

Work after

12 Feb 2020

COPY OF JOB MARKET

Time in reverse

TIMELINE

Feb/March 2021: accept job

Jan/Feb 2021: interview Projects

Nov 2020: submit applications

Oct 2020: job talk / (submit) GLMM ← Reading 2

Aug 31 2020: a very complete draft

Summer 2020: very hard work

writing → meaning math + text

i.e. not refining the question

May 2020: first draft ← Reading 1

try to circulate in Dept.

Branch's RPE (restricted perceptions eqb)

(an undercategory of which is a misspecification eqb, ME) is this model uncertainty/selection issue:
agents consider the set $\tilde{\mathcal{X}}$

$$\tilde{\mathcal{X}} = \left\{ x_t = a + b' \tilde{z}_{t-1} + e_t : \dim(\tilde{z}) < \dim(z) \right\}$$

i.e. each element in $\tilde{\mathcal{X}}$ is a PLM that is misspecified
→ a fitting model which omits a variable or a lag.

Agents then pick the best-performing model in an
FEV/MSE or related concept-sense, like in Cho & Kasa.

→ and I'm also seeing that the Ball-effect

is a negative feedback effect b/c it can
be int-rate, $E(\pi)$ & (π, x) move in
opposite directions

↳ This is what Branch calls a third effect of

exogenous disturbances: 1) direct effect on states x_t
2) induced via expectations 3) policy feedback effect

⇒ joint determination of optimal policy & ME!

Framework

1.1 Optimal State-Contingent Paths

The model equations are summarised by

$$\hat{I} \begin{bmatrix} 2_{t+1} \\ E_t 2_{t+1} \end{bmatrix} = A \begin{bmatrix} 2_t \\ e_t \end{bmatrix} + B i_t + C s_t \quad (1.1)$$

2_t := vector of "nonpredetermined endog vars" (jumps)

e_t := vector of "predetermined endog vars" (endog states)

i_t := policy instrument

s_t := vector of exog disturbances

$$\hat{I} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2_{t+1} \\ E_t 2_{t+1} \end{bmatrix} = |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 2_t \\ e_t \end{bmatrix}$$

$$+ |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + -|\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ C_2 \end{bmatrix} s_t$$

$$\text{Objectives of policy} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \quad (1.2)$$

$$\text{where the period loss } L_t = L_t = \frac{1}{2} (\tau_t - \tau^*)' W (\tau_t - \tau^*) \quad (1.3)$$

τ_t = vector of target variables, $\tau_t = T y_t$

where $y_t = \begin{bmatrix} z_t \\ \bar{z}_t \\ \bar{s}_t \end{bmatrix}$ is the vector of endog. variables.

DEF. A policy rule which is optimal from a timeless perspective in this more general context is a rule such that

1.) jumps z_t can be expressed as a time-invariant function

$$z_t = f_0 + f_2 \bar{z}_t + f_5 \bar{s}_t \quad (1.5)$$

the bars reflect that the fit may involve additional endog. or exog. states relative to the model

2) The e.g. evolution of $\{y_t\}$ (i.e. all endog. vars) minimizes the loss (1.2) for all $t \geq t_0$, subject to the model (1.1)

$$\text{and s.t. } \tilde{E} z_{t_0} = \bar{e} \equiv \tilde{E} [f_0 + f_2 \bar{z}_{t_0} + f_5 \bar{s}_{t_0}] \quad (1.6)$$

which restricts the initial values of the jumps. ("timelessness")

Indeed this method is the same as the one outlined in Chapter 7, just in a more general setting. The solution procedure is to set up the Lagrangian

$$x_{t_0} = E_{t_0} \left\{ \sum_{t=t_0}^{\infty} \beta^{t-t_0} [L(y_t) + r_{t+1}' \tilde{A} y_t - \rho^{-1} r_t' \tilde{I} y_t] \right\} \quad (13)$$

where $\hat{A} = [A \ B]$ $\hat{f} = [\hat{x} \ 0]$

λ -morphisms on the model cgs.

$$y_{t+1} = \begin{cases} s_{t+1} \\ \Xi_t \end{cases} \rightarrow \begin{array}{l} \text{pertains to endogenous states (predetermined at } t) \\ \text{pertains to jumps (set at } t) \end{array}$$

or vice versa :S

Shocks s_1 have been suppressed in L b/c FD's are independent of them anyway.

Note that the last term in \mathcal{L}_1 at t_0 , gives the initial constraints

Time-out: timelessly optimal vs. t_0 -optimal
 time-consistent vs. time-inconsistent optimal commitment

Indeed:

A t_0 -optimal plan minimizes (1.7) s.t. same constraints
 EXCEPT it replaces (1.6) ("timelessness") w/

$$\Xi_{t_0-1} = 0 \quad (1.8)$$

i.e. the "bygones-be-bygones" assumption

FOLs:

$$\tilde{A}' E_t \varphi_{t+1} + T' W (\tau_t - \tau^*) - \beta^{-1} \tilde{I}' \varphi_t' = 0 \quad (1.9)$$

$\forall t \geq t_0$.

Restatement:

t_0 -optimal solution: any bounded processes for $\{\gamma_t\}$ & $\{\varphi_t\}$

that satisfy the FOLs (1.9) s.t. the time-inconsistency constraint (1.8)

timelessly optimal sol.: any bounded processes for $\{\gamma_t\}$ & $\{\varphi_t\}$

that satisfy the FOLs (1.9) s.t. the time-consistency constraint (1.6)

Yet again restate the timelessly optimal solution:

1. E_{t_0} satisfies a time-invariant relation of the form (1.5)
2. $\{y_t\} \forall t \geq t_0$ evolve according to the (by assumption bounded) unique solution to the FOCs (1.9) s.t. the model (1.1)
w/ the given initial conditions and initial L-multiples
 Ξ_{t_0-1} .
3. The initial multipliers are given by a linear rule of the form

$$\Xi_{t_0-1} = g_0 + g_2 \bar{L}_{t_0} + g_5 \bar{S}_{t_0}. \quad (1.11)$$

w/ g_0, g_2, g_5 satisfying

$$\tilde{E}2_+ = e_0 + e_2 \bar{L}_+ + e_5 \bar{S}_+ + e_{\bar{L}} [g_0 + g_2 \bar{L}_+ + g_5 \bar{S}_+] \quad (1.12)$$

$\forall t \geq t_0$

(where what this is saying is that the rule for initial multipliers has to be timelessly optimal (i.e. compatible w/ 1.6) and at the same time conform to the time-invariant rule (1.5))

And note: These are just the optimal plans for $\{y_t\}$.

Implementing this plan is another question.

I suggest now to revisit the simple example in Ch. 7 of deriving the to-/timelandy optimal plans, augmented w/ some review of dynamic systems from Alpha Chiang's wonderful book, before turning to the implementation of plans.

Ch. 7 p. 472 The Ramsey Problem (shh... it's a secret)

$$L = \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2} [\pi_t^2 + \lambda(x_t - x^*)^2] + \rho_t [\pi_{t+1} - kx_t - \beta\pi_t] \right]$$

FOCs:

$$\pi_t: \pi_t + \varphi_t - \varphi_{t-1} = 0 \quad (1.7) \quad t \geq 0$$

$$x_t: \lambda(x_t - x^*) - k\varphi_t = 0 \quad (1.8)$$

$$\text{and for } t=0, \quad \varphi_{-1} = 0 \quad (1.9)$$

This may in fact not be time-inconsistent b/c this is just saying that the model isn't a constraint in $t = -1$.

But initial conditions may be, or stay.

