Chapter 13
Difference Equations

Leonardo di Pisa (c. 1170 – c. 1250) Thomas Robert Malthus (1766– 1834)

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13.1 Difference Equations: Definitions

- We start with a time series $\{y_n\}=\{y_1 , y_2 , y_3 , ..., y_{n-1}, y_n\}$
- Difference Equation – Procedure for calculating a term (y_n) from the preceding terms: y_{n-1}, y_{n-2},... A starting value, y_0, is given.
- **Example**: $y_n = f(y_{n-1}, y_{n-2}, ..., y_{n-k})$, given y_0.
- If $f(.)$ is linear, we have a *linear* difference equation. Our focus.

- First-Order Linear Difference Equation Form:

  $y_n = a\, y_{n-1} + b$ (a & b are constants)

- Second-Order Linear Difference Equation Form:

  $y_n = a\, y_{n-1} + b\, y_{n-2} + c$ (a, b & c are constants)

- Similarly, an Kth-Order Linear Difference equation:

  $y_n = a_{n-1}\, y_{n-1} + a_{n-2}\, y_{n-2} + ...+ a_{n-k}\, y_{n-k} + c$ (a_{n-1}, a_{n-2}, ..., & c are constants)

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13.1 Difference Equations: Famous Example

• Originated in India. It has been attributed to Indian writer Pingala (200 BC). In the West, Leonardo of Pisa (Fibonacci) studied it in 1202.

• Fibonacci studied the (unrealistic) growth of a rabbit population.

• Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, ... (each number represents an additional pair of rabbits.

• This series can be represented as a linear difference equation.

• Let f(n) be the rabbit population at the end of month n. Then,

$$f(n) = f(n-1) + f(n-2), \quad \text{with initial values } f(1)=1, f(0)=0.$$

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13.1 Difference Equations: Example 1

• The number of rabbits on a farm increases by 8% per year in addition to the removal of 4 rabbits per year for adoption. The farm starts out with 35 rabbits.

Let y_n be the population after *n* years. We can write the difference equation:

$$y_n = 1.08y_{n-1} - 4; \qquad y_0 = 35$$

Initial Value

Percentage change every year. (*a*)

What you add or subtract every year. (*b*)

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13.1 Difference Equations: Example 1 – A Few Terms

• Generate the first few terms - This gives us a feeling for how successive terms are generated.

• Graph the terms - Plot the points $(0, y_0)$, $(1, y_1)$, $(2, y_2)$, etc.

Example: $y_n = 1.08\, y_{n-1} - 4$, with $y_0 = 35$

a. Generate $y_0, y_1, y_2, y_3, y_4, \ldots$

$y_0 = 35$

$y_1 = 1.08(35) - 4 = 37.8 - 4 = 33.8$

$y_2 = 1.08(33.8) - 4 = 36.50 - 4 = 32.50$

$y_3 = 1.08(32.50) - 4 = 35.1 - 4 = 31.1$

$y_4 = 1.08(31.1) - 4 = 33.59 - 4 = 29.59$

$y_5 = 1.08(29.59) - 4 = 31.96 - 4 = 27.96$

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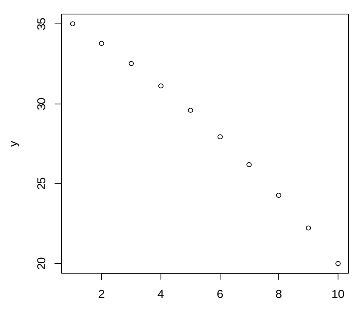
13.1 Difference Equations: Example 1 – R

```
s = 10                      #number of repetitions
> y <- rep(0,10)
> a <- 1.08
> b <- -4
> y[1] = 35                 # initial value
> i=2
> while (i <= reps){
+ y[i] <- a*y[i-1] + b      #generate y
+ i <- i+1
+ }
> y
 [1] 35.00000 33.80000 32.50400 31.10432 29.59267 27.96008 26.19689 24.29264
 [9] 22.23605 20.01493
> plot(y)
```



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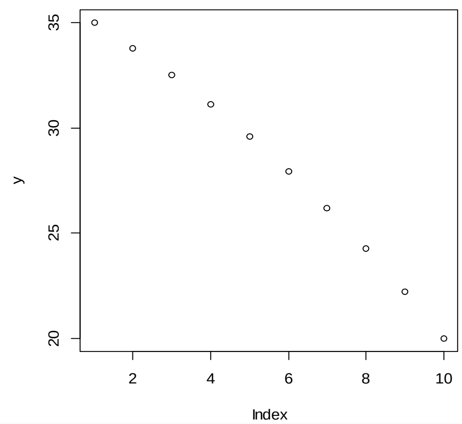
13.1 Difference Equations: Example 1 - Graphing Difference Equations

b. Graph these first few terms

(0, 35) (1, 33.8) (2, 32.5) (3, 31.1) (4, 29.59)



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13.1 Difference Equations: Example 2

- $y_n = 0.5\, y_{n-1} - 1,\; y_0 = 10$

a. Generate y_0, y_1, y_2, y_3, y_4

$y_0 = 10$

$y_1 = 0.5\,(10) - 1 = 5 - 1 = 4$

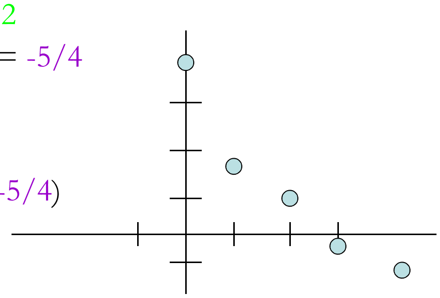
$y_2 = 0.5\,(4) - 1 = 2 - 1 = 1$

$y_3 = 0.5\,(1) - 1 = 0.5 - 1 = -1/2$

$y_4 = 0.5\,(-1/2) - 1 = -0.25 - 1 = -5/4$

b. Graph these first few terms

(0, 10) (1, 4) (2, 1) (3, -1/2) (4, -5/4)



