Materials 25 - Preparing macro lunch

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Overview

$$A_a = \begin{pmatrix} -\frac{(\alpha-1)\beta(\delta\sigma+1)}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\alpha\beta\kappa(\delta\sigma+1)}{\delta\sigma+\kappa\psi\sigma+1} & 0 \\ \frac{(\alpha-1)\beta\sigma\psi}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\alpha\beta\kappa\sigma\psi}{\delta\sigma+\kappa\psi\sigma+1} & 0 \\ -\frac{(\alpha-1)\beta\psi}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\alpha\beta\kappa\psi}{\delta\sigma+\kappa\psi\sigma+1} & 0 \end{pmatrix}$$

$$A_b = \begin{pmatrix} \frac{\kappa\sigma(1-\beta\psi)}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\kappa(\delta\sigma\beta+\beta-1)}{\delta\sigma+\kappa\psi\sigma+1} & 0 \\ \frac{\sigma-\beta\sigma\psi}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\delta\sigma\beta+\beta-1}{\delta\sigma+\kappa\psi\sigma+1} & 0 \\ -\frac{\sigma(\beta\psi-1)(\delta+\kappa\psi)}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{(\delta\sigma\beta+\beta-1)(\delta+\kappa\psi)}{\delta\sigma+\kappa\psi\sigma+1} & 0 \end{pmatrix}$$

$$A_s = \begin{pmatrix} \frac{(\delta\sigma+1)\mathrm{ia}(3,1)+\kappa\sigma(\mathrm{ib}(1,1)-\mathrm{ib}(2,1))}{\delta\sigma+\kappa\psi\sigma+1} & \frac{(\delta\sigma+1)\mathrm{ia}(3,2)+\kappa\sigma(\mathrm{ib}(1,2)-\mathrm{ib}(2,2))}{\delta\sigma+\kappa\psi\sigma+1} & \frac{(\delta\sigma+1)\mathrm{ia}(3,3) \cdots}{\delta\sigma \cdots} \\ -\frac{\sigma(\psi\mathrm{ia}(3,1)-\mathrm{ib}(1,1)+\mathrm{ib}(2,1))}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\sigma(\psi\mathrm{ia}(3,2)-\mathrm{ib}(1,2)+\mathrm{ib}(2,2))}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\sigma(\psi\mathrm{ia}(3,3 \cdots}{\delta \cdots} \\ \frac{\psi\mathrm{ia}(3,1)+\delta\sigma\mathrm{ib}(1,1)+\kappa\sigma\psi\mathrm{ib}(1,1)-\sigma(\delta+\kappa\psi)\mathrm{ib}(2,1)}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\psi\mathrm{ia}(3,2)+\delta\sigma(\mathrm{ib}(1,2)-\mathrm{ib}(2,2)+1)+\kappa\sigma\psi(\mathrm{ib}(1,2)-\mathrm{ib}(2,2)+1)+1}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\psi\mathrm{ia}(3,3)+\delta\sigma\mathrm{ib}(1,3)+ \cdots}{\delta\sigma \cdots} \end{pmatrix}$$

If shocks are iid, ia and ib are identity matrices, and so As becomes:

$$A_s = \begin{pmatrix} \frac{\kappa\sigma}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\kappa\sigma}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\delta\sigma+1}{\delta\sigma+\kappa\psi\sigma+1} \\ \frac{\sigma}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\sigma}{\delta\sigma+\kappa\psi\sigma+1} & -\frac{\sigma\psi}{\delta\sigma+\kappa\psi\sigma+1} \\ \frac{\kappa\psi\sigma+\sigma\delta}{\delta\sigma+\kappa\psi\sigma+1} & \frac{1}{\delta\sigma+\kappa\psi\sigma+1} & \frac{\psi}{\delta\sigma+\kappa\psi\sigma+1} \end{pmatrix}$$

