the arm, where the final segment has a 'hand' that is attached to some joint and also has an orientation. This can be parameterized as $U := S^1$. This combined with the above gives us the *configuration space* $C = U \times V$ of the robot's hand. For simplicity, we will often only be concerned with the final position of the hand and not the orientation. The reason for this is that if the final joint is a revolute joint, you can achieve any angle by rotating only that joint; if there is a prismatic joint attached to the hand, then you can often achieve angle α by setting $\theta_n = \alpha - \sum_{i=1}^{n-1}$ (note this may not always work, but for practical purposes and sake of simplicity, we are only concerned with position).

3 Technical Section



3.1 Forward Kinematics

For every revolute joint j_i, we introduce a local coordinate system x_{i+1}, y_{i+1} where the x_{i+1} axis is parallel to S_{i+1}. The natural question is how to go from the very last coordinate system to the global coordinate system (x_1, y_1). We see that we can apply the rotation matrix R_{θ_i} followed by a translation l_i to move from one to the other, which can be represented in a matrix as

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_i \\ b_i \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & l_i \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} := A_i \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix}$$

Repeating this procedure for all revolute joints, we have a way to go from one to the other as

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \prod_{i=1}^{n} A_i \begin{pmatrix} x_n \\ y_n \\ 1 \end{pmatrix}$$

It is worth noting that $x_n = y_n = 0$ if the last joint is a revolute joint, and $x_n = l_n, y_n = 0$ if it is a prismatic joint. However we would like to turn this into a polynomial system to be able to apply a Grobner basis, so we introduce the substitution

$$c_i = \cos\theta_i, \quad s_i = \sin\theta_i$$

and in order to get sensible results we must also have $c_i^2 + s_i^2 = 1$.
The final form of any system will look like:

$$f(\theta_1, ..., \theta_n, L) = \begin{pmatrix} \sum_{i=1}^{n-1} l_{i+1} \cos(\sum_{j=1}^{i} \theta_j) \\ \sum_{i=1}^{n-1} l_{i+1} \sin(\sum_{j=1}^{i} \theta_j) \\ \sum_{i=1}^{n} \theta_i \end{pmatrix} \tag{1}$$

Where the first two components are the x and y coordinate while the third is the rotation of the hand, and L is the set of variable length segments (i.e those with prismatic joints). This can be proved through a simple induction, using the sine/cosine summation rule in reverse to group attributes (omitted due to space).

3.2 Inverse Kinematics

Now given a point $(x, y) = (a, b)$ we would like to be able to find l_i and θ_j configurations that bring the hand to that point. The result from above leads us to the pair of formulas:

$$\begin{aligned} a &= \sum_{i=1}^{n-1} l_{i+1} \cos(\sum_{j=1}^{i} \theta_j) \\ b &= \sum_{i=1}^{n-1} l_{i+1} \sin(\sum_{j=1}^{i} \theta_j) \end{aligned}$$

But seeing as this is difficult to work with, we can instead replace our A_i as defined above with the substitution,

$$A_i = \begin{pmatrix} c_i & -s_i & l_i \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And work through the matrix multiplications to get equations for for a, b:

$$\begin{aligned} a =& \quad h_a(c_1, s_1, ..., c_n, s_n, l_1, ..., l_n) \\ b =& \quad h_b(c_1, s_1, ..., c_n, s_n, l_1, ..., l_n) \\ 1 =& \quad c_i^2 + s_i^2, \quad \forall i \in [n] \end{aligned}$$

Note that the closed form for the first two equations is not easily defined in a closed way, but can either be worked from equation (1) using sine and cosine or by direct matrix multiplication. Before we calculate the Grobner basis to solve the system, it is worth pointing out we are now in the ring $\mathbb{R}(a, b, l_2, ..., l_n)[s_1, ..., s_n, c_1, ..., c_n]$ and over a field of rational functions, where $a, b, l_1, ..., l_n$ are algebraically independent. This is an important distinction, as there are possible denominators that might lead to restrictions on some variables as shown later.

After calculating the Grobner basis, some natural questions/problems arise:

- When does this basis not have a solution? Notice we can have rational functions of a, b or any l_i, in particular in the denominator so we could have undefined behaviour. What does this look like geometrically?
- When do we have infinite/finite solutions?