Sub out π & x from

$$\pi_t = \kappa x_t + \beta \pi_{t+1} \quad (1.1)$$

using (1.7) & (1.8)

$$p_{t-1} - y_t = \kappa \left[\frac{\kappa}{\lambda} y_t + x^* \right] + \beta (y_t - y_{t+1})$$

⇒

$$p_{t-1} - y_t - \beta y_t - \beta y_{t+1} - \frac{\kappa^2}{\lambda} y_t = \kappa x^*$$

$$\Rightarrow \beta y_{t+1} - \left(1 + \beta + \frac{\kappa^2}{\lambda} \right) p_t + y_{t-1} = \kappa x^* \quad (1.10)$$

OK, this is a difference eq. of order 2. Let's now
revisit Alpha Decay

General method of solving difference equations

(p. 554 book, pdf. p. 561 mac) Ch. 16 / Ch. 17

$$y_{t+1} + a y_t = c \quad (16.6)$$

The general solution = $y_p + y_c$

↑ ↑

particular complementary
solution function

y_p solves the nonhomogen. eq. (16.6), y_c the homogen. eq. :

$$y_{t+1} + a y_t = 0$$

y_p : represents intertemporal eqs level of y
(st. st. 1 guess)

y_c : signifies deviations of the time path from the equilibrium y_p . (business cycles 1 guess)

Focus on y_c first:

$$\text{try } y_t = Ab^t$$

$$\text{then } y_{t+1} = Ab^{t+1}$$

$$\text{and the homogeneous eq } y_{t+1} + ay_t = c$$

$$\text{implies } Ab^{t+1} + aAb^t = 0 \quad | : Ab^t$$

$$\rightarrow b + a = 0 \rightarrow b = -a$$

then the complementary pt becomes

$$\underline{y_c = Ab^t = A(-a)^t}.$$

Then, the particular sol. guess $y_t = k \rightarrow y_{t+1} = b$.

sub into (16.6) $y_{t+1} + ay_t = c$

$$\rightarrow k + ak = c \Rightarrow \underline{k = \frac{c}{1+a}} = y_p$$

Since this y_p is a constant, this is a stationary equilibrium.

Sometimes we want a moving equilibrium (a function of time, like a BGP st. st.), or $a = -1$, so the previous y_p is undefined. Then for the particular sol we can guess $y_t = kt$

$$\rightarrow k(t+1) + ak_t = c$$

$$\rightarrow k = \frac{c}{t+1+at}$$

which, if $a = -1$ implies $k = c$, so now our moving particular sol is $y_p = ct$.

Putting the sol together gives the general sol

$$(16.8) \quad y_t = A(-a)^t + \frac{c}{1+a} \quad \text{if } a \neq -1$$

$$(16.9) \quad y_t = A(-a)^t + c \cdot t \quad \text{if } a = -1$$

$$= A + c \cdot t$$

We still need to determine A.

To determine A, resort to initial conditions $y_+ = y_0$

For (16.8), $t = 0$ implies

$$y_0 = A + \frac{C}{1+a} \Rightarrow A = y_0 - \frac{C}{1+a}$$

thus we obtain the "full-fledged" (1a) general sol

$$y_+ = \left(y_0 - \frac{C}{1+a}\right)(-a)^t + \frac{C}{1+a} \quad \text{for } a \neq -1$$

(16.8')

(analogous procedure for the $a = -1$ case)

Dynamic stability of equilibrium p. 558 / 564 Mac

Dynamic stability is the question of whether the general sol. $y_+ = y_c + y_p$ tends to y_p as $t \rightarrow \infty$.

" $\overset{\uparrow}{\text{bc}}$ " " $\overset{\uparrow}{\text{st. st}}$ "

⇒ In other words, whether $y_c \rightarrow 0$ as $t \rightarrow \infty$.

Dynamic stability depends then on $A(b)^t$, in fact, it depends fully on b!

$b > 1$	explosive
$b = 1$	constant, non-convergent
$b \in (0, 1)$	convergent (stable)
$b = 0$	constant (stable)
$b \in (-1, 0)$	damped oscillations
$b = -1$	uniform oscillations
$b < -1$	explosive oscillations

\Rightarrow Nonoscillatory: $b > 0$ (else oscillatory)

Damped: $|b| < 1$ (else explosive)

Note: $|b|=1$ must be ruled out b/c it's non-convergent.

A: -scales b

- sign of A can invert b.

Higher-order difference equations

The sd for y_p , the particular integral, proceeds exactly the same way. So let's focus on y_c , the complementary function:

Here's a 2nd order diff. eq:

$$(17.1) \quad y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

The homogen. version is

$$(17.3) \quad y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Gives the same thing: $y_t = Ab^t$

$$Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t = 0 \quad | : Ab^t$$

$$b^2 + a_1 b + a_2 = 0 \quad (17.3')$$

(17.3)' is the characteristic equation of (17.3) or (17.1)

Let's repeat: the characteristic equation is the equation you obtain once you plug in the conjecture for the complementary function y_c of $y_t = Ab^t$ and simplify.

It follows that a diff. eq. of order p will have p roots to the characteristic eq., as (17.3') clearly has 2 "characteristic roots"*, b_1 and b_2 .

Ok, but which is the solution?

*characteristic roots are also known as eigenvalues.

Divide side note: take Woodford's 2nd order diff eq:

$$\beta \varphi_{t+1} - (1 + \beta + \frac{\kappa^2}{\lambda}) \varphi_t + \varphi_{t-1} = \kappa x^* \quad (1.10)$$

Take a sol. of the form $A\mu^t$ and plug

$$\beta A\mu^{t+1} - (1 + \beta + \frac{\kappa^2}{\lambda}) A\mu^t + A\mu^{t-1} = \kappa x^*$$

Divide by $A\mu^{t-1}$ and consider the homogen case:

$$\beta \mu^2 - (1 + \beta + \frac{\kappa^2}{\lambda}) \mu + 1 = 0$$

This is how you get eq. (1.11)

Ok, but back to Alpha Albury and the question of what to do w/ the roots.

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Case 1. $a_1^2 - 4a_2 > 0 \Rightarrow$ 2 distinct, real roots

Case 2. $a_1^2 - 4a_2 = 0 \Rightarrow$ Repeated real roots

Case 3. $a_1^2 - 4a_2 < 0 \Rightarrow$ Complex roots

In all cases, all roots need to figure in the y_c b/c they are linearly independent.

Case 1. In this case the complementary function becomes

$$y_c = A_1 b_1 t + A_2 b_2 t$$

(a linear combination of two linearly independent parts)

Case 2. Since $b_1 = b_2$ here, $A_1 b_1 t + A_2 b_2 t$ becomes

$$(A_1 + A_2) b^t = A_3 b^t$$

In this case Alpha Chiang still insists that we need an additional linearly independent term, so he introduces $A_4 \cdot t \cdot b^t$, which is linearly \perp b/c you can't obtain it by multiplying $A_3 b^t$ by some constant coefficient c. (unless $c = A_4 \cdot t$ (guess))

\Rightarrow Thus in this case the complementary sol becomes

$$y_c = A_3 b^t + A_4 \cdot t \cdot b^t$$

Case 3. $b_{1,2} = h \pm vi$ (conjugate complex roots)

$$h = -\frac{a_1}{2} \quad v = \frac{\sqrt{4a - a_1^2}}{2} \quad (\text{ahaa!})$$

$$\Rightarrow y_c = A_5 (h+vi)^t + A_6 (h-vi)^t \quad (\text{ahaa!!})$$

thanks to DeMoivre's theorem (15.28')

$$(h \pm vi)^t = R^t (\cos \theta t \pm i \sin \theta t)$$

where $R = \sqrt{h^2 + v^2} = \sqrt{\alpha_2}$ (always > 0) (radius)

The absolute value of the conjugate complex roots

$$\cos \theta = \frac{h}{R} = \frac{h}{\sqrt{\alpha_2}} \quad \text{and} \quad \sin \theta = \frac{v}{R} = \frac{v}{\sqrt{\alpha_2}}$$

$$\begin{aligned} \Rightarrow y_c &= A_5(h+vi)^t + A_6(h-vi)^t \\ &= A_5 R^t (\cos \theta t + i \sin \theta t) + A_6 R^t (\cos \theta t - i \sin \theta t) \\ &= \underbrace{(A_5 + A_6) R^t \cos \theta t}_{y_c = R^t(A_7 \cos \theta t + A_8 \sin \theta t)} + \underbrace{(A_5 - A_6)i R^t \sin \theta t}_{(17.10)} \end{aligned}$$

Convergence of the time path (dynamic stability) in
the p-order difference equation case

For conciseness, $p=2$ here.

