Materials 3 - Special cases

Laura Gáti

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Overview

1 The models to be simulated

1. Rational expectations NK model (RE)
2. Euler equation approach learning NK model à la Bullard & Mitra (2002) (EE)
3. LR expectations learning NK model à la Preston (2005) (LR)
4. (Eventually: LR expectations learning NK model à la Preston with anchoring à la CEMP)

The difference between these models is 1) in the expectations (rational or not), 2) in the number of horizons of expectations that need to be summed (1 vs. infinite). So what I'm going to do consists of 2 steps:

1. Write a learning rule that takes the form of Preston's, but that nests CEMP, and has a decreasing gain.
2. Write out f_a and f_b as truncated sums of h-period ahead forecasts. When $h = 1$, EE and LR (models (6) and (7)) should coincide.

1.1 RE

$$x_t = \mathbb{E}_t\, x_{t+1} - \sigma(i_t - \mathbb{E}_t\, \pi_{t+1}) + \sigma r_t^n \tag{1}$$

$$\pi_t = \kappa x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} + u_t \tag{2}$$

$$i_t = \bar{i}_t + \psi_\pi \pi_t + \psi_x x_t \tag{3}$$

1.2 EE

$$x_t = \hat{\mathbb{E}}_t x_{t+1} - \sigma(i_t - \hat{\mathbb{E}}_t \pi_{t+1}) + \sigma r_t^n \qquad \text{(Preston, eq. (13))}$$

$$\pi_t = \kappa x_t + \beta \hat{\mathbb{E}}_t \pi_{t+1} + u_t \qquad \text{(Preston, eq. (14))}$$

$$i_t = \bar{i}_t + \psi_\pi \pi_t + \psi_x x_t \qquad \text{(Preston, eq. (27))}$$

1.3 LR

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{Preston, eq. (18)}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{Preston, eq. (19)}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \tag{Preston, eq. (27)}$$

One issue is that if I set $T = t$, I don't think Preston, eq. (18) reduces to Preston, eq. (13), nor does Preston, eq. (19) reduce to Preston, eq. (14).

2 Compact notation

Innovations are summarized as:

$$s_t = Ps_{t-1} + \epsilon_t \quad \text{where} \quad s_t \equiv \begin{pmatrix} r_t^n \\ \bar{i}_t \\ u_t \end{pmatrix} \quad P \equiv \begin{pmatrix} \rho_r & 0 & 0 \\ 0 & \rho_i & 0 \\ 0 & 0 & \rho_u \end{pmatrix} \quad \epsilon_t \equiv \begin{pmatrix} \varepsilon_t^r \\ \varepsilon_t^i \\ \varepsilon_t^u \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_i & 0 \\ 0 & 0 & \sigma_u \end{pmatrix}$$

Let z_t summarize the endogenous variables as

$$z_t \equiv \begin{pmatrix} \pi_t \\ x_t \\ i_t \end{pmatrix} \tag{4}$$

Then I can write the models compactly as

$$z_t = A_p^{RE}\, \mathbb{E}_t\, z_{t+1} + A_s^{RE} s_t \tag{5}$$

$$z_t = A_p^{RE} \hat{\mathbb{E}}_t z_{t+1} + A_s^{RE} s_t \tag{6}$$

$$z_t = A_a^{LR} f_a + A_b^{LR} f_b + A_s^{LR} s_t \tag{7}$$

$$s_t = Ps_{t-1} + \epsilon_t \tag{8}$$

where f_a and f_b capture discounted sum of expectations at all horizons of the endogenous states z (following Preston, I refer to these objects as "long-run expectations"):

$$f_a \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} z_{T+1} \qquad f_b \equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\beta)^{T-t} z_{T+1} \tag{9}$$

and the coefficient matrices are given by:

$$A_p^{RE} = \begin{pmatrix} \beta + \frac{\kappa\sigma}{w}(1-\psi_\pi\beta) & \frac{\kappa}{w} & 0 \\ \frac{\sigma}{w}(1-\psi_\pi\beta) & \frac{1}{w} & 0 \\ \psi_\pi\big(\beta + \frac{\kappa\sigma}{w}(1-\psi_\pi\beta)\big) + \psi_x\frac{\sigma}{w}(1-\psi_\pi\beta) & \psi_x(\frac{1}{w}) + \psi_\pi(\frac{\kappa}{w}) & 0 \end{pmatrix} \tag{10}$$

$$A_s^{RE} = \begin{pmatrix} \frac{\kappa\sigma}{w} & -\frac{\kappa\sigma}{w} & 1 - \frac{\kappa\sigma\psi_\pi}{w} \\ \frac{\sigma}{w} & -\frac{\sigma}{w} & -\frac{\sigma\psi_\pi}{w} \\ \psi_x(\frac{\sigma}{w}) + \psi_\pi(\frac{\kappa\sigma}{w}) & \psi_x(-\frac{\sigma}{w}) + \psi_\pi(-\frac{\kappa\sigma}{w}) + 1 & \psi_x(-\frac{\sigma\psi_\pi}{w}) + \psi_\pi(1 - \frac{\kappa\sigma\psi_\pi}{w}) \end{pmatrix} \tag{11}$$

$$A_a^{LR} = \begin{pmatrix} g_{\pi a} \\ g_{xa} \\ \psi_\pi g_{\pi a} + \psi_x g_{xa} \end{pmatrix} \quad A_b^{LR} = \begin{pmatrix} g_{\pi b} \\ g_{xb} \\ \psi_\pi g_{\pi b} + \psi_x g_{xb} \end{pmatrix} \quad A_s^{LR} = \begin{pmatrix} g_{\pi s} \\ g_{xs} \\ \psi_\pi g_{\pi s} + \psi_x g_{xs} + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{pmatrix} \tag{12}$$

$$g_{\pi a} = (1 - \frac{\kappa\sigma\psi_\pi}{w}) \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{13}$$

$$g_{xa} = \frac{-\sigma\psi_\pi}{w} \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] \tag{14}$$

$$g_{\pi b} = \frac{\kappa}{w} \left[\sigma(1-\beta\psi_\pi), (1-\beta-\beta\sigma\psi_x, 0\right] \tag{15}$$

$$g_{xb} = \frac{1}{w} \left[\sigma(1-\beta\psi_\pi), (1-\beta-\beta\sigma\psi_x, 0\right] \tag{16}$$

$$g_{\pi s} = (1 - \frac{\kappa\sigma\psi_\pi}{w}) \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (I_3 - \alpha\beta P)^{-1} - \frac{\kappa\sigma}{w} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} (I_3 - \beta P)^{-1} \tag{17}$$

$$g_{xs} = \frac{-\sigma\psi_\pi}{w} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (I_3 - \alpha\beta P)^{-1} - \frac{\sigma}{w} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} (I_3 - \beta P)^{-1} \tag{18}$$

$$w = 1 + \sigma\psi_x + \kappa\sigma\psi_\pi \tag{19}$$

3 Learning

In Preston (2005), agents forecast the endogenous variables using the exogenous ones as

$$z_t = a_t + b_t s_t + \epsilon_t \qquad \text{(Preston, p. 101)}$$

which I suspect isn't precise about the timing. Therefore, I write a general PLM of the form

$$z_t = a_{t-2} + b_{t-2} s_{t-1} + \epsilon_t \tag{20}$$

and then $\phi_{t-2} = (a_{t-2}, b_{t-2})$, here 3×4, so that agents learn both a constant and a slope term. Later, I will simplify here so that agents only learn about the constant, i.e. about CEMP's drift term:

$$z_t = \bar{z}_{t-2} + P s_{t-1} + \epsilon_t \tag{21}$$

so that $\phi_{t-2} = (\bar{z}_{t-2}, I_3)$, and $\hat{\mathbb{E}}_t z_{t+1} = \phi_{t-1} \begin{bmatrix} 1 \\ P s_t \end{bmatrix}$. I'm actually quite worried about the assumption that agents only learn about the constant because it seems like a permanent deviation from RE: might it screw up E-stability? I'm also worried about the way P is treated in Preston. I've added it, because I think it makes sense that at time $t-1$, you expect s_t to be $P s_{t-1}$. Or is it the case that the learning of the slope embodies P? I don't think so, because agents know the structure of exogenous states.

Anticipated utility implies that

$$\hat{\mathbb{E}}_{t-1} \phi_{t+h} = \hat{\mathbb{E}}_{t-1} \phi_t \equiv \phi_{t-1} \quad \forall\, h \geq 0 \tag{22}$$

