# Linear Difference Equations

Mario Tirelli

Still a preliminary version

March 2, 2014

Contents

1. Introduction 1
2. Linear difference equations 2
2.1. Equations of first order with a single variable 2
2.2. Equations of first order with $m > 1$ variables (systems of equations) 6
2.3. Higher order difference equations 13
2.4. Solution stability 16
2.5. Stochastic difference equations 19

## 1. Introduction

Dynamic economic models are a useful tool to study economic dynamics and get a better understanding of relevant phenomena such as growth and business cycle. Equilibrium conditions are normally identified by a system of difference equations and a set of boundary conditions (describing limit values of some variables). Thus, studying equilibrium properties requires studying the properties of a system of difference equations. In many interesting cases such difference equations are nonlinear; as you will see, even the textbook, deterministic, neoclassical, growth model is represented by a system of nonlinear equations. Although nonlinearities make the study of dynamic economic systems difficult, we can still hope to derive useful characterizations and results. This is achieved either locally, in a neighborhood of an equilibrium point, or globally for log-linearized systems. In both cases, non-linear systems are studied using the theory of linear difference equations. In these notes we shall summarize some useful results in this theory and apply them to deal with dynamic economic systems.

In these class notes I present some useful material on how to solve linear difference equations and study solution stability. These notes are incomplete; many important questions are left unexplained; but students can find additional, sharp and clear material in Elaydi (2005) "An Introduction to Difference Equations", Springer-Verlag. A general, introductory, reference is Simon & Blume, "Mathematics for Economists", Norton. For issues on linear algebra, my favorite textbook is Serge Lang "Linear Algebra", Springer-Verlag.

## 2. Linear difference equations

2.1. **Equations of first order with a single variable.** Let us start with equations in one variable,

$$x_t + ax_{t-1} = b_t \tag{1}$$

This is a *first-order difference* equation because only one lag of $x$ appears. In this equation, $a$ is a time-independent coefficient and $b_t$ is the *forcing term*. When $b_t = 0$, the difference equation is said to be *homogeneous*, and otherwise *non-homogeneous*. When the forcing term is a constant ($b_t = b$ for all $t$), the difference equation 1 is non-homogeneous and *autonomous* (or time-invariant). Finally, when $b_t$ is time-dependent the equation is said to be *non-autonomous*; this is a more general formulation, allowing, for example, to capture seasonality, deterministic shocks or perturbations. The most general form of linear difference equation is one in which also the coefficient $a$ is time-varying.

2.1.1. *Autonomous equations.* Let us start with a non-homogeneous, autonomous difference equation, with $b_t$ being equal to a time-invariant scalar $b$. A first approach is to solve 1 by successive iteration. Suppose the initial value $x_0$ is given, then

$$\begin{aligned}
x_1 &= b - ax_0 \\
x_2 &= b - a(b - ax_0) \\
&= b - ab + a^2x_0 \\
x_3 &= b - a(b - ab + a^2x_0) \\
&= b - ab + a^2b - a^3x_0 \\
&\cdots\cdots\cdots\cdots \\
x_t &= b(1 + (-a) + (-a)^2, .., +(-a)^{t-1}) + (-a)^t x_0 \\
&= \frac{1 - (-a)^t}{1 - (-a)} b + (-a)^t x_0, \quad \text{if } a \neq -1
\end{aligned}$$

Rearranging, gathering terms in $(-a)^t$,

$$x_t = (-a)^t \left[ x_0 - \frac{b}{1+a} \right] + \frac{b}{1+a} \tag{2}$$

A general method, analogous to the one used for differential equations, is based on the *Superposition Principle* (see theorem [SP] below): the solution of a linear difference equation is the sum of the solution of its homogeneous part, the *complementary solution*, and the *particular solution*. A particular solution is any solution to the non-homogeneous difference equation,

$$x_t = x_t^{co} + x_t^{pa}$$

For autonomous equations, a (very convenient) particular solution is the *steady-state* solution, $x_t^{pa} = x^*$, which is constant over time. Consider the homogeneous equation, $x_t + ax_{t-1}$=0. Recalling differential equations, one may guess a solution to this equation to be, $x_t = ck^t$,

where $c$ is analogous to a constant of integration and $k = -a$.[1] To see the latest, substitute the guessed solution in the equation, $ck^t + ack^{t-1} = 0$; simplifying, $ck^{t-1}(k + a) = 0$, which is satisfied if and only if $k = -a$. To summarize, the complementary solution is,

$$x_t^{co} = c(-a)^t$$

As a particular solution take the steady-state $x^*$; substituting $x_t = x^*$, $x^* + ax^* = b$, hence

$$x_t^{pa} = \frac{b}{1+a}$$

Therefore the general solution is,

$$x_t = x_t^{co} + x_t^{pa} = c(-a)^t + \frac{b}{1+a} \tag{3}$$

This solution is identical to 2, the one found by iterated substitutions, except for $c$ that is still to be determined. Take $t = 0$, then $x_0 = c(-a)^0 + b/(1 + a)$, implying $c = x_0 - b/(1 + a)$, if $a \neq -1$.

It remains to consider the case in which $a = -1$, implying $x_t = x_{t-1} + b$. Incidentally, the method used to find the general solution, and precisely the complementary solution, is called *undetermined coefficients*. This is based on guessing a functional form for $x_t$ and verify that this solves the equation (for some coefficients). Guess, $x_t = \alpha t$, a trend line with initial value at $\alpha$ to be determined. Substituting, $\alpha(t+1) - \alpha t - b = 0$, satisfied for $\alpha = b$. Hence, if $a = -1$,

$$x_t = bt + x_0 \tag{4}$$

You can check that any other guess on functional forms would not work.

We conclude this section establishing the Superposition Principle (SP).

Theorem 2.1 (SP):
*Any solution $x_t$ of the equation $x_t + ax_{t-1} = b_t$ can be written as*

$$x_t = x_t^{co} + x_t^{pa}$$

*with $x_t^{co} = c(-a)^t$, for a particular solution $x_t^{pa}$.*

*Proof.* Let $z_t := x_t - x_t^{pa}$, where $x_t$ solves 1.

$$\begin{aligned} z_{t+1} &= x_{t+1} - x_{t+1}^{pa} \\ &= -ax_t + b_t - (-ax_t^{pa} + b_t) \\ &= -a\left(x_t - x_t^{pa}\right) \\ &= -az_{t-1} \end{aligned}$$

That is, $z_t = c(-a)^t$ solves the the homogeneous equation. Therefore, buy definition of $z_t$, $x_t = z_t + x_t^{pa} = c(-a)^t + x_t^{pa}$ as desired. □

[1] Recall that for homogeneous differential equations of the type, $x'(t) + ax(t) = 0$, the complementary solution is $x^{co}(t) = ce^{(-a)t}$. This can be derived, first, by integrating $\frac{x'}{x} + a = 0$ over $t$, which yields $log[x(t)] = B - at$, with $B$ defined by collecting the two constants of integration. Then, by taking the exponential of both sides and letting $c := e^B$. In the difference equation, the instantaneous rate of change $e^{(-a)t}$ is replaced by $(-a)^t$.

The Superposition Principle is a general result that, as you can see from its proof, extends to non-autonomous equations and to systems of linear equations.

