Final Definitions and Theorems

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$\mathbb{R}^n = \{\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}, x_1, \dots x_n \in \mathbb{R}$

Two vectors $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix}$

Are said to be equal if $x_i = y_i \; for \; i \in [1, n]$

Addition:

$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \dots \\ x_n + y_n \end{bmatrix}$

Scalar multiplication:

$c\vec{x} = \begin{bmatrix} cx_1 \\ \dots \\ cx_n \end{bmatrix}$

Linear combination

For $\overrightarrow{v_1}, \dots, \overrightarrow{v_k} \in \mathbb{R}^n \; and \; c_1, \dots, c_k \in \mathbb{R}$

We call

$c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \dots + c_k\overrightarrow{v_k} \; a \; linear \; combination \; in \; \mathbb{R}^n.$

Theorem (1.1.1)

$if \; \vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n \; and \; c, d \in \mathbb{R}, then$

1. $\vec{x} + \vec{y} \in \mathbb{R}^n$
2. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
4. $\exists \vec{0} \in \mathbb{R}^n, \forall \vec{x} \in \mathbb{R}^n, such \; that \; \vec{x} + \vec{0} = \vec{x}$
5. $for \; every \; \vec{x} \in \mathbb{R}^n, \exists(-\vec{x}) \in \mathbb{R}^n, such \; that \; \vec{x} + (-\vec{x}) = \vec{0}$
6. $c\vec{x} \in \mathbb{R}^n$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
9. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
10. $1\vec{x} = \vec{x}$

Recall for $\overrightarrow{v_1}, \dots, \overrightarrow{v_k} \in \mathbb{R}^n$ and $c_1, \dots, c_k \in \mathbb{R}$ we call $c_1\overrightarrow{v_1} + \dots + c_k\overrightarrow{v_k}$ is a <u>linear combination</u> of $\overrightarrow{v_1}, \dots, \overrightarrow{v_k}$.

Span

Let B $= \{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ be a set of vectors in $\mathbb{R}^n$. We define Span B $= \{c_1\overrightarrow{v_1} + \dots + c_k\overrightarrow{v_k} | c_1, \dots, c_k \in \mathbb{R}\}$.

- B is a <u>spanning set</u> for Span B
- The set Span B is <u>spanned</u> by B

Theorem (1.2.1)

Let $\overrightarrow{v_1}, \dots, \overrightarrow{v_k} \in \mathbb{R}^n$. Some vector $\overrightarrow{v_i}, 1 \leq I \ \leq k$, can be writtn as a linear combination of $\overrightarrow{v_1}, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \overrightarrow{v_k}$ if and only if
$Span\ \{\vec{v_1}, \dots, \vec{v_k}\} = Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \overrightarrow{v_k}\}$.

<u>Simpler proof:</u> Prove $\exists c_1, \dots, c_{k-1} \in \mathbb{R}$ such that $\overrightarrow{v_k} = \sum_{i-1}^{k-1} c_i \overrightarrow{v_i}$ if and only if
$Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\} = Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_{k-1}}\}$

Linearly (In)dependent

A set $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ in $\mathbb{R}^n$ is called

- Linearly dependent if $\exists c_1, \dots, c_k \in \mathbb{R}$, not all zero, such that $\vec{0} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$.
- Linearly independent if the only solution to $\vec{0} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$ is $c_1 = \cdots = c_k = 0$ ("trivial solution")

Theorem 1.2.2

A set of vectors $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ in $\mathbb{R}^n$ is linearly dependent if and only if $\overrightarrow{v_i} \in$ Span $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \dots, \overrightarrow{v_k}\}$ for some $i, 1 \leq i \leq k$.

Theorem 1.2.3

If a set of vectors contains the zero vector, then it is linearly dependent.

Basis

Let S be a subset of $\mathbb{R}^n$.
If $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ such that $S = Span\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$, then the set $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is called a <u>basis</u> for S.
We define a basis for the set $\{0\}$ to be the empty set.

Standard Basis

In $\mathbb{R}^n$, let $\vec{e_i}$ represent the vector whose i-th component is 1 and all other component are 0. The set $\{\overrightarrow{e_1}, \dots, \overrightarrow{e_n}\}$ is called the <u>standard basis</u> for $\mathbb{R}^n$.
Eg: the standard basis for $\mathbb{R}^3$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Theorem 1.2.4

If $\beta = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is a basis for a subset S of $\mathbb{R}^n$, then every $\vec{x} \in$ S can be written as a unique linear combination of vectors in β.

Subspace

A subset $\mathbb{S}$ of $\mathbb{R}^n$ is called a <u>subspace</u> of $\mathbb{R}^n$ if for every $\vec{x}, \vec{y}, \vec{w}, \in \mathbb{S}$ and $c, d \in \mathbb{R}$, we have

1. $\vec{x} + \vec{y} \in \mathbb{S}$
2. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
4. $\exists \vec{0} \in \mathbb{S}, \forall \vec{x} \in \mathbb{S}, such\ that\ \vec{x} + \vec{0} = \vec{x}$
5. $\forall \vec{x} \in \mathbb{S}, \exists(-\vec{x}) \in \mathbb{S}, such\ that\ \vec{x} + (-\vec{x}) = \vec{0}$
6. $c\vec{x} \in \mathbb{S}$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$

9. $c(\vec{x}+\vec{y}) = c\vec{x} + c\vec{y}$
10. $1\vec{x} = \vec{x}$

Theorem 1.3.3 (Subspace Test)
Let $\mathbb{S}$ be a non-empty subset of $\mathbb{R}^n$.
If $\vec{x}+\vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$, then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$.

Theorem 1.3.2
If $\overrightarrow{v_1}, \dots, \overrightarrow{v_k} \in \mathbb{R}^n$, then $\mathbb{S} = Span\,\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is a subspce of $\mathbb{R}^n$.
This theorem says if a subset $\mathbb{S}$ of $\mathbb{R}^n$ has a basis, then $\mathbb{S}$ must be a subspace of $\mathbb{R}^n$.
To find a basis of a $\mathbb{S}$,
- Find a general form of a vector in $\mathbb{S}$
- Use it to find a spanning set of $\mathbb{S}$
- Use theorem 1.2.1 until we get a linearly independent spanning set for $\mathbb{S}$.

Dot product
For $\vec{x} = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \cdots \\ y_n \end{bmatrix} \in \mathbb{R}^n,\ \vec{x}\cdot\vec{y} = \sum_{i=1}^{n} x_i y_i$
If two vectors are orthogonal, then their dot product is 0.