A p-order diff eq converges to st. st if the absolute value
of every characteristic root is less than 1.

Sauer Thm : This holds if all p determinants are positive
(the first p sub-determinants or what) (p. 599 / 589 Mac)

Note that Viète's formulas are also an easy way to relate the roots to each other via their sums & products.

For a quadratic equation, the roots take the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

thus $x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$

□

and $x_1 \cdot x_2 = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2} = \frac{(a+b)(a-b)}{4a^2} = \frac{a^2 - b^2}{4a^2}$

$$\begin{aligned} & (a+b)(a-b) \\ & = a^2 - b^2 \end{aligned}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} \quad \square$$

thus, Viète:

$$x_1 + x_2 = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

Back to Woodford then. The characteristic eq. (1.11)

$$\beta\mu^2 - \left(1 + \beta + \frac{k^2}{\lambda}\right)\mu + 1 = 0$$

has 2 solutions, μ_1 & μ_2

By Vieta,

$$\mu_1 + \mu_2 = -\frac{b}{a} = \frac{1 + \beta + \frac{k^2}{\lambda}}{\beta} > 0$$

$$\mu_1 \cdot \mu_2 = \frac{c}{a} = \frac{1}{\beta} > 0$$

So since $\mu_1 \mu_2 > 0$, $\text{sign}(\mu_1) = \text{sign}(\mu_2)$

Since $\mu_1 + \mu_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$

Now does Woodford know that $\mu_1 < 1 < \mu_2$?

(V1) $\mu_1 + \mu_2 = \frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta} > 1$ (in fact > 2)

(V2) $\mu_1 \cdot \mu_2 = \frac{1}{\beta} > 1$ but close to 1. (around 1.1)

(V2) implies that at least 1 root > 1 b/c if $\mu_1 = \mu_2 \leq 1$,

(V2) ∇ so suppose $\mu_2 > 1$. Why must $\mu_1 < 1$? E.g.

Why can't $\mu_1 = 1$?

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In Woodford's case this is

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - 4\beta}}{2\beta}$$

$$\begin{aligned} \text{Determinant is } & \left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - 4\beta & =: \Delta \\ & = \left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - (2\sqrt{\beta})^2 & a^2 - b^2 = (a+b)(a-b) \\ & = \left(1 + \beta + \frac{k^2}{\lambda} - 2\sqrt{\beta}\right) \left(1 + \beta + \frac{k^2}{\lambda} + 2\sqrt{\beta}\right) \\ & \quad \underbrace{\qquad\qquad\qquad}_{>0} \end{aligned}$$

Ok so for sure there is a value of λ that ensures that (close enough to 0)

Supp $\beta = 0.99$. Then $\sqrt{\beta} = 0.95 \rightarrow 2\sqrt{\beta} = 1.95$

which is miraculously $= 1 + \beta = 1.99$!

$$1 + \beta - 2\sqrt{\beta} \rightarrow 1 + x^2 - 2x \quad \text{where } x = \sqrt{\beta}$$

I can write this as $(1-x)^2 = (1-\sqrt{\beta})^2 \geq 0$ for $\beta < 1$

ok so at least I know I have 2 distinct real roots.

Ok but that means that $\Delta > 0$ so that $\sqrt{\Delta} > 0$

So the 2 roots are

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\Delta}}{2\beta}$$

Is the idea that $\frac{1 + \beta + \frac{k^2}{\lambda}}{2\beta} \approx 1$?

$$= \frac{1}{2\beta} + \frac{1}{2} + \frac{k^2}{2\beta\lambda}$$

$$= \frac{1}{2} \left(\frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta} \right)$$

$$= \frac{1}{2} \left(1 + \frac{1}{\beta} \left(1 + \frac{k^2}{\lambda} \right) \right)$$

The problem is that it seems to me to be a quantitative question, but it should hold for all param values > 0 .

So for $\beta \rightarrow 1$ and $k \rightarrow 0$ the above becomes

$$\frac{1}{2} \left(1 + 1 \left(1 + 0 \right) \right) = \frac{1}{2} (1+1) = 2 \text{ de fine.}$$

And $\frac{k^2}{\lambda} \rightarrow 0$ even if $\lambda \rightarrow \infty$ b/c $k^2 \rightarrow 0$ faster.

I guess \Rightarrow So $0 < \mu_1 < 1 < \mu_2$. And this is important b/c $\mu_2 > 1$ is explosive, so TVC excludes it.

so one side-question that arises is Blanchard & Kahn conditions, which dictates as many explosive eigenvalues as forward-looking vars. why? And what's up w/ eigenvalues vs. roots of characteristic equations here?

→ or: "characteristic roots are also known as eigenvalues." □

Alpha Unidad makes this connection explicit when he discusses dynamic stability of "simultaneous equations (ODE or difference eqs)"

→ Need to work them out the intuition is this. For an n -dimensional difference eq, transform it into a 1-dimensional matrix diff eq, and then proceed as usual.

So for Woodford, the complementary function is

$$\varphi_t = A_1 \mu_1^t + A_2 \mu_2^t$$

I guess TRC sets $A_2 = 0$.

So $\varphi_t = A_1 \mu_1^t$ is the complementary sol, for A_1 , yet to be determined.

For the particular solution, guess $\varphi_t = b$

$$\beta k - (1 + \beta + \frac{k^2}{\lambda})k + k = kx^*$$

$$k = \frac{kx^*}{1 + \beta - 1 - \beta - \frac{k^2}{\lambda}} = \frac{x^*}{-\frac{k}{\lambda}} = -\frac{\lambda x^*}{k}$$

So the general sol is

$$\varphi_t = A_1 \mu_1^t - \frac{\lambda x^*}{k}$$

Now use the initial condition $\varphi_{t=1} |_{t=0} = 0$
(set $t=0$)

$$0 = A_1 \mu_1^0 - \frac{\lambda x^*}{k} \Rightarrow A_1 = \frac{\lambda x^*}{k}$$

So then the general sol is

$$\varphi_t = \frac{\lambda x^*}{k} \mu_1^t - \frac{\lambda x^*}{k} = -\frac{\lambda x^*}{k} [1 - \mu_1^t] \quad (1.12)$$

The only thing I don't know is why $t+1$, not t ? Typo?

$$\bar{\pi}_t = \varphi_{t-1} - \varphi_t \\ = -\frac{\lambda x^*}{K} \left[1 - \mu_1^{t-1} \right] - \left(-\frac{\lambda x^*}{K} \left[1 - \mu_1^t \right] \right)$$

$$= -\cancel{\frac{\lambda x^*}{K}} + \frac{\lambda x^*}{K} \mu_1^{t-1} + \cancel{\frac{\lambda x^*}{K}} - \frac{\lambda x^*}{K} \mu_1^t$$

$$\bar{\pi}_t = \frac{\lambda x^*}{K} \mu_1^{t-1} (1 - \mu_1) \quad (1.13)$$

↑ except that again Woodford has t here.

tomorrow :
1) ↑ check on this

2) complete Alpha (Chiang's) eigenvalue

& simultaneous eq. thing

✓ 3) start reworking draft for midwest

macro, using Ryan's abstract-ish ...

2) Alpha Chiang's connection between matrices & difference equations (& differential eqs) 17 Feb 2020
 Consider the 2nd order diff eq: (p. 602)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \quad (17.21)$$

"convert an artificial new variable $x_t = y_{t+1}$ "

$$x_{t+1} + a_1 x_t + a_2 y_t = c$$

$$y_{t+1} = x_t$$

$$\Rightarrow \begin{aligned} x_{t+1} + a_1 x_t + a_2 y_t &= c \\ y_{t+1} - x_t &= 0 \end{aligned} \quad \left. \right\} \quad (17.21')$$

Suppose then that this looks like

$$x_{t+1} + b x_t + g y_t = h \quad (17.24)$$

$$y_{t+1} - x_t = 0$$

- ① Seek particular integrals (denote them by \tilde{x} & \tilde{y} since they're 1st order values)

- ② Seek complementary functions

① First try stationary sols. If not doesn't work,
try moving sols of the form $x_t = k_1 t$, $y_t = k_2 t$ etc.

