

Work after

12 Feb 2020

## COPY OF JOB MARKET

Time in reverse

### TIMELINE

Feb/March 2021: accept job

Jan/Feb 2021: interview Projects

Nov 2020: submit applications

Oct 2020: job talk / (submit) GLMM ← Reading 2

Aug 31 2020: a very complete draft

Summer 2020: very hard work

writing → meaning math + text

i.e. not refining the question

May 2020: first draft ← Reading 1

try to circulate in Dept.

## Branch's RPE (restricted perceptions eqb)

(an undercategory of which is a misspecification eqb, ME) is this model uncertainty/selection issue:  
agents consider the set  $\tilde{\mathcal{X}}$

$$\tilde{\mathcal{X}} = \left\{ x_t = a + b' \tilde{z}_{t-1} + e_t : \dim(\tilde{z}) < \dim(z) \right\}$$

i.e. each element in  $\tilde{\mathcal{X}}$  is a PLM that is misspecified  
→ a fitting model which omits a variable or a lag.

Agents then pick the best-performing model in an  
FEV/MSE or related concept-sense, like in Cho & Kasa.

→ and I'm also seeing that the Ball-effect

is a negative feedback effect b/c then  
the int-rate,  $E(\bar{x})$  &  $(\bar{x}, x)$  move in  
opposite directions

↳ This is what Branch calls a third effect of

exogenous disturbances: 1) direct effect on states  $x_t$   
2) induced via expectations 3) policy feedback effect

⇒ joint determination of optimal policy & ME!

## Framework

### 1.1 Optimal State-Contingent Paths

The model equations are summarised by

$$\hat{I} \begin{bmatrix} 2_{t+1} \\ E_t 2_{t+1} \end{bmatrix} = A \begin{bmatrix} 2_t \\ e_t \end{bmatrix} + B i_t + C s_t \quad (1.1)$$

$2_t$  := vector of "nonpredetermined endog vars" (jumps)

$e_t$  := vector of "predetermined endog vars" (endog states)

$i_t$  := policy instrument

$s_t$  := vector of exog disturbances

$$\hat{I} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2_{t+1} \\ E_t 2_{t+1} \end{bmatrix} = |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 2_t \\ e_t \end{bmatrix}$$

$$+ |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + -|\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ C_2 \end{bmatrix} s_t$$

$$\text{Objectives of policy} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \quad (1.2)$$

$$\text{where the period loss } L_t = L_t = \frac{1}{2} (\tau_t - \tau^*)' W (\tau_t - \tau^*) \quad (1.3)$$

$\tau_t$  = vector of target variables,  $\tau_t = T y_t$

where  $y_t = \begin{bmatrix} z_t \\ \bar{z}_t \\ \bar{s}_t \end{bmatrix}$  is the vector of endog. variables.

DEF. A policy rule which is optimal from a timeless perspective in this more general context is a rule such that

1.) jumps  $z_t$  can be expressed as a time-invariant function

$$z_t = f_0 + f_2 \bar{z}_t + f_5 \bar{s}_t \quad (1.5)$$

the bars reflect that the fit may involve additional endog. or exog. states relative to the model

2) The e.g.b evolution of  $\{y_t\}$  (i.e. all endog. vars) minimizes the loss (1.2) for all  $t \geq t_0$ , subject to the model (1.1)

$$\text{and s.t. } \tilde{E} z_{t_0} = \bar{e} \equiv \tilde{E} [f_0 + f_2 \bar{z}_{t_0} + f_5 \bar{s}_{t_0}] \quad (1.6)$$

which restricts the initial values of the jumps. ("timelessness")

Indeed this method is the same as the one outlined in Chapter 7, just in a more general writing. The solution procedure is to set up the Lagrangian

$$L_{t_0} = E_{t_0} \left\{ \sum_{t=t_0}^{\infty} \beta^{t-t_0} [L(y_t) + \tilde{\gamma}_{t+1}' \tilde{A} y_t - \tilde{\rho}' \tilde{\gamma}_t' \tilde{I} y_t] \right\} \quad (13)$$

where  $\tilde{A} = [A \ B]$   $\tilde{f} = [\tilde{I} \ 0]$

$\lambda$ -multipliers on the model eqs.

$$\tilde{\gamma}_{t+1} = \begin{bmatrix} \xi_{t+1} \\ \Xi_{t+1} \end{bmatrix} \rightarrow \begin{array}{l} \text{pertains to endog. states (predetermined at } t) \\ \text{pertains to jumps (set at } t) \end{array}$$

or vice versa :S

Shocks  $s_t$  have been suppressed in L b/c FOCs are independent of them anyway.

Note that the last term in  $L$ , at  $t_0$ , gives the initial constraints

$$\tilde{\gamma}_{t_0}' \tilde{I} y_{t_0} = \tilde{\gamma}_{t_0}' \tilde{L}_{t_0} + \Xi_{t_0}' \tilde{E} \tilde{e}_{t_0}$$

$\uparrow$                              $\uparrow$

initial values  
of endog. states                            constraints (1.6)  
(the "timelessness")

Time-out: timelessly optimal vs.  $t_0$ -optimal  
 time-consistent vs. time-inconsistent optimal commitment

Indeed:

A  $t_0$ -optimal plan minimizes (1.7) s.t. same constraints  
 EXCEPT it replaces (1.6) ("timelessness") w/

$$\Xi_{t_0-1} = 0 \quad (1.8)$$

i.e. the "bygones-be-bygones" assumption

FOLs:

$$\tilde{A}' E_+ \varphi_{++} + T' W (\tau_t - \tau^*) - \beta^{-1} \tilde{I}' \varphi'_+ = 0 \quad (1.9)$$

$\forall t \geq t_0$ .

Restatement:

$t_0$ -optimal solution: any bounded processes for  $\{\gamma_t\}$  &  $\{\varphi_t\}$

that satisfy the FOLs (1.9) s.t. the time-inconsistency constraint (1.8)

timelessly optimal sol.: any bounded processes for  $\{\gamma_t\}$  &  $\{\varphi_t\}$

that satisfy the FOLs (1.9) s.t. the time-consistency constraint (1.6)

Yet again restate the timelessly optimal solution:

1.  $E_{t_0}$  satisfies a time-invariant relation of the form (1.5)
2.  $\{y_t\} \forall t \geq t_0$  evolve according to the (by assumption bounded) unique solution to the FOCs (1.9) s.t. the model (1.1)  
w/ the given initial conditions and initial L-multiples  
 $\Xi_{t_0-1}$ .
3. The initial multipliers are given by a linear rule of the form

$$\Xi_{t_0-1} = g_0 + g_2 \bar{L}_{t_0} + g_5 \bar{S}_{t_0}. \quad (1.11)$$

w/  $g_0, g_2, g_5$  satisfying

$$\tilde{E}2_+ = e_0 + e_2 \bar{L}_+ + e_5 \bar{S}_+ + e_{\bar{\Xi}} [g_0 + g_2 \bar{L}_+ + g_5 \bar{S}_+] \quad (1.12)$$

$\forall t \geq t_0$

(where what this is saying is that the rule for initial multipliers has to be timelessly optimal (i.e. compatible w/ 1.6) and at the same time conform to the time-invariant rule (1.5))

And note: These are just the optimal plans for  $\{y_t\}$ .

Implementing this plan is another question.

I suggest now to revisit the simple example in Ch. 7 of deriving the to-/timelandy optimal plans, augmented w/ some review of dynamic systems from Alpha Chiang's wonderful book, before turning to the implementation of plans.

Ch. 7 p. 472 The Ramsey Problem (shh... it's a secret)

$$L = \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} [\pi_t^2 - \lambda(x_t - x^*)^2] + \rho_t [\pi_t - kx_t - \beta\pi_{t+1}] \right]$$

FOCs:

$$\pi_t: \pi_t + \varphi_t - \varphi_{t-1} = 0 \quad (1.7) \quad t \geq 0$$

$$x_t: \lambda(x_t - x^*) - k\varphi_t = 0 \quad (1.8)$$

$$\text{and for } t=0, \quad \varphi_{-1} = 0 \quad (1.9)$$

This may in fact not be time-inconsistent b/c this is just saying that the model isn't a constraint in  $t = -1$ .

But initial conditions may be, or stay.

Sub out  $\pi$  &  $x$  from

$$\pi_t = \kappa x_t + \beta \pi_{t+1} \quad (1.1)$$

using (1.7) & (1.8)

$$p_{t-1} - y_t = \kappa \left[ \frac{\kappa}{\lambda} y_t + x^* \right] + \beta (y_t - y_{t+1})$$

⇒

$$p_{t-1} - y_t - \beta y_t + \beta y_{t+1} - \frac{\kappa^2}{\lambda} y_t = \kappa x^*$$

$$\Rightarrow \beta y_{t+1} - \left( 1 + \beta + \frac{\kappa^2}{\lambda} \right) p_t + y_{t-1} = \kappa x^* \quad (1.10)$$

OK, this is a difference eq. of order 2. Let's now  
revisit Alpha Decay

General method of solving difference equations

(p. 554 book, pdf. p. 561 mac) Ch. 16 / Ch. 17

$$y_{t+1} + a y_t = c \quad (16.6)$$

The general solution =  $y_p + y_c$

↑      ↑

particular      complementary  
solution      function

$y_p$  solves the nonhomogen. eq. (16.6),  $y_c$  the homogen. eq. :

$$y_{t+1} + a y_t = 0$$

$y_p$ : represents intertemporal eqs level of  $y$   
(st. st. 1 guess)

$y_c$ : signifies deviations of the time path from the equilibrium  $y_p$ . (business cycles 1 guess)

Focus on  $y_c$  first:

$$\text{try } y_t = Ab^t$$

$$\text{then } y_{t+1} = Ab^{t+1}$$

$$\text{and the homogeneous eq } y_{t+1} + ay_t = c$$

$$\text{implies } Ab^{t+1} + aAb^t = 0 \quad | : Ab^t$$

$$\rightarrow b + a = 0 \rightarrow b = -a$$

then the complementary pt becomes

$$\underline{y_c = Ab^t = A(-a)^t}.$$

Then, the particular sol. guess  $y_t = k \rightarrow y_{t+1} = b$ .

sub into (16.6)  $y_{t+1} + ay_t = c$

$$\rightarrow k + ak = c \Rightarrow \underline{k = \frac{c}{1+a}} = y_p$$

Since this  $y_p$  is a constant, this is a stationary equilibrium.

Sometimes we want a moving equilibrium (a function of time, like a BGP st. st.), or  $a = -1$ , so the previous  $y_p$  is undefined. Then for the particular sol we can guess  $y_t = kt$

$$\rightarrow k(t+1) + ak_t = c$$

$$\rightarrow k = \frac{c}{t+1+at}$$

which, if  $a = -1$  implies  $k = c$ , so now our moving particular sol is  $y_p = ct$ .

Putting the sol together gives the general sol

$$(16.8) \quad y_t = A(-a)^t + \frac{c}{1+a} \quad \text{if } a \neq -1$$

$$(16.9) \quad y_t = A(-a)^t + c \cdot t \quad \text{if } a = -1$$

$$= A + c \cdot t$$

We still need to determine A.

To determine A, resort to initial conditions  $y_+ = y_0$

For (16.8),  $t = 0$  implies

$$y_0 = A + \frac{C}{1+a} \Rightarrow A = y_0 - \frac{C}{1+a}$$

thus we obtain the "full-fledged" (1a) general sol

$$y_+ = \left(y_0 - \frac{C}{1+a}\right)(-a)^t + \frac{C}{1+a} \quad \text{for } a \neq -1$$

(16.8')

(analogous procedure for the  $a = -1$  case)

Dynamic stability of equilibrium p. 558 / 564 Mac

Dynamic stability is the question of whether the general sol.  $y_+ = y_c + y_p$  tends to  $y_p$  as  $t \rightarrow \infty$ .

" $\overset{\uparrow}{\text{bc}}$ " " $\overset{\uparrow}{\text{st. st}}$ "

⇒ In other words, whether  $y_c \rightarrow 0$  as  $t \rightarrow \infty$ .

Dynamic stability depends then on  $A(b)^t$ , in fact, it depends fully on b!

$b > 1$	explosive
$b = 1$	constant, non-convergent
$b \in (0, 1)$	convergent (stable)
$b = 0$	constant (stable)
$b \in (-1, 0)$	damped oscillations
$b = -1$	uniform oscillations
$b < -1$	explosive oscillations

$\Rightarrow$  Nonoscillatory:  $b > 0$  (else oscillatory)

Damped:  $|b| < 1$  (else explosive)

Note:  $|b|=1$  must be ruled out b/c it's non-convergent.

A: -scales b

- sign of A can invert b.

### Higher-order difference equations

The sd for  $y_p$ , the particular integral, proceeds exactly the same way. So let's focus on  $y_c$ , the complementary function:

Here's a 2<sup>nd</sup> order diff. eq:

$$(17.1) \quad y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

The homogen. version is

$$(17.3) \quad y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Gives the same thing:  $y_t = Ab^t$

$$Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t = 0 \quad | : Ab^t$$

$$b^2 + a_1 b + a_2 = 0 \quad (17.3')$$

(17.3)' is the characteristic equation of (17.3) or (17.1)

Let's repeat: the characteristic equation is the equation you obtain once you plug in the conjecture for the complementary function  $y_c$  of  $y_t = Ab^t$  and simplify.

It follows that a diff. eq. of order  $p$  will have  $p$  roots to the characteristic eq., as (17.3') clearly has 2 "characteristic roots"\*,  $b_1$  and  $b_2$ .

Ok, but which is the solution?

\*characteristic roots are also known as eigenvalues.

Divide side note: take Woodford's 2nd order diff eq:

$$\beta \varphi_{t+1} - (1 + \beta + \frac{\kappa^2}{\lambda}) \varphi_t + \varphi_{t-1} = \kappa x^* \quad (1.10)$$

Take a sol. of the form  $A\mu^t$  and plug

$$\beta A\mu^{t+1} - (1 + \beta + \frac{\kappa^2}{\lambda}) A\mu^t + A\mu^{t-1} = \kappa x^*$$

Divide by  $A\mu^{t-1}$  and consider the homogen case:

$$\beta \mu^2 - (1 + \beta + \frac{\kappa^2}{\lambda}) \mu + 1 = 0$$

This is how you get eq. (1.11)

Ok, but back to Alpha Alarming and the question of what to do w/ the roots.

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Case 1.  $a_1^2 - 4a_2 > 0 \Rightarrow$  2 distinct, real roots

Case 2.  $a_1^2 - 4a_2 = 0 \Rightarrow$  Repeated real roots

Case 3.  $a_1^2 - 4a_2 < 0 \Rightarrow$  Complex roots

In all cases, all roots need to figure in the  $y_c$  b/c they are linearly independent.

Case 1. In this case the complementary function becomes

$$y_c = A_1 b_1 t + A_2 b_2 t$$

(a linear combination of two linearly independent parts)

Case 2. Since  $b_1 = b_2$  here,  $A_1 b_1 t + A_2 b_2 t$  becomes

$$(A_1 + A_2) b^t = A_3 b^t$$

In this case Alpha Chiang still insists that we need an additional linearly independent term, so he introduces  $A_4 \cdot t \cdot b^t$ , which is linearly  $\perp$  b/c you can't obtain it by multiplying  $A_3 b^t$  by some constant coefficient c. (unless  $c = A_4 \cdot t$  (guess))

$\Rightarrow$  Thus in this case the complementary sol becomes

$$y_c = A_3 b^t + A_4 \cdot t \cdot b^t$$

Case 3.  $b_{1,2} = h \pm vi$  (conjugate complex roots)

$$h = -\frac{a_1}{2} \quad v = \frac{\sqrt{4a - a_1^2}}{2} \quad (\text{ahaa!})$$

$$\Rightarrow y_c = A_5 (h+vi)^t + A_6 (h-vi)^t \quad (\text{ahaa!!})$$

thanks to DeMoivre's theorem (15.28')

$$(h \pm vi)^t = R^t (\cos \theta t \pm i \sin \theta t)$$

where  $R = \sqrt{h^2 + v^2} = \sqrt{\alpha_2}$  (always  $> 0$ ) (radius)

The absolute value of the conjugate complex roots

$$\cos \theta = \frac{h}{R} = \frac{h}{\sqrt{\alpha_2}} \quad \text{and} \quad \sin \theta = \frac{v}{R} = \frac{v}{\sqrt{\alpha_2}}$$

$$\begin{aligned} \Rightarrow y_c &= A_5(h+vi)^t + A_6(h-vi)^t \\ &= A_5 R^t (\cos \theta t + i \sin \theta t) + A_6 R^t (\cos \theta t - i \sin \theta t) \\ &= \underbrace{(A_5 + A_6) R^t \cos \theta t}_{y_c = R^t(A_7 \cos \theta t + A_8 \sin \theta t)} + \underbrace{(A_5 - A_6)i R^t \sin \theta t}_{(17.10)} \end{aligned}$$

Convergence of the time path (dynamic stability) in

the p-order difference equation case

For conciseness,  $p=2$  here.

A p-order diff eq converges to st. st if the absolute value of every characteristic root is less than 1.

Sauer Thm: This holds if all p determinants are positive

(the first p sub-determinants or what) (p. 599 / 589 Mac)

Note that Viète's formulas are also an easy way to relate the roots to each other via their sums & products.

For a quadratic equation, the roots take the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

thus  $x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$

□

and  $x_1 \cdot x_2 = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2} = \frac{(a+b)(a-b)}{4a^2} = \frac{a^2 - b^2}{4a^2}$

$$\begin{aligned} &(a+b)(a-b) \\ &= a^2 - b^2 \end{aligned}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} \quad \square$$

thus, Viète:

$$x_1 + x_2 = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

Back to Woodford then. The characteristic eq. (1.11)

$$\beta\mu^2 - \left(1 + \beta + \frac{k^2}{\lambda}\right)\mu + 1 = 0$$

has 2 solutions,  $\mu_1$  &  $\mu_2$

By Vieta,

$$\mu_1 + \mu_2 = -\frac{b}{a} = \frac{1 + \beta + \frac{k^2}{\lambda}}{\beta} > 0$$

$$\mu_1 \cdot \mu_2 = \frac{c}{a} = \frac{1}{\beta} > 0$$

So since  $\mu_1 \mu_2 > 0$ ,  $\text{sign}(\mu_1) = \text{sign}(\mu_2)$

Since  $\mu_1 + \mu_2 > 0$ ,  $\mu_1 > 0$ ,  $\mu_2 > 0$

Now does Woodford know that  $\mu_1 < 1 < \mu_2$ ?

(V1)  $\mu_1 + \mu_2 = \frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta} > 1$  (in fact  $> 2$ )

(V2)  $\mu_1 \cdot \mu_2 = \frac{1}{\beta} > 1$  but close to 1. (around 1.1)

(V2) implies that at least 1 root  $> 1$  b/c if  $\mu_1 = \mu_2 \leq 1$ ,

(V2)  $\nabla$  so suppose  $\mu_2 > 1$ . Why must  $\mu_1 < 1$ ? E.g.

Why can't  $\mu_1 = 1$ ?

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In Woodford's case this is

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - 4\beta}}{2\beta}$$

$$\begin{aligned} \text{Determinant is } & \left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - 4\beta & =: \Delta \\ & = \left(1 + \beta + \frac{k^2}{\lambda}\right)^2 - (2\sqrt{\beta})^2 & a^2 - b^2 = (a+b)(a-b) \\ & = \left(1 + \beta + \frac{k^2}{\lambda} - 2\sqrt{\beta}\right) \left(1 + \beta + \frac{k^2}{\lambda} + 2\sqrt{\beta}\right) \\ & \quad \underbrace{\qquad\qquad\qquad}_{>0} \end{aligned}$$

Ok so for sure there is a value of  $\lambda$  that ensures that (close enough to 0)

Supp  $\beta = 0.99$ . Then  $\sqrt{\beta} = 0.95 \rightarrow 2\sqrt{\beta} = 1.95$

which is miraculously  $= 1 + \beta = 1.99$ !

$$1 + \beta - 2\sqrt{\beta} \rightarrow 1 + x^2 - 2x \quad \text{where } x = \sqrt{\beta}$$

I can write this as  $(1-x)^2 = (1-\sqrt{\beta})^2 \geq 0$  for  $\beta < 1$

ok so at least I know I have 2 distinct real roots.

Ok but that means that  $\Delta > 0$  so that  $\sqrt{\Delta} > 0$

So the 2 roots are

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\Delta}}{2\beta}$$

Is the idea that  $\frac{1 + \beta + \frac{k^2}{\lambda}}{2\beta} \approx 1$ ?

$$= \frac{1}{2\beta} + \frac{1}{2} + \frac{k^2}{2\beta\lambda}$$

$$= \frac{1}{2} \left( \frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta} \right)$$

$$= \frac{1}{2} \left( 1 + \frac{1}{\beta} \left( 1 + \frac{k^2}{\lambda} \right) \right)$$

The problem is that it seems to me to be a quantitative question, but it should hold for all param values  $> 0$ .

So for  $\beta \rightarrow 1$  and  $k \rightarrow 0$  the above becomes

$$\frac{1}{2} \left( 1 + 1 \left( 1 + 0 \right) \right) = \frac{1}{2} (1+1) = 2 \text{ de fine.}$$

And  $\frac{k^2}{\lambda} \rightarrow 0$  even if  $\lambda \rightarrow \infty$  b/c  $k^2 \rightarrow 0$  faster.

I guess  $\Rightarrow$  So  $0 < \mu_1 < 1 < \mu_2$ . And this is important b/c  $\mu_2 > 1$  is explosive, so TVC excludes it.

so one side-question that arises is Blanchard & Kahn conditions, which dictates as many explosive eigenvalues as forward-looking vars. why? And what's up w/ eigenvalues vs. roots of characteristic equations here?

→ or: "characteristic roots are also known as eigenvalues." □

Alpha Unidad makes this connection explicit when he discusses dynamic stability of "simultaneous equations (ODE or difference eqs)"

→ Need to work them out the intuition is this. For an  $n$ -dimensional difference eq, transform it into a 1-dimensional matrix diff eq, and then proceed as usual.

So for Woodford, the complementary function is

$$\varphi_t = A_1 \mu_1^t + A_2 \mu_2^t$$

I guess TRC sets  $A_2 = 0$ .

So  $\varphi_t = A_1 \mu_1^t$  is the complementary sol, for  $A_1$ , yet to be determined.

