Work after

12 Feb 2020

COPY OF JOB MARKET TIMELINE

Time in Reverse

Feb/March 2021: accept job

Jan/Feb 2021: Interview Flyouts

Nov 2020: submit applications

Oct 2020: job talk | (submit) GLMM ← Reading 2

Aug 31 2020: a very complete draft

Summer 2020: very hard work

writing → meaning math + text

i.e. not refining the question

May 2020: First draft ← Reading 1

stay to circulate in Dept.

Branch's RPE (restricted perceptions eqb)

(an undercategory of which is a misspecification eqb, ME) is this model uncertainty / selection issue: agents consider the set $\mathcal{T}$

$$\mathcal{T} = \{ x_t = a + b' \hat{z}_{t-1} + \varepsilon_t : \quad \dim(\hat{z}) < \dim(z) \}$$

i.e. each element in $\mathcal{T}$ is a PLM that is misspecified

-> a testing model which omits a variable or a lag.

Agents then pick the best-performing model in an FEV/MSE or related concept-sense, like in Cho & Kasa.

-> and I'm also seeing that the Ball-effect is a negative feedback effect b/c then the int-rate, $E(\pi)$ & (π, x) move in opposite directions

↳ this is what Branch calls a third effect of exogenous disturbances: 1) direct effect on states x_t 2) indirect via expectations 3) policy feedback effect

=> joint determination of optimal policy & ME!

Woodford Chapter 8 - A General Linear-Quadratic Framework 13 Feb 2020

1.1. Optimal State-Contingent Paths

The model equations are summarized by

$$\hat{I}\begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = A \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + B i_t + C s_t \qquad (1.1)$$

$z_t :=$ vector of "nonpredetermined endog vars" (jumps)

$Z_t :=$ vector of "predetermined endog vars" (endog states)

$i_t :=$ policy instrument

$s_t :=$ vector of exog disturbances

$$\hat{I} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} Z_t \\ z_t \end{bmatrix}$$

$$+ |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} i_t + |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ C_2 \end{bmatrix} s_t$$

Objectives of policy = $E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t$ (1.2)

where the period loss L_t = $L_t = \frac{1}{2} (\tau_t - \tau^*)' W (\tau_t - \tau^*)$ (1.3)

τ_t = vector of target variables, $\tau_t = T y_t$

where $y_t = \begin{bmatrix} z_t \\ z_t \\ i_t \end{bmatrix}$ is the vector of endog. variables.

DEF. A policy rule which is optimal from a timeless perspective in this more general context is a rule such that

1.) jumps z_t can be expressed as a time-invariant function

$$z_t = f_0 + f_z \bar{Z}_t + f_s \bar{s}_t \quad (1.5)$$

the bars reflect that the fct may involve additional endog. or exog states relative to the model

2) The eqb evolution of $\{y_t\}$ (i.e. all endog. vars) minimizes the loss (1.2) for all $t \geq t_0$, subject to the model (1.1) and s.t. $\tilde{E} z_{t_0} = \bar{e} \equiv \tilde{E} \left[f_0 + f_z \bar{z}_{t_0} + f_s \bar{s}_{t_0} \right]$ (1.6)

which restricts the initial values of the jumps. ("timelessness")

Indeed this method is the same as the one outlined in Chapter 7, just in a more general writing. The solution procedure is to set up the Lagrangian

$$\mathcal{L}_{t_0} = E_{t_0}\left\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[L(y_t) + \underbrace{\varphi_{t+1}'}_{\mathcal{L}\text{-multipliers on the model eqs.}}\tilde{A}y_t - \beta^{-1}\varphi_t'\tilde{I}y_t\right]\right\} \quad (1.7)$$

where $\tilde{A} := [A \;\; B]$ $\quad \tilde{I} \equiv [\hat{I} \;\; 0]$

$$\varphi_{t+1} = \begin{bmatrix} \xi_{t+1} \\ \Xi_t \end{bmatrix} \begin{array}{l} \rightarrow \text{pertains to endog states (predetermined at } t) \\ \rightarrow \text{pertains to jumps (set at } t) \end{array}$$

or vice versa :S

Shocks s_t have been suppressed in $\mathcal{L}$ b/c FOCs are independent of them anyway.