Supp. $x_{t+1} = x_t = \bar{x}$ and $y_{t+1} = y_t = \bar{y}$. But

$$\begin{aligned} 7\bar{x} + 3\bar{y} &= 4 \\ \bar{y} - \bar{x} &= 0 \end{aligned} \quad \left. \begin{aligned} \bar{x} &= \bar{y} = \frac{4}{10} = \frac{2}{5} \\ \bar{y} &= \bar{x} \end{aligned} \right\}$$

② Complementary parts:

$$y_t = m b^t \quad y_t = n b^t \quad \text{Sust.}$$

$$x_{t+1} + 6x_t + 3y_t = 0 \Rightarrow mb^{t+1} + 6mb^t + 3nb^t = 4$$

$$y_{t+1} - x_t = 0 \quad nb^{t+1} - mb^t = 0$$

$$\therefore b^t$$

$$\begin{aligned} mb + 6m + 3n &= 0 \\ nb - m &= 0 \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \quad (17.28)$$

looking to solve this for (m, n) .

$$\begin{aligned} (b+6)m + 3n &= 0 \\ -1m + bn &= 0 \end{aligned} \quad \left. \begin{aligned} & \text{cofficient matrix} - \begin{pmatrix} b+6 & 3 \\ -1 & b \end{pmatrix} \end{aligned} \right\}$$

In order to solve (17.28) w/o $m=n=0$, need the determinant of the coff. matrix "to vanish". i.e.

$$\begin{vmatrix} b+6 & 9 \\ -1 & b \end{vmatrix} = 0$$

$$\Rightarrow (b+6)b + 9 = 0 \Rightarrow b^2 + 6b + 9 = 0 \quad (17.29)$$

(17.29) is the characteristic equation, and its roots $b_{1,2}$ are the characteristic roots = eigenvalues of the simultaneous eq. system.

Having b , (17.28) gives us (m, n) . (They usually are equations of the form $m_i = k, n_i$ for each root b_i)

Let's do the same thing w/ matrix notation

Express

$$x_{t+1} + 6x_t - 9y_t = 4 \quad (17.29)$$

$$y_{t+1} - x_t = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=I} \underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}}_{=u} + \underbrace{\begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}}_{=K} \underbrace{\begin{bmatrix} x_t \\ y_t \end{bmatrix}}_{=v} = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{=d} \quad (17.29')$$

$$= I u + K v = d$$

where $u = \underbrace{u}_{=n}$ $v = \underbrace{v}_{=n}$ in Alpha Wang's notation

$$I z_{t+1} + K z_t = d$$

① Particular integral: try constant sols

$$z_{t+1} = z_t = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \Rightarrow (I + K) \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = d$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = (I + K)^{-1} d \quad \text{if } (I + K)^{-1} \text{ exists.}$$

② Complementary functions

$$z_{t+1} = \begin{bmatrix} m b^{t+1} \\ n b^{t+1} \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} \quad z_t = \begin{bmatrix} m \\ n \end{bmatrix} b^t$$

$$\Rightarrow I \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} + K \begin{bmatrix} m \\ n \end{bmatrix} b^t = 0 \quad | : b^t$$

$$(bI + K) \begin{bmatrix} m \\ n \end{bmatrix} = 0 \quad (17.28')$$

It is from this that we wanna find $\begin{bmatrix} m \\ n \end{bmatrix}$

Again we wanna avoid $m=n=0$ type solutions

so we demand $|bI + K| = 0$

which is again the characteristic eq, w/
eigenvalues / characteristic roots as roots.

Further comment on the characteristic equation (p.610)

1. Characteristic eq. of a matrix
2. of a single diff. eq / ODE
3. of system of diff. eqs / ODEs.

What's the connection?

1. If I write an n -order single diff.-eq as a 1^{st} -order system of eqs, the characteristic eq. is the same (as we saw on the previous ex.)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

$$\begin{aligned} \text{CE: } & \frac{b^2 + a_1 b + a_2}{\text{or}} = 0 \\ & \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}}_{= K} = \begin{bmatrix} c \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{CE: } & |bI + K| = \begin{vmatrix} a_1 + b & a_2 \\ -1 & b \end{vmatrix} = (a_1 + b)b + a_2 = 0 \\ & \underline{\underline{b^2 + a_1 b + a_2 = 0}} \end{aligned}$$

2. Connection #2

There's a link between the characteristic eq. of a diff. eq. (or an ODE) to that of a particular matrix, call it D .

$$|bI - D| = 0 \quad (17.38')$$

makes it clear that $D = -K$. D , or $-K$, has a special meaning. Write $I z_{t+1} + K z_t = d$ as $I z_{t+1} = -K z_t$ (setting $d = 0$ for the reduced version of the system).

$$\Rightarrow z_{t+1} = -K z_t$$

$\Rightarrow -K$ is the matrix that can transform the system from z_t to z_{t+1} . (i.e. it's the transition matrix.)

\Rightarrow The characteristic equation is the $\det(\text{com-mat}) = 0$ relation!

NB. If $I = f$ (not identity) so $f z_{t+1} = -K z_t$

then we need to $z_{t+1} = f^{-1}(-K) z_t$ and look at $\det(-f^{-1}K) = 0$. (SVAR!)

Return to Woodford & try to obtain (1.12) again.

Let me write (1.10) as

$$\beta \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = kx^*$$

$$\Rightarrow \beta \mu^2 - \alpha \mu + 1 = 0$$

$$\mu_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}$$

$$Ok, so \alpha = 1 + \beta + \frac{k^2}{\lambda} \in [1 + \beta, \infty]$$

$$So for \alpha = \min(\alpha) = 1 + \beta,$$

$$\begin{aligned} \mu_{1,2} &= \frac{\alpha \pm \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} = \frac{\alpha \pm \sqrt{\beta^2 + 2\beta + 1 - 4\beta}}{2\beta} \\ &= \frac{\alpha \pm \sqrt{(\beta-1)^2}}{2\beta} = \frac{1+\beta \pm (\beta-1)}{2} \quad \begin{array}{l} \frac{1+\beta + \beta-1}{2} = \beta < 1 \\ \frac{1+\beta - \beta+1}{2} = 1 \end{array} \end{aligned}$$

$$If \alpha \rightarrow \infty$$

$$\mu_{1,2} = \frac{\infty \pm \sqrt{\infty^2}}{2\beta} = \frac{\infty + \infty}{2\beta} = \frac{1}{\beta} > 1$$
$$\frac{\infty - \infty}{2\beta} = 0 < 1 \quad Ok!$$

Since, Woodford, $0 \leq \mu_1 < 1 \leq \mu_2$

Ok so before turning a 100% to Woodford, let's think about solutions to DSGE models

In Lect 10, Basu looks for the solutions of the 2-dimensional non-interacting system where the eq. sols are $z'_H = \alpha w_1^+$ and $z'_{D1} = \beta w_2^+$

Since $w_1 > 1$ (and $w_2 \in (0, 1)$), TVC sets $\alpha \neq 0$ (α, β) correspond here to Alpha/Beta's A_1, A_2 etc. which are usually determined by initial conditions.
(what AC calls "definitize the constants A_i ")

For bw-looking vars, TVCs exclude explosive sols.

So maybe the point is that when the economy is a bw-looking eq. system, this is what you do. But if the econ is a fw-looking system, then

$$z_t = w_1 z_{t+1} \quad w_1 < 1, \text{ so } z_t = \frac{1}{w_1} z_{t+1}$$

$\frac{1}{w_1} > 1 \Rightarrow$ transforming the fw-looking one into a bw-looking one requires explosive eqs of the bw-looking system. (Blanchard & Kahn)

Finally back to Woodford, 100%

right, so I've solved the char. eq. to obtain μ_1 .

So my conj must have been $\varphi_t = A\mu^t$

Or did Woodford conjecture $\varphi_t = A\mu^{t+1}$? So that

$$\beta A\mu^{t+2} - \alpha A\mu^{t+1} + A\mu = 0$$
$$\Leftrightarrow \beta\mu^2 - \alpha\mu + 1 = 0$$

But technically it doesn't matter b/c after the conjecture $\varphi_t = A\mu^t \rightarrow \beta A\mu^{t+1} - \alpha A\mu^t + A\mu^{t-1} = 0$

$\beta\mu - \alpha + \frac{1}{\mu} = 0$ leads to the same thing if you multiply by μ ,

$$\beta\mu^2 - \alpha\mu + 1 = 0$$

So taking Woodford's conjecture then, turn to the particular integral:

A stationary sol would involve $\varphi_t = \bar{\varphi} +$

$$\beta\bar{\varphi} - \alpha\bar{\varphi} + \bar{\varphi} = kx^*$$

$$\bar{\varphi}(\beta - \alpha + 1) = kx^*$$

$$\bar{\varphi} = \frac{kx^*}{1 + \beta - \alpha} = \frac{kx^*}{-\frac{1}{K}} = -\frac{\lambda}{K}x^*$$

So w/o yet having definitized the constant, A,
the general sol is

$$Y_+ = A \mu_1^{t+1} + \left(-\frac{\lambda}{K} x^*\right)$$

Definitize A using initial cond: when $t = -1$,

$$Y_{-1} = 0, \quad \text{so} \quad 0 = A - \frac{\lambda}{K} x^*$$

$$\Rightarrow A = \frac{\lambda}{K} x^*$$

So the "full-fledged" general sol then is

$$\begin{aligned} Y_+ &= \frac{\lambda}{K} x^* \mu_1^{t+1} - \frac{\lambda}{K} x^* \\ Y_- &= -\frac{\lambda}{K} x^* (1 - \mu_1^{t+1}) \end{aligned} \tag{1.12}$$

Alright!

And since $\pi_+ = Y_{-1} - Y_+$

$$\begin{aligned} &= -\frac{\lambda}{K} x^* (1 - \mu_1^+) - \left(-\frac{\lambda}{K} x^* (1 - \mu_1^{t+1})\right) \\ &= \frac{\lambda}{K} x^* \left[-1 + \mu_1^+ + 1 - \mu_1^{t+1} \right] \\ &= \frac{\lambda}{K} x^* (\mu_1^+ (1 - \mu_1)) \end{aligned}$$

$$\Rightarrow \pi_+ = (1 - \mu_1) \frac{\lambda}{K} x^* \mu_1^+ \tag{1.13}$$

Consider the timevarying optimal commitment policy. I think that the complementary solution is the same, but λ will be defined differently b/c the initial condition

(19) $y_{-1} = 0$ is replaced by $\pi_+ = \bar{\pi} \quad \forall +$
so that $\pi_0 = \bar{\pi}$.

$$(113) \text{ implies } \pi_+ \stackrel{!}{=} \bar{\pi} = (1 - \mu_1) \frac{\beta}{K} x^* \mu_1^+$$

let me recall that $A = \left(\begin{smallmatrix} -\frac{\beta}{K} x^* \\ 1 \end{smallmatrix} \right)$ so

$$\pi_+ = (1 - \mu_1)(-A)\mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$(\mu_1 - 1) A \mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$\mu_1 A \mu_1^+ - A \mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$\Rightarrow A \stackrel{!}{=} 0 \text{ and } \bar{\pi} \stackrel{!}{=} 0 \quad \text{otherwise } \frac{1}{2}.$$

Optimal responses to shocks p. 488.

Take the same loss (except in $E_{t_0}(\cdot)$) as before

(eq (2.4)). Then clearly you obtain the same diff eq for the multipliers φ_+ (eq (1.10)) except there's an $E(\cdot)$ in front (eq (2.6)) and

The cost-push shock u_t shows up on the RHS

$$\beta E_t y_{t+1} - \alpha p_t + y_{t-1} = \kappa x^* + u_t \quad (2.6)$$

The characteristic eq. is the same, eq. (1.11),

which has 2 roots, $0 < \mu_1 < 1 < \mu_2$.

The question is: how the f^* does Woodford obtain equation (2.7)?

We said that $y_t = A_1 \mu_1^{t-1} + A_2 \mu_2^{t-1}$

but TBC set $A_2 = 0$.

So then the complementary sol is the same,

$p_t = A \mu_1^{t-1}$, except that A may be different.

What about the particular solution, \bar{p} ?

$$\bar{p} (\beta - \alpha - 1) = \kappa x^* + u_t$$

↑ how to treat u_t ?

As long as u_t doesn't have a drift, its LR mean is 0. That would imply the same particular sol as in the non-stochastic case. But if \bar{p} is the same,

then A is also the same.

Ok, so since the μ_1^{t+1} terms are gone, we must have gone somehow from (1.12) :

$$y_t = -\frac{\gamma}{K} x^* (1 - \mu_1^{t+1}) \quad \text{to (2.7).}$$

$$= -\frac{\gamma}{K} x^* (1 - \mu^+ \cdot \mu_1)$$

$$= -\frac{\gamma}{K} x^* (\mu_1 - \mu_1 \mu^+ + 1 - \mu_1)$$

$$= \mu_1 \left[-\frac{\gamma}{K} x^* (1 - \mu^+) \right] - \frac{\gamma}{K} x^* (1 - \mu_1)$$

$$y_t = \mu_1 y_{t-1} - (1 - \mu_1) \frac{\gamma}{K} x^* \quad \text{haha!}$$

The question that remains is how/where do the u_{t+j} terms come from?

I'm wondering if from the perspective of the difference eq u_t isn't an "endog" variable?

Or is Woodford saying that u_t follows its own diff eq which can/is solved?

Wait. Maybe we guess a bit of a different

complementary sol: $y_t = A_1 \mu_1^{t+1} + A_2 u_t$

or, better yet: $y_t = A_1 (\mu_1^{t+1} + u_t)$. But the problem

even w/ this formulation is that it only explains why a single u_{t+j} -term would show up in y_{t+j} . But as long as the characteristic eq is the same and the particular sol too, I don't see how u would enter w/ a sum.

Brown Lect 10 gives some hints: S.3 (p. 3 Mac)

$$z_t = az_{t-1} + m_t \quad |a| < 1 \quad (\text{stable diff eq})$$

Recursive out

$$\begin{aligned} z_t &= a[z_{t-2} + m_{t-1}] + m_t \\ &= a^2 z_{t-2} + am_{t-1} + m_t \\ &= a^k z_{t-k} + \sum_{s=k}^{t-1} a^{t-s} m_s \end{aligned} \quad (\text{exog fctg time } m_t)$$

→

$$z_t = a^t z_0 + \sum_{s=1}^t a^{t-s} m_s$$

Let's reinterpret this a bit

$$z_t = \left(\underbrace{z_0}_{\text{initial condition}} \right) a^t + \underbrace{\sum_{s=0}^t a^{t-s} m_s}_{\text{dynamics from initial cond.}}$$

*exog function
of time*

$z_t = A \cdot \mu^t + \underbrace{\sum_{s=0}^t \mu^{t-s} m_s}_{\text{dynamics from initial cond.}}$

Basin continues w/ unstable diff eqs w/ exog shift:

$$z_t = \alpha z_{t-1} + m_t \quad |\alpha| > 1$$

$$\Rightarrow \alpha z_{t-1} = z_t - m_t \quad \Rightarrow \quad z_{t-1} = \bar{\alpha}^{-1} z_t - \bar{\alpha}^{-1} m_t$$

$$\begin{aligned} \Rightarrow z_{t-1} &= \bar{\alpha}^{-1} [\bar{\alpha} z_{t+1} - \bar{\alpha} m_{t+1}] - \bar{\alpha}^{-1} m_t \\ &= \bar{\alpha}^{-2} z_{t+1} - \bar{\alpha}^{-2} m_{t+1} - \bar{\alpha}^{-1} m_t \\ &= \bar{\alpha}^{-k} z_{t+k} - \sum_{s=0}^k \bar{\alpha}^{-(s+1)} m_{t+s} \end{aligned}$$

