Materials 24 - Implementing the target criterion (TC)

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1 Description of the three steps

Overall goal: find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and implements the target criterion in the simplified anchoring model, equation (B.1).

I proceed in 3 steps:

1) *find an exogenous sequence* $\{i_t\}_{t=1}^T$ *that replaces the Taylor rule as a DGP for* i *and fulfills the other model equations, w/o a target criterion;*

2) find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and fulfills the other model equations *including a simple target criterion from the RE model with discretion;*

3) find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and fulfills the other model equations, *including the anchoring target criterion.*

Variable dimensions in this exercise:

- The # of exogenous sequences to feed in and optimize over: $\{i_t\}$, $\{i_t, x_t\}$ or $\{i_t, x_t, \pi_t\}$.
- The # of equations to consider as residual: none, (A.9), (A.9) and (A.10) or (A.9), (A.10) & TC.

1. "Choosing exogenous sequences for the observables" - the main logic of the exercise

    - The system we are trying to solve can be summarized as:

$$x_t = -\sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$
$$\pi_t = -\kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$
$$i_t = \mathcal{N}(0, \sigma^2) \tag{exog. DGP for i}$$

    - I'll mark the given stuff in blue. In all of this exercise I treat the expectations equations as exactly fulfilled, so the above is

$$x_t = -\sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$
$$\pi_t = -\kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$
$$i_t = \mathcal{N}(0, \sigma_i^2) \tag{exog. DGP for i}$$

    - What this shows is that if the i-DGP is exact, which it has to be, then A9 pins down x_t, and A10 pins down π_t, uniquely. I cannot treat anything as a residual equation, and since all $\{i_t\}$ fulfill this system, the initial guess is the solution, even if expectations blow up.

- So if I want to add "wiggle-room," and make say A9 residual, I need something else to pin down x_t; in other words, I need to feed in (and optimize over) an exogenous sequence of x:

$$res_{A9} = -x_t - \sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$
$$\pi_t = -\kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$
$$i_t = \mathcal{N}(0, \sigma_i^2) \qquad \text{(exog. DGP for } i\text{)}$$
$$x_t = \mathcal{N}(0, \sigma_x^2) \qquad \text{(exog. DGP for } x\text{)}$$

- Similarly, the maximum I can do here is to feed in and optimize over π as well:

$$res_{A9} = -x_t - \sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$
$$res_{A10} = -\pi_t - \kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$
$$i_t = \mathcal{N}(0, \sigma_i^2) \qquad \text{(exog. DGP for } i\text{)}$$
$$x_t = \mathcal{N}(0, \sigma_x^2) \qquad \text{(exog. DGP for } x\text{)}$$
$$pi_t = \mathcal{N}(0, \sigma_\pi^2) \qquad \text{(exog. DGP for } \pi\text{)}$$

2. "Implementing the RE-discretion target criterion"

   - The only thing that changes wrt. point 1 is that I add the TC as a model equation, with its own residual term. In terms of the first equation system above:

$$x_t = -\sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$
$$\pi_t = -\kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$
$$i_t = \mathcal{N}(0, \sigma_i^2) \qquad \text{(exog. DGP for } i\text{)}$$
$$\pi_t = -\frac{\lambda_x}{\kappa} x_t \tag{RE-TC}$$

   $\rightarrow$ the TC has to be a residual equation.

3. "Implementing the simple anchoring target criterion"

   - The only thing that changes wrt. point 1 is that I add the anchoring TC (eq. B.1) as a model equation, with its own residual term. Since this requires a bunch of expected future terms, I evaluate its residual not at each simulation iteration t, but at the end, T. Also, I only evaluate $T - H$ residuals, so I can treat H simulation periods as expectations.

$$x_t = -\sigma i_t + f^1(\text{expectations}) + f^2(\text{exogenous states}) \tag{A9}$$

$$\pi_t = -\kappa x_t + f^3(\text{expectations}) + f^4(\text{exogenous states}) \tag{A10}$$

$$i_t = \mathcal{N}(0, \sigma_i^2) \tag{exog. DGP for i}$$

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \text{stuff} \times \mathbb{E}_t \sum_{h=1}^{H} f(x_{t+h}, \pi_{t+h}, s_{t+h}, k_{t+h}, \bar{\pi}_{t+h}, \mathbf{g}_{\bar{\pi}}(t+h)) \tag{anchoring TC}$$

Questions/notes:

1. I choose $\lambda_x = 0.5$ for these figures.

2. In order not to assume perfect foresight, I write $\mathbb{E}_t\, s_{t+h} = h_x^{h-1} s_t$ in the TC.

3. Could one optimize t-by-t? Normally, I think yes, with anchoring TC, I think no.

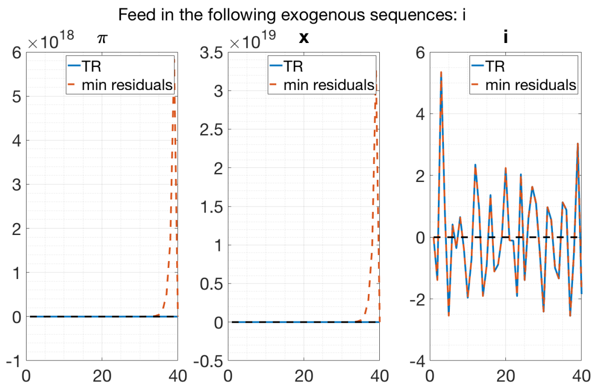
4. Solver stops prematurely (loss on order `e+03` or `e+08`)

5. "Value function iteration-equivalent" solution method?

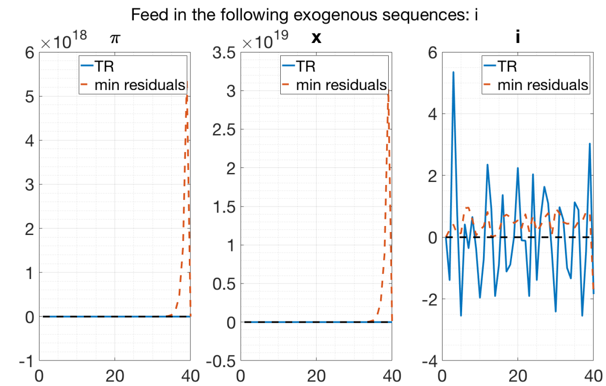
6. "Spline-equivalent" method of finding the optimal functional form that delivers the optimal sequence $\{i_t\}_{t=1}^T$? $\rightarrow$ a numerical approx to the optimal reaction function that replaces the TR.

2 Choosing exogenous sequences for the observables

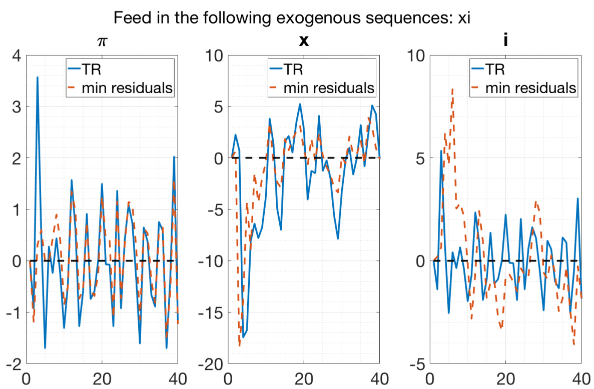
Figure 1: Simulation using Taylor rule against exogenous sequences that minimize equation residuals



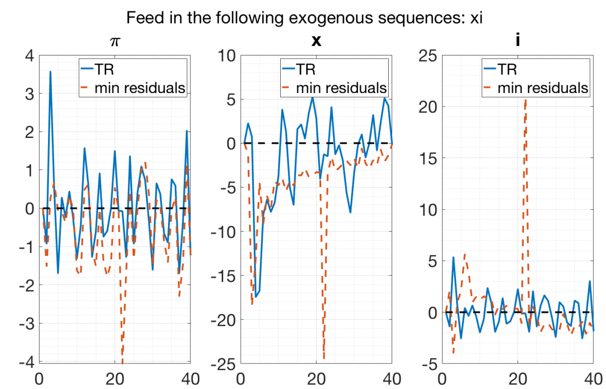
(a) Inputs: i, residual eq.: none; initialized at TR