Note that the last term in $\mathcal{L}$, at t_0, gives the initial constraints

$$\varphi_{t_0}'\tilde{I}y_{t_0} = \underbrace{\xi_{t_0}'z_{t_0}}_{\text{initial values of endog states}} + \underbrace{\Xi_{t_0-1}'\tilde{E}z_{t_0}}_{\text{constraints (1.6) (the "timelessness")}}$$

$\boxed{\begin{array}{l}\text{Time-out: timelessly optimal vs. } t_0\text{-optimal:} \\ \text{time-consistent vs. time-inconsistent optimal commitment}\end{array}}$

Indeed:

A t_0-optimal plan minimizes (1.7) s.t. same constraints EXCEPT it replaces (1.6) ("timelessness") w/

$$\Xi_{t_0-1} = 0 \qquad (1.8)$$

i.e. the "bygones-be-bygones" assumption.

FOCs:

$$\tilde{A}' E_t \varphi_{t+1} + T' W(\tau_t - \tau^*) - \beta^{-1} \tilde{I}' \varphi_t' = 0 \qquad (1.9)$$

$\forall\, t \geq t_0$.

Restatement:

<u>t_0-optimal solution</u>: any bounded processes for $\{y_t\}$ & $\{\varphi_t\}$ that satisfy the FOCs (1.9) s.t. the time-inconsistency constraint (1.8)

<u>timelessly optimal sol.</u>: any bounded processes for $\{y_t\}$ & $\{\varphi_t\}$ that satisfy the FOCs (1.9) s.t. the time-consistency constraint (1.6)

Yet again restate the timelessly optimal solution:

1. Eqb satisfies a time-invariant relation of the form (1.5)
2. $\{y_t\}$ $\forall t \geq t_0$ evolve according to the (by assumption bounded) unique solution to the FOCs (1.9) s.t. the model (1.1) w/ the given initial conditions and initial $\mathcal{L}$-multipliers Ξ_{t_0-1}.
3. The initial multipliers are given by a linear rule of the form

$$\Xi_{t_0-1} = g_0 + g_2 \bar{z}_{t_0} + g_s \bar{s}_{t_0} \qquad (1.11)$$

w/ g_0, g_2, g_s satisfying

$$\tilde{E} z_t = e_0 + e_2 z_t + e_s s_t + e_{\Xi}\left[g_0 + g_2 \bar{z}_t + g_s \bar{s}_t\right] \qquad (1.12)$$

$\forall t \geq t_0$

(where what this is saying is that the rule for initial multipliers has to be timelessly optimal (i.e. compatible w/ 1.6) and at the same time conform to the time-invariant rule (1.5))

And note: these are just the optimal plans for $\{y_t\}$. Implementing this plan is another question.

I suggest now to revisit the simple example in Ch. 7 of deriving the ta-/timelessly optimal plans, augmented w/ some review of dynamic systems from Alpha Chiang's wonderful book, before turning to the implementation of plans.

Ch. 7 p. 472 The Ramsey Problem (shh... it's a secret)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}\left[\pi_t^2 + \lambda(x_t - x^*)^2\right] + \varphi_t\left[\pi_t - k x_t - \beta \pi_{t+1}\right] \right\}$$

FOCs:

π_t: $\pi_t + \varphi_t - \varphi_{t-1} = 0$ (1.7) $\forall t \geq 0$

x_t: $\lambda(x_t - x^*) - k\varphi_t = 0$ (1.8)

and for $t = 0$, $\varphi_{-1} = 0$ (1.9)

This may in fact not be time-inconsistent b/c this is just saying that the model isn't a constraint in $t = -1$.

But initial conditions may be, or stg.