2.1.2. *Non-autonomous equations, lags and leads.* Consider non-autonomous equations, assuming a time-varying term $b_t$.[2] In general, the solutions of these equations will take the functional form of $b_t$. If $b_t$ is an exponential or it is a polynomial of order $p$, then the solution will, respectively, have the form of an exponential or of a $p$-order polynomial in $t$.

A particularly useful method to solve equations is via the introduction of *lag* and *lead* operators. In general, an operator is a linear function between vector spaces. $\mathbb{L}$ and $\mathbb{F}$, respectively, are lag and lead operators if for every sequence $(x_t)$

$$x_{t-1} = \mathbb{L}x_t = \mathbb{F}^{-1}x_t$$

$$x_{t+1} = \mathbb{F}x_t = \mathbb{L}^{-1}x_t.$$

Observe that we can repeatedly apply the operator to shift a variable over time,

$$\mathbb{L}^2 x_t = \mathbb{L}\left(\mathbb{L}x_t\right) = \mathbb{L}x_{t-1} = x_{t-2}$$

and, more generally,

$$x_{t-p} = \mathbb{L}^p x_t = \mathbb{F}^{-p}x_t,$$

$$x_{t+p} = \mathbb{F}^p x_t = \mathbb{L}^{-p}x_t.$$

Consider equation 1 and relabel the coefficient $\alpha := -a$, so that

$$x_t = \alpha x_{t-1} + b_t \tag{5}$$

The solution to the homogeneous part is as above, $x_t^{co} = \alpha^t x_0$. To find the particular solution, we have to distinguish whether or not $\alpha$ is less than one in modulus.

$|\alpha| < 1$

$$(1 - \alpha\mathbb{L})x_t = b_t$$

$$x_t^{pa} = \frac{1}{1 - \alpha\mathbb{L}} b_t$$

Since $|\alpha| < 1$ we know that the fraction is the infinite sum of a geometric sequence of common ratio $\alpha\mathbb{L}$,

$$(1 - \alpha\mathbb{L})^{-1} = \left(1 + \alpha\mathbb{L} + (\alpha\mathbb{L})^2 + (\alpha\mathbb{L})^3 + .....\right)$$

$$\begin{aligned} x_t^{co} &= \left(1 + \alpha\mathbb{L} + (\alpha\mathbb{L})^2 + (\alpha\mathbb{L})^3 + .....\right) b_t \\ &= b_t + \alpha b_{t-1} + \alpha^2 b_{t-2} + \alpha^3 b_{t-3} + ..... \end{aligned}$$

Hence,

$$x_t^{pa} = \sum_{s=-\infty}^{t} \alpha^{t-s} b_s.$$

[2]Non-autonomous equations may also be of the form $x_t = \alpha_{t-1}x_{t-1} + b$, with time-varying coefficients.

In particular, if $x_0$ is a given initial state (i.e. $x_{-t} = 0$ for all $t \geq 0$) the latest reduces to a finite sum,

$$x_t^{pa} = \sum_{s=0}^{t} \alpha^{t-s} b_s,$$

All these infinite sums will not be well defined for $|\alpha| \geq 1$.

$|\alpha| > 1$

In this case it suffices to solve the equation forward, i.e. use the lead operators. Start with $x_{t+1} = \alpha x_t + b_{t+1}$.

$$\begin{aligned} x_{t+1} - \alpha x_t &= b_{t+1} \\ (\mathbb{F} - \alpha)\, x_t &= b_{t+1} \\ \left(1 - \frac{\mathbb{F}}{\alpha}\right) x_t &= -\frac{1}{\alpha} b_{t+1} \end{aligned}$$

with $\gamma := \alpha^{-1}$ is less than one in modulus.[3]

$$\begin{aligned} x_t^{pa} &= -\frac{1}{1-\gamma\mathbb{F}} \gamma b_{t+1} \\ &= -\left(1 + \gamma\mathbb{F} + (\gamma\mathbb{F})^2 + (\gamma\mathbb{F})^3 + .....\right) \gamma b_{t+1} \\ &= -\gamma \left(b_{t+1} + \gamma b_{t+2} + \gamma^2 b_{t+3} + .... + \gamma^s b_{t+1+s} + ....\right) \end{aligned}$$

$$x_t^{pa} = -\frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{1}{\alpha}\right)^s b_{t+1+s}$$

To summarize, whenever $x_0$ is the initial state, the particular solution is

(6) $$x_t^{pa} = \begin{cases} \sum_{s=0}^{t} \alpha^{t-s} b_s & \text{if } |\alpha| < 1 \\ -\frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{1}{\alpha}\right)^s b_{t+1+s} & \text{if } |\alpha| > 1 \end{cases}$$

particular sol of 1-dim non-autonomous diff eq.

You can check that if $b_t = b$ at all dates $t$, we obtain the solution to the autonomous system in 3 and 4 (given that $\alpha := -a$).

**Remark 2.1** (Variable coefficients)**.** *So far we have assumed that the coefficients $a$ or $\alpha$ are constant over time. A non-autonomous equation may also be so because coefficients vary over time.*

$$x_{t+1} = \alpha_{t+1} x_t + b_{t+1}$$

[3]Notice that we could have also proceeded, starting from $x_t = (1 - \alpha\mathbb{L})^{-1} b_t$ as above, and observing that,

$$\frac{1}{1-\alpha\mathbb{L}} = \frac{(\alpha\mathbb{L})^{-1}}{(\alpha\mathbb{L})^{-1}(1-\alpha\mathbb{L})} = \frac{-(\alpha\mathbb{L})^{-1}}{1-(\alpha\mathbb{L})^{-1}} = \frac{-\mathbb{F}}{\alpha} \times \frac{1}{1-\frac{\mathbb{F}}{\alpha}} = \frac{-\gamma\mathbb{F}}{1-\gamma\mathbb{F}}$$

with $\gamma := \alpha^{-1}$ is less than one in modulus.

$$x_t^{pa} = -\frac{1}{1-\gamma\mathbb{F}} \gamma\mathbb{F} b_t = -\gamma \left(1 + \gamma\mathbb{F} + (\gamma\mathbb{F})^2 + (\gamma\mathbb{F})^3 + .....\right) b_{t+1}$$

*You can check, by iterated substitutions (backwards if $|\alpha| < 1$, and forward if $|\alpha| > 1$), that this extension leads to the following particular solutions.*

$$x_t^{pa} = \begin{cases} \sum_{k=1}^{t-1} \left( \prod_{s=0}^{k-1} \alpha_{t-s} \right) b_{t-k} + b_t, & \text{if } |\alpha| < 1; \\ -\sum_{k=0}^{\infty} \left( \prod_{s=0}^{k} \left( \frac{1}{\alpha_{t+1+s}} \right) \right) b_{t+1+k}, & \text{if } |\alpha| > 1. \end{cases} \tag{7}$$

The complementary solution is,

$$x_t^{co} = \left( \prod_{s=0}^{t} \alpha_{s+1} \right) x_0$$

**Example 2.1.** *Consider the individual, sequential, budget constraint,*

$$(1 + r_t)a_t = c_t - e_t + a_{t+1}$$

*where the gross return on assets $a$, $(1 + r_t) > 1$ for all $t$ (i.e. there is positive interest rate in every date) and $(e_t)_t$ is the lifetime endowment profile (e.g. exogenous labor income). Solve this difference equation forward, by iterated substitutions, to get the intertemporal budget constraint, given the limiting condition (transversality+ no-Ponzi)* $\lim_{s\to\infty} \frac{a_{t+1+s}}{\prod_{k=0}^{s-1}(1+r_{t+k})} = 0$. *Alternatively, compute the particular solution applying the above rule, and solve the for the complementary forward, by iterated substitutions.*

2.2. **Equations of first order with $m > 1$ variables (systems of equations).** Consider the following difference equation in $m$ variables, indeed a system of difference equations,

$$\begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{mt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{2m} & \dots & & a_{mm} \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \\ \vdots \\ x_{mt-1} \end{pmatrix} + \begin{pmatrix} b_{1t} \\ b_{2t} \\ \vdots \\ b_{mt} \end{pmatrix}$$

In vector notation,

$$\boldsymbol{x}_t = A\boldsymbol{x}_{t-1} + \boldsymbol{b}_t \tag{8}$$

where $A$ is a square $m-$matrix of coefficients $a_{ij}$.