1 A value function iteration attempt at finding the optimal interest-rate-sequence

The planner chooses $\{\pi_t, x_t, i_t, f_{a,t}, f_{b,t}, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$V(\mathbf{x}_t, t) = \max -\left\{ (\pi_t^2 + \lambda_x x_t^2) + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1) \right\} \tag{1}$$

$$\text{s.t. to model equations} \tag{2}$$

Model equations are:

$$\pi_t = \kappa x_t + (1-\alpha)\beta f_a(t) + \kappa\alpha\beta b_2 (I_3 - \alpha\beta h_x)^{-1} s_t + e_3 (I_3 - \alpha\beta h_x)^{-1} s_t \tag{3}$$

$$x_t = -\sigma i_t + \sigma f_b(t) + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} s_t - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} s_t + \sigma e_1 (I_3 - \beta h_x)^{-1} s_t) \tag{4}$$

$$f_a(t) = \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + b_1 (I_3 - \alpha\beta h_x)^{-1} s_t \tag{5}$$

$$f_b(t) = \frac{1}{1-\beta} \bar{\pi}_{t-1} + b_1 (I_3 - \beta h_x)^{-1} s_t \tag{6}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \big) \tag{7}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \tag{8}$$

Let's substitute out $x_t, f_{a,t}$ and $f_{b,t}$, so that the state vector is simply $\mathbf{x}_t = (\bar{\pi}_{t-1}, k_t^{-1}, r_t^n, u_t)'$.

The problem becomes to choose $\{\pi_t, i_t, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$\begin{aligned}
V(\mathbf{x}_t, t) = \max -\Bigg\{ & \pi_t^2 + \lambda_x \sigma^2 i_t^2 + \lambda_x \frac{\sigma^2}{(1-\beta)^2}\bar{\pi}_{t-1}^2 - \lambda_x \frac{\sigma^2}{1-\beta} i_t \bar{\pi}_{t-1} \\
& - \lambda_x \sigma \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) i_t s_t \\
& + \lambda_x \frac{\sigma}{1-\beta} \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) \bar{\pi}_{t-1} s_t \\
& + \lambda_x \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big)^2 s_t \\
& + \beta \, \mathbb{E}_t V(\mathbf{x}_{t+1}, t+1) \Bigg\} \qquad (9)
\end{aligned}$$

s.t. to model equations

$$\begin{aligned}
\pi_t = & -\kappa\sigma i_t + \left(\kappa\sigma \frac{1}{1-\beta} + \frac{(1-\alpha)\beta}{1-\alpha\beta} \right) \bar{\pi}_{t-1} \\
& + \Big(\kappa\sigma b_1 (I_3 - \beta h_x)^{-1} + \kappa(1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \kappa\sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \kappa\sigma e_1 (I_3 - \beta h_x)^{-1} \\
& + (1-\alpha)\beta b_1 (I_3 - \alpha\beta h_x)^{-1} + \kappa\alpha\beta b_2 (I_3 - \alpha\beta h_x)^{-1} + e_3 (I_3 - \alpha\beta h_x)^{-1} \Big) s_t \qquad (10)
\end{aligned}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \big) \qquad (11)$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \qquad (12)$$

Let's simplify further. Let

$$\Omega_1 \equiv -\lambda_x \sigma \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) \qquad (13)$$

$$\Omega_2 \equiv \lambda_x \frac{\sigma}{1-\beta} \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) \qquad (14)$$

$$\Omega_3 \equiv \lambda_x \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big)^2 \qquad (15)$$

$$\Omega_4 \equiv \left(\kappa\sigma \frac{1}{1-\beta} + \frac{(1-\alpha)\beta}{1-\alpha\beta} \right) \qquad (16)$$

$$\begin{aligned}
\Omega_5 \equiv & \Big(\kappa\sigma b_1 (I_3 - \beta h_x)^{-1} + \kappa(1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \kappa\sigma\beta b_3 (I_3 - \beta h_x)^{-1} + \kappa\sigma e_1 (I_3 - \beta h_x)^{-1} \\
& + (1-\alpha)\beta b_1 (I_3 - \alpha\beta h_x)^{-1} + \kappa\alpha\beta b_2 (I_3 - \alpha\beta h_x)^{-1} + e_3 (I_3 - \alpha\beta h_x)^{-1} \Big) \qquad (17)
\end{aligned}$$