Theorem 1.4.1
If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and θ is an angle between $\vec{x}$ and $\vec{y}$, then $\vec{x}\cdot\vec{y} = \left||\vec{x}|\right|\left||\vec{y}|\right| \cos\theta$.

Theorem 1.4.2
If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, then
1. $\vec{x}\cdot\vec{x} \geq 0$, and $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = 0$
2. $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$
3. $\vec{x}\cdot(s\vec{y}+t\vec{z}) = s(\vec{x}\cdot\vec{y}) + t(\vec{x}\cdot\vec{z})$

Length
Let $\vec{x} \in \mathbb{R}^{\mathrm{n}}$.
The <u>length</u> or <u>norm</u> of $\vec{x}$ is define to be $\left||\vec{x}|\right| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$

Unit Vector
A vector $\vec{x} \in \mathbb{R}^{\mathrm{n}}$ such that $\left||\vec{x}|\right| = 1$ is called a unit vector

Theorem 1.4.3
If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then
1. $\left||\vec{x}|\right| \geq 0$, and $\left||\vec{x}|\right| = 0$ iff $\vec{x} = \vec{0}$
2. $\left||c\vec{x}|\right| = |c|\,\left||\vec{x}|\right|$
3. $\vec{x}\cdot\vec{y} \leq \left||\vec{x}|\right|\,\left||\vec{y}|\right|$
4. $\left||\vec{x}+\vec{y}|\right| \leq \left||\vec{x}|\right| + \left||\vec{y}|\right|$

Orthogonal Set
A set of vectors $\{\overrightarrow{v_1}, \dots \overrightarrow{v_k}\} \in \mathbb{R}^n$ is called an <u>orthogonal set</u> if $\overrightarrow{v_i}\cdot\overrightarrow{v_j} = 0$ for all $i \neq j$.

Theorem 1.4.4
The zero vector $\vec{0} = \mathbb{R}^n$ is orthogonal to every vector $\vec{x} \in \mathbb{R}^n$.

Cross Product
If $\vec{v}, \vec{w} \in \mathbb{R}^3$.

The cross product of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is defined to be

$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$.

Theorem 1.4.5 (Properties of the Cross Product)
Let $\vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

1) If $\vec{n} = \vec{v} \times \vec{w}$, then for any $\vec{y} \in \text{Span}\,\{\vec{v}, \vec{w}\}$, we have $\vec{y} \cdot \vec{n} = 0$
2) $\vec{v} \times \vec{w} = (-\vec{w}) \times \vec{v}$
3) $\vec{v} \times \vec{v} = \vec{0}$
4) $\vec{v} \times \vec{w} = \vec{0}$ iff $\vec{v} = \vec{0}$ or $\vec{w}$ is a scalar multiple of $\vec{v}$
5) $\vec{v} \times (\vec{w} + \vec{x}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{x})$
6) $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$
7) $||\vec{v} \times \vec{w}|| = ||\vec{v}||\,||\vec{w}||\,|\sin\theta|$

Theorem 1.4.6
Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$ with $\{\vec{v}, \vec{w}\}$ being linearly independent and let P be a plane in $\mathbb{R}^3$ with vector eqution $\vec{x} = s\vec{v} + t\vec{w} + \vec{b}$, $s, t \in \mathbb{R}$. If $\vec{n} = \vec{v} \times \vec{w}$, then an equation for the plane is $(\vec{x} - \vec{b})\vec{n} = 0$

Normal Vector, Scalar Equation
Let P be a plane in $\mathbb{R}^3$ through the point $B(b_1, b_2, b_3)$.

If $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3$ is a vector such that

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = b_1 n_1 + b_2 n_2 + b_3 n_3$$

Is an equation for P, then $\vec{n}$ is called for <u>normal vector</u> for P.
We call this equation a <u>scalar equation</u> of P.

Projection onto a Line in $\mathbb{R}^n$
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$. The <u>projection</u> of $\vec{u}$ onto $\vec{v}$ is defined by

$$proj_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\,\vec{v}$$

Perpendicular of a Projection in $\mathbb{R}^n$
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$. The <u>perpendicular</u> of $\vec{u}$ onto $\vec{v}$ is defined by

$$perp_{\vec{v}}(\vec{u}) = \vec{u} - proj_{\vec{v}}(\vec{u})$$

Projection, Perpendicular onto a Plane in $\mathbb{R}^3$

Let P be a plane in $\mathbb{R}^3$ that passes through the origin and has normal vector $\vec{n}$.

- The projection of $x \in \mathbb{R}^3$ onto P is defined by $proj_P(\vec{x}) = perp_{\vec{n}}(\vec{x})$
- The perpendicular of $x \in \mathbb{R}^3$ onto P is defined by $perp_P(\vec{x}) = proj_{\vec{n}}(\vec{x})$

Linear Equation

An equation with n variables (unknowns) $x_1, x_2, \dots, x_n$ that can be written in the form $a_1x_1 + \cdots + a_nx_n = b$ where $a_1, \dots, a_n, b$ are constants is called a linear equation. The constant a_i are called the coefficients of th equation.

System of Linear Equations

A set of m linear equations in the same variables $x_1, \dots, x_n$ is called a system of m linear equations in n variables.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{21}x_n &= b_2 \\
&\dots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Note a_{ij} is the coefficient of x_j in the i-th equation.

Solution of a System

A vector $\vec{s} = \begin{bmatrix} s_1 \\ \dots \\ s_n \end{bmatrix} \in \mathbb{R}^n$ is called a solution of a system of m linear equations in n variables if all m equations are satisfied when we set $x_i = s_i$ for $1 \leq i \leq n$. The set of all solutions of a system of linear equations is called the solution set of the system.

Consistent, Inconsistent

A system of linear equations

- With at least one solution is called consistent
- With no solutions is called inconsistent

Theorem 2.1.1

If the system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{21}x_n &= b_2 \\
&\dots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Has two distinct solutions $\vec{s} = \begin{bmatrix} s_1 \\ \dots \\ s_n \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} t_1 \\ \dots \\ t_n \end{bmatrix}$, then $\vec{y} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

Solution sets of a system of m linear equations in n variables must either be empty, contain exactly one vector, or have infinitely many vectors in it.

