Appendices to "The Role of Learning for Business Cycles and Asset Prices"

Appendix A. Details on the model

Appendix A.1. Adjustment costs and nominal rigidities

Adjustment costs are introduced via capital good producers that operate competitively in input and output markets, producing capital goods using final consumption goods. There is no distinction between new and used capital and depreciation takes place within intermediate firms. The maximization program of capital producers is entirely intratemporal:

$$\max_{I_t} Q_t I_t - \left(I_t + \frac{\psi}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right)$$

Past investment levels I_{t-1} are taken as given when choosing current investment output.[19]

Nominal rigidities are introduced via wholesale and retail firms and labor agencies. Wholesalers (indexed by $i \in [0,1]$) transform the homogeneous intermediate good into differentiated varieties using a one-for-one technology. Each wholesaler enjoys market power in her output market, and sets a nominal price p_{it}. A standard Calvo friction prevents the wholesaler from adjusting her price with probability κ. The wholesaler solves the following optimization:

$$\max_{p_{it}, Y_{it+s}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^{s} \kappa \Lambda_{t+\tau} \right) \left((1+\tau) p_{it} - q_{t+s} p_{t+s} \right) Y_{it+s}$$

$$\text{s.t. } Y_{it+s} = \left(\frac{p_{it}}{p_{t+s}} \right)^{-\sigma} \tilde{Y}_{t+s},$$

where $\tilde{Y}_t$ is aggregate demand for the composite final good and $p_t = \left(\int_0^1 p_{it}^{1-\sigma} \right)^{1/(1-\sigma)}$ is the aggregate price level. It is assumed that the government sets subsidies such that $\tau = 1/(\sigma - 1)$ so that the steady-state markup over marginal cost is zero.

Retailers transform wholesale varieties into the final consumption good according to a stan-

[19] This setup is simpler than the one in Bernanke et al. (1999) where the price of used and new capital goods differ.

dard Dixit-Stiglitz aggregation technology, solving the problem

$$\max_{Y_{it}} \left(\int_0^1 Y_{it}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \text{ s.t. } \int_0^1 p_{it} Y_{it} = p_t \tilde{Y}_t.$$

The labor input of intermediate firms is a CES combination of differentiated labor services $L_{jt} = \left(\int_0^1 L_{jht}^{\frac{\sigma_w - 1}{\sigma_w}} dh \right)^{\frac{\sigma_w}{\sigma_w - 1}}$, which each sell at the nominal wage rate w_{ht}. The real wage index is $w_t = \left(\int_0^1 w_{ht}^{1-\sigma_w} \right)^{1/(1-\sigma_w)}$. Labor agencies transform homogeneous labor input into differentiated labor goods at the nominal price $\tilde{w}_t p_t$ and sell them to intermediate firms at the price w_{ht}, which cannot be adjusted with probability κ_w. Labor agency h solves the following problem:

$$\max_{w_{ht}, L_{ht+s}} \mathbb{E}_t^{\mathcal{P}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^{s} \kappa_w \Lambda_{t+\tau} \right) \left(\left(1+\tau_w\right) w_{ht} - \tilde{w}_{t+s} p_{t+s} \right) L_{ht+s}$$

$$\text{s.t. } L_{ht+s} = \left(\frac{w_{ht}}{\tilde{w}_{t+s}} \right)^{-\sigma_w} \tilde{L}_{t+s}.$$

where $\tilde{L}_t = \int_0^1 L_{jt} dj$ is the aggregate demand for labor by all intermediate firm producers. Again, it is assumed that the government sets wage subsidies $\tau = 1/(\sigma_w - 1)$ such that the steady-state markup over marginal cost is zero.

All profits made by capital goods producers, wholesalers, retailers and labor agencies accrue to lending households. Similarly, the subsidies described above are financed by lump-sum taxes on lending households. Taken together, the term Π_t in the budget constraint (3.1) is

$$\Pi_t = \tilde{Y}_t - q_t Y_t + \tilde{w}_t L_t - w_t \tilde{L}_t + \left(Q_t - 1\right) I_t - \frac{\psi}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2.$$

Appendix A.2. Properties of the rational expectations equilibrium

I consider a rational expectations equilibrium with the following properties that hold in a neighborhood of the non-stochastic steady-state.

1. The expected net discounted return on capital is strictly positive for both investors and lenders: $\beta \mathbb{E}_t R_{t+1}^k > 1$ and $\mathbb{E}_t \Lambda_{t+1} R_{t+1}^k > 1$.
2. At any time t, the stock market valuation P_{jt} of a firm j is linear in its net worth, with a slope that is strictly greater than one.

3. All firms choose the same capital-labor ratio K_{jt}/L_{jt} .
4. All firms can be aggregated. Aggregate debt, capital and net worth are sufficient to describe the intermediate goods sector.
5. Borrowers never default on the equilibrium path and borrow at the risk-free rate, and the lender only accepts debt payments up to a certain limit.
6. If the firm defaults and the lender seizes the firm, it always prefers restructuring to liquidation.
7. The firm always exhausts the borrowing limit.

Here, I derive restrictions on the parameters for existence of such an equilibrium. I first take the first two properties as given and show under which conditions the remaining ones hold, and then derive conditions for the first two properties be verified.

Value functions

An operating firm j enters the period with a predetermined stock of capital and debt. It is convenient to decompose its value function into two stages. The first stage is given by:

$$\Upsilon_1 \left(K_{jt-1}, B_{jt-1}, s_t\right) = \max_{N_{jt}, L_{jt}, D_{jt}, Y_{jt}} \gamma N_{jt} + \left(1-\gamma\right)\left(D_{jt} + \Upsilon_2\left(N_{jt} - D_{jt}, s_t\right)\right)$$

$$\begin{aligned} \text{s.t. } N_{jt} &= q_t Y_{jt} - w_t L_{jt} + \left(1-\delta\right) Q_t K_{jt-1} - R_{t-1} B_{jt} \\ Y_{jt} &= K_{jt-1}^{\alpha}\left(A_t L_{jt}\right)^{1-\alpha} \\ D_{jt} &= \zeta\left(N_{jt} - Q_t K_{jt-1} + B_{jt-1}\right) \end{aligned}$$

The aggregate state of the economy is denoted by s_t. In what follows, I will suppress the time and firm indices for the sake of notation.

After production, the firm exits with probability γ and pays out all net worth as dividends. The second stage of the value function consists in choosing debt and capital levels as well as a strategy in the default game:

$$
\begin{aligned}
\Upsilon_2\left(\tilde{N}, s\right) = \max_{K,B,\text{strategy in default game}} \quad & \beta\mathbb{E}\left[\Upsilon_1\left(K, B, s'\right),\ \text{no default}\right] \\
+ \quad & \beta\mathbb{E}\left[\Upsilon_1\left(K, B^*, s'\right), \text{debt renegotiated}\right] \\
+ \quad & \beta\mathbb{E}\left[0,\ \text{lender seizes firm}\right]
\end{aligned}
$$

$$
\text{s.t. } QK \quad = \quad \tilde{N} + B
$$

Note that, since net worth $\tilde{N}$ is non-negative around the steady state, the firm's debt B cannot exceed its capital stock K.

In the first stage, the first order condition with respect to L equalizes the wage with the marginal revenue. Since there is no firm heterogeneity apart from capital K and debt B and the production function has constant returns to scale, this already implies Property 3 that all firms choose the same capital-labor ratio. Hence the internal rate of return on capital is common across firms:

$$
R^k \quad = \quad \alpha q \left((1-\alpha) \frac{qA}{w} \right)^{\frac{1-\alpha}{\alpha}} + (1-\delta) Q
$$

Taking Property 2 as given for now, Υ_2 is a linear function

$$
\Upsilon_2\left(\tilde{N}, s\right) = \upsilon_s \tilde{N}
$$

with slope $\nu_s > 1$. Then Υ_1 is homogeneous of degree one, and at the steady state (and therefore in a neighborhood):

$$
\begin{aligned}
\Upsilon_1\left(K, B, s\right) \quad &= \quad N + (1-\gamma)\left(D - N + \Upsilon_2\left(N - D, s\right)\right) \\
&= \quad N + (1-\gamma)\left(\upsilon_s - 1\right)\left((1-\zeta) N + \zeta\left(QK - B\right)\right) \\
&> \quad N = R^k K - RB.
\end{aligned}
$$

Limited commitment problem

The second stage involves solving for the subgame-perfect equilibrium of the default game between borrower and lender. Pairings are anonymous, so repeated interactions are ruled out. Also, only the size B and the interest rate $\tilde{R}$ of the loan can be contracted (in equilibrium $\tilde{R} = R$ but this is to be established first). The game is played sequentially:

1. The firm (F) proposes a borrowing contract $\left(B, \tilde{R}\right)$.
2. The lender (L) can accept or reject the contract.
   - A rejection corresponds to setting the contract $\left(B, \tilde{R}\right) = (0, 0)$.
     Payoff for L: 0. Payoff for F: $\beta\mathbb{E}\left[\Upsilon_1\left(\tilde{N}, 0, s'\right)\right]$.
3. F acquires capital and can then choose to default or not.
   - If F does not default, it has to repay in the next period.
     Payoff for L: $\mathbb{E}\Lambda\tilde{R}B - B$. Payoff for F: $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B, s'\right)\right]$.
4. If F defaults, the debt needs to be renegotiated. F makes an offer for a new debt level B^*.[20]
5. L can accept or reject the offer.
   - If L accepts, the new debt level replaces the old one.
     Payoff for L: $\mathbb{E}\Lambda\tilde{R}B^* - B$. Payoff for F: $\beta\mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R}B^*, s'\right)\right]$.
6. If L rejects, then she seizes the firm. A fraction $1 - \xi$ of the firm's capital is lost in the process. Nature decides randomly whether the firm can be "restructured."
   - If the firm cannot be restructured, or it can but the lender chooses not to do so, then the lender has to liquidate the firm.
     Payoff for L: $\mathbb{E}\left[\Lambda Q'\right]\xi K - B$. Payoff for F: 0.
   - If the firm can be restructured and the lender chooses to do so, she retains a debt claim of present value ξB and sells the residual equity claim in the firm to another investor.
     Payoff for L: $\xi B + \beta\mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B, s'\right)\right] - B$. Payoff for F: 0.

[20] That the interest rate on the repayment is fixed is without loss of generality.

Backward induction leads to the (unique) subgame-perfect equilibrium of this game. Start with the possibility of restructuring. L prefers this to liquidation if

$$\xi B + \beta \mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B, s'\right)\right] \geq \mathbb{E}\Lambda \xi Q' K.$$

This holds true at the steady state, as we have $\beta R^k > 1$ (Property 1), $\Lambda = \beta = 1/R$ and $Q' = 1$:

$$\begin{aligned} \beta \Upsilon_1\left(\xi K, \xi B, s\right) &> \beta\left(R^k \xi K - R \xi B\right) \\ &> \xi\left(K - \tilde{\beta} R B\right) \\ &> \xi\left(\beta K - B\right). \end{aligned}$$

Since the inequality is strict, it holds around the steady-state as well. This establishes Property 6.

Next, L will accept an offer B^* if it gives her a better expected payoff (lenders can diversify among borrowers so that their discount factor is invariant to the outcome of the game). The probability of restructuring is given by x. The condition for accepting B^* is therefore that

$$\mathbb{E}\left[\Lambda\right] \tilde{R} B^* \geq x\left(\xi B + \tilde{\beta} \mathbb{E}\left[\Upsilon_1\left(\xi K, \xi B, s'\right)\right]\right) + (1 - x)\, \mathbb{E}\left[\Lambda Q'\right] \xi K.$$

Now turn to the firm F. Among the set of offers B^* that are accepted by L, the firm will prefer the lowest one which satisfies the above restriction with equality. This follows from Υ_1 being a decreasing function of debt. This lowest offer will be made if it leads to a higher payoff than expropriation: $\beta \mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R} B, s'\right)\right] \geq 0$. Otherwise, F offers zero and L seizes the firm.

Going one more step backwards, F has to decide whether to declare default or not. It is preferable to do so if B^* can be set smaller than B or if expropriation is better than repaying, $\beta \mathbb{E}\left[\Upsilon_1\left(K, \frac{\tilde{R}}{R} B, s'\right)\right] \geq 0$.

What is the set of contracts that L accepts in the first place? From the perspective of L, there are two types of contracts: those that will not be defaulted on and those that will. If F does not default ($B^* \geq B$), L will accept the contract simply if it pays at least the risk-free rate, $\tilde{R} \geq R$. If F does default ($B^* < B$), then L accepts if the expected discounted recovery value exceeds the size of the loan—i.e., $\mathbb{E}\left[\Lambda\right] \tilde{R} B^* \geq B$.

Finally, let us consider the contract offer. F can offer a contract $\left(B, \tilde{R}\right)$ on which it will not default. In this case, it is optimal to offer just the risk-free rate $\tilde{R} = R$. Also note that the payoff from this strategy is strictly positive for any non-negative B that does not trigger default, since at the steady-state $R^k > 1/\beta > R$ and therefore

$$\begin{aligned} \beta \mathbb{E}\left[\Upsilon_1\left(K, B, s'\right)\right] &> \beta \mathbb{E}\left[R^k K - RB\right] \\ &= \beta \mathbb{E}\left[R^k \tilde{N} + \left(R^k - R\right) B\right] \\ &> 0. \end{aligned}$$

F therefore prefers this contract to one that leads to default with expropriation. The payoff is increasing in the size of the loan B, since

$$\begin{aligned} &\frac{\partial}{\partial B} \mathbb{E}\left[\Upsilon_1\left(\frac{\tilde{N}+B}{Q}, B, s'\right)\right] \\ =&\beta \mathbb{E}\left[\frac{R^k}{Q} - R + (1-\gamma)\left(\upsilon_{s'} - 1\right)\left((1-\zeta)\left(\frac{R^k}{Q} - R\right) + \zeta\left(\frac{Q'}{Q} - 1\right)\right)\right] \\ >&0. \end{aligned}$$

Therefore, of all values for B that do not lead to default, F will want to choose the largest one, defined as:

$$\bar{B} = \max\left\{ B \left| \begin{array}{c} x\left(\xi B + \beta \mathbb{E}\left[\Upsilon_1\left(\xi\left(\frac{\tilde{N}+B}{Q}, s'\right), \xi B, s'\right)\right]\right) \\ + (1-x)\,\mathbb{E}\left[\Lambda Q'\right] \xi \frac{\tilde{N}+B}{Q} - B \end{array} \geq 0 \right. \right\}.$$

In order for the borrowing constraint to be binding, it must be finite. Since the set above contains $B = 0$, this amounts to the condition that

$$x\left(1 + \beta \frac{\partial}{\partial B} \mathbb{E}\left[\Upsilon_1\left(\frac{\tilde{N}+B}{Q}, B, s'\right)\right]\right) + (1-x)\,\mathbb{E}\left[\Lambda Q'\right] < \frac{1}{\xi} \tag{A.1}$$

which is satisfied for ξ small enough. Because Υ_1 is homogeneous of degree one, the borrowing limit is linear in $\tilde{N}$ and can be written as $\bar{B} = \upsilon_{B,s} \tilde{N}$.

F could also offer a contract $\left(B, \tilde{R}\right)$ that only leads to a default with debt renegotiation. The optimal contract of this type is the solution to the following problem:

$$\max_{\tilde{R},B,B^*} \beta \mathbb{E}\left[\Upsilon_1\left(\tilde{N}+B, \frac{\tilde{R}B^*}{R}, s'\right)\right]$$

$$\begin{aligned} \text{s.t. } \frac{\tilde{R}B^*}{R} &\geq B \\ \frac{\tilde{R}B^*}{R} &= x\left(\xi B + \beta \mathbb{E}\left[\Upsilon_1\left(\xi\frac{\tilde{N}+B}{Q}, \xi B, s'\right)\right]\right) \\ &\quad + (1-x)\,\mathbb{E}\left[\Lambda Q'\right]\xi\frac{\tilde{N}+B}{Q} \end{aligned}$$

It is clear that the value of this problem is solved by setting $\tilde{R} = R$ and $B = B^* = \bar{B}$, which amounts to not defaulting. This establishes Properties 5 and 7.

Linearity of firm value

Since firms do not default and exhaust the borrowing limit $\bar{B}$, the second-stage firm value is

$$\begin{aligned} \Upsilon_2\left(\tilde{N}\right) &= \beta \mathbb{E}\left[\Upsilon_1\left(\frac{\tilde{N}+\bar{B}}{Q}, \bar{B}, s'\right)\right] \\ &= \beta \mathbb{E}\left[\Upsilon_1\left(\frac{1+\upsilon_{B,s}}{Q}\tilde{N}, \upsilon_{B,s}\tilde{N}, s'\right)\right] \\ &= \beta \mathbb{E}\left[\Upsilon_1\left(\frac{1+\upsilon_{B,s}}{Q}, \upsilon_{B,s}, s'\right)\right]\tilde{N}. \end{aligned}$$

We have therefore verified the linearity of Υ_2. To establish Property 2, it remains to show that the slope of Υ_2 is greater than one. At the steady state:

$$\begin{aligned}
\upsilon_s &= \beta\Upsilon_1\left(1+\upsilon_{B,s},\upsilon_{B,s},s\right) \\
&= \beta\left(R^k+\underbrace{\upsilon_{B,s}\left(R^k-R\right)}_{>0}\right)\underbrace{\left(1+(1-\gamma)\overbrace{(\upsilon_s-1)}^{>-1}(1-\zeta)\right)}_{>0}+(1-\gamma)(\upsilon_s-1)\zeta \\
&> \beta R^k\left(1+(1-\gamma)(\upsilon_s-1)(1-\zeta)\right)+(1-\gamma)(\upsilon_s-1)\zeta \\
&> 1+(1-\gamma)(\upsilon_s-1) \\
&> 1.
\end{aligned}$$

Finally, the aggregated law of motion for capital and net worth needs to be established (Property 4). Denoting again by $\Gamma_t\subset[0,1]$ the indices of firms that are alive at the end of period t, we have

$$\begin{aligned}
Q_tK_t=Q_t\int_0^1 K_{jt}dj &= \int_{j\in\Gamma_t}\left(N_{jt}-\zeta E_{jt}+B_{jt}\right)dj \\
&= (1-\gamma)\left(N_t-\zeta E_t\right)+B_t \\
N_t=\int_0^1 N_{jt}dj &= R_t^kK_{t-1}-R_{t-1}B_{t-1} \\
B_t=\int_0^1 B_{jt}dj &= x\xi\left(B_t+P_t\right)+(1-x)\,\xi\mathbb{E}_t\Lambda_{t+1}Q_{t+1}K_t.
\end{aligned}$$

Return on capital

We can now establish a condition under which $\beta R^k>1$ holds (Property 1). From the aggregate equations above, and the definition of earnings $E=N-QK+B$, it follows that in steady state:

$$R^k=\frac{RB}{K}+\frac{1-\zeta(1-\gamma)}{(1-\zeta)(1-\gamma)}\left(1-\frac{B}{K}\right). \tag{A.2}$$

Rearranging the above expression, one obtains that $\beta R^k>1$ holds at the steady state if and only if:

$$\gamma>1-\frac{\beta}{1-\zeta(1-\beta)}. \tag{A.3}$$

Appendix A.3. Conditions to rule out multiple equilibria

Collateral constraints often give rise to multiple equilibria due to their feedback effects: Low asset prices reduce borrowing constraints and activity, which in depress asset prices and so on. This multiplicity appears even in the very early literature (Kiyotaki and Moore, 1997). More recently, Miao and Wang (2018) have shown that when firm borrowing constraints depend on equity value, multiple steady states are possible. In this section, I give conditions under which this type of multiplicity does not arise. These conditions are satisfied for the parameter values at which the model is simulated.

Miao and Wang look for an equilibrium in which firm value $\Upsilon_2\left(\tilde{N}, s\right)$ is not linear but affine in net worth $\tilde{N}$. Even a firm with zero net worth has positive value. This can be an equilibrium: The positive equity value enables the firm to borrow, acquire capital and pay dividends from the returns; those expected dividends can justify the positive equity value.

Suppose that $\Upsilon_2\left(\tilde{N}, s\right) = \upsilon_s\tilde{N} + \vartheta_s$ with $\vartheta_s \geq 0$ and $\upsilon_s > 1$, and that $\beta\mathbb{E}_t R^k_{t+1} > 1$. Then the proof for the existence of an equilibrium satisfying properties 3–7 above still goes through under the same conditions (A.1) and (A.3). The equation determining the coefficient ϑ_s is:

$$\upsilon_s\tilde{N} + \vartheta_s = \beta\mathbb{E}\left[\kappa_{N,s'}\tilde{N} + \kappa_{B,s'}\bar{B} + (1-\gamma)\,\vartheta_{s'}\right]$$

where

$$\kappa_{N,s'} = (1 + (1-\gamma)(\upsilon_{s'}-1)(1-\zeta))\frac{R^k}{Q} - (1-\gamma)(\upsilon_{s'}-1)\zeta\frac{Q'}{Q}$$
$$\kappa_{B,s'} = (1 + (1-\gamma)(\upsilon_{s'}-1)(1-\zeta))\left(\frac{R^k}{Q} - R\right) - (1-\gamma)(\upsilon_{s'}-1)\zeta\left(\frac{Q'}{Q} - 1\right).$$

The borrowing limit $\bar{B}$ depends itself on equity value and therefore on the coefficients υ_s and ϑ_s:

$$\bar{B} = \frac{\mathbb{E}\left[\xi x\beta(1-\gamma)\,\vartheta_{s'} + \xi\tilde{N}\left((1-x)\,\Lambda\frac{Q'}{Q} + x\beta\kappa_{N,s'}\right)\right]}{1 - \xi\mathbb{E}\left[x + (1-x)\,\Lambda\frac{Q'}{Q} + x\beta\kappa_{B,s'}\right]}. \tag{A.4}$$

Comparing coefficients, the equations determining υ_s and ϑ_s are:

$$\upsilon_s = \beta\mathbb{E}\left[\kappa_{N,s'}\right] + \beta\frac{\xi\mathbb{E}\left[(1-x)\Lambda\frac{Q'}{Q} + x\beta\kappa_{N,s'}\right]}{1-\xi\mathbb{E}\left[x+(1-x)\Lambda\frac{Q'}{Q}+x\beta\kappa_{B,s'}\right]}\mathbb{E}\left[\kappa_{B,s'}\right]$$

$$\vartheta_s = \beta(1-\gamma)\left(1+\frac{\xi x\mathbb{E}\left[\kappa_{B,s'}\right]}{1-\xi\mathbb{E}\left[x+(1-x)\Lambda\frac{Q'}{Q}+x\beta\kappa_{B,s'}\right]}\right)\mathbb{E}\left[\vartheta_{s'}\right].$$

Clearly, $\vartheta_s \equiv 0$ is a solution to the second equation and corresponds to the equilibrium considered in this paper. It is the unique solution if the term multiplying $\mathbb{E}\left[\vartheta_{s'}\right]$ is always strictly smaller than one. Around a steady state in which ϑ_s is zero, a sufficient condition to guarantee uniqueness is that

$$\beta(1-\gamma)\left(1+\frac{\xi x\beta\kappa_{B,s}}{1-\xi x+(1-x)\beta+x\beta\kappa_{B,s}}\right)<1. \tag{A.5}$$

This always holds for x small enough. It remains to establish conditions under which a steady state with $\vartheta_s > 0$ can also be ruled out. Such a steady state would necessarily have the term multiplying $\mathbb{E}\left[\vartheta_{s'}\right]$ equal to one at the steady state. From this, it follows that necessarily

$$\begin{aligned}
\kappa_{B,s} &= \frac{1-\beta(1-\gamma)}{\xi\beta x}\left(1-\xi x-\xi(1-x)\beta\right)\\
\upsilon_s &= \frac{1-\beta(1-\gamma)}{\xi\beta x}\frac{1-\xi x}{1-\gamma}\\
R^k &= R+\frac{\kappa_{B,s}}{1+(1-\gamma)(1-\zeta)(\upsilon_s-1)}
\end{aligned} \tag{A.6}$$

holds at the steady state. Note that the values of $\kappa_{B,s}$ and $\kappa_{N,s}$ at the steady-state do not depend on ϑ_s, and that therefore the equilibrium borrowing limit $\bar{B}$ in (A.4) is increasing in ϑ_s for any level of $\tilde{N}$. In particular then, equilibrium leverage B/K is also an increasing in ϑ_s. Since the equilibrium return on capital is a decreasing function of leverage through Equation (A.2), the steady state with $\vartheta_s > 0$ has a lower R^k than in the steady state with $\vartheta_s = 0$. A sufficient condition to guarantee that $\vartheta_s = 0$ is the unique steady state is therefore that the corresponding steady-state value of R^k is higher than the one computed in (A.6).

Appendix A.4. Benchmark model without financial frictions

In the benchmark model without financial frictions, investing households are absent, and firms are owned directly by lending households who hold zero debt and face no financial constraint. Additionally, we introduce a small quadratic adjustment cost to holdings of equity shares away from unity. Because the number of shares outstanding are constant at unity, this cost is zero in any equilibrium. Introducing it is necessary, however, for a perturbation solution to the equilibrium under learning and CMCE to be well-behaved. The lending household's budget constraint (3.1)now becomes

$$\text{s.t. } C_t = \tilde{w}_t L_t + B_t^g - \frac{1+i_{t-1}}{\pi_t} B_{t-1}^g + S_{t-1}(P_t + D_t) - S_t P_t - \frac{\chi}{2}(S_t - 1)^2 + \Pi_t \quad \text{(A.7)}$$

where S_t are stock holdings, and the resulting Euler equation for stocks is

$$P_t = \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^{-\theta} (P_{t+1} + D_{t+1}) - \chi (S_t - 1) . \quad \text{(A.8)}$$

The value for the adjustment costs used in the paper is $\chi = 0.1$. Note that, under rational expectations, the quadratic adjustment cost to S_t is entirely irrelevant because in equilibrium $S_t = 1$.

The problem of intermediate goods producers becomes a standard maximization of the expected discounted value of dividends:

$$\max_{\left(\tilde{L}_t, K_t\right)_{t=0}^{\infty}} \mathbb{E}^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t C_t^{-\theta} D_t$$

$$\text{s.t. } D_t = q_t \alpha K_{t-1}^{\alpha} \left(A_t \tilde{L}_t \right)^{1-\alpha} - w_t \tilde{L}_t + (1-\delta) Q_t K_{t-1} - Q_t K_t .$$

The resulting optimality condition for the choice of capital is standard:

$$Q_t = \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^{-\theta} \left(R_{t+1}^k + (1-\delta) Q_{t+1} \right)$$

where the return on capital R_{t+1}^k is defined as in the baseline model. Goods market clearing requires

$$\tilde{Y}_t = C_t + I_t + \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2.$$

All other equilibrium conditions are identical to those of the baseline model.

Appendix B. Proof of Proposition 4

Under $\mathbb{P}$, u_t maps into endogenous variables y_t (which include P_t) and subjective shocks z_t through the mappings g_t and r_t. Further, the conditional expectation under $\mathcal{P}$ is defined by the mappings h_t.[21] Therefore:

$$\begin{aligned}\mathbb{E}^{\mathcal{P}}\left[y_{t+1} \mid u_0, P_0, \ldots, u_{t+1}, P_{t+1}\right] &= \mathbb{E}^{\mathcal{P}}\left[y_{t+1} \mid u^{(t+1)}, z^{(t+1)}\right] \\ &= h_{t+1}\left(u^{(t+1)}, z^{(t+1)}\right) \\ &= g_{t+1}\left(u^{(t+1)}\right) \\ &= y_{t+1}.\end{aligned}$$

Note that the conditioning set $\left(u^{(t+1)}, z^{(t+1)}\right)$has measure zero under $\mathcal{P}$. Strictly speaking, the conditional expectation in the first line is not uniquely defined on this set. But in Definition 2 the conditional expectation was specified so to satisfy $\mathcal{P}\left[y_t \mid u^{(t)}, z^{(t)}\right] = h\left(u^{(t)}, z^{(t)}\right)$ on $\operatorname{supp}\left(\mathbb{P}_u \otimes \mathcal{P}_z\right)$ instead of just $\mathbb{P}_u \otimes \mathcal{P}_z$-almost surely. This is sometimes called the "canonical version" of the conditional expectation.

[21] Because the equilibrium path has zero measure under subjective expectations, i.e. $\mathcal{P}\left(\operatorname{supp}\left(\mathbb{P}\right)\right) = 0$, it is important that Definition 2 defines the conditional expectation on the entire support of $\mathcal{P}$ rather than $\mathcal{P}$-almost surely.

Appendix C. Full sets of model equations

Appendix C.1. Rational expectations

The full set of model equations under rational expectations is as follows. The intermediate firms block is:

$$Y_t = K_{t-1}^{\alpha}\left(A_t\tilde{L}_t\right)^{1-\alpha} \tag{C.1}$$

$$I_t = K_t - (1-\delta)\,K_{t-1} \tag{C.2}$$

$$w_t = (1-\alpha)\,q_tY_t/L_t \tag{C.3}$$

$$R_t^k = q_t\alpha\frac{Y_t}{K_{t-1}} + Q_t\,(1-\delta)\,K_{t-1} \tag{C.4}$$

$$N_t = R_t^kK_{t-1} - R_{t-1}B_{t-1} \tag{C.5}$$

$$E_t = q_tY_t - w_tL_t - \delta K_{t-1} - (R_{t-1}-1)\,B_{t-1} \tag{C.6}$$

$$Q_tK_t = (1-\gamma)\left((1-\zeta)\,N_t + \zeta\,(B_{t-1} - Q_tK_{t-1})\right) + B_t \tag{C.7}$$

$$B_t = x\mathbb{E}_t\lambda_{t+1}Q_{t+1}\xi K_t + (1-x)\,\xi\,(P_t + B_t) \tag{C.8}$$

$$D_t = \gamma N_t + (1-\gamma)\,\zeta E_t \tag{C.9}$$

$$\log A_t = (1-\rho)\log\bar{A} + \rho\log A_{t-1} + \varepsilon_{At} \tag{C.10}$$

The lending household's budget constraint is:

$$C_t = \tilde{Y}_t - D_t - I_t - \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2. \tag{C.11}$$

Nominal rigidities, and adjustment costs add the following conditions:

$$Q_t = 1 + \psi \left(\frac{I_t}{I_{t-1}} - 1 \right) \tag{C.12}$$

$$\Gamma_{1t} = q_t + \kappa \beta \mathbb{E}_t \left(\frac{C_t}{C_{t+1}} \right)^{\theta} \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} \pi_{t+1}^{\sigma} \tag{C.13}$$

$$\Gamma_{2t} = 1 + \kappa \beta \mathbb{E}_t \left(\frac{C_t}{C_{t+1}} \right)^{\theta} \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} \pi_{t+1}^{\sigma-1} \tag{C.14}$$

$$\frac{\Gamma_{1t}}{\Gamma_{2t}} = \left(\frac{1 - \kappa \pi_t^{\sigma-1}}{1-\kappa} \right)^{\frac{1}{1-\sigma}} \tag{C.15}$$

$$\Delta_t = (1-\kappa) \left(\frac{\Gamma_{1t}}{\Gamma_{2t}} \right)^{-\sigma} + \kappa \pi_t^{\sigma} \Delta_{t-1} \tag{C.16}$$

$$w_t = w_{t-1} + \pi_t^w - \pi_t \tag{C.17}$$

$$\Gamma_{1t}^w = \frac{\eta L_t^{\phi} C_t^{\theta}}{w_t} + \kappa_w \beta \mathbb{E}_t \left(\frac{C_t}{C_{t+1}} \right)^{\theta} \frac{\tilde{L}_{t+1}}{\tilde{L}_t} \pi_{t+1}^{\sigma} \tag{C.18}$$

$$\Gamma_{2t}^w = 1 + \kappa_w \beta \mathbb{E}_t \left(\frac{C_t}{C_{t+1}} \right)^{\theta} \frac{\tilde{L}_{t+1}}{\tilde{L}_t} \left(\pi_{t+1}^w \right)^{\sigma_w - 1} \tag{C.19}$$

$$\frac{\Gamma_{1t}^w}{\Gamma_{2t}^w} = \left(\frac{1 - \kappa_w \left(\pi_{t+1}^w \right)^{\sigma_w - 1}}{1 - \kappa_w} \right)^{\frac{1}{1-\sigma_w}} \tag{C.20}$$

$$\Delta_t^w = (1-\kappa_w) \left(\frac{\Gamma_{1t}}{\Gamma_{2t}} \right)^{-\sigma_w} + \kappa_w \pi_t^{\sigma_w} \Delta_{t-1}^w \tag{C.21}$$

$$\tilde{Y}_t = \frac{Y_t}{\Delta_t} \tag{C.22}$$

$$\tilde{L}_t = \frac{L_t}{\Delta_{wt}} \tag{C.23}$$

$$i_t = \rho_i i_{t-1} + (1-\rho_i) \left(\beta^{-1} + \phi_{\pi} \pi_t + \varepsilon_{it} \right) \tag{C.24}$$

The asset pricing equations are:

$$\lambda_{t+1} = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\theta} \tag{C.25}$$

$$1 = \mathbb{E}_t \lambda_{t+1} \frac{1+i_t}{1+\pi_{t+1}} \tag{C.26}$$

$$1 = \mathbb{E}_t \lambda_{t+1} R_t \tag{C.27}$$

$$P_t = \mathbb{E}_t \beta \left(P_{t+1} + D_{t+1} \right) \tag{C.28}$$

Appendix C.2. Learning with conditionally model-consistent expectations

Under learning with conditionally model-consistent expectations, the subjective law of motion defining beliefs $\mathcal{P}$ is obtained by solving the same set of equations, but replacing the stock pricing equation (C.28) with the subjective law of motion for stock prices. In the main version of the paper, I impose that beliefs about stock price growth are only updated at the end of every period, after the current stock price has been observed. To implement this "lagged belief updating", the subjective forecast error has to be two-dimensional, $z_t = (z_{1t}, z_{2t})$. The subjective law of motion is then:

$$\log P_t = \log P_{t-1} + \hat{\mu}_{t-1} + z_{1t} \tag{C.29}$$

$$\hat{\mu}_t = \hat{\mu}_{t-1} + g z_{2t} \tag{C.30}$$

To compute the equilibrium with lagged belief updating, I impose the stock market clearing condition (C.28) and also update the forecast error in the belief equation to be last period's forecast error: $z_{2t} = z_{1t-1}$.

In Appendix E, I provide a version of the model that does away with lagged belief updating. In this case, the subjective forecast error z_t is one-dimensional. In the subjective law of motion above, z_{1t} and z_{2t} are replaced with just z_t. To compute the learning equilibrium, only the market clearing condition (C.28) needs to be imposed.

Appendix C.3. Alternative belief formation concepts

Adaptive learning

Under adaptive learning, the equations of the rational expectations equilibrium are log-linearized. Letting bars denote steady-state values, and hats denote log deviations steady-state values, the Euler equations for debt and equity are log-linearized as

$$0 = \theta \left(\hat{C}_t - \mathbb{E}_t \hat{C}_{t+1} \right) + \hat{R}_t$$
$$\hat{P}_t = \beta \mathbb{E}_t \hat{P}_{t+1} + \beta \mathbb{E}_t \hat{D}_{t+1}.$$

The remaining equations are log-linearized analogously. To get to the adaptive learning solution, I replace expectations with linear forecating rules, denoted by hats on the expectation operator.

The stock price forecast and its updating rule are kept as in the baseline learning model:

$$\hat{\mathbb{E}}_t \hat{P}_{t+1} = \hat{P}_t + \hat{\mu}_{t-1} \tag{C.31}$$

$$\hat{\mu}_t = \hat{\mu}_{t-1} + g\left(\hat{P}_t - \hat{P}_{t-1}\right) \tag{C.32}$$

197 Other expectations are parameterized with a minimum-state variable forecating rule. The forward-
198 looking variables in the log-linearized model (excluding the stock price) are $Z_t = \left(\hat{C}_t, \hat{\pi}_t, \hat{\pi}_t^w, \hat{D}_t, \hat{Q}_t\right)'$,
199 and the state variables are $X_t = \left(\hat{A}_t, \hat{I}_t, \hat{B}_t, \hat{K}_t, \hat{R}_t, \hat{w}_t, \hat{i}_t, \hat{\mu}_t\right)'$. I define the forecasting rule as

$$\hat{E}_t Z_{t+1} = \hat{A}'_{t-1} X_t \tag{C.33}$$

where the coefficient matrix $\hat{A}_t$ is updated according to the constant-gain least squares formula

$$\hat{A}_t = \hat{A}_{t-1} + \bar{g} R_{t-1}^{-1} X_t \left(Z_t - \hat{A}'_{t-1} X_{t-1}\right) \tag{C.34}$$

$$R_t = R_{t-1} + \bar{g}\left(X_{t-1} X'_{t-1} - R_{t-1}\right). \tag{C.35}$$

200 The gain parameter is set to a standard value of $\bar{g} = 0.01$. Simulations under adaptive learning
201 are carried out with a burn-in of 1,000 periods to eliminate dependency on the initialization value
202 of R_t. Finally, a projection facility is employed to ensure stationarity of the solution: When the
203 log-linear system of model equations using forecating rules has an eigenvalue outside of the unit
204 circle, the coefficient matrix $\hat{A}_t$ is reset to the rational expectations forecast.

205 *"Mixed rational expectations"*

In this version of the model, agents have rational expectations for all variables except for stock prices. This is implemented by taking the rational expectations equations above, and replacing the stock pricing equation (C.28) with

$$P_t = \beta P_t \exp\left(\hat{\mu}_t + \frac{1}{2}\sigma_z^2\right) + \beta \mathbb{E}_t D_{t+1}$$

$$\hat{\mu}_t = \hat{\mu}_{t-1} + g\left(\log P_t - \log P_{t-1}\right).$$

206 *Appendix C.4. Benchmark economy without financial frictions*

In the benchmark model without financial frictions, the intermediate firm block becomes:

$$Y_t = K_{t-1}^{\alpha}\left(A_t\tilde{L}_t\right)^{1-\alpha} \tag{C.36}$$

$$I_t = K_t - (1-\delta)\,K_{t-1} \tag{C.37}$$

$$D_t = R_t^k K_{t-1} - Q_t K_t \tag{C.38}$$

$$R_t^k = q_t \alpha \frac{Y_t}{K_{t-1}} + Q_t\,(1-\delta)\,K_{t-1} \tag{C.39}$$

$$Q_t = \mathbb{E}_t^{\mathcal{P}} \lambda_{t+1} R_{t+1}^k \tag{C.40}$$

$$w_t = (1-\alpha)\,q_t Y_t / L_t \tag{C.41}$$

$$\log A_t = (1-\rho)\log \bar{A} + \rho \log A_{t-1} + \varepsilon_{At}. \tag{C.42}$$

The stock pricing equation (C.28) and lending household budget constraint (C.11) are replaced by:

$$P_t = \mathbb{E}_t \lambda_{t+1}\left(P_{t+1} + D_{t+1}\right) - \chi\left(S_{t-1} - 1\right) \tag{C.43}$$

$$C_t = \tilde{Y}_t - D_t + S_{t-1}\left(P_t + D_t\right) - P_t S_t - \frac{\chi}{2}\left(S_t - 1\right)^2 - \frac{\psi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2. \tag{C.44}$$

207 For a discussion of the adjustment cost on equity holdings with parameter χ see appendix (Appendix A.4).
208 Under rational expectations, the stock market clearing conditions $S_t = 1$ is added to the model
209 equations. Under learning, that condition is instead replaced with the subjective law of motion
210 for stock prices (C.29)–(C.30). The equilibrium under learning is then found by solving in each
211 period for the value of z_t that satisfies $S_t = 1$.

Appendix D. General formulation of CMCE

To set the notation, I start with the solution of the standard rational expectations equilibrium. Denote the n endogenous model variables by y_t and the n_u exogenous shocks by u_t. The exogenous shocks are independent across time with joint distribution F_σ, mean zero and variance $\sigma^2\Sigma_u$. The solution of a (recursive) rational expectations equilibrium satisfies the equilibrium conditions:

$$\mathbb{E}_t\left[f_{-P}\left(y_{t+1}, y_t, y_{t-1}, u_t\right)\right] = 0 \tag{D.1}$$

$$\mathbb{E}_t\left[f_P\left(y_{t+1}, y_t\right)\right] = 0 \tag{D.2}$$

where f_P denotes the stock market clearing condition (C.43) and f_{-P} collects the remaining $n-1$ equilibrium conditions.[22] A recursive solution takes the form:

$$y_t = g_{RE}\left(y_{t-1}, u_t, \sigma\right).$$

By the definition of rational expectations, the expectations in (D.1)–(D.1) are taken under the probability measure induced by g_{RE} and F_σ, so that the policy function g_{RE} itself can be found by solving:

$$\int f_{-P}\begin{pmatrix} g_{RE}\left(g_{RE}\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), \\ g_{RE}\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t \end{pmatrix} dF_\sigma\left(u_{t+1}\right) = 0 \tag{D.3}$$

$$\int f_P\begin{pmatrix} g_{RE}\left(g_{RE}\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), \\ g_{RE}\left(y_{t-1}, u_t, \sigma\right) \end{pmatrix} dF_\sigma\left(u_{t+1}\right) = 0. \tag{D.4}$$

In the learning equilibrium, the probability measure $\mathcal{P}$ used by agents to form expectations does not coincide with the actual probability measure describing the equilibrium outcomes. In particular, agents are not endowed with the knowledge that the stock price is determined by the

[22] There are usually $n+1$ equilibrium conditions in total, but one of the market clearing conditions is redundant due to Walras' law. While under rational expectations, it is immaterial for the computation of the equilibrium which market clearing condition is left out, this choice can matter when constructing the learning equilibrium with conditionally model-consistent expectations. Here I choose to omit the market clearing condition for final consumption goods.

market clearing condition f_P, and instead they form expectations about future prices using a subjective law of motion. This law of motion can be summarized in a function:

$$\phi\left(\tilde{y}_t, \tilde{y}_{t-1}, z_t\right) = \begin{pmatrix} \Delta \log P_t - \hat{\mu}_{t-1} - z_t \\ \Delta \hat{\mu}_t - g z_t \end{pmatrix} = 0$$

where $\tilde{y}_t = (y_t, \hat{\mu}_t)$ incorporates the belief state introduced by the learning process, and z_t is the *subjective forecast error.* In the mind of agents, this forecast error is an exogenous iid shock with distribution G_σ, mean zero and variance $\sigma^2 \Sigma_z$. I assume that agents believe that z and u are mutually independent as well.

I impose discipline on the expectation formation process by requiring that agents have conditionally model-consistent expectations, as defined in the main text. I find such expectations by computing a *subjective policy function*

$$\tilde{y}_t = h\left(\tilde{y}_{t-1}, u_t, z_t, \sigma\right)$$

which satisfies:

$$\int f_{-P} \begin{pmatrix} Ch\left(h\left(\tilde{y}_{t-1}, u_t, z_t, \sigma\right), u_{t+1}, z_{t+1}, \sigma\right), \\ Ch\left(\tilde{y}_{t-1}, u_t, z_t, \sigma\right), C\tilde{y}_{t-1}, u_t \end{pmatrix} dF_\sigma\left(u_{t+1}\right) dG_\sigma\left(z_{t+1}\right) = 0 \tag{D.5}$$

$$\phi\left(h\left(\tilde{y}_{t-1}, u_t, z_t, \sigma\right), \tilde{y}_{t-1}, z_t\right) = 0. \tag{D.6}$$

Here, the matrix C just selects the original model variables y_t from the augmented vector $\tilde{y}_t$ ($y_t = C\tilde{y}_t$). Solving for h effectively amounts to solving a different rational expectations model in which the market clearing condition for the stock market is replaced by the subjective law of motion for stock prices. Once computed, the policy function h together with F_σ and G_σ defines a complete internally consistent probability measure $\mathcal{P}$ on all endogenous model variables. Under $\mathcal{P}$, agents believe that the stock price follows the subjective law of motion ϕ, and $\mathcal{P}$ also satisfies the equilibrium conditions f_{-P}:

$$\mathbb{E}_t^{\mathcal{P}}\left[f_{-P}\left(y_{t+1}, y_t, y_{t-1}, u_t\right)\right] = 0.$$

This subjective belief is very close to rational expectations and preserves as much as possible of its forward-looking, model-consistent logic while allowing for subjective expectations about stock prices.

Now, the subjective policy function h depends on the subjective forecast error z_t, which under $\mathcal{P}$ is believed to be a white noise process. In equilibrium however, z_t is instead determined endogenously by the equilibrium stock price that clears the stock market. That is, the equilibrium value of the subjective forecast error is itself a function of the states and the shocks:

$$z_t = r\left(\tilde{y}_{t-1}, u_t, \sigma\right) \tag{D.7}$$

The function r can be computed by imposing equilibrium in the stock market, represented by the equation $\mathbb{E}_t^{\mathcal{P}}\left[f_P\left(\tilde{y}_{t+1}, \tilde{y}_t, \tilde{y}_{t-1}, z_t\right)\right] = 0$. Substituting the functional forms:

$$\int \psi \left(\begin{array}{c} h\left(h\left(\tilde{y}_{t-1}, u_t, r\left(\tilde{y}_{t-1}, u_t, \sigma\right), \sigma\right), u_{t+1}, z_{t+1}, \sigma\right), \\ h\left(\tilde{y}_{t-1}, u_t, r\left(\tilde{y}_{t-1}, u_t, \sigma\right), \tilde{y}_{t-1}, z_t, \sigma\right) \end{array} \right) dF_\sigma\left(u_{t+1}\right) dG_\sigma\left(z_{t+1}\right) = 0. \tag{D.8}$$

Note that while the current value of the forecast error z_t was substituted out, the future value z_{t+1} was not substituted out, as this value is still taken under the subjective expectation $\mathcal{P}$ which treats it as an exogenous random disturbance.

The final equilibrium of the model is described by the *objective policy function*:

$$\tilde{y}_t = g\left(\tilde{y}_{t-1}, u_t, \sigma\right) = h\left(\tilde{y}_{t-1}, u_t, r\left(\tilde{y}_{t-1}, u_t, \sigma\right), \sigma\right).$$

By construction, this policy function satisfies all equilibrium conditions of the model. This function g together with F_σ defines the equilibrium probability distribution of the model variables. It differs from the subjective distribution $\mathcal{P}$ only in that under $\mathcal{P}$, z_t is an unpredictable exogenous shock, whereas in equilibrium z_t is a function of the state variables and the structural shocks u_t.

It is straightforward to see that the expectations thus constructed satisfy conditional modelconsistency:

$$\begin{aligned}\mathbb{E}_t^{\mathcal{P}}\left[\tilde{y}_{t+1} \mid u_{t+1}, P_{t+1}\right] &= h\left(\tilde{y}_t, u_{t+1}, z_{t+1}, \sigma\right) \\ &= h\left(\tilde{y}_t, u_{t+1}, r\left(\tilde{y}_t, u_{t+1}, \sigma\right), \sigma\right) \\ &= g\left(\tilde{y}_t, u_{t+1}, \sigma\right) \\ &= \tilde{y}_{t+1}.\end{aligned}$$

Appendix D.1. Approximation with perturbation methods

I now describe how to compute an approximation of the objective policy function g around the non-stochastic steady state $\bar{y}$. The procedure has two steps and does not require iteration. The first step consists in deriving a perturbation approximation of the subjective policy function h. This can be done using standard methods, as the system of equations (D.5)–(D.6) can be solved as if it were a standard rational expectations model. The second step consists in finding the derivatives of the function r. Applying the implicit function theorem to Equation (D.8), one can compute the first-order derivatives as:

$$\begin{aligned}r_y &= -A^{-1}\left(\left(\frac{\partial\psi}{\partial\tilde{y}_{t+1}}h_y + \frac{\partial\psi}{\partial\tilde{y}_t}\right)h_y + \frac{\partial\psi}{\partial\tilde{y}_{t-1}}\right) \\ r_u &= -A^{-1}\left(\frac{\partial\psi}{\partial\tilde{y}_{t+1}}h_y + \frac{\partial\psi}{\partial\tilde{y}_t}\right)h_u \\ r_\sigma &= -A^{-1}\left(\frac{\partial\psi}{\partial\tilde{y}_{t+1}}h_y + \frac{\partial\psi}{\partial\tilde{y}_t}\right)h_\sigma\end{aligned}$$

where the matrix A is given by $A = \left(\frac{\partial\psi}{\partial\tilde{y}_{t+1}}h_y + \frac{\partial\psi}{\partial\tilde{y}_t}\right)h_z + \frac{\partial\psi}{\partial z_t}$. This matrix needs to be invertible for the learning equilibrium to exist. The first-order derivatives of the actual policy function g can be obtained by applying the chain rule:

$$\begin{aligned}g\left(\tilde{y}_{t-1}, u_t, \sigma\right) &\approx g\left(\bar{y}, 0, 0\right) + g_y\left(\tilde{y}_{t-1} - \bar{y}\right) + g_u u_t + g_\sigma\sigma \\ g_y &= h_y + h_z r_y \\ g_u &= h_u + h_z r_u \\ g_\sigma &= h_\sigma + h_z r_\sigma\end{aligned}$$

The certainty-equivalence property holds for the subjective policy function h, hence $h_\sigma = 0$. This implies that $r_\sigma = 0$ and $g_\sigma = 0$ as well, so certainty equivalence also holds under learning.

Second- and higher-order perturbation approximations of g can be computed analogously. As in first order, only invertibility of the matrix A is required for a unique local solution under learning.

Code to compute perturbation solutions up to third order is available at www.fabianwinkler.com/research.

Appendix E. Results for estimated version of the model

Here, I present results for an alternative version of the learning model which tries to match the data more closely by estimating a number of parameters through a method of moments. This estimated version makes small changes in specification that allow the model to fit the data better than the version in the main text of the paper.

Appendix E.1. Specification and choice of parameters

Compared to the baseline version of the learning model, the following three changes are being made to the model. First, the dividend policy of firms is changed to simply read $D_{jt} = \zeta E_{jt}$. Second, the belief updating process is changed such that beliefs about asset price growth are updated simultaneously, instead of the "lagged belief updating" used in the main text of the paper.

Third, the probability of keeping the firm as a going concern in the event of default is increased from $x = 0.03$ to $x = 0.093$. This is the fraction of US business bankruptcy filings in 2006 that filed for Chapter 11 instead of Chapter 7, and that subsequently emerged from bankruptcy with an approved restructuring plan.

As in the baseline model, γ and ξ are chosen such that the non-stochastic steady state of the model jointly matches the average investment share in output of 18 percent and aestima-tionn average ratio of debt to assets of one (the sample average in the Fed flow of funds). The corresponding parameter values are $\gamma = 0.0155$ and $\xi = 0.3094$.

The remaining six parameters are the standard deviation of the technology shock (σ_A), the degree of nominal price and wage rigidities (κ, κ_w), the size of investment adjustment costs (ψ), the dividend payout ratio (ζ), and the learning gain (g). I estimate these six parameters to minimize the distance to a set of seven moments in quarterly U.S. data (1962Q1–2012Q4): The standard deviation of output; the standard deviations of consumption, investment, hours worked, and dividends relative to output; and the standard deviations of inflation and stock returns (see Tables E.2 and E.3 for the value of the data moments and estimated standard errors). All variables are are HP-filtered both in the data and in the simulations (cf. Gorodnichenko and Ng, 2010) except for stock returns, which are unfiltered.[23] Table E.1 contains the SMM estimates for both the learning and rational expectations (RE) version of the model.

[23]The set of estimated parameters $\theta \in \mathcal{A}$ minimizes the distance of simulated moments $m(\theta)$ to estimated mo-

Table E.1: Estimated parameters.

parameter	σ_A	κ	κ_w	ψ	ζ	g
learning	.660%	.408	.961	25.28	.562	.00498
	(.091%)	(.024)	(.020)	(1.98)	(.007)	(.00005)
RE re-estimated	1.11%	.645	.982	0	.548	-
	(.13%)	(.020)	(.002)	(.015)	(.146)	

Parameters as estimated by simulated method of moments. Asymptotic standard errors in parentheses are adjusted for boundary constraints on the parameters following Andrews (1999). Targeted data moments and estimated standard errors in Tables E.2 and E.3.

The size of the shock σ_A is larger under rational expectations than under learning, despite the fact that investment adjustment costs are much weaker. This already points to a larger degree of endogenous amplification of shocks under learning.

The Calvo price adjustment parameter κ under learning implies retailers adjust their prices about every five months on average, while they do so about every eight months for the rational expectations estimation. Under learning, movements in productivity and strong aggregate demand effects from stock price movements are counteracting forces on inflation, which means that inflation responds less to productivity shocks under learning than under rational expectations. By consequence, a lower degree of price rigidity is needed to match the volatility of inflation in the data.

The degree of nominal wage rigidities κ_w is estimated at a relatively high value both under learning and rational expectations, which is needed in order to match the relative volatility of employment in the data, which is about as high as that of output. The degree of wage ridigidy is at the upper end of the DSGE literature, but this feature helps the amplification mechanism of the learning model, as it ensures that changes in subjective expectations generate strong positive comovement of consumption, investment and employment.

Investment adjustment costs ψ are notably stronger under learning than under rational expectations. The reason is simply that investment is affected by stock price movements, and the

ments $\hat{m}$ in the data, using a weighting matrix W. I choose $W = \text{diag}\left(\hat{\Sigma}\right)^{-1}$ where $\hat{\Sigma}$ is the covariance matrix of the data moments, estimated using a Newey-West kernel with optimal lag order. This choice of W leads to a consistent estimator that places more weight on moments which are more precisely estimated in the data. The set $\mathcal{A}$ restricts the parameter values to $\sigma_A, \psi, g \in [0, \infty[$ and $\kappa, \kappa_w, \zeta \in [0, 1[$. The estimates are robust to using alternative filtering methods that capture business-cycle frequencies in the data, including the Christiano and Fitzgerald (2003) and Hamilton (2017) filters and linear detrending.

amount of stock price volatility under learning is much larger than under rational expectations. To simulatenously match investment and stock return volatility, then, strong adjustment costs are needed. In a sense, the amplification mechanism from learning would be much too strong without these adjustment costs.

The dividend payout ratio ζ is estimated at around 55 percent under both learning and rational expectations, which is in the ballpark of the historical average for the S&P500 (49 percent). Finally, the learning gain g implies that agents believe the amount of predictability in stock price growth to be small.

Appendix E.2. Results

Tables E.2 and E.3 repeat the business cycle and asset price moment statistics from the main text for the estimated model, and also add moments for the rational expectations version with re-estimated parameters in Column (5).

By construction, the estimated model matches the data better than the calibrated model because it includes more matched moments. But it also does at least as well on the non-matched moments such as the correlation of business cycle aggregates with output or the predictability of stock returns. It is worthwile noting that the re-estimated model under rational expectations is able to match the business cycle moments equally well, but fails to produce a sizeable amount of stock return volatility, despite the fact that this moment is explicitly targeted by the estimation.

Figure E.1 plots impulse responses to a productivity shock for the estimated model. As in the baseline model, one can see clearly how learning amplifies the movements of dividends, which feeds back into the learning dynamics and forms a two-sided feedback loop, and produces co-movement of output, inflation, consumption and employment.

Figure E.2 reproduces the forecast error predictability patterns for the estimated version of the model. The fit to the data is roughly comparable to that of the estimated version of the model.

Finally, Figure E.4 reproduces the table on sensitivity to monetary policy rules. Just as in the calibrated version, a reaction to asset price growth in the policy rule in Column (4) is strongly stabilizing. In fact, here it achieves the same reduction in inflation volatility as the output growth reaction in Column (3), while at the same time lowering output volatility.

Table E.2: Business cycle statistics, estimated version.

	moment	(1) data	(2) learning	(3) RE	(4) no fin. fric., RE	(5) RE re-estimated
output volatility	$\sigma_{hp}(Y_t)$	1.43% (0.14%)	1.40%*	.73	.57	1.41%*
volatility rel. to output	$\sigma_{hp}(I_t)/\sigma_{hp}(Y_t)$	2.90 (.12)	2.96*	.30	.17	2.78*
	$\sigma_{hp}(C_t)/\sigma_{hp}(Y_t)$	.60 (.035)	.59*	.97	1.34	.60*
	$\sigma_{hp}(L_t)/\sigma_{hp}(Y_t)$	1.13 (.061)	1.15*	.54	.26	1.25*
	$\sigma_{hp}(D_t)/\sigma_{hp}(Y_t)$	3.00 (.489)	3.08*	.31	1.45	1.96*
correlation with output	$\rho_{hp}(I_t, Y_t)$	.95 (.0087)	.73	.89	.20	0.91
	$\rho_{hp}(C_t, Y_t)$	.94 (.0087)	.81	.93	.99	0.63
	$\rho_{hp}(L_t, Y_t)$	.85 (.035)	.93	.72	.09	0.75
	$\rho_{hp}(D_t, Y_t)$	.56 (.080)	.55	.54	-.20	0.41
inflation	$\sigma_{hp}(\pi_t)$	.27% (.047%)	.30%*	.25%	.27%	.30%*
nominal rate	$\sigma_{hp}(i_t)$	.37% (.046%)	.09%	.09%	.10%	.12%

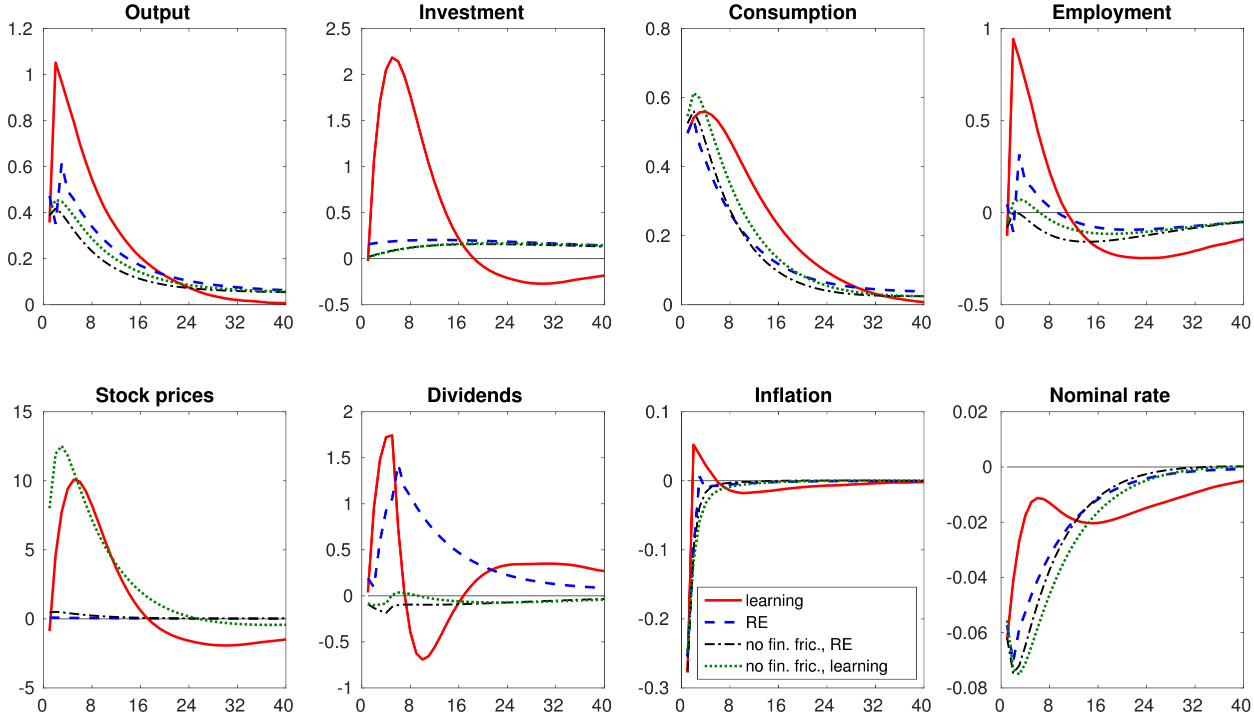
Quarterly U.S. data 1962Q1–2012Q4. Standard errors in parentheses. π_t is quarterly CPI inflation. i_t is the federal funds rate. All following variables are in logarithms. L_t is total non-farm payroll employment. Consumption C_t consists of services and non-durable private consumption. Investment I_t consists of private non-residential fixed investment and durable consumption. Output Y_t is the sum of consumption and investment. Dividends D_t are four-quarter moving averages of S&P 500 dividends. $\sigma_{hp}(\cdot)$ is the standard deviation and $\rho_{hp}(\cdot,\cdot)$ is the correlation coefficient of HP-filtered data (smoothing coefficient 1600). Moments used in the SMM estimation are marked with an asterisk.