Sub out π & x from

$$\pi_t = \kappa x_t + \beta \pi_{t+1} \qquad (1.1)$$

using (1.7) & (1.8)

$$\varphi_{t-1} - \varphi_t = \kappa \left[\frac{\kappa}{\lambda} \varphi_t + x^* \right] + \beta \left(\varphi_t - \varphi_{t+1} \right)$$

$\Leftrightarrow$

$$\varphi_{t-1} - \varphi_t - \beta \varphi_t + \beta \varphi_{t+1} - \frac{\kappa^2}{\lambda} \varphi_t = \kappa x^*$$

$$\Rightarrow \beta \varphi_{t+1} - \left(1 + \beta + \frac{\kappa^2}{\lambda} \right) \varphi_t + \varphi_{t-1} = \kappa x^* \qquad (1.10)$$

Ok, this is a difference eq. of order 2. Let's now revisit Alpha Chiang

General method of solving difference equations

(p. 554 book, pdf. p. 561 mac) Ch. 16 / Ch. 17

$$y_{t+1} + a y_t = c \qquad (16.6)$$

The general solution $= \underbrace{y_p}_{\text{particular solution}} + \underbrace{y_c}_{\text{complementary function}}$

y_p solves the nonhomogen. eq. (16.6), y_c the homogen. eq.:

$$y_{t+1} + a y_t = 0$$

y_p: represents intertemporal eq'b level of y
(st. st. I guess)

y_c: signifies deviations of the time path from the equilibrium y_p. (business cycles I guess)

Focus on y_c first:

Try $y_t = Ab^t$

Then $y_{t+1} = Ab^{t+1}$

and the homogeneous eq $y_{t+1} + a y_t = c$

implies $Ab^{t+1} + aAb^t = 0 \quad | : Ab^t$

$$\rightarrow \quad b + a = 0 \quad \rightarrow \quad b = -a$$

then the complementary fct becomes

$$\underline{y_c = Ab^t = A(-a)^t} .$$

Then, the particular sol. Guess $y_t = k \rightarrow y_{t+1} = k$.

Sub into (16.6) $y_{t+1} + a y_t = c$

$$\rightarrow \quad k + ak = c \quad \Rightarrow \quad \underline{\underline{k = \frac{c}{1+a} = y_p}}$$

Since this y_p is a constant, this is a stationary equilibrium.

Sometimes we want a moving equilibrium (a function of time, like a BGP st. st.), or $a = -1$, so the previous y_p is undefined. Then for the particular sol we can guess $y_t = kt$

$\rightarrow \quad k(t+1) + akt = c$

$$\rightarrow \quad k = \frac{c}{t+1+at}$$

which, if $a = -1$ implies $k = c$, so now our moving particular sol is $y_p = c \cdot t$.

Putting the sol together gives the general sol

$$y_t = A(-a)^t + \frac{c}{1+a} \qquad \text{if } a \neq -1 \tag{16.8}$$

$$y_t = A(-a)^t + c \cdot t \qquad \text{if } a = -1 \tag{16.9}$$

$$= A + c \cdot t$$

We still need to determine A.

To determine A, resort to initial conditions $y_t = y_0$

For (16.8), $t=0$ implies

$$y_0 = A + \frac{c}{1+a} \implies A = y_0 - \frac{c}{1+a}$$

Thus we obtain the "full-fledged" (Lit) general sol

$$y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a} \qquad \text{for } a \neq -1 \qquad (16.8')$$

(analogous procedure for the $a=-1$ case)

<u>Dynamic stability of equilibrium</u> p. 558 / 564 Mac

Dynamic stability is the question of whether the general sol. $y_t = \underset{\text{"BC"}}{\underset{\uparrow}{y_c}} + \underset{\text{"st. st"}}{\underset{\uparrow}{y_p}}$ tends to y_p as $t \to \infty$.

$\Leftrightarrow$ In other words, whether $y_c \to 0$ as $t \to \infty$.

Dynamic stability depends then on $A(b)^t$, in fact, it depends fully on b!

$b > 1$	explosive
$b = 1$	constant, non-convergent
$b \in (0,1)$	convergent (stable)
$b = 0$	constant (stable)
$b \in (-1,0)$	damped oscillations
$b = -1$	uniform oscillations
$b < -1$	explosive oscillations

$\Rightarrow$ Nonoscillatory: $b > 0$ (else oscillatory)

Damped: $|b| < 1$ (else explosive)

Note: $|b| = 1$ must be ruled out b/c it's non-convergent.

A: - scales b

- sign of A can <u>mirror</u> b.

