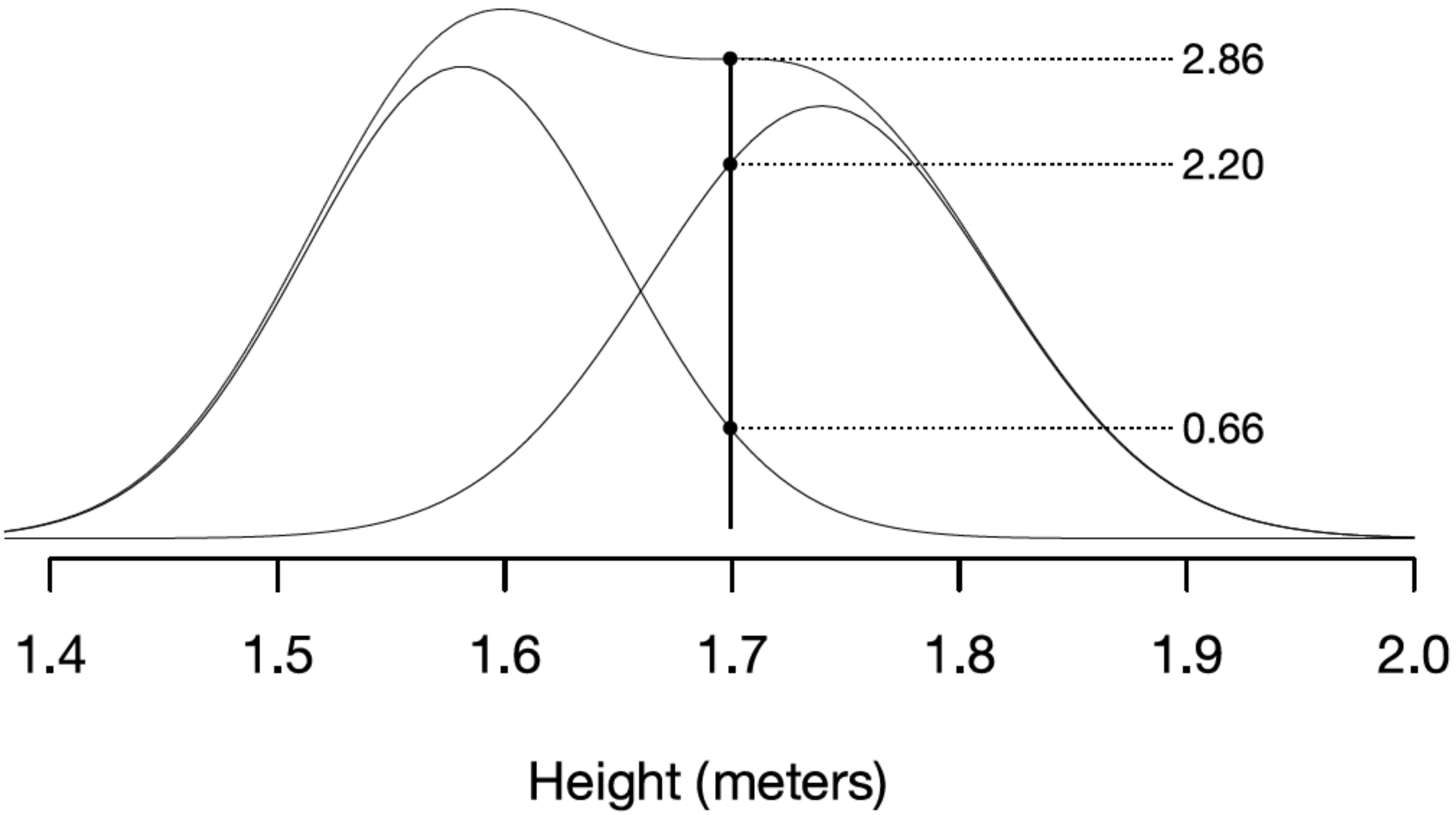
Latent class analysis

Bayes rule

DL Oberski & L Boeschoten


2.86
2.20
0.66
1.4
1.5
1.6
1.7
1.8
1.9
2.0
Height (meters)

