Math 239 Formula/Theorem Sheet

Enumeration

Sets	**Cartesian Product**
Unordered: $\{1,1,3\} = \{1,3,1\}$ Ordered: $\{1,1,3\} \neq \{1,3,1\}$ $\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|$	$A \times B = \{(a,b), a \in A, b \in B\}$ $\|A \times B\| = \|A\|\|B\|$ $\|A^k\| = \|A\|^k$
Binomial Coefficient Sets	**Binomial Theorem**
$S_{k,n} = \{\Omega \subseteq \{1,2,\dots,n\} : \|\Omega\| = k\}$ $\|S_{k,n}\| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $A_{k,n} = \{(z_1, z_2, \dots, z_n) : z_1 + z_2 + \dots + z_n = k\}$ $\|Ak, n\| = \binom{n}{k}$	$\binom{n}{k} = \binom{n}{n-k}$ $(1+x)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k$ $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$ $\binom{n+k}{n} = \sum\limits_{i=0}^{k} \binom{n+i-1}{n-1}$ - **Hockey Stick Identity**
Bijection	**Bijection Theorems**
$S \to T$ 1. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ - 1-1/injective 2. $\forall y \in T, \exists x \in S, f(x) = y$ - onto/surjective	Suppose S, T are finite sets and $f : S \to T$ (i) If f is 1-1 $\Rightarrow \|S\| \leq \|T\|$ (ii) If f is onto $\Rightarrow \|S\| \geq \|T\|$ (iii) if f is bijection $\Rightarrow \|S\| = \|T\|$ f has inverse if and only if f is a bijection
Generating Series	**Generating Series Theorems**
S is a set with weight function $w : S \to \{0,1,2,\dots\}$ $\Phi_s(x) = \sum\limits_{\sigma \in S} x^{w(s)}$	$\Phi_s(x) = \sum\limits_{k \geq 0} a_k x^k$ $\|S\| = \Phi_s(1)$ $\sum\limits_{\sigma \in S} w(\sigma) = \frac{d\Phi_S}{dx}\|_{x=1}$ Average Weight = $\frac{\frac{d\Phi_S}{dx}}{\Phi_s(1)}$
Formal Power Series	**FPS Operations**
$C(x) = c_0 + c_1 x + c_2 x^2 + \dots + \sum\limits_{k \geq 0}^{\infty} c_k x^k$ where $(c_0, c_1, \dots)$ are rational numbers. $[x^n]C(x) = c_n$ - Coefficient of x^n	$A(x) = \sum\limits_{k=0}^{\infty} a_k x^k$, $B(x) = \sum\limits_{k=0}^{\infty} b_k x^k$ **Addition**: $A(x) + B(x) = \sum\limits_{k=0}^{\infty} (a_k + b_k) x^k$ **Equality**: $A(x) = B(x) \iff a_k = b_k$ **Multiplication**: $A(x)B(x) = \sum\limits_{n=0}^{\infty} (\sum\limits_{j=0}^{n} a_j b_{n-j}) x^n$
Inverse/Composition of FPS (not all FPS have inverses)	**Inverse/Recurrence Theorems**
$A(x), B(x)$ are fps satisfying $A(x)B(x) = 1$ $\Rightarrow \frac{1}{(1-x)^k} = \sum\limits_{k=0}^{\infty} x^k$ $\Rightarrow [x^0]A(x)B(x) = 1$ - **Inverse**	$\frac{1}{(1-x)^k} = \sum\limits_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$ - **Negative Binomial Theorem** Let $A(x) = \sum\limits_{j=0}^{\infty} a_j x^j$. $A(x)$ has inverse $\iff [x^0]A(x) = a_0 \neq 0$ Let $A(x), C(x)$ be fps.

Let $A(x), B(x)$ be fps.
If $b_0 = 0 \Rightarrow A(B(x)) = \sum_{j=0}^{\infty} a_j (B(x))^j$ - **Composition**

If $[x^0]A(x) \neq 0 \Rightarrow$ there exists fps $B(x)$ where
$A(x)B(x) = C(x)$
Let $A(x), C(x)$ be fps with $a_0 \neq 0$. Let $B(x) = \frac{C(x)}{A(x)}$.
$\Rightarrow [x^n]B(X) = b_n = \frac{1}{a_0}(c_n - \sum_{j=0}^{n-1} a_{n-j} b_j)$

Let $A(x), B(x)$ be fps s.t. $[x^0]A(x) \neq 0$ and $[x^0]B(x) = 0$
$\Rightarrow (A(B(x)))^{-1} = A^{-1}(B(x))$

Sum and Product Lemmas for Generating Series

Let $S = A \cup B, A \cap B = \varnothing, w : S \to \{0, 1, \dots\}$
$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$ - **Sum lemma**

Let A, B be sets with weight functions:
$\alpha : A \to \{0, 1, \dots\}$
$\beta : B \to \{0, 1, \dots\}$
$w : A \times B \to \{0, 1, \dots\}$: $w((a, b)) = \alpha(a) + \beta(b)$
$\Rightarrow \Phi_{A \times B}(x) = \Phi_A(x)\Phi_B(x)$ - **Product Lemma**

Geometric Series

Geometric Series:
$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ -

Composition of Geometric Series:
$[x^0]A(x) = 0 \Rightarrow \sum_{j=0}^{\infty} (A(x))^j = \frac{1}{1-A(x)}$

Integer Composition

A composition of n is a *tuple* of **positive integers**
$(a_1, a_2, \dots, a_k)$ s.t. $a_1 + a_2 + \dots + a_k = n$
k is the number of parts of the composition.

Ambiguous/Unambiguous Binary Strings

Let A, B be sets of binary strings.

AB is **ambiguous** if:
There exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ s.t.
$a_1 b_1 = a_2 b_2$

$A \cup B$ is **unambiguous** if $A \cap B \neq \varnothing$

Expression with several concatenation/union operations is **unambiguous** if 1 of the concatenations or unions is.

Unambiguous/Ambiguous Binary Strings Expressions

A^k is **ambiguous** if there are distinct k-tuples and the concatenations are equal.

A^* is unambiguous $\iff$
1. $A^k \cap A^j = \varnothing$
2. A^k is unambiguous for each $k \geq 0$

Unambiguous Expressions for set of all binary strings:
1. $S = \{0, 1\}^*$
2. $S = \{0\}^*(\{1\}\{0\}^*)^*$
3. $S = \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$

Unambiguous Binary String Operations

Theorem 2.6.1:
A, B are sets of unambiguous binary strings.
(i) $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$
(ii) $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$
(iii) $\Phi_{A*}(x) = \frac{1}{1-\Phi_A(x)}$

Partial Fractions Theorems

f, g are polynomials where $deg(f) < deg(g)$ and
$g(x) = (1 - r_1 x)^{e_1} \dots (1 - r_k x)^{e_k}$
where $r_1, \dots, r_k \in \mathbb{C}, e_1, \dots, e_k \in \mathbb{Z}^+$

Partial Fractions Theorem:
$\Rightarrow \frac{f(x)}{g(x)} = \frac{c_{1,1}}{(1-r_1 x)} + \dots$
$+ \frac{c_{1,e_1}}{(1-r_1 x)^{e_1}} + \frac{c_{2,1}}{(1-r_2 x)} + \dots$
$+ \frac{c_{2,e_2}}{(1-r_x)^{e_2}} + \dots + \frac{c_{k,1}}{(1-r_k x)} + \dots + \frac{c_{k,e_k}}{(1-r_k x)^{e_k}}$

Theorem 3.1.4:

Homogeneous Linear Recurrences

Theorem 3.2.1:
Let $\{c_n\}_{n \geq 0}$ satisfy the linear recurrence:
$c_n + q_1 c_{n-1} + \dots + q_k c_{n-k} = 0$
Define $g(x) = 1 + q_1 x + \dots + q_k x^k$, $C(x) = \sum_{n=0}^{\infty} c_n x^n$
$\Rightarrow$ There exists polynomial $f(x)$ s.t. $deg(f) < k$
$C(x) = \frac{f(x)}{g(x)}$

There exist polynomials $P_1, \ldots, P_k$ s.t. $deg(P_i) < e_i$ $\Rightarrow [x^n]\frac{f(x)}{g(x)} = P_1(n)r_1^n + P_2(n)r_2^n + \ldots + P_k(n)r_k^n$	
Characteristic Polynomial	
$h(x) = q_k + q_{k-1}x + \ldots + q_1x^{k-1} + x^k$ for recurrence: $c_n + q_1c_{n-1} + \ldots + q_kc_{n-k} = 0$ **Theorem**: Roots: $r_1, r_2, \ldots, r_L \in \mathbb{C}$ Multiplicities: $e_1, e_2, \ldots, e_L \geq 1$ $\Rightarrow$ Then there exists polynomials $P_1, \ldots, P_L$ s.t. $deg(f_j) = e_j - 1$ and $c_n = P_1(n)r_1^n + \ldots + P_L(n)r_L^n$	

