Materials 4 - DW prep

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Overview

1 Timeline in the learning models

<u>$t = 0$</u>: Initialize learning coefficients $\phi_{t-1} = \phi_0$ at the RE solution. For each t:

1. Evaluate expectations $t + s$ (the one-period ahead, ($s = 1$) or the full 1 to ∞-period ahead ($s = 1, \ldots, \infty$)) given ϕ_{t-1} and states dated t
2. Evaluate ALM given expectations: "today's observables are a function of expectations and today's state"
3. Update learning: $\phi_t =$ RLS of ϕ_{t-1} and fcst error between today's data and yesterday's forecast

2 The models

2.1 Rational expectations NK model (RE)

$$x_t = \mathbb{E}_t \, x_{t+1} - \sigma(i_t - \mathbb{E}_t \, \pi_{t+1}) + \sigma r_t^n \tag{1}$$

$$\pi_t = \kappa x_t + \beta \, \mathbb{E}_t \, \pi_{t+1} + u_t \tag{2}$$

$$i_t = \bar{i}_t + \psi_\pi \pi_t + \psi_x x_t \tag{3}$$

2.2 NK model with ∞-horizon forecasts (LR) with learning the constant of π only

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \big((1-\beta) x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n \big) \tag{Preston, eq. (18)}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \big) \tag{Preston, eq. (19)}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \tag{Preston, eq. (27)}$$

2.3 Anchoring: same as LR, just anchoring instead of decreasing gain

3 Compact notation

Exogenous states are summarized as:

$$s_t = P s_{t-1} + \epsilon_t \quad \text{where} \quad s_t \equiv \begin{pmatrix} r_t^n \\ \bar{i}_t \\ u_t \end{pmatrix} \quad P \equiv \begin{pmatrix} \rho_r & 0 & 0 \\ 0 & \rho_i & 0 \\ 0 & 0 & \rho_u \end{pmatrix} \quad \epsilon_t \equiv \begin{pmatrix} \varepsilon_t^r \\ \varepsilon_t^i \\ \varepsilon_t^u \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_i & 0 \\ 0 & 0 & \sigma_u \end{pmatrix}$$

Let z_t summarize the endogenous variables as

$$z_t \equiv \begin{pmatrix} \pi_t \\ x_t \\ i_t \end{pmatrix} \tag{4}$$

Then I can write the models compactly as

$$z_t = A_p^{RE}\,\mathbb{E}_t\, z_{t+1} + A_s^{RE} s_t \tag{5}$$

$$z_t = A_p^{RE}\hat{\mathbb{E}}_t z_{t+1} + A_s^{RE} s_t \tag{6}$$

$$z_t = A_a^{LR} f_a(t) + A_b^{LR} f_b(t) + A_s^{LR} s_t \tag{7}$$

$$s_t = P s_{t-1} + \epsilon_t \tag{8}$$

where f_a and f_b capture discounted sums of expectations at all horizons of the endogenous states z (following Preston, I refer to these objects as "long-run expectations"):

$$f_a(t) \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} z_{T+1} \qquad\qquad f_b(t) \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\beta)^{T-t} z_{T+1} \tag{9}$$

and the coefficient matrices are given by:

$$A_p^{RE} = \begin{pmatrix} \beta + \frac{\kappa\sigma}{w}(1-\psi_\pi\beta) & \frac{\kappa}{w} & 0 \\ \frac{\sigma}{w}(1-\psi_\pi\beta) & \frac{1}{w} & 0 \\ \psi_\pi\big(\beta + \frac{\kappa\sigma}{w}(1-\psi_\pi\beta)\big) + \psi_x\frac{\sigma}{w}(1-\psi_\pi\beta) & \psi_x(\frac{1}{w}) + \psi_\pi(\frac{\kappa}{w}) & 0 \end{pmatrix} \tag{10}$$

$$A_s^{RE} = \begin{pmatrix} \frac{\kappa\sigma}{w} & -\frac{\kappa\sigma}{w} & 1 - \frac{\kappa\sigma\psi_\pi}{w} \\ \frac{\sigma}{w} & -\frac{\sigma}{w} & -\frac{\sigma\psi_\pi}{w} \\ \psi_x(\frac{\sigma}{w}) + \psi_\pi(\frac{\kappa\sigma}{w}) & \psi_x(-\frac{\sigma}{w}) + \psi_\pi(-\frac{\kappa\sigma}{w}) + 1 & \psi_x(-\frac{\sigma\psi_\pi}{w}) + \psi_\pi(1 - \frac{\kappa\sigma\psi_\pi}{w}) \end{pmatrix} \tag{11}$$

