You know, I don't know about that. <u>16 March 2020</u>

- I wanna pause on for a sec b/c I can't seem to get it to work and I'm confused about how to get it time-varying anyway. So let's turn to estimation.

Estimation of the anchoring function

The issue is that we wanna estimate the anchoring fct together w/ the model. On the top of my head I can think of 3 ways of doing that:

1) IR-matching
2) Likelihood-based (either MLE or Bayesian)
3) VAR-representation ⇒ exist? est. that!
   ↳ it would be a time-varying one.

↳ I'm leaning toward #2 b/c 1) it's sexier
2) it's more general than conditional on shocks
3) a TV-VAR sounds challenging

The thing is: 17 March 2020

need to derive the log-likelihood of my model

Take the midsimple model w/ the TR.

In materials 21, this is eq. (5)-(10) + TR

$$\pi_t = \kappa x_t - (1-\alpha)\beta f_a(t) + \left[-\kappa\alpha\beta(I_3-\alpha\beta h x)^{-1} - e_3(I_3-\alpha\beta h x)^{-1}\right]s_t$$

$$x_t = -\sigma i_t - \sigma f_b(t) + \left[-(1-\beta)b_2(I_3-\beta h x)^{-1} + \sigma\beta b_3(I_3-\beta h x)^{-1} - \sigma e_1(I_3-\beta h x)^{-1}\right]s_t$$

$$f_a(t) = \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} - b_1(I_3-\alpha\beta h x)^{-1}s_t$$

$$f_b(t) = \frac{1}{1-\beta}\bar{\pi}_{t-1} - b_1(I_3-\beta h x)^{-1}s_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\right)$$

$$k_t^{-1} = k_{t-1}^{-1} + d\left(\pi_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})\right) + c$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t$$

This is a state-space model (believe it or not)

and I'm gonna eliminate some variables

$$\pi_t = \kappa x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} - (1-\alpha)\beta b_1(I_3-\alpha\beta h x)^{-1}s_t$$
$$+ \left[-\kappa\alpha\beta(I_3-\alpha\beta h x)^{-1} - e_3(I_3-\alpha\beta h x)^{-1}\right]s_t$$

$$x_t = -\sigma\psi_\pi\pi_t - \sigma\psi_x x_t - \sigma\bar{i}_t - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1} - \sigma b_1(I_3-\beta h x)^{-1}s_t$$
$$+ \left[-(1-\beta)b_2(I_3-\beta h x)^{-1} + \sigma\beta b_3(I_3-\beta h x)^{-1} - \sigma e_1(I_3-\beta h x)^{-1}\right]s_t$$

$$\pi_t = \kappa x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h_x)^{-1} S_1 - \kappa\alpha\beta (I_3 - \alpha\beta h_x)^{-1} - e_3 (I_3 - \alpha\beta h_x)^{-1}\right] S_t$$

$$x_t = -\sigma\psi_\pi \pi_t - \sigma\psi_x x_t - \sigma \bar{i}_t - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1} - \sigma b_1 (I_3 - \beta h_x)^{-1} S_t$$

$$+ \left[-(1-\beta) b_2 (I_3 - \beta h_x)^{-1} + \sigma\beta b_3 (I_3 - \beta h_x)^{-1} - \sigma e_1 (I_3 - \beta h_x)^{-1}\right] S_t$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t \qquad - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1}$$

$$\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h_x)^{-1} - (1-\beta) b_2 (I_3 - \beta h_x)^{-1} + \sigma\beta b_3 (I_3 - \beta h_x)^{-1} - \sigma e_1 (I_3 - \beta h_x)^{-1}\right] S_t$$

$$\Rightarrow \quad x_t = -\frac{\sigma\psi_\pi}{1+\sigma\psi_x}\pi_t - \frac{1}{1+\sigma\psi_x}\frac{\sigma}{1-\beta}\bar{\pi}_{t-1}$$

$$+ \frac{1}{1+\sigma\psi_x}\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h_x)^{-1} - (1-\beta) b_2 (I_3 - \beta h_x)^{-1} + \sigma\beta b_3 (I_3 - \beta h_x)^{-1} - \sigma e_1 (I_3 - \beta h_x)^{-1}\right] S_t$$

Can even sub x_t out!

