ELSEVIER

Journal of Economic Dynamics and Control
19 (1995) 1113–1140

JOURNAL OF
Economic
Dynamics
& Control

Induction and the Ramsey policy

In-Koo Cho*,[a], Akihiko Matsui[b]

[a] *Department of Economics, University of Chicago, Chicago, IL 60637, USA*
[b] *Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA*

(Received September 1993; final version received March 1994)

Abstract

We examine the classic policy problem of Ramsey (1927) modeled as an infinitely repeated game (Stokey, 1991) between the government and the private sector where the private sector consists of a continuum of identical households. We assume that the private sector is boundedly rational in the sense that its behavior is dictated by a forecasting scheme which is 'inductive'. Instead of assigning a particular forecasting scheme to the players, we let each player choose one from a class of 'inductive' forecasting schemes. By focusing on the actions dictated by forecasting schemes selected by the player, we identify the Ramsey policy as the unique solution of the game where players do not discount future payoffs. When players discount future payoffs, the Ramsey policy is the only solution which is robust against small changes in the discount factor in the neighborhood of one (Kalai, Samet, and Stanford, 1988). If we modify the model so that the private sector incurs memory costs in forecasting the action of the government, we obtain that the only equilibrium outcome is to play the Ramsey outcome every period.

Key words: Ramsey policy; Credible policy; Inductive forecasting schemes; Bounded rationality
JEL classification: C73; E61

*Corresponding author.

This paper is a revised version of the paper previously entitled 'Learning and the Ramsey Policy'. We would like to thank Klaus Schmidt and two anonymous referees for comments that improved the content of this paper. Financial support from National Science Foundation, Division of Social Sciences at the University of Chicago, and the Alfred P. Sloan Foundation is gratefully acknowledged.

SSDI 0165188994008 21 X

1. Introduction

The Ramsey policy is the government's scheme that maximizes its objective function if the government behaves as a Stackelberg leader. If the government's objective function is the aggregate utility of the private sector, then it is the Ramsey policy that maximizes social welfare subject to the constraint that the private sector responds optimally to the government's policy. Once the private sector believes that the government will implement the Ramsey policy and responds accordingly, however, the government may have an incentive to deviate from the Ramsey policy, since the Ramsey policy need not be an optimal response to the private sector's action. In this sense, the Ramsey policy may be 'time-inconsistent' (Kydland and Prescott, 1977) unless the government has a credible technology of commitment to the Ramsey policy.

Chari and Kehoe (1990) and Stokey (1991) demonstrate that a government without a credible commitment technology may have an incentive to implement the Ramsey policy if the game is repeated infinitely many times, and if the players are sufficiently patient. By exploiting the results from infinitely repeated games, we can construct a strategy where any deviation by the government from the Ramsey policy will trigger punishment by the private sector, which is severe enough to wipe out any gain from one time deviation. The expected loss from future punishment forces the government to stay with the Ramsey policy, even though the government is fully aware of the short-run gain from deviation. If the government has an incentive to follow its announcement at the beginning of the game, we call the policy a *credible* policy (Stokey, 1991). An important drawback of this approach is the multiplicity of credible policies. By applying the same method, we can construct a plethora of credible policies. In particular, any policy that generates an average payoff higher than a Nash equilibrium of the component game but lower than the Ramsey outcome payoff to the government can be made credible. If the government's objective is to maximize social welfare, then the Ramsey policy is the most plausible credible policy. In order to support our intuition that the Ramsey policy should be implemented by the government, we need a theory that selects the Ramsey outcome as the only solution to the model.

This paper follows the tradition of learning models that have been extensively applied to rational expectations models of competitive economy (e.g., Lucas, 1986; Bray and Kreps, 1987; Marcet and Sargent, 1989). Since a learning rule such as the least squares forecasting scheme is generally intuitive, a rational expectations equilibrium to which a learning dynamics converges has a good reason to be considered as a salient equilibrium.

In these types of models, they restrict the way each player forecasts the unknown parameter values of the economy. Traditionally, simplicity and intuition are the main criteria for selecting a forecasting scheme. Once a scheme is selected, each agent is 'tied' to the forecasting scheme exogenously specified by the modeler, and each player is reduced to a machine programmed to play a certain action in response to its forecast. This is the main reason why it may not be appropriate to apply the conventional learning models to the policy problem where the government can manipulate the outcome of the game in order to take advantage of the learning process of the private sector.

For example, suppose that the private sector is assigned a forecasting scheme which can store only the two most recent moves of the government for generating a forecast. Also, assume that after observing the same action y for two consecutive periods, the forecast becomes y. Given this sort of forecasting scheme, the government has an obvious incentive to mislead the private sector by playing the Ramsey policy in two consecutive periods and then deviating from it for one period in order to maximize its long-run average payoff.

However, the private sector can deduce that the government has such an incentive from the fact that the government knows the forecasting scheme of the private sector, and that it maximizes its long-run payoff. In this case, it is not clear whether the private sector has to 'believe' the forecast because the past observations may be contaminated by the strategic manipulation of the government. The main contribution of this paper is the formalization of a criterion that delineates the class of forecasting schemes that dictate the strategic choices of the agents. The key innovation is that we let players choose their own forecasting schemes instead of assigning specific schemes. Here, the modeler specifies only the *set* of forecasting schemes instead of assigning a specific scheme to the players. A selected forecasting scheme must be optimal against the other players' schemes in the sense that the best response based on the forecast is indeed a best response to the actual actions by the other players.

The present model should be regarded as a game with bounded rationality instead of a learning model. Following the tradition of Rubinstein (1986) and Abreu and Rubinstein (1988), we impose a specific restriction on the set of feasible strategies of the households. In particular, we assume that a representative household's forecast about the future policy of the government is determined by an 'inductive' forecasting rule, instead of perfect foresight, coupled with optimal response. By being inductive, roughly, we mean that the forecasting scheme will generate a forecast y after the agent observes the same y sufficiently many times. Since the private sector consists of a large number of households, an individual household has no strategic power to influence the outcome of the game. Therefore, it is natural to assume that a representative household responds optimally to its belief about how the government will play. In other words, the representative household's belief, and hence behavior, deviates from that under perfect foresight only when the same action has been observed for a sufficient number of periods.

By a game with inductive forecast, we mean a repeated game between the government and the private sector, where each household must choose a repeated game strategy consistent with inductive behavior. However, since a representative household chooses a best response to the forecast, it is much more convenient to treat the class of forecasting schemes as strategies of a representative household. In this way, we can focus on the implication of the restrictions on the forecasting capability of the private sector. For this reason, we will treat a specific inductive forecasting scheme as a strategy of a household, assuming that the house responds optimally to the forecast.

We examine symmetric Nash equilibria of the policy game. If every household uses an inductive forecasting rule, then the government has an obvious incentive to manipulate the opponents by repeating the same action. Therefore, in order to avoid being manipulated, a representative household must carefully select an inductive forecasting scheme. The strategic pressure combined with the restriction to inductive forecasting rule dramatically reduces the set of equilibirum outcomes of the policy game. When the players do not discount future payoff, any forecasting scheme which survives this test must support the Ramsey policy. Though the private sector behaves *as if* its forecasting scheme were 'inductive', it is sophisticated enough to respond to the government's incentive for manipulation. Since the selected forecasting scheme is an optimal choice, the private sector has a reason to behave according to an inductive forecasting rule.

Critical to the analysis is the government's ability to manipulate the belief of the private sector by repeating the same action many times. When the government discounts future payoff, its ability of manipulating beliefs is substantially reduced. In particular, if it takes too many repetition to manipulate the belief of the private sector, then the government may end up with a policy which is inferior to the Ramsey policy. By the same token, the game with inductive schemes and discounting admits plethora of Nash equilibria.

However, the Ramsey policy is the only Nash equilibrium outcome which is robust against the perturbation of discount factor in the neighborhood of unity. Since a forecast requires substantial patience among players, it is reasonable to study the model with a discount factor close to unity. In this case, this result provides at least a partial justification for studying the Ramsey policy.