Answering these questions will lead into kinematic singularities, as well as bring up interesting questions about finding bases that are specialization preserving. First, we look at some toy examples to have better intuition.

3.3 Geometric Toy Problems

We can get a bound on points that obviously do not have a solution geometrically. In particular any point (a, b) with distance $> \sum_{i=1}^{n} l_i$ can not be reached and will have no solution. We can also find a case where we have infinite solutions. Consider two prismatic joints with $l_1 = l_2 = 1$. Note by convention we have that these joints are oriented along the positive y-axis, so a point $(0, b), 0 \leq b \leq 1$ could potentially have infinite solutions. To see this, we simply set $l_1 = b + \delta$ and $l_2 = b + \delta$ where $\delta > 0$ leading to infinite solutions. Furthermore, we can learn about an upper bound when there are n revolute joints. We will get an equation with maximum degree n and will therefore have up to n possible solutions to our joint settings (once we add prismatic joints there will most likely be infinite solutions).

3.4 Specialization

When substituting in real values for our arm's length, we may get that our basis is no longer a Grobner basis. We explore which specializations are bad. It is also worth mentioning comprehensive Grobner bases that remain a Grobner basis under all specializations. We refer to [Wei92] for a proof of existence and construction of such bases (outside the scope of this survey).

We will however prove two questions from the textbook exercises, in particular 6.3.6 and 6.3.7 dealing with the conditions of when specialization is preserved and when it does not preserve the Grobner basis properties. To see how solving over this rational field can lead us to believe there is no solution, despite there actually being a Grobner basis for that specialization, we solve this example from the textbook:

Question 6: This exercise will explore a more subtle example of what can go wrong during a specialization. Consider the ideal $I = \langle x + ty, x + y \rangle \subset \mathbb{R}(t)[x, y]$, where t is a symbolic parameter. We will use lex order $x > y$.

a) Show that $\{x, y\}$ is a reduced Grobner basis of I.

Proof. We calculate the Syzygy as $S(x, y) = 0$ and so it is 0 when we perform division with remainders thus it is a Grobner basis. We see $LC(p) = 1$ for both, and no leading term lies in $LT(g) \setminus \{p\}$ for x and y so it is a reduced Grobner basis. $\square$

b) Let $t = 1$ and show that $\{x + y\}$ is a Grobner basis for the specialized ideal $\bar{I} \subset \mathbb{R}(t)[x, y]$.

Proof. We use the definition of the Grobner basis, and see that the leading monomials (x in this case), generate the ideal $LM(\bar{I}) = (LM(x+y)) = (x)$ which is true. $\square$

c) To see why $t = 1$ is special, express the Grobner basis $\{x, y\}$ in terms of the original basis $\{x + ty, x + y\}$.

Proof. We rewrite x and y as $x = f \cdot (x+y) + g \cdot (x+ty)$ for some f, g and likewise for y and solve:

$$x = \frac{t}{t-1}(x+y) + \frac{-1}{t-1}(x+ty), \quad y = \frac{1}{1-t}(x+y) - \frac{1}{1-t}(x+ty)$$

This becomes an issue because we can not specialize here, but we know that such a Grobner basis exists when $t = 1$. $\square$

We now look at conditions where the form of the basis does not change under specialization.

Question 7: Consider the ideal $I = \langle f_i(t_1, ..., t_m, x_1, ..., x_n) : 1 \leq i \leq s \rangle$ in $\mathbb{F}(t_1, ..., t_m)[x_1, ..., x_n]$ and fix a monomial order. By dividing each f_i by it's leading coefficients, we may assume $LC(f_i) = 1$. Then let $\{g_1, ..., g_t\}$ be a Grobner basis for I. Thus the leading coefficients of the g_i are also 1. Finally, let $(t_1, ..., t_m) \mapsto (a_1, ..., a_m) \in \mathbb{F}^m$ be a specialization of the parameters such that none of the denominators of the f_i or g_i vanish at $(a_1, ..., a_m)$. Let $\bar{h}$ represent any polynomial h that is specialized in this way.

a) If we use the division algorithm to find $A_{ij} \in \mathbb{F}(t_1, ..., t_m)[x_1, ..., x_n]$ such that $f_i = \sum_{j=1}^{t} A_{ij} g_j$ then show that none of the denominators of A_{ij} vanish at $(a_1, ..., a_m)$.