Susanto writes the index differently, but this way of writing it at least corresponds to Woodford

Under uncertainty, just tack on " E_t " to the exog m_{t+s}

$$z_{t-1} = \bar{\alpha}^{-k} z_{t+k} - \sum_{s=0}^k \bar{\alpha}^{-(s+1)} E_t m_{t+s}$$

so at least this is exactly

$$\text{where } \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}$$

comes from; and μ_2 is used

$$\underline{b/c} \quad \mu_2 > 1$$

and $\bar{\beta}^1$ will come from $\beta E_t \varphi_{t-1}$

But this seems to indicate, like Baum writes,

$$\varphi_t = a^+ b_0 - \sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{s-t} m_s$$

$$\text{or } \varphi_t = \underbrace{-\frac{\beta}{K} x^* (1 - \mu_1^{t+1})}_{\text{from (1.12)}}$$

effect of initial cond
general sol

effect of shock for
fwd-diff eq.

$$\mu_1 \varphi_{t+1} - (1 - \mu_1) \frac{\beta}{K} x^*$$

So it seems like when there's a function of time on the RHS, you need to add a sum, fwd or bw-looking depending on the eq. or the roots.

But how do we know which sum (fwd- or bw-looking) to add?

$$\text{Supp: } \beta E_t \varphi_{t+1} = a \varphi_t + u_t$$

$$a \varphi_t = \beta E_t \varphi_{t+1} - u_t$$

$$\varphi_t = \frac{\beta}{a} E_t \varphi_{t+1} - \frac{1}{a} u_t$$

$$y_t = \frac{\beta}{\alpha} E_t y_{t+1} - \frac{1}{\alpha} u_t$$

$$= \frac{\beta}{\alpha} \left[\frac{\beta}{\alpha} E_t y_{t+2} - \frac{1}{\alpha} E_t u_{t+1} \right] - \frac{1}{\alpha} u_t$$

$$= \left(\frac{\beta}{\alpha}\right)^2 E_t y_{t+2} - \frac{1}{\alpha} u_t - \frac{\beta}{\alpha} E_t u_{t+1} - \frac{\beta^2}{\alpha^3} E_t u_{t+2} + \dots$$

$$= \dots - \frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^s u_{t+s}$$

Ok, now this raises the issue of what exactly we're doing. In the Barn Lect. Notes, an AR(1) process's root is the AR-coeff. Maybe since we have a 2nd order process, we have a distinction between the AR-coeff & the roots.

\Rightarrow Indeed: for a one-dim AR(1), $y_{t+1} = a y_t + c$
a is the root! (Alpha Chiang, p. 554)

So I'm pulling on Blume p. 585 ft (Mac, 559 ft) to study difference eqs. in general.

Note: Blume calls Barn's method of making the system non-interacting "decoupling" or "change of coordinates" (Mac, 564)

On p. 567 (Mac), Blume then says: The uncoupled system

is $\mathbf{z}'_{n+1} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & & \\ & & \ddots & \\ & & & r_k \end{pmatrix} \mathbf{z}'_n$

where r_i are the eigenvalues of $A_{k \times k}$, where the orig system was

$$\mathbf{z}_{n+1} = A \mathbf{z}_n$$

then the decoupled (non-interacting) z 's are

$$z'_{1,n} = c_1 r_1^n$$

⋮

$$z'_{k,n} = c_k r_k^n$$

So the original z -variables's solutions are

$$\mathbf{z}_n = c_1 r_1^n \mathbf{v}_1 + c_2 r_2^n \mathbf{v}_2 + \dots + c_k r_k^n \mathbf{v}_k$$

where the \mathbf{v}_i are eigenvectors

i.e. a sum of terms of the form

$$c_i \cdot (\text{eigenvalue})^n \cdot \text{eigenvector}$$

where the c_i are constants determined from the initial conditions.

I'm on econdse.org and find the remark:

"DSE-Maths-
pdf" in literature

"non-autonomous equations"

$$x_t = a x_{t-1} + b_t \quad (2)$$

- The complementary function will be the same as for the autonomous eq. $x_t = a x_{t-1} + b$

- But the particular sol won't be the same b/c

trying a constant value of x won't work, so
no x_t and x_{t-1} can be the same. Good point!

There are 2 methods for arriving at the particular sol:

backward sol & forward sol

See also "diffeqn.pdf" → This is a really good note!

In particular, this note shows that finding eigenvalues
is really equal to factoring polynomials. E.g. p.17

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + b_t$ can be written as

$(1 - \phi_1 L - \phi_2 L^2) y_t = b_t$. Now, this we can

write as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)y_+ = b_+$$

These are the eigens.

We know that since L behaves like z^{-1} , we can write

$$1 - \phi_1 L - \phi_2 L^2 = 0 \quad \text{as} \quad 1 - \phi_1 z^{-1} - \phi_2 z^{-2} = 0$$

and multiply by z^2 to write the "associated polynomial"

$$z^2 - \phi_1 z - \phi_2 = 0 \quad \text{which we can solve for}$$

$$z_{1,2} = \lambda_{1,2}$$

So $y_+ = \frac{b_+}{(1 - \lambda_1 L)(1 - \lambda_2 L) \dots (1 - \lambda_p L)}$

So if both eigenvalues are less than 1, then

$$y_+ = \hat{c}_1 \lambda_1^+ + \hat{c}_2 \lambda_2^+ + \frac{1}{(1 - \lambda_1 L)(1 - \lambda_2 L)} b_+ \quad \text{eq. (27)}$$

(P. 14)

In terms of Woodford, what I still don't know is why he solves it forward (why not solve bw using $\mu_1 < 1$?) and where B^{-1} is coming from?

Ok I don't know: the result of today's hard work is that in the case of a non-autonomous diff. eq., the sol. takes the form

$$y_t = y_c + y_p$$

↑
same as for
autonomous eq

↑ not the same as for nonhom.
eq. instead, this will be a
forward or backward sum

of b_t (τ_1 exog. part), w/
the eigenvalues as coeffs.

What I don't know is how to determine which eigenvalue will be coefficient.

- If the system is univariate, there's only one eig. λ and the fwd/bwd-nature of the sum depends on whether $|\lambda| \geq 1$
- If the system is p -variate, but all eigs $|\lambda_i| < 1$

then $y_p = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L) \dots (1-\lambda_p L)}$ (quen... :S)

• If all eigs $|\lambda_i| > 1$ then $y_p = \frac{1}{(1-\lambda_1^{-1}L)(1-\lambda_2^{-1}L) \dots (1-\lambda_p^{-1}L)}$

What I don't know is what when one eig $\mu_1 < 1$
the other $|\mu_2| > 1$.

The lecture notes by Tirotti "linear...Tirotti.pdf"
seem to provide the answer. (not here.)

Actually, they *almost* do
- but they also don't say what to do w/ "alternating"
eigenvalues.

So I'll pause the issue of (2.7) here for a while.
I wanna go on to see what's going about implementation.

Note on nonneutral p. 407 : purely fwd-looking policy, i.e.
maximizes the same utility as under Ramsey but
subject to the condition that endog. vars evolve
according to $\bar{z}_t = \bar{z} + f' s_t$ (purely fwd-looking)
i.e. the optimal plan is purely fwd-looking.

It's mean b/c Woodford has actually introduced one
year before he officially does starting p. 507.

What I said just now isn't quite right: splitting
the same loss into L^{det} & L^{stab} serves the purpose
of making one 1) time-consistent 2) still optimal.
The point is that whatever path a nation has $\pi_t = \bar{\pi} + f_S$
gives you is not time-consistent (b/c it is purely
forward-looking), but simply imposing the timelessly
optimal plan from before is not optimal for $\pi_t = \bar{\pi} + f_S$.
So you split the problem in two, allowing the same
time- or timelessly optimal policy to minimize L^{stab}
(b/c responses to shocks optimally are the same)
which is independent of the econ's state at any time t .
And shit: the LR averages are exactly NOT more than min
 L^{det} ; instead you just adopt those from a timelessly
optimal policy.

p. 516: Woodford also says that if a simple TR is found that is the optimal one, then a TR that involves past does not improve upon the baseline TR. The reason is that already the baseline TR was purely forward-looking, so you haven't changed that. In fact, as Gersbach & Woodford (2002b) show, for frosts of horizons large enough, the cgs becomes indeterminate.