Our selection of the class of forecasting schemes is arbitrary. However, this exercise can give some sense of why the Ramsey policy scheme should be selected in the repeated game. Our approach is close to that used by Rubinstein (1986) and Abreu and Rubinstein (1988) where each agent is choosing a 'machine' which will dictate the strategic choice in the repeated game. The difference here lies in the way we specify the bounded rationality of each agent. The main consideration of Rubinstein (1986) is the complexity of the machine. Here, we choose the class of inductive forecasting schemes because they are simple and intuitive. But, our 'simple' forecasting schemes may have infinite complexity according to the criterion of Rubinstein (1986). The present paper is related to Matsui (1989) in which one player has an opportunity to commit to his repeated game strategy and reveal it to the opponent. In our model, every player chooses his own learning scheme simultaneously before the repeated game begins.

Consequently, no player has an opportunity to reveal his own forecasting scheme completely, nor to observe the opponent's choice while the game proceeds. For example, the only way for the government to convince other players that it will take a certain action is to repeat that action many times.

The rest of the paper is organized as follows. In order to simplify presentation, we start with the case with no discounting. The next section examines a simple example based upon Chari and Kehoe (1990) to motivate our approach. Section 3 reviews the equilibrium model of the policy problem to lay out the ground work. We choose Stokey (1991) since it provides the most general framework. Section 4 defines the policy game with inductive forecast, restricting the strategy space of the private sector. Section 5 analyzes this game and states the main results. A nonequilibrium approach is also examined. Section 6 examines the case in which agents discount the future. We present two approaches that select the Ramsey outcome. The first is to pay attention to equilibria which are robust against small changes in the discount factor in the neighborhood of unity. The second is to introduce a cost of memory. Section 7 concludes the paper.

2. Example

Before plunging into technical analysis, let us examine a simple example based on Chari and Kehoe (1990). Here, there are two agents, a representative consumer and the government. They play an infinitely repeated game where the component game has three stages: the morning, noon, and the afternoon. At the beginning of each day $t = 1, 2, 3, \dots$, the consumer receives a unit of divisible endowment, which lasts only one period. In the morning, the consumer decides how much to consume before noon and how much to save for the afternoon. Let k^t be the saving in period t, and let R be the return from the saving of one unit. Hence, by saving k^t in the morning of date t, the consumer receives Rk^t in the afternoon as an income before tax. Assume that $R = 2$. At noon, the government decides on the tax rates for capital income and labor income. Let θ^t denote the tax rate for the saving (capital income), and let τ^t be that for labor income. Assume that the government can choose one of two tax rates for the capital income, $\theta^t \in \{0.5, 1\}$, and any nonnegative rate for the labor income, $\tau^t \in \mathbb{R}_+$. We write $U = 1$ (high rate) and $D = 0.5$ (low rate). The government must raise two units of income in each period. In the afternoon, given (θ^t, τ^t), the consumer decides on the amount of labor supply which generates labor income. Let $L = 10$ be the maximum amount of leisure, and let $l \in [0, L]$ denote the amount of labor supplied by the consumer. The wage is 1. The game is repeated infinitely many times, and the government is infinitely patient.[1] The private sector

[1] Assume that the government evaluates the payoff stream according to the limit of the mean criterion.

consists of a continuum of identical consumers, and in each period, the government can observe only the aggregate values of the private sector.

The consumptions in the morning and in the afternoon are perfect substitutes. Hence, we can represent the consumer's preference by a utility function over the total amount of consumption, c, and the labor supply, l,

$$U(c, l) = \log c(L - l).$$

If the consumer decides to save k^t, the government's tax rates are (θ^t, τ^t) and the labor supply is l^t, then the gross consumption in period t is

$$c^t = (1 - k^t) + (1 - \theta^t)Rk^t + (1 - \tau^t)l^t.$$

The government must satisfy the budget constraint:

$$\theta^t Rk^t + \tau^t l^t = 2. \tag{2.1}$$

The Ramsey outcome of this game is characterized by the highest capital tax rate θ_r^t that still encourages the consumer to save his entire endowment in the morning, and by the lowest labor tax τ_r^t that raises enough revenue to meet the budget constraint (2.1): $k_r^t = 1, \theta_r^t = 0.5 = D, l_r^t = 4.351$, and $\tau_r^t = 0.230$ for every $t \geq 1$. The average payoff to the consumer and the government is $\mathscr{U}^r = \log 24.435$. If every consumer decides to save his entire endowment $k^t = 1$ in every period, then the government can improve the payoff even further by charging $\theta^t = 1$ and $\tau^t = 0$. In this case, the payoff to the consumer is $\mathscr{U}^d = \log 25$. Notice that the government's manipulation leads to a higher payoff to the private sector than $\mathscr{U}^r$. On the other hand, the best response of the consumer against the government's policy $(\theta^t, \tau^t) = (1, 0)$ is to consume the entire endowment in the morning and to finance consumption in the afternoon by labor income. Put $\theta^t = 1 = U$. In this case, the consumer receives $\mathscr{U}^p = \log 30.25$.

Assume that the government can choose either low capital tax $D = 0.5$ or high capital tax $U = 1$. Since the capital tax rate must be considered before making consumption decision, the representative household needs to make a forecast about the government's tax policy.

Now, suppose that the representative consumer must choose a forecasting scheme from

$$\mathscr{F} = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\},$$

in which

$$\varphi_0(U, U) = \varphi_0(U, D) = \varphi_0(D, U) = U, \quad \varphi_0(D, D) = D,$$

$$\varphi_1(U, U) = \varphi_1(D, U) = U, \quad \varphi_1(U, D) = \varphi_1(D, D) = D,$$

$$\varphi_2(U,U) = U, \quad \varphi_2(D,U) = \varphi_2(U,D) = \varphi_2(D,D) = D,$$

$$\varphi_3(U,U) = \varphi_3(U,D) = U, \quad \varphi_3(D,U) = \varphi_3(D,D) = D.$$

Every forecasting scheme has an important common feature. If a household observes the same action twice in a sequence, then it expects that the same action will be taken in the next round. The only difference among the four forecasting schemes is the speed of response to the change. For example, in φ_0, it takes two consecutive D's to shift the forecast from U to D, while the remaining forecasting rule responds to D by quickly changing its forecast to D. (φ_3 may not fit into this description. But this is a minor point.)

We regard each forecasting scheme φ_i as a firm which is hired by the private sector to make a forecast about the government's policy and recommend an action. For example, following two consecutive U's, φ_1 says: 'Based on our observation, we believe that the government will choose U (high capital tax) next round with probability 1, and therefore, we recomend each household to take a best response to the high labor tax policy (i.e., no saving).' If the consumer believes what the forecasting firm says, then he will save zero dollars in the next day. The question is under what circumstances the consumer is willing to believe the forecast. The answer to this question is not obvious since the government has an incentive to manipulate the forecasting rules in $\mathscr{F}$.

Since the private sector consists of a continuum of households, it makes sense that each household behave myopically by choosing the one-period best response to the expected government policy. By $b(y^e)$, we mean the function that assigns the one-period best response to the forecast y^e. By combining the two functions b and φ_i we obtain a function $b \circ \varphi_i$ that maps each sequence of government's policies to an action by the representative household. In this sense, choosing a forecasting rule can be interpreted as a selection of a repeated game strategy of the private sector.

The bounded rationality is incorporated in φ_i. Each forecasting scheme can process only two most recent policies by the government. On the other hand, each household simply assume that the forecast φ_i is correct, and everybody else has the same forecast. The forecasting rule φ_i is 'inductive' since two consecutive actions let φ_i forecast the same action in the following period. The behavior rule is 'naive' in the sense that the representative household do not raise any question about the precision of the forecast.

The important point is that the government knows that the private sector must use a forecasting scheme from $\mathscr{F}$. Consequently, the government knows the scope of strategic choices available to the private sector. The government is willing to mislead the private sector if the manipulation can improve the social welfare (indeed, it can improve since $\mathscr{U}^d > \mathscr{U}^r$). This whole information is common knowledge.

We examine a game between the private sector and the government, in which the private sector chooses a forecasting scheme from $\mathscr{F}$ and the government chooses a policy. Each household is trying to maximize its own payoff subject to the constraint that its forecasting technology is constrained by $\mathscr{F}$.