$$\pi_t = -\frac{\kappa\sigma\psi_\pi}{1+\sigma\psi_x}\pi_t - \frac{\kappa}{1+\sigma\psi_x}\frac{\sigma}{(1-\beta)}\bar{\pi}_{t-1}$$

$$+ \frac{\kappa}{1+\sigma\psi_x}\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h_x)^{-1} - (1-\beta) b_2 (I_3 - \beta h_x)^{-1} + \sigma\beta b_3 (I_3 - \beta h_x)^{-1} - \sigma e_1 (I_3 - \beta h_x)^{-1}\right] S_t$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h_x)^{-1} S_1 - \kappa\alpha\beta (I_3 - \alpha\beta h_x)^{-1} - e_3 (I_3 - \alpha\beta h_x)^{-1}\right] S_t$$

$$\Leftrightarrow \left(1 + \frac{\kappa\sigma\psi_\pi}{1+\sigma\psi_x}\right)\pi_t = -\left(\frac{\kappa\sigma}{1+\sigma\psi_x(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right)\bar{\pi}_{t-1} + \text{stuff} \cdot S_t$$

$$\left(1+\frac{k\vartheta\psi_\pi}{1+\vartheta\psi_x}\right)\pi_t = -\left(\frac{k\vartheta}{1+\vartheta\psi_x(1-\beta)}+\frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right)\bar{\pi}_{t-1} \quad +$$

$$\Big\{\frac{k}{1+\vartheta\psi_x}\left[-\vartheta e_2-\vartheta b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\vartheta\beta b_3(f_3-\beta h x)^{-1}-\vartheta e_1(f_3-\beta h x)^{-1}\right]$$

$$+\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Big\}s_t$$

$$\frac{1+\vartheta\psi_x+k\vartheta\psi_\pi}{1+\vartheta\psi_x}\pi_t = \text{same}$$

$$\Rightarrow \pi_t = -\frac{\cancel{1+\vartheta\psi_x}}{1+\vartheta\psi_x+k\vartheta\psi_\pi}\left(\frac{k\vartheta(1-\alpha\beta)+(1-\alpha)\beta(1-\beta)(1+\vartheta\psi_x)}{\cancel{(1+\vartheta\psi_x)}(1-\beta)(1-\alpha\beta)}\right)\bar{\pi}_{t-1}$$

$$+\Big\{\frac{k}{1+\vartheta\psi_x+k\vartheta\psi_\pi}\left[-\vartheta e_2-\vartheta b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\vartheta\beta b_3(f_3-\beta h x)^{-1}-\vartheta e_1(f_3-\beta h x)^{-1}\right]$$

$$+\frac{1+\vartheta\psi_x}{1+\vartheta\psi_x+k\vartheta\psi_\pi}\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Big\}$$

$$\Rightarrow \pi_t = -\frac{k\vartheta(1-\alpha\beta)+\beta(1-\alpha)(1-\beta)(1+\vartheta\psi_x)}{(1+\vartheta\psi_x+k\vartheta\psi_\pi)(1-\beta)(1-\alpha\beta)}\bar{\pi}_{t-1} \quad + \qquad \cdot s_t$$

$$\Big\{\frac{k}{1+\vartheta\psi_x+k\vartheta\psi_\pi}\left[-\vartheta e_2-\vartheta b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\vartheta\beta b_3(f_3-\beta h x)^{-1}-\vartheta e_1(f_3-\beta h x)^{-1}\right]$$

$$+\frac{1+\vartheta\psi_x}{1+\vartheta\psi_x+k\vartheta\psi_\pi}\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Big\}s_t$$

Damn dam! So we have $\equiv A$

$$\pi_t = -\frac{\kappa\sigma(1-\alpha\beta)+\beta(1-\alpha)(1-\beta)(1+\sigma\psi_x)}{(1+\sigma\psi_x+\kappa\sigma\psi_\pi)(1-\beta)(1-\alpha\beta)}\bar{\pi}_{t-1} +$$

$$\Big\{\frac{\kappa}{1+\sigma\psi_x+\kappa\sigma\psi_\pi}\Big[-\sigma e_2-\sigma b_1(f_3-\beta h_x)^{-1}-(1-\beta)b_2(f_3-\beta h_x)^{-1}+\sigma\beta b_3(f_3-\beta h_x)^{-1}-\sigma e_1(f_3-\beta h_x)^{-1}\Big]$$

$$+\frac{1+\sigma\psi_x}{1+\sigma\psi_x+\kappa\sigma\psi_\pi}\Big[-(1-\alpha)\beta b_1(f_3-\alpha\beta h_x)^{-1}s_1-\kappa\alpha\beta(f_3-\alpha\beta h_x)^{-1}-e_3(f_3-\alpha\beta h_x)^{-1}\Big]\Big\}s_t$$