Proof. Assume that such a A_{ij} vanishes at $(a_1, ..., a_m)$, meaning the denominator is $h_{ij} \in \mathbb{F}(t_1, ..., t_m)$ where $h_{ij}(a_1, ..., a_m) = 0$. We know that f_i does not vanish at $(a_1, ..., a_m)$, which means that the denominator must be cancelled out in g_j. But notice $LC(g_j) = 1$ so there can not be a factor of h_{ij} in g_j, and note there can not be another g_i' that cancels g_j, as they are linearly independent and so f_i must vanish, a contradiction. $\square$

b) We also know that g_j can be written as $g_j = \sum_{i=1}^{s} B_{ji} f_i$ for some $B_{ij} \in \mathbb{F}(t_1, ..., t_m)[x_1, ..., x_n]$. As exercise 6 shows, none of the denominators of the B_{ij} vanish under specialization $(t_1, ..., t_m) \mapsto (a_1, ..., a_m)$. Let $\bar{I}$ denote the ideal in $\mathbb{F}[x_1, ..., x_n]$ generated by the specialized f_i. Under these assumptions, prove that the specialized g_j form a basis of $\bar{I}$.

Proof. Let $h \in \bar{I}$. Then $h = h_1 \bar{f}_1 + ... + h_s \bar{f}_s$. We use the above to get

$$h = h_1 \sum_{j=1}^{t} \bar{A_{1j}} \bar{g_j} + ... + h_s \sum_{j=1}^{t} \bar{A_{sj}} \bar{g_j} = (\sum_{i=1}^{s} \bar{h_1} \bar{A_{1i}}) \bar{g_1} + ... + (\sum_{i=1}^{s} \bar{h_s} \bar{A_{si}}) \bar{g_t}$$

This is valid because none of the A_{ij} (by part a) or g_j (by definition of question) have vanishing denominators, so it is a valid substitution. Therefore any $h \in \bar{I}$ is generated by $\langle \bar{g_1}, ..., \bar{g_t} \rangle$ and so it forms a basis. $\square$

c) Show that the specialized g_j form a Grobner basis for $\bar{I}$. Hint: the monomial order used to compute $\bar{I}$ only deals with terms in the variables x_j. The parameters t_j are constants as far as ordering is concerned.

Proof. We see that the fixed ordering does not change under specialization, so the leading monomials of g_i remain the same. We show that the $LM(\bar{I})$ is similar to $LM(I)$, in the way that if $h \in I$, we look at h under specialization $\bar{h}$, and examine $LT(\bar{h})$. If $LC(\bar{h})$ has no t terms in the denominator, then $LC(\bar{h}) \in \mathbb{R}$ and can still be generated by $LM(\bar{g_1}, ..., \bar{g_t})$. If there was at least one t term in the denominator, we see under specialization it is still defined, since $\{\bar{g_1}, ..., \bar{g_t}\}$ is a basis for $\bar{I}$ so $\bar{h} = \sum_{i=1}^{t} \alpha_i g_t$, and since none of the denominators of g_t are zero under specialization and $\alpha_i \in \mathbb{R}$, we have that $LC(\bar{h})$ has a nonzero value, and so is still defined and in $\mathbb{R}$. Therefore, $LM(\{\bar{g_1}, ..., \bar{g_t}\})$ generates $LM(\bar{I})$ and so they form a Grobner basis under specialization. □

d) Let $d_1, ..., d_M \in \mathbb{F}[t_1, ..., t_m]$ be all denominators that appear among f_i, g_j, and B_{ji}, and let $W = V(d_1 d_2 \cdots d_M) \subset \mathbb{F}^m$. Conclude that the g_j remain a Grobner basis for the f_i under all specializations $(t_1, ..., t_m) \mapsto (a_1, ..., a_m) \in \mathbb{F}^m - W$.

