Materials 3 - Special cases

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Overview

1 The models to be simulated

1. Rational expectations NK model (RE)

2. Euler equation approach learning NK model à la Bullard & Mitra (2002) (EE)

3. LR expectations learning NK model à la Preston (2005) (LR)

4. (Eventually: LR expectations learning NK model à la Preston with anchoring à la CEMP)

The difference between these models is 1) in the expectations (rational or not), 2) in the number of horizons of expectations that need to be summed (1 vs. infinite). So what I'm going to do consists of 2 steps:

1. Write a learning rule that takes the form of Preston's, but that nests CEMP, and has a decreasing gain.

2. Write out f_a and f_b as truncated sums of h-period ahead forecasts. When $h = 1$, models (6) and (7) should coincide.

1.1 RE

$$x_t = \mathbb{E}_t\, x_{t+1} - \sigma(i_t - \mathbb{E}_t\, \pi_{t+1}) + \sigma r_t^n \tag{1}$$

$$\pi_t = \kappa x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} + u_t \tag{2}$$

$$i_t = \bar{i}_t + \psi_\pi \pi_t + \psi_x x_t \tag{3}$$

1.2 EE

$$x_t = \hat{\mathbb{E}}_t x_{t+1} - \sigma(i_t - \hat{\mathbb{E}}_t \pi_{t+1}) + \sigma r_t^n \tag{Preston, eq. (13)}$$

$$\pi_t = \kappa x_t + \beta\hat{\mathbb{E}}_t \pi_{t+1} + u_t \tag{Preston, eq. (14)}$$

$$i_t = \bar{i}_t + \psi_\pi \pi_t + \psi_x x_t \tag{Preston, eq. (27)}$$

1.3 LR

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \big((1-\beta) x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n \big) \qquad \text{(Preston, eq. (18))}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \big) \qquad \text{(Preston, eq. (19))}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(Preston, eq. (27))}$$

2 Compact notation

Innovations are summarized as:

$$s_t = P s_{t-1} + \epsilon_t \quad \text{where} \quad s_t \equiv \begin{pmatrix} r_t^n \\ \bar{i}_t \\ u_t \end{pmatrix} \quad P \equiv \begin{pmatrix} \rho_r & 0 & 0 \\ 0 & \rho_i & 0 \\ 0 & 0 & \rho_u \end{pmatrix} \quad \epsilon_t \equiv \begin{pmatrix} \varepsilon_t^r \\ \varepsilon_t^i \\ \varepsilon_t^u \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_i & 0 \\ 0 & 0 & \sigma_u \end{pmatrix}$$

Let z_t summarize the endogenous variables as

$$z_t \equiv \begin{pmatrix} \pi_t \\ x_t \\ i_t \end{pmatrix} \tag{4}$$

Then I can write the models compactly as

$$z_t = A_p^{RE} \, \mathbb{E}_t \, z_{t+1} + A_s^{RE} s_t \tag{5}$$

$$z_t = A_p^{RE} \hat{\mathbb{E}}_t z_{t+1} + A_s^{RE} s_t \tag{6}$$

$$z_t = A_a^{LR} f_a + A_b^{LR} f_b + A_s^{LR} s_t \tag{7}$$

$$s_t = P s_{t-1} + \epsilon_t \tag{8}$$

where f_a and f_b capture discounted sum of expectations at all horizons of the endogenous states z (following Preston, I refer to these objects as "long-run expectations"):

$$f_a \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} z_{T+1} \qquad\qquad f_b \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\beta)^{T-t} z_{T+1} \tag{9}$$

and the coefficient matrices are given by:

$$
A_p^{RE} = \begin{pmatrix} \beta + \frac{\kappa\sigma}{w}(1 - \psi_\pi\beta) & \frac{\kappa}{w} & 0 \\ \frac{\sigma}{w}(1 - \psi_\pi\beta) & \frac{1}{w} & 0 \\ \psi_\pi\left(\beta + \frac{\kappa\sigma}{w}(1 - \psi_\pi\beta)\right) + \psi_x\frac{\sigma}{w}(1 - \psi_\pi\beta) & \psi_x(\frac{1}{w}) + \psi_\pi(\frac{\kappa}{w}) & 0 \end{pmatrix} \tag{10}
$$

$$
A_s^{RE} = \begin{pmatrix} \frac{\kappa\sigma}{w} & -\frac{\kappa\sigma}{w} & 1 - \frac{\kappa\sigma\psi_\pi}{w} \\ -\frac{\sigma}{w} & -\frac{\sigma}{w} & -\frac{\sigma\psi_\pi}{w} \\ \psi_x(\frac{\sigma}{w}) + \psi_\pi(\frac{\kappa\sigma}{w}) & \psi_x(-\frac{\sigma}{w}) + \psi_\pi(-\frac{\kappa\sigma}{w}) + 1 & \psi_x(-\frac{\sigma\psi_\pi}{w}) + \psi_\pi(1 - \frac{\kappa\sigma\psi_\pi}{w}) \end{pmatrix} \tag{11}
$$