Coefficient Matrix and Augmented Matrix

The system of equations

$$2x_1 + 3x_2 = 17$$
$$7x_1 - 2x_2 = 22$$

Has coefficient matrix

$$A = \begin{bmatrix} 2 & 3 \\ 7 & -2 \end{bmatrix}$$

And augmented matrix

$$A = \left[\begin{array}{cc|c} 2 & 3 & 17 \\ 7 & -2 & 22 \end{array}\right]$$

Elementary Row Operations

The three elementary row operations are

1. Multiplying a row by a non-zero scalar (eg. $3R_2$)
2. Adding a multiple of one row to another (eg. $3R_2 + R_1$)
3. Swapping two rows (eg. $R_2 \leftrightarrow R_3$)

Row Equivalent

Matrices A, B are called row equivalent, writte $A \sim B$, if there exists a sequence of elementary row operations that transform A into B

Theorem 2.2.1

If the augmented matrices $\left[A_1 \middle| \overrightarrow{b_1}\right]$ and $\left[A \middle| \vec{b}\right]$ are row equivalent, then the systems of linear equations associated with each system are equialent.
Goal: solve a system by using EROs to find a row equivalent matrix for which we can identify the solution.

Reduced Row Echelon Form

A matrix R is said to be in reduced row echelon form (RREF) if:

1. Any row containing only zeros are at the bottom
2. The first non-zero entry in each non-zero row is 1, called a leading one
3. The leading one in each non-zero row is to the right of the leading one in any row above it
4. A leading one is the only non-zero entry in its column

If A is row equivalent to matrix R in RREF, then we say that R is the RREF of A.

Theorem 2.2.2

If A is a matrix, then A has a unique reduced row echelon form R.

Algorithm 11.1 Gauss-Jordan to reduce matrix to RREF

1. Use EROs to get a leading one in the top of the first non-zero column.
2. Use the ERO "add a multiple of one row to another" to make all entries beneath the leading one into a 0.
3. Consider the submatrix consisting of columns to the right of the column just modified and the rows beneath the row that just got a leading one. Use EROs to get a leading one in the top left of this submatrix.
4. Use the ERO "add a multiple of one row to another" to make all other entries in the column (for the whole matrix) containing the new leadings one into a 0.
5. Repeat steps 3 and 4 until the matrix is in RREF.

Free Variable

Let R be the RREF of a coefficient matrix for a system of linear equations. If the j-th column of R does not contain a leading one, then we call x_j a free variable.

Homogeneous System

A system of linear equations is called a homogeneous system if the right hand side only contains zeros That is, has the form $[A|\vec{0}]$.
We omit writing the 0 column from a homogeneous system since all EROs give 0 in this column.

Thm 2.2.3

The solution set of a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$, and is called the solution space of the system.

Rank

The rank of a matrix A, denoted rank A, is the number of leading ones in the RREF of the matrix.

Theorem 2.2.4

For any $m \times n$ matrix A we have rank $A \le \min(m, n)$

Theorem 2.2.5 (System Rank)

Let A be the coefficient matrix of a system of m linear equations in n unknowns $[A|\vec{b}]$.

1. The rank of A is less than the rank of the augmented matrix $[A|\vec{b}]$ if and only if the system is inconsistent.
2. If the system $[A|\vec{b}]$ is consistent, the number of free variables is (n −rank A).
3. The system $[A|\vec{b}]$ is consistent for every $\vec{b} \in \mathbb{R}^n$ if and only if rank $A = m$.

Theorem 2.2.6 (Solution Thm)

Let $[A|\vec{b}]$ be a consistent system of m linear equations in n variables. If rank $A = k < n$, then a vector equation of the solution set of $[A|\vec{b}]$ has the form

$$\vec{x} = \vec{d} + t_1\overrightarrow{v_1} + \cdots + t_{n-k}\overrightarrow{v_{n-k}}, \qquad t_1, \dots, t_{n-k} \in \mathbb{R}$$

Where $\vec{d} \in \mathbb{R}$ and $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_{n-k}}\}$ is a linearly independent set in $\mathbb{R}^n$.
The solution set of $[A|\vec{b}]$ is an $(n-k)$-flat in $\mathbb{R}^n$.

Theorem 2.2.7

A set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if it spans $\mathbb{R}^n$.

Matrix, $\mathrm{M}_{\mathrm{m}\times\mathrm{n}}(\mathbb{R})$

An $m \times n$ matrix A is a rectangular array

- A has m rows and n columns
- $a_{ij} = (A)_{ij}$ is the entry in the i-th row and j-th column
- Two $m \times n$ matrices A and B are equal if $a_{ij} = b_{ij}$ for all $1 \le i \le m, 1 \le j \le n$.
- The set of all $m \times n$ matrices with real entries is denoted $M_{m\times n}(\mathbb{R})$

Addition, Scalar Multiplication

Let $A, B \in M_{m\times n}(\mathbb{R})$ and $c \in \mathbb{R}$. We define $(A + B)$ and cA by

$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$

$(cA)_{ij} = c(A)_{ij}$

Theorem 3.1.1

If $A, B, C \in M_{m\times n}(\mathbb{R})$ and $c, d \in \mathbb{R}$ then

1. $A + B \in M_{m\times n}(\mathbb{R})$
2. $(A + B) + C = A + (B + C)$
3. $A + B = B + A$
4. There exists a matrix $O_{m,n} \in M_{m\times n}(\mathbb{R})$ such that $A + O_{m,n} = A$ for all A.
   $O_{m,n}$ is the zero matrix, the $m \times n$ matrix with all entries 0.
5. For every $A \in M_{m\times n}(\mathbb{R})$ there exists $(-A) \in M_{m\times n}(\mathbb{R})$ such that $A + (-A) = O_{m,n}$
6. $cA \in M_{m\times n}(\mathbb{R})$
7. $c(dA) = (cd)A$
8. $(c + d)A = cA + dA$
9. $c(A + B) = cA + cB$
10. $1A = A$

Transpose

The <u>transpose</u> of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose ij-th entry is the ji-th entry of A. That is,

$(A^T)_{ij} = (A)_{ji}$

Theorem 3.1.2

If $A, B \in M_{m\times n}(\mathbb{R})$ and $c \in \mathbb{R}$ then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = c(A^T)$

Matrix Vector Multiplication

Let A be an $m \times n$ matrix whose rows are denoted $\vec{a}_i^T$ for $1 \le i \le m$. For any $\vec{x} \in \mathbb{R}^{\mathrm{n}}$, we define $A\vec{x} = \begin{bmatrix} \overrightarrow{a_1} \cdot \vec{x} \\ \dots \\ \overrightarrow{a_2} \cdot \vec{x} \end{bmatrix}$.

