Materials 31 - Parameterized expectations (PE) against value iteration (VFI)

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The value function problem: the planner chooses $\{i_t\}_{t=t_0}^{\infty}$ to minimize

$$V(\mathbf{x}_t, t) = \min\left\{(\pi_t^2 + \lambda_x x_t^2) + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1)\right\} \tag{1}$$

$$\text{s.t. to model equations, (A.1)-(A.8)} \tag{2}$$

where $\mathbf{x} \equiv (k^{-1}, \bar{\pi}, r^n, u)$. V is thus $\mathcal{R}^4 \to \mathcal{R}$.

Questions:

- How to compare the results? For VFI, the policy function is `size(grid)` while for PE, it is $1 \times T$, where T is the length of a given simulated time series of shocks. I'm trying to pass the same time series of states from the PE result (X^{pe}), obtain the value function, and try to back out the associated $1 \times T$ policy from there. That is

$$V(X_t^{pe}) = L(i) + \beta\, \mathbb{E}_t\, V(X_{t+1}^{pe}) \tag{3}$$

  Since I had a hard time taking the $\mathbb{E}$, I cheated by computing

$$L(i) = (1 - \beta)V(X_t^{pe}) \tag{4}$$

  $$\text{Given} \quad L(i), \quad i_t = L^{-1}(i)$$

- If I have $L(i)$, I need to solve $L(i_t) = \pi_t(i_t)^2 + \lambda_x x_t(i_t)^2$ at each t, which is a quadratic in i_t. Difficult to discriminate between the roots, which sometimes, for some t, are complex.

Notes for me:

- Since equations (A.5)-(A.7) involve past shocks, I actually have 2 additional states. If I instead take expectations regarding what node of the state-space we're coming from ("backward expectations"), then:

- I'm using a $4 \times 4 \times 2 \times 2$ grid, and cubic spline interpolation for the value function. This takes 260 seconds. I'm sure I could speed things up by doing policy iteration instead.

Figure 1: Parameterized expectations against value iteration solution, 4 states (ignore past states)

(a)

(b)
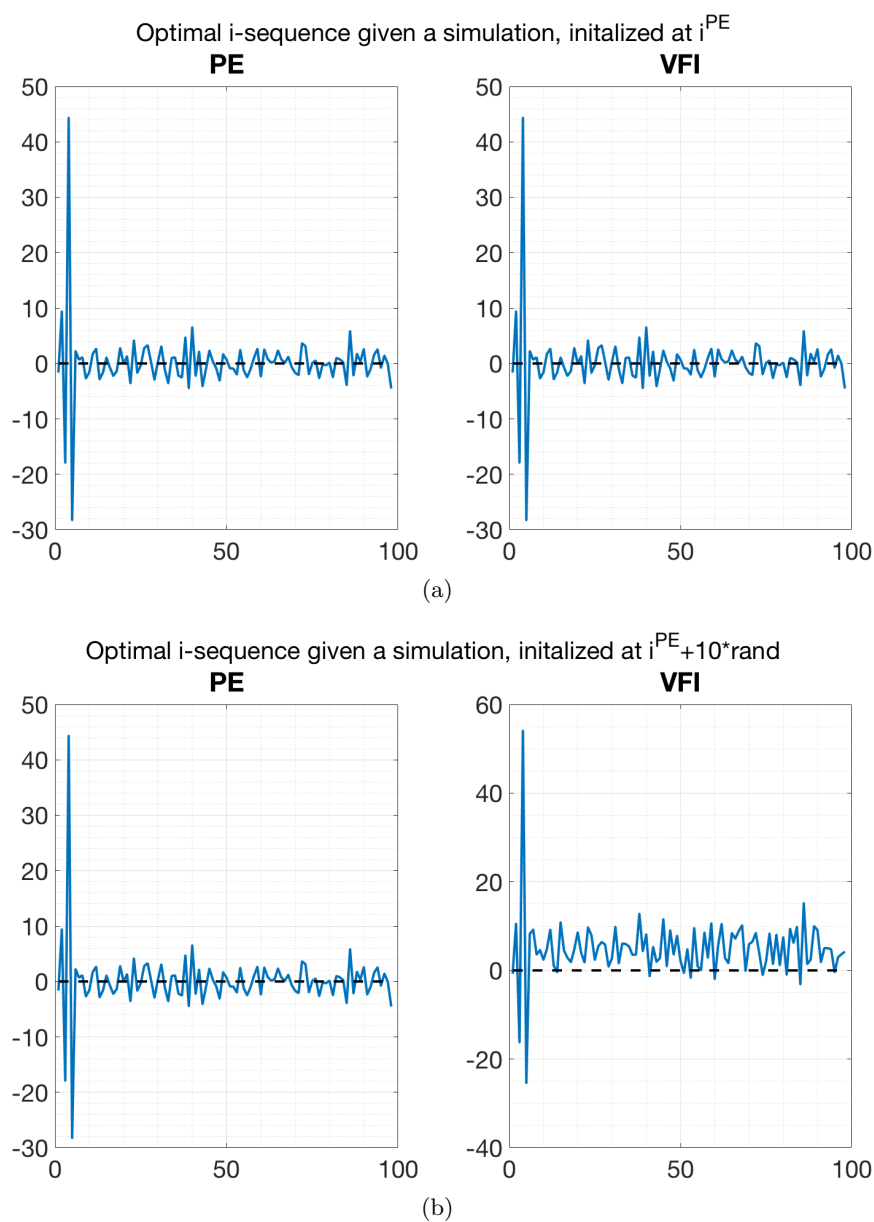
A Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \quad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + bh_x^{h-1}s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \quad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + bs_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \quad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \quad fe_{t-1} = z_t - (a_{t-1} + bs_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \quad f_a(t) = \frac{1}{1-\alpha\beta}a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1}s_t \qquad f_b(t) = \frac{1}{1-\beta}a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1}s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \big(\bar{\pi}_t \quad 0 \quad 0\big)'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix} \sigma & 1-\beta & -\sigma\beta \end{bmatrix} f_b + \sigma \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1}s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix} (1-\alpha)\beta & \kappa\alpha\beta & 0 \end{bmatrix} f_a + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1}s_t \tag{A.10}$$

B Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t) \right) \right.$$

$$\left. \left(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j)) \right) \right\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^{0} \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa}x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t) \right) \mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i}$$

$$-\frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t) \right)\left(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j) \right) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs