

Solutions to a Class of Linear Expectational Difference Equations

1. Introduction

These notes provide an informal characterization of the solution to a class of expectational difference equations. I first consider a very simple example in order to illustrate the main results. I then turn to the more general case without uncertainty. After that I consider the case with uncertainty. The previous two sections assumed that a certain matrix is singular, an assumption that is not typically satisfied in practice. The next section shows how to address this problem using the QZ decomposition. The last section has an extended example to illustrate the various points in detail.

2. A Simple Example to Illustrate Some of the Basic Points

The intertemporal Euler equation associated with the neoclassical growth model, after that equation has been linearized about steady state, has the following representation:

$$k_{t+2} - \phi k_{t+1} + \frac{1}{\beta} k_t = 0, \quad (2.1)$$

for $t = 0, 1, 2, \dots$. Here, k_{t+1} denotes the value of the capital stock selected at time t , expressed in deviation from steady state. Also, \bar{k}_0 represents the deviation of the initial stock of capital from steady state. We assume $\bar{k}_0 \neq 0$. Also, $0 < \beta < 1$ and

$$\phi > 1 + \frac{1}{\beta}. \quad (2.2)$$

We refer to a sequence, $\{k_t\}_{t=1}^{\infty}$, which satisfies the above sequence of difference equations as well as the initial condition, as a *solution*. (The notion of a solution is slightly different here than it was in Econ411). It is easy to see how many solutions there are. Consider the sequence of equations, (2.1):

$$\begin{aligned} k_2 - \phi k_1 + \frac{1}{\beta} k_0 &= 0 \\ k_3 - \phi k_2 + \frac{1}{\beta} k_1 &= 0 \\ &\dots \end{aligned}$$

If we arbitrarily set a value for k_1 , the first equation can be solved for k_2 . Then the second equation can be solved for k_3 and in this way we can find an entire sequence, $\{k_t\}_{t=1}^{\infty}$. Since this sequence is completely determined by the selected value of k_1 , we conclude that the set of solutions, $\{k_t\}_{t=1}^{\infty}$, has one dimension, indexed by the value of k_1 .

Unfortunately, this way of characterizing the set of solutions to (2.1) is not very convenient. There is little else one can say about that set. An alternative characterization is more convenient. It represents the solutions in the space of roots of the polynomial equation associated with (2.1):

$$f(\lambda) = \lambda^2 - \phi\lambda + \frac{1}{\beta} \quad (2.3)$$

Let λ_1 and λ_2 satisfy $f(\lambda_i) = 0$ for $i = 1, 2$. Under the assumption that λ_1 and λ_2 are distinct (a condition that we verified in Econ 411), the complete set of solutions to (2.1) can be represented as follows:

$$k_t = (\bar{k}_0 - a)\lambda_1^t + a\lambda_2^t, \quad (2.4)$$

where a is arbitrary. It is easy to verify that (2.4) satisfies (2.1) and the initial condition for every possible value of a .

From (2.4) we see, like before, that the set of solutions to (2.1) is one-dimensional and is indexed by the scalar, a . There are two other characteristics of the space of solutions that we can see from (2.4). First, there exist two solutions in which k_t solves a first order difference equation. The two solutions are the ones associated with $a = \bar{k}_0$ and $a = 0$, respectively, that is:

$$k_t = \bar{k}_0\lambda_2^t, \quad k_t = \bar{k}_0\lambda_1^t.$$

In each case, divide the expression by itself evaluated at $t - 1$, to obtain:

$$k_t = \lambda_2 k_{t-1}, \quad k_t = \lambda_1 k_{t-1}.$$

This verifies that in the case of these two solutions, the sequence of k_t 's which satisfy the second order difference equation, (2.1), also satisfy a first order difference equation. It is natural to call these solutions minimal state variable (MSV) solutions, because in each case every capital stock in the sequence can be expressed as a function of only one previous value of the capital stock. This is a 'natural' solution if we think of k_t as being decided at time $t - 1$. At that time, k_{t-1} is sufficient information to characterize the current situation and the value of capital in period $t - 2$ or earlier seems superfluous. For example, to understand the nature of production opportunities at date $t - 1$ one needs only know k_{t-1} . We can see from (2.4) that there are exactly two MSV solutions. All the other solutions make the capital stock a non-trivial function of two eigenvalues, and so the capital stock follows a second order difference equation. These are not MSV solutions because k_t at any point in the solution cannot be determined exclusively as a function of k_{t-1} .

We can also define a solution as being 'non-explosive' if

$$k_t \rightarrow 0$$

as $t \rightarrow \infty$. From (2.4) we can see three cases.

- Case 1: $|\lambda_1| > 1$ and $|\lambda_2| < 1$. In this case, there is precisely one solution that is non-explosive, the one associated with $a = \bar{k}_0$.
- Case 2: $|\lambda_1|, |\lambda_2| > 1$. In this case there is no solution that satisfies non-explosiveness.
- Case 3: $|\lambda_1|, |\lambda_2| < 1$. In this case, all solutions are non-explosive.

Note that in case 1, when there is exactly one non-explosive solution, that solution is a MSV.¹

These properties of solutions are quite general. (i) The number of solutions correspond to the number of points in a finite-dimensional Euclidean space (in the example, it's R^1). (ii) There is a finite number of MSV solutions and that set is easy to characterize. (iii) The number of non-explosive solutions requires comparing the absolute value of eigenvalues with unity, and if there is only one solution that is non-explosive, then that solution is MSV.

3. Deterministic Case, Invertible a

We now develop a matrix version of the analysis in the previous section. Suppose that (2.1) is actually

$$\alpha_0 k_{t+2} + \alpha_1 k_{t+1} + \alpha_2 k_t = 0, \quad t = 0, 1, 2, \dots \quad (3.1)$$

where k_{t+1} is the $n \times 1$ vector of time t endogenous variables, expressed in deviation from steady state. Also, α_i are known $n \times n$ matrices for $i = 0, 1, 2$. The '0' after the equality in (3.1) is an $n \times 1$ vector of zeros.² The initial conditions, k_0 are given. It is convenient to express (3.1) as a first order difference equation system:

$$aY_{t+1} + bY_t = 0, \quad t \geq 0, \quad (3.2)$$

¹It is readily verified that the neoclassical model falls in case 1. Dividing f in (3.1) by λ , we find that the zero condition corresponds to

$$g(\lambda) = \phi,$$

where

$$g(\lambda) = \lambda + \frac{1}{\beta\lambda}.$$

The graph of g against λ has a 'U' shape, and reaches a minimum at $\lambda = \sqrt{1/\beta}$, when g takes on a value of $2\sqrt{\beta}$. Note

$$0 < \left(1 - \sqrt{1/\beta}\right)^2 = 1 + \frac{1}{\beta} - 2\sqrt{1/\beta},$$

so that $1 + \frac{1}{\beta} > 2\sqrt{1/\beta}$. It follows from (2.2) that the ϕ line cuts the g curve above the point where g reaches its minimum. As a result, roots of f are definitely distinct. In addition, it is easily verified that one root is less than unity and the other is greater than unity.

²In the case, $n = 1$, (3.1) could be the linearized intertemporal Euler equation for the neoclassical growth model. Thus, suppose the problem is to maximize $\sum_{t=0}^{\infty} \beta^t \log c_t$ subject to $c_t + k_{t+1} - (1 - \delta)k_t \leq k_t^\alpha$, $\beta, \alpha, \delta \in (0, 1)$ and k_0 given. Then the first order condition is

$$\frac{1}{c_t} - \beta \frac{1}{c_{t+1}} [\alpha k_{t+1}^\alpha + 1 - \delta] = 0,$$

for $t \geq 0$. Substituting out for c_t and c_{t+1} in this expression from the resource constraint, we obtain

$$v(k_t, k_{t+1}, k_{t+2}) = 0, \quad t = 0, 1, 2, \dots$$

Compute k^* such that $v(k^*, k^*, k^*) = 0$ (i.e., the steady state). Then, (3.6) represents the above condition in which the v function has been replaced by its first order Taylor series expansion about steady state. In (3.6), k_t stands for $k_t - k^*$. I do not adopt a new piece of notation to express deviations from steady state in order to keep the notation simple and because this should not cause confusion in this setting.

where

$$Y_t = \begin{pmatrix} k_{t+1} \\ k_t \end{pmatrix}, \quad a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}.$$

The first set of n equations in (3.2) reproduce the n equations in (3.1), while the second set of n equations in (3.2) capture the fact that the second set of variables in Y_{t+1} coincide with the first set of variables in Y_t .

Here we assume that a is invertible (i.e., we assume that α_0 is invertible). This assumption is rarely satisfied in practice, although it is satisfied in the simple example of the previous section when $n = 1$. It is convenient to separate the question of how to deal with the singularity in a from other aspects of model solution. For this reason we defer addressing the singular a case until a later section.

Solving (3.2), we find:

$$Y_t = \Pi^t Y_0, \quad \Pi = -a^{-1}b. \quad (3.3)$$

According to this equation, a solution is completely determined by the initial condition, Y_0 . We assume that the eigenvalues of Π are distinct, which guarantees that Π has the following eigenvector-eigenvalue decomposition:

$$\Pi = P \Lambda P^{-1}. \quad (3.4)$$

Write

$$P = \begin{pmatrix} P_1 & \cdots & P_{2n} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \tilde{P}_1 \\ \vdots \\ \tilde{P}_{2n} \end{pmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{2n} \end{bmatrix},$$

where Λ is a diagonal matrix. The elements along the diagonal of Λ are the eigenvalues of the matrix, Π . The column vectors, P_i , are the right eigenvectors of Π :

$$\Pi P_i = \lambda_i P_i, \quad i = 1, \dots, 2n.$$

After premultiplying (3.4) by P^{-1} , we see that \tilde{P}_i are the left eigenvectors of Π :

$$\tilde{P}_i \Pi = \lambda_i \tilde{P}_i, \quad i = 1, \dots, 2n.$$

The left eigenvectors of Π play a fundamental role in shaping the dynamics of the system.

From (3.3), we see that the space of solutions is n -dimensional. This is because Y_0 has n ‘free parameters’ in it, namely the n elements of k_1 (recall, k_0 is given). That is, the set of solutions corresponds to the n -dimensional Euclidean space, R^n . This is the analog of result (i) in the previous section. We now proceed to the analogs of results (ii) and (iii).

It is easy to verify that

$$\Pi^j = P \Lambda^j P^{-1}. \quad (3.5)$$

So, multiplying both sides of (3.3) by P^{-1} we obtain:

$$\tilde{Y}_t = \Lambda^t \tilde{Y}_0, \quad \tilde{Y}_t \equiv P^{-1} Y_t.$$

The equations that determine the evolution of \tilde{Y}_t are completely independent, and can be expressed as follows:

$$\tilde{Y}_{it} = \lambda_i^t \tilde{Y}_{i,0}, \quad \text{for } i = 1, 2, \dots, 2n, \quad (3.6)$$

where,

$$\tilde{Y}_t = \begin{pmatrix} \tilde{Y}_{1,t} \\ \vdots \\ \tilde{Y}_{2n,t} \end{pmatrix}.$$

Equation (3.6) shows that if $\tilde{Y}_{i,0} = 0$, then $\tilde{Y}_{i,t} = 0$ for all t . Put differently,

$$\tilde{P}_i Y_0 = 0 \rightarrow \tilde{P}_i Y_t = 0, \text{ for } t = 1, 2, \dots. \quad (3.7)$$

That is, if Y_0 is orthogonal to the i^{th} left eigenvector, then Y_t will be orthogonal to that eigenvector too, for $t \geq 1$. This results will be useful in what comes next.

We define a candidate MSV solution as a solution, Y_0, Y_1, \dots , having the property that there exists an $n \times n$ real matrix, A with the property

$$\begin{bmatrix} I & : & -A \end{bmatrix} Y_t = 0 \text{ for } t = 0, 1, 2, \dots.$$

Equation (3.7) suggests how one can find a candidate MSV. Select n left eigenvectors of Π and use them to construct an $n \times 2n$ matrix D . Then,

$$DY_0 = \begin{bmatrix} D_1 & : & D_2 \end{bmatrix} \begin{pmatrix} k_1 \\ k_0 \end{pmatrix} = D_1 k_1 + D_2 k_0.$$

If

$$D_1 \text{ invertible, } A \equiv -D_1^{-1} D_2 \text{ real,} \quad (3.8)$$

then the candidate MSV is an MSV (an *actual* MSV, for emphasis). The MSV is written

$$k_1 = A k_0.$$

When the candidate MSV is an actual MSV, then Y_0 is uniquely determined:

$$Y_0 = \begin{pmatrix} A \\ I \end{pmatrix} k_0,$$

where I is the $n \times n$ identity matrix. Note that there exist

$$\begin{pmatrix} 2n \\ n \end{pmatrix} \quad (3.9)$$

candidate MSV's, since this is the number of ways of choosing n left eigenvectors from the set of $2n$ left eigenvectors. The set of actual MSV's is smaller than what is indicated in (3.9) if there are candidate MSV's which generate a complex A matrix and/or D_1 that is not invertible. (We discuss the complex case by way of an example below.) The key result is that there is a finite number of isolated MSV solutions. This is analogous to result (ii) in the previous section.

To determine the set of non-explosive solutions, note that $\tilde{Y}_t \rightarrow 0$ if, and only if, $Y_t \rightarrow 0$. Because of (3.6) it is convenient to consider the convergence properties of \tilde{Y}_t . According to (3.6), a non-explosive solution must have the property that the eigenvalues greater than unity

in absolute value have been ‘extinguished’ from the system. If $|\lambda_i| > 1$ (i.e., λ_i is ‘explosive’) this eigenvalue is extinguished by choosing Y_0 so that $\tilde{P}_i Y_0 = 0$. The word, ‘extinguished’, is appropriate here because - according to (3.6) - λ_i has no impact on solution dynamics when $\tilde{P}_i Y_0 = 0$. Consider the same three cases delineated in the previous section. Let q denote the number of explosive eigenvalues.

- Case 1: $n = q$. In this case a candidate non-explosive solution is found by constructing a D matrix containing the n left eigenvectors associated with the explosive eigenvalues and determining the Y_0 such that $DY_0 = 0$. This candidate is an actual non-explosive solution if (3.8) is satisfied. The candidate non-explosive solution is unique.
- Case 2: $n < q$. In this case, every solution is explosive.
- Case 3: $n > q$. In this case, construct an $n \times 2n$ D matrix by including the left eigenvectors of Π associated with the q explosive eigenvalues. The remaining $n - q$ rows of D can be constructed by selecting some combination of other left eigenvectors of Π and arbitrarily selected row vectors. If (3.8) is satisfied for this D matrix, then the D matrix corresponds to a solution. If D is composed only of left eigenvectors of Π , then the computed non-explosive solution is MSV. Otherwise, it is not MSV.

4. Stochastic Case, Invertible a

In the stochastic case, (3.1) is written

$$E[\alpha_0 k_{t+2} + \alpha_1 k_{t+1} + \alpha_2 k_t + \beta_0 s_{t+1} + \beta_1 s_t | s_t] = 0, \quad (4.1)$$

for each $k_t \in R^n$, $s_t \in R^{n_s}$. Here, s_t is the vector of exogenous shocks and is assumed to have the following time series representation:

$$s_t = P s_{t-1} + \varepsilon_t,$$

where the eigenvalues of P are all less than unity in absolute value.³ It is convenient to rewrite (4.1):

$$\alpha_0 k_{t+2} + \alpha_1 k_{t+1} + \alpha_2 k_t + \beta_0 s_{t+1} + \beta_1 s_t = \xi_{t+1}, \quad (4.2)$$

³This example, in the case $n = 1$, could arise from a stochastic version of the neoclassical model in the previous footnote in which the production function is replaced by $k_t^\alpha \exp(\theta_t)^{1-\alpha}$. If $\theta_t - \theta = \rho(\theta_{t-1} - \theta) + u_t$, where u_t is *iid* and uncorrelated with past θ_t and $|\rho| < 1$, then $s_t = \theta_t$, $P = \rho$, $\varepsilon_t = u_t$. When $n_s > 1$ the notation accommodates more complicated θ_t processes and/or other shocks. For example, suppose

$$\theta_t - \theta = \rho_1(\theta_{t-1} - \theta) + \rho_2(\theta_{t-2} - \theta) + u_t + \gamma u_{t-1}$$

then, $s_t = [\theta_t - \theta \quad \theta_{t-1} - \theta \quad u_t]'$, $\varepsilon_t = [u_t \quad 0 \quad u_t]'$ and

$$F = \begin{bmatrix} \rho_1 & \rho_2 & \gamma \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that the roots of $\lambda^2 - \rho_1 \lambda - \rho_2$ are less than unity in absolute if and only if the eigenvalues of F are less than unity in absolute value.

where ξ_{t+1} is a stochastic process satisfying

$$E_t \xi_{t+1} = 0, \quad (4.3)$$

for all t . Note that the set of ξ_{t+1} satisfying (4.3) is very large. For example,

$$\xi_{t+1} \sim N(0, 1 - \cos(t)),$$

is a possibility.

As before, we proceed by expressing the system as a first order process. If

$$Y_t = \begin{pmatrix} k_{t+1} \\ k_t \\ s_t \end{pmatrix}, \quad a = \begin{bmatrix} \alpha_0 & 0 & \beta_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1 & \alpha_2 & \beta_1 \\ -I & 0 & 0 \\ 0 & 0 & -P \end{bmatrix}, \quad \omega_t = \begin{pmatrix} \xi_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix},$$

then (4.2) and the law of motion of s_t are summarized as follows:

$$aY_{t+1} + bY_t = \omega_{t+1}. \quad (4.4)$$

As in the previous section, we assume α_0 is non-singular, so that a is invertible. Then, the analog of (3.3) is

$$Y_t = \Pi^t Y_0 + a^{-1} \omega_t + \Pi a^{-1} \omega_{t-1} + \cdots + \Pi^{t-1} a^{-1} \omega_1.$$

We see that now the $\{Y_t\}$ that solves the system is a stochastic process. We can still use the sort of approach of the previous section here in characterizing the set of solutions. The set of solutions is now indexed by Y_0 and a stochastic process for $\{\xi_{t+1}\}$ that satisfies (??). The other stochastic process, $\{\varepsilon_t\}$, is treated like k_0 in that it is given by the problem. Thus, the space of solutions is characterized by the choice of Y_0 , $\{\xi_{t+1}\}$. This is a very large space. The vector, Y_0 , has n free elements, the ones corresponding to k_1 . The vector stochastic process, $\{\xi_{t+1}\}$, also has n free elements.

To further characterize the set of solutions, it is convenient to make use of (3.5) and premultiply (4.4) by P^{-t} :

$$\tilde{Y}_t = \Lambda^t \tilde{Y}_0 + P^{-1} a^{-1} \omega_t + \Lambda P^{-1} a^{-1} \omega_{t-1} + \cdots + \Lambda^{t-1} P^{-1} a^{-1} \omega_1, \quad (4.5)$$

or,

$$\tilde{Y}_{it} = \lambda_i^t \tilde{Y}_{i,0} + v_{i,t} + \lambda_i v_{i,t-1} + \cdots + \lambda_i^{t-1} v_{i,1}, \quad v_{i,t} \equiv \tilde{P}_i a^{-1} \omega_t,$$

for $i = 1, \dots, 2n + n_s$. Note, selecting Y_0 to be orthogonal to the left eigenvector of λ_i , \tilde{P}_i , no longer ensures that Y_t remains orthogonal to that eigenvector forever after. For this to be the case, we must also set $v_{i,t} = 0$ for each t . In fact we have the degrees of freedom to do this because of the n free stochastic processes in ξ_t . To see this, write

$$v_{i,t} = \begin{bmatrix} \delta_1^{(i)} & \delta_2^{(i)} \\ 1 \times 1 & 1 \times (n-1) \end{bmatrix} \begin{pmatrix} \xi_t^1 \\ \xi_t^2 \end{pmatrix} + \gamma^{(i)} \varepsilon_t, \quad \begin{bmatrix} \delta_1^{(i)} & \delta_2^{(i)} & 0 & \gamma^{(i)} \\ & & 1 \times n & \end{bmatrix} = \tilde{P}_i a^{-1},$$

where

$$\xi_t \equiv \begin{pmatrix} \xi_t^1 \\ 1 \times 1 \\ \xi_t^2 \\ (n-1) \times 1 \end{pmatrix}.$$

We suppose that $\delta_1^{(i)} \neq 0$. If ξ_t^2 is any given process that satisfies (4.3), then choose ξ_t^1 to satisfy

$$\xi_t^1 = \frac{\delta_2^{(i)} \xi_t^2 + \gamma^{(i)} \varepsilon_t}{\delta_1^{(i)}}.$$

Note that with ξ_t constructed in this way, (4.3) is satisfied. Thus, it is possible to find a solution (i.e., a Y_0 and a $\{\xi_t\}$ that satisfies (4.3)) with the property that $\tilde{P}_i Y_t = 0$ for all t .

As before, we use this result to construct a set of MSV solutions. Here, we define a candidate MSV solution as a solution, $(Y_0, \{\xi_t\})$, having the property that there exists a $n \times (2n + n_s)$ matrix D such that

$$DY_t = 0, \quad t = 0, 1, 2, \dots$$

Note that if the analog of (3.8) holds (i.e., the left $n \times n$ block of D is invertible and the analog of A is real), an MSV solution has the property that k_{t+1} can be determined uniquely knowing only k_t and s_t . In this case, we say the candidate MSV solution is an actual MSV. As before, there are many candidate MSV solutions in the space of solutions. They can be found by constructing D matrices by selecting n left eigenvectors from the set of $2n + n_s$ eigenvectors of Π and by selecting the n free stochastic processes in ξ_t so that

$$Da^{-1}\omega_t = \begin{matrix} 0 \\ n \times 1 \end{matrix}, \quad (4.6)$$

for each $t = 0, 1, 2, \dots$. Note that by selecting ξ_t in this way, we make ξ_t an exact function of ε_t . Such a solution is sometimes called by economists a ‘fundamental’ solution because it makes the stochastic processes driving the system, ξ_t, ε_t exclusively a function of the exogenous disturbances impacting on preferences and technology, namely the ε_t ’s.⁴

We now define a non-explosive solution as one having the property:

$$\begin{aligned} E_0 Y_t &\rightarrow 0, \\ \text{Var}_0(Y_t) &\text{ bounded.} \end{aligned}$$

Note that the first condition is not enough, because it does not restrict the ξ_t ’s at all. Without restricting the ξ_t ’s, the first condition could be satisfied while the Y_t ’s have exploding variances...hardly ‘non-explosive’.

As before, the non-explosive solutions are the ones in which the explosive eigenvalues have been suppressed. To suppress an explosive eigenvalue, λ_i , $a^{-1}\omega_t$ and Y_0 must be selected so that $\tilde{P}_i a^{-1}\omega_t = 0$ for all t and $\tilde{P}_i Y_0 = 0$. There are n degrees of freedom in setting ξ_t and in setting Y_0 . As a result, we obtain the same three cases considered in the previous two sections. Let q denote the number of explosive eigenvalues.

- Case 1: if $q = n$, the number of candidate non-explosive solutions is unique and it is an MSV if the analog of (3.8) is satisfied.
- Case 2: if $q > n$, all solutions are explosive.

⁴I say ‘economists’ here, because the word, ‘fundamental’ means something different in other areas, for example, time series analysis.

- Case 3: if $q < n$, there are many candidate non-explosive solutions. Some may be MSV's. In addition, non-explosive solutions that are not MSV's also exist.

It is worthwhile to elaborate a little on case 3. To identify a candidate non-explosive MSV solution, construct the first q rows of an $n \times (2n + n_s)$ D matrix using the q left eigenvectors of Π associated with the explosive eigenvalues. One could fill out the bottom $n - q$ rows of D with left eigenvectors of Π associated with the non-explosive eigenvalues. There are obviously several ways of doing this. For each D constructed in this way, choose Y_0 so that $DY_0 = 0$ and choose the n elements of ξ_t so that (4.6) is satisfied. Each resulting solution, $(Y_0, \{\xi_t\})$, is a candidate non-explosive MSV solution in which only fundamental shocks appear. There are also non-MSV solutions. Fill out the bottom $n - q$ rows of D vectors other than left eigenvectors of Π . Choose them so that D has rank n . There is obviously a continuum of ways of doing this. Choose the ξ_t 's any way you want, subject only to (4.3) and the requirement that $\tilde{P}_t a^{-1} \omega_t = 0$ for the explosive eigenvalues. There is obviously a continuum of ways of choosing ξ_t 's to satisfy these conditions. To the extent that the ξ_t 's are not a function of ε_t , these candidate non-MSV solutions are also non-fundamental. The elements of ξ_t that are not a function of ε_t are called 'sunspots'.

5. The Non-Invertible a Case

Now consider the case when a is not invertible. This case arises when α_0 is non-singular. It occurs because in practice equilibrium conditions are different in terms of how many future variables they include. An intertemporal Euler equation includes variables stretching relatively far into the future while a resource constraint or an intratemporal equation involves variables that extend less far in the future. A consequence of this is that rows of α_0 can be zero. A general procedure for handling this case is to substitute out variables in a way that makes the system smaller, and have the property that α_0 in that system is non-singular. A simple example is the neoclassical growth model with an hours worked decision. In that case, the system involves the intra- and inter- temporal Euler equations (I'm assuming the resource constraint has used to substitute out consumption). This system has a singular α_0 . However, hours worked may be substituted out from the linearized intra-temporal Euler equation into the linearized inter temporal Euler equation. The resulting system has a non-singular α_0 . In effect, we 'decouple' hours worked from the system and solve the smaller system. Once a solution for the smaller system is found, we can solve for the variable, hours worked, that has been substituted out.

Although manual substituting along the lines of the previous paragraph works in some examples, in general it is tedious. Fortunately, there is a simple matrix procedure based on the QZ decomposition that allows us to make the system smaller by decoupling some variables. To my knowledge, it was first suggested in Sims in a 1989 working paper, which was subsequently published in a 2001 issue of *Computational Economics*.

We only consider the deterministic case, because the stochastic case is a straightforward extension. Let the dimension of Y_t be denoted $m = 2n$. Suppose that the rank of a is $l < m$. The QZ decomposition of matrices, a and b , is a set of orthonormal matrices Q and Z , and upper triangular matrices H_0 and H_1 with the properties:

$$QaZ = H_0, \quad QbZ = H_1. \quad (5.1)$$

It is possible to order the rows of H_0 so that the l zeros on its diagonal are located in the lower right part of H_0 . Denote the upper $(m-l) \times (m-l)$ block of H_0 by G_0 . This matrix must be non-singular. Let the corresponding upper left $(m-l) \times (m-l)$ block in H_1 be denoted G_1 . By construction, the l terms on the lower right part of the diagonal of G_0 are zero. I assume that the diagonal terms in the lower right $l \times l$ block of H_1 are non-zero. Also, it is useful to partition Z' as follows:

$$Z' = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad (5.2)$$

where L_1 is $(m-l) \times m$ and L_2 is $l \times m$.

Inserting $ZZ' (= I)$ before Y_{t+1} and Y_t in (3.2), defining $\gamma_t \equiv Z'Y_t$, and pre-multiplying (3.2) by Q , (3.2) becomes:

$$H_0\gamma_{t+1} + H_1\gamma_t = 0, \quad t = 0, 1, \dots \quad (5.3)$$

Partition γ_t as follows:

$$\gamma_t = \begin{pmatrix} \gamma_t^1 \\ \gamma_t^2 \end{pmatrix}, \quad (5.4)$$

where γ_t^1 is $(m-l) \times 1$ and γ_t^2 is $l \times 1$. It is easy to verify that (5.3) implies $\gamma_t^2 = 0$, $t \geq 0$, i.e.,⁵

$$L_2 Y_t = 0, \quad t = 0, 1, \dots \quad (5.5)$$

With (5.5) imposed, the last l equations in (5.3) are redundant, so (5.3) can be written

$$G_0\gamma_{t+1}^1 + G_1\gamma_t^1 = 0, \quad t = 0, 1, \dots \quad (5.6)$$

In effect, we have now ‘decoupled’ γ_t^2 from the system. The set of solutions to the reduced sized system, (5.6), can be expressed as $\gamma_t^1 = (-G_0^{-1}G_1)^t \gamma_0^1$, $t \geq 0$, or,

$$P^{-1}\gamma_t^1 = \Lambda^t P^{-1}\gamma_0^1, \quad (5.7)$$

where $P\Lambda P^{-1} = -G_0^{-1}G_1$ is the eigenvector, eigenvalue decomposition of $-G_0^{-1}G_1$.

The set of candidate MSV solutions is constructed by putting $n-l$ of the left eigenvectors of $-G_0^{-1}G_1$ into a matrix D and filling out the last l rows of D with the rows of L_2 . In this case, $DY_t = 0$ for all t . There are obviously several ways to construct D in this way. There

⁵To see this, let us temporarily adopt a simpler notation. Let the lower right $l \times l$ block of H_0 be denoted Γ and let the corresponding block of H_1 be denoted W . Write $\Gamma = [\Gamma_{ij}]$ and $W = [W_{ij}]$. The matrices, Γ and W , are upper triangular, with the former having zeros along its diagonal and the latter having non-zero terms along its diagonal. Also, write $x_t = \gamma_t^2$, with $x_t = [x_{1t}, \dots, x_{lt}]'$. Then we have $\Gamma x_{t+1} + W x_t = 0$ for $t = 0, 1, 2, \dots$. Note that the last row of Γ is composed of zeros, so that the last row of this system of equations is $W_{l,l}x_{lt} = 0$ for all t . Since $W_{l,l} \neq 0$, this implies $x_{lt} = 0$ for all t . Now consider the $l-1$ equation:

$$\Gamma_{l-1,l}x_{l,t+1} + W_{l-1,l-1}x_{l-1,t} + W_{l-1,l}x_{lt} = 0,$$

for $t = 0, 1, 2, \dots$. But, since $x_{.,t} = 0$ for all t , this implies $W_{l-1,l-1}x_{l-1,t} = 0$ for all t . Since $W_{l-1,l-1} \neq 0$, this in turn implies $x_{l-1,t} = 0$ for $t = 0, 1, 2, \dots$. Proceeding in this way, we establish recursively that $x_{j,t} = 0$ for all t , for $j = l, l-1, \dots, 1$.

are thus several candidate MSV solutions. A candidate MSV solution is an actual MSV solution if D satisfies condition (3.8).

Now consider the set of non-explosive solutions, $Y_t \rightarrow 0$. The γ_t^1 that solve (5.7) converge to zero asymptotically if, and only if, $\tilde{p}\gamma_0^1 = 0$, where \tilde{p} is composed of the rows of P^{-1} corresponding to diagonal terms in Λ that exceed 1 in absolute value. This condition is:

$$\tilde{p}L_1Y_0 = 0. \quad (5.8)$$

There are many candidate non-explosive solutions if q , the number of explosive eigenvalues of $-G_0^{-1}G_1$, is less than $n-l$. There is exactly one if $q = n-l$ and there are none if $q > n-l$.

Consider the case, $q = n-l$. Recall that the number of free elements in Y_0 is n . Equation (5.5) for $t = 0$ represents l restrictions on Y_0 , so that to pin Y_0 down uniquely, $n-l$ more restrictions are required. Construct the matrix \tilde{p} using the $n-l$ left eigenvectors associated with the explosive eigenvalues in $-G_0^{-1}G_1$. Then, define

$$D = \begin{bmatrix} \tilde{p}L_1 \\ L_2 \end{bmatrix}. \quad (5.9)$$

Under the assumption that D satisfies (3.8), the unique non-explosive solution corresponds to the MSV associated with the Y_0 satisfying $DY_0 = 0$.

6. ‘Candidate’ Versus ‘Actual’ MSV’s

In the previous sections, we have frequently referred to candidate versus actual MSV solutions. Here, we develop an example to make concrete the difference between them. Consider the standard neo-Keynesian model:

$$\begin{aligned} \delta\pi_{t+1} + \lambda y_t - \pi_t &= 0 \\ y_{t+1} - y_t - \frac{1}{\sigma}(r_t - \pi_{t+1}) &= 0 \\ \rho r_{t-1} + (1-\rho)\beta\pi_{t+1} + (1-\rho)\gamma y_t - r_t &= 0. \end{aligned}$$

The first equation is the neo-Keynesian Phillips curve, according to which current inflation, π_t , is a function of expected future inflation and current output, y_t . The second equation is a log-linear approximation of the household intertemporal Euler equation for holding bonds. It says that the expected growth rate in output, $y_{t+1} - y_t$, is proportional to the expected real rate of interest, equal to the nominal interest rate, r_t , minus expected inflation. The last equation is the monetary policy rule, which makes the nominal rate of interest a weighted average of a target rate of interest (this is a linear function of expected inflation and output) and the lagged nominal rate of interest. All the variables are expressed relative to their steady state values.

The system can be expressed in our canonical form as follows:

$$\begin{bmatrix} \delta & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 \\ (1-\rho)\beta & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \\ r_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & -\frac{1}{\sigma} \\ 0 & (1-\rho)\gamma & -1 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ r_t \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \\ r_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or,

$$\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0,$$

in obvious notation. There is no uncertainty. A minimal state variable solution is a matrix A such that

$$z_t = Az_{t-1},$$

where A satisfies

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0. \quad (6.1)$$

We will now describe the set of candidate and actual MSV solutions. The parameter values that we adopt for the model are:

$$\delta = 0.99, \sigma = 1, \lambda = 0.3, \gamma = 0.15, \rho = 0.5, \beta = 1.5.$$

First, we set up the system in first-order form:

$$aY_{t+1} + bY_t = 0,$$

where

$$a = \begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix}, \quad b = \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix}, \quad Y_t = \begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix}.$$

In this example, α_0 is 3×3 and has rank 2. As a result, the rank of a is 5. To study the set of solutions, we must apply the QZ decomposition. Thus, we find Q and Z such that

$$QaZ = H_0, \quad QbZ = H_1.$$

Here, $QQ' = I$, $ZZ' = I$, where here and throughout “ $'$ ” denotes the Hermetian transpose (e.g., transposition and conjugation) and

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -0.22 - 0.0003i & -0.70 - 0.001i & -0.17 - 0.0002i & 0 & 0 & 0.65 + 0.0009i \\ 0.36 + 0.36i & 0.42 - 0.09i & 0.27 + 0.27i & 0 & 0 & 0.64 + 0.10i \\ 0.58 - 0.009i & -0.47 + 0.32i & 0.44 - 0.007i & 0 & 0 & -0.20 + 0.34i \\ -0.60 & 0 & 0.80 & 0 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 & -0.25 + 0.0004i & 0.59 - 0.3i & 0.32 - 0.29i & -0.56 \\ 0 & 0 & -0.55 + 0.0008i & 0.26 + 0.34i & -0.65 - 0.28i & 0.11 \\ 0 & 0 & 0.27 - 0.0004i & 0.51 - 0.17i & 0.13 - 0.28i & 0.74 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.75 - 0.001i & 0.21 + 0.21i & -0.42 - 0.20i & -0.37 \end{bmatrix}.$$

Also, the upper 5×5 block of H_0 , G_0 , and the upper 5×5 block of H_1 , G_1 , are

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.14 & -0.66 + 0.21i & -0.15 + 0.37i \\ 0 & 0 & 0 & 0.96 + 0.26i & -0.09 + 0.37i \\ 0 & 0 & 0 & 0 & 0.77 - 0.21i \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0.25 - 0.0004i & -0.59 + 0.3i & -0.32 + 0.29i \\ 0 & 0 & 0.55 - 0.0008i & -0.26 - 0.34i & -0.65 + 0.28i \\ 0 & 0 & -0.41 & 0.38 + 0.09i & -0.27 - 0.28i \\ 0 & 0 & 0 & -1.19 & -0.25 + 0.19i \\ 0 & 0 & 0 & 0 & -0.95 \end{bmatrix}.$$

Let

$$Z' = \begin{bmatrix} \underbrace{L_1}_{5 \times 6} \\ \underbrace{L_2}_{1 \times 6} \end{bmatrix}, \quad \gamma_t^1 = L_1 Y_t, \quad \gamma_t^2 = L_2 Y_t.$$

As explained in the previous section, the system ‘decouples’ with $\gamma_t^2 = 0$ for $t \geq 0$:

$$L_2 Y_t, \text{ all } t, \tag{6.2}$$

and

$$G_0 \gamma_{t+1}^1 + G_1 \gamma_t^1 = 0, \tag{6.3}$$

or,

$$\gamma_{t+1}^1 = -G_0^{-1} G_1 \gamma_t^1 = \Pi \gamma_t^1.$$

Interestingly, the variables, γ_t^1 , are complex and so is Π :

$$\Pi = \begin{bmatrix} 0 & 0 & -0.25 + 0.0004i & 0.59 - 0.30i & 0.32 - 0.29i \\ 0 & 0 & -0.55 + 0.0008i & 0.26 + 0.34i & -0.65 - 0.28i \\ 0 & 0 & 0.35 & 0.27 - 0.47i & 0.70 - 0.03i \\ 0 & 0 & 0 & 1.15 - 0.31i & 0.31 + 0.19i \\ 0 & 0 & 0 & 0 & 1.15 + 0.31i \end{bmatrix}$$

Still, we have no problem concluding that the class of solutions here is given by

$$\gamma_t^1 = \Pi^t \gamma_0^1, \tag{6.4}$$

where γ_0^1 has 2 ‘free parameters’. These are composed of the first three elements of Y_0 , net of the one restriction on Y_0 implied by (6.2). Thus, (6.2) represents a 2 parameter space of solutions. The eigenvector-eigenvalue decomposition of Π ,

$$\Pi = P \Lambda P^{-1}, \tag{6.5}$$

plays an important role in the dynamics of the solutions, (6.4). The eigenvalues of Π (i.e., the 5 terms on the diagonal of the diagonal matrix, Λ) are:

$$0, 0, 0.35, 1.15 \pm 0.31i.$$

Note that there are two explosive eigenvalues and three non-explosive. Also, although there are two repeating eigenvalues, it is nevertheless the case that the eigenvector-eigenvalue decomposition, (6.5) exists in this example.

We can compute candidate MSV solutions for the system as follows. Choose the 2×5 matrices, B , such that $B\gamma_0^1 = 0$ and B is composed of left eigenvectors of π . There are 10 ways to construct a B matrix in this way and each corresponds to a candidate MSV solution. This is because for any B constructed in this way we can construct a 3×6 matrix, D such that $DY_0 = 0$, from

$$D = \begin{bmatrix} BL_1 \\ L_2 \end{bmatrix}.$$

Whether a given candidate MSV solution is an actual solution requires that there exist a 3×3 real matrix A such that

$$A = -D_1^{-1}D_2,$$

where

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

That is, we require that D_1 be invertible and that $D_1^{-1}D_2$ be real. In this case,

$$z_t = Az_{t-1}$$

is an actual MSV solution.

It is easy to confirm that in each of the 10 candidate MSV solutions, D_1 is invertible. However, in only four cases $-D_1^{-1}D_2$ is real. Thus, among the 10 candidate MSV's there are just four actual MSV's. These are A_1, A_2, A_3, A_4 :

$$\begin{aligned} \underbrace{A_1}_{(0.35, 1.15 \pm 0.31i)} &= \begin{bmatrix} 1.01 & -0.3 & 0. \\ -1.01 & 1.30 & 1.0 \\ 0.92 & -0.43 & 0.35 \end{bmatrix}, \quad \underbrace{A_2}_{(0, 1.15 \pm 0.31i)} = \begin{bmatrix} -61.85 & 0 & 45.18 \\ 269.30 & 0 & -193.28 \\ -87.87 & 0 & 64.16 \end{bmatrix}, \\ \underbrace{A_3}_{(0, 1.15 \pm 0.31i)} &= \begin{bmatrix} 0 & -0.30 & 0.73 \\ 0 & 1.30 & 0.27 \\ 0 & -0.42 & 1.01 \end{bmatrix}, \quad \underbrace{A_4}_{(0, 0, 0.35)} = \begin{bmatrix} 0 & 0 & -0.34 \\ 0 & 0 & -0.74 \\ 0 & 0 & 0.31 \end{bmatrix}, \end{aligned}$$

where numbers in parentheses beneath A_j are the three eigenvalues of A_j , $j = 1, \dots, 4$.

Note that the MSV solutions all look very different. The entries in A_2 are a couple of orders of magnitude different in size from the corresponding entries in the other A matrices. Note, too, that there is only one A matrix which has all its eigenvalues less than unity in absolute value. The latter is to be expected. There are two explosive eigenvalues in the decoupled system, (6.3). The fourth MSV extinguishes both eigenvalues by working with B constructed using the two associated left eigenvectors of Π .

An interesting special case of the model sets $\rho = 0$. In this case, (6.1) reduces to:

$$(\alpha_0 A + \alpha_1) A = 0.$$

We can see one MSV solution right away, without using the technology based on the QZ decomposition. In particular,

$$A = 0$$

represents a solution to the system. When we apply the QZ decomposition approach to this case, we find - like when $\rho \neq 0$ - that there are 10 candidate MSV's, but only 4 actual MSV's. Among these four MSV's, only the one in which $A = 0$ has all eigenvalues less than unity.

Another interesting special case occurs when $\beta = 0.8$ and ρ is held at its benchmark value of 0.5. In this case, Π has only one explosive eigenvalue. Thus, we can expect that there are many non-explosive solutions, possibly even many non-explosive MSV's. As before, there are 10 candidate MSV's. It turns out that each one is an actual MSV because each satisfies the invertibility condition and the requirement that A be real. A puzzling feature of the set of MSV's is that it does not include $A = 0$. Finally, four MSV's are non-explosive. It is of interest to display these:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & -0.40 \\ 0 & 0 & -0.85 \\ 0 & 0 & 0.38 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.54 & 0.82 \\ 0 & 0.46 & 0.18 \\ 0 & 0.20 & 0.82 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 & 1.04 \\ 0 & 0 & 0.37 \\ 0 & 0 & 0.90 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.65 & 0 & 0.29 \\ 0.54 & 0 & -0.26 \\ 0.24 & 0 & 0.63 \end{bmatrix}. \end{aligned}$$

Since each matrix is different and each has only non-explosive eigenvalues, each represents a valid equilibrium. One may attempt to impose an equilibrium selection. For example, based on the economics of the model one might suppose it 'implausible' for the state to include lagged inflation or lagged output. But, two of the MSV solutions satisfy this plausibility criterion because the first two columns of A_1 and A_2 are zero. Another possible criterion is that A should 'resemble' the A computed when $\beta = 1.5$ and there is a unique, non-explosive solution. This criterion selects A_1 as the solution. However, it is not clear why this selection is appealing.