First, notice that if $A$ were a diagonal matrix, then the system would consist in $m$ independent variables, with equations of the type

$$x_{it} = a_{ii}x_{it-1} + b_{it}$$

In this lucky case a solution to the system is just the one in 6 for each single equation.

A second case is when we can transform 8 into a system whose coefficient matrix is indeed diagonal, or closed to diagonal, solve the new system and map back the solutions to attain solutions of the original system. We shall illustrate this methodology next, starting from the simplest case of an autonomous system.

2.2.1. *Autonomous systems.* Let $\boldsymbol{b}_t = \boldsymbol{b}$ and, without loss of generality, assume $m = 2$.

First derive the eigenvalues of $A$. The *eigenvalues* $\lambda$ of $A$ are the $m$ roots of the following polynomial (or characteristic) equation

$$\begin{aligned} p(\lambda) &= det\,(A - \lambda I_m) \\ &= det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \end{aligned}$$

$p(\lambda) = 0$ if and only if $(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$. Simplifying, $\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$ or

$$\lambda^2 - \lambda tr + det = 0$$

with $tr$ denoting the trace of $A$, and $det$ its determinant. Notice that the eigenvalues of $A$ are the roots of the polynomial $p(\cdot) = 0$; hence,[4]

$$\lambda_1, \lambda_2 = \frac{tr \pm \sqrt{tr^2 - 4det}}{2} \tag{9}$$

A really useful shortcut.

The *discriminant* is $\Delta := (tr)^2 - 4det$. If $\Delta > 0$ we have distinct, real roots; if $\Delta < 0$ we have distinct, complex roots; if $\Delta = 0$ roots are coincident, both equal to $tr/2$.

Second, derive the eigenvectors of A. The $i^{th}$ *eigenvector* of $A$ (associated to eigenvalue $\lambda_i$) is the vector $\boldsymbol{v}_i$ in $\mathbb{R}^m$, such that

$$(A - \lambda_i I_m)\,\boldsymbol{v}_i = 0 \tag{10}$$

Observe that the latest is homogeneous of degree 0 in $\boldsymbol{v}_i$ (i.e. if $\boldsymbol{v}_i$ solves 10, $c\boldsymbol{v}_i$ does also solve 10 for any scalar $c \neq 0$). Thus, we can normalize $\boldsymbol{v}_i$, for example, by taking one of its components equal to unity (i.e. $c = 1/v_m$ if $v_m \neq 0$),

$$\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{\nu}_i \\ 1 \end{pmatrix}$$

For the case $m > 2$, $\boldsymbol{\nu}_i$ is a vector of $m - 1$ components. Expression 10 is a system in $m = 2$ equations and one unknown, the scalar $\nu_i$. To find a solution, we write the system as,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \nu_i \\ 1 \end{pmatrix} - \begin{pmatrix} \nu_i \\ 1 \end{pmatrix} \lambda_i = 0$$

Consisting in two equations,

$$a_{11}\nu_i + a_{12} - \nu_i\lambda_i = 0$$

$$a_{21}\nu_i + a_{22} - \lambda_i = 0$$

that, solved by substitution, yield,

$$\frac{a_{12}}{\lambda_i - a_{11}} = \nu_i = \frac{\lambda_i - a_{22}}{a_{21}}$$

[4] Consider a matrix $A$ with $m$ eigenvalues $(\lambda_i)$. Two useful properties of eigenvalues are:

*i)* $\prod_{i=1}^{m} \lambda_i = det(A)$

*ii)* $\sum_{i=1}^{m} \lambda_i = tr(A)$.

To check this is a viable solution, recall that $\lambda_i$ is an eigenvalue; that is $\lambda_i$ is such that $p(\lambda_i) = 0$, implying $a_{21}a_{12} - (\lambda_i - a_{22})(\lambda_i - a_{11}) = 0$. Therefore, we conclude the $i^{th}$ eigenvector is,

$$\begin{pmatrix} \nu_i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_i - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$$

We start by deriving a particular solution for the autonomous system, with a $2-$dimensional square matrix $A$.

Particular solution

The particular solution is the steady-state, if it exists. Let $\boldsymbol{x}_t = \boldsymbol{x}^*$ for all $t$. Then, the system reduces to

$$\boldsymbol{x}^* = A\boldsymbol{x}^* + \boldsymbol{b} \iff (I_2 - A)\boldsymbol{x}^* = \boldsymbol{b}$$

If $(I_2 - A)$ is invertible,

$$\boldsymbol{x}_t^{pa} := \boldsymbol{x}^* = (I_2 - A)^{-1}\boldsymbol{b}$$

$$(I_2 - A)^{-1} = \frac{1}{p(1)} \begin{pmatrix} 1 - a_{22} & a_{12} \\ a_{21} & 1 - a_{11} \end{pmatrix}$$

which is well defined as long as $p(1) = det(I - A) \neq 0$ or, equivalently, if and only if $1 - tr(A) + det(A) \neq 0$.[5]

Complementary and general solutions

Consider the homogeneous difference equation, $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$. Let the matrix of eigenvalues and eigenvectors of $A$ be,

$$\Lambda := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P := [\boldsymbol{v}_1, \boldsymbol{v}_2] = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix}$$

*Case 1: distinct eigenvalues.*

Observe that $P$ is of full rank (hence invertible) provided eigenvalues are distinct, $\lambda_1 \neq \lambda_2$. Hence, by definition 10 (see also the following Remark),

$$AP = P\Lambda \iff A = P\Lambda P^{-1} \iff P^{-1}AP = \Lambda$$

diagonalization is the "decoupling"

Distinct eigenvalues allow for a *diagonalization* of $\Lambda$ (i.e to transform $A$ into a diagonal matrix $\Lambda$). Substituting into the homogeneous system, $\boldsymbol{x}_t = P\Lambda P^{-1}\boldsymbol{x}_{t-1}$ or

$$P^{-1}\boldsymbol{x}_t = \Lambda P^{-1}\boldsymbol{x}_{t-1}$$

We can now perform a change of variables, $\boldsymbol{z}_t := P^{-1}\boldsymbol{x}_t$,

$$\boldsymbol{z}_t = \Lambda \boldsymbol{z}_{t-1}$$

The strategy is to derive a solution for $\boldsymbol{z}_t$ to latter recover the solution of the original variable $\boldsymbol{x}_t$. The solution is easily determined since the matrix $\Lambda$ is diagonal: given initial conditions, $\boldsymbol{z}_0 := P^{-1}\boldsymbol{x}_0$,

$$\boldsymbol{z}_t = \begin{pmatrix} z_{10}\lambda_1^t \\ z_{20}\lambda_2^t \end{pmatrix}$$

[5] You should verify that $det(I - A) = 1 - tr(A) - det(A)$, which is the characteristic equation of $A$, $p(\lambda)$ evaluated at 1.