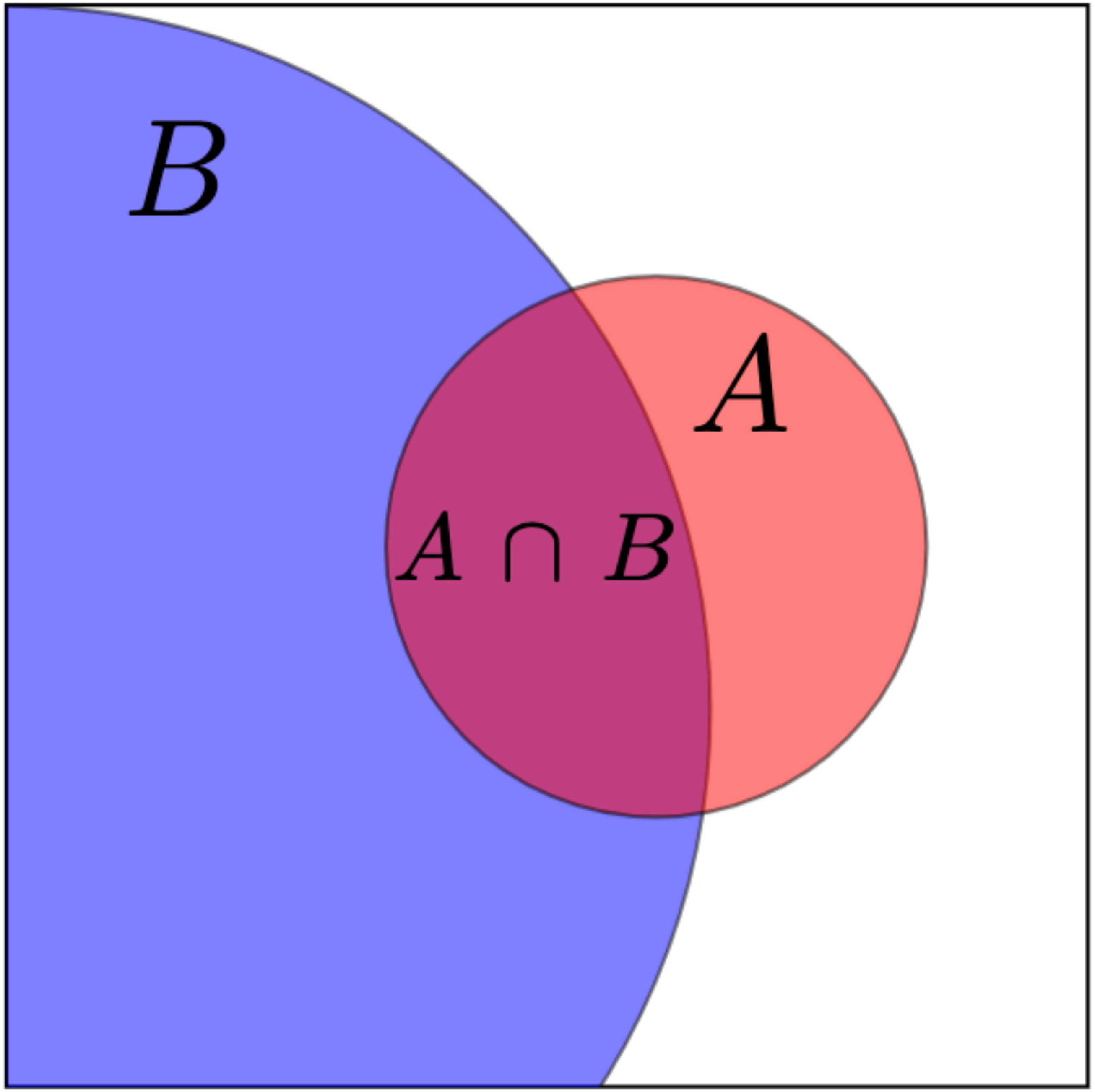
Learning goals

- Apply Bayes rule to calculate posterior probabilities in a simple setting

Question

“Steve is very shy and withdrawn, invariably helpful, but with little interest in people, or in the world of reality. A meek and tidy soul, he has a need for order and structure and a passion for detail.”

- What is the probability that Steve is a **librarian**?
- What is the probability that Steve is a **farmer**?



Source: MIT

Bayes' rule

$$\overbrace{P(X|\,Y)}^{\textit{Posterior}} = \frac{\overbrace{P(Y|X)}^{\textit{Likelihood}}\;\overbrace{P(X)}^{\textit{Prior}}}{\underbrace{P(Y)}_{\textit{Marginal}}}$$

Derivation of Bayes' Theorem

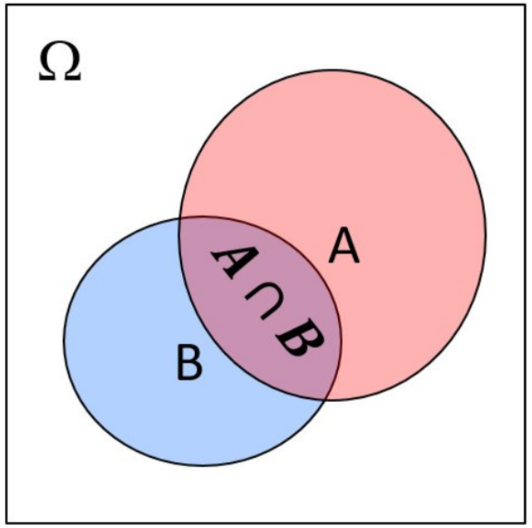
Michael Pyrcz, University of Texas at Austin (Twitter @GeostatsGuy)

Bayes' Theorem is central to Bayesian Statistics. It allows for: (1) the updating prior probability distribution with a likelihood function based on new information, and (2) the calculation of the conditional probability, P(A|B), given another calculated conditional probability, P(B|A). Did you know that is it derived from basic probability logic?