For the particular solution, guess  $\varphi_t = b$

$$\beta k - (1 + \beta + \frac{k^2}{\lambda})k + k = kx^*$$

$$k = \frac{kx^*}{1 + \beta - 1 - \beta - \frac{k^2}{\lambda}} = \frac{x^*}{-\frac{k}{\lambda}} = -\frac{\lambda x^*}{k}$$

So the general sol is

$$\varphi_t = A_1 \mu_1^t - \frac{\lambda x^*}{k}$$

Now use the initial condition  $\varphi_{t=1} |_{t=0} = 0$   
(set  $t=0$ )

$$0 = A_1 \mu_1^0 - \frac{\lambda x^*}{k} \Rightarrow A_1 = \frac{\lambda x^*}{k}$$

So then the general sol is

$$\varphi_t = \frac{\lambda x^*}{k} \mu_1^t - \frac{\lambda x^*}{k} = -\frac{\lambda x^*}{k} [1 - \mu_1^t] \quad (1.12)$$

The only thing I don't know is why  $t+1$ , not  $t$ ? Typo?

$$\bar{\pi}_t = \varphi_{t-1} - \varphi_t \\ = -\frac{\lambda x^*}{K} \left[ 1 - \mu_1^{t-1} \right] - \left( -\frac{\lambda x^*}{K} \left[ 1 - \mu_1^t \right] \right)$$

$$= -\cancel{\frac{\lambda x^*}{K}} + \frac{\lambda x^*}{K} \mu_1^{t-1} + \cancel{\frac{\lambda x^*}{K}} - \frac{\lambda x^*}{K} \mu_1^t$$

$$\bar{\pi}_t = \frac{\lambda x^*}{K} \mu_1^{t-1} (1 - \mu_1) \quad (1.13)$$

↑ except that again Woodford has t here.

tomorrow : 1) ↑ check on this

2) complete Alpha (Chiang's) eigenvalue

& simultaneous eq. thing

✓ 3) start reworking draft for midwest

macro, using Ryan's abstract-ish ...

2) Alpha Chiang's connection between matrices & difference equations ( & differential eqs) 17 Feb 2020  
 Consider the 2<sup>nd</sup> order diff eq: (p. 602)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \quad (17.21)$$

"concoct an artificial new variable  $x_t = y_{t+1}$ "

$$x_{t+1} + a_1 x_t + a_2 y_t = c$$

$$y_{t+1} = x_t$$

$$\Rightarrow \begin{aligned} x_{t+1} + a_1 x_t + a_2 y_t &= c \\ y_{t+1} - x_t &= 0 \end{aligned} \quad \left. \right\} \quad (17.21')$$

Suppose then that this looks like

$$x_{t+1} + b x_t + g y_t = h \quad (17.24)$$

$$y_{t+1} - x_t = 0$$

- ① Seek particular integrals (denote them by  $\tilde{x}$  &  $\tilde{y}$  since they're 1<sup>st</sup> order values)

- ② Seek complementary functions

① First try stationary sols. If not doesn't work,  
try moving sols of the form  $x_t = k_1 t$ ,  $y_t = k_2 t$  etc.

Supp.  $x_{t+1} = x_t = \bar{x}$  and  $y_{t+1} = y_t = \bar{y}$ . But

$$\begin{aligned} 7\bar{x} + 3\bar{y} &= 4 \\ \bar{y} - \bar{x} &= 0 \end{aligned} \quad \left. \begin{aligned} \bar{x} &= \bar{y} = \frac{4}{10} = \frac{2}{5} \\ \bar{y} &= \bar{x} \end{aligned} \right\}$$

② Complementary parts:

$$y_t = m b^t \quad y_t = n b^t \quad \text{Sust.}$$

$$x_{t+1} + 6x_t + 3y_t = 0 \Rightarrow mb^{t+1} + 6mb^t + 3nb^t = 4$$

$$y_{t+1} - x_t = 0 \quad nb^{t+1} - mb^t = 0$$

|:  $b^t$

$$\begin{aligned} mb + 6m + 3n &= 0 \\ nb - m &= 0 \end{aligned} \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \quad (17.28)$$

looking to solve this for  $(m, n)$ .

$$\begin{aligned} (b+6)m + 3n &= 0 \\ -1m + bn &= 0 \end{aligned} \quad \left. \begin{aligned} & \text{cofficient matrix} - \begin{pmatrix} b+6 & 3 \\ -1 & b \end{pmatrix} \end{aligned} \right\}$$

In order to solve (17.28) w/o  $m=n=0$ , need the determinant of the coff. matrix "to vanish". i.e.

$$\begin{vmatrix} b+6 & 9 \\ -1 & b \end{vmatrix} = 0$$

$$\Rightarrow (b+6)b + 9 = 0 \Rightarrow b^2 + 6b + 9 = 0 \quad (17.29)$$

(17.29) is the characteristic equation, and its roots  $b_{1,2}$  are the characteristic roots = eigenvalues of the simultaneous eq. system.

Having  $b$ , (17.28) gives us  $(m, n)$ . (They usually are equations of the form  $m_i = k, n_i$  for each root  $b_i$ )

Let's do the same thing w/ matrix notation

Express

$$x_{t+1} + 6x_t - 9y_t = 4 \quad (17.29)$$

$$y_{t+1} - x_t = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=I} \underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}}_{=u} + \underbrace{\begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}}_{=K} \underbrace{\begin{bmatrix} x_t \\ y_t \end{bmatrix}}_{=v} = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{=d} \quad (17.29')$$

$$= I u + K v = d$$

where  $u = \underbrace{u}_{=n}$   $v = \underbrace{v}_{=n}$  in Alpha Wang's notation

$$I z_{t+1} + K z_t = d$$

① Particular integral: try constant sols

$$z_{t+1} = z_t = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \Rightarrow (I + K) \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = d$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = (I + K)^{-1} d \quad \text{if } (I + K)^{-1} \text{ exists.}$$

② Complementary functions

$$z_{t+1} = \begin{bmatrix} m b^{t+1} \\ n b^{t+1} \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} \quad z_t = \begin{bmatrix} m \\ n \end{bmatrix} b^t$$

$$\Rightarrow I \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} + K \begin{bmatrix} m \\ n \end{bmatrix} b^t = 0 \quad | : b^t$$

$$(bI + K) \begin{bmatrix} m \\ n \end{bmatrix} = 0 \quad (17.28')$$

It is from this that we wanna find  $\begin{bmatrix} m \\ n \end{bmatrix}$

Again we wanna avoid  $m=n=0$  type solutions

so we demand  $|bI + K| = 0$

which is again the characteristic eq, w/  
eigenvalues / characteristic roots as roots.

## Further comment on the characteristic equation (p.610)

1. Characteristic eq. of a matrix
2. of a single diff. eq / ODE
3. of system of diff. eqs / ODEs.

What's the connection?

1. If I write an  $n$ -order single diff.-eq as a  $1^{\text{st}}$ -order system of eqs, the characteristic eq. is the same (as we saw on the previous ex.)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

$$\underline{\text{CE:}} \quad \underline{\frac{b^2 + a_1 b + a_2}{= 0}} \quad \text{or} \quad \underbrace{K}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}}$$

$$\underline{\text{CE:}} \quad \left| bI + K \right| = \begin{vmatrix} a_1 + b & a_2 \\ -1 & b \end{vmatrix} = (a_1 + b)b + a_2 = 0$$

$$b^2 + a_1 b + a_2 = 0$$


---

## 2. Connection #2

There's a link between the characteristic eq. of a diff. eq. (or an ODE) to that of a particular matrix, call it  $D$ .

$$|bI - D| = 0 \quad (17.38')$$

makes it clear that  $D = -K$ .  $D$ , or  $-K$ , has a special meaning. Write  $I z_{t+1} + K z_t = d$  as  $I z_{t+1} = -K z_t$  (setting  $d = 0$  for the reduced version of the system).

$$\Rightarrow z_{t+1} = -K z_t$$

$\Rightarrow -K$  is the matrix that can transform the system from  $z_t$  to  $z_{t+1}$ . (i.e. it's the transition matrix.)

$\Rightarrow$  The characteristic equation is the  $\det(\text{com-mat}) = 0$  relation!

NB. If  $I = f$  (not identity) so  $f z_{t+1} = -K z_t$

then we need to  $z_{t+1} = f^{-1}(-K) z_t$  and look at  $\det(-f^{-1}K) = 0$ . (SVAR!)

Return to Woodford & try to obtain (1.12) again.

Let me write (1.10) as

$$\beta \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = kx^*$$

$$\Rightarrow \beta \mu^2 - \alpha \mu + 1 = 0$$

$$\mu_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}$$

$$Ok, so \alpha = 1 + \beta + \frac{k^2}{\lambda} \in [1 + \beta, \infty]$$

$$So for \alpha = \min(\alpha) = 1 + \beta,$$

$$\begin{aligned} \mu_{1,2} &= \frac{\alpha \pm \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} = \frac{\alpha \pm \sqrt{\beta^2 + 2\beta + 1 - 4\beta}}{2\beta} \\ &= \frac{\alpha \pm \sqrt{(\beta-1)^2}}{2\beta} = \frac{1+\beta \pm (\beta-1)}{2} \quad \begin{array}{l} \frac{1+\beta + \beta-1}{2} = \beta < 1 \\ \frac{1+\beta - \beta+1}{2} = 1 \end{array} \end{aligned}$$

$$If \alpha \rightarrow \infty$$

$$\mu_{1,2} = \frac{\infty \pm \sqrt{\infty^2}}{2\beta} = \frac{\infty + \infty}{2\beta} = \frac{1}{\beta} > 1$$
$$\frac{\infty - \infty}{2\beta} = 0 < 1 \quad Ok!$$

Since, Woodford,  $0 \leq \mu_1 < 1 \leq \mu_2$

Ok so before turning a 100% to Woodford, let's think about solutions to DSGE models

In Lect 10, Basu looks for the solutions of the 2-dimensional non-interacting system where the eq. sols are  $z'_H = \alpha w_1^+$  and  $z'_{D1} = \beta w_2^+$

Since  $w_1 > 1$  (and  $w_2 \in (0, 1)$ ), TVC sets  $\alpha \neq 0$  ( $\alpha, \beta$ ) correspond here to Alpha/Beta's  $A_1, A_2$  etc. which are usually determined by initial conditions.  
(what AC calls "definitize the constants  $A_i$ ")

For bw-looking vars, TVCs exclude explosive sols.

So maybe the point is that when the economy is a bw-looking eq. system, this is what you do. But if the econ is a fw-looking system, then

$$z_t = w_1 z_{t+1} \quad w_1 < 1, \text{ so } z_t = \frac{1}{w_1} z_{t+1}$$

$\frac{1}{w_1} > 1 \Rightarrow$  transforming the fw-looking one into a bw-looking one requires explosive eqs of the bw-looking system. (Blanchard & Kahn)

Finally back to Woodford, 100%

right, so I've solved the char. eq. to obtain  $\mu_1$ .

So my conj must have been  $\varphi_t = A\mu^t$

Or did Woodford conjecture  $\varphi_t = A\mu^{t+1}$ ? So that

$$\beta A\mu^{t+2} - \alpha A\mu^{t+1} + A\mu = 0$$
$$\Leftrightarrow \beta\mu^2 - \alpha\mu + 1 = 0$$

But technically it doesn't matter b/c after the

conjecture  $\varphi_t = A\mu^t \rightarrow \beta A\mu^{t+1} - \alpha A\mu^t + A\mu^{t-1} = 0$

$\beta\mu - \alpha + \frac{1}{\mu} = 0$  leads to the same thing

if you multiply by  $\mu$ ,

$$\beta\mu^2 - \alpha\mu + 1 = 0$$

So taking Woodford's conjecture then, turn to the particular integral:

A stationary sol would involve  $\varphi_t = \bar{\varphi} +$

$$\beta\bar{\varphi} - \alpha\bar{\varphi} + \bar{\varphi} = kx^*$$

$$\bar{\varphi}(\beta - \alpha + 1) = kx^*$$

$$\bar{\varphi} = \frac{kx^*}{1 + \beta - \alpha} = \frac{kx^*}{-\frac{1}{K}\alpha} = -\frac{\lambda}{K}x^*$$

So w/o yet having definitized the constant, A,  
the general sol is

$$Y_+ = A \mu_1^{t+1} + \left(-\frac{\lambda}{K} x^*\right)$$

Definitize A using initial cond: when  $t = -1$ ,

$$Y_{-1} = 0, \quad \text{so} \quad 0 = A - \frac{\lambda}{K} x^*$$

$$\Rightarrow A = \frac{\lambda}{K} x^*$$

So the "full-fledged" general sol then is

$$\begin{aligned} Y_+ &= \frac{\lambda}{K} x^* \mu_1^{t+1} - \frac{\lambda}{K} x^* \\ Y_- &= -\frac{\lambda}{K} x^* (1 - \mu_1^{t+1}) \end{aligned} \tag{1.12}$$

Alright!

And since  $\pi_+ = Y_{-1} - Y_+$

$$\begin{aligned} &= -\frac{\lambda}{K} x^* (1 - \mu_1^+) - \left(-\frac{\lambda}{K} x^* (1 - \mu_1^{t+1})\right) \\ &= \frac{\lambda}{K} x^* \left[ -1 + \mu_1^+ + 1 - \mu_1^{t+1} \right] \\ &= \frac{\lambda}{K} x^* (\mu_1^+ (1 - \mu_1)) \end{aligned}$$

$$\Rightarrow \pi_+ = (1 - \mu_1) \frac{\lambda}{K} x^* \mu_1^+ \tag{1.13}$$

Consider the timevarying optimal commitment policy. I think that the complementary solution is the same, but  $\lambda$  will be defined differently b/c the initial condition

(19)  $y_{-1} = 0$  is replaced by  $\pi_+ = \bar{\pi} \quad \forall +$   
so that  $\pi_0 = \bar{\pi}$ .

$$(113) \text{ implies } \pi_+ \stackrel{!}{=} \bar{\pi} = (1 - \mu_1) \frac{\beta}{K} x^* \mu_1^+$$

let me recall that  $A = \left( \begin{smallmatrix} -\frac{\beta}{K} x^* \\ 1 \end{smallmatrix} \right)$  so

$$\pi_+ = (1 - \mu_1)(-A)\mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$(\mu_1 - 1) A \mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$\mu_1 A \mu_1^+ - A \mu_1^+ \stackrel{!}{=} \bar{\pi} \quad \forall +$$

$$\Rightarrow A \stackrel{!}{=} 0 \text{ and } \bar{\pi} \stackrel{!}{=} 0 \quad \text{otherwise } \frac{1}{2}.$$

Optimal responses to shocks p. 488.

Take the same loss (except in  $E_{t_0}(\cdot)$ ) as before

(eq (2.4)). Then clearly you obtain the same diff eq for the multipliers  $\varphi_+$  (eq (1.10)) except there's an  $E(\cdot)$  in front (eq (2.6)) and

The cost-push shock  $u_t$  shows up on the RHS

$$\beta E_t y_{t+1} - \alpha p_t + y_{t-1} = \kappa x^* + u_t \quad (2.6)$$

The characteristic eq. is the same, eq. (1.11),

which has 2 roots,  $0 < \mu_1 < 1 < \mu_2$ .

The question is: how the  $f^*$  does Woodford obtain equation (2.7)?

We said that  $y_t = A_1 \mu_1^{t-1} + A_2 \mu_2^{t-1}$

but TBC set  $A_2 = 0$ .

So then the complementary sol is the same,

$p_t = A \mu_1^{t-1}$ , except that  $A$  may be different.

What about the particular solution,  $\bar{p}$ ?

$$\bar{p} (\beta - \alpha - 1) = \kappa x^* + u_t$$

↑ how to treat  $u_t$ ?

As long as  $u_t$  doesn't have a drift, its LR mean is 0. That would imply the same particular sol as in the non-stochastic case. But if  $\bar{p}$  is the same,

then A is also the same.

Ok, so since the  $\mu_1^{t+1}$  terms are gone, we must have gone somehow from (1.12) :

$$y_t = -\frac{\gamma}{K} x^* (1 - \mu_1^{t+1}) \quad \text{to (2.7).}$$

$$= -\frac{\gamma}{K} x^* (1 - \mu^+ \cdot \mu_1)$$

$$= -\frac{\gamma}{K} x^* (\mu_1 - \mu_1 \mu^+ + 1 - \mu_1)$$

$$= \mu_1 \left[ -\frac{\gamma}{K} x^* (1 - \mu^+) \right] - \frac{\gamma}{K} x^* (1 - \mu_1)$$

$$y_t = \mu_1 y_{t-1} - (1 - \mu_1) \frac{\gamma}{K} x^* \quad \text{haha!}$$

The question that remains is how/where do the  $u_{t+j}$  terms come from?

I'm wondering if from the perspective of the difference eq  $u_t$  isn't an "endog" variable?

Or is Woodford saying that  $u_t$  follows its own diff eq which can/is solved?

Wait. Maybe we guess a bit of a different

complementary sol:  $y_t = A_1 \mu_1^{t+1} + A_2 u_t$

or, better yet:  $y_t = A_1 (\mu_1^{t+1} + u_t)$ . But the problem

even w/ this formulation is that it only explains why a single  $u_{t+j}$ -term would show up in  $y_{t+j}$ . But as long as the characteristic eq is the same and the particular sol too, I don't see how  $u$  would enter w/ a sum.

Brown Lect 10 gives some hints: S.3 (p. 3 Mac)

$$z_t = az_{t-1} + m_t \quad |a| < 1 \quad (\text{stable diff eq. w/ exog. fctg. time } m_t)$$

Recursive sol

$$\begin{aligned} z_t &= a[z_{t-2} + m_{t-1}] + m_t \\ &= a^2 z_{t-2} + am_{t-1} + m_t \\ &= a^k z_{t-k} + \sum_{s=k}^{t-1} a^{t-s} m_s \end{aligned}$$

$\rightarrow$

$$z_t = a^t z_0 + \sum_{s=1}^t a^{t-s} m_s$$

Let's reinterpret this a bit

$$z_t = \left( z_0 \right) a^t + \sum_{s=0}^t a^{t-s} m_s$$

$$z_t = A \cdot \mu^t + \sum_{s=0}^t \mu^{t-s} m_s$$

initial condition

exog. function  
of time

dynamics from initial val.

Basin continues w/ unstable diff eqs w/ exog shift:

$$z_t = \alpha z_{t-1} + m_t \quad |\alpha| > 1$$

$$\Rightarrow \alpha z_{t-1} = z_t - m_t \quad \Rightarrow \quad z_{t-1} = \bar{\alpha}^{-1} z_t - \bar{\alpha}^{-1} m_t$$

$$\begin{aligned} \Rightarrow z_{t-1} &= \bar{\alpha}^{-1} [\bar{\alpha} z_{t+1} - \bar{\alpha}^1 m_{t+1}] - \bar{\alpha}^{-1} m_t \\ &= \bar{\alpha}^{-2} z_{t+1} - \bar{\alpha}^{-2} m_{t+1} - \bar{\alpha}^{-1} m_t \\ &= \bar{\alpha}^{-k} z_{t+k} - \sum_{s=0}^{k-1} \bar{\alpha}^{-(s+1)} m_{t+s} \end{aligned}$$

Susanto writes the index differently, but this way of writing it at least corresponds to Woodford

Under uncertainty, just tack on " $E_t$ " to the exog  $m_{t+s}$

$$z_{t-1} = \bar{\alpha}^{-k} z_{t+k} - \sum_{s=0}^{k-1} \bar{\alpha}^{-(s+1)} E_t m_{t+s}$$

so at least this is exactly

$$\text{where } \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}$$

comes from; and  $\mu_2$  is used

$$\underline{b/c} \quad \mu_2 > 1$$

and  $\bar{\beta}^1$  will come from  $\beta E_t \varphi_{t-1}$

But this seems to indicate, like Baum writes,

$$\varphi_t = a^+ b_0 - \sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{s-t} m_s$$

$$\text{or } \varphi_t = \underbrace{-\frac{\beta}{K} x^* (1 - \mu_1^{t+1})}_{\text{from (1.12)}}$$

effect of initial cond  
general sol

effect of shock for  
fwd-diff eq.

$$\mu_1 \varphi_{t+1} - (1 - \mu_1) \frac{\beta}{K} x^*$$

So it seems like when there's a function of time on the RHS, you need to add a sum, fwd or bw-looking depending on the eq. or the roots.

But how do we know which sum (fwd- or bw-looking) to add?

$$\text{Supp: } \beta E_t \varphi_{t+1} = a \varphi_t + u_t$$

$$a \varphi_t = \beta E_t \varphi_{t+1} - u_t$$

$$\varphi_t = \frac{\beta}{a} E_t \varphi_{t+1} - \frac{1}{a} u_t$$

$$y_t = \frac{\beta}{\alpha} E_t y_{t+1} - \frac{1}{\alpha} u_t$$

$$= \frac{\beta}{\alpha} \left[ \frac{\beta}{\alpha} E_t y_{t+2} - \frac{1}{\alpha} E_t u_{t+1} \right] - \frac{1}{\alpha} u_t$$

$$= \left(\frac{\beta}{\alpha}\right)^2 E_t y_{t+2} - \frac{1}{\alpha} u_t - \frac{\beta}{\alpha} E_t u_{t+1} - \frac{\beta^2}{\alpha^3} E_t u_{t+2} + \dots$$

$$= \dots - \frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^s u_{t+s}$$

ok, now this raises the issue of what exactly we're doing. In the Barn Lect. Notes, an AR(1) process's root is the AR-coeff. maybe since we have a 2nd order process, we have a distinction between the AR-coeff & the roots.

$\Rightarrow$  Indeed: for a one-dim AR(1),  $y_{t+1} = a y_t + c$   
a is the root! (Alpha Chiang, p. 554)

So I'm pulling on Blume p. 585 ft (Mac, 559 ft) to study difference eqs. in general.

Note: Blume calls Barn's method of making the system non-interacting "decoupling" or "change of coordinates" (Mac, 564)

On p. 567 (Mac), Blume then says: The uncoupled system

is  $\mathbf{z}'_{n+1} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & & \\ & & \ddots & \\ & & & r_k \end{pmatrix} \mathbf{z}'_n$

where  $r_i$  are the eigenvalues of  $A_{k \times k}$ , where the orig system was

$$\mathbf{z}_{n+1} = A \mathbf{z}_n$$

then the decoupled (non-interacting)  $z$ 's are

$$z'_{1,n} = c_1 r_1^n$$

⋮

$$z'_{k,n} = c_k r_k^n$$

So the original  $z$ -variables's solutions are

$$\mathbf{z}_n = c_1 r_1^n \mathbf{v}_1 + c_2 r_2^n \mathbf{v}_2 + \dots + c_k r_k^n \mathbf{v}_k$$

where the  $\mathbf{v}_i$  are eigenvectors

i.e. a sum of terms of the form

$$c_i \cdot (\text{eigenvalue})^n \cdot \text{eigenvector}$$

where the  $c_i$  are constants determined from the initial conditions.