$\equiv B$

$$\bar{\pi}_t = \bar{\pi}_{t-1}+k_t^{-1}(\pi_t-(\bar{\pi}_{t-1}+b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1}+d(\pi_t-(\bar{\pi}_{t-1}-b_1 s_{t-1}))+c$$

↳ 1 jump (π_t), 3 exog states ($s_t = \begin{bmatrix} r_t^n \\ \bar{i}_t \\ u_t \end{bmatrix}$) and 2 endog states $\xi_t = \begin{bmatrix} \bar{\pi}_t \\ k_t^{-1} \end{bmatrix}$ (or $\begin{bmatrix} \bar{\pi}_{t-1} \\ k_t^{-1} \end{bmatrix}$) so $X_t = \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$ (states)

$$\pi_t = A\bar{\pi}_{t-1}+Bs_t = \begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1}+k_t^{-1}(\pi_t-(\bar{\pi}_{t-1}+b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1}+d\big(\underbrace{\pi_t-(\bar{\pi}_{t-1}-b_1 s_{t-1})}_{\text{call this the state (?) } fe_{t-1}}\big)+c$$

$\pi_t = A\bar{\pi}_{t-1} + B s_t = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$ $\qquad Y_t = gx \cdot X_t$

$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} fe_{t-1}$ $\qquad X_{t+1} = hx\, X_t + \eta\, \varepsilon_t$

$k_t^{-1} = k_{t-1}^{-1} + d \cdot fe_{t-1} + c$

$fe_{t-1} = \pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}$

Several issues y'all:

1) fe state or jump? depends on π_t (a jump)

→ there's gotta be some trick (like π_t in LEMP)

to make it a pure state

2.) $\bar{\pi}_t$ nonlinear LOM!

↙

ok – that's troubling. But let's pause it for a sec & let's read what Lütkepohl has to say about MLE & log-likelihoods.

Supp. our VAR(p) looks like

$$y_t - \mu = A_1(y_{t-1} - \mu) + \dots + A_p(y_{t-p} - \mu) + u_t \quad (3.3.1)$$

p. 75

Then if the VAR(p) is Gaussian, that is

$u \equiv vec(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim N(0, I_T \otimes \Sigma_u)$, which equivalently means that the prob. density of u is

$$f_u(u) = \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp\left[-\frac{1}{2} u' (I_T \otimes \Sigma_u^{-1}) u \right]$$

then we can use the fact that $u = y - \mu^* - (X' \otimes I_K)\alpha$

where $\underset{k^2p \times 1}{\alpha} := vec(A)$, $A := (A_1, \ldots, A_p)$ $\quad k \times kp$

$Y^0 := (y_1 - \mu, \ldots, y_T - \mu) \qquad k \times T$

$Y_t^0 := \begin{bmatrix} y_t - \mu \\ \vdots \\ y_{t-p+1} - \mu \end{bmatrix} \qquad (kp \times 1)$

$X := (Y_0^0, \ldots, Y_{T-1}^0)$,

to write

$$f_y(y) = \left| \frac{\partial u}{\partial y} \right| f_u(u)$$

$$= \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp\left[-\frac{1}{2} \left(y - \mu^* - (X' \otimes I_K)\alpha \right)' \left(I_T \otimes \Sigma_u^{-1} \right) \right.$$

$$\left. \cdot \left(y - \mu^* - (X' \otimes I_K)\alpha \right) \right] \qquad (3.4.4)$$

and

$$\ln l(\mu, \alpha, \Sigma_u) = -\frac{kT}{2} \ln(2\pi) - \frac{T}{2} \ln|\Sigma_u| - \frac{1}{2} tr (Y^0 - AX)' \Sigma_u^{-1} (Y^0 - AX)$$

is the log-likelihood. $\qquad (3.4.5)$

Given that the updating eq. w/ endog. gain <u>18 March 2020</u>
introduces non-linearities, I'm afraid that even a "simple" & quick estimation has to involve some form of particle filter. But let's see whether Lütkepohl has anything interesting to say about 1) state-space models 2) non-linearities

Lütkepohl, State-space models (Ch. 13, p. 415 ff.)