Note: if A is an $m \times n$ matrix, then $A\vec{x}$ is only defined if $\vec{x} \in \mathbb{R}^{\mathrm{n}}$.

If $\vec{x} \in \mathbb{R}^{\mathrm{n}}$, then $A\vec{x} \in \mathbb{R}^{\mathrm{m}}$

Matrix times Vector, Coefficient format

Let $A = [\overrightarrow{a_1}, \dots, \overrightarrow{a_n}]$ be an $m \times n$ matrix. For any $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

we define $A\vec{x} = x_1\overrightarrow{a_1} + \dots + x_n\overrightarrow{a_n}$.

Theorem 3.1.3

If $\overrightarrow{e_i}$ is the i-th standard basis vector and $A = [\overrightarrow{a_1}, \dots, \overrightarrow{a_n}]$, then $A\overrightarrow{e_i} = \overrightarrow{a_i}$.

Theorem 3.1.4

If $A \in M_{m\times n}(\mathbb{R}), \vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

2. $c(A\vec{x}) = (cA)\vec{x} = A(c\vec{x})$
3. If $\vec{x}, \vec{y} \in \mathbb{R}^n$ then
$$\vec{x}^T\vec{y} = [x_1 \quad \dots \quad x_n]\begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix} = [x_1y_1 + \dots + x_ny_n] = \vec{x} \cdot \vec{y}$$
4. We represent a system of linear equations
$[A|\vec{b}]$ as $A\vec{x} = \vec{b}$

Matrix Multiplication

For an $m \times n$ matrix A and an $n \times p$ matrix $B = [\overrightarrow{b_1} \dots \overrightarrow{b_2}]$ we define AB to be the $m \times p$ matrix $AB = A[\overrightarrow{b_1} \dots \overrightarrow{b_2}] = [A\overrightarrow{b_1} \dots A\overrightarrow{b_2}]$.
That is $(AB)_{ij} = \overrightarrow{a_i} \cdot \overrightarrow{b_j}$.

If $A = B$ then $DA = DB$ and $AD = BD$.
Do not assume $AD = DB$ or $DA = BD$.

Do not assume that if $AC = BC$ then $A = B$, that is do not cancel out the C.

Theorem 3.1.5

If A, B, C are matrices of the correct size so that the required products are defined, and $t \in \mathbb{R}$, then
1. $A(B + C) = AB + AC$
2. $t(AB) = (tA)B = A(tB)$
3. $(AB)^T = B^TA^T$

Matrices Equal (Theorem 3.1.6)

If A, B are $m \times n$ matrices such that $A\vec{x} = B\vec{x}$ for every $x \in \mathbb{R}$, then $A = B$.

Theorem 3.1.7

If $I = [\overrightarrow{e_1} \dots \overrightarrow{e_n}]$, then for any $n \times n$ matrix A we have $AI = A = IA$.

Theorem 3.1.8

The multiplicative identity for $M_{n\times n}(\mathbb{R})$ is unique.

Identity Matrix

The $n \times n$ identity matrix denoted I or I_n is the matrix such that $(I)_{jj} = 1$, for $1 \leq j \leq n$
And $(I)_{ij} = 0$ whenever $i \neq j$
Equivalently, $I = [\overrightarrow{e_1} \quad \dots \quad \overrightarrow{e_n}]$.

If A is an $m \times n$ matrix, we can define a function $f: \mathbb{R}^n \to \mathbb{R}^m$ by $f(\vec{x}) = A\vec{x}$, called a **matrix mapping**. Sometimes we write $\vec{x}$ as a row vector.

Linear Mappings

A function $L: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear mapping if it has the property that
$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$
For every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$.

Notes
1. Linear transformation and linear mapping mean the same thing
2. A linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ is sometimes called a linear operator

3. If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then
$$L(t_1\vec{v_1} + \cdots + t_k\vec{v_k}) = t_1L(\vec{v_1}) + \cdots + t_kL(\vec{v_k})$$
For all $\vec{v_i} \in \mathbb{R}$ and $t_1, \ldots, t_k \in \mathbb{R}$.

Theorem 3.2.2
If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then L can be represented as a matrix mapping with the corresponding $m \times n$ matrix $[L]$ given by
$$[L] = [L(\vec{e_1}) \quad L(\vec{e_2}) \quad \ldots \quad L(\vec{e_k})]$$

Standard Matrix
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The matrix $[L] = [L(\vec{e_1}) \quad L(\vec{e_2}) \quad \ldots \quad L(\vec{e_k})]$ is called the standard matrix of L and has the property that $L(\vec{x}) = [L]\vec{x}$

Rotations in $\mathbb{R}^2$
Let $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ denote the function that rotates a vector $\vec{x} \in \mathbb{R}^2$ about the origin through an angle θ.
The standard matrix of R_θ is
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Theorem 3.2.3
If $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation with rotation matrix $A = [R_\theta]$, then the columns of A are orthogonal unit vectors.

Reflections
Let $\mathrm{refl}_P: \mathbb{R}^n \to \mathbb{R}^n$ denote the mapping that sends a vector $\vec{x} \in \mathbb{R}^n$ to its mirror image in the hyperplane P with normal vectors $\vec{n}$.
The reflection is given by
$$\mathrm{refl}_P(\vec{x}) = \vec{x} - 2\mathrm{proj}_{\vec{n}}(\vec{x}).$$

Range
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The range of L is defined by
$$\mathrm{Range}(L) = \{L(\vec{x}) \in \mathbb{R}^m | x \in \mathbb{R}^n\}$$

Lemma (Theorem 3.3.1)
If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $L(\vec{0}) = \vec{0}$.

Theorem 3.3.2
If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\mathrm{Range}(L)$ is a subspace of $\mathbb{R}^m$.

Kernel
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. The kernel (nullspace) of L is the set of all vectors in the domain which are mapped to the zero vector in the codomain.
That is,
$$\mathrm{Ker}(L) = \{\vec{x} \in \mathbb{R}^n \big| L(\vec{x}) = \vec{0}\}.$$

Theorem 3.3.3
If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\text{Ker}(L)$ is a subspace of $\mathbb{R}^n$.

Theorem 3.3.4
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping with standard matrix $[L]$. Then $\vec{x} \in \text{Ker}(L)$ if and only if $[L]\vec{x} = \vec{0}$.

Corollary 3.3.5
Let $A \in M_{m \times n}(\mathbb{R})$. The set $\{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.