I'm on econdse.org and find the remark:

"DSE-Maths-  
pdf" in literature

### "non-autonomous equations"

$$x_t = a x_{t-1} + b_t \quad (2)$$

- The complementary function will be the same as for the autonomous eq.  $x_t = a x_{t-1} + b$

- But the particular sol won't be the same b/c trying a constant value of  $x$  won't work, so no  $x_t$  and  $x_{t-1}$  can be the same. Good point!

There are 2 methods for arriving at the particular sol:  
backward sol & forward sol

See also "diffeqn.pdf" → This is a really good note!

In particular, this note shows that finding eigenvalues is really equal to factoring polynomials. E.g. p.17

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + b_t$  can be written as

$(1 - \phi_1 L - \phi_2 L^2) y_t = b_t$ . Now, this we can

write as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)y_+ = b_+$$

These are the eigens.

We know that since  $L$  behaves like  $z^{-1}$ , we can write

$$1 - \phi_1 L - \phi_2 L^2 = 0 \quad \text{as} \quad 1 - \phi_1 z^{-1} - \phi_2 z^{-2} = 0$$

and multiply by  $z^2$  to write the "associated polynomial"

$$z^2 - \phi_1 z - \phi_2 = 0 \quad \text{which we can solve for}$$

$$z_{1,2} = \lambda_{1,2}$$

So  $y_+ = \frac{b_+}{(1 - \lambda_1 L)(1 - \lambda_2 L) \dots (1 - \lambda_p L)}$

So if both eigenvalues are less than 1, then

$$y_+ = \hat{c}_1 \lambda_1^+ + \hat{c}_2 \lambda_2^+ + \frac{1}{(1 - \lambda_1 L)(1 - \lambda_2 L)} b_+ \quad \text{eq. (27)}$$

(P. 14)

In terms of Woodford, what I still don't know is why he solves it forward (why not solve bw using  $\mu_1 < 1$ ?) and where  $B^{-1}$  is coming from?

Ok I don't know: the result of today's hard work is that in the case of a non-autonomous diff. eq., the sol. takes the form

$$y_t = y_c + y_p$$

↑  
same as for  
autonomous eq

↑ not the same as for nonhom.  
eq. instead, this will be a  
forward or backward sum

of  $b_t$  ( $\tau_1$  exog. part), w/  
the eigenvalues as coeffs.

What I don't know is how to determine which eigenvalue will be coefficient.

- If the system is univariate, there's only one eig.  $\lambda$  and the fwd/bwd-nature of the sum depends on whether  $|\lambda| \geq 1$
- If the system is  $p$ -variate, but all eigs  $|\lambda_i| < 1$

then  $y_p = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L) \dots (1-\lambda_p L)}$  (quen... :S)

• If all eigs  $|\lambda_i| > 1$  then  $y_p = \frac{1}{(1-\lambda_1^{-1}L)(1-\lambda_2^{-1}L) \dots (1-\lambda_p^{-1}L)}$

What I don't know is what when one eig  $\mu_1 < 1$   
the other  $|\mu_2| > 1$ .

The lecture notes by Tirotti "linear...Tirotti.pdf"  
seem to provide the answer. (not here.)

Actually, they \*almost\* do  
- but they also don't say what to do w/ "alternating"  
eigenvalues.

So I'll pause the issue of (2.7) here for a while.  
I wanna go on to see what's going about implementation.

Note on nonneutral p. 407 : purely fwd-looking policy, i.e.  
maximizes the same utility as under Ramsey but  
subject to the condition that endog. vars evolve  
according to  $\bar{z}_t = \bar{z} + f' s_t$  (purely fwd-looking)  
i.e. the optimal plan is purely fwd-looking.

It's mean b/c Woodford has actually introduced one  
year before he officially does starting p. 507.

What I said just now isn't quite right: splitting  
the same loss into  $L^{\text{det}}$  &  $L^{\text{stab}}$  serves the purpose  
of making one 1) time-consistent 2) still optimal.  
The point is that whatever path a nation has  $\pi_t = \bar{\pi} + f\pi_t$   
gives you is not time-consistent (b/c it is purely  
forward-looking), but simply imposing the timelessly  
optimal plan from before is not optimal for  $\pi_t = \bar{\pi} + f\pi_t$ .  
So you split the problem in two, allowing the same  
time- or timelessly optimal policy to minimize  $L^{\text{stab}}$   
(b/c responses to shocks optimally are the same)  
which is independent of the econ's state at any time  $t$ .  
And shit: the LR averages are exactly NOT more than min  
 $L^{\text{det}}$ ; instead you just adopt those from a timelessly  
optimal policy.

p. 516: Woodford also says that if a simple TR is found that is the optimal one, then a TR that involves past does not improve upon the baseline TR. The reason is that already the baseline TR was purely forward-looking, so you haven't changed that. In fact, as Gersbach & Woodford (2002b) show, for frosts of horizons large enough, the cgs becomes indeterminate.

By the way: given a  $\pi$ -admissible optimal set for the Ramsey problem, p. 518 calculates the opt. time paths for  $\pi$  &  $i$  (eqs. (4.1) & (4.2))

Then there's the problem of solving for optimal plans for  $(\pi, x, i)$  and using the plan for  $i$  as a reaction function. But that's indeterminate.

⇒ So the optimal time path isn't a good policy rule, but the purely forward-looking TR isn't fully optimal. ⇒

Are targeting rules the solution? p. 521

A targeting rule is a commitment of the CB to adjust its instrument such that a target criterion is projected to be fulfilled at all points in time.

One target criterion which is optimal from a timeline perspective is

$$\pi_t + \frac{\lambda}{k} (x_t - x_{t-1}) = 0 \quad (\text{S.1})$$

It is not just optimal, but it is also feasible (b/c an REE w/ it exists), moreover it's unique (not indeterminate)

$\Delta x_t \rightarrow$  reflects the history-dependence of optimal commitment (p. 525)

But you know, there are other robustly optimal target criteria. (p. 526-27)

But we still don't know how to implement the targeting rule, all we have is an optimal target criterion.

→ But you can use the target criterion together w/ model equations & Laws of Motion to derive a reaction

function for it.

As a first-pass, you obtain the **fundamentals-based reaction function** of E&H (5.4)

(You've assumed the private sector's expectations are RE)

The problem though is that for many parameter values, this again isn't determinate.

So instead of using expectations that "ought to be", plug in expectations of the private sector the CB actually desires: **"expectations-based reaction function"** (w/o solving for 'em) (5.5)

This is determinate according to Prop 7.17.

→ so an interest rate rule (5.5) [=reaction function] is equivalent to committing to the  $\pi$ -target (5.1). If implements it, this (5.5) is a forward-looking TR, however, it does have history-dependence from

$$x_{t-1} \cdot w \mid \frac{(k^e + \gamma) - \gamma}{\alpha(k^e + \gamma)} = \frac{k^e}{\alpha(k^e + \gamma)} = \phi_x \text{ in LR.}$$

Looking at Gaspar et al 2011 & Preston 2008,  
it's not quite clear to me what the Ramsey problem  
should look like.

Ok: supposing that the indexation parameter  $\gamma = 0$ ,  
at least the period loss  $L_t$  takes the same form  
in Gaspar et al 2011 & Preston 2008:

$$L_t = \pi_t^2 + \gamma x_t^2$$

$$W = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \quad \leftarrow \text{overall loss fn.}$$

And this is also the loss Molnár & Santoro (2016)  
consider (except they call  $\gamma$   $d$  for a charge)  
and they only claim to do discretionary policy (but  
actually, they seem to be solving the commitment  
problem).

And indeed: Preston takes Woodford's equations and  
replaces the eigenvalue  $\mu_2 > 1$  w/  $\mu_1 < 1$   
(see my comments in Preston 2008, p. 8 Mac)

Ok, brace yourselves, I'm approaching the "approach":

①  $\min \mathcal{L} \text{ s.t. ALM, LDM (beliefs)}$

$$\pi_t, x_t, i_t, \phi_t \\ ?$$

this will give you FOCs that, given eigs, can be solved for the evolution of  $\mathcal{L}$ -multipliers, which can be subbed back to obtain paths for

$$\{\pi_t, x_t, i_t, \phi_t\}$$

- ② sub out  $\mathcal{L}$ -multipliers from FOCs to obtain a target criterion:  $\pi_t$  in terms of  $x$  ( $x_1, x_{t+1}, \dots$ ).  
(Analogue of (5.1))

- ③ Use the ALMs to figure out what  $\hat{E}^H$  &  $\hat{E}^X$  are and confront them w/ the criterion to obtain the i-rate rule / reaction function.

In materials 17, I have written out

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both the big - & - ugly Lagrangian for the full system,  
as well as that of a simplified one (eq 20-25)  
(or 26-31)

Take FOCS of the simplified one.

$$\pi: 2\pi_+ + \gamma_{1,+} - \gamma_{5,+}\bar{g}^{-1} = 0 \quad (1)$$

$$x: 2\lambda x_+ - \gamma_{1,+}k + p_{2,+} = 0 \quad (2)$$

$$i: p_{2,+} b = 0 \rightarrow p_{2,+} = 0 \Rightarrow \text{interesting: } 1/m$$

also finding that 1S - curve isn't binding.

$$fa: -\gamma_{1,+}(1-\alpha)\beta + \gamma_{3,+} = 0 \quad (3)$$

$$fb: -p_{2,+}b + p_{4,+} = 0 \quad (4)$$

$$\begin{aligned} \tilde{\pi}_+: & -E_T \beta \gamma_{3,+1} \frac{1}{1-\alpha\beta} - E_T \beta \gamma_{3,+11} \frac{1}{1-\beta} \\ & + \gamma_{5,+} + E_T \beta \gamma_{5,+11} (-1 + \bar{g}^{-1}) = 0 \end{aligned} \quad (5)$$

The good news is that these FOCS look similar to  
the ones Gaspar et al (2011) get.

In particular, using that  $p_{2,+}=0$ , I can combine (1) & (2)

where I've rewritten the rational expectations:

$$\text{If you can form } E(x)^{\text{RE}}, \text{ then } E_T x_{T+1} = g_2 E_T s_{T+1} \\ = g_2 h_x s_T$$

$$E_T x_{T+k} = g_2 h_x^k s_T \rightarrow E_T x_T = g_2 h_x^{T-t} s_T \\ = b_2 h_x^{T-t-1} s_T$$

$$\rightarrow E_T x_{T+1} = g_2 h_x^{T-t+1} s_T = b_2 h_x^{T-t} s_T$$

$$\kappa \alpha \beta E_T \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} x_{T+1} = \kappa \alpha \beta E_T \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} b_2 h_x^{T-t} s_T \\ = \kappa \alpha \beta b_2 \sum_{T=t}^{\infty} (\alpha \beta)^{T-t} h_x^{T-t} s_T \\ = \kappa \alpha \beta b_2 (I_3 - \alpha \beta h_x)^{-1} s_T$$

$$2\pi_t + \gamma_{1,t} - \gamma_{5,t} \bar{g}^{-1} = 0 \quad (1)$$

$$2\lambda x_t - \gamma_{1,t+k} = 0 \quad (2)$$

$$(1): \gamma_{1,t} = 2\frac{\lambda}{\kappa} x_t$$

$$\Rightarrow (1): 2\pi_t + 2\frac{\lambda}{\kappa} x_t - \gamma_{5,t} \bar{g}^{-1} = 0$$

$$\Rightarrow x_t = \frac{k}{\lambda} \left( \frac{\gamma_{5,t}}{\alpha} \bar{g}^{-1} - \pi_t \right)$$

$$\Rightarrow x_t = -\frac{k}{\lambda} \left( \pi_t - \frac{\gamma_{5,t}}{\alpha} \bar{g}^{-1} \right) = \text{Gaspar et al's (22)!}$$

Now turn to (3)-(5): (using  $\varphi_{2,t} = 0$ )

$$-\varphi_{1,t} (1-\alpha) \beta + \varphi_{3,t} = 0 \quad (3)$$

$$\varphi_{4,t} = 0 \quad (4)$$

$$-E_t \beta \varphi_{3,t+1} \frac{1}{1-\alpha \beta} - E_t \beta \varphi_{5,t+1} \frac{1}{1-\beta} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0 \quad (5)$$

$\rightarrow \varphi_{4,t} = 0$  b/c  $f_\beta$  only shows up in the NKIS, which isn't binding.

$$(3): \varphi_{3,t} = (1-\alpha) \beta \varphi_{1,t}$$

$$(5) \quad -E_t \beta \frac{(1-\alpha) \beta \varphi_{1,t+1}}{(1-\alpha) \beta} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0$$

But we know  $\varphi_{1,t} = 2 \frac{\partial}{K} x_{t+1}$  so (5) becomes

$$-E_t \beta 2 \frac{\partial}{K} x_{t+1} + \varphi_{5,t} + E_t \beta \varphi_{5,t+1} (-1 + \bar{g}^{-1}) = 0 \quad (5')$$

Now let's solve (5') for  $\varphi_{5,t}$  by iterating fwd

$$\varphi_{5,t} = 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta E_t \varphi_{5,t+1}$$

$$= 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta \left[ 2 \beta \frac{\partial}{K} E_t x_{t+2} + (1 - \bar{g}^{-1}) \beta E_t \varphi_{5,t+2} \right]$$

$$= 2 \beta \frac{\partial}{K} E_t x_{t+1} + (1 - \bar{g}^{-1}) \beta 2 \beta \frac{\partial}{K} E_t x_{t+2} + ((1 - \bar{g}^{-1}) \beta)^2 E_t \varphi_{5,t+2}$$

$$\varphi_{5,t} = 2 \beta \frac{\partial}{K} \sum_{i=0}^{\infty} (1 - \bar{g}^{-1})^i E_t x_{t+1+i} \quad \text{except for the sign,}$$

it's the same as Gaspar et al (2011).

So now plug the sol for  $\varphi_{S,t}$  into

$$x_t = -\frac{\lambda}{\kappa} \left( \pi_t - \frac{\beta \varphi_{S,t}}{2} \bar{g}^{-1} \right), \quad \text{expressed for } \pi_t$$

$$-\frac{\lambda}{\kappa} x_t + \frac{\bar{g}^{-1}}{2} \varphi_{S,t} = \pi_t$$

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \frac{\bar{g}^{-1}}{2} \left[ 2\beta \sum_{i=0}^{\infty} (1-\bar{g}^{-1})^i E_t x_{t+i} \right]$$

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \bar{g}^{-1} \beta \sum_{i=0}^{\infty} (1-\bar{g}^{-1})^i E_t x_{t+i}$$

$$\pi_t = -\frac{\lambda}{\kappa} \left( x_t - \beta \bar{g}^{-1} E_t \sum_{i=0}^{\infty} (1-\bar{g}^{-1}) x_{t+i} \right) \quad (24)$$

$\nearrow$   
Gaspar et al's

Allows you to get the nice interpretation of Gaspar et al 2011

when the gain is 0, (24) boils down to

$$\pi_t = -\frac{\lambda}{\kappa} x_t \quad (= \text{discretionary RE solution})$$

This is what Gaspar et al call an "intertemporal tradeoff" (or actually, Molnár & Santoro do)

If the gain  $> 0$ , then there is an additional, inter-temporal tradeoff between  $\pi$  &  $x$  b/c whatever  $\pi$

you bring about today, it affects  $\pi$ -expectations tomorrow, presenting you w/ an interesting tradeoff tomorrow.

Mendoza & Santoro derive an interest rate rule!

they say: plug  $ARM(\pi)$  &  $ARM(x)$  (eq 18 & 19) into the NKIS

Their ARMs take the following form

$$\pi_t = c_\pi a_t + d_\pi u_t \quad \text{and} \quad \hat{E}_t \pi_{t+1} = a_t$$

$$x_t = c_x a_t + d_x u_t \quad \hat{E}_t x_{t+1} = b_t$$

$$\text{NKIS: } x_t = \hat{E}_t x_{t+1} - b^{-1} (r_t - \hat{E}_t \pi_{t+1})$$

$$\Leftrightarrow b x_t = b \hat{E}_t x_{t+1} - r_t + \hat{E}_t \pi_{t+1}$$

$$r_t = b \hat{E}_t x_{t+1} + \hat{E}_t \pi_{t+1} - b x_t$$

$$= b b_t + a_t - b(c_x a_t + d_x u_t)$$

$$= \underbrace{(1 - b c_x)}_{\delta_\pi} a_t + \underbrace{b b_t}_{\delta_x} - \underbrace{b d_x u_t}_{\delta_u}$$

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u u_t$$

which is what M&S obtain if you plug in  $c_x$  &  $d_x$ .

→ This is their expectation-based reaction function

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_y a_t \quad (20)$$

Ryan meeting

19 Feb 2020

Mantra:

Question of the paper: "How does a concern for the anchoring of expectations affect the conduct of monetary policy?"

- 1) Take simple pencil & paper: how does it change (32)? An extra term ...
- 2.) Numerically solve the full-fledged model
- 3.) Estimate the gain function  $k(\theta_t)$   
"My est. suggests that  $E(\cdot)$  are likely to be anchored when people are surprised by 1%." → CB-ers would be interested to hear that. It'd be a great ending to the abstract.

"There should be a gentlemanly distance between assumptions and results." (Stephanie Schmalzle) For me, this means that if I ask "does anchoring matter for policy?" then it's too easy to critique my work by saying "well, the way you set up the model, of course it does".

The mantra is mainly a marketing object: it doesn't assume that there is a concern for anchoring a priori, but given that I find that there is, how anchoring changes the policy problem, I avoid that kind of critique.

The reason Woodford might advocate an approach like Preston (2008) where the RE optimum should be implemented under learning is b/c he might say that "it is not obvious that a policymaker can credibly commit to a non-RE optimum."

Big issue: why is RE-discretion the alternative here? Did I not accidentally write down the discretion problem?

Ryan: discretion is when I choose a single it, commitment when I choose a whole plan of {it}.

↳ check discretion is commitment in a very simple problem!

Work after

20 Feb 2020

So I think that discretion

comes about here b/c there aren't really lagged multipliers.

Ryan made a remark that w/ commitment, you get lagged multipliers and it's intuitive that you do - when you choose sequences (today's and tomorrow's values) one multiplier will be date t, the other t-1. Which is in fact exactly why Woodford's (1.7) is

$$\pi_t + \gamma_t - p_{t-1} = 0$$

Ok but this goes to the heart of my unease w/  
the way I treat  $f_C$  &  $f_D$

Take a simplified version of Woodford's Ramsey problem

(p. 472)

$$\mathcal{L} = E \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + p_t (\pi_t - \underline{\beta \pi_{t+1}}) \right\} \quad \begin{matrix} \text{Commitment} \\ \text{discretion} \end{matrix}$$

vs.

$$\mathcal{L} = E_T \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + y_t (\pi_t - \beta f_t) \right\} \quad \begin{matrix} \text{where } f_t \text{ is taken as} \\ \text{given} \end{matrix}$$

FDCs:

$$\pi_t + y_t - y_{t-1} = 0 \quad \begin{matrix} \text{Commitment} \\ \text{discretion} \end{matrix}$$

vs.

$$\pi_t + y_t = 0 \quad \begin{matrix} \text{discretion} \\ \text{discretion} \end{matrix}$$

What I'm scared of is that in learning,  $f_t$  becomes a new variable that is bw-looking. So you can't comment?

peter

anticipated utility shouldn't be applied to the planner

Consider 2 models for CB

1. aut. n. would be like CB has 2 rules for 1 period & doesn't care about W of future CB
2. an  $\infty$ -hor CB that could commit to forming a policy rule today  $\rightarrow$  would be a diff.

( $\hookrightarrow$  constrained at all periods)

Can argue that  $P_{t+1}$

CMP isn't a model of reputation-building

b/c it  $\rightarrow$  discretionary RE, that would support egf under commitment.

Expectations, Credibility ... Ireland, 2000, Maastricht

multiple egfs based on reputation, CB-ers want

the one w/ credible → see if it's been acted  
→ Cho & Matsui (1995) JEDC

E-forming scheme when all they ass  
is that if the CB does the same thing,  
sooner or later ppl get that that's how  
it's gonna be

→ CB "if I do the right thing now, it will  
have a cost, but it'll be worth it"

details of the transition to be worked out  
↳ could be an extended learning scheme  
which gives the CB incentives to tilt expectations  
toward commitment fol.

Lucas & Sargent : PC w/ adaptive  $E(\cdot)$

→ need to be sensitive to that point; my  
policy isn't to exploit  $E(\cdot)$  → no one is  
fooled all of the time in this model!

My work is less about the st. st., more about the transition like Ball (1994).

2 approaches to opt policy

Taylor → see what are welfare of simple rules  
→ I see an interesting tradeoff.

Woodford, Wübbe & Schmitt-Grohé

opt sequential choice

I see interesting imp b/w lag & future multipliers → mon. pol. shapes  $E(\cdot)$

is similar in certain aspects &

diff. in others

This isn't a model of building credibility for opt. commitment. In p'tile work, one could think of alternative E-formation schemes.

Work after

21 Feb 2020

Reading Cho & Matsui (1995) & Peter Ireland (2000)  
kind of at the same time.

Cho & Matsui have a nice def of Ramsey policy  
and a nice sum-up of the problem:

Ramsey policy = the gov's scheme which maximizes  
its objective function if the gov behaves as a Stackelberg  
leader

↳ leader moves first and the followers move sequentially

If the gov's objective fn = social welfare (diff to count)  
fn, then the Ramsey policy max social welfare  
s.t. the private sector responding optimally.

The problem: once PS believes that gov follows Ramsey,  
the gov may have an incentive to deviate b/c Ramsey  
is (or may not be) its best-response. (Holy shit!)

→ Ramsey pol may be "time-inconsistent" (Rydland & Rents  
1977)

UNLESS the gov has a credible commitment strategy  
to the Ramsey policy.

Cho & Matsui say that one way to rationalize the  
Ramsey policy is by building reputation, meaning  
by repeating the same action many times in order  
to convince agents that that is what I'll do.

→ my confusion here is that adaptive learners are  
the same thing!

No, they are not: they are machines that take an  
action given a pre-programmed forecasting rule.

Inductive forecasters, in Cho & Matsui's terms, however  
select the forecast optimally. I think this is a way  
to make them "strategic" in the way the gov is,  
and in the way they used to be w/ RE.

Peter's 2000 paper says that time-consistent policy  
futures multiple ergon (is indeterminate), and he argues

¶ This is due to RE. He says that RE is not good for modeling reputational cbs.

So what he does is:

- 1)  $E(\pi)$  needs to move w/  $\pi$
  - 2)  $E(\pi) \xrightarrow{!} \pi$ , that is  
 $E(\pi)$  needs to be inductive à La Cho & Matsui.
  - 3)  $E(\pi) = f(\pi_{t+1}, \pi_{t+2}, \dots)$  if continuous, diff-ble
- (1) & (2) allow the CB to build credibility

This discussion seems to be related to the conclusion that learning can be a model selection criterion.