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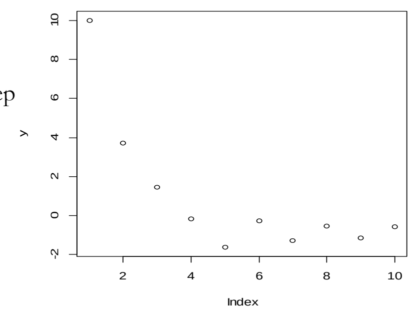
13.1 Difference Equations: Example 3 (in R)

• In economics we think of data as realizations of random variables. We modify Example 2 by introducing a random error term, ε. That is, in time series terminology, we have an autoregressive model, an AR(1):

$$y_n = 0.5\, y_{n-1} - 1 + \varepsilon_n, \qquad \varepsilon_n \sim N(0,1)$$

• We generate the first 10 terms and graph them:

```
> reps=10                  #number of repetitions
> y <- rep(0,10)
> a <-.5; b <- -1;
> ep <- rnorm(10,0,1)      #generate errors, ep
> y[1] = 10
> i=2
> while (i <= reps){
+ y[i] <- a*y[i-1] + b + ep[i]   #generate y
+ i <- i+1 }
> y
[1] 10.0000 3.71451 1.45935 -0.16354 -1.61239 -0.26786 -1.2914981 -0.53511 -1.13426 -0.5712
```



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13.1 Difference Equations: More Examples

Example: The population of a country is currently 70 million, but is declining at the rate of 1% per year. Let y_n be the population after *n* years. Difference equation showing how to compute y_n from y_{n-1}:

$$y_n = .99\, y_{n-1}, \quad \text{with } y_0 = 70{,}000{,}000 \text{ (initial value)}$$

Example: We borrow $150,000 at 6% APR compounded monthly for 30 years to purchase a home. The monthly payment is determined to be $899.33. The difference equation for the loan balance (y_n) after each monthly payment has been made:

$$y_n = 1.005\, y_{n-1} - 899.33, \qquad \text{with } y_0 = 150{,}000$$

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13.1 Difference Equations: Jokes

- **Order of Fibonaccos**

  Customer: "How much is a large order of Fibonaccos?"
  Cashier: "It's the price of a small order plus the price of a medium order."

- **Exponential Growth**

  I have been dabbling with mathematics for many years. As a matter of fact, the first time I became quite annoyed with math was the day I turned 2 (that's how far back I go with number crunching). For you see, the day I turned 2 I realized that in one year my age doubled, which led me to conclude that by the time I was 7 I'd really be 64!!!

13.1 Difference Equations: The Steady State

• The *steady state* or *long-run* value represents an equilibrium, where there is no more change in y_n. We call this value y_∞ :

$$y_n = ay_{n-1} + b \quad \Rightarrow \; y_\infty = \frac{b}{1-a}; \quad a \neq 1.$$

- **Example 1**: $y_n = 1.08y_{n-1} - 4$,

  $\Rightarrow y_\infty = b/(1 - a) = -4/(1 - 1.08) = 50$

  Check: $y_n = 1.08\ (50) - 4 = 50$

- **Example 2**: $y_n = 0.5\ y_{n-1} - 1$,

  $\Rightarrow y_\infty = b/(1 - a) = -1/(1 - 0.5) = -2$

  Check: $y_n = 0.5\ (-2) - 1 = -2$

13.2 Solving Difference Eq's – Repeated Iteration

- We want to generate a formula from which we can directly calculate *any* term without first having to calculate all the terms preceding it.

- Repeated Iteration Method (*Backward* Solution):

$$\begin{aligned} y_n &= ay_{n-1} + b = a(ay_{n-2} + b) + b = a^2 y_{n-2} + ab + b = \\ &= a^2(ay_{n-3} + b) + ab + b = a^3 y_{n-3} + a^2 b + ab + b = \\ &= a^n y_0 + a^{n-1} b + a^{n-2} b + \ldots + ab + b \\ &= a^n y_0 + \left(\frac{1 - a^n}{1 - a}\right) b; \qquad a \neq 1 \end{aligned}$$

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13.2 Solving Difference Eq's – Repeated Iteration

- *Solution:* $y_n = a^n y_0 + \left(\frac{1 - a^n}{1 - a}\right) b; \qquad a \neq 1$

  $or \quad y_n = a^n y_0 + (1 - a^n) y_\infty .$

- The steady state is:

$$y_\infty = \lim_{n \to \infty} y_n = \lim_{n \to \infty} a^n y_0 + \lim_{n \to \infty} \left(\frac{1 - a^n}{1 - a}\right) b$$

- We have 3 cases:
  - a) If $|a| < 1$ $\Rightarrow y_\infty = b/(1-a)$ = finite; y_n converges
  - b) If $|a| > 1$ $\Rightarrow y_\infty$ indefinite; y_n diverges
  - c) If $|a| = 1$ $\Rightarrow y_\infty$ indefinite; y_n diverges

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13.2 Solving Difference Eq's – Forward Solution

- Solve for y_{n-1} in $y_n = a\,y_{n-1} + b$ $\Rightarrow y_{n-1} = (1/a)\,y_n - b/a$.
- Or $y_n = (1/a)\,y_{n+1} - b/a$.

$$y_n = \frac{1}{a}y_{n+1} - \frac{b}{a} = \frac{1}{a}(\frac{1}{a}y_{n+2} - \frac{b}{a}) - \frac{b}{a} = (\frac{1}{a})^2 y_{n+2} - [(\frac{1}{a})^2 b + \frac{1}{a}b] =$$

$$= (\frac{1}{a})^t y_{n+t} - [(\frac{1}{a})^t b + (\frac{1}{a})^{t-1} b + \ldots + (\frac{1}{a})^2 b + \frac{b}{a}]$$

$$= \theta^t y_{n+t} - \left(\frac{1-\theta^{t+1}}{1-\theta}\right) b; \qquad \text{where } \theta = \frac{1}{a};\ a \neq 1$$

- If $|\theta| = |(1/a)| < 1$ $\Rightarrow y_n$ converges $(|a|>1)$
- When $|a|>1$, equation is divergent, the forward solution works.