$$A_a^{LR} = \begin{pmatrix} g_{\pi a} \\ g_{xa} \\ \psi_\pi g_{\pi a} + \psi_x g_{xa} \end{pmatrix} \quad A_b^{LR} = \begin{pmatrix} g_{\pi b} \\ g_{xb} \\ \psi_\pi g_{\pi b} + \psi_x g_{xb} \end{pmatrix} \quad A_s^{LR} = \begin{pmatrix} g_{\pi s} \\ g_{xs} \\ \psi_\pi g_{\pi s} + \psi_x g_{xs} + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{pmatrix} \tag{12}$$

$$g_{\pi a} = (1 - \frac{\kappa\sigma\psi_\pi}{w}) \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{13}$$

$$g_{xa} = \frac{-\sigma\psi_\pi}{w} \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{14}$$

$$g_{\pi b} = \frac{\kappa}{w} \left[\sigma(1-\beta\psi_\pi), (1-\beta-\beta\sigma\psi_x, 0\right] \tag{15}$$

$$g_{xb} = \frac{1}{w} \left[\sigma(1-\beta\psi_\pi), (1-\beta-\beta\sigma\psi_x, 0\right] \tag{16}$$

$$g_{\pi s} = (1 - \frac{\kappa\sigma\psi_\pi}{w}) \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (I_3 - \alpha\beta P)^{-1} - \frac{\kappa\sigma}{w} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} (I_3 - \beta P)^{-1} \tag{17}$$

$$g_{xs} = \frac{-\sigma\psi_\pi}{w} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (I_3 - \alpha\beta P)^{-1} - \frac{\sigma}{w} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} (I_3 - \beta P)^{-1} \tag{18}$$

$$w = 1 + \sigma\psi_x + \kappa\sigma\psi_\pi \tag{19}$$

4 Learning

Use Ryan's timing notation, in which the time index designated the time in which the coefficient is *realized*, which is always one less than the period in which it is *used*. Agents only learn about the constant, and only about the constant of inflation, i.e. about CEMP's drift term, but forecast exogenous states rationally:

$$z_t = \begin{bmatrix} \bar{\pi}_{t-2} \\ 0 \\ 0 \end{bmatrix} + b s_{t-1} + \epsilon_t \qquad b = gx\ hx \tag{20}$$

which is equivalent to saying that their expectations about x and i are rational.

Anticipated utility implies that

$$\hat{\mathbb{E}}_{t-1}\bar{\pi}_{t+h} = \hat{\mathbb{E}}_{t-1}\bar{\pi}_t \equiv \bar{\pi}_{t-1} \quad \forall\ h \geq 0 \tag{21}$$

Agents today mistakenly believe that they will not update the forecasting rule. Moreover, the constant $\bar{\pi}$ agents will use in period t is the one they updated to in $t-1$. Assuming RE about the exogenous process and anticipated utility, h-horizon forecasts are constructed as:

$$\hat{\mathbb{E}}_t z_{t+h} = \begin{bmatrix} \bar{\pi}_{t-1} \\ 0 \\ 0 \end{bmatrix} + bP^{h-1}s_t \quad \forall h \geq 1 \tag{22}$$

and the regression coefficient is updated using an RLS algorithm (b_1 is the first row of b):

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \underbrace{\left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\right)}_{\text{fcst error using (22)}} \tag{23}$$

where k_t is either always $k_{t-1}+1$ (decreasing gain), or it's described by the CEMP anchoring mechanism:

$$k_t = \mathbb{I} \times \frac{1}{k_{t-1}+1} + (1-\mathbb{I}) \times \bar{g} \tag{24}$$

$$\mathbb{I} = \begin{cases} 1 & \text{if } \theta_t \leq \bar{\theta} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

$$\theta_t = |\hat{\mathbb{E}}_{t-1}\pi_t - \mathbb{E}_{t-1}\,\pi_t|/\sigma_s \tag{26}$$

5 ALMs

5.1 RE

With some abuse of terminology, call the state-space representation the ALM of the RE model:

$$x_t = hx\ x_{t-1} + \eta e_t \tag{27}$$

$$z_t = gx\ x_t \tag{28}$$

Then I can write the "ALM" as

$$z_t = gx\ hx\ x_{t-1} + gx\ \eta e_t \tag{29}$$

Since this ALM implies no constant, I initialize $\bar{\pi}_0 = 0$.

5.2 LR

Evaluate analytical "LR expectations" (9) using the PLM (22):

$$f_a(t) = \frac{1}{1-\alpha\beta}\bar{z}_{t-1} + b(I_3 - \alpha\beta P)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta}\bar{z}_{t-1} + b(I_3 - \beta P)^{-1} s_t \tag{30}$$

6 Plan for DW presentation

6.1 Contrast LR model w/ and w/o anchoring

6.2 Simulations w/ different TR coefficients