Graph Theory

Handshaking Lemma	**Corollary 4.3.2 (# of odd degrees)**
$\sum\limits_{v \in v(G)} deg(v) = 2\lvert E(G)\rvert$	Number of vertices of odd degree is even.
Corollary 4.3.3 (Average Vertex Degree)	**Isomorphic**
Avg vertex degree = $\frac{2\lvert E(G)\rvert}{\lvert V(G)\rvert}$	2 graphs G_1, G_2 are isomorphic if there exists a bijection $f: V(G_1) \rightarrow V(G_2)$ s.t. $uv \in E(G) \iff f(u)f(v) \in E(G_2)$
Isomorphic Equivalence Relation	**Complete Graphs** (K_n)
1. A graph is isomorphic to itself (reflexive) 2. If $G_1 \cong G_2 \Rightarrow G_2 \cong G_1$ (symmetric) 3. If $G_1 \cong G_2$ and $G_2 \cong G_3 \Rightarrow G_1 \cong G_3$ (transitive)	Every pair of vertices is an edge. # of edges = $\binom{n}{2}$
Regular Graphs (k-regular)	**Bipartite Graphs**
Graph where every vertex has degree k. # of edges = $\frac{nk}{2}$	Graph where there exists partition of vertices (A, B) s.t. each edge of G joins 1 vertex of B with a vertex in A.
Complete Bipartite Graphs $(K_{m,n})$	**N-cube**
Graph with partition (A,B) where all possible edges joining vertex in A with vertex in B. $\lvert A\rvert = m, \lvert B\rvert = n$ where $m, n > 0$ # of edges = mn	Graph where $V(G) = \{0,1\}^n$, and 2 strings are adjacent $\iff$ they differ in 1 position. # of vertices = 2^n # of edges = $n2^{n-1}$ - Regular, bipartite
Theorem 4.5.2 (Walk, Path)	**Corollary 4.6.3 (Path Transitive)**
If there is a u, v-walk in $G \Rightarrow$ there is a u, v-path in G.	If there exists u, v-path and v, w-path $\Rightarrow$ there exists u, w-path.

Theorem 4.6.4 (Cycle, degree)	Girth
If every vertex in G has $deg \geq 2 \Rightarrow G$ has a cycle.	Length of graph's shortest cycle. If G has no cycles $\Rightarrow$ girth = ∞ - Measure of graph density - High girth = lower average degree (usually)
Spanning Cycle/Hamiltonian Cycle	**Connectedness**
Cycle that uses every vertex in the graph	Connected if there is a u, v-path for any pair of vertices u, v.
Theorem 4.8.2 (Connectedness)	**Cuts**
Let $u \in V(G)$ If u, v-path exists for each $v \in V(G) \Rightarrow G$ is connected.	Let $x \subseteq V(G)$. **Cut induced by x** in G is the set of all edges in G with exactly 1 end in x.
Theorem 4.8.5 (Disconnectedness)	**Eulerian Circuit/Tour**
G is disconnected $\iff$ There exists a nonempty proper subset x of $V(G)$ s.t. the cut induced by x is $\emptyset$.	**Closed** walk that uses every edge of G exactly once. - Connected unless it has isolated vertices
Theorem 4.9.2 (Eulerian Circuit Vertices)	**Bridge**
G is connected. G has an Eulerian circuit $\iff$ every vertex in G has even degree.	An edge e (or cut-edge) in G if $G - e$ has more components than G.
Lemma 4.10.2 (Bridge Component)	**Theorem 4.10.3 (Bridge, Cycle)**
If $e = uv$ is a bridge in G and H is the component containing e $\Rightarrow H - e$ has exactly 2 components (hence $G - e$ has 1 more component than G) Moreover, u and v are in different components of $H - e$. (hence in $G - e$)	An edge e is a bridge of $G \iff e$ is not in any cyce of G.
Tree	**Forest**
Connected graph with no cycles # of edges = $n - 1$ by **Theorem 5.1.5** - Bipartite by **Lemma 5.1.4**	Graph with no cycles. # of edges = $n - k$, with n vertices and k components
Lemma 5.1.4 (Tree, Bridge)	**Leaf**
Every edge in a tree/forest is a bridge.	Tree with vertex degree 1.
Theorem 5.1.8 (Tree Leaves)	**Lemma 5.1.3 (Unique Path Tree)**
Every tree with ≥ 2 vertices has ≥ 2 leaves.	There is a unique path between any 2 vertices in a tree.