$$\underset{N\times 1}{z_{t+1}} = \underset{N\times N}{B_t}\, z_t + \underset{N\times m}{F_t}\, x_t + \underset{N\times 1}{w_t} \qquad t = 0,1,2,\dots \qquad (13.2.1a)$$

noise

$$\underset{k\times 1}{y_t} = \underset{k\times N}{H_t}\, z_t + \underset{k\times M}{G_t}\, \underset{M\times 1}{x_t} + \underset{k\times 1}{v_t} \qquad t = 1,2,\dots \qquad (13.2.2)$$

measurement error

inputs/ instruments/ policy vars

transition matrix

measurement matrix

input matrices

and $\begin{bmatrix} w_t \\ v_t \end{bmatrix} \sim WN\left[0, VC_t\right]$ $\qquad VC_t = \begin{bmatrix} \Sigma_{w_t} & \Sigma_{w_t v_t} \\ \Sigma_{v_t w_t} & \Sigma_{v_t} \end{bmatrix}$

Nonlinear state-space models

$$z_{t+1} = b_t(z_t, x_t, w_t, \delta_1)$$

$$y_t = h_t(z_t, x_t, v_t, \delta_2)$$

vectors of params

Example of nonlin state-space is the "bilinear" model:

$y_t = \alpha y_{t-1} + u_t + \beta y_{t-1} u_{t-1}$ p. 427 w/ Refs.

$\hookrightarrow$ $$z_{t+1} = B z_t + w_t + C \operatorname{vec}(z_t z_t') \quad (13.2.33)$$

$$y_t = [I_K \; 0 \ldots 0] \, z_t \quad (13.2.34)$$

$\Rightarrow$ Bunch of refs on bilinear systems, univariate & multivariate on p. 427 bottom.

MLE of state-space models p. 434

Gather the time-invariant params from $B, F, H_t, G, \Sigma_w, \Sigma_v$ and Σ_0 & μ_0 in δ as

$$\delta = \begin{bmatrix} \operatorname{vec}[v, A_1, \ldots, A_p] \\ \operatorname{vech}(\Sigma_u) \end{bmatrix}$$

where vech = "half-vectorization", $\operatorname{vech}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{bmatrix} a \\ b \\ d \end{bmatrix}$.

The log-likelihood for the Gaussian state-space model is:

$$\ln l(\delta | y_1, \ldots, y_T) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln |\Sigma_y(t|t-1)|$$
$$- \frac{1}{2} \sum_{t=1}^{T} (y_t - y_{t|t-1})' \Sigma_y(t|t-1)^{-1} (y_t - y_{t|t-1}) \quad (13.4.1)$$

Denoting the fcst error $e_t(\delta) := y_t - y_{t|t-1}$

and $\Sigma_t(\delta) := \Sigma_y(t|t-1)$, we can rewrite this as

$$\ln \ell(\delta) = -\frac{kT}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln|\Sigma_t(\delta)| + e_t'(\delta)\Sigma_t(\delta)^{-1}e_t(\delta)\right] \tag{13.4.3}$$

which makes explicit

1) the dependence of $\ln \ell$ on δ,
2) that all quantities in the $\ln \ell$ all functions of δ and can (must) be computed using the Kalman filter.

locally identified: when in a subspace of the param space, δ is uniquely determined.

vs. **globally identified**: when δ is uniquely determined in the entire param space.

↳ identification: we need a min of $-\ln \ell$, so we need some sort of Hessian = pos. def. ⟹ the information matrix, = E[Hessian] = $E\left[\frac{\partial^2(-\ln \ell)}{\partial\delta\partial\delta}\Big|_{\delta_0}\right]$

A quick note on DSEMs (dynamic simultaneous equations models a.k.a. "linear systems") p. 323

essentially, these are linear VARMAX(p,s,q) models

$$A_0 y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + B_1 x_{t-1} + \dots + B_s x_{t-s} + w_t \qquad (10.1.1)$$

- VARMAX(p,s,q) is w_t is MA(q)
- VARX(p,s) if $w_t \sim WN$.

VAR(p) models w/ time-varying coefficients p. 391ff.

periodic VARs → e.g. w/ seasonal dummies

intervention models → DGP_1 is replaced by DGP_2 at time τ.

$$y_t = v_t + A_{1t} y_{t-1} + \dots + A_{pt} y_{t-p} + u_t \qquad (12.2.1)$$

↳ $WN(0, \Sigma_t)$

also time-varying

(not identically distrib!)