This is a little tricky. It doesn't only mean that agents today mistakenly believe that they will not update the forecasting rule. It also implies that the belief about ϕ_t was formed at $t-1$. Assuming RE about the exogenous process and anticipated utility, this implies that h-horizon forecasts are constructed as:

$$\hat{\mathbb{E}}_t z_{t+h} = \bar{z}_{t-1} + P^h s_t \quad \forall h \geq 1 \tag{23}$$

and the regression coefficients are updated using (for now) a decreasing gain RLS algorithm:

$$\phi_t = \phi_{t-1} + t^{-1} \mathbf{R_t^{-1} s_{t-1}} \left(z_{t-1} - \phi_{t-1} \begin{bmatrix} 1 \\ s_{t-1} \end{bmatrix} \right) \tag{24}$$

$$R_t = R_{t-1} + t^{-1} \left(\begin{bmatrix} 1 \\ s_{t-1} \end{bmatrix} \begin{bmatrix} 1 & s_{t-1} \end{bmatrix} - R_{t-1} \right) \tag{25}$$

R_t is 4×4 and ϕ_t is 3×4. Two questions:

1. Can this formulation capture the special case that agents only learn about the constant?

2. The bold $R_t^{-1} s_{t-1}$ indicates a difference to CEMP's learning algorithm: these terms are missing in CEMP.

And a note: CEMP is a special case of this model, with the gain switching between decreasing and constant according to the anchoring mechanism. I'm leaving that out for the time being.

4 ALMs

4.1 RE

With some abuse of terminology, call the state-space representation the ALM of the RE model:

$$x_t = hx\ x_{t-1} + \eta s_t \tag{26}$$

$$z_t = gx\ x_t \tag{27}$$

Then I can write the "ALM" as

$$z_t = gx\ hx\ x_{t-1} + gx\ \eta s_t \tag{28}$$

Since this ALM implies no constant, I initialize $\bar{z}_0$ as a 3×1 zero vector, and thus $\phi_0 = \begin{bmatrix} \bar{z}_0 & P \end{bmatrix}$ (and $P = I_3\ hx$). Analogously, I initialize R as a $R_0 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Sigma_x \end{bmatrix}$, where Σ_x is the VC matrix of the states from the RE solution.

4.2 EE

I just need to use (23) to evaluate one-period ahead forecasts, and plug those into (6).

4.3 LR

Evaluate analytical "LR expectations" (9) using the PLM (23),

$$f_a = \frac{1}{1-\alpha\beta}\bar{z}_{t-1} + P(I_3 - \alpha\beta P)^{-1} s_t \qquad f_b = \frac{1}{1-\beta}\bar{z}_{t-1} + P(I_3 - \beta P)^{-1} s_t \tag{29}$$

and plug them into (7). In the general case, where agents learn the slope too, b_{t-1} replaces P in the above expression:

$$f_a = \frac{1}{1-\alpha\beta} a_{t-1} + b_{t-1}(I_3 - \alpha\beta P)^{-1} s_t \qquad f_b = \frac{1}{1-\beta} a_{t-1} + b_{t-1}(I_3 - \beta P)^{-1} s_t \tag{30}$$

Alternatively I can evaluate each h-period forecast individually using (23), and then sum H of these terms, discounting appropriately. Earlier, it seemed that already a $H = 100$ is not a bad approximation of ∞-horizons, but now that only holds for f_a. For f_b to be accurate, I need at least $H = 10000$. Why?

5 Timeline in the learning models

$\underline{t = 0}$: Initialize $\phi_{t-1} = \phi_0$ at the RE solution.

For each t:

1. Evaluate expectations $t + s$ (the one-period ahead, ($s = 1$) or the full 1 to ∞-period ahead ($s = 1, \ldots, \infty$)) given ϕ_{t-1} and states dated t
2. Evaluate ALM given expectations: "today's observables are a function of expectations and today's state"
3. Update learning: ϕ_t = RLS of ϕ_{t-1} and fcst error between today's data and yesterday's forecast

6 Special cases towards general case: procedure

1. Simulate RE model ✓
2. Simulate EE model where agents learn both slope and constant ✓ In this case, expectations take the easier form:

$$\hat{\mathbb{E}}_t z_{t+h} = \phi_{t-1} \begin{bmatrix} 1 \\ P^{h-1} s_t \end{bmatrix} \quad \forall h \geq 1 \tag{31}$$

   - Simulate using the "implicit ALM": rearranging the expectational matrix equation that underlies the solution to the model, you obtain the simulated observables z_t without explicitly writing out the ALM ✓
   - Simulate using the "explicit ALM", equation (6), plugging in expectations evaluated separately. ✓

The cool thing is: when I do the above two steps, I obtain the same simulated observables, so I know I'm doing it correctly.

3. Simulate LR model where agents learn both slope and constant, extend horizons from 1 to infinity

4. Simulate EE model where agents learn only the constant

5. Simulate LR model where agents learn only the constant, extend horizons from 1 to infinity