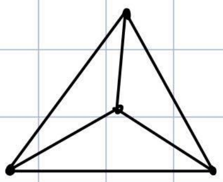
Theorem 5.2.1 (Connectedness, Spanning tree)	**Corollary 5.2.2 (Connected, tree)**
G is connected $\iff$ G has a spanning tree.	If G is connected with n vertices and $n-1$ edges $\Rightarrow G$ is a tree.
Corollary (Tree-Graph equivalence)	**Corollary 5.2.3 (Spanning Tree, Cycle)**
If any of 2 the following 3 conditions hold $\Rightarrow G$ is a tree: 1. G is connected 2. G has no cycles. 3. G has $n-1$ edges.	If T is a spanning tree of G and e is an edge in G that is not in T $\Rightarrow T+e$ has exactly 1 cycle. Moreover, if e' is an edge in C $\Rightarrow T+e-e'$ is a spanning tree of G.
Corollary 5.2.4 (Spanning Tree, Component)	**Bipartite Characterization Theorem**
If T is a spanning tree of G and e is an edge in T $\Rightarrow T-e$ has 2 components (e is a bridge in T). If e' is an edge in the cut induced by the vertices of 1 component $\Rightarrow T-e+e'$ is a spanning tree of G.	G is bipartite $\iff$ G has no odd cycles.
Minimum Spanning Tree (MST) Problem	**Prim's Algorithm Theorem**
Given connected graph G and weight function on edges $w: E(G) \to \mathbb{R}$, find minimum spanning tree in G whose total edge weight in minimized.	Prim's algorithm produces a MST.
Complement	**Planar**
If 2 vertices are adjacent in G $\iff$ The same 2 vertices are not adjacent in $\bar{G}$.	A graph is planar $\iff$ Each component is planar
Face	**Boundary**
A face of a planar embedding is a connected region on the plane. 2 faces are adjacent $\iff$ The faces share ≥ 1 edge in their boundaries.	Subgraph of all vertices and edges that touch a face.
Boundary Walk	**Handshaking Lemma for Faces**
Closed walk once around the perimeter of the face boundary. Degree of face = length of boundary walk	Let G be a planar graph with planar embedding where F is the set of all faces. $\sum_{f \in F} deg(f) = 2\|E(G)\|$
Lemma L7-1	**Jordan Curve Theorem**
In a planar embedding, an edge e is a bridge $\iff$ The 2 sides of e are in the same face	Every planar embedding of a cycle separates the plane into 2 parts: 1 on the inside, 1 on the outside

Euler's Formula	Platonic
Let G be a connected planar graph. Let n = # of vertices in G. Let m = # of edges in G. Let s = # of faces in a planar embedding of G. $\Rightarrow n - m + s = 2$ **Note:** If G has C components $\Rightarrow$ $n - m + s = 1 + C$ **Result:** All planar embeddings of a graph have the same number of faces.	A connected planar graph is **platonic** if it has a planar embedding where every vertex has same degree (≥ 3) AND every face has same degree (≥ 3)
Lemma 7.5.2	**Lemma 7.5.1**
Let G be a planar graph with n vertices and m edges. If there is a planar embedding of G where every face has degree ≥ 3 $\Rightarrow m \leq \frac{d(n-2)}{d-2}$	If G contains a cycle $\Rightarrow$ In any planar embedding of G, every face boundary contains a cycle.
Theorem 7.5.3	**Corollary 7.5.4**
Let G be a planar graph with $n \geq 3$ vertices $\Rightarrow m \leq 3n - 6$	K_5 is not planar.
Theorem 7.5.6	**Corollary 7.5.7**
Let G be a bipartite planar graph with ≥ 3 vertices and m edges. $\Rightarrow m \leq 2n - 4$	$K_{3,3}$ is not planar.
Edge Subdivision	**Kuratowski's Theorem**
Edge subdivision of G is obtained by replacing each edge of G with a new path of length ≥ 1	A graph is planar $\iff$ Graph does not have an edge subdivision of K_5 or $K_{3,3}$ as a subgraph.
K-colouring	**Theorem 7.7.2**
If C is a set of size k ("colours") $\Rightarrow f : V(G) \to C$ s.t. $f(u) \neq f(v)\ \forall uv \in E(G)$ **Note**: A k-colouring does not need to use all k colours.	G is 2-colourable $\iff$ G is bipartite
Theorem 7.7.3	**6-Colour Theorem**
K_n (complete graph) is n-colourable, and not k-colourable for any $k < n$	Every planar graph is 6-colourable.
Corollary 7.5.4	**5-Colour Theorem**
Every planar graph has a vertex of degree ≤ 5	Every planar graph is 5-colourable.
Contraction Result L7-7	**4-Colour Theorem**