Rewrite the VAR(p) as a VAR(1)

$$Y_t = v_t + A_t Y_{t-1} + U_t \qquad (12.2.2)$$

$$\underset{kp\times 1}{Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}}, \quad \underset{kp\times 1}{v_t := \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \quad \underset{kp\times kp}{A_t := \begin{bmatrix} A_{1,t} & \cdots & A_{p-1,t} & A_{p,t} \\ I_k & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_k & 0 \end{bmatrix}}, \quad \underset{kp\times 1}{U_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

And by recursive sub we get

$$Y_t = \left[\prod_{j=0}^{h-1} A_{t-j}\right] Y_{t-h} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j}\right] v_{t-i} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j}\right] U_{t-i} \qquad (12.2.3)$$

Defining $J := [I_k \ \ 0]$ such that $y_t = JY_t$, we can premultiply (12.2.3) by J, define

$\Phi_{it} := J\left[\prod_{j=0}^{i-1} A_{t-j}\right] J'$ to get

$$y_t = \mu_t + \sum_{i=0}^{\infty} \Phi_{it} u_{t-i} \qquad (12.2.6)$$

where $\mu_t = E[y_t]$

$\Rightarrow$ then the MSE (or FEV of the FE $y_{t+h} - y_t(h)$) is

$$\Sigma_t(h) := \sum_{i=0}^{h-1} \Phi_{i,t+h} \Sigma_{t+h-i} \Phi'_{i,t+h} \qquad (12.2.10)$$

MLE of TV-VAR   p. 394

Write (12.2.1) as $y_t = B_t Z_{t-1} + u_t$ (12.2.11)

where $B_t := [\nu_t, A_{1t}, \dots, A_{pt}]$, $Z_{t-1} := (1, Y_{t-1}')'$

B_t depend on the vector $\underset{N\times 1}{\gamma}$ of time-invariant params.

Σ_t depend on $\underset{M\times 1}{\sigma}$ of fixed params.

If $u_t \sim N(0, \Sigma_t)$, the log-likelihood is

$$\ln l(\gamma, \sigma) = -\frac{KT}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\ln|\Sigma_t| - \frac{1}{2}\sum_{t=1}^{T} u_t' \Sigma_t^{-1} u_t \quad (12.2.12)$$

where initial conditions have been ignored.

You can also derive an info-matrix.

So it seems like you just min $-\ln l(\gamma, \sigma)$!

It also seems like for certain special cases you can even derive the estimators in closed-form!

Ryan meeting 18 March 2020

Not yet convinced that the proc procedure is sensible

→ the only procedure that can work is

$f_{z,t}$ and more RHS-variables: k^{-1}, $\bar{\pi}$

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

things in $f_{z,t}$ will be 0 or simply described by the LOMs

It may be too restrictive.

A computationally intensive exercise is:

- conj. path for the int. rate (exog.)
- solve the model for that int. rate
- check the target criterion, compute the resid
- fmincon to min that resid

⇒ find simulated optimal plan.

- Simulate the Ramsey model
- Simulate the model w/ Taylor-rule
  - ⇒ see how close you can get

It might then be that most results won't be pencil & paper.

Comments:

- Concerning w/ linearness of (10)
  - → "negative surprises cause me to be unanchored" shouldn't be "big mistakes" → so take the square
    - ↳ like smooth one better than jumpy
- Estim: filter the data → App paper w/ Robert:
  - match the params to moments cors of data
  - ⇒ would give you results

They est an NK model: HP-filter both data and model, compute moments and try to match those.

Work after

It's the primal economy paper w/ Robert that Ryan meant → chahrourWorkit.pdf in "Literature."

⇒ They 1) take data on real per-capita output, inflation, nom interest rates & per-capita hours, 2) (BK) high-pass filter them, 3) Use GMM to estimate the process $\hat{\tau}_t$ that best-matches the autocov. structure of the data.

App. D p.46 (Mac).

Wedges are an MA(14):

$$\tau_t = \Phi_\varepsilon(L)\,\varepsilon_t + \Phi_u(L)\,u_t$$

ε_t: $WN(0, \sigma_\varepsilon)$ u_t: $2\times 1\ WN(0, I_2)$

⇒ params to be estimated are

$$\gamma_{ma} \equiv (\Phi_\varepsilon, \Phi_u, \sigma_\varepsilon)$$

I think that the following is the target: $k=8$

$$\tilde{\Omega}_T = vech\{Var[\tilde{q}_t^{data}, \dots, \tilde{q}_{t-k}^{data}]\} \quad q: y_t, \pi_t, i_t, n_t$$

Filtering: Baxter-King filter w/ truncation horizon 32, lag-length 12

→ $\tilde{q}_t \equiv BK_{32}(q_t)$, so $\tilde{q}_t$ is filtered data.