Nullspace
Let A be an $m \times n$ matrix. The nullspace (kernel) of A is defined by

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$$

Theorem 3.3.6
If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping with standard matrix $[L] = A = [\vec{a_1} \quad \dots \quad \vec{a_n}]$, then $\text{Range}(L) = Span\,\{\vec{a_1}, \dots, \vec{a_n}\}$.

Column Space
Let $A = [\vec{a_1} \quad \dots \quad \vec{a_n}]$.
The column space of A is the subspace of R^m defined by

$$\text{Col}(A) = Span\,\{\vec{a_1}, \dots, \vec{a_n}\} = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}.$$

Row Space
Let A be an $m \times n$ matrix. The row space of A is the subspace of $\mathbb{R}^n$ defined by

$$\text{Row}(A) = \{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}.$$

Left Nullspace
Let A be an $m \times n$ matrix. The left nullspace of A is the subspace of $\mathbb{R}^m$ defined by

$$\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}.$$

Note: $A^T\vec{x} = \vec{0} \Leftrightarrow \vec{x}^T A = \vec{0}$.

Fundamental Subspaces
For any $m \times n$ matrix A, we call the nullspace, column space, row space, and left nullspace the four fundamental subspaces of A.

Addition and Scalar Multiplication
Let $L: \mathbb{R}^n \to \mathbb{R}^m$ and $M: \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. We define $L + M: \mathbb{R}^n \to \mathbb{R}^m$ and $cL: \mathbb{R}^n \to \mathbb{R}^m$ by

$$(L + M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$$

And

$$(cL)(\vec{x}) = cL(\vec{x}).$$

Theorem 3.4.1
If $L, M: \mathbb{R}^n \to \mathbb{R}^m$ are linear mappings and $c \in \mathbb{R}$, then $L + M: \mathbb{R}^n \to \mathbb{R}^m$ and $cL: \mathbb{R}^n \to \mathbb{R}^m$ are linear

mappings.
Moreover, we have

$$[L+M]=[L]+[m] \text{ and } [cL]=c[L].$$

Composition

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ and $M: \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings. The composition of M and L is the function $(M \circ L): \mathbb{R}^n \to \mathbb{R}^p$ defined by

$$(M \circ L)(\vec{x}) = M(L(\vec{x}))$$

Theorem 3.4.3

If $L: \mathbb{R}^n \to \mathbb{R}^m$ and $M: \mathbb{R}^m \to \mathbb{R}^p$ are linear mappings, then $(M \circ L): \mathbb{R}^n \to \mathbb{R}^p$ is a linear mapping and

$$[M \circ L] = [M][L].$$

Vector Space

A set $\mathbb{V}$ with an operation of addition denoted $\vec{x}+\vec{y}$ and an operation of scalar multiplication denoted $c\vec{x}$ is called a vector space over $\mathbb{R}$ if for every $\vec{v}, \vec{x}, \vec{y} \in \mathbb{V}$ and $c, d \in \mathbb{R}$,

1. $\vec{x}+\vec{y} \in \mathbb{V}$
2. $(\vec{x}+\vec{y})+\vec{v} = \vec{x}+(\vec{y}+\vec{v})$
3. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
4. $\exists \vec{0} \in \mathbb{V}, \forall \vec{x} \in \mathbb{V}, such\ that\ \vec{x}+\vec{0}=\vec{x}$
5. $\forall \vec{x} \in \mathbb{V}, \exists (-\vec{x}) \in \mathbb{V}, such\ that\ \vec{x}+(-\vec{x})=\vec{0}$
6. $c\vec{x} \in \mathbb{V}$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. $(c+d)\vec{x} = c\vec{x}+d\vec{x}$
9. $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$
10. $1\vec{x} = \vec{x}$

We call elements of $\mathbb{V}$ vectors (even if they are matrices or polynomials).

Theorem 4.1.1

If $\mathbb{V}$ is a vector space, then

1. $0\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{V}$
2. $(-\vec{x}) = (-1)\vec{x}$ for all $\vec{x} \in \mathbb{V}$

<u>Proof:</u>

1. $0\vec{x} = 0\vec{x}+\vec{0}$ $(V4)$
   $0\vec{x} = 0\vec{x}+[\vec{x}+(-\vec{x})]$ $(V5)$
   $0\vec{x} = 0\vec{x}+[1\vec{x}+(-\vec{x})]$ $(V10)$
   $0\vec{x} = [0\vec{x}+1\vec{x}]+(-\vec{x})$ $(V2)$
   $0\vec{x} = (0+1)\vec{x}+(-\vec{x})$ $(V8)$
   $0\vec{x} = 1\vec{x}+(-\vec{x})$
   $0\vec{x} = \vec{x}+(-\vec{x})$ $(V10)$
   $0\vec{x} = \vec{0}$ $(V5)$

Subspace of $\mathbb{V}$

Let $\mathbb{V}$ be a vector space. If $\mathbb{S}$ is a subset of $\mathbb{V}$ and $\mathbb{S}$ is a vector space under the same operations as $\mathbb{V}$, then $\mathbb{S}$ is called a <u>subspace</u> of $\mathbb{V}$.

Subspace Test (Theorem 4.1.2)

If $\mathbb{S}$ is a non-empty subset of $\mathbb{V}$ such that $\vec{x}+\vec{y} \in \mathbb{S}$ and $c\vec{x} \in \mathbb{S}$ for all $\vec{x}, \vec{y} \in \mathbb{S}$ and $c \in \mathbb{R}$ under the operations of $\mathbb{V}$, then $\mathbb{S}$ is a subspace of $\mathbb{V}$.

Span

Let $B = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_k}\}$ be a set of "vectors" in a vector space $\mathbb{V}$. We define the span of B by
$Span\ B = \{c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k} | c_1, \dots, c_k \in \mathbb{R}\}$

Theorem 4.1.3

If $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is a set of "vectors" in a vector space $\mathbb{V}$, then $Span\ B$ is a subspace of $\mathbb{V}$.

Theorem 4.1.4

Let $\mathbb{V}$ be a vector space and $\overrightarrow{v_1}, \dots, \overrightarrow{v_k} \in \mathbb{V}$.
Then $\overrightarrow{v_i} \in Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \dots, \overrightarrow{v_k}\}$ if and only if
$Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\} = Span\ \{\overrightarrow{v_1}, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \dots, \overrightarrow{v_k}\}$.

Linearly (In)dependent

A set of vectors $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ in a vector space $\mathbb{V}$ is said to be

- Linearly dependent if $\exists c_1, \dots, c_k$ not all zero such that
  $\vec{0} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$
- Linearly independent if the only solution to
  $\vec{0} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}$ is the trivial solution $c_1 = \cdots = c_k = 0$.