Then I can rewrite the problem as

$$\begin{aligned} V(\mathbf{x}_t, t) = \max -\Big\{ & \pi_t^2 + \lambda_x \sigma^2 i_t^2 + \lambda_x \frac{\sigma^2}{(1-\beta)^2} \bar{\pi}_{t-1}^2 - \lambda_x \frac{\sigma^2}{1-\beta} i_t \bar{\pi}_{t-1} \\ & + \Omega_1 i_t s_t + \Omega_2 \bar{\pi}_{t-1} s_t + \Omega_3 s_t + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1) \Big\} \end{aligned} \tag{18}$$

s.t.

$$\pi_t = -\kappa\sigma i_t + \Omega_4 \bar{\pi}_{t-1} + \Omega_5 s_t \tag{19}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\big) \tag{20}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \tag{21}$$

to sub out π_t and get

$$\begin{aligned} V(\mathbf{x}_t, t) = \max -\Big\{ & (\lambda_x \sigma^2 + (\kappa\sigma)^2) i_t^2 + (\lambda_x \frac{\sigma^2}{(1-\beta)^2} + \Omega_4^2) \bar{\pi}_{t-1}^2 + (-\lambda_x \frac{\sigma^2}{1-\beta} - \kappa\sigma\Omega_4) i_t \bar{\pi}_{t-1} \\ & + (\Omega_1 - \kappa\sigma\Omega_5) i_t s_t + (\Omega_2 + \Omega_4\Omega_5) \bar{\pi}_{t-1} s_t + (\Omega_3 + \Omega_5^2) s_t + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1) \Big\} \end{aligned} \tag{22}$$

s.t.

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\big(- \kappa\sigma i_t + (\Omega_4 - 1)\bar{\pi}_{t-1} + \Omega_5 s_t - b_1 s_{t-1})\big) \tag{23}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(-\kappa\sigma i_t + (\Omega_4 - 1)\bar{\pi}_{t-1} + \Omega_5 s_t - b_1 s_{t-1}) \tag{24}$$

Let's introduce more Ωs. Let

$$\Omega_6 \equiv (\lambda_x \sigma^2 + (\kappa\sigma)^2) \tag{25}$$

$$\Omega_7 \equiv (\lambda_x \frac{\sigma^2}{(1-\beta)^2} + \Omega_4^2) \tag{26}$$

$$\Omega_8 \equiv (-\lambda_x \frac{\sigma^2}{1-\beta} - \kappa\sigma\Omega_4) \tag{27}$$

$$\Omega_9 \equiv (\Omega_1 - \kappa\sigma\Omega_5) \tag{28}$$

$$\Omega_{10} \equiv (\Omega_2 + \Omega_4\Omega_5) \tag{29}$$

$$\Omega_{11} \equiv (\Omega_3 + \Omega_5^2) \tag{30}$$

$$\Omega_{12} \equiv \Omega_4 - 1 \tag{31}$$

Then the problem takes its final form:

$$V(\mathbf{x}_t, t) = \max_{i_t} -\left\{\Omega_6 i_t^2 + \Omega_7 \bar{\pi}_{t-1}^2 + \Omega_8 i_t \bar{\pi}_{t-1} + \Omega_9 i_t s_t + \Omega_{10} \bar{\pi}_{t-1} s_t + \Omega_{11} s_t + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1)\right\} \tag{32}$$

s.t.