→ backward-looking  $\Rightarrow$  model selection

→ learning: too machine-like (not strategic)  
 $\Rightarrow$  time-inconsistent

It seems like what induces the multiplicity is that under RE, expectations are a jump variable... of course! Stable cbs induce indeterminacy if  $\pi$  is far-lookin...

Ok so the vicious circle we're having is this:

- Ramsey sol under RE is cool but may be indeterminate (in fact is indeterminate - unless you do Woodford's target criterion. That one is determinate, but it the target criterion (and Neustein rule) isn't unique )
- You can intro learning to make it bw-looking and make it determinate "
- But learning takes away the commitment tech. b/c agents are machines (not strategic) and so you get the discretionary sol "
- Cho & Matsui (1995) try to get the best of both worlds by having inductive expectations which are both bw-looking and optimal (i.e. strategic).

And this is what Peter does too. koby shd.

Materials 18 : very simple problem

$$\check{Y} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \pi_t^2 + \varphi_{1,t} (\pi_t - \beta f_t) + \varphi_{2,t} (f_t - f_{t-1} - k^{-1}(\pi_t - f_{t-1})) \right\}$$

FOG

$$\pi: 2\pi_t + \varphi_{1,t} - \varphi_{2,t} k^{-1} = 0 \quad (1)$$

$$f_t: -\beta \varphi_{1,t} + \varphi_{2,t} + E_t \varphi_{2,t+1} (-1 + k^{-1}) = 0 \quad (2)$$

$$(1) \quad \varphi_{1,t} = \varphi_{2,t} k^{-1} - 2\pi_t$$

$$(2) \quad 2\beta\pi_t - \beta \varphi_{2,t} k^{-1} + \varphi_{2,t} + E_t \varphi_{2,t+1} (-1 + k^{-1}) = 0$$

$$2\beta\pi_t + \varphi_{2,t} (1 - \beta k^{-1}) - E_t \varphi_{2,t+1} (1 - k^{-1}) = 0$$

$$\varphi_{2,t} = E_t \varphi_{2,t+1} \frac{1 - k^{-1}}{1 - \beta k^{-1}} - \frac{2\beta}{1 - \beta k^{-1}} \pi_t$$

$$= \frac{1 - k^{-1}}{1 - \beta k^{-1}} \left( E_t \varphi_{2,t+2} \frac{1 - k^{-1}}{1 - \beta k^{-1}} - \frac{2\beta}{1 - \beta k^{-1}} E_t \pi_{t+1} \right) - \frac{2\beta}{1 - \beta k^{-1}} \pi_t$$

$$= \left( \frac{1 - k^{-1}}{1 - \beta k^{-1}} \right)^n E_t E \varphi_{2,t+2} - \sum_{s=0}^{n-1} \left( \frac{2\beta}{1 - \beta k^{-1}} \right)^s E_t \pi_{t+s}$$

$\hookrightarrow 0$  as  $n \rightarrow \infty$

$$1 - k^{-1} \leq 1 - \beta k^{-1} \rightarrow \beta k^{-1} \geq k^{-1} \rightarrow \beta \geq 1$$

Ok, so should turn to optimal policy in my simplified model w/ smooth anchoring.

So return to system (26)-(31) of materials 17.

Need to replace  $\bar{g}^{-1}$  in (31) w/  $k_t^{-1}$   
and add an equation like  $k_t = f(\theta_t)$   
and one for  $f(\theta_t)$ .

Suppose for simplicity that there's only the cost-push shock, so  $\Sigma = \eta\eta'$  is reduced to  $\mathbb{R}^n$ .

Then the CUSUM-criterion is much simpler:

$$\omega_t = \omega_{t-1} + \tilde{\kappa} k_{t-1}^{-1} (f_t^2 - \omega_{t-1})$$

↳ scalar b/c we're only  
leaving  $\pi$  now

→  $\omega_t$  is a scalar

$$\theta_t = \theta_{t-1} + \tilde{\kappa} k_{t-1}^{-1} \left( \frac{f_t^2}{\omega_t} - \theta_{t-1} \right)$$

(was also previously a scalar)

But now, as time goes by,  $w_t \rightarrow b_u^e$   
 and as agents are learning, the period FE  $f_t^2$   
 is on avg. going to be  $\rightarrow b_u^2$  as well.

$\Rightarrow \theta_t \rightarrow 1$  from above on average

So suppose then the gain is

$$\underline{b_t} = f(\theta_t) = c + d \cdot \theta_t$$

Since  $\theta_t \rightarrow 1$ ,  $c = -d$ , so we could also

$$\text{write this as } -d + d\theta_t = d(-1 + \theta_t)$$

and simply set  $d=1$  and write  $f(\theta_t) = \theta_t - 1$

My only concern is that occasionally it may happen

that  $f_t^2 < w_t$  so  $\theta_t < 1$ . So maybe perhaps

we could write  $f(\theta_t) = \underline{\theta_t - \gamma}$  where  $\gamma < 1$  but close.

So in this case, the system to solve for optimal policy is eq (26)-(31) and the gain, so:

Materials 18 sets up the Lagrangian

22 Feb 2020

Fols: (ignoring E<sub>t</sub>'s for simplicity)

$$\pi_+: 2\pi_T + \varphi_{1,+} - \varphi_{5,+} k_t^{-1} - \varphi_{9,+} = 0 \quad (1)$$

$$x_T: -\varphi_{1,+} k_t + 2\lambda x_+ = 0 \Rightarrow \varphi_{1,+} = 2\frac{\partial}{\partial k_t} x_+ \quad (2)$$

Let me not take FOCs wrt i<sub>T</sub> & f<sub>B</sub> b/c  $\varphi_{2,+} = \varphi_{4,+} = 0$  anyway.

$$f_{9,+}: -\varphi_{1,+}(1-\alpha)\beta + \varphi_{3,+} = 0 \Rightarrow \varphi_{3,+} = (1-\alpha)\beta 2\frac{\partial}{\partial k_t} x_T \quad (3)$$

$$\bar{\pi}_t: -\varphi_{3,++1} \frac{1}{1-\alpha\beta} + \varphi_{5,+} - \varphi_{5,++1} (1-k_t^{-1}) + \varphi_{9,+} = 0 \quad (4)$$

$$k_t: -\varphi_{5,+} (-1) k_t^{-2} (\pi_+ - (\bar{\pi}_{+-1} + b\varsigma_{+-1})) + \varphi_{6,+} \\ - \varphi_{7,++1} \tilde{\kappa} (-1) k_t^{-2} (f_{t+1}^2 / \omega_{t+1} - \theta_t) \\ - \varphi_{8,++1} \tilde{\kappa} (-1) k_t^{-2} (f_{t+1}^2 - \omega_t) = 0 \quad (5)$$

$$\theta_t: -\varphi_{6,+} d + \varphi_{7,+} - \varphi_{7,++1} (1 - \tilde{\kappa} k_t^{-1}) = 0 \quad (6)$$

$$\omega_t: -\varphi_{7,+} \tilde{\kappa} k_{t+1}^{-1} \frac{f_t^2}{\omega_t^2} (-1) + \varphi_{8,+} - \varphi_{8,++1} (1 - \tilde{\kappa} k_t^{-1}) = 0 \quad (7)$$

$$f_t: -\varphi_{7,+} \tilde{\kappa} k_{t+1}^{-1} \cdot 2 \frac{f_t}{\omega_t} - \varphi_{8,+} \tilde{\kappa} k_{t+1}^{-1} 2f_t + \varphi_{9,+} = 0 \quad (8)$$

Let me rewrite in a simplified manner

$$\pi_t: 2\pi_t + 2\frac{\beta}{\kappa}x_t - \varphi_{S,t} k_t^{-1} - \varphi_{g,t} = 0 \quad (1)$$

$$\bar{\pi}_t: -\frac{(1-\alpha)\beta}{1-\alpha\beta} 2\frac{\beta}{\kappa}x_{t+1} + \varphi_{S,t} - \varphi_{S,t+1}(1-k_t^{-1}) + \varphi_{g,t} = 0 \quad (2)$$

$$k_t: -\varphi_{S,t}(-1)k_t^{-2}(\pi_t - (\bar{\pi}_{t-1} + b\varphi_{S,t-1})) + \varphi_{6,t}$$

$$- \varphi_{7,t+1} \tilde{\kappa}(-1)k_t^{-2} \left( f_{t+1}^2 / \omega_{t+1} - \theta_t \right)$$

$$- \varphi_{8,t+1} \tilde{\kappa}(-1)k_t^{-2} \left( f_{t+1}^2 - \omega_t \right) = 0 \quad (5)$$

$$\theta_t: -\varphi_{6,t} d + \varphi_{7,t} - \varphi_{7,t+1}(1 - \tilde{\kappa}k_t^{-1}) = 0 \quad (6)$$

$$\omega_t: -\varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1 - \tilde{\kappa}k_t^{-1}) = 0 \quad (7)$$

$$f_t: -\varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \cdot 2 \frac{f_t}{\omega_t} - \varphi_{8,t} \tilde{\kappa} k_{t-1}^{-1} 2f_t + \varphi_{g,t} = 0 \quad (8)$$

The approach will be sthg like Ans: solve (1) & (2) for  $\varphi_{S,t} \rightarrow \varphi_{S,t}$

Solve (6), (7) for  $\varphi_{7,t}$  &  $\varphi_{8,t}$  as a function of  $\varphi_{6,t}$

Do sthg w/ (5) & (8).

$$(1) \quad \varphi_{S,t} = 2\pi_t + 2\frac{\beta}{\kappa}x_t - \varphi_{S,t}k_t^{-1}$$

$$\Rightarrow (2) \quad -2\frac{(1-\alpha)}{1-\alpha\beta} \frac{\beta}{\kappa}x_{t+1} + \varphi_{S,t} - \varphi_{S,t+1}(1-k_t^{-1}) + 2\pi_t + 2\frac{\beta}{\kappa}x_t - \varphi_{S,t}k_t^{-1} = 0$$

$$\Leftrightarrow -2\pi_t - 2\frac{\beta}{\kappa} \left( x_t - \frac{1-\alpha}{1-\alpha\beta} x_{t+1} \right) + (1-k_t^{-1}) \varphi_{S,t+1} = (1-k_t^{-1}) \varphi_{S,t}$$

$$\Rightarrow \varphi_{S,t} = \frac{1}{1-k_t^{-1}} \left( -2\pi_t - 2\frac{\beta}{\kappa} \left( x_t - \frac{1-\alpha}{1-\alpha\beta} x_{t+1} \right) + \varphi_{S,t+1} \right)$$

Darn. Neither fwd nor bw-looking. It's b/c the gain is endog.

Pause that: focus on (6) & (7)

$$-\varphi_{6,t+d} + \varphi_{7,t} - \varphi_{7,t+1} (1 - \tilde{\kappa} k_t^{-1}) = 0 \quad (6)$$

$$\varphi_{7,t} = \underbrace{\varphi_{7,t+1} (1 - \tilde{\kappa} k_t^{-1})}_{< 1} + \varphi_{6,t} \cdot d$$

so solve fwd:

$$\begin{aligned} \varphi_{7,t} &= (1 - \tilde{\kappa} k_t^{-1}) \left[ (1 - \tilde{\kappa} k_t^{-1}) \varphi_{7,t+2} - \varphi_{6,t+1} \cdot d \right] + \varphi_{6,t} \cdot d \\ &= (1 - \tilde{\kappa} k_t^{-1})^j \varphi_{7,t+j} + d \sum_{i=0}^{j-1} (1 - \tilde{\kappa} k_t^{-1})^i \varphi_{6,t+i} \end{aligned}$$

$$\varphi_{7,t} = d \sum_{i=0}^{\infty} (1 - \tilde{\kappa} k_t^{-1})^i \varphi_{6,t+i}$$

$\overbrace{\hspace{10em}}$

Analogously: (7)

$$-\varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2} (-1) + \varphi_{8,t} - \varphi_{8,t+1} (1 - \tilde{\kappa} k_t^{-1}) = 0 \quad (7)$$

$$\varphi_{8,t} = (1 - \tilde{\kappa} k_t^{-1}) \varphi_{8,t+1} - \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2} \varphi_{7,t}$$

$$\Rightarrow \varphi_{8,t} = -\tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2} \sum_{i=0}^{\infty} (1 - \tilde{\kappa} k_t^{-1}) \varphi_{7,t+i}$$

$\overbrace{\hspace{10em}}$

Now to rewrite  $\varphi_{8,t}$  in terms of  $\varphi_6$ ?

try something different: plug in all multipliers that only show up w/ a  $t$ -subscript:

(8) in (1) & (4):

$$2\pi_t + 2\frac{\beta}{\alpha}x_t - \varphi_{5,t} k_t^{-1} - \varphi_{7,t} \tilde{K} k_{t-1}^{-1} \cdot 2\frac{f_t}{\omega_t} - \varphi_{8,t} \tilde{K} k_{t-1}^{-1} 2f_t = 0 \quad (1)$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta} 2\frac{\beta}{\alpha}x_{t+1} + \varphi_{5,t+1} - \varphi_{5,t+1}(1-k_t^{-1})$$

$$+ \varphi_{7,t} \tilde{K} k_{t-1}^{-1} \cdot 2\frac{f_t}{\omega_t} + \varphi_{8,t} \tilde{K} k_{t-1}^{-1} 2f_t = 0 \quad (4)$$

(6) in (5):

$$-\varphi_{5,t}(-1)k_t^{-2}(\pi_t - (\bar{\pi}_{t-1} + b\varsigma_{t-1})) + \frac{1}{2}\varphi_{7,t} - \frac{(1-\tilde{K}k_t^{-1})}{d}\varphi_{7,t+1}$$

$$- \varphi_{7,t+1} \tilde{K}(-1)k_t^{-2}(f_{t+1}^2/\omega_{t+1} - \theta_t)$$

$$- \varphi_{8,t+1} \tilde{K}(-1)k_t^{-2}(f_{t+1}^2 - \omega_t) = 0 \quad (5)$$

And we still have (7)

$$-\varphi_{7,t} \tilde{K} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1 - \tilde{K}k_t^{-1}) = 0 \quad (7)$$

having a hard time w/ this system, I don't really know why.

What if we said for a sec that all stays the way it was w/ an exog again, except that  $k_t = f(\cdot)$  some function. Then eqs (15) - (17) are gone and (14) is just  $\varphi_{6,t} (k_t - f(\cdot))$ . Suppose  $f(\cdot) = f(\bar{\pi}_t, \bar{\pi}_{t-1}, k_{t-1})$  and the derivatives of  $f$  are  $f_{\pi}$ ,  $f_{\bar{\pi}}$  and  $f_k$ .

FDCs :  $\varphi_{6,t} (k_t - f(\bar{\pi}_t, \bar{\pi}_{t-1}, \underline{k_{t-1}}))$

$$\pi_t: 2\pi_t + \varphi_{1,t} - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_{\pi} = 0 \quad (1)$$

$$x_t: 2x_t - \varphi_{1,t} k_t = 0 \quad \underbrace{\varphi_{1,t} = 2 \frac{\beta}{K} x_t}_{(2)}$$

$$f_{\alpha,t}: -\varphi_{1,t} (1-\alpha)\beta + \varphi_{3,t} = 0 \quad \underbrace{\varphi_{3,t} = 2(1-\alpha)\beta \frac{\beta}{K} x_t}_{(3)}$$

$$\bar{\pi}_t: -\varphi_{3,t+1} \frac{1}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1} (1-k_t^{-1}) + \varphi_{6,t} f_{\bar{\pi}} \quad (4)$$

$$k_t: \varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \underbrace{\varphi_{6,t+1} f_k}_{\substack{\text{also here the} \\ \text{same prob}}} = 0 \quad (5)$$

$k_{t+1}$  in the endog gain is intro - ing the difficulty.

$$2\pi_t + 2 \frac{\beta}{K} x_t - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_{\pi} = 0 \quad \begin{matrix} \text{(I)} & \text{(2)} \end{matrix}$$

$$\frac{2(1-\alpha)\beta \frac{\beta}{K} x_{t+1}}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1} (1-k_t^{-1}) + \underbrace{\varphi_{6,t} f_{\bar{\pi}}}_{\varphi_{6,t+1}} = 0 \quad \begin{matrix} \text{(II)} & \text{(2)} \end{matrix}$$

$$\varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1} f_k = 0 \quad (\text{III})$$

Gauge solves (II) for  $\varphi_{\text{mgs}, \text{itr}(I)}$ . Maybe I can solve (II) for  $\varphi_{6,+}$ , plug into (II), solve that for  $\varphi_{5,+}$ , and plug that into (I).

$$(III) \varphi_{5,1+} k_t^{-2} + \varphi_{6,1+} - \varphi_{6,1++} f_k = 0$$

$$\varphi_{6,k} = f_k \varphi_{6,1+k} - k_t^{-2} \varphi_{5,1+}$$

↑ have to assume  $f_k < 1$

$$\varphi_{6,1+} = k_t^{-2} \sum_{i=0}^{\infty} f_k^i \varphi_{5,1+i}$$

a problem...

$$(II) \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\partial}{\partial x} x_{t+1} + \varphi_{5,1+} - \varphi_{5,1++} (1-k_t^{-1}) + \varphi_{6,1+} \cdot f_{\bar{n}} = 0$$

$$\varphi_{5,1+} = (1-k_t^{-1}) \varphi_{5,1++} - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\partial}{\partial x} x_{t+1} - f_{\bar{n}} \varphi_{6,1+}$$

$$\varphi_{5,1+} = - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\partial}{\partial x} \sum_{j=0}^{\infty} (1-k_t^{-1})^j x_{t+j} - f_{\bar{n}} \sum_{j=0}^{\infty} (1-k_t^{-1})^j \varphi_{6,1+j}$$

$$(I): 2\pi_+ + 2 \frac{\partial}{\partial x} x_+ - \varphi_{5,1+} k_t^{-1} + \varphi_{6,1+} f_{\bar{n}} = 0$$

$$\pi_+ = - \frac{\partial}{\partial x} x_+ + \frac{k_t^{-1}}{2} \varphi_{5,1+} - \frac{f_{\bar{n}}}{2} \varphi_{6,1+}$$

$$y \quad \varphi_{6,1+} = - \varphi_{5,1+} k_t^{-2}, \text{ then (I) is } \pi_+ = - \frac{\partial}{\partial x} x_+ + \frac{k_t^{-1}}{2} \varphi_{5,1+} + \frac{k_t^{-2} f_{\bar{n}}}{2} \varphi_{5,1+}$$

$$\Rightarrow \pi_+ = - \frac{\partial}{\partial x} x_+ + k_t^{-1} \frac{(1+k_t^{-1} f_{\bar{n}})}{2} \varphi_{5,1+} \quad (I')$$

If  $f(\cdot)$  didn't depend on  $k_{t+1}$ , then (II) would be  $\psi_{5,t} + k_t^{-2} + \psi_{6,t} = 0$   
 $\rightarrow \psi_{6,t} = -\psi_{5,t} - k_t^{-2}$  Even this isn't right b/c here's a  $\psi_{6,t+1}$  still...

$$(II) \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} x_{t+1} + \psi_{5,t} - \psi_{5,t+1}(1-k_t^{-1}) - \psi_{5,t} \cdot k_t^{-2} \cdot f_{\bar{n}} = 0$$

$$\rightarrow \psi_{5,t} (1 - k_t^{-2} f_{\bar{n}}) = (1 - k_t^{-1}) \psi_{5,t+1} - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} x_{t+1}$$

$$\rightarrow \psi_{5,t} = \frac{1 - k_t^{-1}}{1 - k_t^{-2} f_{\bar{n}}} \psi_{5,t+1} - \frac{2(1-\alpha)\beta}{(1 - k_t^{-2} f_{\bar{n}})(1 - \alpha\beta)} \frac{\lambda}{k} x_{t+1} \quad (II')$$

$$k_t^{-2} < k_t^{-1} \text{ since } \frac{1}{k_t^2} < \frac{1}{k_t} \Leftrightarrow \frac{1}{k_t} < 1 .$$

So whether  $\frac{1 - k_t^{-1}}{1 - k_t^{-2} f_{\bar{n}}} \geq 1$  depends on  $f_{\bar{n}}$ . If  $f_{\bar{n}} = k_t$

then  $\frac{1 - k_t^{-1}}{1 - k_t^{-2} f_{\bar{n}}} = 1$ . So I'll assume  $f_{\bar{n}} < k_t$ . A stronger ass. that also works is  $f_{\bar{n}} < 1$ .

Rewrite

$$\psi_{5,t} = \frac{1 - k_t^{-1}}{1 - k_t^{-2} f_{\bar{n}}} \psi_{5,t+1} - \frac{2(1-\alpha)\beta}{(1 - k_t^{-2} f_{\bar{n}})(1 - \alpha\beta)} \frac{\lambda}{k} x_{t+1} \quad (II')$$

as

$$\underbrace{\psi_{5,t} = \frac{1 - k_t^{-1}}{1 - k_t^{-2} f_{\bar{n}}} \psi_{5,t+1} - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} x_{t+1}}_C \quad \underbrace{\frac{1}{1 - k_t^{-2} f_{\bar{n}}}}_{b_t} \quad (II'')$$

$\therefore a_t$

$$\begin{aligned}
Y_{S,t} &= a_t Y_{S,t+1} - c \cdot b_t x_{t+1} \\
&= a_t [a_{t+1} Y_{S,t+2} - c b_{t+1} x_{t+2}] - c b_t x_{t+1} \\
&= a_t \cdot a_{t+1} Y_{S,t+2} - a_t c b_{t+1} x_{t+2} - c b_t x_{t+1} \\
&= a_t a_{t+1} [a_{t+2} Y_{S,t+3} - c b_{t+2} x_{t+3}] - a_t c b_{t+1} x_{t+2} - c b_t x_{t+1} \\
&= a_1 a_{t+1} a_{t+2} Y_{S,t+3} - a_1 a_{t+1} c b_{t+2} x_{t+3} - a_1 c b_{t+1} x_{t+2} - c b_t x_{t+1} \\
&= \prod_{i=0}^{k-1} a_{t+i} Y_{S,t+k} - c \sum_{j=0}^k \prod_{l=0}^{j-1} a_{t+l} b_{t+j} x_{t+1+j}
\end{aligned}$$

Omg - even this is ugly ...

Can we make a case for  $\prod_{i=0}^{\infty} a_{t+i} \rightarrow 0$ ?

We've assumed  $f_{\bar{\pi}} < 1$  so  $a_t < 1$ . the problem is

that as  $t \rightarrow \infty$ ,  $k_t^{-1} \rightarrow 0$  so  $a_t \rightarrow 1$ . So  $\prod_{i=0}^{\infty} a_{t+i}$  is

slimy like  $0.5 \cdot 0.6 \cdot 0.7 \cdot 0.8 \dots$  I'm not entirely sure what the limit of this is (1?) but it sure ain't 0.