Theorem 4.1.5

A set of vectors $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\} \in \mathbb{V}$ is linealy dependent if and only if $\overrightarrow{v_i} \in Span\ \{v_1, \dots, \overrightarrow{v_{i-1}}, \overrightarrow{v_{i+1}}, \dots, \overrightarrow{v_k}\}$
For some i, $1 \leq i \leq k$.

Theorem 4.1.6

Any set of vectors in a vector space $\mathbb{V}$ which contains the zero vector is linearly dependent.

Basis

Let $\mathbb{V}$ be a vector space.

- The set B is called a basis for $\mathbb{V}$ if B is a linearly independent spanning set for $\mathbb{V}$.
- We define a basis for $\{\vec{0}_{\mathbb{V}}\}$ to be the empty set.

Theorem 4.2.1

Let $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ be a basis for a vector space $\mathbb{V}$ and let $C = \{\overrightarrow{w_1}, \dots, \overrightarrow{w_k}\}$ be a set in $\mathbb{V}$. If $k > n$ then C is linearly depedent.

Theorem 4.2.2

If $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ and $C = \{\overrightarrow{w_1}, \dots, \overrightarrow{w_k}\}$ are bases for a vector space $\mathbb{V}$ then $k = n$.

Dimension

If $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ is a basis for a vector space $\mathbb{V}$, then $\dim \mathbb{V} = n$.

- If $\mathbb{V}$ is the trivial vector space, then $\dim \mathbb{V} = 0$.
- If $\mathbb{V}$ does not have a basis with a finite number of vectors in it, then $\mathbb{V}$ is said to be infinite dimensional.

Theorem 4.2.3

If $\mathbb{V}$ is an n-dimensional vector space with $n > 0$ then

1. A set of more than n vectors in $\mathbb{V}$ must be linearly dependent
2. A set of fewer than n vectors in $\mathbb{V}$ cannot span $\mathbb{V}$.
3. A set of n vectors in $\mathbb{V}$ is linearly independent if and only if it spans $\mathbb{V}$.

Theorem 4.2.4
If $\mathbb{V}$ is an n-dimensional vector space and $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}\}$ is a linearly independent set in $\mathbb{V}$ with $k < n$, then there exists $\overrightarrow{w_{k+1}}, \dots, \overrightarrow{w_n}$ in $\mathbb{V}$ such that $\{\overrightarrow{v_1}, \dots, \overrightarrow{v_k}, \overrightarrow{w_{k+1}}, \dots, \overrightarrow{w_n}\}$ is a basis for $\mathbb{V}$.

Corollary 4.2.5
If $\mathbb{S}$ is a subspace of a finite dimensional vector space $\mathbb{V}$, then $\dim \mathbb{S} \leq \dim \mathbb{V}$.

Theorem 4.3.1
If $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ is a basis for a vector space $\mathbb{V}$, then every $\vec{v} \in \mathbb{V}$ can be written as a unique linear combination of the vectors in B.

B-coordinates
Let $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ be a basis for a vector space $\mathbb{V}$. If $\vec{v} = b_1\overrightarrow{v_1} + \cdots + b_n\overrightarrow{v_n}$, then $b_1, \dots, b_n$ are called the B-coordinates of $\vec{v}$, and we define the B-coordinate vector by

$$[\vec{v}]_B = \begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix}.$$

Theorem 4.3.2
If $\mathbb{V}$ is a vector space with basis $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$, then for any $\vec{v}, \vec{w} \in \mathbb{V}$ and $s, t \in \mathbb{R}$ we have $[s\vec{v} + t\vec{w}]_B = s[\vec{v}]_B + t[\vec{w}]_B$.

Change of Coordinates Matrix
Let $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ and C be bases for vector space $\mathbb{V}$. The change of coordinates matrix from B-coordinates to C-coordinates is defined by ${}_CP_B = [[\overrightarrow{v_1}]_C \quad \dots \quad [\overrightarrow{v_n}]_C]$.
For any $\vec{x} \in \mathbb{V}$ we have $[\vec{x}]_C = {}_CP_B[\vec{x}]_B$

Theorem 4.3.3
If B and C are bases for an n-dimensional vector space $\mathbb{V}$, then the change of coordinate matrices satisfy
${}_CP_B\,{}_BP_C = I = {}_BP_C\,{}_CP_B$

Left and Right Inverse
Let A be an $m \times n$ matrix.
- If B is an $n \times m$ matrix such that $AB = I_m$, then B is called a **right inverse** of A.
- If C is an $n \times m$ matrix such that $CA = I_n$, then C is called a **left inverse** of A.

Theorem 5.1.1
If A is an $m \times n$ matrix with $m > n$, then A cannot have a right inverse.

Corollary 5.1.2
If A is an $m \times n$ matrix with $m < n$, then A cannot have a left inverse.

Square Matrix

An $n \times n$ matrix is called a **square matrix**.

Theorem 5.1.3
If A, B, C are $n \times n$ matrices such that $AB = I = CA$, then $B = C$
Proof: $B = IB = (CA)B = C(AB) = CI = C$

Matrix Inverse, Invertible
Let A be an $n \times n$ matrix. If B is a matrix such that $AB = I = BA$, then B is called the inverse of A. We write $B = A^{-1}$ and we say that A is **invertible**.

Theorem 5.1.4
If A and B are $n \times n$ matrices such that $AB = I$, then A and B are invertible and $rank\ A = rank\ B = n$.

Theorem 5.1.5
If A and B are invertible matrices and $c \in \mathbb{R}$ with $c \neq 0$, then:
1. $(cA)^{-1} = \frac{1}{c}A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 5.1.6
If A is an $n \times n$ matrix such that $rank\ A = n$, then A is invertible.

Inverse of $M_{2\times 2}(\mathbb{R})$
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$
- If A is invertible then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Invertible Matrix Theorem
For any $n \times n$ matrix A, the following are equivalent:
1. A is invertible
2. The RREF of A is I
3. $rank\ A = n$
4. For all $\vec{b} \in \mathbb{R}^n$, the system of equtions $A\vec{x} = \vec{b}$ is consistent with a unique solution.
5. The nullspace of A is $\{\vec{0}\}$.
6. The columns of A form a basis for $\mathbb{R}^n$.
7. The rows of A form a basis for $\mathbb{R}^n$.
8. A^T is invertible.