If G is planar $\Rightarrow G/e$ is planar.	Every planar graph is 4-colourable.
Dual Graph Results L7-8(G^*)	**Lemma 8.2.1**
Key: Colouring faces is equivalent to colouring vertices of dual graph. The dual graph is planar. 1. If G is connected $\Rightarrow (G^*)^* = G$ 2. A vertex in G corresponds to a face in G^* of the same degree. 3. A face in G corresponds to a vertex in G^* of the same degree. 4. The dual of a platonic graph is platonic. 5. If we draw a closed curve on the plane $\Rightarrow$ the faces are 2-colourable.	If M is any matching of G and C is any cover in G $\Rightarrow \lvert M\rvert \leq \lvert C\rvert$
Lemma 8.2.2	**Konig's Theorem**
If M is a matching of G and C is a cover of G where $\lvert M\rvert = \lvert C\rvert$ $\Rightarrow M$ is a maximum matching of G and C is a minimum cover of G.	In a bipartite graph, the size of a maximum matching = size of a minimum cover.
Augmenting Path Results L8-2	**Lemma 8.1.1**
Let M be a matching. Let P be an augmenting path. Let M' be a new matching from P. $\Rightarrow M' = [M \cup (E(P) \setminus M)] \ \setminus (E(P) \cap M)$ **Notes**: - P is always odd length (L8-2 for proof) - P always starts and ends in different parts of the bipartition.	If a matching M has an augmenting path $\Rightarrow M$ is not a maximum.
Bipartite Matching Algorithm	**Corollary L8-4**
Minimum cover = $Y \cup (A \setminus X)$	A bipartite graph G with m edges and maximum degree d has a matching of size $\geq \frac{m}{d}$
Neighbour Set ($N_G(D)$ or $N(D)$)	**Hall's Theorem**
Let $D \subseteq V(G)$. Neighbour set of D is set of all vertices adjacent to at least 1 vertex in D.	A bipartite graph G with bipartition (A, B) has a matching that saturates all vertices in A $\iff \ \forall\, D \subseteq A, \lvert N(D)\rvert \geq \lvert D\rvert$ (Aka Hall's Condition)
Corollary 8.6.2	**Corollary L8-6**
If G is a k-regular bipartite graph with $k \geq 1$ $\Rightarrow G$ has a perfect matching.	The edges of a k-regular bipartite graph can be partitioned into k perfect matchings.

The Only Possible Platonic Graphs

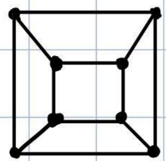
① $d_v = 3$, $d_f = 3$



tetrahedron

$n = 4$, $m = 6$, $s = 4$

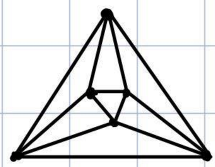
② $d_v = 3$, $d_f = 4$



cube

$n = 8$, $m = 12$, $s = 6$

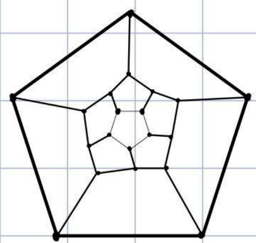
③ $d_v = 4$, $d_f = 3$



octahedron

$n = 6$, $m = 12$, $s = 8$

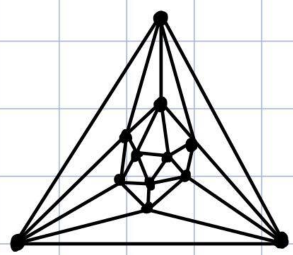
④ $d_v = 3$, $d_f = 5$



decahedron

$n = 20$, $n = 30$, $s = 12$

⑤ $d_v = 5$, $d_f = 3$



icosahedron

$n = 12$, $m = 30$, $s = 20$