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13.2 Solving Difference Eq's – General Solution

- Steps:
  1) Get a solution to the homogenous equation (b=0)
  2) Get a particular solution, for example y_∞
  3) General solution: Add both solutions
- **Step 1**) Homogenous equation: $y_n = a\,y_{n-1}$,
  - Guess a solution: $y_n = A\,k^n$,
  - Check the guessed solution: $y_n = A\,k^n$

$$= a\,y_{n-1} = a\,(A\,k^{n-1}) \Rightarrow a = k$$

$$= A\,a^n$$

- **Step 2**) Particular solution: $y_\infty = b/(1-a)$, $a \neq 1$
- **Step 3**) *General Solution*: $y_n = Aa^n + y_\infty = Aa^n + \dfrac{b}{1-a}$

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13.2 Solving Difference Eq's – General Solution

- **Step 3**) *General Solution*: $y_n = Aa^n + y_\infty = Aa^n + \frac{b}{1-a}$

- We can determine A, if we have some values for y_t. Say y_0.

$$y_0 = Aa^0 + y_\infty = A + \frac{b}{1-a} \qquad \Rightarrow A = y_0 - y_\infty = y_0 - \frac{b}{1-a}$$

- We replace A in the general solution to get a *definite solution*, with no unknown values:

$$y_n = (y_0 - \frac{b}{1-a})a^n + \frac{b}{1-a} \qquad (\textit{definite}\ \text{solution})$$

which is just the backward solution!

$$y_n = (y_0 - \frac{b}{1-a})a^n + \frac{b}{1-a} = a^n y_0 + (1-a^n)y_\infty$$

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13.2 Solving Difference Eq's – General Solution

- **Example**: Solve the difference equation:

$$y_n = 0.5\, y_{n-1} - 1, \qquad y_0 = 10$$

Steady state: $y_\infty = b/(1\text{-}a) = -1/.5 = -2$

Solution:
$$y_n = y_\infty + (y_o - y_\infty)a^n$$
$$= -2 + (10-(-2))(.5)^n = -2 + 12(.5)^n$$

Q: What is the value of y at n=10?

$$y_{n=10} = -2 + 12(.5)^{10} = \text{-}1.988281$$

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13.2 Special Case - *a*=1 ("Random Walk")

- In the difference equation $y_n = a\, y_{n-1} + b$, let $a = 1$

  $\Rightarrow y_n = y_{n-1} + b$

- *Solution* (Repeated Iteration): $y_n = y_0 + b\, n$

  There is only a change in b (constant change per period).

- **Example**: Solve $y_n = y_{n-1} + 5$, with $y_0 = 10$.

  Solution: $y_n = 10 + 5\, n$

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13.2 Simple Financial Difference Equations

- Simple Interest: $y_n = y_{n-1} + (y_0\, i)$
- Compound Interest: $y_n = (1 + i)\, y_{n-1}$
- Increasing Annuities: $y_n = (1 + i)\, y_{n-1} + b$ *(PMT)*
- Decreasing Annuities: $y_n = (1 + i)\, y_{n-1} - b$ *(PMT)*
- Loans: $y_n = (1 + i)\, y_{n-1} - b$ *(PMT)*

- Compound Interest *Solution:* $y_n = y_0\, (1 + i)^n$

  This equation is the same as $FV = PV * (1 + i)^n$

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13.3 Graphing Difference Eq's: Definitions

- Vertical Direction – The up-and-down motion of successive terms.
  - *Monotonic:* The graph heads in one direction (up-increasing, down-decreasing)
  - *Oscillating:* The graph changes direction with every term.
  - *Constant:* The graph always remains at the same height.

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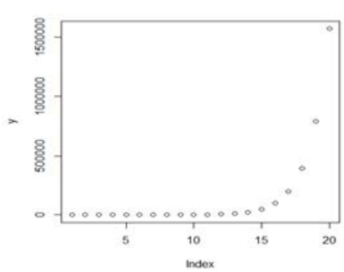
13.3 Graphing Difference Eq's: Vertical Direction

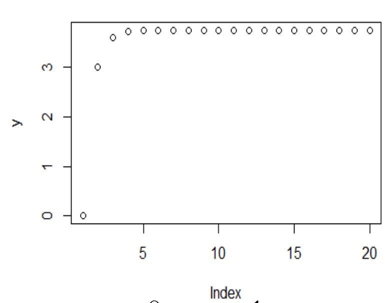
- *Monotonic:* The graph heads in one direction (up – increasing, down – decreasing). The constant a is positive ($a > 0$).
- **Example**:

$y_n = 2\, y_{n-1} + 3,\;\; y_0 = 0$

$y_n = 0.2\, y_{n-1} + 3,\;\; y_0 = 10$



$a > 1$

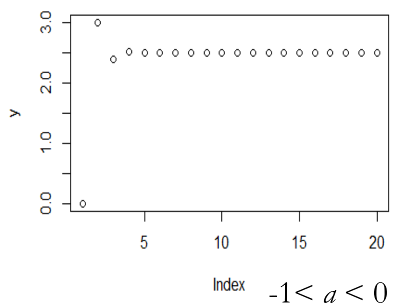


$0 < a < 1$

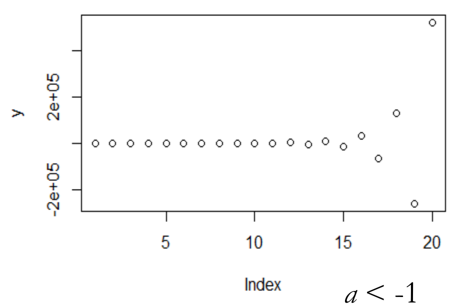
13.3 Graphing Difference Eq's: Vertical Direction

- *Oscillating*: The graph changes direction with every term. The constant a is negative ($a < 0$).
- **Example**:

$y_n = -0.2\, y_{n-1} + 3, \;\; y_0 = 0$ $\qquad$ $y_n = -2\, y_{n-1} + 3, \;\; y_0 = 0$



$-1 < a < 0$



$a < -1$

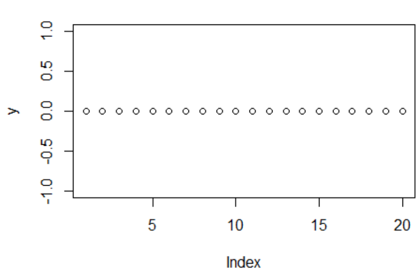
13.3 Graphing Difference Eq's: Vertical Direction