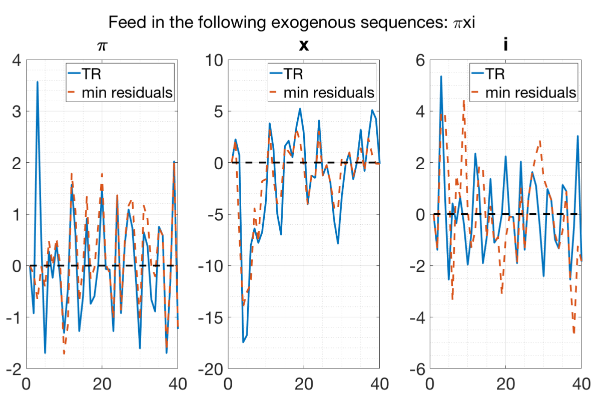
(b) Same as (a), initialized at random sequence(s)



(c) Inputs: x, i, residual eq.: A9; initialized at TR



(d) Same as (c), initialized at random sequence(s)



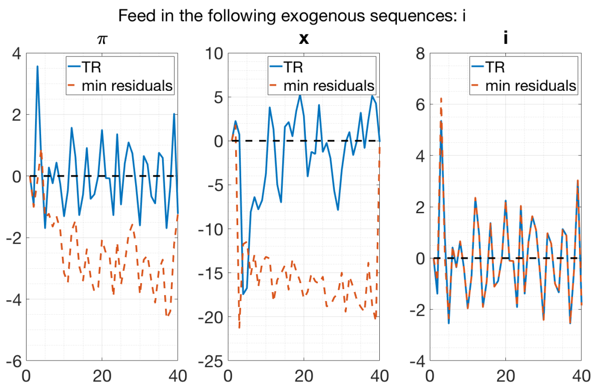
(e) Inputs: π, x, i, residual eq. A9, A10; initialized at TR

(f) Same as (e), initialized at random sequence(s)

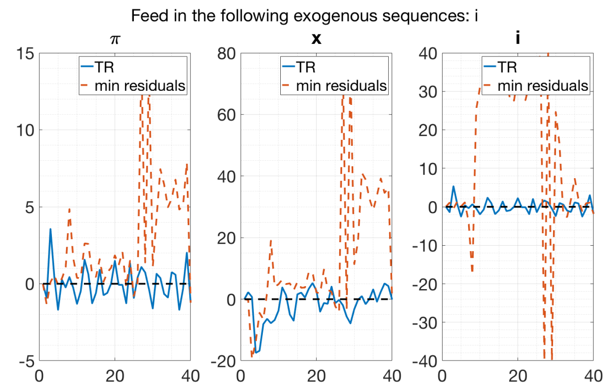
$\rightarrow$ I can implement the Tayor-rule-outcome without using a Taylor rule. (Conditional on initial sequences being the Taylor-rule-sequences.)

3 Implementing the RE-discretion target criterion

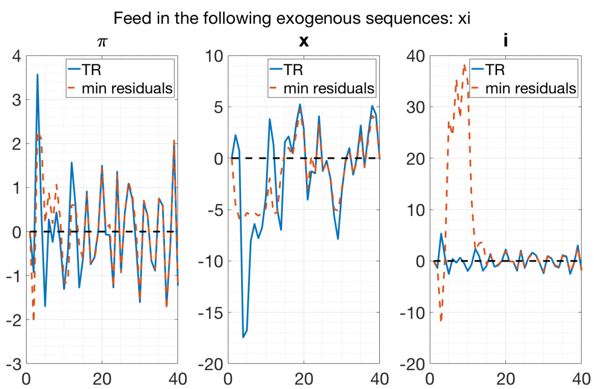
Figure 2: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including RE discretion target criterion



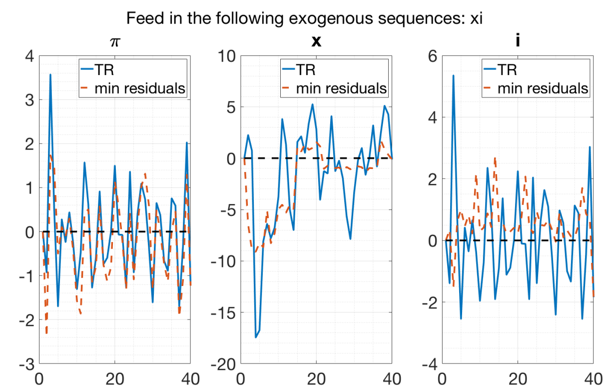
(a) Inputs: i, residual eq.: TC.