Then, they do a trick in converting γ_{ma} into γ_{ar} to finally estimate γ as:

$$\hat{\gamma}_{ar} = \underset{\gamma_{ar}}{\operatorname{argmin}} \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{ar})\right)' W^{-1} \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{ar})\right)$$

- W is a diagonal matrix w/ the bootstrapped variances of $\tilde{\Omega}_T$ along the main diagonal.
- The model analogue $\tilde{\Omega}(\gamma_{ar})$ is computed after the model data has been similarly filtered as the data.

↳ Hold it there, now I know what I need to do.

↳ That and

- the numerical implementation of the target criterion ⟹ both are things to do once I have the big screen. So now give a last try to do the time-varying one.

The ori - a first shot

Ryan wrote:

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

I think he meant $z_t = \bar{z}_t + h_{z,t} z_{t-1} + f_{z,t} u_t$

Let me ignore r_t^n b/c it just blows things up and I just wanna see if it works. Ignore $\bar{z}_t$ too.

$$h_{\pi,t} \pi_{t-1} + f_{\pi,t} u_t - \kappa (h_{x,t} x_{t-1} + f_{x,t} u_t)$$
$$- (1-\alpha)\beta \left(h_{fa,t} fa(t-1) + f_{fa,t} u_t \right) + exog_1 \cdot u_t = 0 \quad (9)$$

$$h_{x,t} x_{t-1} + f_{x,t} u_t + \sigma (h_{i,t} i_{t-1} + f_{i,t} u_t)$$
$$- \sigma \left(h_{fb,t} fb(t-1) + f_{fb,t} u_t \right) + exog_2 \cdot u_t = 0 \quad (10)$$

$\Leftrightarrow$

$$h_{\pi,t} \pi_{t-1} - \kappa h_{x,t} x_{t-1} - (1-\alpha)\beta h_{fa,t} fa(t-1)$$
$$+ \underbrace{\left(f_{\pi,t} - \kappa f_{x,t} - (1-\alpha)\beta f_{fa,t} + exog_1 \right)}_{\text{this is exactly what I have in materials 21}} u_t = 0 \quad (10)$$

$$h_{x,t} x_{t-1} + \sigma h_{i,t} i_{t-1} - \sigma h_{fb,t} fb(t-1)$$
$$+ \left(f_{x,t} + \sigma f_{i,t} - \sigma f_{fb,t} + exog_2 \right) u_t = 0$$

$$z_t = \bar{z}_t + h_{z,t}\, z_{t-1} + f_{z,t}\, u_t$$

$$h_{fa,t}\, f_a(t-1) + f_{fa,t}\, u_t - \frac{1}{1-\alpha\beta}\left(h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + f_{\bar{\pi},t-1}\, u_{t-1}\right) + exog_3\, u_t$$

$$\underbrace{h_{fa,t}\, f_a(t-1) - \frac{1}{1-\alpha\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + (f_{fa,t} + exog_3)\, u_t - \frac{1}{1-\alpha\beta} f_{\bar{\pi},t-1}\, u_{t-1} = 0}_{\text{same as in materials 21}} \quad (13)$$

$$h_{fb,t}\, f_b(t-1) - \frac{1}{1-\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + (f_{fb,t} + exog_3)\, u_t - \frac{1}{1-\beta} f_{\bar{\pi},t-1}\, u_{t-1} = 0 \quad (14)$$

Honestly, I don't even think I will complete this b/c we can see that if, in say (14), $h_{fb,t} = 0$ and $h_{\bar{\pi},t-1} = 0$ then we're back to exactly the system I had in mat 21. So suppose $h_{fb,t} \neq 0$. But then in each equation we have an earlier iteration of itself. Now I might argue $h_\pi = h_x = h_i = 0$, but if I ass that for $h_{fa,t} = h_{fb,t} = 0$ then again the link between (9)-(10) and (15)-(16) is broken. Then (14) would give an additional constraint

$$h_{fb,t}\, f_b(t-1) = \frac{1}{1-\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2}$$

In two unknowns. Mm... You'd get a proliferation of unknowns...! I'll stop here, this wasn't for endog states, merde!