By the way: given a π -admissible optimal set for the Ramsey problem, p. 518 calculates the opt. time paths for π & i (eqs. (4.1) & (4.2))

Then there's the problem of solving for optimal plans for (π, x, i) and using the plan for i as a reaction function. But that's indeterminate.

\Rightarrow So the optimal time path isn't a good policy rule, but the purely forward-looking TR isn't fully optimal. \Rightarrow

Are targeting rules the solution? p. 521

A targeting rule is a commitment of the CB to adjust its instrument such that a target criterion is projected to be fulfilled at all points in time.

One target criterion which is optimal from a timeline perspective is

$$\pi_t + \frac{\lambda}{k} (x_t - x_{t-1}) = 0 \quad (\text{S.1})$$

It is not just optimal, but it is also feasible (b/c an REE w/ it exists), moreover it's unique (not indeterminate)

$\Delta x_t \rightarrow$ reflects the history-dependence of optimal commitment (p. 525)

But you know, there are other robustly optimal target criteria. (p. 526-27)

But we still don't know how to implement the targeting rule, all we have is an optimal target criterion.

→ But you can use the target criterion together w/ model equations & Laws of Motion to derive a reaction

function for it.

As a first-pass, you obtain the **fundamentals-based reaction function** of E&H (5.4)

(You've assumed the private sector's expectations are RE)

The problem though is that for many parameter values, this again isn't determinate.

So instead of using expectations that "ought to be", plug in expectations of the private sector the CB actually desires: **"expectations-based reaction function"** (w/o solving for 'em) (5.5)

This is determinate according to Prop 7.17.

→ so an interest rate rule (5.5) [=reaction function] is equivalent to committing to the π -target (5.1). If implements it, this (5.5) is a forward-looking TR, however, it does have history-dependence from

$$x_{t-1} \cdot w \mid \frac{(k^e + \gamma) - \gamma}{\alpha(k^e + \gamma)} = \frac{k^e}{\alpha(k^e + \gamma)} = \phi_x \text{ in LR.}$$

Looking at Gaspar et al 2011 & Preston 2008,
it's not quite clear to me what the Ramsey problem
should look like.

Ok: supposing that the indexation parameter $\gamma = 0$,
at least the period loss L_t takes the same form
in Gaspar et al 2011 & Preston 2008:

$$L_t = \pi_t^2 + \gamma x_t^2$$

$$W = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \quad \leftarrow \text{overall loss fn.}$$

And this is also the loss Molnár & Santoro (2016)
consider (except they call γ d for a charge)
and they only claim to do discretionary policy (but
actually, they seem to be solving the commitment
problem).

And indeed: Preston takes Woodford's equations and
replaces the eigenvalue $\mu_2 > 1$ w/ $\mu_1 < 1$
(see my comments in Preston 2008, p. 8 Mac)

Ok, brace yourselves, I'm approaching the "approach":

① $\min \mathcal{L} \text{ s.t. ALM, LDM (beliefs)}$

$$\pi_t, x_t, i_t, \phi_t \\ ?$$

this will give you FOCs that, given eigs, can be solved for the evolution of \mathcal{L} -multipliers, which can be subbed back to obtain paths for

$$\{\pi_t, x_t, i_t, \phi_t\}$$

- ② sub out \mathcal{L} -multipliers from FOCs to obtain a target criterion: π_t in terms of x (x_1, x_{t+1}, \dots).
(Analogue of (5.1))

- ③ Use the ALMs to figure out what \hat{E}^H & \hat{E}^X are and confront them w/ the criterion to obtain the i-rate rule / reaction function.

In materials 17, I have written out

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both the big - & - ugly Lagrangian for the full system,
as well as that of a simplified one (eq 20-25)
(or 26-31)

Take FOCS of the simplified one.

$$\pi: 2\pi_+ + \gamma_{1,+} - \gamma_{5,+}\bar{g}^{-1} = 0 \quad (1)$$

$$x: 2\lambda x_+ - \gamma_{1,+}k + p_{2,+} = 0 \quad (2)$$

$$i: p_{2,+} b = 0 \rightarrow p_{2,+} = 0 \Rightarrow \text{interesting: } 1/m$$

also finding that 1S - curve isn't binding.

$$fa: -\gamma_{1,+}(1-\alpha)\beta + \gamma_{3,+} = 0 \quad (3)$$

$$fb: -p_{2,+}b + p_{4,+} = 0 \quad (4)$$

$$\begin{aligned} \tilde{\pi}_+: & -E_T \beta \gamma_{3,+} \frac{1}{1-\alpha\beta} - E_T \beta \gamma_{4,+} \frac{1}{1-\beta} \\ & + \gamma_{5,+} + E_T \beta \gamma_{5,+} (-1 + \bar{g}^{-1}) = 0 \end{aligned} \quad (5)$$

The good news is that these FOCS look similar to
the ones Gaspar et al (2011) get.

In particular, using that $p_{2,+}=0$, I can combine (1) & (2)

where I've rewritten the rational expectations:

$$\text{If you can form } E(x)^{\text{RE}}, \text{ then } E_T x_{T+1} = g_2 E_T s_{T+1} \\ = g_2 h_x s_T$$

$$E_T x_{T+k} = g_2 h_x^k s_T \rightarrow E_T x_T = g_2 h_x^{T-t} s_T \\ = b_2 h_x^{T-t-1} s_T$$

$$\rightarrow E_T x_{T+1} = g_2 h_x^{T-t+1} s_T = b_2 h_x^{T-t} s_T$$

$$\kappa \alpha \beta E_T \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} x_{T+1} = \kappa \alpha \beta E_T \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} b_2 h_x^{T-t} s_T \\ = \kappa \alpha \beta b_2 \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} h_x^{T-t} s_T \\ = \kappa \alpha \beta b_2 (I_3 - \alpha \beta h_x)^{-1} s_T$$

$$2\pi_t + \gamma_{1,t} - \gamma_{5,t} \bar{g}^{-1} = 0 \quad (1)$$

$$2\lambda x_t - \gamma_{1,t+k} = 0 \quad (2)$$

$$(1): \gamma_{1,t} = 2\frac{\lambda}{\kappa} x_t$$

$$\Rightarrow (1): 2\pi_t + 2\frac{\lambda}{\kappa} x_t - \gamma_{5,t} \bar{g}^{-1} = 0$$

$$\Rightarrow x_t = \frac{k}{\lambda} \left(\frac{\gamma_{5,t}}{\alpha} \bar{g}^{-1} - \pi_t \right)$$

$$\Rightarrow x_t = -\frac{k}{\lambda} \left(\pi_t - \frac{\gamma_{5,t}}{\alpha} \bar{g}^{-1} \right) = \text{Gaspar et al's (22)!}$$

Now turn to (3)-(5): (using $\varphi_{2,t} = 0$)

$$-\varphi_{1,t} (1-\alpha) \beta + \varphi_{3,t} = 0 \quad (3)$$

$$\varphi_{4,t} = 0 \quad (4)$$

$$-E_t \beta \varphi_{3,t+1} \frac{1}{1-\alpha \beta} - E_t \beta \varphi_{5,t+1} \frac{1}{1-\beta} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0 \quad (5)$$

$\rightarrow \varphi_{4,t} = 0$ b/c f_β only shows up in the NKIS, which isn't binding.

$$(3): \varphi_{3,t} = (1-\alpha) \beta \varphi_{1,t}$$

$$(5) \quad -E_t \beta \frac{(1-\alpha) \beta \varphi_{1,t+1}}{(1-\alpha) \beta} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0$$

But we know $\varphi_{1,t} = 2 \frac{\partial}{K} x_{t+1}$ so (5) becomes

$$-E_t \beta 2 \frac{\partial}{K} x_{t+1} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0 \quad (5')$$

Now let's solve (5') for $\varphi_{5,t}$ by iterating fwd

$$\varphi_{5,t} = 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta E_t \varphi_{5,t+1}$$

$$= 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta \left[2 \beta \frac{\partial}{K} E_t x_{t+2} + (1 - \bar{g}^{-1}) \beta E_t \varphi_{5,t+2} \right]$$

$$= 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta 2 \beta \frac{\partial}{K} E_t x_{t+2} + ((1 - \bar{g}^{-1}) \beta)^2 E_t \varphi_{5,t+2}$$

$$\varphi_{5,t} = 2 \beta \frac{\partial}{K} \sum_{i=0}^{\infty} (1 - \bar{g}^{-1})^i E_t x_{t+1+i} \quad \text{except for the sign,}$$

it's the same as Gaspar et al (2011).