Or is it? So this is a decreasing series ... hm.

I'm wondering if this shows up when you do this for

again learning as in Mohr & Santor. There, it doesn't seem to show up.

Let's see Molnár & Santoro : FOCs

$$2\pi_t - \gamma_{2,t} + \delta_{t+1} \gamma_{3,t} = 0 \quad (11)$$

$$2\alpha x_t + k \gamma_{2,t} = 0 \rightarrow \gamma_{2,t} = -\frac{2\alpha}{k} x_t \quad (12)$$

$$\beta^2 \gamma_{2,t+1} + \beta(1-\delta_{t+2}) \gamma_{3,t+1} = \gamma_{3,t} \quad (13)$$

(10) & (12)

$$2\pi_t + \frac{2\alpha}{k} x_t - \delta_{t+1} \gamma_{3,t} = 0 \quad (26)$$

Molnár & Santoro already use this to conclude that once beliefs have converged, you're in discretion.

Let's look at my analogue of this equation :

$$2\pi_t + 2\frac{\gamma}{k} x_t - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_\pi = 0 \quad (\pm) \stackrel{\text{Gains}}{(22)}$$

As  $\gamma \rightarrow \infty$   $k_t^{-1} \rightarrow 0$  and  $f_\pi \rightarrow 0$  as the gain changes less and less in response to  $\pi$ . I guess. In that case I also get the same result that in the LR, you have discretion.

Let's now iterate their (13) fwd

$$-\frac{2\alpha\beta^2}{K}x_{t+1} + \beta(1-\gamma_{t+2})\gamma_{3t+1} = \gamma_{3t} \quad (13)$$

$$\Rightarrow \gamma_{3t+1} = \beta(1-\gamma_{t+2}) \left[ \beta(1-\gamma_{t+3})\gamma_{3t+2} - \frac{2\alpha\beta^2}{K}x_{t+2} \right] - \frac{2\alpha\beta^2}{K}x_{t+1}$$

$$= \underbrace{\beta(1-\gamma_{t+2})(1-\gamma_{t+3})}_{\text{they get the same feature as I do}} \gamma_{3t+2} - \beta(1-\gamma_{t+2}) \frac{2\alpha\beta^2}{K}x_{t+2} - \frac{2\alpha\beta^2}{K}x_{t+1}$$

they get the same feature as I do

hm. In the appendix, where they let  $\gamma_t = \frac{1}{t}$ ,

somewhat only very neat things show up ( $\frac{1}{t+1}$  only)

although here I get  $(1 - \frac{1}{t+2})(1 - \frac{1}{t+3}) \dots$

$$= \left( \frac{t+2-1}{t+2} \right) \left( \frac{t+3-1}{t+3} \right) \dots = \left( \frac{t+1}{t+2} \right) \left( \frac{t+2}{t+3} \right) \dots \text{happens!}$$

so we're left w/  $\frac{t+1}{t+\infty}$  so that way  $\gamma_{3t+2}$  does disappear

over time. As for the  $x_t$  elements, here we have (ignoring the constants  $-\frac{2\alpha\beta^2}{K}$ )

$$x_{t+1} + \left(1 - \frac{1}{t+2}\right)x_{t+2} + \left(1 - \frac{1}{t+2}\right)\left(1 - \frac{1}{t+3}\right)x_{t+3} + \dots$$

$$= x_{t+1} + \left(\frac{t+2-1}{t+2}\right)x_{t+2} + \left(\frac{t+2-1}{t+2}\right)\left(\frac{t+3-1}{t+3}\right)x_{t+3} + \dots$$

$$= x_{t+1} + \left(\frac{t+1}{t+2}\right)x_{t+2} + \left(\frac{t+1}{t+2}\right)\left(\frac{t+1}{t+3}\right)x_{t+3} + \left(\frac{t+1}{t+2}\right)\left(\frac{t+1}{t+3}\right)\left(\frac{t+1}{t+4}\right)x_{t+4} + \dots$$

$$= (t+1) \left[ \frac{1}{t+1} x_{t+1} + \frac{1}{t+2} x_{t+2} + \frac{1}{t+3} x_{t+3} + \dots \right]$$

$$= (t+1) \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i}$$

$$\text{so then } \gamma_{3,+} = -\frac{2\alpha\beta^2}{\kappa} (t+1) \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i}$$

which for their (16) gives

$$2\pi_t + \frac{2\alpha}{\kappa} x_t - \frac{2\alpha\beta^2}{\kappa} \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i} = 0 \quad (16)$$

Nice!

The issue is that in my case, even the 29 Feb 2020

again won't be this simple b/c if  $k_{t+1} = k_t + 1$

then the series  $k_{t+2} k_{t+1} k_t = (k_{t+1}-1)(k_t+1) k_t$

$$= (k_t + 1 + 1)(k_t + 1) k_t \quad \text{oh that's not too bad}$$

$$= (k_t + 2)(k_t + 1) k_t$$

$$= \sum_{i=0}^{\infty} (k_t + i) \quad \text{ok, that's not too bad. The problem is}$$

that the endog. gain is gonna break this.

Let's go back to (II').

$$\varphi_{S,t} = \underbrace{\frac{1-k_t^{-1}}{1-k_t^{-2}f_{\bar{\pi}}}}_{=:a_t} \varphi_{S,t+1} - \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{1}{k}}_c \underbrace{\frac{1}{1-k_t^{-2}f_{\bar{\pi}}}}_{b_t} x_{t+1} \quad (\text{II}'')$$

$$a_t = \frac{1 - \frac{1}{k_t}}{1 - \frac{f_{\bar{\pi}}}{k_t^2}} = \frac{\frac{k_t - 1}{k_t}}{\frac{k_t^2 - f_{\bar{\pi}}}{k_t^2}} = \frac{k_t - 1}{\frac{k_t^2 - f_{\bar{\pi}}}{k_t}} = \frac{(k_t - 1) k_t}{k_t^2 - f_{\bar{\pi}}} \quad \text{Not useful.}$$

so let me instead rewrite it like this

$$\varphi_{S,t} = \frac{1-k_t^{-1}}{d_t} \varphi_{S,t+1} - c \frac{1}{d_t} x_{t+1} \quad \text{with } d_t = 1 - k_t^{-2} f_{\bar{\pi}}$$

The problem is still: what to do w/  $k_{t+2}, k_{t+3}, k_t$

$f(\bar{\pi}_t, \dots)$

ok, I just noticed that this is not right either b/c  
due to  $\bar{\pi}_{t-1}$  we have a  $\varphi_{6,t+1}$  floating around...

→ so what I first need to check is how to solve  
these damn systems of nonautonomous diff. eqs.

So the system is:

$$2\pi_t + 2 \frac{\beta}{k} x_t - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_{\bar{n}} = 0 \quad (\text{I}) \quad (2)$$

$$\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{k} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1} (1-k_t^{-1}) + \varphi_{6,t+1} f_{\bar{n}} = 0 \quad (\text{II}) \quad (2)$$

$$\varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1} f_k = 0 \quad (\text{III})$$

Is the following dumb? Express  $\varphi_{6,t}$  from (III) plug in (I).

then: express  $\varphi_{6,t+1}$  from (I), plug in (II). Solve for  $\varphi_{5,t}$ .

Yeah it's dumb.

What if instead we express  $\varphi_{5,t}$  from (II) and plug in (I) & (I)

$$\varphi_{5,t} = \frac{f_k \varphi_{6,t+1} - \frac{1}{k} \varphi_{6,t}}{k_t^{-2}}$$

$$\Rightarrow (\text{I}): 2\pi_t + 2 \frac{\beta}{k} x_t - k_r f_k \varphi_{6,t+1} - k_t \varphi_{6,t} + f_{\bar{n}} \varphi_{6,t} = 0$$

$$(\text{II}) \quad \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{k} x_{t+1} + \frac{f_k}{k_t^{-2}} \varphi_{6,t+2} - \frac{1}{k_t^{-2}} \varphi_{6,t+1}$$

$$- (1-k_t^{-1}) \left[ \frac{f_k}{k_{t+1}^{-2}} \varphi_{6,t+2} - \frac{1}{k_{t+1}^{-2}} \varphi_{6,t+1} \right] + f_{\bar{n}} \varphi_{6,t+1} = 0$$

$$(I): 2\pi_t + 2\frac{\partial}{K}x_t - k_r f_k \varphi_{6,t+1} + (f_{\bar{\pi}} - k_+) \varphi_{6,t+2} = 0$$

$$(II) \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\partial}{K}x_{t+1} - k_r^2 \varphi_{6,t+1} + (k_r^2 f_k + (1-k_r^{-1})k_{t+1}^2 - f_{\bar{\pi}}) \varphi_{6,t+2} - k_{t+1}^2 (1-k_r^{-1}) f_k \varphi_{6,t+3} = 0$$

I think I can solve any of (I) or (II) for  $\varphi_{6,t+1}$  (and  $\varphi_{6,t+2}$ ) and plug in the other. I could also have done it the other way: expressed for  $\varphi_{6,t+1}$  and have these two eqs for  $\varphi_{5,t+1}$ . The economic interpretation is interesting in either case: in the system w/  $x$  and  $\varphi_6$ , we'll relate the  $\pi$ - $x$  tradeoff to the shadow value of the anchoring constraint; in the system w/  $x$  and  $\varphi_5$ , we relate the  $\pi$ - $x$  tradeoff to the shadow value of the expectation of LR inflation,  $\bar{\pi}$ .

But before we can solve any of those, we need to deal w/ these  $k_r$ 's. For that reason, I suggest to focus on (II) for a sec, and b/c it has an interesting interpretation: it says that  $\varphi_5$ , the shadow value of the LR  $\pi$ -expectation,  $\bar{\pi}$ , depends on all future ones of

the shadow value of the anchoring constraint:  
The other way around!

$$\gamma_{5,t} k_t^{-2} + \gamma_{6,t} - \gamma_{6,t+1} f_k = 0 \quad (\text{III})$$

$$k_t^{-2} \gamma_{5,t} = f_k \gamma_{6,t+1} - \gamma_{6,t}$$

$$\gamma_{5,t} = k_t^2 f_k \gamma_{6,t+1} - k_t^2 \gamma_{6,t}$$

This is saying that that the extent of today's LR-expectation constraint depends on the change in how the anchoring constraint binds. In fact,

$$\gamma_{5,t} = k_t^2 (f_k \gamma_{6,t+1} - \gamma_{6,t})$$

$\gamma_{5,t} = 0$  when  $\gamma_{6,t} = \gamma_{6,t+1} = 0$ , i.e. it's not enough that there's a single period where the anchoring doesn't bind. This makes intuitive sense b/c you want the learning to have converged. Another way to do it is to have

$$\gamma_{6,t} = f_k \gamma_{6,t+1} \Rightarrow \gamma_6 \text{ is increasing } (f_k < 1) ?!$$

or  $k_t = 0$ . But I guess  $\gamma_6$  can't increase if  $f_k \downarrow$ .

I kinda feel that the anchoring constraint binds as long as  $(k_t, \bar{t}_t, f_t(\cdot))$  is non-zero. That is, as long as  $\gamma_5 \neq 0$ .  
└ time-varying.

One thing this points to which I've ignored so far is that  $f(\bar{a}_t, \bar{a}_{t+1}, k_{t+1})$  is clearly time-varying, so that I should call it  $f(t)$ , and its derivatives are likely also time-varying  $f_k(t)$ . (If  $f(t)$  isn't linear in its arguments, then the derivatives will indeed be time-varying.)

$$\text{So } y_{5,t} = k_t^2 (f_k(t) y_{6,t+1} - y_{6,t})$$

That is one way  $y_{5,t} = 0$  is

$f_k(t) > 1$  but  $y_{6,t+1} < y_{6,t} \rightarrow \varphi_6 \downarrow$  (as 1)

the other:  $f_k(t) < 1$  but  $y_{6,t+1} > y_{6,t} \rightarrow \varphi_6 \uparrow$  (as 2)

Take Case 1: tomorrow's anchoring constraint binds less than today's, but the anchoring function is changing a lot in response to  $k_{t+1}$

$\rightarrow$  I'm not sure I understand what this means!

$$\text{You can rewrite as } y_{5,t} = k_t^2 f_k(t) \left( \varphi_{6,t+1} - \frac{\varphi_{6,t}}{f_k(t)} \right)$$

If  $f_k(t) = 0$  the gain isn't changing,  $k_t = 0$ .

Case 1:  $f_k(t)$  large,  $\varphi_{6,t} > \varphi_{6,t+1} \rightarrow f_k(t)$  large

means / implies that  $\gamma_{0,t}$  is large b/c when the gain changes a lot, that's when the constraint binds more.

So if  $\gamma_{0,t+1} < \gamma_{0,t}$ , this implies that  $E f_k(t+1) < f_k(t) \Rightarrow$  so when anchoring is expected to happen, then learning may not be a binding constraint

Case 2:  $f_k(t)$  small,  $\gamma_{0,t+1} > \gamma_{0,t}$

If seems like if anchoring or unanchoring is expected to happen is a particular ratio, then  $P_{S,t} = 0$ .

I'm still unsure about this interpretation and don't know if it's just a mathematical awkwardness or if it really carries a deep economic meaning.

Now let's finally turn to solving this:

$$k_t^{-2} \varphi_{5,t} = f_k(t) \varphi_{6,t+1} - \varphi_{6,t} \quad (\text{II})$$

$$\varphi_{6,t} = f_k(t) \varphi_{6,t+1} - k_t^{-2} \varphi_{5,t}$$

Let me call  $\beta_t = k_t^{-1}$

$$\varphi_{6,t} = f_k(t) \varphi_{6,t+1} - \beta_t^2 \varphi_{5,t}$$

Wait one sec: this is a non-autonomous diff eq.

w/ time-varying coeffs  $f_k(t)$  and  $\beta_t^2$ . But for  $\beta_t^2$ , we can map it to a non-autonomous diff eq. by introducing  $b_t = -\beta_t^2 \varphi_{5,t}$ . Calling  $f_k(t) = a_t$ , we can write like Tirrelli (p. 5)

$$\varphi_{6,t} = a_t \varphi_{6,t+1} + b_t \quad \text{or redifining}$$

$$\text{dubiously: } x_{t+1} = \alpha_{t+1} x_t + b_t$$

Fuck! Tirrelli just states that

$$x_t^{(\alpha)} = \begin{cases} \sum_{k=1}^{t-1} \prod_{s=0}^{k-2} \alpha_{t-s} b_{t-k} + b_t & \text{if } |\alpha| < 1 \\ - \sum_{k=0}^{\infty} \left( \prod_{s=0}^k \left( \frac{1}{\alpha_{t+s}} \right) b_{t+k} \right) & \text{if } |\alpha| > 1 \end{cases}$$

which is what I've found but that doesn't help a lot!

$$y_{6,t} = f_k(t) y_{6,t+1} + b_t$$

$$b_t \equiv -\gamma^2 y_{51t}$$

$$y_{6,t} = f_k(t) [f_k(t+1) y_{6,t+2} + b_{t+1}] + b_t$$

$$= f_k(t) f_k(t+1) y_{6,t+2} + f_k(t) b_{t+1} + b_t$$

$$= f_k(t) f_k(t+1) [f_k(t+2) y_{6,t+3} + b_{t+2}] + f_k(t) b_{t+1} + b_t$$

$$= f_k(t) f_k(t+1) f_k(t+2) y_{6,t+3} + f_k(t) f_k(t+1) b_{t+2}$$

$$+ f_k(t) b_{t+1}$$

$$+ b_t$$

I assume that all  $f_k(t) < 1 \quad \forall t$ , so the

stuff  $y_{6,t+k} \rightarrow 0 \quad \text{as } k \rightarrow \infty$ . At least.

$$y_{6,t} = b_t + f_k(t) b_{t+1} + f_k(t) f_k(t+1) b_{t+2} + f_k(t) f_k(t+1) f_k(t+2) b_{t+3}$$

$$= b_t + \sum_{i=1}^{\infty} b_{t+i} (f_k(t) \dots f_k(t+i-1)) + \dots$$

$$= b_t + \sum_{i=1}^{\infty} b_{t+i} \prod_{j=0}^{i-1} f_k(t+j)$$

$$\Rightarrow y_{6,t} = -\gamma^2 y_{51t} - E_t \sum_{i=1}^{\infty} \gamma^2 y_{51,t+i} \prod_{j=0}^{i-1} f_k(t+j)$$

The anchoring constraint binds today to the extent that the learning constraint binds today and is expected to bind in the future.

Ok, now go back to the 2 equations:

$$(I): 2\pi_t + 2\frac{\beta}{K}x_t - k_r f_K \varphi_{6,t+1} + (f_{\bar{N}} - k_r) \varphi_{6,t} = 0$$

$$(II) \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{1}{K} x_{t+1} - k_r^2 \varphi_{6,t} + \left( k_r^2 f_K + (1-k_r^{-1}) k_{r+1}^2 + f_{\bar{N}} \right) \varphi_{6,t+1} \\ - k_{r+1}^2 (1-k_r^{-1}) f_K \varphi_{6,t+2} = 0$$

Solve (II) for  $\varphi_{6,t}$  to then plug back  $\varphi_{6,t}$  &  $\varphi_{6,t+1}$  in (I)

$$\varphi_{6,t} = \frac{\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{1}{K} x_{t+1} + \frac{k_r^2 f_K + (1-k_r^{-1}) k_{r+1}^2 + f_{\bar{N}}}{k_r^2} \varphi_{6,t+1}}{k_r^2}$$

$$- \frac{k_{r+1}^2}{k_r^2} (1-k_r^{-1}) f_K \varphi_{6,t+2} = 0$$

Sigh. Have to solve a 2nd order non-autonomous eq.

Call  $\varphi_{6,t+1} \equiv y_t$

$$\varphi_{6,t} = \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{1}{K} \frac{1}{k_r^2} x_{t+1} + \frac{k_r^2 f_K + (1-k_r^{-1}) k_{r+1}^2 + f_{\bar{N}}}{k_r^2} \varphi_{6,t+1}}_{(*)}$$

$$- \frac{k_{r+1}^2}{k_r^2} (1-k_r^{-1}) f_K y_{t+1} = 0$$

$$y_t = \varphi_{6,t+1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{6,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{k_r^2 f_K + (1-k_r^{-1}) k_{r+1}^2 + f_{\bar{N}}}{k_r^2} & - \frac{k_{r+1}^2}{k_r^2} (1-k_r^{-1}) f_K \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{6,t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} (*) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_{0,+} \\ y_+ \end{bmatrix} = \begin{bmatrix} \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^2 + f_{\bar{n}}}{k_+^2} ; & -\frac{k_{++1}^2}{k_+^2} (1-k_+^{-1}) f_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{0,+m} \\ y_{+-1} \end{bmatrix} + \begin{bmatrix} (*) \\ 0 \end{bmatrix} x_{++1}$$

$$\begin{bmatrix} \psi_{0,+} \\ y_+ \end{bmatrix} = \begin{bmatrix} \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^2 + f_{\bar{n}}}{k_+^2} ; & -\frac{k_{++1}^2}{k_+^2} (1-k_+^{-1}) f_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{0,+m} \\ y_{+-1} \end{bmatrix} + \begin{bmatrix} (*) & 0 \\ 0 & 0 \end{bmatrix} x_{++1}$$

= A

So to solve this, we need to make this non-interacting.

→ need to diagonalize A!

I'm using Tibelli here, but could use Susanto's lecture notes too.

The characteristic polynomial here is

$$\det \begin{bmatrix} \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^2 + f_{\bar{n}} - \lambda}{k_+^2} ; & -\frac{k_{++1}^2}{k_+^2} (1-k_+^{-1}) f_k \\ 1 & -\lambda \end{bmatrix} = 0$$

Take an extra step just b/c it gives nice intuition:

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ -a_{21} & a_{22} - \lambda \end{bmatrix} = 0 \Rightarrow (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$\Rightarrow a_{11}a_{22} - \lambda a_{11} - \lambda a_{22} + \lambda^2 - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$$

Useful!

$$\text{tr}(A) = a_{11} + a_{22} = \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^{-2} + f_{\bar{n}}}{k_+^2}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = 0 - \left( -\frac{k_{++1}^{-2}}{k_+^2} (1-k_+^{-1}) f_k \right)$$

so the eigenvalues are

$$\lambda_{1,2} = \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^{-2} + f_{\bar{n}}}{k_+^2} \pm \sqrt{\left( \frac{k_+^2 f_k + (1-k_+^{-1}) k_{++1}^{-2} + f_{\bar{n}}}{k_+^2} \right)^2 - 4 \frac{k_{++1}^{-2}}{k_+^2} (1-k_+^{-1}) f_k}$$

$$\text{Under the } \sqrt{\cdot}, \quad a_+ := \frac{(1-k_+^{-1}) k_{++1}^{-2}}{k_+^2}, \quad \text{then}$$

$$\left( f_k + a_+ + \frac{f_{\bar{n}}}{k_+^2} \right)^2 - 4 f_k a_+ = f_k^2 + a_+^2 + \frac{f_{\bar{n}}^2}{k_+^4} + 2 f_k a_+ + 2 f_k \frac{f_{\bar{n}}}{k_+^2}$$

$$+ 2 a_+ \frac{f_{\bar{n}}}{k_+^2} (-4 f_k a_+)$$

$$= f_k^2 + a_+^2 + \frac{f_{\bar{n}}^2}{k_+^4} - 2 f_k a_+ + 2 f_k \frac{f_{\bar{n}}}{k_+^2} + 2 a_+ \frac{f_{\bar{n}}}{k_+^2} \quad \text{Dann!}$$

Ok so at least establish sthg about signs & sizes of  
 $\lambda_1, \lambda_2$ .

$$\lambda_1 \cdot \lambda_2 = \det(A)$$

$$\lambda_1 + \lambda_2 = \text{trace}(A)$$

$$\lambda_1 \cdot \lambda_2 = \frac{k_{\pi\bar{\pi}}^2}{k_\pi^2} (1 - k_\pi^{-2}) f_\pi$$

$$\lambda_1 + \lambda_2 = \frac{k_\pi^2 f_\pi + (1 - k_\pi^{-2}) k_{\pi\bar{\pi}}^2 + f_{\bar{\pi}}}{k_\pi^2}$$

Somg guys, even this is unwieldy. Step back a little:  
assume a little more structure on  $f$ :

$$k_1 = f = f(\pi_t - \bar{\pi}_{t-1} - b\delta_{t-1})$$

$$\Rightarrow \frac{\partial f}{\partial \pi_t} = f_{\bar{\pi}} < 1 \text{ but } f_{\bar{\pi}} > 0.$$

$$\frac{\partial f}{\partial \bar{\pi}} = -f_{\bar{\pi}} \text{ where } f_{\bar{\pi}} \in (0, 1) \text{ also.}$$

In fact, since  $FE = \pi_t - \bar{\pi}_{t-1} - b\delta_{t-1}$ , and  $f = f(FE)$ ,  
it's likely that  $f_\pi = f_{\bar{\pi}}$ . Can impose if necessary.

In that case the FOCs are: was wrong previously anyway

FOCs : <sup>Also I'm now treating  $k_t^{-1}$  as a variable.</sup>  $\varphi_{6,t} + \varphi_{6,t} \left( k_t^{-1} - f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \right)$

$$\pi_t: 2\pi_t + \varphi_{1,t} - \varphi_{5,t} k_t^{-1} - \varphi_{6,t} f_{\pi}(t) = 0 \quad (1)$$

$$x_t: 2\lambda x_t - \varphi_{1,t} K = 0 \quad \underline{\varphi_{1,t} = 2\frac{\lambda}{K} x_t} \quad (2)$$

$$f_{\alpha t}: -\varphi_{1,t} (1-\alpha)\beta + \varphi_{3,t} = 0 \quad \underline{\varphi_{3,t} = 2(1-\alpha)\beta \frac{\lambda}{K} x_t} \quad (3)$$

$$\bar{\pi}_t: -\varphi_{3,t+1} \frac{1}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1} (1-k_t^{-1}) + \varphi_{6,t+1} f_{\bar{\pi}}(t) \quad (4)$$

$$\overset{1}{k_t}: -\varphi_{5,t} (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) + \varphi_{6,t} = 0 \quad (5)$$

$\Rightarrow$  this becomes the following 3 equations:

$$2\pi_t + 2\frac{\lambda}{K} x_t - \varphi_{5,t} k_t^{-1} - \varphi_{6,t} f_{\pi}(t) = 0 \quad (I)$$

$$-\frac{2(1-\alpha)\beta\lambda}{1-\alpha\beta} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1} (1-k_t^{-1}) - \varphi_{6,t+1} f_{\bar{\pi}}(t) \quad (II)$$

$$= 0$$

$$\varphi_{6,t} = (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \varphi_{5,t} \quad (III)$$

The nice thing is that this makes the relationship between  $\varphi_6$  &  $\varphi_5$  much more clear: one is FE · the other.

Plug  $\varphi_{6,t}$  into (I) & (II)

$$2\pi_t + 2\frac{\lambda}{K} x_t - \varphi_{5,t} k_t^{-1} - ((\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \varphi_{5,t}) f_{\pi}(t) = 0$$

$$2\pi_t + 2\frac{\lambda}{K} x_t - \varphi_{5,t} (k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_{\bar{\pi}}(t)) = 0$$

$$2\bar{\pi}_+ + 2\frac{\beta}{K}x_+ - \psi_{S,+} \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - b\varsigma_{+-1})f_{\bar{\pi}}(+)\right) = 0 \quad (I')$$

And (II):

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{K} x_{++1} + \psi_{S,+} - \psi_{S,+1} (1-k_+^{-1})$$

$$+ f_{\bar{\pi}}(+) (\pi_{+1} - \bar{\pi}_+ - b\varsigma_+) \psi_{S,++1} = 0$$

$$\psi_{S,+} = \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{K} x_{++1} + \psi_{S,+1} \left[ (1-k_+^{-1}) - f_{\bar{\pi}}(+) (\pi_{+1} - \bar{\pi}_+ - b\varsigma_+) \right] \quad (II')$$

so we're down to 2 equations, (I') & (II') Gaspar et al's 22 821

$$2\bar{\pi}_+ + 2\frac{\beta}{K}x_+ - \psi_{S,+} \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - b\varsigma_{+-1})f_{\bar{\pi}}(+)\right) = 0 \quad (I')$$

$$\psi_{S,+} = \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{K} x_{++1} + \psi_{S,+1} \left[ 1 - k_+^{-1} - f_{\bar{\pi}}(+) (\pi_{+1} - \bar{\pi}_+ - b\varsigma_+) \right] \quad (II')$$

$$\psi_{S,+} = Cx_{++1} + \psi_{S,+1} \cdot \alpha_{++1} \quad \text{(assuming } \alpha_t < 1 \text{ at least for most } t \dots \text{)}$$

$$= Cx_{++1} + \alpha_{++1} [Cx_{++2} + \psi_{S,+2} \cdot \alpha_{++2}]$$

$$= Cx_{++1} + C\alpha_{++1} x_{++2} + \alpha_{++1} \alpha_{++2} \psi_{S,+2}$$

$$= (x_{++1} + \alpha_{++1} x_{++2}) + \alpha_{++1} \alpha_{++2} [Cx_{++3} + \psi_{S,+3} \alpha_{++3}]$$

$$= ((x_{++1} + \alpha_{++1} x_{++2} + \alpha_{++1} \alpha_{++2} x_{++3} + \dots))$$

$$\psi_{S,+} = Cx_{++1} + C \sum_{i=2}^{\infty} x_{++i} \prod_{j=1}^{i-1} \alpha_{++j}$$

Which means that  $(\dagger')$  becomes:

$$2\bar{\pi}_+ = -2\frac{\beta}{K}x_+ + \left(k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - bS_{+1})f_{\bar{\pi}}(+) \right)\varphi_{S,+}$$

$$\pi_+ = -\frac{\beta}{K}x_+ + \frac{1}{2} \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - bS_{+1})f_{\bar{\pi}}(+) \right)$$

$$\left( Cx_{+1} + C \sum_{i=2}^{\infty} x_{+1+i} \prod_{j=1}^{i-1} \alpha_{+1+j} \right)$$

$$\pi_+ = -\frac{\beta}{K}x_+ + \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - bS_{+1})f_{\bar{\pi}}(+) \right)$$

$$\frac{(1-\alpha)\beta}{1-\alpha\beta} \frac{\beta}{K} \left( x_{+1} + \sum_{i=2}^{\infty} x_{+1+i} \prod_{j=1}^{i-1} \alpha_{+1+j} \right)$$

$$\pi_+ = -\frac{\beta}{K} \left\{ x_+ - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - bS_{+1})f_{\bar{\pi}}(+) \right) \cdot \right.$$

$$\left. \left( x_{+1} + \sum_{i=2}^{\infty} x_{+1+i} \prod_{j=1}^{i-1} (1 - k_{+1+j}^{-1} + f_{\bar{\pi}}(t+j)) (\pi_{+1+j} - \bar{\pi}_{+1+j} - bS_{+1+j}) \right) \right\}$$

Bagpar et al's (24).

In materials 18, I write this using  $\sum_{j=1}^{\infty} \pi_j = 1$  as

$$\pi_+ = -\frac{\beta}{K} \left\{ x_+ - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_+^{-1} + (\pi_+ - \bar{\pi}_{+1} - bS_{+1})f_{\bar{\pi}}(+) \right) \right\}$$

$$\left( \sum_{i=1}^{\infty} x_{+1+i} \prod_{j=1}^{i-1} (1 - k_{+1+j}^{-1} + f_{\bar{\pi}}(t+j)) (\pi_{+1+j} - \bar{\pi}_{+1+j} - bS_{+1+j}) \right) \right\}$$

Interpretation: we have "No grim trigger result" is here too.

$$\pi_+ = -\frac{\lambda}{K} x_+ + \frac{\lambda}{K} \cdot \text{stuff} . \quad \text{Clearly, stuff} \rightarrow 0 \text{ if } k_i^{-1} & f_n(k) \\ \pi_+ \text{ still can be discussed as:} \quad \rightarrow 0 .$$

that stuff can be decomposed as:

As you learn things,

$$\pi_{+}^{-1} \rightarrow 0$$

The more anchored you are,

The more  $f_{\overline{n}}(t) \rightarrow 0$ .

Note that the FE doesn't go to 0 b/c exog shrinks.

The FE only goes to 0 in expectation.

So in theory a situation may happen that  $k_+^{-1} \approx 0$

but  $f_+$  is large and you become unanchored.

So the "no grim trigger" result is here, is however qualified. Why? B/c in a situ where  $f=f(\text{FE})$  (so  $\theta^{\text{CEMP}}$  doesn't work,  $\theta^{\text{casum}}$  does), you can at any time in history be hit by a shock large enough such that  $f_A(+)$  is big! This is "as if" the grim trigger

was back, but not really in the sense that PS is still a machine and not strategic: has no recourse to threats. It's more like nature can discipline the CB by spitting out a shock large enough to restart the learning.

Trying to simplify prod:

$$(1-a)(1-b)(1-c)$$

$$(1-b)(1-c) - a(1-b)(1-c)$$

$$1-c-b+bc - a(1-c-b+bc)$$

$$1-c-b+bc - a + ac - ab - abc$$

$$1-a-b-c + ac - ab + bc - abc$$

Unfortunately that's not much simpler.

The next step is to solve for the interest rate rule.

As a first step, let's work them how Molnár & Santoro do it (Proof of Prop. 2, in App B., p. 19(Mac))

Eqs (10) - (13) and (15) are :  $(10) \lambda_{1,t} = 0 \quad (15) \lambda_{4,t} = 0$

$$2\bar{\pi}_t - \lambda_{2,t} + \gamma_{t+1} \lambda_{3,t} = 0 \quad (11)$$

$$2\alpha x_t + k \lambda_{2,t} = 0 \rightarrow \underline{\lambda_{2,t} = -\frac{2\alpha}{k} x_t} \quad (12)$$

$$E_t \beta^2 \lambda_{2,t+1} + \beta(1 - \gamma_{t+2}) \lambda_{3,t+1} = \lambda_{3,t} \quad (13)$$

$$2\bar{\pi}_t + \frac{2\alpha}{k} x_t + \gamma_{t+1} \lambda_{3,t} = 0 \quad (11') \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$-E_t \beta^2 \frac{2\alpha}{k} x_{t+1} + \beta(1 - \gamma_{t+2}) \lambda_{3,t+1} = \lambda_{3,t} \quad (13')$$

Now impose  $\gamma_t = \frac{1}{t}$

$$2\bar{\pi}_t + \frac{2\alpha}{k} x_t + \frac{1}{t+1} \lambda_{3,t} = 0 \quad (11'')$$

$$-E_t \beta^2 \frac{2\alpha}{k} x_{t+1} + \underbrace{\beta \left(1 - \frac{1}{t+2}\right)}_{<1} \lambda_{3,t+1} = \lambda_{3,t} \quad (13'')$$

$$\text{So } \lambda_{3,t} = \beta \left( \frac{t+1}{t+2} \right) \left[ \beta \left( \frac{t+2}{t+3} \right) \lambda_{3,t+1} - E_t \beta^2 \frac{2\alpha}{k} x_{t+2} \right] - E_t \beta^2 \frac{2\alpha}{k} x_{t+1}$$

$$\lambda_{3,t} = -E_t \beta^2 \frac{2\alpha}{k} \left[ \beta \left( \frac{t+1}{t+2} \right) x_{t+2} + x_{t+1} \right]$$

$$= -E_t \beta^2 \frac{2\alpha}{k} \left[ \beta^2 \left( \frac{t+2}{t+3} \right) \left( \frac{t+1}{t+2} \right) x_{t+3} + \beta \left( \frac{t+1}{t+2} \right) x_{t+2} + x_{t+1} \right]$$

$$= -E_t \beta^2 \frac{2\alpha}{K} \left[ \beta^2 \left( \frac{t+1}{t+3} \right) x_{t+3} + \beta \left( \frac{t+1}{t+2} \right) x_{t+2} + x_{t+1} \right]$$

$$\gamma_{3,t} = -E_t \beta^2 \frac{2\alpha(t+1)}{K} \sum_{i=0}^{\infty} \beta^i \left( \frac{1}{t+1+i} \right) x_{t+1+i}$$

(1'') was  $2\pi_t + \frac{2\alpha}{K} x_t + \frac{1}{t+1} \gamma_{3,t} = 0$ , so it becomes

$$2\pi_t + \frac{2\alpha}{K} x_t = \frac{2\alpha}{K} \beta^2 E_t \sum_{i=0}^{\infty} \beta^i \left( \frac{1}{t+1+i} \right) x_{t+1+i}$$

which is nowhere near Molnár & Santoro's expression!

(p. 19)

Suppose for a sec that their expression is right, i.e.  
 ↳ for the target criterion  
 given by

$$\frac{K}{\alpha} \pi_t + u_t = \beta E_t \left[ \beta \frac{1}{t+1} x_{t+1} + \frac{K}{\alpha} \pi_{t+1} + x_{t+1} \right]$$

$$\text{Use (2)} : x_t = \frac{1}{K} \pi_t - \frac{\beta}{K} E_t^* \pi_{t+1} - \frac{1}{K} u_t \quad (\text{NKPC})$$

to sub out  $x$ :

$$\begin{aligned} \frac{K}{\alpha} \pi_t + \frac{1}{K} \pi_t - \frac{\beta}{K} E_t^* \pi_{t+1} - \frac{1}{K} u_t &= \beta E_t \frac{K}{\alpha} \pi_{t+1} \\ &+ \beta E_t \left[ \left( \beta \frac{1}{t+1} + 1 \right) \left( \frac{1}{K} \pi_{t+1} - \frac{\beta}{K} E_t^* \pi_{t+2} - \frac{1}{K} u_{t+1} \right) \right] \end{aligned}$$

Note: they ass.  $u_t \sim iid$ , so  $E_t u_{t+k} = 0 \quad \forall k > 0$

$$\frac{k}{\alpha} \pi_t + \frac{1}{k} \pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \beta E_t \frac{k}{\alpha} \pi_{t+1}$$

$$+ \beta E_t \left[ \left( \beta \frac{1}{t+1} + 1 \right) \left( \frac{1}{k} \pi_{t+1} - \frac{\beta}{k} E_{t+1}^* \pi_{t+2} - \frac{1}{k} u_{t+1} \right) \right]$$

$$\Rightarrow \left( \frac{k}{\alpha} + \frac{1}{k} \right) \pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \beta E_t \frac{k}{\alpha} \pi_{t+1} + \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{1}{k} E_t \pi_{t+1} \\ - \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{\beta}{k} E_t E_{t+1}^* \pi_{t+2}$$

$$\Leftrightarrow \left( \frac{k}{\alpha} + \frac{1}{k} \right) \pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \left( \beta \frac{k}{\alpha} + \frac{\beta}{k} \left( \beta \frac{1}{t+1} + 1 \right) \right) E_t \pi_{t+1} \\ - \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{\beta}{k} E_t E_{t+1}^* \pi_{t+2}$$

$$\text{Now sub } \pi - E(\cdot) : E_t^* \pi_{t+1} = a_t = a_{t-1} + t^{-1} (\pi_{t-1} - a_{t-1})$$

$$\Rightarrow \left( \frac{k}{\alpha} + \frac{1}{k} \right) \pi_t - \frac{\beta}{k} a_t - \frac{1}{k} u_t = \left( \beta \frac{k}{\alpha} + \frac{\beta}{k} \left( \beta \frac{1}{t+1} + 1 \right) \right) E_t \pi_{t+1}$$

$$- \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{\beta}{k} \left( a_t + \frac{1}{t+1} (\pi_t - a_t) \right)$$

$$\Leftrightarrow \left( \frac{k}{\alpha} + \frac{1}{k} \right) \pi_t - \frac{\beta}{k} a_t - \frac{1}{k} u_t = \left( \beta \frac{k}{\alpha} + \frac{\beta}{k} \left( \beta \frac{1}{t+1} + 1 \right) \right) E_t \pi_{t+1}$$

$$- \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{\beta}{k} \left( 1 - \frac{1}{t+1} \right) a_t - \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{1}{t+1} \pi_t$$

$$\Leftrightarrow \left( \frac{k}{\alpha} + \frac{1}{k} + \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{1}{t+1} \right) \pi_t + \left( \beta \left( \beta \frac{1}{t+1} + 1 \right) \frac{\beta}{k} \left( 1 - \frac{1}{t+1} \right) - \frac{\beta}{k} \right) a_t - \frac{1}{k} u_t$$

$$= \left( \beta \frac{k}{\alpha} + \frac{\beta}{k} \left( \beta \frac{1}{t+1} + 1 \right) \right) E_t \pi_{t+1}$$

$$\text{So } E_t \pi_{t+1} = A_{11,t} \pi_t + A_{12,t} a_t + P_{1,t} u_t$$

where

$$A_{11,t} = \frac{\left(\frac{k}{\alpha} + \frac{1}{k} + \beta \left(\beta \frac{1}{t+1} + 1\right) \frac{1}{t+1}\right)}{\left(\beta \frac{k}{\alpha} + \frac{\beta}{k} \left(\beta \frac{1}{t+1} + 1\right)\right)}$$

$$A_{12,t} = \frac{\left(\beta \left(\beta \frac{1}{t+1} + 1\right) \frac{1}{k} \left(1 - \frac{1}{t+1}\right) - \frac{\beta}{k}\right)}{\left(\beta \frac{k}{\alpha} + \frac{\beta}{k} \left(\beta \frac{1}{t+1} + 1\right)\right)}$$

and  $P_{1,t} = \frac{-\frac{1}{k}}{\left(\beta \frac{k}{\alpha} + \frac{\beta}{k} \left(\beta \frac{1}{t+1} + 1\right)\right)}$

Let's rewrite the denom:  $\alpha k \rightarrow \frac{1}{\alpha k} \underbrace{\left(\beta k^2 + \alpha \beta \left(\beta \frac{1}{t+1} + 1\right)\right)}$   
*this is what they have.*

→ So multiply numer's by  $\alpha k$ :

$$\rightarrow \text{num}(P_{1,t}) = -\alpha \quad \checkmark \quad \text{Got it!}$$

$$\rightarrow \text{num}(A_{11,t}) = k^2 + \alpha - \underbrace{\alpha k \beta}_{\text{This is diff, they have } \beta^2 \text{ here - may be a typo.}} \left(\beta \frac{1}{t+1} + 1\right) \frac{1}{t+1}$$

$$\rightarrow \text{num}(A_{12,t}) = \alpha k \frac{\beta}{k} \left[ k \left(\beta \frac{1}{t+1} + 1\right) \left(1 - \frac{1}{t+1}\right) - 1 \right]$$

$$\Rightarrow -\alpha\beta \left[ -K \left( \beta \frac{1}{t+1} + 1 \right) \left( 1 - \frac{1}{t+1} \right) + 1 \right]$$

$$= -\alpha\beta \left[ 1 - \left( 1 - \frac{1}{t+1} \right) \left( 1 + \beta \frac{1}{t+1} \right) \cdot K \right]$$

↑  
this guy again tuned  
the  $\beta$  for them,  
otherwise we're good.

Ok so we have (more or less) their eq. (31).

$$\text{So having } E_t \pi_{t+1} = A_{11,t} \pi_t + A_{12,t} a_t + P_{1,t} u_t$$

they write a big system using the LOMs of beliefs

$$y_t = \begin{bmatrix} \pi_t \\ a_t \\ b_t \end{bmatrix}$$

and they need to solve that somehow

$$\text{for } \pi_t = C_{\pi,t} a_t + d_{\pi,t} u_t \quad (23)$$

then (maybe from the NKPC somehow?) (Prop. 2)

$$\text{they get } x_t = \frac{C_{\pi,t} - \beta}{K} a_t + \frac{d_{\pi,t} - 1}{K} u_t \quad (24)$$

Then from the NKIS they back out the inf. rate rule

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u \cdot u_t .$$

Ok - what I'm trying to understand is what this method does exactly and how that's related to what Woodford does. So in words: Molnár & Santoro:

- Take FOCs, get target criterion, solve for  $E_t \pi_{t+1}$
- derive  $\pi_t$  as a fct of  $a_t, u_t$  (optimal path)
- derive  $x_t$  as a fct of  $a_t, u_t$  (optimal path) (from NKPC)
- use NKIS to get it as a fct of  $a_t, b_t, u_t$

Woodford:

The econ is characterized by:

$$(2.1): \quad \pi_t = \beta E_t \pi_{t+1} + \kappa x_t \Rightarrow x_t = \frac{1}{\kappa} \pi_t - \frac{\beta}{\kappa} E_t \pi_{t+1}$$

$$(2.23): \quad x_t = E_t x_{t+1} - \beta [i_t - E_t \pi_{t+1} - r_t^n]$$

$$(5.1): \quad \pi_t = -\frac{\gamma}{\kappa} (x_t - x_{t-1})$$

Woodford says that you calc  $E_t x_{t+1}$  &  $E_t \pi_{t+1}$  and plug into (5.1).

Can you simply plug NKPC & NKIS into (5.1)?

→ No b/c it's recursive! It looks as if Woodford also derived the optimal plans for  $\pi_t, x_t$ . He does so to eval the  $E(\cdot)$ .

It looks like Woodford does stay like this:

$$\text{if we're solved for } E_t \pi_{t+1} = a_x x_t + a_u \cdot u_t$$

$$\text{and } E_t x_{t+1} = b_x x_t + b_u \cdot u_t$$

and sub these into NKIS & NKPC:

$$\pi_t = \beta E_t \pi_{t+1} + k x_t$$

$$x_t = E_t x_{t+1} + \beta E_t \pi_{t+1} - \beta (i_t - r_t^n)$$

$$\Rightarrow \pi_t = \beta a_x x_t + \beta a_u u_t + k x_t$$

$$x_t = b_x x_t + b_u \cdot u_t + \beta a_x x_t + \beta a_u u_t - \beta (i_t - r_t^n)$$

$$\Rightarrow \pi_t = (\beta a_x + k) x_t + \beta a_u u_t$$

$$x_t = (b_x + \beta a_x) x_t + (b_u + \beta a_u) u_t - \beta (i_t - r_t^n)$$

$$\hookrightarrow x_t = \frac{b_u + \beta a_u}{1 - b_x - \beta a_x} u_t - \frac{\beta}{1 - b_x - \beta a_x} (i_t - r_t^n)$$

$$\Rightarrow \pi_t = \frac{(\beta a_x + k)(b_u + \beta a_u)}{1 - b_x - \beta a_x} u_t - \frac{\beta (\beta a_x + k)}{1 - b_x - \beta a_x} (i_t - r_t^n) + \beta a_u u_t$$

$$\pi_t = \left( \frac{(\beta a_x + k)(b_u + \beta a_u) + \beta a_u}{1 - b_x - \beta a_x} \right) u_t - \frac{\beta (\beta a_x + k)}{1 - b_x - \beta a_x} (i_t - r_t^n)$$

So we get  $x_+$  &  $\pi_+$  as fns of  $u_+$  &  $(i_+ - r_+^n)$ :

$$\pi_+ = \frac{(\beta_{ax} + k)(b_n + \beta a_n) + \beta a_n}{1 - b_x - \beta a_x} u_+ - \frac{\beta / \beta_{ax} + k}{1 - b_x - \beta a_x} (i_+ - r_+^n)$$

$$x_+ = \frac{b_n + \beta a_n}{1 - b_x - \beta a_x} u_+ - \frac{\beta}{1 - b_x - \beta a_x} (i_+ - r_+^n)$$

Then do we plug these into the target criterion?

$$\pi_+ + \frac{\gamma}{k} x_+ = \frac{\gamma}{k} x_+ - 1 \quad \text{given?}$$

$$\alpha_u u_+ + \alpha_i (i_+ - r_+^n) + \frac{\gamma}{k} \beta_u u_+ + \frac{\gamma}{k} \beta_i (i_+ - r_+^n) = \frac{\gamma}{k} x_+ - 1$$

$$(\alpha_i + \frac{\gamma}{k} \beta_i) i_+ + \alpha_u u_+ - \alpha_i r_+^n + \frac{\gamma}{k} \beta_u u_+ - \frac{\gamma}{k} \beta_i r_+^n = \frac{\gamma}{k} x_+ - 1$$

$$(\alpha_i + \frac{\gamma}{k} \beta_i) i_+ = -(\alpha_u + \frac{\gamma}{k} \beta_u) u_+ + (\alpha_i + \frac{\gamma}{k} \beta_i) r_+^n + \frac{\gamma}{k} x_+ - 1$$

$$i_+ = \frac{-(\alpha_u + \frac{\gamma}{k} \beta_u) u_+ + (\alpha_i + \frac{\gamma}{k} \beta_i) r_+^n + \frac{\gamma}{k} x_+ - 1}{\alpha_i + \frac{\gamma}{k} \beta_i} \quad \text{oh yeah!}$$

Ahright! That looks ok, in particular b/c it's indeterminate ;)

So Woodford: 1) Solve for optimal plan as a fn of  $x_+$  & shocks

2) Use these plans to eval  $E(\cdot)$  in NKIS & NKPC. Solve for  $\pi_+$  &  $x_+$  as a function of shocks only.