Rule of Multiplication:

$$P(B \cap A) = P(A|B)\,P(B)$$

$$P(A \cap B) = P(B|A)\,P(A)$$



Venn Diagram - Marginal and joint probability

It follows that:

$$P(B \cap A) = P(A \cap B)$$

Therefore when we substitute we get:

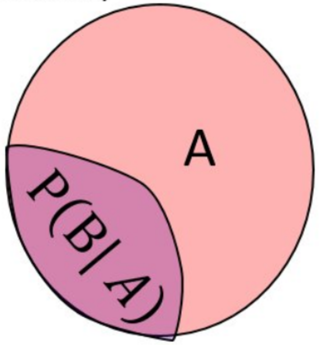
$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

Now we have Bayes' Theorem!

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

The terms are known as:

$$\text{Posterior} = \frac{\text{Likelihood x Prior}}{\text{Evidence}}$$



Venn Diagram - Marginal and conditional probability

Bayes' rule

$$P(X = x \mid Y = y) = \frac{P(Y = y \mid X = x)\, P(X = x)}{P(Y = y)}$$


RAB 4-8L/OB
MI 1.2
Fetal Medicine Centre
fmc53522 GA=12w6d
11.8cm / 20Hz
TIs 0.1
04.11.2008
03:29:22 PM
1. Trim.
Har-high
Pwr 100 %
Gn -7
C9 / M7
P2 / E3
SRI II 2
fetus2
fetus1

Example

$$P(X = x \mid Y = y) = \frac{P(Y = y \mid X = x)\, P(X = x)}{P(Y = y)}$$

- X : type of twin (x_0 means identical, x_1 means fraternal)
- $Y = y$: both children are girls
- Of all twins, 25% (say) are identical
- If the ultrasound shows two girls, what is the probability the twins are identical?

$$P(Y) = P(Y = y|X = x_0)\, P(X = x_0) + \cdots + P(Y = y|X = x_K)\, P(X = x_K)$$

Bayes' rule

$$P(X = x_0 \mid Y = y)$$
$$= \frac{P(Y = y|X = x_0)\, P(X = x_0)}{P(Y = y|X = x_0)\, P(X = x_0) + \cdots + P(Y = y|X = x_K)\, P(X = x_K)}$$

Bayes' rule

$$P(X|\,Y) = \frac{P(Y|X)\;P(X)}{\sum_X P(Y|X)P(X)}$$

Question

Of the one million rabbits on the GESIS campus most are good-natured. But one hundred of them are pure evil! An enterprising student develops an "Evil Rabbit Alarm" which she offers to sell to GESIS. GESIS decides to test the reliability of the alarm by conducting trials.

Source: MIT

- When presented with an evil rabbit, the alarm goes off 99% of the time.
- When presented with a good-natured rabbit, the alarm goes off 1% of the time.

(a) If a rabbit sets off the alarm, what is the probability that it is evil?

(b) Should GESIS co-opt the patent rights and employ the system?

Source: MIT

One solution
(This is a base rate fallacy problem)

We are given:

$$P(\text{nice}) = 0.9999, \qquad P(\text{evil}) = 0.0001 \text{ (base rate)}$$

$$P(\text{alarm} \mid \text{nice}) = 0.01, \quad P(\text{alarm} \mid \text{evil}) = 0.99$$

$$P(\text{evil} \mid \text{alarm}) = \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm})}$$

$$= \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm} \mid \text{evil}) P(\text{evil}) + P(\text{alarm} \mid \text{nice}) P(\text{nice})}$$

$$= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)}$$

$$\approx 0.01$$

Source: MIT

Bayes rule for LCA with covariates z

$$P(x \mid \mathbf{Y}_i = \mathbf{y}, \mathbf{z}_i) = \frac{P(x \mid \mathbf{z}_i) \prod_{t=1}^{T} P(Y_{it} = y_t \mid x, \mathbf{z}_i)}{\sum_{x=1}^{K} P(x \mid \mathbf{z}_i) \prod_{t=1}^{T} P(Y_{it} = y_t \mid x, \mathbf{z}_i)}$$