① GMM of midsimple — 19 March 2020

② Numerical implementation of target criterion

① GMM of midsimple

sim_learnLH.m w/ PLM = constant_only

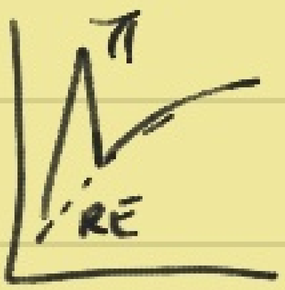
→ Need to:

1) add smooth function as a third criterion

2) To have "midsimple", we need that x (and i) aren't learned, so input $a = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$, $b = b^{RE}$

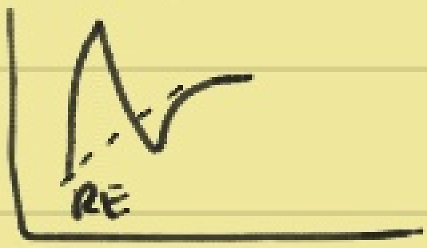
↳ ok, I've learned one thing (comparing w/ IRFs of materials 9)

⇒ when only π is learned, IRFs are less oscillatory in the sense that they overshoot only once:



vs.

when $(\pi, x, i)'$ learned



→ did I ever conclude this? I thought so, but now it doesn't seem like it. In materials12 I just state that vector vs. scalar learning is pretty much the same

...And it almost is, just not quite.

⇒ so mental note!

Learning only LOM(π) dampens the oscillations b/c $E(\cdot)$ in NKIS & NKPC are moving less, and therefore $E(i_{t+k})$ is moving less. I think more importantly since the bulk of action is in x, the fact that FE(x) is now cut out gets rid of the FE that was most oscillatory.

Ok I just wasted an hour trying to customize legends in Matlab...

Back to 1) the anchoring function:

$$k_t^{-1} - k_{t-1}^{-1} = c + d(FE)$$

has the interpretation that the gain decreases when $FE < 0$. Ryan is right: the easiest thing that makes Δk_t^{-1} big when FE is big in absolute value is FE^2

$\rightarrow$ $k_t^{-1} - k_{t-1}^{-1} = c + d\,FE_t^2$

↑ does this even make sense?

> Not really: it's saying that the gain always changes, even if the $FE^2 = 0$.

↳ So then gather data and implement

- HP
- BK
- Hamilton filters

But there is a problem: for this current form

$$k_t^{-1} - k_{t-1}^{-1} = c + d \cdot FE^2,$$

y always explodes, even more so if $c=0$.
And it's b/c $k \to 0$, so I guess k^{-1} explodes.

↳ Need to be smart about this!

Normally, a dgain would be <u>20 March 2020</u>

$k_{t+1} = k_t + 1$

So I could just do $k_t = k_{t-1} + \frac{1}{FE^2}$

so that if in the limit $FE^2 \to \infty$, $k_t = k_{t-1}$ (cgain)

Then one could have

$$k_t = k_{t-1} + d\frac{1}{FE^2}$$ in this case I guess d small $d<1$

or $$k_t = k_{t-1} + \frac{1}{d\,FE^2}$$ in this case I guess d big $d>1$

↳ both of them work and exactly the way I thought

or even $k_t = k_{t-1} + \frac{1}{(dFE)^2} = k_{t-1} + (dFE)^{-2}$

works.

HP-filter

A time series y_t (in logs) is

$$y_t = \underset{\substack{\uparrow \\ \text{growth} \\ \text{component}}}{g_t} + \underset{\substack{\uparrow \\ \text{cyclical component}}}{c_t}$$

Obtain g_t as

$$\min_{\{g_t\}_{t=-1}^{T}} \left\{ \sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=1}^{T} \left[(g_t - g_{t-1}) - (g_{t-1} - g_{t-2}) \right]^2 \right\}$$

where $\lambda = 1600$.

Hamilton provides a closed-form solution to this problem as $g^* = (H'H + \lambda Q'Q)^{-1} H' y$ (2)

where $\tilde{T} := T + 2$

$$\underset{T \times 1}{y} := (y_T, y_{T-1}, \dots, y_1)' \qquad \underset{\tilde{T} \times 1}{g} = (g_T, g_{T-1}, \dots, g_0, g_{-1})'$$

and $\underset{T \times \tilde{T}}{H} = \begin{bmatrix} \underset{T \times T}{I_T} & \underset{T \times 2}{0} \end{bmatrix}$ and $\underset{T \times \tilde{T}}{Q} = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \ddots & & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{bmatrix}$

Hamilton filter

21 March 2020

Idea: $c = y_{t+h} - E[y_{t+h} \mid y_t]$

choose $h=8$ for quarterly data.