- *Constant:* The graph always remains at the same height

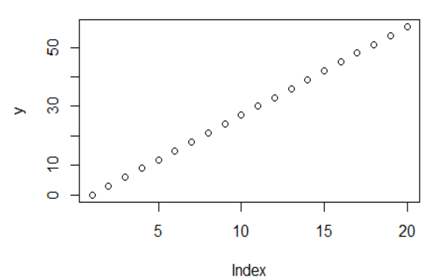
$\Rightarrow y_n = y_\infty$ (a variation, constant trend)

- **Example**:

$y_n = y_{n-1} + 0, \;\; y_0 = 0$ $\qquad$ $y_n = y_{n-1} + 3, \;\; y_0 = 0$



$a = 1;\, b = 0$



$a = 1;\, b = 3$

13.3 Graphing Difference Eq's: Definitions 2

- Long-run Behavior – The eventual behavior of the graph.
  - *Attracted or Stable:* The graph approaches a horizontal line (asymptotic or attracted to the line).
  - *Repelled or Unstable:* The graph goes infinitely high or infinitely low (unbounded or repelled from the line).

- In general, we say a system is *stable* if its long-run behavior is not sensitive to the initial conditions. Some "unstable" system maybe "stable" by chance: when $y_0=y_\infty$.

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13.3 Graphing Difference Eq's: Long-run

Attracted (Stable)

Example: $y_n = 0.5y_{n-1}+300$, $y_0 = 400$

monotonic, increasing, stable

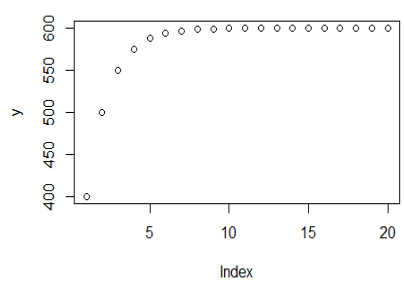
$|a|<1$; $y_0 < b/(1 - a)$

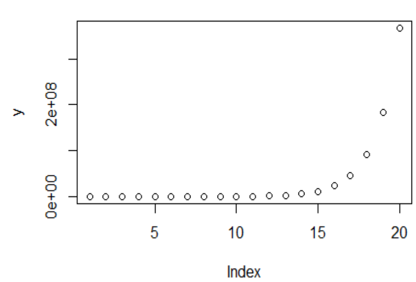
Repelled (Unstable)

$y_n = 2y_{n-1}+300$, $y_0 = 300$

monotonic, increasing, unstable

$|a|>1$; $y_0 > b/(1 - a)$





13.3 Graphing Difference Eq's: Long-run

Summary:
- $|a| > 1$ unstable or unbounded –repelled from line $[b/(1-a)]$
- $|a| < 1$ stable or bounded –attracted or convergent to $[b/(1-a)]$
- $a < 0$ oscillatory
- $a > 0$ monotonic
- $a = -1$ bounded oscillatory
- $a = 1$, $b = 0$ constant
- $a = 1$, $b > 0$ constant increasing
- $a = 1$, $b < 0$ constant decreasing

<u>Note</u>: All of this can be deduced from the solution:

$$y_n = (y_0 - \frac{b}{1-a})a^n + \frac{b}{1-a}$$

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Figure 13.1 Stable Difference Equations (13.2) and (13.3)

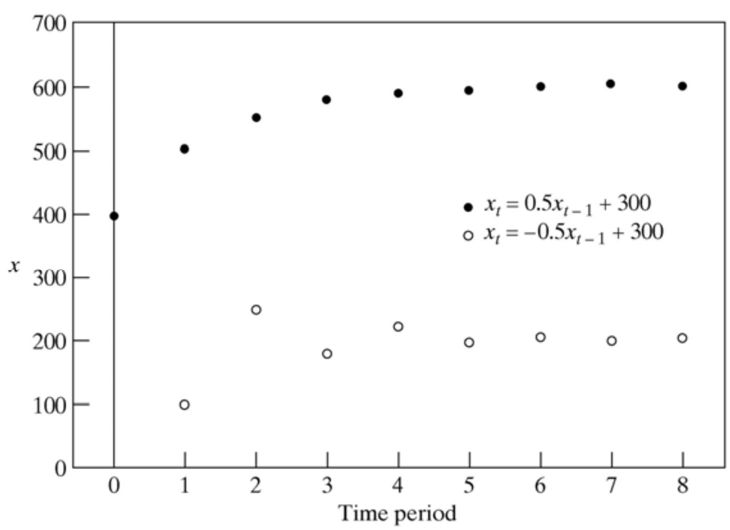

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Figure 13.2 Unstable Difference Equations (13.5) and (13.6)

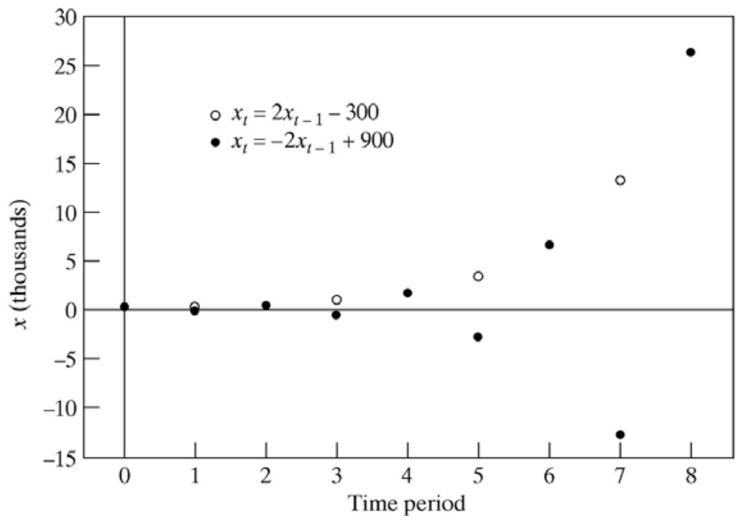

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Figure 13.3 Phase Diagram for Equation (13.2)