Therefore,

$$\boldsymbol{x}_t^{co} = P\boldsymbol{z}_t = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_{10}\lambda_1^t \\ z_{20}\lambda_2^t \end{pmatrix} \tag{11}$$

More generally,

$$x_{it} = x_i^* + \sum_{j=1}^{m} \nu_{ij} z_{j0} \lambda_j^t, \quad \text{for all} \quad i = 1, .., m-1 \tag{12}$$

$$x_{mt} = x_m^* + \sum_{j=1}^{m} z_{j0} \lambda_j^t$$

In matrix form, the general solution is

$$\mathbf{x}_t = \mathbf{x}^* + P\Lambda^t \boldsymbol{z}_0, \quad \boldsymbol{z}_0 := P^{-1}\boldsymbol{x}_0 \tag{13}$$

**Remark 2.2** (Matrix diagonalization)**.** *In general, an $m \times m$ matrix is diagonalizable if and only if it has $m$ linearly independent eigenvectors (these vectors form a $m-$dimensional, complete, basis). To see why, observe that, by definition, $P := [\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m]$, with $\boldsymbol{v}_i$ being the $ith-$column of $P$. Since $A$ is diagonalizable, $P^{-1}AP = \Lambda$; implying that $P$ is nonsingular, i.e. its columns (and hence the eigenvectors ($\boldsymbol{v}_i$) of $A$) are linearly independent. Conversely, if there are $k$ linearly independent eigenvectors of a $m-$square matrix $A$, $P$ is invertible, hence $A$ is diagonalizable.*

*Eigenvectors are not all linearly independent when some eigenvalue $\lambda_i$ that is repeated, with an algebraic multiplicity $r_i > 1$ and it has associated eigenvectors who form a basis of dimension $m_i < r_i$ (i.e. has geometric multiplicity $m_i$ less than the algebraic multiplicity $r_i$). We are going to cover this in case 2 below.*

*Case 2: repeated eigenvalues.*

Suppose $A$ is not already a diagonal matrix; then, it is not diagonalizable whenever it has repeated eigenvalues; indeed, in this case not all eigenvectors are linearly independent. For example, matrices such as,

$$\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 2 & -4 \\ 1 & 4 & -3 \\ 1 & 1 & 0 \end{pmatrix}$$

are not diagonalizable. The first has both eigenvalues equal to 3 (its discriminant is zero). The second matrix has two repeated eigenvalues equal to 3 and one equal to 2. In all these cases, of multiple eigenvalues, some eigenvectors $[\boldsymbol{v}_1, .., \boldsymbol{v}_m]$ are linearly dependent; hence, $P$ fails to be invertible and $A$ to be diagonalizable. Yet we can still transform $A$ so as to simplify our system of difference equations, appealing to the *Jordan canonical form*. This transformation is attained as follows:

(1) form the characteristic polynomial $p(\lambda)$ and derive the eigenvalues ($\lambda_i$) of $A$; determine the "algebraic multiplicity" of $\lambda_i$, which is by convention 1 when all eigenvalues are distinct;

(2) compute the associated eigenvectors: for each distinct $\lambda_i$, find the normalized $m-$vector $\boldsymbol{v_i}$ that solves equation 10 above; instead, when $\lambda_i$ has algebraic multiplicity $r_i > 1$, compute the *generalized eigenvectors* of the repeated roots, as we are going to explain below; the number of $m_i$ linearly independent eigenvectors defines the *geometric multiplicity* of the associated eigenvalue (for a distinct eigenvalue, both multiplicity indexes are one);
(3) next, form $P$ with the derived (linearly independent) eigenvectors to attain the *Jordan canonical form* of $A$, $J := P^{-1}AP$;
(4) finally, proceed as above with a change of variables in the original system of difference equations.

We now provide a definition of generalized eigenvectors. Suppose that an eigenvalue $\lambda_i$ of $A$ has an algebraic multiplicity $r_i > 1$. The associated *generalized eigenvector* $\boldsymbol{v}_i$ is a nonzero vector such that $(A - \lambda_i I)\boldsymbol{v}_i \neq 0$ but $(A - \lambda_i I)^h \boldsymbol{v}_i = 0$, for some $h > 0$. Further, suppose that some eigenvalue of $A$, say $\lambda^i$ is repeated $r^i > 1$ times. The (standard) $r^i$ eigenvectors associated to $\lambda^i$ are linearly dependent (identical if normalized), that is to say that there is a unique linearly independent eigenvector, $\boldsymbol{v}_i$ corresponding to $\lambda_i$, satisfying $(A - \lambda_i I)\boldsymbol{v}_i = 0$. Next, we form a sequence of $r_i - 1$ generalized eigenvectors $(\boldsymbol{v}_2, ..., \boldsymbol{v}_{r_i})$ corresponding to $\lambda_i$, defined as vectors solving the following difference equations,

$$(A - \lambda_i I_m)\boldsymbol{v}_i = \boldsymbol{v}_{i-1}, i = 2, ..., r_i \iff \begin{cases} (A - \lambda_i I_m)\boldsymbol{v}_2 = \boldsymbol{v}_1 \\ (A - \lambda_i I_m)^2 \boldsymbol{v}_3 = \boldsymbol{v}_1 \\ ..... \\ (A - \lambda_i I_m)^{r_i - 1} \boldsymbol{v}_{r_i} = \boldsymbol{v}_1 \end{cases}$$

Clearly, $(A - \lambda_i I_m)\boldsymbol{v}_i \neq 0$. Moreover, pre-multiplying $(A - \lambda_i I_m)^{r_i - 1}\boldsymbol{v}_{r_i} = \boldsymbol{v}_1$ by $(A - \lambda_i I_m)$ we can derive a more general formulation of a the $j - th$ generalized eigenvector associated to an eigenvalue $\lambda^i$ with multiplicity $r^i$,

$$(A - \lambda_i I_m)^{r_i}\boldsymbol{v}_{r_i} = 0$$

This computation is reiterated stepwise for each eigenvalue $i$ with multiplicity $r_i > 1$.

To better illustrate computations, consider the following example.