3) Plug the latter functions into the target criterion. Solve for  $i_+$ .

→ But for learning, we don't need to eval these  $E(\cdot)$  b/c we have the LOMs of  $E(\cdot)$ . The problem is that those aren't in functions of shocks only. Which is why Molnár & Santoro solve for  $\begin{bmatrix} E_{t+1} \\ a_t \\ b_t \end{bmatrix}$  as a function of shocks only.

Most likely whether you eval NKIS first or target criterion, doesn't matter.

→ Tomorrow: try to understand Molnár & Santoro's target criterion & their solution procedure.

One thing I wanted to check is what if we 25 Feb 2020  
iterate materials 18, eq(16) backwards? No, there's no way  
we can do that b/c  $(1 - k_t^{-1}) \leq 1$

Molnar & Santoro have solved for

$$\pi_t = C_{\pi,t} a_t + d_{\pi,t} u_t$$

$$x_t = C_{x,t} a_t + d_{x,t} u_t$$

$$\text{and NKIS} = x_t = \underbrace{E_t^* x_{t+1}}_{=b_t} - b^{-1} r_t + b^{-1} \underbrace{E_t^* \pi_{t+1}}_{=a_t}$$

$$\Rightarrow r_t = -b x_t + b b_t + a_t$$

$$r_t = -b (C_{x,t} a_t + d_{x,t} u_t) + b b_t + a_t$$

$$r_t = (1 - b C_{x,t}) a_t + b b_t - b d_{x,t} u_t$$

which is what they get.

So the procedure is: solve for optimal plans (and expectations if necessary) and plug into the NKIS.

So in either case you solve for the optimal plans. What I'm a little confused about is the following: if you have optimal plans for  $(x_t, \pi_t)$  and you've solved for expressions for  $E_t \pi_{t+1}$  &  $E_t x_{t+1}$  then the NKIS should give you the interest rate you wanna set; the only one consistent w/ this path. So why need the target?

Stop for a sec - investigate diff. eqs w/ Mathematica  
(Mathematica calls them recurrence equations)

### materials 18.nb

#### 1<sup>st</sup> order homogeneous diff. eq.

$$y_{t+1} + a y_t = 0$$

$$y |a| < 1, \quad y_{t+1} = -a y_t = -a (-a) y_{t-1} \dots$$

$$\Rightarrow y_t = (-a)^k y_{t-k} \quad \text{or} \quad y_t = (-a)^t y_0$$

#### 1<sup>st</sup> order nonhomogeneous diff eq

$$y_{t+1} + a y_t = c$$

$$\rightarrow \text{Alpha Chiang p. 556, (16.8')} \quad y_t = \left( y_0 - \frac{c}{1+a} \right) (-a)^t + \frac{c}{1+a}$$

is exactly what Mathematica gets. Neat!

#### 1<sup>st</sup> order nonhomogeneous, nonautonomous diff eq.

$$y_{t+1} + a y_t = x_{t+1}$$

↑ It seems like there is no way to tell Mma that  $|a| < 1$ ,

so assume  $|a| > 1$ .

$$y_t = -\frac{1}{a} y_{t-1} + \frac{1}{a} x_{t+1}$$

$$\rightarrow y_t = -\frac{1}{a} \left[ \left( -\frac{1}{a} \right) y_{t-2} + \frac{1}{a} x_{t-2} + \frac{1}{a} x_{t-1} \right]$$

$$= \left( -\frac{1}{a} \right)^2 y_{t-2} + \frac{1}{a} \left[ \left( -\frac{1}{a} \right) x_{t-2} + x_{t-1} \right]$$

$\vdots \rightarrow 0$

$$y_t = (-a)^t y_0 + \frac{1}{a} \sum_{j=0}^{t-1} \left( -\frac{1}{a} \right)^j x_{t-1-j}$$

↑  
complementary  
sol is the same

particular

Mma gets  $\Rightarrow$  here

and  $(-a)^t$  is multiplying everything for Mma. Why?

In diffgn. pdf, we have this exact example:

$$x_t = a x_{t-1} + b_t$$

$$\begin{aligned} \text{if } |a| < 1 \quad x_t &= a \left[ a x_{t-2} + b_{t-1} \right] + b_t && \text{Base, Lec 10} \\ &= a^2 x_{t-2} + a b_{t-1} + b_t && \sum_{s=1}^t a^{t-s} b_s \\ &= a^k x_{t-k} + \sum_{j=0}^{k-1} a^j b_{t-k} && \text{Tirelli too!} \\ &\vdots \\ &= \sum_{j=0}^{\infty} a^j b_{t-j} &= \sum_{s=-\infty}^t a^{t-s} b_s & \downarrow \quad (2) \end{aligned}$$

$$\text{if } |a| > 1 \quad a x_{t-1} = x_t - b_t \Rightarrow x_{t-1} = \frac{1}{a} x_t - \frac{1}{a} b_t$$

$$x_{t-1} = \frac{1}{a} x_t - \frac{1}{a} b_t$$

$$= \frac{1}{a} \left( \frac{1}{a} x_{t+1} - \frac{1}{a} b_{t+1} \right) - \frac{1}{a} b_t$$

$$x_{t-1} = \left(\frac{1}{a}\right)^2 x_{t+1} - \frac{1}{a} \left( \frac{1}{a} b_{t+1} + b_t \right)$$

$$x_t = \left(\frac{1}{a}\right)^2 x_{t+2} - \frac{1}{a} \left( \frac{1}{a} b_{t+2} + b_{t+1} \right)$$

$\hookrightarrow 0$

$$\vdots = -\frac{1}{a} \sum_{i=0}^{k-1} \left(\frac{1}{a}\right)^i b_{t+i+1}$$

$$= -\frac{1}{a} \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i b_{t+i+1} = -\frac{1}{a} \sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{t+1-s} b_s$$

$$\hookrightarrow = -\sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^{i+1} b_{t+i+1}$$

Troll gets this yeah!

not quite what diff eqn.  
says.

$$= -\sum_{j=1}^{\infty} \left(\frac{1}{a}\right)^j b_{t+j}$$

Basn, Lect 20:

$$z_t = az_{t-1} + m_t \quad |a| > 1$$

$$az_{t-1} = z_t - m_t$$

$$z_{t-1} = \frac{1}{a} z_t - \frac{1}{a} m_t$$

He writes

$$z_t = -\sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{t-s} m_s$$

this doesn't quite look right  
to me

Some messy work, but I think I'm fine.

So for the diff. eq.  $y_{t+1} = a y_t + b_{t+1}$

$$\text{I get: } y_t^P = -\frac{1}{a} \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i b_{t+i+1}$$

$$\text{Mma gets: } y_t = a^{t-1} \left( a y_0 + \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1} - a b_0 \right)$$

$$\therefore y_t = \underbrace{a^t y_0}_{\text{this is fine}} - \underbrace{a^t b_0}_{\text{this is also}} + a^{t-1} \underbrace{\sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1}}_{y^P} \\ \text{ignore. fine, ignore.} \rightarrow y^c$$

Mma is saying that this system has evolved since  $t=0$

$$\text{for } t \text{ periods only (not } \infty) : a^t \left( y_0 + a^{t-1} \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^{k+1} a b_{k+1} - b_0 \right)$$

$$y_t = a^t \left[ y_0 - b_0 + \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^{k+1} b_{k+1} \right]. \text{ Focus on } y^P \text{ & let } s = k+1$$

$$y_t^P = -b_0 + \sum_{s=0}^t \left(\frac{1}{a}\right)^s b_s = -b_0 + \left(\frac{1}{a}\right) \sum_{s=0}^t \left(\frac{1}{a}\right)^{s-1} b_s$$

$$= -b_0 + b_0 + \left(\frac{1}{a}\right) \sum_{s=1}^t \left(\frac{1}{a}\right)^{s-1} b_s$$

$$= \frac{1}{a} \sum_{j=0}^t \left(\frac{1}{a}\right)^j b_{j+1} \quad \text{Yeah baby! Up to the sign } \wedge$$

So when MMA outputs

$$y^P = \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1}, \quad \text{you can convert that to}$$

$$y^P = \uparrow \frac{1}{a} \sum_{j=0}^{\infty} \left(\frac{1}{a}\right)^j b_{t+1+j}$$

and multiply by  $(-1)$ . It seems like MMA isn't getting that right.

1<sup>st</sup> order nonhomogeneous nonautonomous diff. eq.

w/ variable coefficients

$$x_{t+1} = \alpha_{t+1} x_t + b_{t+1} \quad \text{Tirelli, p. 5.}$$

Supp  $|x| < 1$

$$\begin{aligned} x_{t+1} &= \alpha_{t+1} (\alpha_t x_{t-1} + b_t) + b_{t+1} \\ &= \alpha_{t+1} \alpha_t x_{t-1} + \alpha_{t+1} b_t + b_{t+1} \\ &= \alpha_{t+1} \alpha_t (\alpha_{t-1} x_{t-2} + b_{t-1}) + \alpha_{t+1} b_t + b_{t+1} \\ &= \alpha_{t+1} \alpha_t \alpha_{t-1} x_{t-2} + \alpha_{t+1} \alpha_t b_{t-1} + \alpha_{t+1} b_t + b_{t+1} \\ y_t^C &= \prod_{s=0}^t \alpha_{s+1} x_0 \quad \text{as Tirelli notes} \end{aligned}$$

$$\begin{aligned}
 y^P &: \alpha_{t+1} \alpha_t b_{t-1} + a_{t+1} b_t + b_{t+1} + \dots \\
 &= \alpha_t \alpha_{t-1} b_{t-2} + a_t b_{t-1} + b_t + \dots \\
 &= \alpha_t \alpha_{t-1} \alpha_{t-2} b_{t-3} + \alpha_t \alpha_{t-1} b_{t-2} + \alpha_t b_{t-1} + b_t \\
 &- \sum_{k=0}^{\infty} b_{t-k} (\alpha_{t-0} \dots \alpha_{t-k}) \\
 &= b_t + \sum_{k=1}^{\infty} b_{t-k} (\alpha_{t-0} \dots \alpha_{t-k+1}) \\
 &\quad \alpha_{t-(k-1)} \\
 &= b_t + \sum_{k=1}^{\infty} b_{t-k} \prod_{s=0}^{k-1} \alpha_{t-s} \quad \text{ok, good. This is what} \\
 &\quad \text{Tirelli gets.}
 \end{aligned}$$

Now if  $|\alpha| > 1$

$$x_{t+1} = \alpha_{t+1} x_t + b_{t+1}$$

$$\Leftrightarrow x_t = \frac{1}{\alpha_{t+1}} x_{t+1} - \frac{1}{\alpha_{t+1}} b_{t+1}$$

$$\begin{aligned}
 \Rightarrow x_t &= \frac{1}{\alpha_{t+1}} \left[ \frac{1}{\alpha_{t+2}} x_{t+2} - \frac{1}{\alpha_{t+2}} b_{t+2} \right] - \frac{1}{\alpha_{t+1}} b_{t+1} \\
 &= \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} x_{t+2} - \left( \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1} \right)
 \end{aligned}$$

$$y^c = \prod_{s=1}^k \left( \frac{1}{\alpha_{t+s}} \right) x_{t+k}$$

$$y^P = - \left( \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1} \right)$$

$$= - \left( \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} \frac{1}{\alpha_{t+3}} b_{t+3} + \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1} \right)$$

:

$$= - \sum_{k=1}^{\infty} b_{t+k} \prod_{s=1}^k \left( \frac{1}{\alpha_{t+s}} \right)$$

$$= - \sum_{k=0}^{\infty} b_{t+1+k} \prod_{s=0}^k \frac{1}{\alpha_{t+1+s}}$$

Good. Torelli gets  
this, eq. 7, part 2.

Mumma gets

$\prod (y_0 - b_0)$  and then the following term:

$$y^P(\text{I hope}) = \sum_{k_2=-1}^{t-1} b_{1+k_2} \prod_{k_1=0}^{k_2} \left( \frac{1}{\alpha_{1+k_1}} \right)$$

so again  $b_0$  get lost, but otherwise adding it, and  $t$  we get the correct thing up to the sign:

$$y^P = (-1) \sum_{k_2=0}^{\infty} b_{t+1+k_2} \prod_{k_1=0}^{k_2} \left( \frac{1}{\alpha_{t+1+k_1}} \right)$$

## 1<sup>st</sup> order simultaneous nonhomogeneous diff. eq.

Finally! Diagonalizing to solve diff. eqs!

$$x_t = Ax_{t-1} + b \quad (3)$$

In DSE Notes.

Particular sol: Try  $x_t = x + t+$

$$x = Ax + b$$

$$x - Ax = b$$

$$\Leftrightarrow (I - A)x = b$$

$$x = (I - A)^{-1}b \quad \rightarrow \text{the unique st. st!}$$

(if it exists)

Complementary sol: sol to  $x_t = Ax_{t-1}$

We need to diagonalize A (decouple the equations)

$\Rightarrow$  We're looking for  $M^{-1}AM = \Lambda$ , where  $\Lambda$  is a diagonal matrix w/  $\lambda_i$  on the diags (eigenvals of A)  
(and actually, M contains the eigenvectors)

$$M^{-1}AM = \Lambda \Rightarrow AM = M\Lambda \Rightarrow A = M\Lambda M^{-1} \quad | \text{ Sub.}$$

$$x_t = M\Lambda M^{-1}y_{t-1} \quad | \quad M^{-1} \text{ from left} \Rightarrow \underbrace{M^{-1}x_t}_{y_t} = \Lambda \underbrace{M^{-1}y_{t-1}}_{y_{t-1}}$$

$$y_t = \Lambda y_{t-1}$$

→ decoupled!

Then the sol. of the orig system is just

$$x_t = M y_t + (\mathbb{I} - A)^{-1} b$$

Suppose the st. st. is just 0 (homogeneous eq.)

$$x_t = M \Lambda^t$$

$$\text{e.g. } x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} \quad M = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \quad \text{Cigenvectors} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$x_{1,t} = c_1 e_{11} \lambda_1^t + c_2 e_{12} \lambda_2^t$$

$$x_{2,t} = c_1 e_{21} \lambda_1^t + c_2 e_{22} \lambda_2^t \quad (\text{w/ constant } c \text{ to be pruned down})$$

Let's work thru the example in diffegn.pdf (3.2) p. 7 Mac.

$$x_t = A x_{t-1} + b$$

$$\text{where } \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{t-1}^1 \\ x_{t-1}^2 \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}, \quad \text{so we're aiming}$$

$$\text{for } \begin{pmatrix} \hat{x}_t^1 \\ \hat{x}_t^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \hat{x}_{t-1}^1 \\ \hat{x}_{t-1}^2 \end{pmatrix} + \begin{pmatrix} \hat{b}^1 \\ \hat{b}^2 \end{pmatrix}.$$

Step 1. Find eigenvalues:

$$\det[A - \lambda \cdot I_n] = 0$$

In the  $2 \times 2$  case:  $\lambda_{1,2} = \frac{\text{tr}(A)}{2} \pm \frac{1}{2} \sqrt{(\text{tr}(A))^2 - 4 \det(A)}$

Step 2. Find eigenvectors:

Def. Eigenvector  $e_i$  is defined as

$$A e_i = e_i \lambda_i \quad (13)$$

(Note that (13) doesn't pin down a unique  $e_i$ . But this is actually cool b/c we can choose our favorite one.)

So we pick the eigenvector with  $e_{i2} = 1$  ( $y$ -coordinate...)

$$A \begin{pmatrix} e_i \\ 1 \end{pmatrix} = \begin{pmatrix} e_i \\ 1 \end{pmatrix} \lambda_i \quad (\text{some abuse of notation...})$$

So we only need to find the  $x$ -coordinate (hmm...)

Write this out:

$$a_{11} e_i + a_{12} = e_i \lambda_i \rightarrow e_i (a_{11} - \lambda_i) = -a_{12} \rightarrow e_i = -\frac{a_{12}}{a_{11} - \lambda_i}$$

$$a_{21} e_i + a_{22} = \lambda_i \rightarrow e_i = \frac{\lambda_i - a_{22}}{a_{21}}$$

These are apparently identical equations...

If that's true then the two sols should equal:

$$e_i = -\frac{a_{12}}{a_{11} - \lambda_i} = \frac{\lambda_i - a_{22}}{a_{21}}$$

$$\Leftrightarrow -a_{12}a_{21} = (\lambda_i - a_{22})(a_{11} - \lambda_i)$$

$(a_{11} - \lambda_i)(a_{22} - \lambda_i) - a_{12}a_{21} = 0$  Ok. this is the condition  $\det(A - \lambda I_n) = 0$ . ✓

So then  $e_i = \begin{pmatrix} \frac{\lambda_i - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$

Step 3. Put things together

thus:  $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$   $P = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} = (e_1, e_2)$

If both eigenvals are distinct  $\nearrow$ ,  $P$  isn't singular, and thus  $P^{-1}$  exists.  $\tilde{P}^T A P = \Lambda \Rightarrow A = P \Lambda P^{-1}$

$$x_t = A x_{t-1} \Rightarrow x_t = P \Lambda P^{-1} x_{t-1}$$

$$\hat{x}_t = \Lambda \hat{x}_{t-1} \quad (\text{decoupled})$$

And the thing is WE KNOW the sol to this: it's " $c \Lambda^t \hat{x}_0$ "

thus  $\hat{x}_+ = \wedge \hat{x}_{+-1}$  has the complementary sol

$$\begin{pmatrix} \hat{x}_+^1 \\ \hat{x}_+^2 \end{pmatrix} = \begin{bmatrix} c_1 \lambda_1^t \hat{x}_0^1 \\ c_2 \lambda_2^t \hat{x}_0^2 \end{bmatrix}$$

And since  $x_+ = P \hat{x}_+$ , we have

$$\begin{pmatrix} x_+^1 \\ x_+^2 \end{pmatrix} = \begin{bmatrix} \frac{\lambda_1 - a_{21}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{22}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \lambda_1^t \hat{x}_0^1 \\ c_2 \lambda_2^t \hat{x}_0^2 \end{bmatrix}$$

$+ \bar{x}$

By the superposition principle, the general sol just adds  $\bar{x}$

here, where  $\bar{x} = (I_2 - A)^{-1} b$

$$\rightarrow x_+^1 = \left( \frac{\lambda_1 - a_{21}}{a_{21}} \right) c_1 \lambda_1^t \hat{x}_0^1 + \left( \frac{\lambda_2 - a_{22}}{a_{22}} \right) c_2 \lambda_2^t \hat{x}_0^2$$

$$x_+^2 = c_1 \lambda_1^t \hat{x}_0^1 + c_2 \lambda_2^t \hat{x}_0^2$$

Voilà!

Now I'm wondering whether we can get the plans for  $\pi_t, x_t$  using this method from (15)-(17) in materials 18.

Woodford subs the FOCs into the NKPC. I'm not sure I can do that though (+ into the other model equations) b/c everything is so damn backward-looking and GE...

I think what I need to do is simplify the system as much as possible, and then solve the system of diff eqs. As it seems to me, it may be 5-dimensional, easily ...

We have:

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- NKPC, LOM( $f_a$ ), LOM( $\bar{\pi}$ ), LOM( $k$ ), eq. (15), (16), (17) (7)
- $\pi_t, x_t, f_a, \bar{\pi}_{t+1}, k_t^{-1}, y_5, y_6$  (7)

Ok we should be good.