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\big(-\kappa\sigma i_t + \Omega_{12}\bar{\pi}_{t-1} + \Omega_5 s_t - b_1 s_{t-1})\big) \tag{33}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(-\kappa\sigma i_t + \Omega_{12}\bar{\pi}_{t-1} + \Omega_5 s_t - b_1 s_{t-1}) \tag{34}$$

A Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \qquad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \qquad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + b s_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \qquad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \qquad fe_{t-1} = z_t - (a_{t-1} + b s_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \qquad f_a(t) = \frac{1}{1-\alpha\beta} a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta} a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1} s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \begin{pmatrix}\bar{\pi}_t & 0 & 0\end{pmatrix}'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix}\sigma & 1-\beta & -\sigma\beta\end{bmatrix} f_b + \sigma \begin{bmatrix}1 & 0 & 0\end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1} s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix}(1-\alpha)\beta & \kappa\alpha\beta & 0\end{bmatrix} f_a + \begin{bmatrix}0 & 0 & 1\end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1} s_t \tag{A.10}$$

B Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\Bigg\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t)\Big)$$
$$\Big(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j))\Big)\Bigg\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^{0} \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i}$$
$$-\frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\Big(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j)\Big) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs

C A target criterion system for an anchoring function specified for gain changes

$$k_t = k_{t-1} + \mathbf{g}(fe_{t|t-1}) \tag{C.1}$$

Turns out the k_{t-1} adds one $\varphi_{6,t+1}$ too many which makes the target criterion unwieldy. The FOCs of the Ramsey problem are

$$2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t^{-1}\varphi_{5,t} - \mathbf{g}_\pi(t)\varphi_{6,t} = 0 \tag{C.2}$$

$$cx_{t+1} + \varphi_{5,t} - (1 - k_t^{-1})\varphi_{5,t+1} + \mathbf{g}_{\bar{\pi}}(t)\varphi_{6,t+1} = 0 \tag{C.3}$$

$$\varphi_{6,t} + \varphi_{6,t+1} = fe_t\varphi_{5,t} \tag{C.4}$$

where the red multiplier is the new element vis-a-vis the case where the anchoring function is specified in levels ($k_t^{-1} = \mathbf{g}(fe_{t-1})$, as in App. B), and I'm using the shorthand notation

$$c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa} \tag{C.5}$$

$$fe_t = \pi_t - \bar{\pi}_{t-1} - bs_{t-1} \tag{C.6}$$

(C.2) says that in anchoring, the discretion tradeoff is complemented with tradeoffs coming from learning ($\varphi_{5,t}$), which are more binding when expectations are unanchored (k_t^{-1} high). Moreover, the change in the anchoring of expectations imposes an additional constraint ($\varphi_{6,t}$), which is more strongly binding if the gain responds strongly to inflation ($\mathbf{g}_\pi(t)$). One can simplify this three-equation-system to:

$$\varphi_{6,t} = -cfe_t x_{t+1} + \left(1 + \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1}) - fe_t\mathbf{g}_{\bar{\pi}}(t)\right)\varphi_{6,t+1} - \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1})\varphi_{6,t+2} \tag{C.7}$$

$$0 = 2\pi_t + 2\frac{\lambda}{\kappa}x_t - \left(\frac{k_t^{-1}}{fe_t} + \mathbf{g}_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} \tag{C.8}$$

Unfortunately, I haven't been able to solve (C.7) for $\varphi_{6,t}$ and therefore I can't express the target criterion so nicely as before. The only thing I can say is to direct the targeting rule-following central bank to compute $\varphi_{6,t}$ as the solution to (C.8), and then evaluate (C.7) as a target criterion. The solution to (C.8) is given by:

$$\varphi_{6,t} = -2\,\mathbb{E}_t\sum_{i=0}^{\infty}(\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i})\prod_{j=0}^{i-1}\frac{\frac{k_{t+j}^{-1}}{fe_{t+j}}}{\frac{k_{t+j}^{-1}}{fe_{t+j}} + \mathbf{g}_\pi(t+j)} \tag{C.9}$$

Interpretation: the anchoring constraint is not binding ($\varphi_{6,t} = 0$) if the CB always hits the target ($\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i} = 0 \quad \forall i$); or expectations are always anchored ($k_{t+j}^{-1} = 0 \quad \forall j$).