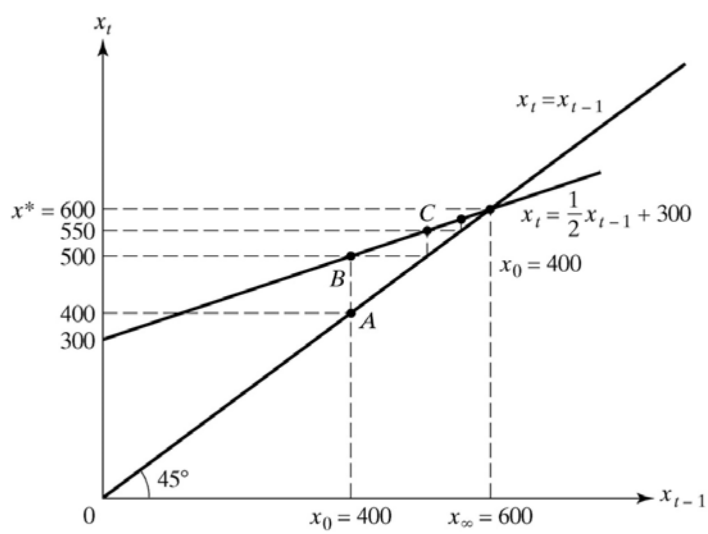

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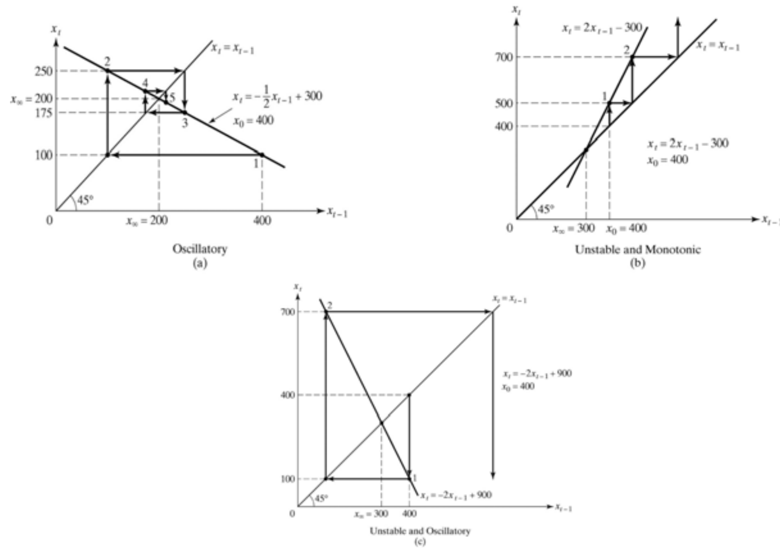
Figure 13.4 Phase Diagrams for Difference Equations (13.3), (13.5), and (13.6)



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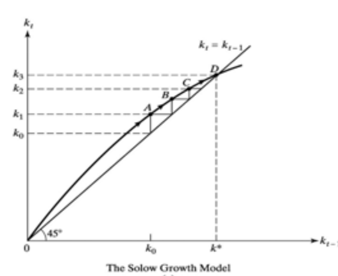
13.3 Difference Equations: Application 1

- *Solow's Growth Model*
- k_t: capital per capita (K/L)
- y_t: income/production per capita: $f(k_t) = A\,(k_t)^{\alpha}$
- δ: depreciation
- i_t: investment per capita: capital accumulation: $k_t - (1-\delta)\,k_{t-1}$
- s_t: savings per capita: $\sigma\, f(k_t)$ (σ: propensity to save)
- Equilibrium condition: $s_t = i_t$ $\Rightarrow k_t - (1-\delta)\,k_{t-1} = \sigma\, f(k_t)$
- Difference equation: $k_t - \sigma\, f(k_t) = (1-\delta)\,k_{t-1}$



13.3 Difference Equations: Application 2

• *Half-life PPP*

Half-life: how long it takes for the initial deviation from y_0 and y_∞ to be cut in half.

- r_t: real exchange rate (= $S_t P_d/P_f$)
- r_t follows an AR(1) process: $r_t = a\, r_{t-1} + b$
- $r_H = (r_0 + r_\infty)/2$

• Recall solution to r_t: $r_t = a^t r_0 + (1 - a^t) r_\infty;\qquad r_\infty = \dfrac{b}{1-a}; a \neq 1$

$r_H = a^H r_0 + (1 - a^H)\, r_\infty$ $\Rightarrow (r_0 + r_\infty)/2 = a^H r_0 + (1 - a^H)\, r_\infty$

$\Rightarrow (1 - 2a^H)\, r_0 = (1 - 2a^H)\, r_\infty$

$\Rightarrow 1 - 2a^H = 0 \qquad 1 = 2a^H$

$\Rightarrow H = -\ln(2)/\ln(a)$

• Interesting cases: If $a = 0.9 \Rightarrow$ H=- ln(2)/ln(0.9) =6.5763

If $a = 0.95 \Rightarrow$ H=- ln(2)/ln(0.95) =13.5135

If $a = 0.99 \Rightarrow$ H=- ln(2)/ln(0.95) =68.9675

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13.4 2nd-Order Difference Equations: Example

- We want a *general solution* to $y_n = a_1\, y_{n-1} + a_2\, y_{n-2} + c$
- Steps:
  1) Guess a solution to the homogenous equation (c=0)
  2) Get a particular solution, for example y_∞
  3) General solution: Add both solutions
- To get a definite solution –i.e., with no unknowns-, we need initial values.

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13.4 2nd-Order Difference Equations: Example

- **Step 1**: Homogenous equation: $y_n = a_1\, y_{n-1} + a_2\, y_{n-2}$
  Guess a solution: $y_n = k^n$
  – Check the guessed solution: $k^n = a_1\, k^{n-1} + a_2\, k^{n-2}$
  $\Rightarrow (k^2 - a_1\, k^1 - a_2)\, k^{n-2} = 0$ (quadratic equation)
  $k_1, k_2 = ½\, (a_1 \pm [a_1^2 + 4\, a_2]^{1/2})$

  - 3 cases: $a_1^2 + 4\, a_2 > 0$ $\Rightarrow k_1, k_2$ are real and distinct.
  $a_1^2 + 4\, a_2 = 0$ $\Rightarrow k_1 = k_2$ real and repeated.
  $a_1^2 + 4\, a_2 < 0$ $\Rightarrow k_1, k_2$ are complex and distinct.