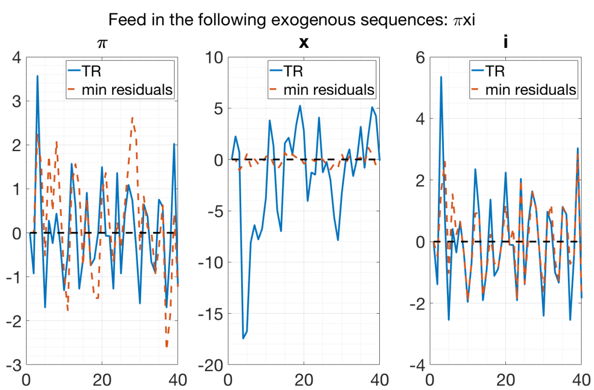
(b) Same as (a), initialized at random sequence(s)



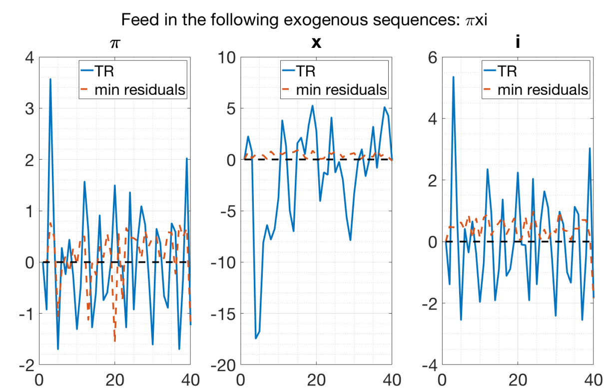
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



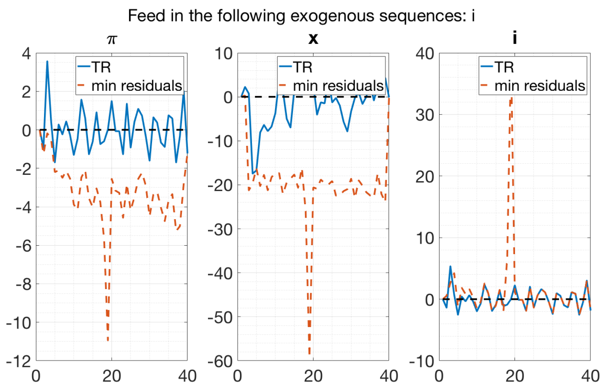
(e) Inputs: π, x, i, residual eq. A9, A10, TC.



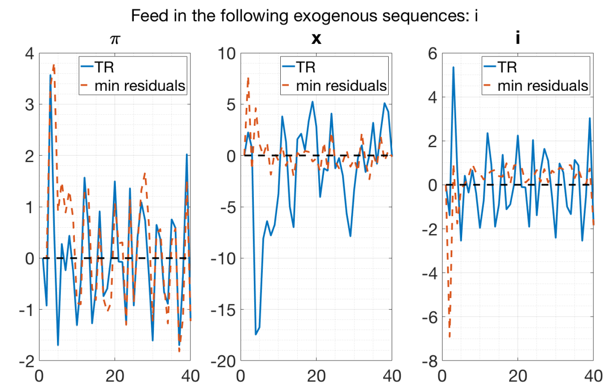
(f) Same as (e), initialized at random sequence(s)

4 Implementing the simple anchoring target criterion

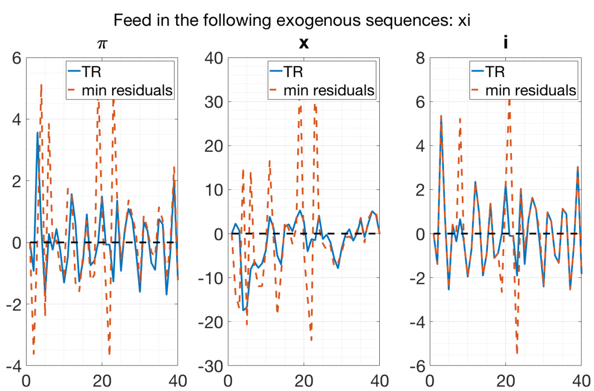
Figure 3: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including the simple anchoring target criterion