$$
A_a^{LR} = \begin{pmatrix} g_{\pi a} \\ g_{xa} \\ \psi_\pi g_{\pi a} + \psi_x g_{xa} \end{pmatrix} \quad A_b^{LR} = \begin{pmatrix} g_{\pi b} \\ g_{xb} \\ \psi_\pi g_{\pi b} + \psi_x g_{xb} \end{pmatrix} \quad A_s^{LR} = \begin{pmatrix} g_{\pi s} \\ g_{xs} \\ \psi_\pi g_{\pi s} + \psi_x g_{xs} + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{pmatrix} \tag{12}
$$

$$
g_{\pi a} = (1 - \frac{\kappa\sigma\psi_\pi}{w})\left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{13}
$$

$$
g_{xa} = \frac{-\sigma\psi_\pi}{w}\left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{14}
$$

$$
g_{\pi b} = \frac{\kappa}{w}\left[\sigma(1 - \beta\psi_\pi), (1 - \beta - \beta\sigma\psi_x, 0\right] \tag{15}
$$

$$
g_{xb} = \frac{1}{w}\left[\sigma(1 - \beta\psi_\pi), (1 - \beta - \beta\sigma\psi_x, 0\right] \tag{16}
$$

$$
g_{\pi s} = (1 - \frac{\kappa\sigma\psi_\pi}{w})\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}(I_3 - \alpha\beta P)^{-1} - \frac{\kappa\sigma}{w}\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}(I_3 - \beta P)^{-1} \tag{17}
$$

$$
g_{xs} = \frac{-\sigma\psi_\pi}{w}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}(I_3 - \alpha\beta P)^{-1} - \frac{\sigma}{w}\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}(I_3 - \beta P)^{-1} \tag{18}
$$

$$
w = 1 + \sigma\psi_x + \kappa\sigma\psi_\pi \tag{19}
$$

3 Learning

In Preston (2005), agents forecast the endogenous variables using the exogenous ones as

$$z_t = a_t + b_t s_t + \epsilon_t \qquad \text{(Preston, p. 101)}$$

and then ϕ_t summarizes (a, b), so that agents learn both a constant and a slope term. I'm going to simplify here a bit so that agents only learn about the constant, i.e. about CEMP's drift term:

$$z_t = \phi_{t-1} + s_{t-1} + \epsilon_t \qquad (20)$$

To clarify, the $t-1$ index on ϕ stands for the period *in which* agents form the expectation. Anticipated utility implies that

$$\hat{\mathbb{E}}_t \phi_{t+h} = \hat{\mathbb{E}}_t \phi_t = \phi_t \quad \forall\, h > 0 \qquad (21)$$

Assuming RE about the exogenous process and anticipated utility, this implies that h-horizon forecasts are constructed as:

$$\hat{\mathbb{E}}_t z_{t+h} = \phi_t + P^h s_t \quad \forall h \geq 1 \qquad (22)$$

and the regression coefficients are updated using (for now) a decreasing gain RLS algorithm:

$$\phi_t = \phi_{t-1} + t^{-1} \mathbf{R_t^{-1} s_{t-1}} \big(z_{t-1} - (\phi_{t-1} + s_{t-1}) \big) \qquad (23)$$

$$R_t = R_{t-1} + t^{-1} (s_{t-1} s'_{t-1} - R_{t-1}) \qquad (24)$$

R_t is 3×3 and ϕ_t is 3×1. The bold $R_t^{-1} s_{t-1}$ indicates a difference to CEMP's learning algorithm: these terms are missing in CEMP. CEMP is a special case of this model, with the gain switching between decreasing and constant according to the anchoring mechanism. I'm leaving that out for the time being.

4 ALMs

4.1 RE

With some abuse of terminology, call the state-space representation the ALM of the RE model:

$$x_t = hx\, x_{t-1} + \eta s_t \qquad (25)$$

$$z_t = gx\, x_t \qquad (26)$$

Then I can write the "ALM" as

$$z_t = gx\ hx\ x_{t-1} + gx\ \eta s_t \tag{27}$$

Since this ALM implies no constant, I will initialize ϕ_0 as a 3×1 zero vector. Analogously, I will initialize R as a 3×3 identity matrix.

4.2 EE

Using (22),

$$\hat{\mathbb{E}}_t z_{t+1} = \phi_t + P s_t \tag{28}$$

Evaluating this in (6), I obtain the ALM of the EE model as

$$z_t = A_p^{RE} \phi_t + \left(A_p^{RE} P + A_s^{RE} \right) s_t \tag{29}$$

4.3 LR

Evaluation "LR expectations" (9) using the PLM (22), I get

$$f_a = \frac{1}{1 - \alpha\beta}\phi_t + P(I_3 - \alpha\beta P)^{-1} s_t \qquad f_b = \frac{1}{1 - \beta}\phi_t + P(I_3 - \beta P)^{-1} s_t \tag{30}$$

Then the ALM of the LR model is (7) with (30) subbed in (I suppress LR-superscripts):

$$z_t = \left(A_a \frac{1}{1 - \alpha\beta} + A_b \frac{1}{1 - \beta} \right)\phi_t + \left(A_a P(I_3 - \alpha\beta P)^{-1} + A_b P(I_3 - \beta P)^{-1} + A_s \right) s_t \tag{31}$$

5 Timeline in the learning models

$\underline{t = 0}$: Initialize $\phi_t = \phi_0$ at the RE solution.

For each t:

1. Evaluate expectations (the one-period ahead, or the full 1 to ∞-period ahead)

2. Evaluate ALM given expectations

3. Update ϕ

Timing issues, as always!