Note: Similar to the 1st-order case, the stability of the equation depends on the roots, k_1 & k_2.

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13.4 2nd-Order Difference Equations: Example

- **Case 1**: If $a_1^2 + 4\, a_2 > 0$ $\Rightarrow k_1, k_2$ are real and distinct.
The general solution of the homogeneous equation is:
$A\, k_1^t + B\, k_2^t$, where k_1 and k_2 are the two roots.

Stability: If $|k_1| > 1$ or $|k_2| > 1$, the equation is divergent.

- **Case 2**: If $a_1^2 + 4\, a_2 = 0$ $\Rightarrow k_1 = k_2$ real and repeated.
The general solution of the homogeneous equation is
$(A + Bt)\, k^t$, where $k = -(1/2)\, a_1$ is the root.

Stability: If $|k| > 1$.

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13.4 2nd-Order Difference Equations: Example

• **Case 3**: If $a_1^2 + 4\,a_2 < 0$ $\Rightarrow k_1, k_2$ are complex and distinct.
The general solution of the homogeneous equation is

$$A r^t \cos(\theta t + \omega),$$

where A and ω are constants, $r = \sqrt{-a_2}$, and $\cos\theta = -a_1/(2\sqrt{-a_2})$,

Alternatively: $C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t)$,
where $C_1 = A\cos\omega$
$C_2 = -A\sin\omega$
(using the formula that $\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$.

Stability: If $|r|>1$, the equation is divergent.

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13.4 2nd-Order Difference Equations: Examples

Example 1: $x_{t+2} + x_{t+1} - 2x_t = 0$.
k_1, k_2: 1, −2 (real and distinct). The solution is: $A\,k_1^t + B\,k_2^t$.

$$\Rightarrow x_t = A\,(1)^t + B(-2)^t = A + B(-2)^t.$$

Example 2: $x_{t+2} + 6x_{t+1} + 9x_t = 0$.
k_1, k_2: −3 (real and repeated). The solution is: $(A + Bt)\,k^t$

$$\Rightarrow x_t = (A + Bt)(-3)^t.$$

Example 3: $x_{t+2} - x_{t+1} + x_t = 0$.
k_1, k_2: complex, with $r = 1$ & $\cos\theta = 1/2$, so $\theta = (1/3)\pi$. The solution is: $A r^t \cos(\theta t + \omega)$

$$\Rightarrow x_t = A\cos((1/3)\pi t + \omega).$$

The frequency is $(\pi/3)/2\pi = 1/6$ and the growth factor is 1, so the oscillations are undamped.

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13.4 2nd-Order Difference Equations: Example

- **Step 2**: Get a particular solution, for example, y_∞
- **Step 3**: General Solution: Add homogeneous solution to particular solution.

Example: $y_t = -6y_{t-1} - 9y_{t-2} + 16$.

Solution to homogeneous equation: $y_t = (A + Bt)(-3)^t$.

Particular solution: $y_\infty = 16/(1+6+9) = 1$

Solution: $y_t = (A + Bt)(-3)^t + 1$

Note: If we have y_0 and y_1, we can solve for A and B.

Say: $y_0 = 1$ and $y_1 = 2$

$y_0 = 1 = (A + B\,0)(-3)^0 + 1 = A + 1 \quad \Rightarrow A=0$

$y_1 = 2 = (A + B\,1)(-3)^1 + 1 = -3\text{x}0 - 3B + 1 \quad \Rightarrow B=-1/3$

Definite Solution: $y_t = (-1/3t)(-3)^t + 1$

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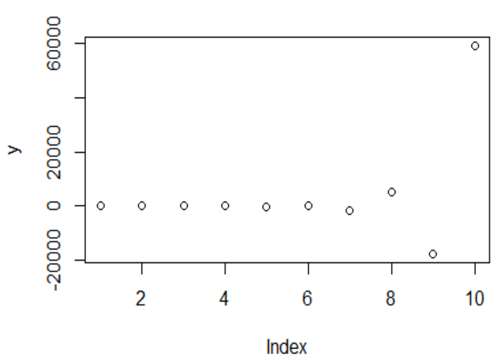
13.4 2nd-Order Difference Equations: Example

- In R

```
> reps=10 #number of repetitio
> y <- rep(0,10)
> a1 <- -6
> a2 <- -9
> b <- 16
> y[1] = 1
> y[2] = 2
> i=3
> while (i <= reps){
+ y[i] <- a1*y[i-1] + a2*y[i-2] + b
+ i <- i+1
+ }
> y
 1    2   -5    28  -107    406  -1457   5104 -17495  59050
```



Note: Explosive series.

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13.5 System of Equations: VAR(1)

• Now, we have a system

$$y_t = a\, y_{t-1} + b\, x_{t-1} + m$$
$$x_t = c\, y_{t-1} + d\, x_{t-1} + n$$

• Let's rewrite the system using linear algebra. We have a vector autoregressive model with one lag, or VAR(1):

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix} = \mathbf{A} z_{t-1} + \kappa$$

• Let's introduce the lag operator, L: $L^q\, y_t = y_{t-q}$
Then, $L\, y_t = y_{t-1}$.

Now we can write: $\boldsymbol{z}_t = \mathbf{A}L\, \boldsymbol{z}_t + \boldsymbol{\kappa}$ $\Rightarrow (\mathbf{I} - \mathbf{A}L)\, \boldsymbol{z}_t = \boldsymbol{\kappa}$

Assuming $(\mathbf{I} - \mathbf{A})$ is non-singular $\Rightarrow \boldsymbol{z}_\infty = (\mathbf{I} - \mathbf{A})^{-1}\, \boldsymbol{\kappa}$

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13.5 System of Equations: VAR(1)

• $\boldsymbol{z}_\infty = (\mathbf{I} - \mathbf{A})^{-1}\, \boldsymbol{\kappa}$ is the long-run solution to the system.

• The dynamics of the VAR(1) depend on the properties $\mathbf{A}$, which can be understood from the eigenvalues.