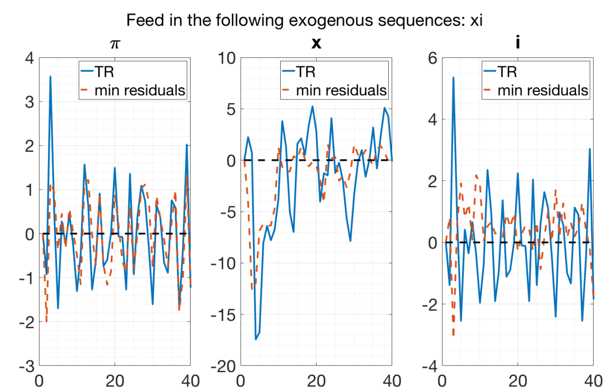
(a) Inputs: i, residual eq.: TC.



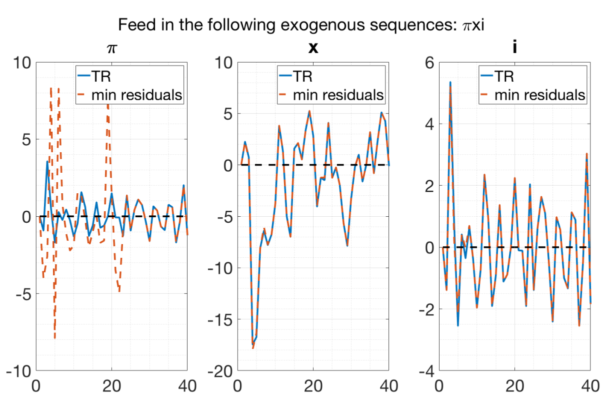
(b) Same as (a), initialized at random sequence(s)



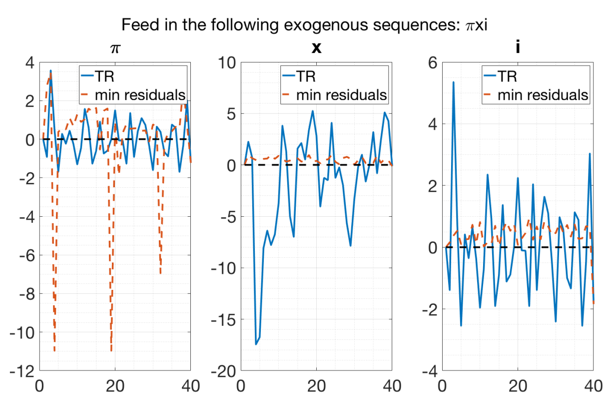
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



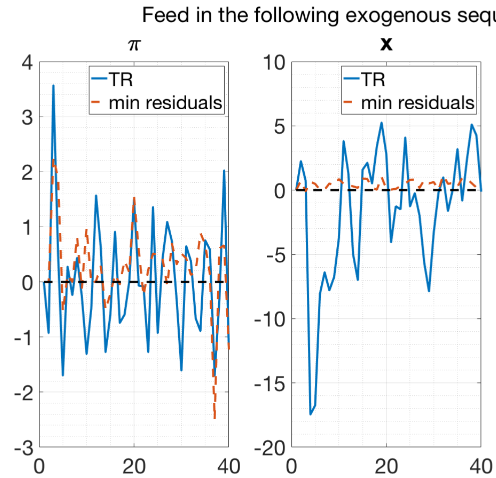
(e) Inputs: π, x, i, residual eq. A9, A10, TC.



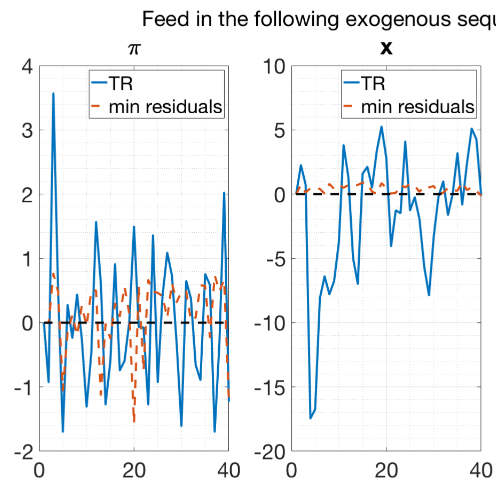
(f) Same as (e), initialized at random sequence(s)