**Example 2.2.** *Let $A$ be the $3 \times 3$ matrix above, with an eigenvalue of 2 and two eigenvalues of 3 (i.e. an eigenvalue with algebraic multiplicity of 2).*

$$\lambda_1 = 2, \ \boldsymbol{v}_1 \text{ solves } (A - \lambda_1 I_3)\boldsymbol{v}_1 = 0$$
$$\lambda_2 = 3 =: \lambda, \ \boldsymbol{v}_2 \text{ solves } (A - \lambda I_3)\boldsymbol{v}_2 = 0$$
$$\lambda_3 = \lambda, \ \boldsymbol{v}_3 \text{ solves } (A - \lambda I_3)\boldsymbol{v}_3 = \boldsymbol{v}_2 \iff (A - \lambda I_3)^2 \boldsymbol{v}_3 = 0$$

*Carrying out computations,*

$$P := [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

*$P$ is nonsingular (with a determinant equal to 1); hence, the Jordan matrix is well defined:*

$$J := P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

*Notice that $J$ is almost (but not really) diagonal, with $J_1 = \lambda_1 = 2$ and $J_2$ that is almost diagonal of the form,*

$$J_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda I_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*Then,*

$$A = PJP^{-1}$$

*which, substituted into the homogeneous difference equation system, gives $\boldsymbol{x}_t = PJP^{-1}\boldsymbol{x}_{t-1}$ or*

$$P^{-1}\boldsymbol{x}_t = JP^{-1}\boldsymbol{x}_{t-1}$$

*We can now perform a change of variables, $\boldsymbol{z}_t := P^{-1}\boldsymbol{x}_t$,*

$$\boldsymbol{z}_t = J\boldsymbol{z}_{t-1}$$

*Notice, that everything is identical to the case in which $A$ were diagonalizable except that the Jordan matrix $J$ replaces the diagonal matrix of eigenvalues $\Lambda$. Yet, since $J$ is not diagonal, something changes in the solution. We illustrate so in the context of the latest example.*

$$\boldsymbol{z}_t = J\boldsymbol{z}_{t-1} \iff \begin{cases} z_{1t} = 2z_{1t-1} \\ z_{2t} = 3z_{2t-1} + z_{3t-1} \\ z_{3t} = 3z_{3t-1} \end{cases}$$

*The first and third equations have a solution of the form $z_{it} = z_{i0}\lambda_i^t$. The second equation is of the form, $x_t = \alpha x_{t-1} + b_t$ with $|\alpha_j| > 1$, $b_t := x_0\alpha^{t-1}$. In the present context, $x_t = z_3 t$, $\alpha = \lambda = 3$, $x_0 = z_{30}$. Looking back to the particular solution in 6, one sees that it does not work here, because $b_t$ is exponential (the particular solution in 6 would not be well defined). Instead, one can find an alternative particular using undetermined coefficients. For an exponential term $b_t$ one should guess an exponential functional form for the variable. Hence, let $z_{2t} = k\lambda^t$, for an unknown (undetermined) coefficient $k$, and substitute into the equation to check whether or not the guess is correct: $k\lambda^t = \alpha k\lambda^{t-1} + z_{30}\lambda^{t-1}$; dividing through by $\lambda^{t-1}$, the equation is satisfied for $k\lambda = 3k + z_{30}$, or $k = z_{30}/(\lambda - 3)$. The latest is not well defined since $\lambda = 3$. So, guess an alternative functional form, $z_{2t} = kt\lambda^t$ and verify:*

$$kt\lambda^t = \lambda\left(k(t-1)\lambda^{t-1}\right) + z_{30}\lambda^{t-1}$$

*simplifying, $k = z_{30}/\lambda$, with $\lambda = 3$. So,*

$$z_{2t}^{pa} = z_{30}t\lambda^{t-1}$$

*By the Superimposition Principle, the general solution is,*

$$z_{2t} = z_{20}\lambda^t + z_{30}t\lambda^{t-1} = (z_{20}\lambda + z_{30})\,\lambda^{t-1}$$

*where the complementary solution is the solution of the homogeneous equation, $z_{2t} = 3z_{2t-1}$. Finally, it is easy to derive a solution for the original system in $x$.*

To conclude, the general solution of an autonomous system is analogous to 13, with the novelty that $\Lambda$ is replaced by $J$ and the matrix $P$ contains generalized eigenvectors:

$$\mathbf{x}_t = \mathbf{x}^* + PJ^t\boldsymbol{z}_0, \quad \boldsymbol{z}_0 := P^{-1}\boldsymbol{x}_0 \tag{14}$$

Finally, we point out that the Jordan form, which we derived in the above example, posses the following general representation: form $g$ distinct eigenvalues,

$$J = diag\left(J_1, ..., J_g\right),$$

with a typical (Jordan) block,

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{pmatrix} = \lambda_i I + \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

Each block $J_i$ is a $r_i \times r_i$ matrix, where $r_i$ is the algebraic multiplicity of $\lambda_i$ and $\sum_{i=1}^{g} r_i = m$ (i.e. $J$, as a $m-$square matrix, conforms to $A$). In our example above, $r_1 = 1$ and $r_2 = 2$.

Clearly, for ordinary matrices, with distinct eigenvalues, $g = m$, there are as many $1 \times 1$ Jordan blocks as the eigenvalues, and $J = \Lambda$.

2.2.2. *Non-autonomous systems.* Let us go back to the general case, with $b_t$ that is time-varying. The complementary solution is identical to the one just derive for the autonomous system. It remains to derive a particular solution and apply the Superposition Principle. We can do so using the method of undetermined coefficients. To guess a solution, we use the form of the solution found for single difference equations in section 2.1.2: we guess,

$$\boldsymbol{x}_t^{pa} = \sum_{s=0}^{t} (A)^{t-s} \boldsymbol{b}_s \tag{15}$$

Let $R_{s,t} := \prod_{i=s}^{t-1} A^i$, so that $\boldsymbol{x}_t^{pa} = \sum_{s=0}^{t} R_{s,t}\boldsymbol{b}_s$; and observe that $R_{s,t} = AR_{s,t-1}$, $R_{t,t} = A^0 = I$ for all $t$.

$$\begin{aligned} \boldsymbol{x}_{t+1}^{pa} &= \sum_{s=0}^{t+1} R_{s,t+1}\boldsymbol{b}_s \\ &= \sum_{s=0}^{t} R_{s,t+1}\boldsymbol{b}_s + R_{t+1,t+1}\boldsymbol{b}_{t+1} \\ &= A\sum_{s=0}^{t} R_{s,t}\boldsymbol{b}_s + \boldsymbol{b}_{t+1} \\ &= A\boldsymbol{x}_t^{pa} + \boldsymbol{b}_{t+1} \end{aligned}$$

Although this is fully correct, it may not be computationally clever. In fact, we may have a much easier problem to solve if, rather than considering the original system, we pick the diagonalized one. For simplicity, consider the case in which $A$ has distinct eigenvalues, in the diagonal matrix $\Lambda$, and thus independent eigenvectors in $P$. The transformed system is,

$$\boldsymbol{z}_t = \Lambda \boldsymbol{z}_{t-1} + \boldsymbol{c}_t$$

where $\boldsymbol{c}_t := P^{-1}\boldsymbol{b}_t$.[6] Therefore, for $m = 2$, we attained two independent, difference equations of the first order,

$$z_{it} = \lambda_i z_{it-1} + c_{it}, \quad i = 1, 2$$

whose solutions are in 6. Alternatively, we can also express the particular solution as 15,

$$\boldsymbol{z}_t^{pa} = \sum_{s=0}^{t} (\Lambda)^{t-s} \boldsymbol{c}_s$$

with the advantage that $\Lambda^n = diag[\lambda_1^n, \lambda_2^n]$ is easy to compute.[7]

Finally, we can retrieve the solution of the original system by using $\boldsymbol{x}_t = Pz_t$.