How to calc $E[y_{t+h} \mid y_t]$?

just regress y_{t+h} on $y_t, y_{t-1}, y_{t-2}, y_{t-3}$ and take the fitted value from that reg.

In fact, Hamilton's eq (21) gives us c directly:

$$c = \hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

So: $Y = y_{t+h}$, $X = \overset{4\times 1}{\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}}$ $\Rightarrow \underset{1\times 4}{\beta} = \underset{1\times 1}{(\underset{1\times 4}{X'}\underset{4\times 1}{X})^{-1}} \underset{1\times 4}{X'}\underset{1\times 1}{Y}$

and $\hat{v}_{t+h} = y_{t+h} - \beta X$

Baxter & King filter (BK filter, bandpass filter)

Def: business cycle = cyclical components of a ts that last between 6 and 32 quarters in duration

Notation: $BP_K(p,q)$ = filter that passes cycles between $q=32$ and $p=6$ cycle length, and is truncated at leads/lags K (→ recommend $K=12$)

↳ Such a filter is a MA(K).

So we're interested in constructing the filtered series

$$y_t^* = \sum_{k=-K}^{K} a_k \, y_{t-k} \qquad (1)$$

and the question is how to obtain a_k?

(where the MA is symmetric, i.e. $a_k = a_{-k}$)

Eq. (10) gives the answer:

$$a_h^{BP} = (\bar{b}_h - \underline{b}_h) + (\bar{\theta} - \underline{\theta})$$

where the bars denote things pertaining to low-pass filters that only allow stuff below the frequency $\bar{\omega}$ & $\underline{\omega}$, i.e periodicities $\bar{p}$ & $\underline{p}$,

where $p = \frac{2\pi}{\omega}$

θ is a normalizing constant (coming from truncation)

$$\theta = \left(1 - \sum_{h=-k}^{k} b_h\right) \frac{1}{2k+1}$$

and $b_0 = \frac{\omega}{\pi}$ and $b_h = \frac{\sin(h\underline{\omega})}{h\pi}$ for $h = 1, 2, \ldots$ (7)

$q = 32 \qquad p = 6$

Ok, so we have $\underline{p} = 32$, $\bar{p} = 6$

$\Rightarrow \underline{\omega} = \frac{2\pi}{\underline{p}} = \frac{2 \cdot \pi}{32} = \frac{\pi}{16}$, $\bar{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}$

Autocovariances of VARs (Lütkepohl, p. 21)

For a stationary VAR(1)

$$y_t = v + A_1 y_{t-1} + u_t$$
$$\hookrightarrow N(0, \Sigma_u)$$

define $\Gamma_y(h) \equiv E[(y_t - \mu)(y_{t-h} - \mu)']$

where $\mu := E[y_t]$. Then if you know $\Gamma_y(0)$,

you can compute

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) \quad \text{recursively!} \quad (2.1.31)$$

If you know A_1 and Σ_u, you can compute

$$\operatorname{vec}\Gamma_y(0) = (I_{k^2} - A_1 \otimes A_1)^{-1} \operatorname{vec}\Sigma_u \quad (2.1.32)$$

But I don't want to est a VAR... I just want the empirical auto-covariance matrix of the data and of the simulated data.

So, the GMM of the anchoring fit works 23 March 2020

so far. Issues.

1.) I'm not sure if I should model the raw data w/ a time series. So e.g. if they were a VAR(1) then I could estimate a RF-VAR, I could bootstrap using the residuals and I could estimate the autocov structure as in (2.1.32) & (2.1.31) in Lütkepohl.

This issue shows up for $W = \begin{bmatrix} \hat{\sigma}^2_{acf_1} & & \\ & \ddots & \\ & & \hat{\sigma}^2_{acf_{15}} \end{bmatrix}$ b/c $\hat{\sigma}^2_{acf,i}$ are super small, thus W^{-1} is huge, and therefore the estimation does not move.

2) Filters may not be working?

→ I think now they are!

↳ check bootstrap! ✓ It's fine.

Write up materials22_peter.

Read "Stochastic".