Table E.3: Asset price statistics, estimated version.

	moment	(1) data	(2) learning	(3) RE	(4) no fin. fric., RE	(5) RE re-estimated
excess volatility	$\sigma\left(R_{t,t+1}\right)$	32.56% (2.44%)	35.53%*	0.37%	1.90%	0.13%*
	$\sigma\left(\frac{P_t}{D_t}\right)$	41.08% (6.11%)	31.49%	3.23%	1.70%	5.16%
return predictability	$\rho\left(\frac{P_t}{D_t},\, R_{t,t+4}\right)$	-.297 (.092)	-.495	.101	.018	.226
	$\rho\left(\frac{P_t}{D_t},\, R_{t,t+20}\right)$	-.585 (.132)	-.759	.082	-.012	.147
	$\rho\left(\frac{P_t}{D_t},\, \frac{P_{t+4}}{D_{t+4}}\right)$	.904 (.056)	.672	.753	.791	.677
negative skewness	skew $\left(R_{t,t+1}\right)$	-.897 (.154)	-.147	.004	-.009	.181
heavy tails	kurt $\left(R_{t,t+1}\right)$	1.57 (.62)	2.56	.01	-.02	0.27
risk-free rate	$\mathbb{E}\left(R_t^f\right)$	1.99% (.61%)	1.99%	1.99%	1.99%	1.99%
	$\sigma\left(R_t^f\right)$	2.34% (.29%)	0.56%	0.55%	.56%	0.65%
equity premium	$\mathbb{E}\left(R_{t,t+1} - R_t^f\right)$	4.06% (1.93%)	0.00%	0.00%	0.00%	0.00%
price correlation with output	$\rho_{hp}\left(P_t,\, Y_t\right)$	.458 (.115)	.704	.939	.985	.468

Quarterly U.S. data 1962Q1–2012Q4. Standard errors in parentheses. Dividends D_t are four-quarter moving averages of S&P 500 dividends. The stock price index P_t is the S&P 500. Stock returns $R_{t,t+s}$ are annualized s-quarter ahead real returns of the S&P 500. Risk-free returns R_t^f are 3-month real Treasury yields. $\sigma(\cdot)$ is the standard deviation; $\rho(\cdot,\cdot)$ is the correlation coefficient; $\rho_{hp}(\cdot,\cdot)$ is the correlation coefficient of HP-filtered data (smoothing coefficient 1600); skew $(\cdot)$ is skewness; kurt $(\cdot)$ is excess kurtosis. Moments used in the SMM estimation are marked with an asterisk.

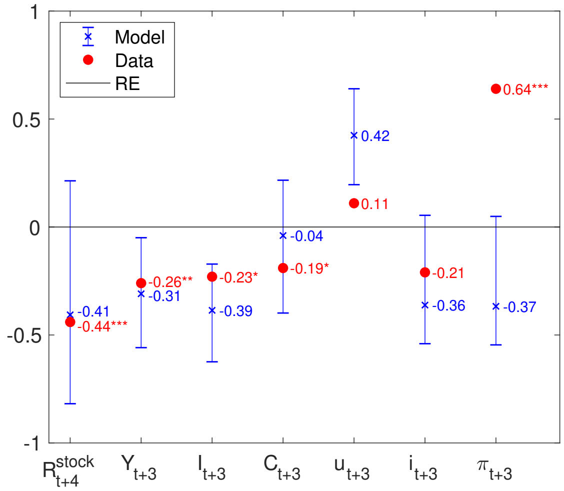
Figure E.1: Impulse responses to a productivity shock, estimated version.



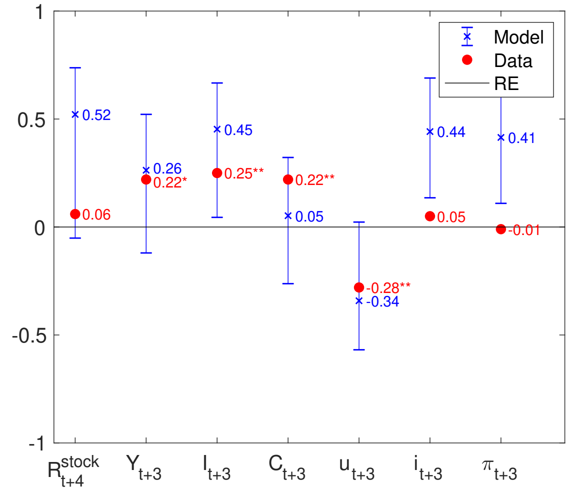
Impulse responses to a one-standard deviation innovation in ε_t. Responses averaged over 5,000 random shock paths with a burn-in of 1,000 periods. Stock prices, dividends, output, investment, consumption, and employment are in 100*log deviations. Inflation and the nominal interest rate are in percentage point deviations.

Figure E.2: Forecast error predictability, estimated version.

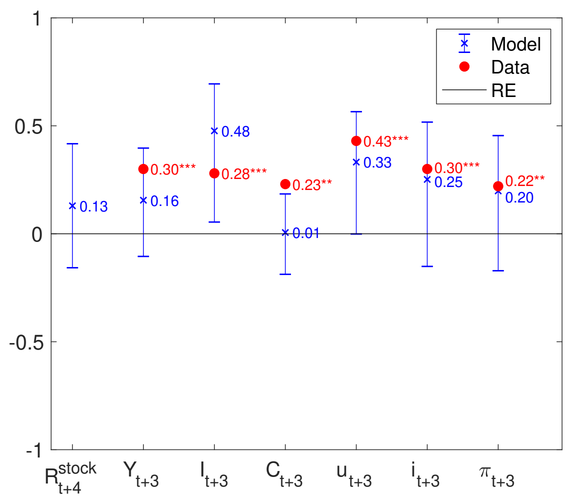
(a) Correlation of forecast error with $\log PD_t$.



(b) Correlation of forecast error with $\Delta \log PD_t$.



(c) Correlation of forecast error with forecast revision.



Red dots show correlation coefficients for mean forecast errors on one year-ahead nominal stock returns (Graham-Harvey survey) and three quarters-ahead real output growth, investment growth, consumption growth, unemployment rate, CPI inflation and 3-month treasury bill (SPF). Regressors: Panel (a) is the S&P 500 P/D ratio and Panel (b) is its first difference. Panel (c) is the forecast revision as in Coibion and Gorodnichenko (2015), which is only available in the SPF. Data from Graham-Harvey covers 2000Q3–2012Q4. Data for the SPF covers 1981Q1–2012Q4. *, **, and *** indicate significance at the 1, 5, and 10 percent level, respectively, using Newey-West standard errors. Blue crosses show corresponding correlation coefficients in the model, computed using a simulation of length 50,000, where subjective forecasts are computed using a second-order approximation to the subjective belief system on a path in which no more future shocks occur, starting at the current state in each period. Unemployment in the model is taken to be $u_t = 1 - L_t$. Stock returns in the model $R^{stock}_{t,t+4}$ are quarterly nominal aggregate market returns. Blue lines show 95% confidence bands of the correlation coefficients in the model in small samples of the same size as the data (123 quarters in the SPF and 49 quarters in the Graham-Harvey survey) from 5,000 simulations with a burn-in period of 1,000 periods.

Table E.4: Alternative monetary policy rules, estimated version.

	(1)	(2)	(3)	(4)
ϕ_π	1.5	3.0	1.5	1.5
ϕ_Y			0.5	
ϕ_P				0.5
$\sigma(Y)$	2.27%	3.03%	1.97%	1.32%
$\sigma(\pi)$	0.29%	0.13%	0.31%	0.31%
$\sigma(P)$	22.78%	28.76%	17.90%	7.10%
$\sigma(i)$	0.12%	0.15%	0.08%	0.12%
$\sigma(Y)/\sigma(Y_{RE})$	1.46	1.27	1.26	0.85

Standard deviations of output, stock prices, inflation, and interest rates (unfiltered) under learning in percent. The standard deviation of output under RE $\sigma(Y_{RE})$ is calculated at the same parameter values as the learning solution. The interest rate smoothing coefficient is kept at $\rho_i = 0.85$ for all rules considered.