Proof. We know that $(a_1, ..., a_m) \in W$ if $d_1 \cdots d_M = 0$. Since we are in a field, this is true iff at least one of the d_i vanishes on $(a_1, ..., a_m)$. Therefore we conclude that any points that vanish on any denominators in f_i, g_j or B_{ji} are in W. Thus we use the results from above to say that since none of the denominators vanish on $\mathbb{F}^m - W$ then the g_j remain a Grobner basis under all specializations in $\mathbb{F}^m - W$. □

3.5 Kinematic Singularities

At some joint setting $j \in \mathcal{J}$ we can define the best linear approximation of the function f by using the Jacobian, defined as

$$(J_f(\theta_1, ..., \theta_r))_{ij} := \frac{\partial f_j}{\partial \theta_i}$$

Near j, f and $J_f(j)$ have roughly the same behaviour so $J_f(j)$ represents the derivative of the mapping f at $j \in \mathcal{J}$. It is also helpful to introduce the notion of dimensions of $\mathcal{J}$ and configuration space $\mathcal{C}$ for the robot. $dim(\mathcal{J})$ is the number of joints and $dim(\mathcal{C})$ is the number of independent degrees of freedom in the configuration, which in the 2D case is 3 (two for position, one for rotation of the hand). Another important idea to include is the rank of a matrix which is defined as the maximal number of linearly dependent rows or columns. The matrix $J_f(j)$ has maximal rank if it's rank is $min(m, n)$ (the largest possible value), otherwise $J_f(j)$ is said to have deficient rank. When $J_f(j)$ has deficient rank, it's kernel is larger and image is smaller thus indicating some singular behaviour of f near the point j. We now define a kinematic singularity.

Definition: A **kinematic singularity** for a robot is a point $j \in \mathcal{J}$ such that $J_f(j)$ has rank strictly less than $\min(dim(\mathcal{J}), dim(\mathcal{C}))$.

We can think of this as having some kind of 'singular' behaviour at these points. It intuitively makes sense that in our 2D case, whenever we have more than 3 revolute joints, we will have kinematic singularities.

Proposition: Let $f : \mathcal{J} \to \mathcal{C}$ be the configuration mapping for a planar robot with $n \geq 3$ revolute joints. Then there exist kinematic singularities $j \in \mathcal{J}$.

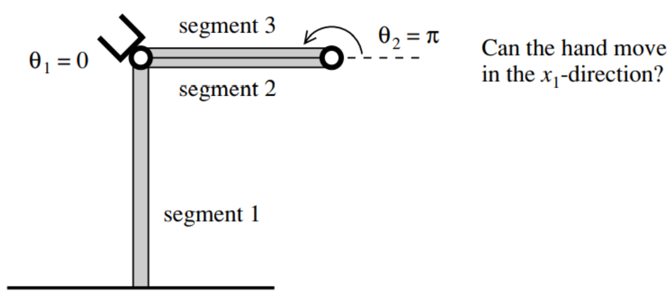
Proof. (for full proof, see textbook). The intuition here is that when all of our $\theta_i \in \{0, \pi\}$, we either have straight lines or it folds back in on itself, where our derivative is 0, as well as creating a column of all zeros in the Jacobian. □

3.6 Motion Planning

Another important section of robotics is motion planning and specifically in our case is finding a parameterized path $c(t) \in \mathcal{C}$ to move from one configuration to another, or to find a path $j(t) \in \mathcal{J}$ such that $f(j(t)) = c(t)$ for all t. There are three other restrictions on a path we would like to impose:

1. Closed paths, i.e the start and end of a path is the same, useful for machines doing repetitive tasks.
2. Limiting joint speeds so as to not damage the parts.
3. Minimize total joint movement to perform a motion.

Seeing as kinematic singularities describe the derivative of f, it can be thought of as the velocity of the joints in different dimensions. Suppose we have a configuration space path $c(t)$ where the joint space path $j(t)$ passes through a kinematic singularity. We can differentiate $c(t) = f(j(t))$ as $c'(t) = J_f(j_t) \cdot j'(t)$, where $c'(t)$ is now the velocity. If we have a singularity, this equation might not have any solution, meaning there are no smooth joint paths to move in certain directions.



At a Kinematic Singularity

In the above example (taken from the textbook), we see that it needs a very large joint space velocity to move it slightly. Further development on this topic is not explained in the book.