We claim that φ_0 will not be selected by the representative household. Given φ_0, the government's best response is to alternate between U and D one after the other in order to let $\varphi_0(\cdot) = U$ constantly. By systematically manipulating the private sector's learning scheme, the government achieves $\mathscr{U}^r$ and $\mathscr{U}^d$ in every other period, and the average payoff of the government is $(\mathscr{U}^r + \mathscr{U}^d)/2$. However, under the assumption that the government alternates between U and D and every other consumer uses φ_0, a typical consumer finds φ_3 generates a better payoff. φ_3 perfectly forecasts the government's next move in each period, $\varphi_3(U, D) = U$ and $\varphi_3(D, U) = D$, and by choosing the best response to the forecast by φ_3 the household can play the actual best against the government's actual policy. By employing φ_3, he can entertain $\mathscr{U}^p$ and $\mathscr{U}^r$ every other period, since the single deviation does not change the aggregate value k^t, and therefore, the government's action will not be affected. By employing φ_3, a typical household can receive $(\mathscr{U}^r + \mathscr{U}^p)/2$ which is strictly larger than $(\mathscr{U}^r + \mathscr{U}^d)/2$. In this case, it is not convincing that the private sector's action will be dictated by φ_0.

On the other hand, for each φ_i $(i = 1, 2, 3)$, the government's best response is to play the Ramsey policy in each period. Since the best outcome against the Ramsey outcome is $\mathscr{U}^r$ which can be obtained only by following the recommendation from φ_i and b, the private sector has reason to follow the learning scheme induced by φ_i and b. There is more than one symmetric Nash equilibrium of this learning game, but every symmetric Nash equilibrium induces the Ramsey policy.

We have demonstrated that if a representative household is smart enough to consider the strategic response of the government, then it must choose one of $\{\varphi_1, \varphi_2, \varphi_3\}$. And the best response of the government to any one of the forecasting rules is to implement the Ramsey policy. By restricting the feasible strategies of the private sector, we identify the Ramsey policy as a unique solution of this model.

We need some restriction on the strategy spaces of the players in order to reduce the set of solutions. However, if we impose too strong a restriction, then not only does the restriction become unintuitive, but also the Ramsey policy may no longer be a solution. For example, if the only available forecasting rule is φ_0, then the household has no choice but being manipulated by the government, which takes advantage of the slow response of φ_0. On the other hand, if we impose too little restriction, then every individual rational outcome can be a solution. In order to select the Ramsey policy, the set of learning schemes must be large enough so that the private sector can avoid being systematically manipulated by the government. At the same time, the set must be small enough to guarantee the government the Ramsey outcome payoff. We will elaborate on this issue to obtain the 'right' restriction on the strategy spaces.

3. Basic model

Our model is a simplified version of Stokey (1991). Let us first describe a component (one-shot) game. There are a unit mass of identical households uniformly distributed on [0, 1] and the government. Each household is indexed by $i \in [0, 1]$ and chooses action $x \in X$, where X is a common action space for every household. The government chooses action $y \in Y$. We assume that X and Y are compact intervals of $\mathbb{R}$. A measurable choice configuration $x_p = \{x_i\}_{i \in [0,1]} \in X_p \equiv X^{[0,1]}$ represents the distribution of actions of the private sector. Given $x_p \in X_p$, $x_a = \int_0^1 x_i \,\mathrm{d}i \in X$ is the aggregate value of the private sector. Note that if all the households take the same action $x \in X$, then $x_a = x$ holds as well. Let u: $X \times X \times Y \to \mathbb{R}$ be utility function of every household; ($u(x_i, x_a, y)$ is household i's utility if i takes x_i, the aggregate action is x_a, and the government action is y. We assume that u is strictly concave with respect to the first argument and continuous with respect to the second argument. We admit randomized strategies.

The government's utility function is given by

$$w: \quad X_p \times Y \to \mathbb{R}_+.$$

For all $y \in Y$, there exists a unique best response of the private sector irrespective of the aggregate value of x.[2] Given $y \in Y$, let $b(y)$ be the unique symmetric Nash equilibrium strategy of the representative household. That is, for each $y \in Y$, there exists a unique $b(y)$ such that, for all $x' \neq b(y)$,

$$u(b(y), b(y), y) > u(x', b(y), y)$$

holds. We often use $\{x\}$ to represent the action profile in which a representative household takes x. The government and the private sector choose actions simultaneously.

This model contains some important classes of economic problems. For example, in a typical economic model, each household faces a budget constraint. Let $h(x_i, y) \geq 0$ denote the budget constraint for household i determined by its own action x_i and the government policy y. Household i's payoff is zero if

[2]This assumption does not exclude the possibility that the aggregate action of the private sector affects a household's payoff. An appropriate example is the one in which one always prefers to pay less tax, though he prefers others paying more.

$h(x_i, y) < 0$. If $h(x_i, y)$ is a weakly increasing continuous function of y, then the same analysis applies. Also included in the present formulation is the model where the government has a 'loose' budget constraint. We imagine a government with a very tight credit limit to finance the budget deficit. Suppose that the budget constraint is given by $R(x_a, y) \geq E$, where E is given expenditures and $R(x_a, y)$ is the revenue for $x_a \in X$ and $y \in Y$. In order for u to satisfy continuity, we have to perturb this budget constraint a little. For example, given $\eta > 0$, let $u(x_i, x_a, y) = v(x_i)$ with $v(0) = 0$ and $w(x_p, y) = w(\{x_i\}, y)$ be given by

$$w(x_p, y) = \begin{cases} \int_0^1 u(x_i, x_a, y)\,\mathrm{d}i & \text{if} \quad R(x_a, y) \geq E, \\ \frac{1}{\eta}[R(x_a, y) + \eta - E] \int_0^1 u(x_i, x_a, y)\,\mathrm{d}i & \text{if} \quad E > R(x_a, y) > E - \eta, \\ 0 & \text{otherwise,} \end{cases}$$

where $x_a = \int_0^1 x_i\,\mathrm{d}i$ as defined before.

Then this model conforms to our formulation. Note that in this example, if the budget is balanced, then the government's utility is the same as the average utility of the households. There are two special outcomes worth mentioning. A Ramsey outcome is a pair (x^r, y^r) satisfying $x^r = b(y^r)$ and

$$y^r \in \arg\max_{y \in Y} w(\{b(y)\}, y). \tag{3.1}$$

Let $w^r = w(x^r, y^r)$ denote the government's payoff from the Ramsey outcome. This is the government's payoff if the government can commit itself to a particular action as a Stackelberg leader. On the other hand, the noncommitment outcome, denoted by (x^n, y^n), is a Nash equilibrium of the simultaneous move game between the government and the private sector, i.e., $x^n = b(y^n)$ and

$$y^n \in \arg\max_{y \in Y} w(\{x^n\}, y).$$

Let w^n denote the government's payoff from a noncommitment outcome (x^n, y^n). Obviously, $w^r \geq w^n$. For the sake of simplicity, we assume that there exists a unique noncommitment outcome.

Now, consider the supergame obtained by repeating the above component game infinitely many times. Remember that $x_p = (x_i)_{i \in [0,1]} \in X_p$. In each period, the outcome from the game is $(x_p, y) \in X_p \times Y$. Assume that the aggregate value x_a^t of the private sector's activities and the government's action y are publicly observable. A sequence $(x_a^1, y^1; \ldots; x_a^t, y^t)$ of aggregate variables is a public history at period $t+1$. Define $Z = X \times Y$. Let Z^t be the set of all (public) histories in period $t+1$. Let Z^0 denote the null history in the initial round.

We assume that at the end of each period, the government observes only the aggregate value of the private sector's actions. Similarly, we assume that an individual household can observe only the government's action and the aggregate value of the private sector as opposed to the individual actions taken by the other households. Therefore, a strategy of the government is represented by a mapping

$$\phi\colon \bigcup_{t=1}^{\infty} Z^{t-1} \to Y,$$

which dictates a strategic choice $\phi(z^{t-1})$ in period t conditioned on the public history z^{t-1}. Let Φ be the set of all repeated game strategies of the government.

Since each household is infinitesimal, we may focus on the *public strategy* of each household:

$$\sigma_i\colon \bigcup_{t=1}^{\infty} Z^{t-1} \to X,$$

which specifies household i's choice in period t conditioned on each public history. Let Σ_i be the set of repeated game strategies of household i. Since all the households are identical, we drop a subscript from Σ_i, i.e., Σ denotes the set of repeated game strategies of a household.

Given σ and ϕ, there is a sequence of outcomes induced by the strategy profile. Define $\theta_i^t(\sigma, \phi)$ to be household i's action in period t induced by (σ, ϕ) and $\theta_g^t(\sigma, \phi)$ to be the government's action in period t induced by the strategy profile. We write $\theta^t(\sigma, \phi)$ for the profile of actions induced by (σ, ϕ) in period t. By the outcome induced by (σ, ϕ), we mean the sequence $\theta(\sigma, \phi) = \{\theta^t(\sigma, \phi)\}_{t=1}^{\infty}$. Define $\theta_p^t(\sigma, \phi) = (\theta_i^t(\sigma, \phi))_{i \in [0,1]}$. Let $\theta_a^t(\sigma, \phi) = \int \theta_i(\sigma, \phi)\,\mathrm{d}i$ be the aggregate variable of the private sector in period t induced by (σ, ϕ).

The players evaluate the outcome path according to the limit of the mean criterion. Given a profile (σ, ϕ) of strategies, define for each i

$$v_i(\sigma, \phi) = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} u(\theta_i^t(\sigma, \phi), \theta_a^t(\sigma, \phi), \theta_g^t(\sigma, \phi))$$

as the long-run average payoff of household i and

$$v_g(\sigma, \phi) = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} w(\theta_p^t(\sigma, \phi), \theta_g^t(\sigma, \phi))$$

as the government's long-run average payoff. By the repeated policy game, we mean

$$G^{\infty} = \langle \Phi, \Sigma; v_i, v_g \rangle.$$

Notice that we drop the subscript from Σ, since we assume that every household has the same set of repeated game strategies. We also assume that all the households have the same payoff function.

In order to simplify notation, we shall focus on symmetric equilibria. It must be emphasized, however, that this simplification has no impact on the main results of this paper. From now on, by σ, we mean a strategy profile of the private sector in which every household i uses strategy σ. A strategy profile (σ, ϕ) is a (symmetric) Nash equilibrium if for each household σ is a best response to the government's strategy ϕ, while every other household uses σ, and the government's strategy ϕ is a best response to σ.

Given a history $z^t \in Z^t$, $\sigma(z^t)$ represents the action of a typical household dictated by σ following z^t. Let $\phi|_{z^t}$ be the strategy of the government in the continuation game following z^t. We define $\sigma|_{z^t}$ in the same way. Let

$$v_g(\sigma, \phi\colon z^t) = v_g(\sigma|_{z^t}, \phi|_{z^t})$$

be the long-run average payoff of the government induced by (σ, ϕ) following history z^t. It is straightforward to define Nash equilibrium in the continuation game following history z^t. A strategy profile (σ, ϕ) is a (symmetric) subgame-perfect equilibrium if it induces a Nash equilibrium following each z^t.

A government's policy ϕ is a *credible* policy (Stokey, 1991) if there exists $\sigma \in \Sigma$ such that for each history z^t,

$$\sigma(z^t) \in \arg\max_x u(x, \sigma(z^t), \phi(z^t)) \tag{3.2}$$

and

$$\phi(z^t) \in \arg\max_\phi v_g(\sigma, \phi\colon z^t). \tag{3.3}$$

Since the government cannot identify the action of a single household, in any equilibrium, a typical household solves a sequence of a short-run maximization problem. In this sense, (3.2) means that every household uses the same strategy σ, and responds to the government's action optimally in each period. On the other hand, (3.3) implies that the government responds to the actions of private sector optimally following every history z^t. Indeed, a credible policy is a symmetric subgame-perfect equilibrium strategy of the government in which every household uses the same repeated game strategy. For this reason, we often call a credible policy a symmetric subgame-perfect equilibrium strategy of the government.

By the Ramsey outcome of G^{∞} we mean a sequence of outcomes where each household chooses x^r and the government chooses y^r for every $t \geq 1$. Obviously, the long-run average payoff of the government from the Ramsey outcome is w^r. A central result from the standard equilibrium model (Stokey, 1991; Chari and Kehoe, 1990) is that the Ramsey outcome can be supported as a credible policy, which requires rational expectation of the individual household following *every* history.

Theorem 3.1 (*Stokey, 1991*). *There exists a credible policy of the government which induces the Ramsey outcome.*

4. Game with inductive forecast

Suppose that the private sector forecasts that the government's policy is $y^e \in \Delta(Y)$. Because the private sector consists of a continuum of small households and the government can only observe the aggregate actions of the private sector instead of the individual household's behavior, an optimal action of the representative household is $b(y^e)$ which satisfies

$$[\text{BR}] \qquad b(y^e) \in \arg\max_{x \in X} u_i(x, b(y^e), y^e),$$

where we extend the domain of $b(\cdot)$ to include mixed strategies of the government.

Rational expectations presume that the representative household perfectly foresees the government's action so that y^e must be the actual policy of the government folowing every history. While formally correct, we feel this aspect of rational expectations is rather unintuitive. Instead of perfect foresight we assume that the forecast by the representative household is determined by an 'inductive' forecasting scheme: after observing the same policy y by the government for a long time consecutively, it generates a forecast that the next period's government's policy will be y with high probability. Clearly, inductive forecasting schemes are open to strategic manipulation by the government in principle.

All we require is that the representative household chooses an action 'as if' it had an inductive forecasting rule so that the private sector ends up choosing a best response to a policy that has been repeated many times. For this reason, we will not be specific about the functional form of forecasting schemes. But, in order to demonstrate that familiar learning rules induce the inductive behavior, we examine two examples, which will give some idea about how general the class of inductive forecasting rule is.

Example 1. Suppose that the representative household uses the weighted average of the most recent K actions of the government as the forecast. That is, there exist $\alpha_1, \ldots, \alpha_K > 0$ satisfying

$$\sum_{k=1}^{K} \alpha_k = 1, \tag{4.1}$$

such that the forecast about the government's action in $t+1$ after observing $y^1, \ldots, y^t$ is $\sum_{k=1}^{K} \alpha_k y^{t-K+k}$. Given K, the government can convince the private sector by repeating the same policy K times. Let $W(K)$ be the set of all forecasting rules that can be represented as (4.1). The class of inductive forecasting rules contains $\bigcup_{K \geq 1} W(K)$.

Example 2. Inductive behaviors can incorporate Bayesian learning rule if we specify the uncertainty of the model following Fudenberg and Levine (1989). Assume first that the number of actions available to every player is finite. Suppose that the representative household has slight uncertainty about the rationality of the government. The representative household believes that with probability $1-\varepsilon$ the government is fully rational, but with probability ε the government is committed to play a particular policy $y \in \mathbb{R}$ throughout the game. Let $\xi(y)$ be the probability distribution over all 'commitment type' which has a full support over the space of the government's policy. That is, $\varepsilon\xi(y)$ represents the likelihood that the government is committed to policy y. If the representative household updates its belief according to the Bayes rule, then the following many repetitions of the same policy y^* its posterior conjecture must assign large probability to the 'y^* commitment type'. As a result, if the government repeats y^* sufficiently many times, then the representative household chooses an optimal response to y^* in finitely many periods.[3] Thus, for any $\varepsilon > 0$ and ξ with the full support over the space of the policies, the representative household's behavior induced by the Bayesian learning rule is inductive.

Although the inductive behavior includes Bayesian learning scheme in some models, it must be emphasized that our model does not follow the Harsanyi doctrine, which requires that the uncertainty of the model must be captured by the 'types' of the players, and every player in the game has a common prior about the uncertainty. In our model, the players need not have the same prior about the uncertainty. All we need is that the representative household's behavior must be inductive, and this fact is common knowledge among players. The government need not know the origin of the inductive behavior nor the belief of representative household about the government's rationality.

[3] For the formal proof, see Fudenberg and Levine (1989).

Remember that at the end of each period, every household observes the aggregate value of the private sector's actions and the government's action. Recall that $Z = X \times Y$ is the set of all variables that can be publicly observable and that Z^t is the set of all t publicly observable variables. We call a generic element of Z^t a public history of period $t + 1$. The representative household summarizes a public history according to

$$\varphi_i\colon \ \bigcup_{t=1}^{\infty} Z^{t-1} \to \Delta(Z),$$

where $Z^0 = \{z^0\}$ is the set of the null history and $\varphi_i(z^0)$ is the initial forecast by household i before the beginning of the game.

Given a public history z, let $\varphi(z)(x, y) = (\varphi_1(z)(x), \varphi_2(z)(y))$ be the probability assessment for the private sector's outcome x and the government's policy y. Since the representative household responds to the forecast by taking an optimal response according to [BR], the expected response of the private sector would be $b(\varphi_2(z))$. Hence $\varphi_1(z)$ must be concentrated at $b(\varphi_2(z))$. One can interpret $\varphi(z)(x, y)$ as 'based on history z, we (economists working for the forecasting company φ) believe that the government will take policy y with probability $\varphi_2(z)(y)$, and we recommend a typical household in the private sector to take an action $b(\varphi_2(z))$'.

We impose a specific restriction on the length of history z which φ can process, and on the way z influences the outcome of φ. We assume that the households can exploit only the most recent finitely many observations in order to forecast the government's action in the next round. Moreover, for any $\varepsilon > 0$, there exists K such that if the representative household observes the same policy y during the last K periods consecutively, then it is convinced that the government will take y in the following period with the probability of more than $1 - \varepsilon$. Let

$$\mathscr{F}^{K\varepsilon} = \left\{\varphi\colon \begin{array}{l} \forall (z; x^1, y, \ldots, x^k, y) \in Z, \quad \forall k \geq K \\ \varphi(z; x^1, y, \ldots, x^k, y)(b(y), y) \geq 1 - \varepsilon \end{array}\right\}. \tag{4.2}$$

This assumption is summarized as follows.

[I] $\quad \forall i \in [0, 1], \quad \forall \varepsilon > 0, \quad \exists K \quad \text{such that} \quad \varphi_i \in \mathscr{F}^{K\varepsilon}.$

Let

$$\mathscr{F} = \bigcap_{\varepsilon > 0} \bigcup_{K \geq 1} \mathscr{F}^{K\varepsilon}$$

be the set of all inductive forecasting rules. This assumption basically restricts the set of strategies of the households to a certain extent. In this sense, the present approach is a behavioral approach as opposed to the Bayesian approach with restricted beliefs.

The important question is whether the representative household is willing to follow the recommendation of the economists. Since the inductive forecasting rule is easily manipulated by the government and every household knows that the government has an incentive to do so, a sophisticated household should not follow the recommended action $b(\varphi_2(z))$ unless it is convinced that the recommended action is optimal. This sort of strategic consideration by the representative household is the key to our result.

Combining an 'inductive' forecasting scheme $\varphi = (\varphi_1, \varphi_2) \in \Delta(X) \times \Delta(Y)$ and the behavior scheme b, we obtain a repeated game strategy $\sigma(z) = b(\varphi_2(z))$ which maps each history $z \in Z$ into an action. Let

$$\Psi_p = \{\sigma \in \Sigma \colon \exists \varphi = (\varphi_1, \varphi_2) \in \mathscr{F},\ \forall z \in Z,\ \sigma(z) = b(\varphi_2(z))\}.$$

Let Ψ_p denote the set of repeated game strategies by a representative household, which can be induced by some forecasting scheme [I] and naive behavior rule satisfying [BR]. We restrict the feasible set of strategies of player i to Ψ_p, which is a small subset of the collection of all repeated game strategies.

We impose no restriction on the strategy space of the government and let Ψ_g denote the set of all of the government's strategies for the repeated game. We write a generic element of Ψ_g as ψ_g. It must be emphasized that one can impose the same kind of restrictions as [I] and [BR] on the government's strategy spaces and still derive the same results of this paper.

By the game with inductive forecast, we mean the policy game obtained by restricting the strategy spaces of the representative household and the government to Ψ_p and Ψ_g. Set

$$G_L = \langle \Psi_p, \Psi_g; v_i, v_g \rangle.$$

Since every household is identical, we drop subscript i from Ψ_p.

A (symmetric) Nash equilibrium of G_L is a profile (ψ_p^*, ψ_g^*) in which, $\forall \psi_g \in \Psi_g$,

$$v_g(\psi_p^*, \psi_g^*) \geq v_g(\psi_p^*, \psi_g),$$

and there is no household i with ψ_i satisfying

$$v_i(\psi_p^*, \psi_g^*) \geq v_i(\psi_p, \psi_g^*),$$

where $\psi_p = ((\psi_j^*)_{j \neq i}, \psi_i)$. By $SNE(G_L)$, we mean that the set of all (symmetric) Nash equilibria of the policy game G_L with inductive forecast.

A profile (ψ_p, ψ_g) of forecasting schemes *essentially induces* the Ramsey outcome if $\forall \varepsilon > 0$ there exists T such that $\forall t \geq T$

$$|\theta_g^t(\psi_p, \psi_g) - y^r| < \varepsilon$$

and

$$Pr(\{i \in [0, 1]: |\theta_i^t(\psi_p, \psi_g) - x^r| < \varepsilon\}) > 1 - \varepsilon,$$

where Pr is the Lebesgue measure over [0, 1]. Note that this definition is weaker than the definition of induction of the Ramsey outcome given in the previous section.

5. Analysis

First, we claim that the restrictions [I] and [BR] are mild[4] enough to allow each household to choose a forecasting scheme that almost perfectly foresees the government's actual action in almost every period.[5]

Lemma 5.1. $\forall (\psi_p^*, \psi_g^*) \in SNE(G_L)$, $\forall \varepsilon > 0$, *and* $\forall i \in [0, 1]$,

$$\liminf_{T \to \infty} \frac{1}{T} \# \left\{ t \leq T : \left| \theta_i^t(\psi_p^*, \psi_g^*) - \arg\max_{x \in X} u(x, \theta_a^t(\psi_p^*, \psi_g^*), \theta_g^t(\psi_p^*, \psi_g^*)) \right| < \varepsilon \right\} = 1. \tag{5.1}$$

Proof. See the Appendix.

Since (5.1) must hold for each household, this lemma implies that the private sector's action must be arbitrarily close to the equilibrium conditioned on the *actual* action of the government in almost every period. As long as the private

[4]Another reason why assumptions [I] and [BR] are mild is that these restrictions do not exclude *a priori* any average payoff vectors from the repeated game. Given any payoff vector in the convex hull of payoff vectors of the component game, we can select a profile of forecasting schemes that induces the same payoff vector as the average payoff in the repeated game. The proof is straightforward and left to the readers.

[5]In the Appendix, in order to achieve a full generality, we establish the result for all Nash equilibria instead of symmetric Nash equilibria.

sector responds optimally to the government's action, the best payoff for the government is the Ramsey outcome payoff. By Lemma 5.1, in almost every period, the government's payoff is at most the Ramsey equilibrium payoff w^r. Therefore, in any equilibrium of the policy game with inductive forecast, the government's long-run average payoff is at most w^r.

Lemma 5.2. $\forall(\psi_p^*, \psi_g^*) \in SNE(G_L)$, $v_g(\psi_p^*, \psi_g^*) \leq w^r$.

Next, we construct a Nash equilibrium that induces (*a fortiori* essentially induces) the Ramsey outcome in the game with inductive forecast.

Proposition 5.3. There exists $(\psi_p^*, \psi_g^*) \in SNE(G_L)$, *where* $\forall i \in [0, 1]$, $\theta_i^t(\psi_p^*, \psi_g^*) = x^r$ *and* $\theta_g^t(\psi_p^*, \psi_g^*) = y^r$ *in every* $t \geq 1$.

Proof. See the Appendix.

Fix a (symmetric) Nash equilibrium (ψ_p^*, ψ_g^*) of G_L. Since ψ_p^* must satisfy [I], for every household there exists K_p such that if the government repeats y^r as many as K_p times, then a representative household forecasts y^r and chooses x^r as a best response to the forecast. We can construct an increasing sequence $\{I^t\}$ converging to $I = [0, 1]$, where I^t is the set of households that starts to play x^r following t repetitions of y^r. Let ψ_g^r be the strategy of the government which plays y^r following every history. Since the government's objective function is continuous, we can write

$$w(\theta_p^t(\psi_p^*, \psi_g^r), \theta_g^t(\psi_p^*, \psi_g^r)) = w^r - \eta^t,$$

and $\eta^t \downarrow 0$ as $t \to \infty$. Therefore, the government's average payoff

$$\frac{1}{T}\sum_{t=1}^{T} w(\theta_p^t(\psi_p^*, \psi_g^r), \theta_g^t(\psi_p^*, \psi_g^r)) = w^r - \frac{\sum_{t=1}^{T} \eta^t}{T} \to w^r$$

as $T \to \infty$. The next lemma summarizes this observation.

Lemma 5.4. $\forall(\psi_p^*, \psi_g^*) \in SNE(G_L)$, $v_g(\psi_p^*, \psi_g^*) \geq w^r$.

Since the only way to achieve w^r is to play the Ramsey outcome in almost every period, Lemma 5.4 proves that any (symmetric) Nash equilibrium of the game with inductive forecast must essentially induce the Ramsey outcome.

Proposition 5.5. *Every $(\psi_p^*, \psi_g^*) \in SNE(G_L)$ essentially induces the Ramsey outcome in G_L.*

In the present framework, our main result does not rely on equilibrium approach. In an equilibrium a player's belief is adjusted in such a way that it is consistent with the opponent's actual strategy. Once the belief is adjusted in this way (by an outside modeler) the agent takes the optimal strategy based on this belief. But where does this belief come from? Why should the agent have such a belief? An equilibrium model does not answer to these questions. By separating the forecasting scheme from the behavior rule, we can reveal the problem which is inherent to equilibrium approach in general.

Indeed, one can justify the Ramsey outcome in a different way. Assume that the government is prudent in the sense that it maximizes the long-run average payoff under the assumption that the private sector is trying to minimize the government's payoff. A prudent strategy ψ_g of the government is given by

$$\psi_g \in \arg\max_{\psi_g'} \min_{\psi_p} v_g(\psi_p, \psi_g').$$

If the government plays y^r in every period from the beginning for the rest of the game, then it can achieve at least w^r. By the same reason, a government's strategy that essentially induces the Ramsey outcome against any learning scheme of the private sector is prudent.

On the other hand, in order to be prudent, the government's strategy must achieve at least w^r against any ψ_p, since the government has a feasible strategy ψ_g that achieves w^r against any private sector's learning scheme. Furthermore, no prudent strategy can induce more than w^r as a long-run average payoff since the best response outcome against the private sector's learning scheme constructed in Lemma 5.2 is w^r.

This reasoning proves that the Ramsey outcome is equivalent to the set of outcomes induced by the prudent strategies of the learning game.

Proposition 5.6. *A strategy ψ_g of the government is prudent if and only if for each $\psi_p \in \Psi_p$, (ψ_p, ψ_g) essentially induces the Ramsey outcome in G_L.*

6. Discounting

The analysis in the previous section relies on the fact that the players do not discount their future utility. The ability of the government to influence the forecast of the private sector depends crucially on the length of memory K in $\mathscr{F}^{K\varepsilon}$. Since we impose no upper bound for the feasible memory size of the private sector, an impatient government may not be willing to spend a long period in order to manipulate the private sector's action. As a result, if the discount factor is strictly less than 1, we can construct many equilibria other than the Ramsey policy from the policy game with bounded rationality.[6]

However, the Ramsey policy is a salient outcome in the model with discounting as long as the discount factor is close to 1. Given $\delta < 1$, the policy game $G_L(\delta)$ with inductive forecast and discounting is defined as

$$G_L(\delta) = \langle \Psi_p, \Psi_g; v_i^\delta, v_g^\delta \rangle,$$

where for each $i \in [0, 1]$,

$$v_i^\delta(\psi_p, \psi_g) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u(\theta_i^t(\psi_p, \psi_g), \theta_a^t(\psi_p, \psi_g), \theta_g^t(\psi_p, \psi_g))$$

and

$$v_g^\delta(\psi_p, \psi_g) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} w(\theta_p^t(\psi_p, \psi_g), \theta_g^t(\psi_p, \psi_g)).$$

We can define a symmetric Nash equilibrium of $G_L(\delta)$ in the same way as we do in G_L. Let $SNE(G_L(\delta))$ be the set of symmetric Nash equilibria of $G_L(\delta)$.

It is crucial that the players be sufficiently patient to learn and to manipulate the learning process. Consequently, our main interest is the equilibrium behavior when δ is close to 1. Given our research objective, it is reasonable to examine those equilibria which are robust against small changes in the discount factor in the neighborhood of 1. Following the spirit of the selection criterion of Kalai, Samet, and Stanford (1988), we will eliminate the equilibria of $G_L(\delta)$ which collapse for some $\delta' > \delta$. The main result is to show that any symmetric

[6] As we demonstrated earlier (without proof), the class of forecasting schemes $\mathscr{F}$ includes the Bayesian learning as a special case if there are only finitely many actions. The only restriction on $\mathscr{F}$ is [I], which takes effect only when the same action is observed for a long time from the beginning. Thus, we can construct a forecasting scheme that follows the Bayes rule whenever two different actions of the government appear in the past. What we have to show now is that if the prior distribution assigns positive probability, say, $\eta/|Y|$ where $|Y|$ is the number of actions in Y, then the Bayesian updating satisfies [I]. Indeed, for each prior probability distribution μ over the types of the government, Fudenberg and Levine (1989) demonstrate that $\forall \varepsilon, \exists K(\varepsilon)$ such that following $k \geq K(\varepsilon)$ repetition of the same action, the posterior must assign probability at least as much as $1-\varepsilon$. This implies that the Bayesian updating rule with the special prior mentioned above satisfies [I]. Of course, we admit more diverse forecasting rules than the Bayesian learning. Since, however, we do not impose any restriction on K, we do not have a uniform upper bound as Fudenberg and Levine (1989); they assume a common prior across short-run players, which assures that for any ε and any player there exists a uniform upper bound for K. This is the main reason why we cannot derive the analogous result to Fudenberg and Levine (1989).

Nash equilibrium from $G_L(\delta)$ satisfying this additional requirement must induce the Ramsey policy.

In any Nash equilibrium, every household optimally responds to the government's actual policy in every period.

Lemma 6.1. $\forall(\psi_p^*, \psi_g^*) \in SNE(G_L(\delta))$, $\forall i \in [0, 1]$, *and* $\forall t \geq 1$,

$$\theta_i^t(\psi_p^*, \psi_g^*) = \arg\max_{x \in X} u(x, \theta_a^t(\psi_p^*, \psi_g^*), \theta_g^t(\psi_p^*, \psi_g^*)).$$

Proof. Apply Lemma 5.1.

Lemma 6.1 implies that the average discount payoff of the government is at most w^r. Next, we prove the existence of such an equilibrium. In particular, the Ramsey outcome can be achieved if the government is patient. Remember that ψ_g^r is the government's strategy that plays y^r following every history. Define

$$\psi_i^r(h) = \begin{cases} x^r & \text{if} \quad h = (x^1, y^r; \ldots; x^t, y^r) \quad \text{for some } (x^1, \ldots, x^t), \\ x^n & \text{otherwise.} \end{cases}$$

where x^n is the unique Nash equilibrium strategy of the component game conditioned on the noncommitment outcome $y = y^n$. Let $\psi_p^r = (\psi_i^r)_{i \in [0,1]}$.

Lemma 6.2. $\exists \underline{\delta} < 1$, *such that* $\forall \delta > \underline{\delta}$, (ψ_p^r, ψ_g^r) *is a Nash equilibrium of* $G_L(\delta)$.

Proof. Apply Proposition 5.3.

This shows that the Ramsey outcome is robust against small changes in the discount factor in the neighborhood of 1:

$$(\psi_p^r, \psi_g^r) \in \bigcup_{\delta < 1} \bigcap_{\delta' \geq \delta} NE(G_L(\delta')) \neq \emptyset.$$

Our next step is that (ψ_p^r, ψ_g^r) is the only Nash equilibrium which is robust against small changes in the discount factor.

Proposition 6.3. $\forall(\psi_p^*, \psi_g^*) \in \bigcup_{\delta < 1} \bigcap_{\delta' \geq \delta} NE(G_L(\delta')) \liminf_{\delta \uparrow 1} v_g^\delta(\psi_p^*, \psi_g^*) = w^r$.

Proof. See the Appendix.