An extension to the case in which $A$ has repeated eigenvalues and to those in which it has time-variable coefficients should be straightforward.

2.3. **Higher order difference equations.** The order $p$ of a difference equation refers to the higher order lag with which a variable is represented. A difference equation of order $p$ in one variable take the form,

$$x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2} - ..... - \alpha_p x_{t-p} = b_t \tag{16}$$

The strategy to solve higher order, linear, difference equations is to transform them into systems with more variables of order one and exploit the techniques illustrated in section 2.2.

For expositional simplicity, we illustrate the procedure for the case of $p = 2$. We can rewrite 16 as

$$\begin{cases} x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + b_t \\ x_{t-1} = x_{t-1} \end{cases}$$

or

$$\begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ x_{t-2} \end{pmatrix} + \begin{pmatrix} b_t \\ 0 \end{pmatrix}$$

Looking at the system in vector notation, we can write

$$\boldsymbol{x}_t = A\boldsymbol{x}_{t-1} + \boldsymbol{b}_t, \tag{17}$$

which is of first order. We can solve this system applying the techniques learned above for equations of first order with $m$ variables.

Complementary solution

The eigenvalues $(\lambda_1, \lambda_2)$ of $A$ can be derived applying formula 9, with

$$tr = \alpha_1, \ det = -\alpha_2$$

[6] You can check that for $m = 2$, $c_{1t} = (b_{1t} - \nu_2 b_2)/(\nu_1 - \nu_2)$ and $c_{2t} = (-b_{1t} + \nu_1 b_{2t})/(\nu_1 - \nu_2)$.

[7] Here however one should be careful to distinguish cases in which the eigenvalues are bigger or less than one in absolute value.

Hence,

$$\lambda_1, \lambda_2 = \frac{\alpha_1}{2} \pm \frac{1}{2}\sqrt{\alpha_1^2 + 4\alpha_2}$$

The associated eigenvectors are

$$v_i = \begin{pmatrix} \lambda_i \\ 1 \end{pmatrix}, \ i = 1, 2.$$

Using the same notation as above,

$$P := (v_1, v_2) = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

which, again, is invertible if $\lambda_1 \neq \lambda_2$. Finally, use equation 11,

$$\begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = P\boldsymbol{z}_t = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_{10}\lambda_1^t \\ z_{20}\lambda_2^t \end{pmatrix}$$

with

$$\boldsymbol{z}_0 := P^{-1}\boldsymbol{x}_0 = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{-1} \end{pmatrix} \tag{18}$$

which, assuming $x_{-1} = 0$, yields, $z_{10} = x_0/(\lambda_1 - \lambda_2)$ and $z_{20} = -x_0/(\lambda_1 - \lambda_2)$

Extracting the solution for $x_t$,

$$x_t^{co} = \lambda_1 z_{10}\lambda_1^t + \lambda_2 z_{20}\lambda_2^t \tag{19}$$

This, for equations of order $p$, generalizes to,

$$x_t^{co} = \sum_{i=1}^{p} \lambda_i z_{i0}\lambda_i^t$$

Particular solution

Here we can simply refer to the particular solution of a non-autonomous system 15 derived above. An alternative procedure, specific to single difference equations of order $p$ is to derive the particular solution exploiting the Fundamental Theorem of Algebra (FTA); this solution method is known as *factorization*. FTA says that any polynomial of order $p$ can be factor into exactly $p$ terms, each of which contains one characteristic root. That is, the polynomial in $z$,

$$z^p + a_1 z^{p-1} + ... + a_{p-1}z + a_p = 0$$

can equivalently be written (or factorized) as,

$$(\gamma_1 - z)(\gamma_2 - z) \cdots (\gamma_p - z) = 0,$$

where $(\gamma_i)$ are the (possibly non-distinct) characteristic roots of the polynomial.

Again, for simplicity, consider the $p = 2$ version of the difference equation 16,

$$x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2} = b_t$$

Using the lag operator,

$$(1 - \alpha_1 \mathbb{L} - \alpha_2 \mathbb{L}^2)x_t = b_t \tag{20}$$

The left hand side of this equation is,

$$(1 - \alpha_1 \mathbb{L} - \alpha_2 \mathbb{L}^2) x_t$$

equivalent to

$$(1 + a_1 \mathbb{L} + a_2 \mathbb{L}^2) x_t$$

with $a_i := -\alpha_i$ for all $i$. Now we can apply the FTA and factor the polynomial in $\mathbb{L}$, as

$$p(\mathbb{L}) = (1 - \lambda_1 \mathbb{L})(1 - \lambda_2 \mathbb{L}),$$

where $(\lambda_1, \lambda_2)$ are its characteristic roots. This is not immediately obvious. Indeed, it is key to notice the difference between 2.3 and 2.3. The form of the polynomial 2.3 is analogous to the one obtained by dividing 2.3 through by $z^p$,

$$1 + a_1 z^{-1} + ... + a_{p-1} z^{1-p} + a_p z^{-p} = 0$$

This, can factorized as,

$$(1 - \gamma_1 z^{-1})(1 - \gamma_2 z^{-1}) \cdots (1 - \gamma_p z^{-1}) = 0,$$

and the equivalence between the two forms can easily be checked for the case of $p = 2$. By analogy, treating $\mathbb{L}$ as $z^{-1}$, we have explained the form of 2.3.

Since, the characteristic roots, by definition, satisfy $p(\lambda_i) = 0$ for all $i$. So,

$$(1 - \lambda_1 \mathbb{L})(1 - \lambda_2 \mathbb{L}) x_t = b_t$$

Therefore,

$$x_t^{pa} = \frac{b_t}{(1 - \lambda_1 \mathbb{L})(1 - \lambda_2 \mathbb{L})}$$

with

$$\lambda_1, \lambda_2 = \frac{\alpha_1}{2} \pm \frac{1}{2} \sqrt{\alpha_1^2 + 4\alpha_2}$$

Suppose that either one of the roots, say $\lambda_2$ is less than one in modulus, then a way to solve the equation is as follows,

$$(1 - \lambda_1 \mathbb{L}) x_t^{pa} = \frac{b_t}{1 - \lambda_2 \mathbb{L}}$$

yielding,

$$x_t = \lambda_1 x_{t-1} + c_t$$

which is a simple equation of first order, with the last term,

$$c_t := \frac{b_t}{1 - \lambda_2 \mathbb{L}} = \sum_{s=-\infty}^{t} \lambda_2^{t-s} b_s$$

The extension to equations of order $p$ yields,

$$x_t^{pa} = \frac{b_t}{(1 - \lambda_1 \mathbb{L})(1 - \lambda_2 \mathbb{L}) \cdots (1 - \lambda_p \mathbb{L})}$$

To determine $\lambda's$. Go back to the original system representation 17 of a generic difference equation of order $p$. $(\lambda_i)$ are the $p$ eigenvalues of matrix

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_p \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

Finally, a general solution can simply be found using, $x_t = x_t^{co} + x_t^{pa}$.

2.4. **Solution stability.** One of the main objectives in the study of a dynamical system is to analyze the behavior of its solutions near a steady-state. This is the concern of stability theory. In this subsection, we introduce the basic concepts and results of stability theory for linear difference equations. These will latter be extended to analyze local stability of non-linear equations.