• Diagonalizing the system:

$$\mathbf{H}^{-1} \boldsymbol{z}_t = \mathbf{H}^{-1} \mathbf{A}\, (\mathbf{H}\, \mathbf{H}^{-1})\, \boldsymbol{z}_{t-1} + \mathbf{H}^{-1} \boldsymbol{\kappa}$$
$$\mathbf{H}^{-1} \mathbf{A}\, \mathbf{H} = \boldsymbol{\Lambda}$$
$$\mathbf{H}^{-1} \boldsymbol{\kappa} = \boldsymbol{s}$$
$$\mathbf{H}^{-1} \boldsymbol{z}_t = \boldsymbol{u}_t \qquad (\text{or } \boldsymbol{z}_t = \mathbf{H}\boldsymbol{u}_t)$$

Each $\boldsymbol{z}_t$ is a linear combination of the $\boldsymbol{u}$'s.

• Now, $\boldsymbol{u}_t = \boldsymbol{\Lambda}\, \boldsymbol{u}_{t-1} + \boldsymbol{s}$

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13.5 System of Equations: VAR(1)

• Diagonalized system: $\boldsymbol{u}_t = \boldsymbol{\Lambda}\ \boldsymbol{u}_{t-1} + \boldsymbol{s}$

$$u_t = \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 u_{1,t-1} + s_1 \\ \lambda_2 u_{2,t-1} + s_2 \end{bmatrix}$$

• To solve the system, we need to solve the eigenvalue equation:

$\lambda^2 - (a + d)\ \lambda + (ad - cb) = 0 \qquad (\lambda^2 - tr(\mathbf{A})\ \lambda + |\mathbf{A}| = 0)$

• Stability:

- $|\lambda_1|,|\lambda_2| < 1$ ⇒ Stable system ("stationary," or $\boldsymbol{z}_t$ is I(0)).
- $|\lambda_i| > 1$ ⇒ Unstable system ("explosive"). Not typical of macro/finance time series.
- $|\lambda_i| = 1$ ⇒ Unit root system. Common in macro/finance time series.

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13.5 System of Equations: VAR(1) - Example

• Now, we have a system

$y_t = 4\ y_{t-1} + 5\ x_{t-1} + 2$

$x_t = 5\ y_{t-1} + 4\ x_{t-1} + 4$

• Let's rewrite the system using linear algebra:

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

• Eigenvalue equation: $\lambda^2 - 8\ \lambda - 9 = 0 \qquad \Rightarrow \lambda_1, \lambda_2 = (9,-1)$

• Transformed univariate equations:

$u_{1,t} = 9\ u_{1,t-1} + s_1$ (unstable equation)

$u_{2,t} = -1\ u_{2,t-1} + s_2$ (unstable equation)

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13.5 System of Equations: VAR(1) - Example

• Two eigenvalues: $\lambda_1, \lambda_2 = (9,-1)$

• Transformed univariate equations:

$u_{1,t} = 9\, u_{1,t-1} + s_1$ (unstable equation)

$u_{2,t} = -1\, u_{2,t-1} + s_2$ (unstable equation)

• Recall solution for linear first-order equation:

$$y_n = a^t y_0 + \left(\frac{1-a^t}{1-a}\right)b; \qquad a \neq 1$$

• *Solution for transformed univariate equations:*

$u_{1,t} = 9^t\, u_{1,0} + (1-9^t)/(-8)\, s_1$

$u_{2,t} = (-1)^t\, u_{2,0} + (1-(-1)^t)/(2)\, s_2$

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13.5 System of Equations: VAR(1) - Example

- Use the eigenvector matrix, **H**, to transform the system back. (1) From $\mathbf{H}^{-1}\boldsymbol{\kappa} = \boldsymbol{s}$, get the values for s_1 and s_2:

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad H^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}(-1/2)$$

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = H^{-1}\kappa = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

- Plug these values into $u_{1,t}$ and $u_{2,t}$:

$u_{1,t} = 9^t\, u_{1,0} + (1-9^t)/(-8)\, s_1 = 9^t\, u_{1,0} - 3\,(1-9^t)/8$

$u_{2,t} = (-1)^t\, u_{2,0} + (1-(-1)^t)/(2)\, s_2 = (-1)^t\, u_{2,0} - 1\,(1-(-1)^t)/2$

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13.5 System of Equations: VAR(1) - Example

(2) From $\mathbf{H}^{-1}\, \boldsymbol{z}_t = \boldsymbol{u}_t$, get the solution in terms of $\mathbf{z}_t$ –i.e., x_t and y_t:

$$z_t = Hu_t = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} u_{1,t} + u_{2,t} \\ u_{1,t} - u_{2,t} \end{bmatrix}$$

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} [9^t u_{1,0} - 3\frac{1-9^t}{8}] + [(-1)^t u_{2,0} - \frac{1-(-1)^t}{2}] \\ [9^t u_{1,0} - 3\frac{1-9^t}{8}] - [(-1)^t u_{2,0} - \frac{1-(-1)^t}{2}] \end{bmatrix}$$

• If we are given y_0 and x_0, we can solve for $u_{1,0}$ and $u_{2,0}$ (2x2 system):

$y_0 = u_{1,0} + u_{2,0}$

$x_0 = u_{1,0} - u_{2,0}$

$\Rightarrow$

$u_{1,0} = (x_0 + y_0)/2$

$u_{2,0} = (y_0 - x_0)/2$

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13.5 System of Equations: VAR(1) - Cointegration

• Q: Suppose we have a *unit root* system, with $\lambda_1 = 1$ and $|\lambda_2| < 1$. Can we have *cointegration*? That is, is there a linear combination of $\boldsymbol{z}_t$'s that is "stationary" (stable)?

Consider $u_t = H^{-1} z_t = \begin{bmatrix} h^*_{11} & h^*_{12} \\ h^*_{21} & h^*_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} \Rightarrow \begin{bmatrix} u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} = \begin{bmatrix} h^*_{11} y_{t-1} + h^*_{12} x_{t-1} \\ h^*_{21} y_{t-1} + h^*_{22} x_{t-1} \end{bmatrix}$

• We know that $u_{2,t}$ is stable, we call $[h^*_{21}\ h^*_{22}]$ a *cointegrating (CI) vector*.