In the real situation, people are more credulous than to disbelieve the government when it has taken the Ramsey outcome many times. In particular, if a household incurs a small cost of keeping a record of the government's past actions (i.e., memory), then the situation changes dramatically even if we do not focus on the robustness of equilibria. Given a strategy of household i, $\psi_i = (\varphi_i, b)$, the government's action y, and $\varepsilon > 0$, let $K_i(y, \varepsilon)$ be the smallest integer such that $\forall t \geq K_i(y, \varepsilon)$

$$|\varphi_i(x^1, y; \ldots; x^t, y) - (b(y), y)| < \varepsilon$$

holds. If such an integer does not exist, then set $K_i(y, \varepsilon) = \infty$. Assume that $K_i(y, \varepsilon)$ is a measurable function of y. Define $c(\psi_i, \varepsilon)$ as the memory cost of implementing ψ_i:

$$c(\psi_i, \varepsilon) = c \int_{y \in Y} K_i(y, \varepsilon)\,\mathrm{d}y,$$

where c is a small positive number.

Now, given $\delta < 1$ and $c > 0$, the game $G_L(\delta, c, \varepsilon)$ with inductive forecast, discounting and cost of memory is defined as

$$G_L(\delta, c, \varepsilon) = \langle (\Psi_i)_{i \in [0,1]}, \Psi_g; (v_i^{\delta, c, \varepsilon})_{i \in [0,1]}, v_g^{\delta, c, \varepsilon} \rangle,$$

where

$$v_i^{\delta, c, \varepsilon}(\psi_p, \psi_g) = v_i^{\delta}(\psi_p, \psi_g) - c(\psi_i, \varepsilon)$$

and

$$v_g^{\delta, c, \varepsilon}(\psi_p, \psi_g) = v_g^{\delta}(\psi_p, \psi_g).$$

We can show that for a given discount factor $\delta < 1$, in any Nash equilibrium of $G_L(\delta, c, \varepsilon)$, the government's payoff is at least $(1 - \delta)\min_{x \in X} w(x, y^r) + \delta w^r$, which converges to w^r as $\delta \uparrow 1$.

Proposition 6.4. *Given $c, \varepsilon > 0$ and $\delta < 1$, the government receives at least*

$$(1 - \delta)\min_{x \in X} w(x, y^r) + \delta w^r,$$

in any Nash equilibrium of $G_L(\delta, c, \varepsilon)$.

Proof. See the Appendix.

7. Conclusion

We examine the classic policy problem of Ramsey modeled as an infinitely repeated game between the government and the private sector. Assuming bounded rationality of the households, we identify the Ramsey outcome as the unique solution of the game where players do not discount future payoffs.

Several remarks are in order. First, our approach is characterized as a behavioral approach in the sense that we restrict the strategy space, as opposed to prior, of the agents. Unlike some other attempts in the literature on learning, we impose only mild restrictions on how players learn from the past. In our view, even Bayesian updating rule is a special case, especially when one specifies a prior of agents.

Second, if we only have an equilibrium result, then we suffer from the criticism which applies to equilibrium approach in general, and equilibrium approach with bounded rationality in particular. In models of bounded rationality, it is controversial to use equilibrium concept since it typically requires full rationality for an agent to calculate an equilibrium. In our model, we have another result that states that the Ramsey policy is the unique policy if the government faces uncertainty as to which strategies are chosen by the private sector and behaves as a maximin player. In this result, we do not have to let the agents choose their learning schemes but simply assume that they are assigned to ones, which may be different across agents, and the government is not informed of which beliefs the private sector has.[7]

Appendix

In order to maintain full generality of analysis, we prove the results in terms of Nash equilibrium instead of symmetric Nash equilibrium. Let $NE(G)$ be the set of all Nash equilibria of game G.

Proof of Lemma 5.1

Suppose that $\exists(\psi_p^*, \psi_g^*) \in NE(G_L)$, $\exists\varepsilon > 0$, and $\exists i$ such that

[7] If we take this approach, one may argue that we lose the initial motivation which makes forecasting schemes strategic choices. Further scrutiny, however, assures that this is not the case for the following reason. We suffer much less from arbitrariness of selection of forecasting scheme than usual approach with fixed beliefs. If we, the modelers, choose specific learning scheme, then the government can often manipulate the forecast. In our present analysis, the government's expectation that the private sector may have completely different beliefs than the government imagines prevents the government from manipulating the private sector. In other words, a large set of learning schemes is still needed for the result.

$$\liminf_{T\to\infty}\frac{1}{T}\#\left\{t\le T:\ \left|\theta_i^t(\psi_p^*,\psi_g^*)-\arg\max_{x\in X}u(x,\theta_a^t(\psi_p^*,\psi_g^*),\theta_g^t(\psi_p^*,\psi_g^*))\right|<\varepsilon\right\}\le 1-2\varepsilon.$$

Then there exists $\{T_k\}_{k=1}^{\infty}$ such that for every $k\ge 1$,

$$\frac{1}{T_k}\#\left\{t\le T_k:\ \left|\theta_i^t(\psi_p^*,\psi_g^*)-\arg\max_{x\in X}u(x,\theta_a^t(\psi_p^*,\psi_g^*),\theta_g^t(\psi_p^*,\psi_g^*))\right|<\varepsilon\right\}\le 1-\varepsilon.$$

Define

$$\tau^k(\varepsilon)=\left\{t\le T_k:\ \left|\theta_i^t(\psi_p^*,\psi_g^*)-\arg\max_{x\in X}u(x,\theta_a^t(\psi_p^*,\psi_g^*),\theta_g^t(\psi_p^*,\psi_g^*))\right|\ge\varepsilon\right\}.$$

By hypothesis,

$$\lim_{k\to\infty}\frac{\#\tau^k(\varepsilon)}{T_k}\ge\varepsilon.$$

Since the action space of the individual household is finite, there exists $\eta>0$ such that

$$u(\theta_i^t(\psi_p^*,\psi_g^*),\theta_a^t(\psi_p^*,\psi_g^*),\theta_g^t(\psi_p^*,\psi_g^*))$$
$$\le\max_{x\in X}u(x,\theta_a^t(\psi_p^*,\psi_g^*),\theta_g^t(\psi_p^*,\psi_g^*))-\eta,$$

for each $t\in\tau^k(\varepsilon)$ and for each $k\ge 1$. Let $\{(x^t,y^t)\}_{t=1}^{\infty}$ be the sequence of aggregate actions of the private sector and the government's actions. Let φ_i^* denote household i's forecasting scheme satisfying $\psi_i^*=b\circ\varphi_i^*$. Choose a forecasting rule

$$\tilde\varphi(h)=\begin{cases}(x^t,y^t) & \text{if}\quad h=(x^1,y^1;\dots;x^{t-1},y^{t-1}),\\ \varphi_i^*(h) & \text{otherwise.}\end{cases}$$

Let $\tilde\psi(h)=b(\tilde\varphi(h))$ for each h.

By construction, this strategy satisfies [BR]. In order to prove that $\tilde{\varphi}$ satisfies [I], we need to establish a short lemma.

Lemma A.1. *Suppose that $y^t = y$ for all $t \geq 1$ for some $y \in Y$ along the aggregate outcome path $\{(x^t, y^t)\}_{t=1}^{\infty}$. Then, $\forall \varepsilon > 0$, $\exists K'$ such that for all $k \geq K'$, $|x^k - b(y)| < \varepsilon$.*

Proof. For $\eta > 0$, define

$$J^k(\eta) = \{ j \in [0,1]: |\varphi_j^*(x^1, y^1), \ldots, (x^k, y^k)) - (b(y), y)| < \eta \}.$$

Since every φ_j^* satisfies [I], $\liminf_{k \to \infty} J^k(\eta) = [0,1]$. Since b is a continuous function, $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\forall j \in J^k(\eta)$

$$|\psi_j^*(x^1, y, \ldots, x^k, y) - b(y)| < \frac{\varepsilon}{2}.$$

Since X is compact, there exists K' such that $\forall k \geq K'$

$$\int_{[0,1] \setminus J^k(\eta)} \mathrm{d}j < \frac{\varepsilon}{2 \sup X}.$$

Hence, for all $k \geq K'$

$$|x^k - b(y)| = \left| \int \psi_j(x^1, y, \ldots, x^k, y) \, \mathrm{d}j - b(y) \right|$$

$$\leq \int_{J^k(\eta)} |\psi_j(x^1, y, \ldots, x^k, y) - b(y)| \, \mathrm{d}j + \sup X \int_{[0,1] \setminus J^k(\eta)} \mathrm{d}j$$

$$\leq \varepsilon. \qquad \text{Q.E.D.}$$

Recall that φ_i^* satisfies [I]. Hence, given $\varepsilon > 0$, $\exists K''$ such that $\forall k \geq K''$

$$|\varphi_i(x^1, y, \ldots, x^k, y) - (b(y), y)| < \varepsilon.$$

We must consider two different cases. First, suppose that there exists K^* such that $y^1 \neq y^{K^*}$. Then, choose $K = \max(K'', K^*) + 1$. Since any history h involving K repetitions of the same policy y is a part of aggregate outcome path, $\tilde{\varphi}(h) = \varphi_i^*(h)$. Since φ_i^* satisfies [I], we have

$$|\tilde{\varphi}(h) - (b(y), y)| < \varepsilon.$$

Next, assume that $y^t = y$ for some $y \in Y$ for all $t \geq 1$. Choose $K = \max(K'', K') + 1$, where K' is selected according to Lemma A.1. Fix a history h that involves K repetitions of the same policy y'. If $y' \neq y$, then the conclusion follows from the first case. If $y' = y$, then the conclusion follows from Lemma A.1. This proves that $\tilde{\varphi}$ satisfies [I].