<u>Higher-order difference equations</u>

The sol for y_p, the particular integral, proceeds exactly the same way. So let's focus on y_c, the complementary function:

Here's a 2nd order diff. eq:

(17.1) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$

The homogen. version is

(17.3) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

Guess the same thing: $y_t = Ab^t$

$$A b^{t+2} + a_1 A b^{t+1} + a_2 A b^t = 0 \quad | : Ab^t$$

$$b^2 + a_1 b + a_2 = 0 \qquad (17.3')$$

(17.3)' is the characteristic equation of (17.3) or (17.1)

Let's repeat: the characteristic equation is the equation you obtain once you plug in the conjecture for the complementary function y_c of $y_t = Ab^t$ and simplify.

It follows that a diff. eq. of order p will have p roots to the characteristic eq., as (17.3') clearly has 2 "characteristic roots"*, b_1 and b_2.

Ok, but which is the solution?

*characteristic roots are also known as eigenvalues.

Quick side note: take Woodford's 2nd order diff eq:

$$\beta \varphi_{t+1} - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\varphi_t + \varphi_{t-1} = \kappa x^* \quad (1.10)$$

Take a sol. of the form $A\mu^t$ and plug

$$\beta A \mu^{t+1} - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)A\mu^t + A\mu^{t-1} = \kappa x^*$$

Divide by $A\mu^{t-1}$ and consider the homogen case:

$$\beta \mu^2 - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\mu + 1 = 0$$

This is how you get eq. (1.11)

Ok, but back to Alpha Chiang and the question of what to do w/ the roots.

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Case 1. $a_1^2 - 4a_2 > 0 \Rightarrow$ 2 distinct, real roots

Case 2. $a_1^2 - 4a_2 = 0 \Rightarrow$ Repeated real roots

Case 3. $a_1^2 - 4a_2 < 0 \Rightarrow$ Complex roots

In all cases, all roots need to figure in the y_c b/c they are linearly independent.

Case 1. In this case the complementary function becomes

$$y_c = A_1 b_1^t + A_2 b_2^t$$

(a linear combination of two linearly independent parts)

Case 2. Since $b_1 = b_2$ here, $A_1 b_1^t + A_2 b_2^t$ becomes

$$(A_1 + A_2) b^t = A_3 b^t$$

In this case Alpha Chiang still insists that we need an additional linearly independent term, so he introduces $A_4 \cdot t \cdot b^t$, which is linearly ⊥ b/c you can't obtain it by multiplying $A_3 b^t$ by some constant coefficient c. (unless $c = A_4 \cdot t$ I guess)

⇒ thus in this case the complementary sol becomes

$$y_c = A_3 b^t + A_4 \cdot t \cdot b^t$$

Case 3. $b_{1,2} = h \pm vi$ (conjugate complex roots)

$$h = -\frac{a_1}{2} \qquad v = \frac{\sqrt{4a - a_1^2}}{2} \qquad \text{(ahaa!)}$$

$$\Rightarrow y_c = A_5 (h + vi)^t + A_6 (h - vi)^t \qquad \text{(ahaa!!)}$$

thanks to DeMoivre's theorem (15.28')

$$(h \pm vi)^t = R^t(\cos\theta t \pm i \sin\theta t)$$

where $R = \sqrt{h^2+v^2} = \sqrt{a_2}$ (always > 0) (radius)

the absolute value of the conjugate complex roots

$$\cos\theta = \frac{h}{R} = \frac{h}{\sqrt{a_2}} \quad \text{and} \quad \sin\theta = \frac{v}{R} = \frac{v}{\sqrt{a_2}}$$

$$\Rightarrow y_c = A_5(h+vi)^t + A_6(h-vi)^t$$

$$= A_5 R^t(\cos\theta t + i\sin\theta t) + A_6 R^t(\cos\theta t - i\sin\theta t)$$

$$= \underbrace{(A_5+A_6)}_{A_7} R^t \cos\theta t + \underbrace{(A_5-A_6)i}_{A_8} R^t \sin\theta t$$

$$y_c = R^t(A_7 \cos\theta t + A_8 \sin\theta t) \qquad (17.10)$$

<u>Convergence of the time path (dynamic stability) in the p-order difference equation case</u>

For concreteness, $p=2$ here.

A p-order diff eq converges to st.st if the absolute value of every characteristic root is less than 1.

<u>Schur THM</u>: this holds if all p determinants are positive (the first p subdeterminants or what) (p. 599/589 Mac)

Note that Viète's formulas are also an easy way to relate the roots to each other via their sums & products.