If A is invertible, the solution to $A\vec{x} = \vec{b}$ comes from

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

Elementary Matrix
An $n \times n$ matrix E is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix by performing exactly one elementary row operation.

Theorem 5.2.1
If E is an elementary matrix, then E is invertible and E^{-1} is also an elementary matrix.

Theorem 5.2.2
If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix corresponding to the row operation $R_i + cR_j$
for $i \neq j$, then EA is the matrix obtained from A by performing the row operation $R_i + cR_j$ on A.

Theorem 5.2.3
If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix for the row operation cR_i, then EA is the matrix obtained from A by performing the row operation cR_i on A.

Theorem 5.2.4
If A is an $m \times n$ matrix and E is the $m \times m$ elementary matrix for the row operation $R_i \leftrightarrow R_j$, for $i \neq j$, then EA is the matrix obtained from A by performaing the row operation $R_i \leftrightarrow R_j$ on A.

Corollary 5.2.5
If A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix, then $rank\ EA = rank\ A$.

Theorem 5.2.6
If A is an $m \times n$ matrix with reduced row echelon form R, then there exists a sequence $E_1, \ldots E_k$ of $m \times m$ elementary matrices such that $E_k \ldots E_2 E_1 A = R$. In particular, $A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} R$.

Theorem 5.2.7
If A is an $n \times n$ invertible matrix, then A and A^{-1} can be written as a product of elementary matrices.

2×2 Determinant
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
The determinant of A is
$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Cofactor
Let A be an $n \times n$ matrix with $n \geq 2$. let $A(i, j)$ be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column. The cofactor of a_{ij} is
$C_{ij} = (-1)^{i+j} \det A(i, j)$.

n × n Determinant
Let A be an $n \times n$ matrix with $n \geq 2$. The determinant of A is

$$\det A = \sum_{i=1}^{n} a_{i1} C_{i1}$$

Where the determinant of a 1×1 matrix is defined by $\det[c] = c$.

Theorem 5.3.1
Let A be an $n \times n$ matrix. For any i with $1 \leq i \leq n$,

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik}$$
(the cofactor expansion across the i-th row),
OR for any j with $1 \leq j \leq n$,
$$\det A = \sum_{k=1}^{n} a_{kj} C_{kj}$$
(the cofactor expansion across the j-th column).

Upper Triangular, Lower Triangular

An $m \times n$ matrix U is said to be upper triangular if $u_{ij} = 0$ whenever $i > j$.
An $m \times n$ matrix L is said to be lower triangular if $l_{ij} = 0$ whenever $i < j$.

Theorem 5.3.2

If an $n \times n$ matrix A is upper triangular or lower triangular, then $\det A = a_{11} a_{22} \dots a_{nn}$.

Theorem 5.3.3

If A is an $n \times n$ matrix and B is the matrix obtained from A by multiplying one row of A by $c \in \mathbb{R}$, then $\det B = c \det A$.

Theorem 5.3.4

If A is an $n \times n$ matrix and B is the matrix obtained from A by swapping two rows of A, then $\det B = -\det A$

Corollary 5.3.5

If an $n \times n$ matrix A has two identical rows, then $\det A = 0$

Theorem 5.3.6

If A is an $n \times n$ matrix and B is the matrix obtained from A by adding a multiple of one row of A to another row, then $\det B = \det A$.

Corollary 5.3.7

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then
$\det EA = \det E \det A$

Addition to the Invertible Matrix Theorem (5.3.8)

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$

Theorem 5.3.9

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.
Proof: by theorem 5.2.6, there exists a sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that $A = E_1 E_2 \dots E_k R$ where R is the RREF of A.

- If $\det A \neq 0$:
  Then A is invertible, and $R = I$. Using Corollary 5.3.7,
  $\det(AB) = \det(E_1 \dots E_k B) = \det(E_1 \dots E_k) \det B = \det A \det B$
- If $\det A = 0$:
  $R \neq I$ and R contains at least one row of zeros.
  $\det(AB) = \det(E_1 \dots E_k RB) = \det E_1 \dots \det E_k \det(RB)$
  Since R contains a row of zeros, so does RB.
  Thus $\det(RB) = 0$ and $\det(AB) = 0 = 0 \det B = \det A \det B$
  ■

Corollary 5.3.10
If A is an invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

Theorem 5.3.11
If A is an $n \times n$ matrix, then $\det A = \det A^T$.
Since $\det A = \det A^T$, we can do elementary column operations as well as EROS when evaluating a determinant.

Lemma 5.4.1
If A is an $n \times n$ matrix with cofactors C_{ij} and $i \neq j$, then

$$\sum_{k=1}^{n} (A)_{ik} C_{jk} = 0$$

Proof:

Let B be the matrix obtained from A by replacing the j-th row of A by the i-th row of A
That is,

$$b_{lk} = a_{lk} \text{ for all } l \neq j$$

And $b_{jk} = a_{ik}$.
Then B has two identical rows, thus $\det B = 0$.
Since the cofactors of the j-th row of the B are the same as the cofactors of the j-th row of A.

$$0 = \det B = \sum_{k=1}^{n} b_{jk} C_{jk} = \sum_{k=1}^{n} a_{ik} C_{jk}$$

Theorem 5.4.2
If A is an invertible $n \times n$ matrix, then

$$(A^{-1})_{ij} = \frac{1}{\det A} C_{ji}$$

Proof:

Let the cofactors of a_{ij} be C_{ij}.
Let B be the $n \times n$ matrix defined by $(B)_{ij} = \frac{1}{\det A} C_{ji}$
Then for $1 \leq j \leq n$,

$$(AB)_{ii} = \sum_{k=1}^{n} (A)_{ik} \cdot (B)_{ki} = \frac{1}{\det A} \sum_{k=1}^{n} (A)_{ik} C_{ik} = \frac{1}{\det A} \det A = 1$$

For $(AB)_{ij}$ with $i \neq j$ we have

$$(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik} \cdot (B)_{kj} = \frac{1}{\det A} \sum_{k=1}^{n} (A)_{ik} C_{jk} = 0 \qquad \text{(Lemma 5.4.1)}$$

Thus $AB = I$ so $B = A^{-1}$

Cofactor Matrix of A
Let A be an $n \times n$ matrix. Then $(\operatorname{cof} A)_{ij} = C_{ij}$.

Adjugate of A
Let A be an $n \times n$ matrix. Then $(\operatorname{adj} A)_{ij} = C_{ji}$.
In particular, $\operatorname{adj} A = (\operatorname{cof} A)^T$.