Comparison of the three exercises with favorite specification

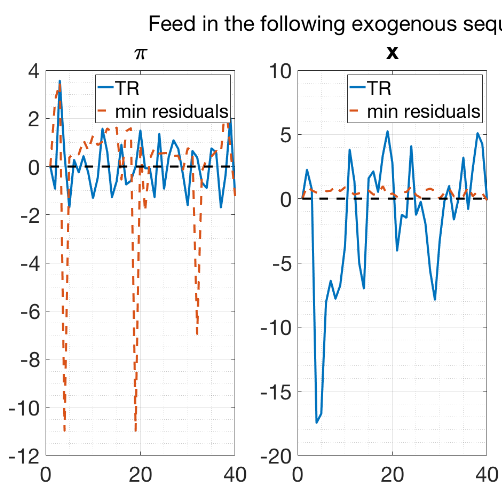
Figure 4: Optimizing over $\{\pi_t, x_t, i_t\}$, initialized at random sequences



(a) No TC



(b) RE-discretion TC



(c) Anchoring TC

5 A value function iteration attempt at finding the optimal interest-rate-sequence

The planner chooses $\{\pi_t, x_t, i_t, f_{a,t}, f_{b,t}, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$V(\mathbf{x}_t, t) = \max -\Big\{(\pi_t^2 + \lambda_x x_t^2) + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1)\Big\} \tag{1}$$

$$\text{s.t. to model equations} \tag{2}$$

Model equations are:

$$\pi_t = \kappa x_t + (1-\alpha)\beta f_a(t) + \kappa\alpha\beta b_2(I_3 - \alpha\beta h_x)^{-1} s_t + e_3(I_3 - \alpha\beta h_x)^{-1} s_t \tag{3}$$

$$x_t = -\sigma i_t + \sigma f_b(t) + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} s_t - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} s_t + \sigma e_1 (I_3 - \beta h_x)^{-1} s_t\big) \tag{4}$$

$$f_a(t) = \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} + b_1(I_3 - \alpha\beta h_x)^{-1} s_t \tag{5}$$

$$f_b(t) = \frac{1}{1-\beta}\bar{\pi}_{t-1} + b_1(I_3 - \beta h_x)^{-1} s_t \tag{6}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\big) \tag{7}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \tag{8}$$

Let's substitute out $x_t, f_{a,t}$ and $f_{b,t}$, so that the state vector is simply $\mathbf{x}_t = (\bar{\pi}_t, k_t^{-1}, r_t^n, u_t)'$.

The problem becomes to choose $\{\pi_t, i_t, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$
\begin{aligned}
V(\mathbf{x}_t, t) = \max -\Bigg\{ & \pi_t^2 + \lambda_x \sigma^2 i_t^2 + \lambda_x \frac{\sigma^2}{(1-\beta)^2} \bar{\pi}_{t-1}^2 - \lambda_x \frac{\sigma^2}{1-\beta} i_t \bar{\pi}_{t-1} \\
& - \lambda_x \sigma \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) i_t s_t \\
& + \lambda_x \frac{\sigma}{1-\beta} \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) \bar{\pi}_{t-1} s_t \\
& + \lambda_x \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big)^2 s_t \\
& + \beta \, \mathbb{E}_t V(\mathbf{x}_{t+1}, t+1) \Bigg\} \qquad (9)
\end{aligned}
$$

s.t. to model equations

$$
\begin{aligned}
\pi_t = & -\kappa \sigma i_t + \left(\kappa \sigma \frac{1}{1-\beta} + \frac{(1-\alpha)\beta}{1-\alpha\beta} \right) \bar{\pi}_{t-1} \\
& + \Big(\kappa \sigma b_1 (I_3 - \beta h_x)^{-1} + \kappa (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \kappa \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \kappa \sigma e_1 (I_3 - \beta h_x)^{-1} \\
& + (1-\alpha) \beta b_1 (I_3 - \alpha \beta h_x)^{-1} + \kappa \alpha \beta b_2 (I_3 - \alpha \beta h_x)^{-1} + e_3 (I_3 - \alpha \beta h_x)^{-1} \Big) s_t \\
\bar{\pi}_t = & \ \bar{\pi}_{t-1} + k_t^{-1} \big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \big) \qquad (10) \\
k_t^{-1} = & \ k_{t-1}^{-1} + \mathbf{g} (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \qquad (11)
\end{aligned}
$$

A Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \qquad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \qquad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + b s_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \qquad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \qquad fe_{t-1} = z_t - (a_{t-1} + b s_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \qquad f_a(t) = \frac{1}{1-\alpha\beta} a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta} a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1} s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \begin{pmatrix} \bar{\pi}_t & 0 & 0 \end{pmatrix}'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix} \sigma & 1-\beta & -\sigma\beta \end{bmatrix} f_b + \sigma \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1} s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix} (1-\alpha)\beta & \kappa\alpha\beta & 0 \end{bmatrix} f_a + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1} s_t \tag{A.10}$$

B Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\Bigg\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t)\Big) \\ \Big(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} (1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j))\Big)\Bigg\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^{0} \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \\ -\frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\Big(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} (k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j)\Big) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs

C A target criterion system for an anchoring function specified for gain changes

$$k_t = k_{t-1} + \mathbf{g}(fe_{t|t-1}) \tag{C.1}$$

Turns out the k_{t-1} adds one $\varphi_{6,t+1}$ too many which makes the target criterion unwieldy. The FOCs of the Ramsey problem are

$$2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t^{-1}\varphi_{5,t} - \mathbf{g}_\pi(t)\varphi_{6,t} = 0 \tag{C.2}$$

$$cx_{t+1} + \varphi_{5,t} - (1 - k_t^{-1})\varphi_{5,t+1} + \mathbf{g}_{\bar{\pi}}(t)\varphi_{6,t+1} = 0 \tag{C.3}$$

$$\varphi_{6,t} + \varphi_{6,t+1} = fe_t\varphi_{5,t} \tag{C.4}$$

where the red multiplier is the new element vis-a-vis the case where the anchoring function is specified in levels ($k_t^{-1} = \mathbf{g}(fe_{t-1})$, as in App. B), and I'm using the shorthand notation

$$c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa} \tag{C.5}$$

$$fe_t = \pi_t - \bar{\pi}_{t-1} - bs_{t-1} \tag{C.6}$$

(C.2) says that in anchoring, the discretion tradeoff is complemented with tradeoffs coming from learning ($\varphi_{5,t}$), which are more binding when expectations are unanchored (k_t^{-1} high). Moreover, the change in the anchoring of expectations imposes an additional constraint ($\varphi_{6,t}$), which is more strongly binding if the gain responds strongly to inflation ($\mathbf{g}_\pi(t)$). One can simplify this three-equation-system to:

$$\varphi_{6,t} = -cfe_t x_{t+1} + \left(1 + \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1}) - fe_t\mathbf{g}_{\bar{\pi}}(t)\right)\varphi_{6,t+1} - \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1})\varphi_{6,t+2} \tag{C.7}$$

$$0 = 2\pi_t + 2\frac{\lambda}{\kappa}x_t - \left(\frac{k_t^{-1}}{fe_t} + \mathbf{g}_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} \tag{C.8}$$

Unfortunately, I haven't been able to solve (C.7) for $\varphi_{6,t}$ and therefore I can't express the target criterion so nicely as before. The only thing I can say is to direct the targeting rule-following central bank to compute $\varphi_{6,t}$ as the solution to (C.8), and then evaluate (C.7) as a target criterion. The solution to (C.8) is given by:

$$\varphi_{6,t} = -2\,\mathbb{E}_t\sum_{i=0}^{\infty}(\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i})\prod_{j=0}^{i-1}\frac{\frac{k_{t+j}^{-1}}{fe_{t+j}}}{\frac{k_{t+j}^{-1}}{fe_{t+j}} + \mathbf{g}_\pi(t+j)} \tag{C.9}$$

Interpretation: the anchoring constraint is not binding ($\varphi_{6,t} = 0$) if the CB always hits the target ($\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i} = 0 \quad \forall i$); or expectations are always anchored ($k_{t+j}^{-1} = 0 \quad \forall j$).