Since $\tilde{\varphi}$ perfectly foresees the government's action in the next period, $\tilde{\psi}$ induces the best response to the government's policy in every period. Hence, $\forall k$ and $\forall t \in \tau^k(\varepsilon)$, the individual household receives at least η more payoff than ψ^*, given that the other households use ψ^*. Therefore, the average payoff of a typical household after T_k periods is at least $\#\tau^k(\varepsilon)\eta/T_k$ more than the candidate equilibrium payoff. By hypothesis, $\lim_{k\to\infty} \#\tau^k(\varepsilon)\eta/T_k \geq \varepsilon$. Hence, the long-run average payoff from $\tilde{\psi}$ is at least $\varepsilon\eta$ higher than $v_i(\psi_p^*, \psi_g^*)$ which contradicts the hypothesis that (ψ_p^*, ψ_g^*) is a (symmetric) Nash equilibrium of the game. Q.E.D.

Proof of Proposition 5.3

Clearly, $w^r \geq w^n$. If $w^r = w^n$, then w^n can be achieved by a symmetric Nash equilibrium of the game where each player chooses the Nash equilibrium of the component game following every period.

Suppose that $w^r > w^n$, and let y^r be the Ramsey policy of the government. Define x^r as the private sector's optimal response to y^r in the component game,

$$\varphi_i(h) = \begin{cases} y^r & \text{if} \quad h = (y^r, \ldots, y^r), \\ y^n & \text{otherwise.} \end{cases}$$

Let $\psi_i = b \circ \varphi_i$ and $\psi_p = \prod_i \psi_i$. Define $\psi_g(h) = y^r$ for all h. In order to see that (ψ_p, ψ_g) forms a Nash equilibrium, note that each household's action is optimal with respect to ψ_g. The government entertains w^r as the equilibrium payoff. Since any potential gain from deviating from the assigned action is followed by noncommitment payoff, the government has no incentive to employ another forecasting scheme. Since each household forecast y^r in each period along the path, x^r is optimal. Q.E.D.

Proof of Proposition 6.3

By way of contradiction, suppose that $\exists (\psi_p^*, \psi_g^*) \in \bigcup_{\delta<1} \bigcap_{\delta' \geq \delta} NE(G_L(\delta'))$ and $\exists \{\delta_n\}_{n=1}^{\infty}$ converging to 1 such that

$$\lim_{n\to\infty} v_g^{\delta_n}(\psi_p^*, \psi_g^*) \leq w^r - \varepsilon,$$

for some $\varepsilon > 0$. In particular, $\exists N \geq 1$ such that $\forall n \geq N$,

$$v_g^{\delta_n}(\psi_p^*, \psi_g^*) < w^r - \varepsilon/2. \tag{A.1}$$

Given ψ_p^*, there exists $K > 0$ such that $\forall t \geq K$

$$w(\theta_a^t(\psi_p^*, \psi_g^r), \theta_g^t(\psi_p^*, \psi_g^r)) \geq w^r - \frac{\varepsilon}{4}.$$

Hence, the average equilibrium payoff to the government is at least $(1 - \delta^K)\underline{w} + \delta^K(w^r - \varepsilon/4)$, where $\underline{w} = \inf_{x,y} w(x, y)$. Since K is completely determined by ψ_p^* and $\delta \uparrow 1$, $\exists \delta^*$ such that $\forall \delta > \delta^*$

$$(1 - \delta^K)\underline{w} + \delta^K(w^r - \varepsilon/4) > w^r - \varepsilon/2. \tag{A.2}$$

Since $\delta_n \to 1$, we can choose N^* such that $\forall n \geq N^*$

$$(1 - (\delta_n)^K)\underline{w} + (\delta_n)^K(w^r - \varepsilon/4) > w^r - \varepsilon/2.$$

This contradicts (A.1), since

$$v_g^{\delta_n}(\psi_p^*, \psi_g^*) \geq (1 - (\delta_n)^K)\underline{w} + (\delta_n)^K(w^r - \varepsilon/4) > w^r - \varepsilon/2.$$

The conclusion follows from this contradiction. Q.E.D.

Proof of Proposition 6.4

Fix a Nash equilibrium (ψ_p, ψ_g) of $G_L(\delta, c, \varepsilon)$. Note that for any $\varepsilon > 0$, $K_i(y, \varepsilon) \geq 1$. Since the memory cost increases as $K_i(y, \cdot)$ increases for some set of y's with positive measure, each household must set $K_i(y, \varepsilon) = 1$ for almost all y's unless y is used by the government with positive probability along the path of a Nash equilibrium. Recall that Y is convex. By Fubini's theorem, for any $\eta > 0$, there exists y_η in η neighborhood of y^r such that $K_i(y_\eta, \varepsilon) = 1$ for all $i \in [0, 1]$ but a null set. By repeating y_η after every history, the government can receive at least

$$(1 - \delta)w(\theta_p^1(\psi_p, \psi_g), y_\eta) + \delta w(b(y_\eta), y_\eta), \tag{A.3}$$

since almost all households forecast y_η after observing a single y_η. Since w is continuous and $\eta > 0$ is arbitrary, (A.3) implies that in any Nash equilibrium (ψ_p, ψ_g) the government must receive at least

$$(1 - \delta)w(\theta_p^1(\psi_p, \psi_g), y^r) + \delta w^r,$$

which must be greater than

$$(1-\delta)\min_{x\in X} w(x, y^r) + \delta w^r. \qquad \text{Q.E.D.}$$

References

Abreu, Dilip and Ariel Rubinstein, 1988, The structure of Nash equilibrium in repeated games with finite automata, Econometrica 56, 1259–1281.

Bray, Margaret and David M. Kreps, 1987, Rational learning and rational expectations, in: George Feiwel, ed., Arrow and the ascent of modern economic theory (New York University Press, New York, NY) 597–625.

Chari, V.V. and Patrick J. Kehoe, 1990, Sustainable plans, Journal of Political Economy 98, 783–802.

Fudenberg, Drew and David K. Levine, 1989, Reputation and equilibrium selection in games with a patient player, Econometrica 57, 759–778.

Kalai, Ehud, Dov Samet, and William Stanford, 1988, A note on reactive equilibria in the discounted prisoner's dilemma and associated games, International Journal of Game Theory 17, 177–186.

Kydland, Finn E. and Edward J. Prescott, 1977, Rules rather than discretion: Inconsistency of optimal plan, Journal of Political Economy 85, 473–491.

Lucas, Robert E., 1986, Adaptive behavior and economic theory, Journal of business 59, 401–426.

Marcet, Albert and Thomas J. Sargent, 1989, Convergence of least squares learning mechanisms in self-referential linear stochastic models, Journal of Economic Theory 48, 337–368.

Matsui, Akihiko, 1989, Information leakage forces cooperation, Games and Economic Behavior 1, 94–115.

Ramsey, Frank P., 1927, A contribution to the theory of taxation, Economic Journal 37, 47–61.

Rubinstein, Ariel, 1986, Finite automata play the repeated prisoners' dilemma, Journal of Economic Theory 39, 83–96.

Stokey, Nancy L., 1991, Credible public policy, Journal of Economic Dynamics and Control 15, 627–656.