Note: $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

Cramer's Rule (Theorem 5.4.3)

If A is an $n \times n$ invertible matrix, then the solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{\det A_i}{\det A}, \qquad 1 \leq i \leq n$$

Where A_i is the matrix obtained from A by replacing the i-th column of A by $\vec{b}$.

Formula

The area of the parallelogram induced by $\vec{u}$ and $\vec{v}$ is

$$Area(\vec{u}, \vec{v}) = |\det[\vec{u} \quad \vec{v}]|$$

Formula

The volume of a parallelepiped induced by $\vec{u}, \vec{v}, \vec{w}$ is

$$Volume(\vec{u}, \vec{v}, \vec{w}) = \left| \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \right|$$

B-matrix

Let $B = \{\overrightarrow{v_1}, \dots, \overrightarrow{v_n}\}$ be a basis for $\mathbb{R}^n$ and let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator.
The B-matrix of L is defined to be $[L]_B = [[L(\overrightarrow{v_1})]_B \quad \dots \quad [L(\overrightarrow{v_n})]_B]$.
If satisfies $[L(\vec{x})]_B = [L]_B[\vec{x}]_B$.

Diagonal Matrix

An $n \times n$ matrix D is said to be a diagonal matrix if $d_{ij} = 0$ for all $i \neq j$.
We denote a diagonal matrix by $\operatorname{diag}(d_{11}, d_{22}, \dots, d_{nn})$.

Theorem 6.1.1

If A and B are $n \times n$ matrices such that $P^{-1}AP = B$ for some invertible matrix P, then

1. $\operatorname{rank} A = \operatorname{rank} B$
2. $\det A = \det B$
3. $\operatorname{tr} A = \operatorname{tr} B$ where $\operatorname{tr} A$ is defined by
$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

Similar matrices

If A and B are $n \times n$ matrices such that $P^{-1}AP = B$ for some invertible matrix P, then A is said to be similar to B.

Eigenvalues, Eigenvectors, Eigenpair

Let A be an $n \times n$ matrix. If there exists a vector $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda\vec{v}$,
then λ is called an eigenvalue of A and $\vec{v}$ is called an eigenvector of A corresponding to λ.
The pair $(\lambda, \vec{v})$ is called an eigenpair.

Eigenvalues, Eigenvectors

Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If there exists a vector $\vec{v} \neq \vec{0}$ such that $L(\vec{v}) = \lambda\vec{v}$, then λ is called an eigenvalue of L and $\vec{v}$ is called an eigenvector of L corresponding to λ.

Characteristic Polynomial

Let A be an $n \times n$ matrix. The characteristic polynomial of A is the n-th degree polynomial $C(\lambda) = \det(A - \lambda I)$.

Theorem 6.2.1

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if $C(\lambda) = 0$.

Eigenspace

Let A be an $n \times n$ matrix with eigenvalue λ. We call the nullspace of $A - \lambda I$ the eigenspace of λ.
The eigenspace is denoted E_λ.
The set S of eigenvectors with eigenvalue λ is

$$S = E_\lambda - \{\vec{0}\}.$$

Theorem 6.2.2

If A is an $n \times n$ upper or lower triangular matrix, then the eigenvalues of A are the diagonal entries of A.

Algebraic/Geometric Multiplicity

Let A be an $n \times n$ matrix with eigenvalue λ_1.
The algebraic multiplicity of λ_1, denoted a_{λ_1}, is the number of times that λ_1 is a root of the characteric polynomial $C(\lambda)$. That is, if
$C(\lambda) = (\lambda - \lambda_1)^k C_1(\lambda)$, where $C_1(\lambda_1) \neq 0$, then $a_{\lambda_1} = k$.
The geometric multiplicity of λ_1, denoted g_{λ_1}, is the dimension of its eigenspace.
So $g_{\lambda_1} = \dim(E_{\lambda_1})$

Lemma 6.2.3

If A and B are similar matrices, then A and B have the same characteristic polynomial and hence the same eigenvalues.

Theorem 6.2.4

If A is an $n \times n$ matrix with eigenvalue λ_1, then $1 \leq g_{\lambda_1} \leq a_{\lambda_1}$.

Diagonalizable

An $n \times n$ matrix A is said to be diagonalizable if A is similar to a diagonal matrix D.
If $P^{-1}AP = D$, then we say that P diagonalizes A.

Theorem 6.3.1

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of A.

Theorem 6.3.2

If A is n $n \times n$ matrix with eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \dots, (\lambda_k, \vec{v}_k)$ where $\lambda_i \neq \lambda_j$ for $i \neq j$,
Then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

Theorem 6.3.3

If A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and $B_i = \left\{\vec{v}_{i,1}, \ldots, \vec{v}_{i,g_{\lambda_i}}\right\}$ is a basis for the eigenspace of
λ_i for $1 \leq i \leq k$, then
$B_1 \cup B_2 \cup \cdots \cup B_k$ is a linearly independent set.

Corollary 6.3.4

If A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then A is diagonalizable if and only if $g_{\lambda_i} = a_{\lambda_i}$ for $1 \leq i \leq k$.

Corollary 6.3.5

If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable.

Theorem 6.3.6

If $\lambda_1, \ldots, \lambda_n$ are all the eigenvalues of an $n \times n$ matrix A, then $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$.

Algorithm 35.2

To diagonalize an $n \times n$ matrix A, or show that A is not diagonalizable.

1. Find and factor the characteristic polynomial
   $C(\lambda) = \det(A - \lambda I)$
2. Let $\lambda_1, \ldots, \lambda_n$ denote the n-roots of $C(\lambda)$ (repeated according to multiplicity). If any of the eigenvalues λ_i are not real, then A is not diagonalizable over $\mathbb{R}$.
3. Find a basis for the eigenspace of each λ_i by finding a basis for the nullspace of $A - \lambda_i I$.
4. If $g_{\lambda_i} < a_{\lambda_i}$ for any λ_i, then A is not diagonalizable.
   Otherwise, form a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of A by using theorem 3.
   Let $P = [\vec{v}_1 \quad \ldots \quad \vec{v}_n]$.
   Then $P^{-1}AP = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i is an eigenvalue corresponding to the eigenvector $\vec{v}_i$ for $1 \leq i \leq n$.

Theorem 6.4.1

Let A be an $n \times n$ matrix. If there exists a matrix P and diagonal matrix D such that $P^{-1}AP = D$, then
$A^k = PD^kP^{-1}$.