The equations are :  $m_i$  ( $i=1, \dots$ ) are coeffs on  $\log S_t$ .

$$\pi_t - kx_t - (1-\alpha)\beta f_{\alpha}(t) = m_1 \cdot s_t \quad (9)$$

$$f_{\alpha}(t) = \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + m_2 \cdot s_t \quad (11)$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (13)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (14)$$

$$2\pi_t + 2\frac{\lambda}{K} x_t - \psi_{S,t} k_t^{-1} - f_{\pi}(t)(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \psi_{S,t} = 0 \quad (15+17)$$

$$- \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K} x_{t+1} + \psi_{S,t+1} - (1-k_t^{-1}) \psi_{S,t+1} + f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - b s_t) \psi_{S,t+1} = 0 \quad (16+17)$$

6 in 6. Sub (11) into (5)

$$\pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} - (1-\alpha)\beta m_2 \cdot s_t = m_1 \cdot s_t$$

$$\Leftrightarrow \pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} = \underbrace{(m_1 + (1-\alpha)\beta m_2)}_{=: m_3} s_t$$

$$\pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} = m_3 s_t \quad \pi_t, x_t, \bar{\pi}_{t-1}, k_t^{-1}, \psi_{S,t} \quad (1)$$

$$\bar{\pi}_t - (1-k_t^{-1}) \bar{\pi}_{t-1} - k_t^{-1} \pi_t + b s_{t-1} k_t^{-1} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\pi_t + 2\frac{\lambda}{K} x_t - (k_t^{-1} + f_{\pi}(t))(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \psi_{S,t} = 0 \quad (4)$$

$$- \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K} x_{t+1} + \psi_{S,t+1} - (1-k_t^{-1} - f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - b s_t)) \psi_{S,t+1} = 0 \quad (5)$$

$$\pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} = m_3 s_t \quad \pi_t, x_t, \bar{\pi}_{t-1}, k_t^{-1}, \varphi_{5,t}$$

$$\bar{\pi}_t - (1 - k_t^{-1}) \bar{\pi}_{t-1} - k_t^{-1} \pi_t + b s_{t-1} k_t^{-1} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\pi_t + 2 \frac{\lambda}{K} x_t - (k_t^{-1} + f_\pi(t)) (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \varphi_{5,t} = 0 \quad (4)$$

$$- \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K} x_{t+1} + \varphi_{5,t+1} - (1 - k_t^{-1} - f_{\bar{\pi}}(t)) (\pi_{t+1} - \bar{\pi}_t - b s_t) \varphi_{5,t+1} = 0 \quad (5)$$

Express  $x_t$  from (1) & sub into (4) & (5)

$$x_t = \frac{1}{K} \pi_t - \frac{1}{K(1-\alpha\beta)} \bar{\pi}_{t-1} - \frac{m_3}{K} s_t$$

$$\bar{\pi}_t - (1 - k_t^{-1}) \bar{\pi}_{t-1} - k_t^{-1} \pi_t + b s_{t-1} k_t^{-1} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\pi_t + 2 \frac{\lambda}{K} \left( \frac{1}{K} \pi_t - \frac{1}{K(1-\alpha\beta)} \bar{\pi}_{t-1} - \frac{m_3}{K} s_t \right) - (k_t^{-1} + f_\pi(t)) (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \varphi_{5,t} = 0 \quad (4)$$

$$- \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K} \left( \frac{1}{K} \pi_{t+1} - \frac{1}{K(1-\alpha\beta)} \bar{\pi}_t - \frac{m_3}{K} s_{t+1} \right) + \varphi_{5,t+1} - (1 - k_t^{-1} - f_{\bar{\pi}}(t)) (\pi_{t+1} - \bar{\pi}_t - b s_t) \varphi_{5,t+1} = 0 \quad (5)$$

Now we're down to 4 eqs in  $\pi_t, \bar{\pi}_t, k_t^{-1}, \varphi_{5,t}$

$$\bar{\pi}_t - (1 - k_t^{-1}) \bar{\pi}_{t-1} - \underline{k_t^{-1} \pi_t + b s_{t-1} k_t^{-1}} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\left(1 + \frac{2}{K^2}\right)\pi_t - 2\frac{2}{K^2} \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} - 2\frac{2}{K^2} m_3 s_t - k_t^{-1} - f_{\bar{\pi}}(t) \underline{\pi_t \psi_{S,t}} + f_{\pi}(t) \underline{\bar{\pi}_{t-1} \psi_{S,t}} + f_{\bar{\pi}}(t) \underline{b s_{t-1} \psi_{S,t}} = 0 \quad (4)$$

$$- 2\frac{2}{K^2} \frac{(1-\alpha)\beta}{1-\alpha\beta} \pi_{t+1} + 2\frac{2}{K^2} \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_t + 2\frac{2}{K^2} \frac{(1-\alpha)\beta}{1-\alpha\beta} m_3 s_{t+1} + \psi_{S,t} - \psi_{S,t+1} + \underline{k_t^{-1} \psi_{S,t+1}} + f_{\bar{\pi}}(t) \underline{\pi_{t+1} \psi_{S,t+1}} - f_{\bar{\pi}}(t) \underline{\bar{\pi}_t \psi_{S,t+1}} - f_{\bar{\pi}}(t) \underline{b s_t \psi_{S,t+1}} = 0 \quad (5)$$

The problem is the presence of these interactions.

They seem to be coming from  $\psi_S$ . But not only. Also from  $k_t^{-1}$   
 $\rightarrow$  are they breaking the linearity ...?

The other issue is eq. (3) b/c I can't separate  $\pi_t$  &  $\bar{\pi}_{t-1}$  elements. One could "drop it up" as

$$f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) = g(\pi_t) - g(\bar{\pi}_{t-1}) - g(b s_{t-1}) \text{ in which}$$

case we have an eq. system in  $\pi_t$ ,  $\bar{\pi}_t$  &  $\psi_{S,t}$ . Ignore the interactions for a sec. Define  $y_+ = \begin{bmatrix} \bar{\pi}_t \\ \pi_t \\ \psi_{S,t} \end{bmatrix}$

then we have  $A E_t y_{t+1} + B y_+ + C y_{t-1} = D E_t s_{t+1} + E s_t + F s_{t-1}$

let  $z_+ = E_t y_{t+1}$ ,  $e_+ = E_t s_{t+1}$ ,  $e_t = s_{t-1}$

$$\rightarrow A y_t + B z_t + C y_{t-1} = D \ell_t + E s_t + F \varepsilon_t$$

Call  $x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$  and  $m_t = \begin{bmatrix} s_t \\ \ell_t \\ \varepsilon_t \end{bmatrix}$

$$\Rightarrow [B \ A] x_t = -[C]_{t-1} + [E \ D \ F] m_t$$

$$x_t = \underbrace{-[B \ A]^{-1} C}_{=: A} x_{t-1} + \underbrace{[E \ D \ F]}_{=: M} m_t$$

$$x_t = A x_{t-1} + M m_t$$

Ryan meeting

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- no credible threats  $\rightarrow$  Melé, Molnár & Saitoro, 2019
- eq (6)-(7) if  $k_t$  endog, does that break linearity?
- if optimal plan, doesn't NKIS suffice to get  $i$ ? Why target out?
- commitment = discretion under learning
- $\Delta k = f(FE)$   $\leftarrow$  It's not obvious in (14) that level of  $k$  should be a pct of the post error.  
 $\uparrow$  change in  $k$
- The interactions have to be there. Maybe one can't get anything done from bmt eq. (18) is good to describe the tradeoff. Try the simple model!

## Work after

the simple model is (5)-(7), but of course we'd need to add a constraint governing  $k_t^{-1}$ . I'm rephrasing the problem in Materials19 so that  $x_t$  shows up too, so that we can get an interest rate rule.

FOCs:

$$\pi_t: 2\pi_t + \varphi_{1,t} - \varphi_{2,t} k_t^{-1} - \varphi_{3,t} g_\pi = 0 \quad (1)$$

$$x_t: 2\lambda x_t - k \varphi_{1,t} = 0 \rightarrow \varphi_{1,t} = 2\frac{\lambda}{k} x_t \quad (2)$$

$$f_t: -\beta \varphi_{1,t} + p_{2,t} - (1 - k_t^{-1}) \varphi_{2,t+1} + \varphi_{3,t+1} g_f = 0 \quad (3)$$

$$k_t^{-1}: -\varphi_{2,t} (\pi_t - f_{t-1}) + \varphi_{3,t} = 0 \quad (4)$$

$$\rightarrow \varphi_{3,t} = (\pi_t - f_{t-1}) \varphi_{2,t}$$

$$\Rightarrow 2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{2,t} k_t^{-1} - g_\pi (\pi_t - f_{t-1}) p_{2,t} = 0 \quad (1)$$

$$-2\beta \frac{\lambda}{k} x_t + \varphi_{2,t} - (1 - k_t^{-1}) \varphi_{2,t+1} + g_f (\pi_{t+1} - f_t) p_{2,t+1} = 0 \quad (2)$$

$$\pi_t = k x_t + \beta f_t + u_t \quad (3)$$

$$f_t = f_{t-1} - k_t^{-1} (\pi_t - f_{t-1}) \quad (4)$$

$$k_t^{-1} = g(\pi_t - f_{t-1}) \quad (5)$$

In  $\pi_t, x_t, f_t, k_t^{-1}$  and  $\varphi_{2,t}$

This guy, eq. (4), is what is causing the trouble.

$$f_t = f_{t-1} + k_t^{-1}(\pi_t - f_{t-1})$$

What if we "loglinearized" this?

$$\ln f_t = \ln(f_{t-1} + k_t^{-1}(\pi_t - f_{t-1})) \quad | \text{Total diff.}$$

$$\frac{1}{f} \cdot df_t = \frac{1}{f + k_t^{-1}(\pi_t - f)} \left[ (1 - k_t^{-1}) df_{t-1} + (\pi_t - f) dk_t + k_t^{-1} d\pi_t \right]$$

$\downarrow$   
 $\approx 0$  inst. st.

$$\hat{f}_t = \frac{1}{f} \left[ f \cdot \hat{f}_{t-1} + (\pi - f) \hat{k}_t^{-1} \right]$$

$\uparrow$   
just call  $dk_t$   
 $\hat{k}_t$

$$\rightarrow \hat{f}_t = \hat{f}_{t-1} + \underbrace{(\pi - f)}_F \hat{k}_t^{-1}$$

$\hat{k}_t^{-1} = \text{st. st. } \frac{\text{FE}}{\text{Expectation}}$

Suppose we replace

$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) \quad w/$$

$$\hat{f}_t - \hat{f}_{t-1} - \varepsilon \hat{k}_t^{-1}$$

in the opti problem. Then FOCs become:

$$\pi_t: 2\pi_t + \varphi_{1,t} - \varphi_{3,t} + g_\pi = 0 \quad (1)$$

$$x_t: 2\lambda x_t - k\varphi_{1,t} = 0 \rightarrow \varphi_{1,t} = 2\frac{\lambda}{k}x_t \quad (2)$$

$$f_t: \varphi_{2,t} - \varphi_{2,t+1} + \varphi_{3,t} + g_f = 0 \quad (3)$$

$$k_t^{-1}: -\varphi_{2,t} - \varphi_{3,t} = 0 \rightarrow \varphi_{3,t} = -\varphi_{2,t} \quad (4)$$

$\Rightarrow$

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varepsilon g_\pi \varphi_{2,t} = 0 \quad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \quad (2)$$

$$\varphi_{2,t+1} = \underbrace{(1 + \varepsilon g_f)}_{>1} \varphi_{2,t} \rightarrow \varphi_{2,t} = \frac{1}{1 + \varepsilon g_f} \varphi_{2,t+1}$$

Problem: this is explosive  $\varphi_{2,t+1} = (1 + \varepsilon g_f)^t \varphi_0$  explodes.

But there is some hope of at least deriving the optimal paths

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varepsilon g_\pi \varphi_{2,t} = 0 \quad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \quad (2)$$

$$\pi_t = kx_t + \beta f_t + u_t \quad (3)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon \hat{k}_t^{-1} \quad (4)$$

$$k_t^{-1} = g(\pi_t - f_{t-1}) \quad (5)$$

$$2\pi_t + 2\frac{\beta}{K}x_t - \varepsilon g_\pi y_{2,t} = 0 \quad (1)$$

$$y_{2,t} - y_{2,t+1} + \varepsilon g_f y_{2,t} = 0 \quad (2)$$

$$\pi_t = Kx_t + \beta f_t + u_t \rightarrow x_t = \frac{1}{K}\pi_t - \frac{\beta}{K}f_t - \frac{1}{K}u_t \quad (3)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon g(\pi_t - f_{t-1}) \quad (4)$$

$$2\left(1 + \frac{2}{K^2}\right)\pi_t - 2\beta\frac{2}{K^2}f_t - 2\frac{2}{K^2}u_t - \varepsilon g_\pi y_{2,t} = 0 \quad (1)$$

$$y_{2,t} - y_{2,t+1} + \varepsilon g_f y_{2,t} = 0 \quad (2)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon g(\pi_t - f_{t-1}) \quad (4)$$

$\hat{g} =$

~~so~~ A system in  $\pi_t, f_t, y_{2,t}$ . Call  $y_2 = \varphi$ ,  $g(\pi_t) + g(-f_{t-1})$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{A}} \begin{bmatrix} \pi_{t+1} \\ f_{t+1} \\ y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 2\left(1 + \frac{2}{K^2}\right) & -2\beta\frac{2}{K^2} & -\varepsilon g_\pi \\ 0 & 0 & 1 + \varepsilon g_f \\ g(\pi_t) & 0 & 0 \end{bmatrix}}_{\sqrt{\varepsilon b}} \begin{bmatrix} \pi_t \\ f_t \\ y_t \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 + g(-f_{t-1}) & 0 \end{bmatrix}}_{\Xi C} \begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 2\frac{2}{K^2} \\ 0 \\ 0 \end{bmatrix} u_t$$

$\Xi C$

Scalar analogy:

$$a x_{t+1} + b x_t + c x_{t-1} = u_t \quad \text{Intro: } x_{t+1} = x_{t-1}$$

$$\Rightarrow a x_{P_t} + b x_t + c x_{P_{t-1}} = u_t$$

$$x_{P_{t-1}} = x_t$$

$$\begin{bmatrix} b & a \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_{P_t} \end{bmatrix} + \begin{bmatrix} c & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{P_{t-1}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t =: D$$

So:

$$\begin{bmatrix} 2(1-\frac{\lambda^2}{K^2}) & -2\beta\frac{\lambda}{K^2} & -\varepsilon g_{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\varepsilon g_f & 0 & 0 & -1 & \\ g(\pi_t) & 0 & 0 & 0 & 0 & 0 & \\ -1 & 0 & 0 & & & & \\ 0 & -1 & 0 & 0 & & & \\ 0 & 0 & -1 & & & & \\ \end{bmatrix} =: E \quad \begin{bmatrix} \pi_t \\ f_t \\ y_t \\ \pi_{P_t} \\ f_{P_t} \\ y_{P_t} \end{bmatrix} = \begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ y_{t-1} \\ \pi_{P_{t-1}} \\ f_{P_{t-1}} \\ y_{P_{t-1}} \end{bmatrix} = \begin{bmatrix} 2\frac{\lambda}{K^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t$$

$$D y_t + E y_{t-1} = \begin{bmatrix} \text{shift} \end{bmatrix} u_t$$

$$y_t = -D^{-1} E y_{t-1} + D^{-1} \begin{bmatrix} \text{shift} \\ 6 \times 1 \end{bmatrix} u_t$$

$$y_t = A y_{t-1} + M u_t$$

$6 \times 1 \quad 6 \times 6 \quad 6 \times 1 \quad 6 \times 1 \quad 1 \times 1$

The question is whether  $D^{-1}$  exists. Of course not:

The 4<sup>th</sup> & 5<sup>th</sup> column are not lin. independent.

Yo bigmouth! What if you roll eq. (2) bw?

$$2\left(1 + \frac{\beta}{K^2}\right)\pi_t - 2\beta\frac{\beta}{K^2}f_t - 2\frac{\beta}{K^2}u_t - Eg_\pi Y_{2,t} = 0 \quad (1)$$

$$\hat{Y}_{2,t} - Y_{2,t-1} + Eg_f Y_{2,t} = 0 \rightarrow \hat{Y}_{2,t} = (1 + Eg_f) \hat{Y}_{2,t-1} \quad (2)$$

$$\hat{f}_t = \hat{f}_{t-1} + Eg(\pi_t - f_{t-1}) \quad (3)$$

$$\Rightarrow 2\left(1 + \frac{\beta}{K^2}\right)\pi_t - 2\beta\frac{\beta}{K^2}f_t - Eg_\pi \hat{Y}_{2,t} = 2\frac{\beta}{K^2}u_t \quad (1)$$

$$\hat{Y}_{2,t} = (1 + Eg_f) \hat{Y}_{2,t-1} \quad (2)$$

$$-Eg(\pi_t) + \hat{f}_t = \hat{f}_{t-1} + Eg(-f_{t-1}) \quad (3)$$

$$\underbrace{\begin{bmatrix} 2\left(1 + \frac{\beta}{K^2}\right) & -2\beta\frac{\beta}{K^2} & -Eg_\pi \\ 0 & 0 & 1 \\ -Eg(\pi_t) & 1 & 0 \end{bmatrix}}_D \underbrace{\begin{bmatrix} \pi_t \\ f_t \\ Y_t \end{bmatrix}}_y = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 + Eg_f \\ 0 & 1 + Eg(-f_{t-1}) & 0 \end{bmatrix}}_E \underbrace{\begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ Y_{t-1} \end{bmatrix}}_{y_{t-1}} + \underbrace{\begin{bmatrix} 2\frac{\beta}{K^2} \\ 0 \\ 0 \end{bmatrix}}_{u_t}$$

$$y_+ = A_+ y_{+-1} + M_+ u_+ \quad (\text{see Mathematica, materials 19.nb})$$

Then  $\hat{y}_+^c = P \hat{g}_+$  ( $P$  = matrix of eigenvectors)

is the complementary solution where

$$\hat{y}_+^i = c_i \lambda_i^t \hat{g}_0^i \quad i = 1, 2, 3$$

the eigenvalues are (see MMA)

$$\lambda_1 = 0, \quad \lambda_2 = \frac{(k^2 + \lambda)(1 + \varepsilon g(-f_{+-1}))}{k^2 + \lambda - \beta \varepsilon \lambda g(\pi_+)}, \quad \lambda_3 = 1 + \varepsilon g_f < 1 \qquad > 1$$

$$\lambda_2 = \frac{1 + \varepsilon g(-f_{+-1})}{1 - \frac{\beta \varepsilon \lambda g(\pi_+)}{k^2 + \lambda}} \quad \text{looks like } \mu_5 \text{ is } > 1 \text{ as well.}$$

But since we have 2 bw-looking vars, we might have

2 TVCs which set  $c_2 = c_3 = 0$  so we're left w/  
the only stable eig. I don't like it that it's 0.

But allow the system to evolve for long enough  
such that the initial condition disappears.

And turn to the particular sol:

$$y_+ = A_+ [A_{+-1} y_{+-2} + M_{+-1} u_{+-1}] + M_+ u_+$$

$$\begin{aligned}
 y_+ &= A_+ A_{+-1} y_{+-2} + A_+ M_{+-1} u_{+-1} + M_+ u_+ \\
 &= A_+ A_{+-1} [A_{+-2} y_{+-3} + M_{+-2} u_{+-2}] + A_+ M_{+-1} u_{+-1} + M_+ u_+ \\
 &= \underbrace{A_+ A_{+-1} A_{+-2} y_{+-3}}_{\text{I wonder how on earth this is supposed to vanish}} + A_+ A_{+-1} M_{+-2} u_{+-2} + A_+ M_{+-1} u_{+-1} + M_+ u_+
 \end{aligned}$$

But  $A^{-1}$  doesn't exist so can do nothing else.

Or were we supposed to do this for the diagonal system

$$\hat{y}_+ = 1 y_{+-1} \dots \text{No b/c that's just the homogen. part.}$$

So

$$y_+^P = M_+ u_+ + \sum_{i=1}^n M_{+-i} u_{+-i} (A_{+-0} A_{+-1} \dots A_{+-i+1})$$

$$y_+^P = M_+ u_+ + \sum_{i=1}^n M_{+-i} u_{+-i} \prod_{k=0}^{i-1} A_{+-k}$$

Ok - I think we had this confusion before:

diffeqn.pdf says that for an  $n$ -dim system;

$$y_+^P = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L) \dots (1-\lambda_n L)}$$

$$\text{where } \frac{1}{1-\lambda_i L} = \sum_{s=-\infty}^t \lambda_i^{t-s} b_s \text{ if } |\lambda_i| < 1$$

$$\text{and } \frac{1}{1-\lambda_i L} = \text{sum of future } b_s \text{ if } |\lambda_i| > 1$$

But many questions. First of all, does that mean that

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \sum_{s=-\infty}^t \lambda_1^{t-s} b_s \cdot \sum_{s=-\infty}^t \lambda_2^{t-s} b_s ?$$

(if both roots  $> 1$ ?) Second, if  $|\lambda_1| < 1$  but  $|\lambda_2| > 1$ , then

$$= \sum_{s=-\infty}^t \lambda_1^{t-s} b_s \sum_{j=t}^{\infty} \left(\frac{1}{\lambda_2}\right)^{j-t} b_j ?$$

Third, how does this relate to iterating fwd/bwd?

→ I think I know. In the scalar case,  $\text{eig}(A) = A_{1 \times 1}$

So the sum is a sum of eigens. In the matrix case, we go directly to the eigens.

Fourth,  $A$  is time-varying. If it wasn't, then the particular sol would have been:

$$y_t^P = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L)} M u_t \xrightarrow{\quad}$$

With  $A$  time-varying, it is (maybe)

$$y_t^P = \frac{1}{(1-\lambda_{1,t} L)(1-\lambda_{2,t} L)(1-\lambda_{3,t} L)} M u_t$$

Fifth,  $M_t$  is time-varying too.

→ This may not be correct. Look at Tirotli p. 6 Mac

$$x_+^{PA} = \begin{cases} \sum_{k=1}^t \left( \prod_{s=0}^{k-1} \alpha_{t-s} \right) b_{t-k} + b_+ & \text{if } |\alpha| < 1 \\ - \sum_{k=0}^{\infty} \left( \prod_{s=0}^k \left( \frac{1}{\alpha_{t+s}} \right) \right) b_{t+1+k} & \text{if } |\alpha| > 1 \end{cases}$$

So for a 2-dim system, this should be something like:

$$\left( \sum_{k=1}^t \left( \prod_{s=0}^{k-1} \lambda_{1,t-s} \right) b_{t-k} + b_+ \right) \left( - \sum_{k=0}^{\infty} \left( \prod_{s=0}^k \left( \frac{1}{\lambda_{2,t+s}} \right) \right) b_{t+1+k} + b_2 \right)$$

where  $|\lambda_1| < 1$ ,  $|\lambda_2| > 1$ .

Tirotli also says on p. 13 Mac

$$z_+^{PA} = \sum_{s=0}^t \lambda^{t-s} c_s$$

where  $z_+ = P x_+$  and the transformed system is

$$z_t = \lambda z_{t-1} + c_t \quad \text{w/ } c_1 = P^{-1} b_+$$

$\Lambda^+ = \text{diag}(\lambda_1^n, \lambda_2^n)$  which implies that you simply sum separately as  $\sum \lambda_1^+ + \sum \lambda_2^+$

$$x_t = Ax_{t-1} + b_t$$

$$x_t = P^T \Lambda P x_{t-1} + b_t$$

$$Px_t = \Lambda Px_{t-1} + Pb_t$$

$$z_t = \Lambda z_{t-1} + c_t \quad \text{so } c_t \text{ should be } Pb_t.$$

So  $y_t^P = \sum \pi_i \gamma_i$  <sup>forward</sup>  $+ \sum \pi_i \gamma_i$  <sup>forward</sup>.

but since  $\gamma_1 = 0$   
that drops

- To do tomorrow: search online for particular sol. of non-autonomous diff eq. w/ time-varying coeffs.
- Only that way can you characterize the sol. for the single model w/ "loglin-ed"  $\text{Lm}(f_t)$ .
- If that doesn't work, drop the "loglin"  $\text{Lm}(f_t)$ , and return to trying to deal w/ the interacting system.