• Let's subtract $\boldsymbol{z}_{t-1}$ from $\boldsymbol{z}_t = \mathbf{A}\mathrm{L}\, \boldsymbol{z}_t + \boldsymbol{\kappa}$:

$$\boldsymbol{z}_t - \boldsymbol{z}_{t-1} = \Delta \boldsymbol{z}_t = (\mathbf{I} - \mathrm{L})\, \boldsymbol{z}_t = \boldsymbol{\kappa} - (\mathbf{I} - \mathbf{A})\, \boldsymbol{z}_{t-1} = \boldsymbol{\kappa} - \boldsymbol{\Pi}\, \boldsymbol{z}_{t-1}$$

The eigenvalues of $\boldsymbol{\Pi}$ are the complements of the λ's from $\mathbf{A}$: $\mu_i = 1-\lambda_i$; then $\mu_1 = 0$ & $\mu_2 = 1 - \lambda_2$. $\Rightarrow \boldsymbol{\Pi}$ is singular with rank 1!

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13.5 System of Equations: VAR(1) - Cointegration

• We decompose Π:

$$\Pi = (\mathbf{I} - \mathbf{A}) = \mathbf{H}\,\mathbf{H}^{-1} - \mathbf{H}\,\boldsymbol{\Lambda}\,\mathbf{H}^{-1} = \mathbf{H}(\mathbf{I} - \boldsymbol{\Lambda})\,\mathbf{H}^{-1}$$

Or

$$\Pi = H\begin{bmatrix} 0 & 0 \\ 0 & 1-\lambda_2 \end{bmatrix} H^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1-\lambda_2 \end{bmatrix}\begin{bmatrix} h^*_{11} & h^*_{12} \\ h^*_{21} & h^*_{22} \end{bmatrix} = \begin{bmatrix} 0 & h_{12}(1-\lambda_2) \\ 0 & h_{22}(1-\lambda_2) \end{bmatrix}\begin{bmatrix} h^*_{11} & h^*_{12} \\ h^*_{21} & h^*_{22} \end{bmatrix}$$

$$= \begin{bmatrix} h_{12}(1-\lambda_2)h^*_{21} & h_{12}(1-\lambda_2)h^*_{22} \\ h_{22}(1-\lambda_2)h^*_{21} & h_{22}(1-\lambda_2)h^*_{22} \end{bmatrix} = \begin{bmatrix} h_{12}(1-\lambda_2) \\ h_{22}(1-\lambda_2) \end{bmatrix}\begin{bmatrix} h^*_{21} & h^*_{22} \end{bmatrix} = \alpha\beta'$$

• Π is factorized into the product of a row vector and a column vector, called an outer product:

- The row vector: β = the CI vector.

- The column vector: α = the loading matrix = the weights with which the CI vector enters into each equation of the VAR.

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13.5 System of Equations: VAR(1) - Cointegration

• Replacing Π into $\boldsymbol{z}_t - \boldsymbol{z}_{t-1} = \Delta\boldsymbol{z}_t = \boldsymbol{\kappa} - \Pi\,\boldsymbol{z}_{t-1}$:

$$\begin{bmatrix} \Delta y_{1,t} \\ \Delta y_{2,t} \end{bmatrix} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} - \begin{bmatrix} h_{12}(1-\lambda_2)h^*_{21} & h_{12}(1-\lambda_2)h^*_{22} \\ h_{22}(1-\lambda_2)h^*_{21} & h_{22}(1-\lambda_2)h^*_{22} \end{bmatrix}\begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} =$$

$$= \begin{bmatrix} \kappa_1 - h_{12}(1-\lambda_2)(h^*_{21}y_{1,t-1} + h^*_{22}y_{2,t-1}) \\ \kappa_2 - h_{22}(1-\lambda_2)(h^*_{21}y_{1,t-1} + h^*_{22}y_{2,t-1}) \end{bmatrix} = \begin{bmatrix} \kappa_1 - h_{12}(1-\lambda_2)u_{2,t-1} \\ \kappa_2 - h_{22}(1-\lambda_2)u_{2,t-1} \end{bmatrix}$$

• All variables here are stationary: Δy's and $u_{2,t}$. This reformulation is called the *vector error correction model of the VAR* (or VECM).

• $u_{2,t}$ is the *error correction term*. It measures the extent to which y's deviate from their equilibrium long-run value.

<u>Note</u>: If $\lambda_1=\lambda_2=1$, we cannot do what we have done above! ($\boldsymbol{z}_t$ is I(2)).

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13.5 System of Equations: CI VAR(1) - Example

• Now, we have a system:

$$y_t = 1.2\, y_{t-1} + 0.2\, x_{t-1} + e_{y,t}$$
$$x_t = 0.6\, y_{t-1} + 0.4\, x_{t-1} + e_{x,t}$$

• We find the eigenvalues of **A**:

$$|A-\lambda I| = \begin{vmatrix} 1.2-\lambda & -0.2 \\ 0.6 & 0.4-\lambda \end{vmatrix} = (1.2-\lambda)(0.4-\lambda)+0.12 = 0 \quad \Rightarrow \lambda_1 = 1; \lambda_2 = 0.6$$

• Eigenvectors are: $H = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}; \quad H^{-1} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$

• We can rewrite the VAR(1) into VECM form:

$$\Delta z_t = \begin{bmatrix} \kappa_1 - h_{12}(1-\lambda_2)(h_{21}^* y_{1,t-1} + h_{22}^* y_{2,t-1}) \\ \kappa_2 - h_{22}(1-\lambda_2)(h_{21}^* y_{1,t-1} + h_{22}^* y_{2,t-1}) \end{bmatrix} = \begin{bmatrix} e_{y,t-1} - 1(0.4)(-0.5 y_{1,t-1} + 0.5 y_{2,t-1}) \\ e_{x,t-1} - 3(0.4)(-0.5 y_{1,t-1} + 0.5 y_{2,t-1}) \end{bmatrix}$$

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13.5 System of Equations: CI VAR(1) - Example

• The VECM:

$$\Delta z_t = \begin{bmatrix} e_{y,t-1} - 0.4(-0.5 y_{1,t-1} + 0.5 y_{2,t-1}) \\ e_{x,t-1} - 1.2(-0.5 y_{1,t-1} + 0.5 y_{2,t-1}) \end{bmatrix}$$

• Then, the CI loading and the CI vector are:

$$\alpha = \begin{bmatrix} -0.4 \\ -1.2 \end{bmatrix}; \qquad \beta = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}.$$

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