4 Example

It is informative to work through a full example using Macaulay2 and we will be doing the example as shown in the textbook, with two revolute joints (plans for doing textbook examples yielded either trivial solutions that could be solved by regular algebra, or unwieldy examples of 33 equations). Using the closed form of equation (1) and some sine/cosine laws, we get the following equations:

$$l_3(c_1c_2 - s_1s_2) + l_2c_1 = a$$

$$l_3(s_1c_2 + c_1s_2) + l_2s_1 = b$$

We will however be deviating from the textbook and solving when $l_2 \neq l_3$, as this produces more complex equations. In particular, we choose $l_2 = 2, l_3 = 1$.

```
K = frac(QQ[a, b, l2, l3]);
R = K[c2,s2,c1,s1, MonomialOrder=> Lex];
use(R);

eq1 = l3*(c1*c2 - s1*s2) + l2*c1-a;
eq2 = l3*(s1*c2 + c1*s2) + l2*s1-b;
-- Bounding our ci and si
eq3 = c1^2+s1^2-1;
eq4 = c2^2+s2^2-1;

I = ideal(eq1, eq2, eq3, eq4);
result = mutableMatrix( gens gb I)
for i from 0 to #result do (
    result_(0,i) = sub(result_(0,i),{l2=>2,l3=>1})
)
```

Using our work from earlier, we see that all specializations work except when $a = 0$ or $a^2 + b^2 = 1$. We calculate the Grobner basis as:

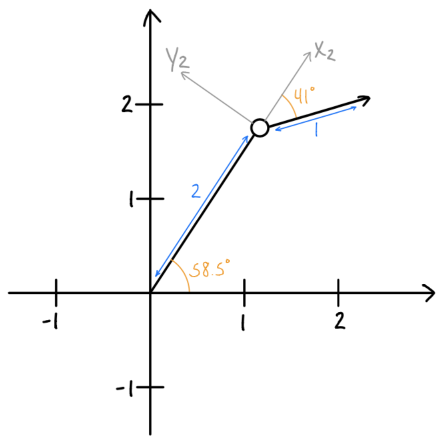
$$s_1^2 + \frac{-a^2b-b^3-3\,b}{2\,a^2+2\,b^2}\,s_1 + \frac{a^4+2\,a^2b^2+b^4-10\,a^2+6\,b^2+9}{16\,a^2+16\,b^2}, \qquad c_1 + \frac{b}{a}\,s_1 + \frac{-a^2-b^2-3}{4\,a}$$

$$s_2 + \frac{a^2+b^2}{a}\,s_1 + \frac{-a^2b-b^3-3\,b}{4\,a}, \qquad c_2 + \frac{-a^2-b^2+5}{4}$$

After rearranging we get the following:

$$\begin{aligned} c_2 &= \frac{a^2+b^2-5}{4} \\ s_2 &= \pm\sqrt{1-c_2^2} \\ s_1 &= \frac{a}{a^2+b^2}\left(-\frac{-a^2b-b^3-3b}{4a} - s_2\right) \\ c_1 &= \pm\sqrt{1-s_1^2} \end{aligned} \tag{2}$$

Lastly we need to check when we get real solutions. The first two equations give $1 \leq a^2 + b^2 \leq 9$, which is what we would expect for a maximum and minimum range of the arm. We test by giving it the point $(2, 2)$ to get $c_2 = \frac{3}{4}$, $s_2 = \pm\frac{\sqrt{7}}{4}$, $s_1 = \frac{1}{16}(-11 \pm \sqrt{7})$ which is enough to solve for $\theta_2 = 41.4, -41.4$ and $\theta_1 = -31.5, -58.5$. Note that when drawing we had to take the negative of these angles to get our desired position in regular space (some phase correction might be needed if coordinate systems are chosen differently).



5 Conclusion and Open Problems

As we have explored, symbolic computation is a powerful tool used to model real world and mathematical problems. It was used to solve the forward kinematic problem, and Grobner bases were used to find solutions to the inverse kinematic problem. Topics from calculus such as the Jacobian matrix are important for real world robotics, helping us find 'problem spots' called singularities. Lastly, parameterization techniques can be used for a single variable equation for motion planning. The notions introduced here to solving polynomial systems can be applied in a more general setting for geometric reasoning and theorem proving. Automatic theorem proving for non-geometric spaces is possible, as shown in [CEI00] for proofs of unsatisfiability in computer science.

An interesting open problem we came across is the involvement of Neural Networks in the inverse kinematic problem (Mahajan, Singh, and Sukavanam 2017) as well as some surprising applications in physics with proton scattering [PSS$^+$21]. The study of efficiently selecting the monomial order of Grobner bases for non-redundant open-chain robots is also an important topic further discussed in inverse kinematics [GGFAR21].

Acknowledgments

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