So now plug the sol for $\varphi_{S,t}$ into

$$x_t = -\frac{\lambda}{\kappa} \left(\pi_t - \frac{\beta \varphi_{S,t}}{2} \bar{g}^{-1} \right), \quad \text{expressed for } \pi_t$$

$$-\frac{\lambda}{\kappa} x_t + \frac{\bar{g}^{-1}}{2} \varphi_{S,t} = \pi_t$$

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \frac{\bar{g}^{-1}}{2} \left[2\beta \sum_{i=0}^{\infty} (1-\bar{g}^{-1})^i E_t x_{t+i+1} \right]$$

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \bar{g}^{-1} \beta \sum_{i=0}^{\infty} (1-\bar{g}^{-1})^i E_t x_{t+i+1}$$

$$\pi_t = -\frac{\lambda}{\kappa} \left(x_t - \beta \bar{g}^{-1} E_t \sum_{i=0}^{\infty} (1-\bar{g}^{-1}) x_{t+i+1} \right) \quad (24)$$

Gaspar et al's

Allows you to get the nice interpretation of Gaspar et al 2011

when the gain is 0, (24) boils down to

$$\pi_t = -\frac{\lambda}{\kappa} x_t \quad (= \text{discretionary RE solution})$$

This is what Gaspar et al call an "intertemporal tradeoff" (or actually, Molnár & Santoro do)

If the gain > 0 , then there is an additional, inter-temporal tradeoff between π & x b/c whatever π

you bring about today, it affects π -expectations tomorrow, presenting you w/ an interesting tradeoff tomorrow.

Mendoza & Santoro derive an interest rate rule!

they say: plug $ARM(\pi)$ & $ARM(x)$ (eq 18 & 19) into the NKIS

Their ARMs take the following form

$$\pi_t = c_\pi a_t + d_\pi u_t \quad \text{and} \quad \hat{E}_t \pi_{t+1} = a_t$$

$$x_t = c_x a_t + d_x u_t \quad \hat{E}_t x_{t+1} = b_t$$

$$\text{NKIS: } x_t = \hat{E}_t x_{t+1} - b^{-1} (r_t - \hat{E}_t \pi_{t+1})$$

$$\Leftrightarrow b x_t = b \hat{E}_t x_{t+1} - r_t + \hat{E}_t \pi_{t+1}$$

$$r_t = b \hat{E}_t x_{t+1} + \hat{E}_t \pi_{t+1} - b x_t$$

$$= b b_t + a_t - b(c_x a_t + d_x u_t)$$

$$= \underbrace{(1 - b c_x)}_{\delta_\pi} a_t + \underbrace{b b_t}_{\delta_x} - \underbrace{b d_x u_t}_{\delta_u}$$

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u u_t$$

which is what M&S obtain if you plug in c_x & d_x .

→ This is their expectation-based reaction function

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_y a_t \quad (20)$$

Ryan meeting

19 Feb 2020

Mantra:

Question of the paper: "How does a concern for the anchoring of expectations affect the conduct of monetary policy?"

- 1) Take simple pencil & paper: how does it change (32)? An extra term ...
- 2.) Numerically solve the full-fledged model
- 3.) Estimate the gain function $k(\theta_t)$
"My est. suggests that $E(\cdot)$ are likely to be anchored when people are surprised by 1%." → CB-ers would be interested to hear that. It'd be a great ending to the abstract.

"There should be a gentlemanly distance between assumptions and results." (Stephanie Schmalzle) For me, this means that if I ask "does anchoring matter for policy?" then it's too easy to critique my work by saying "well, the way you set up the model, of course it does".

The mantra is mainly a marketing object: it doesn't assume that there is a concern for anchoring a priori, but given that I find that there is, how anchoring changes the policy problem, I avoid that kind of critique.

The reason Woodford might advocate an approach like Preston (2008) where the RE optimum should be implemented under learning is b/c he might say that "it is not obvious that a policymaker can credibly commit to a non-RE optimum."

Big issue: why is RE-discretion the alternative here? Did I not accidentally write down the discretion problem?

Ryan: discretion is when I choose a single it, commitment when I choose a whole plan of {it}.

↳ check discretion is commitment in a very simple problem!

Work after

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So I think that discretion

comes about here b/c there aren't really lagged multipliers.

Ryan made a remark that w/ commitment, you get lagged multipliers and it's intuitive that you do - when you choose sequences (today's and tomorrow's values) one multiplier will be date t, the other t-1. Which is in fact exactly why Woodford's (1.7) is

$$\pi_t + \gamma_t - p_{t-1} = 0$$

OK but this goes to the heart of my unease w/
the way I treat f_C & f_D

Take a simplified version of Woodford's Ramsey problem

(p. 472)

$$\mathcal{L} = E \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + p_t (\pi_t - \underline{\beta \pi_{t+1}}) \right\} \quad \begin{matrix} \text{Commitment} \\ \text{discretion} \end{matrix}$$

vs.

$$\mathcal{L} = E_T \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + y_t (\pi_t - \beta f_t) \right\} \quad \begin{matrix} \text{where } f_t \text{ is taken as} \\ \text{given} \end{matrix}$$

FDCs:

$$\pi_t + y_t - y_{t-1} = 0 \quad \begin{matrix} \text{Commitment} \\ \text{discretion} \end{matrix}$$

vs.

$$\pi_t + y_t = 0 \quad \begin{matrix} \text{discretion} \\ \text{discretion} \end{matrix}$$

What I'm scared of is that in learning, f_t becomes a new variable that is bw-looking. So you can't comment?

peter

anticipated utility shouldn't be applied to the planner

Consider 2 models for CB

1. aut. n. would be like CB has 2 rules for 1 period & doesn't care about W of future CB
2. an ∞ -hor CB that could commit to forming a policy rule today \rightarrow would be a diff.

(\hookrightarrow constrained at all periods)

Can argue that P_{t+1}

CMP isn't a model of reputation-building

b/c it \rightarrow discretionary RE, that would support egf under commitment.

Expectations, Credibility ... Ireland, 2000, Maastricht

multiple egfs based on reputation, CB-ers want

the one w/ credible → see if it's been acted
→ Cho & Matsui (1995) JEDC

E-forming scheme when all they ass
is that if the CB does the same thing,
sooner or later ppl get that that's how
it's gonna be

→ CB "if I do the right thing now, it will
have a cost, but it'll be worth it"

details of the transition to be worked out
↳ could be an extended learning scheme
which gives the CB incentives to tilt expectations
toward commitment fol.

Lucas & Sargent : PC w/ adaptive $E(\cdot)$

→ need to be sensitive to that point; my
policy isn't to exploit $E(\cdot)$ → no one is
fooled all of the time in this model!

My work is less about the st. st., more about the transition like Ball (1994).

2 approaches to opt policy

Taylor → see what are welfare of simple rules
→ I see an interesting tradeoff.

Woodford, Wübbe & Schmitt-Grohé

opt sequential choice

I see interesting imp b/w lag & future multipliers → mon. pol. shapes $E(\cdot)$

is similar in certain aspects &

diff. in others

This isn't a model of building credibility for opt. commitment. In p'tile work, one could think of alternative E-formation schemes.