2.4.1. *First order equations.* Let us start with a first order difference equation as 5,

$$x_t = \alpha x_{t-1} + b$$

A steady-state $x^*$ is such that $x_t = x^*$ at all $t$.

**Definition 1** (Stability)**.** *The steady-state $x^*$ of 5 is **stable** if given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x_0 - x^*| < \delta \implies |x_t - x^*| < \epsilon, \text{ for all } t > 0.$$

*If $x^*$ is not stable, then it is called **unstable**.*

A stable solution is represented in figure 1,[8] where $n = t$, the initial period is labeled as $n_0$, the horizontal axis crosses the vertical at the steady-state, the $\delta$-neighborhood contains the initial value of $x_t$, there exists an $\epsilon$ neighborhood containing the values of $x_t$ for all $t = n \geq n_0$

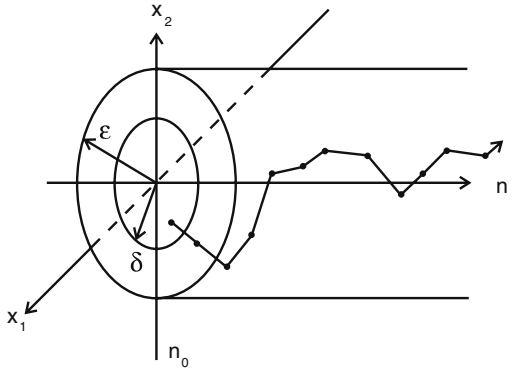


FIGURE 1. $x^*$ is stable

[8]This figure is taken from Elaydi (2005), p.177.

**Definition 2** (Asymptotic stability)**.** *The steady-state $x^*$ of 5 is* ***asymptotically stable*** *(or* attracting*) if there exists a $\eta > 0$ such that*

$$|x_0 - x^*| < \eta \implies \lim_{t\to\infty} x_t = x^*.$$

*If $\eta = \infty$, $x^*$ is* globally asymptotically stable *($x^*$ is a* global attractor*).*

Loosely speaking, a $x^*$ is asymptotically stable if every solution $x_t$ that starts (at $t = 0$) near $x^*$ converges to $x^*$. It is globally asymptotically stable if just about every solution $x_t$ tends to $x^*$ over time. $x^*$ is stable if every solution $x_t$ starting (at $t = 0$) close to $x^*$ (i.e. in ball of ray $\delta$ centered in $x^*$), does also remain close to it over time (i.e. it will stay in an $\epsilon-$ball as $t$ goes to infinity). Notice that asymptotic stability is a stronger requirement; it implies stability, but it is not implied by it.

Theorem 2.2 (Stability 1):
*A steady-state $x^*$ is* ***asymptotically stable*** *if $|\alpha| < 1$ and* ***asymptotically unstable*** *if $|\alpha| > 1$. If $|\alpha| = 1$ the steady-state $x^*$ is* ***stable.***

The proof of the theorem goes as follows. First, it is clear that the system has no dynamics if $\alpha = 0$, since then $x_t = x^* = b$. Next, suppose $|\alpha| < 1$, but $\alpha \neq 0$. Notice that, however distant from $x^*$ is an initial state $x_0$, the system will converge to the steady state: a solution to equation 5 is $x_t = \alpha^t x_0 + x^*$, hence as $t$ goes to infinity $x_t$ approaches $x^*$ (because $\alpha^t \to 0$). The sign of $\alpha$ determines the transitional dynamics. If negative, the sequence will oscillate around $x^*$ until convergence (in fact, when $t$ is even -odd-, $a^t$ is positive -negative-). Conversely, when $|\alpha| > 1$, but $\alpha \neq 0$, the dynamics is explosive; and transitional dynamics is either oscillatory or not, depending on whether $a$ is negative or positive. It remains to consider the case in which $|\alpha| = 1$. If $\alpha = 1$ we have $x_t = x_0$, hence constant. If $\alpha = -1$, $x_t = \pm x_0$, respectively, in even and odd times $t$, hence a stable oscillatory dynamics.

A diagrammatic analysis is also instructive. In a Cartesian diagram $(x_t, x_{t+1})$ one can graph equation 5 as time passes. The intersection of 5 with the $45°-$line is the steady state $x^*$. Then, take as initial state $x_0$ any point to the right, or to the left, of $x^*$ and draw the *phase diagram* as follows. From $x_0$ on the horizontal axis recover $x_1(x_0)$ from the graph of equation 5 and placed it on the vertical axis; then, use the $45°-$line to map $x_1(x_0)$ into the horizontal axis. Reiterate: given the new state $x_1(x_0)$, find $x_2(x_1(x_0))$ from the graph of equation 5 and placed it on the vertical axis; then, use the $45°-$line to map $x_2(x_1(x_0))$ into the horizontal axis. Each of these steps either brigs the value of the sequence closer to $x^*$ (asymptotical stable case) or it pushes it away from it (asymptotically unstable case). You can check that $x^*$ is asymptotically stable if the slope of the phase diagram of equation 5, at $x^*$, is less than one (crossing the $45^o$-line from below with a slope less than 1 or from above with a grater than $-1$).

2.4.2. *Systems of equations.* We deal directly with first-order systems because, as we have learned above, this does also allow to study higher order difference equations. Consider the autonomous system 8 in $m$ variables and equations,

$$\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + \boldsymbol{b}$$

Again, we aim at studying the stability the solution of 8, which we recall being 12

$$x_{it} = x_i^* + \sum_{j=1}^{m} v_{ij} z_{j0} \lambda_j^t, \quad i = 1, .., m,$$

where $(\lambda_i)$ are the eigenvalues of $A$ and $v_{ij}$ are elements of the matrix $P$, whose columns are formed by the eigenvectors of $A$, and $(z_0)$ are initial conditions. Notice that what drives the dynamics, essentially, are the eigenvalues. If they are all less than one in absolute values, then the system solution will tend to $\boldsymbol{x}^*$. Instead, if even one of the eigenvalues is larger than one in absolute value, the system solution will be asymptotically unstable: if $|\lambda_i| > 1$ for some $i$, and $z_{i0} \neq 0$, some variable of the system will tend to diverge from its steady-state value (even if the $ith-$row of $P$, $\boldsymbol{v}_i$, is zero, $x_{mt}$ will be asymptotically unstable).

We summarize these considerations in the following,

Theorem 2.3 (Stability 2):
*Let $\boldsymbol{x}^*$ be a steady-state of the system on $\mathbb{R}^m$.*

(1) *$\boldsymbol{x}^*$ is **asymptotically stable** if all the eigenvalues $(\lambda_i)$ of $A$ are strictly less than one in modulus. The solution is said to be a sink.*
(2) *$\boldsymbol{x}^*$ is **asymptotically unstable** if at least one of the eigenvalues of $A$ has modulus grater than one. The solution is said to define a **source** or a **saddle**, respectively, depending if $|\lambda_i| > 1$ for all $i$ or for some (but not all) $i$.*
(3) *$\boldsymbol{x}^*$ is **stable** if all the eigenvalues of $A$ have modulus weakly less than one, and some has modulus equal to one.*

In the special case in which the characteristic polynomial is of order 2 (i.e. $m = 2$ as in our growth model), then there is a simple stability test one can carry out. Recall the characteristic polynomial of $A$ is $p(\lambda) = \lambda^2 - tr\lambda + det$; its zeros, $\lambda_1, \lambda_2$, are the eigenvalues of $A$. This can be also seen a $2nd$-order polynomial in $\lambda$ that can be factor using its roots (corresponding to the eigenvalues of $A$),

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

so that, by definition $\lambda = \lambda_i$ satisfies $p(\lambda_i) = 0$ for all $i = 1, 2$.

Next, for stability analysis, it is interesting to evaluate $p(\lambda)$ at 1. If $\lambda = 1$ solves 2.4.2,

$$p(1) = (1 - \lambda_1)(1 - \lambda_2) = 0,$$

i.e. if $det = tr - 1$, then the system has either one of the eigenvalues equal to unity. Similarly, when

$$p(-1) = (-1 - \lambda_1)(-1 - \lambda_2) = 0 \iff (1 + \lambda_1)(1 + \lambda_2) = 0$$

i.e. $det = -(tr + 1)$, then either $\lambda_1$ or $\lambda_2$ are equal to $-1$. This is also useful to verify if the eigenvalues are both ling above or below one. When $p(1) > 0$, $(1 - \lambda_1)(1 - \lambda_2) > 0$, occurring when either $\lambda_1, \lambda_2 \geq 1$ or $\lambda_1, \lambda_2 \geq 1$, both with strict inequality if eigenvalues are repeated $(\lambda_1 = \lambda_2)$. In contrast, when $p(1) < 0$, $(1 - \lambda_1)(1 - \lambda_2) < 0$, the eigenvalues lie on opposite sides of unity. We summarize the latest results in the following.

**Remark 2.3** (Testing stability)**.** *Let* $p(z) := (z - \lambda_1)(z - \lambda_2)$.
$p(1) = 0$ *iff* $\lambda_1$ *or* $\lambda_2$ *are equal to* 1 *iff* $det = tr - 1$.
$p(-1) = 0$ *iff* $\lambda_1$ *or* $\lambda_2$ *are equal to* $-1$ *iff* $det = -(tr + 1)$.
$p(1) > 0 (< 0)$ *iff* $\lambda_1, \lambda_2$ *lie on the same (opposite) side of* 1, *iff* $\lambda_1, \lambda_2 > 1$ *or* $\lambda_1, \lambda_2 < 1$ *(*$\lambda_i < 1, \lambda_{-i} > 1,\ i = 1, 2$*),* $det > tr - 1$ *(*$det < tr - 1$*).*
$p(-1) > 0 (< 0)$ *iff* $\lambda_1$ *and* $\lambda_2$ *lie on the same (opposite) side of* $-1$, *iff* $det > -(tr + 1)$ *(*$det < -(tr + 1)$*).*

Notice that if one knows that either $p(1) < 0$ or $p(-1) < 0$, it means that $|\lambda_i| < 1$ for some (at least one) $i$; i.e. there is at least a stable eigenvalue, hence the solution is either a saddle or a sink. Instead, if one knows that $p(1) > 0$ or $p(-1) > 0$ and $|\lambda_i| < 1$ for some $i$, also the other eigenvalue $\lambda_{-i}$ satisfies $|\lambda_{-i}| < 1$ and the solution is a sink.

## 2.5. **Stochastic difference equations.**

2.5.1. *Definitions.* Stochastic models are those in which some variable is uncertain, i.e. it is regarded as random. Suppose $x$ is such variable,

$$x = (x_0, x_1, ....., x_t, ....)$$

it is an infinite sequence of random variables, a *stochastic process*. At each date $t$ the $t-$element of this process, $x_t$, is a random variable whose realization is on some domain $X_t$. A realization of $x$ is an infinite sequence of realizations, each for every $t-$element. A finite, $t$, realization of $x$, $x^t = (x_0, x_1, .., x_t)$ is said to be a *history* up to $t$, or a *time-series*.

A stochastic process is said to be *strictly stationary* if for all $k \geq 0$ the sequences $(x_0, x_1, ....., x_t, ....)$ and $(x_k, x_{k+1}, ....., x_{k+t}, ....)$ have the same distribution. A stochastic process is said to be *mean stationary* if the mean of its elements is time invariant: $\mathbb{E}(x_t) = \mathbb{E}(x_{t+k})$ for all $k \geq 0$. A stochastic process is said to be *covariance stationary* if the covariance of its elements depends only on theirs distance: $Cov(x_t, x_{t+k}) = Cov(x_T, x_{T+k})$ for all $t, T, k \geq 0$. A stochastic process is said to be *independently and identically distributed (iid)* if each and every one of its elements, $x_t$, has the same marginal distribution. This is different concept from stationarity that also place restrictions on the joint distributions; iid implies stationarity if the process is also *serially independent*, i.e. if every collection of sequence coordinate (or history of random variables) is totally independent.

A stochastic process has the *Markov* property if for all $s \geq 1$,

$$Prob(x_{t+1}|x_t, x_{t-1}, ..., x_{t-s}) = Prob(x_{t+1}|x_t)$$

We assume that agents have the same information (degree of uncertainty); that is, they all know the probability space governing the system (space of possible events or state space, probability distributions of stochastic processes, realizations of past events). At each time $t$ agents, having observed the current and past realizations of each variable $x_t$, form their expectations on $x_{t+1}$ rationally, i.e. by computing the conditional expectation, ${}_tx_{t+1} := \mathbb{E}_t(x_{t+1}|\Omega_t)$, where $\Omega_t$ represents the realizations of current and past variables. If the process is (at least) mean stationary, ${}_tx_{t+1} := \mathbb{E}(x_{t+1}|\Omega_t)$. If it is also Markovian, one takes $\Omega_t = x_t$, i.e. ${}_tx_{t+1} := \mathbb{E}(x_{t+1}|x_t)$.

Law of Iterated Expectations: If $\Omega_t \subseteq \Omega_{t+1}$, $\mathbb{E}\left(\mathbb{E}\left(x_t|\Omega_{t+1}\right)|\Omega_t\right) = \mathbb{E}\left(x_t|\Omega_t\right)$

2.5.2. *Solving stochastic difference equations.* Solutions can basically be found applying the techniques of section 2.2 on deterministic difference equations. We clarify this with a simple example. For simplicity assume that stochastic processes are mean stationary, and let $\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot|\Omega_t)$.

**Example 2.3.** *Consider the following first order equation,*

$$\mathbb{E}_t x_{t+1} = \alpha x_t + \mathbb{E}_t u_{t+1},\ |\alpha| > 1.$$

*Apply the operator* $\mathbb{L}$*.*

$$\mathbb{L}\mathbb{E}_t x_{t+1} = \alpha \mathbb{L} x_t + \mathbb{L}\mathbb{E}_t u_{t+1}$$

*and find,*

$$\mathbb{E}_t x_t = \alpha x_{t-1} + \mathbb{E}_t u_t$$

*Observe that* $\mathbb{E}_t x_t = x_t$*. Hence,*

$$x_t = \alpha x_{t-1} + \mathbb{E}_t u_t$$

*Find the general solution applying 6,*

$$x_t = \alpha^t x_0 - \frac{1}{\alpha} \sum_{s=t+1}^{\infty} \left(\frac{1}{\alpha}\right)^{s-t} \mathbb{E}_t u_s$$