For a quadratic equation, the roots take the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

thus $x_1 + x_2 = \dfrac{-b + \sqrt{b^2-4ac} \; -b - \sqrt{b^2-4ac}}{2a} = -\dfrac{b}{a}$ $\square$

and $x_1 \cdot x_2 = \dfrac{\left(-b + \sqrt{b^2-4ac}\right)\left(-b - \sqrt{b^2-4ac}\right)}{4a^2}$

$(a+b)(a-b)$
$= a^2 - b^2$

$$= \frac{b^2 - (b^2-4ac)}{4a^2} = \frac{c}{a} \quad \square$$

thus, Viète:

$$x_1 + x_2 = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a}$$

Back to Woodford then. The characteristic eq. (1.11)

$$\beta\mu^2 - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\mu + 1 = 0$$

has 2 solutions, μ_1 & μ_2.

By Viete,

$$\mu_1+\mu_2 = -\frac{b}{a} = \frac{1+\beta+\frac{\kappa^2}{\lambda}}{\beta} > 0$$

$$\mu_1\cdot\mu_2 = \frac{c}{a} = \frac{1}{\beta} > 0$$

So since $\mu_1\mu_2 > 0$, $\text{sign}(\mu_1) = \text{sign}(\mu_2)$.

Since $\mu_1+\mu_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$.

How does Woodford know that $\mu_1 < 1 < \mu_2$?

(V1) $\mu_1+\mu_2 = \frac{1}{\beta} + 1 + \frac{\kappa^2}{\lambda\beta} > 1$ (in fact > 2)

(V2) $\mu_1\cdot\mu_2 = \frac{1}{\beta} > 1$ but close to 1. (around 1.1)

(V2) implies that at least 1 root > 1 b/c if $\mu_1 = \mu_2 \leq 1$,

(V2) ↯ So suppose $\mu_2 > 1$. Why must $\mu_1 < 1$? E.g.

Why can't $\mu_1 = 1$?

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

In Woodford's case this is

$$\mu_{1,2} = \frac{\left(1+\beta+\frac{k^2}{\lambda}\right) \pm \sqrt{\left(1+\beta+\frac{k^2}{\lambda}\right)^2 - 4\beta}}{2\beta}$$

Determinant is $\left(1+\beta+\frac{k^2}{\lambda}\right)^2 - 4\beta \quad =: \Delta$

$= \left(1+\beta+\frac{k^2}{\lambda}\right)^2 - \left(2\sqrt{\beta}\right)^2 \qquad a^2-b^2 = (a+b)(a-b)$

$= \left(1+\beta+\frac{k^2}{\lambda} + 2\sqrt{\beta}\right)\underbrace{\left(1+\beta+\frac{k^2}{\lambda} - 2\sqrt{\beta}\right)}_{\overset{!}{>}0}$

ok so for sure there is a value of λ that ensures that (close enough to 0).

Supp $\beta = 0.99$. Then $\sqrt{\beta} = 0.95 \rightarrow 2\sqrt{\beta} = 1.99$

which is miraculously $= 1+\beta = 1.99$!

$1+\beta-2\sqrt{\beta} \Rightarrow 1+x^2-2x$ where $x = \sqrt{\beta}$

I can write this as $(1-x)^2 = (1-\sqrt{\beta})^2 > 0$ for $\beta < 1$

ok so at least I know I have 2 distinct real roots.

Ok but that means that $\Delta > 0$ so that $\sqrt{\Delta} > 0$

So the 2 roots are

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\Delta}}{2\beta}$$

Is the idea that $\frac{1 + \beta + \frac{k^2}{\lambda}}{2\beta} \approx 1$?

$$= \frac{1}{2\beta} + \frac{1}{2} + \frac{k^2}{2\lambda\beta}$$

$$= \frac{1}{2}\left(\frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\beta}\left(1 + \frac{k^2}{\lambda}\right)\right)$$

The problem is that it seems to me to be a quantitative question, but it should hold for all param values > 0.

So for $\beta \to 1$ and $k \to 0$ the above becomes

$\frac{1}{2}\left(1 + 1(1 + 0)\right) = \frac{1}{2}(1+1) = 2$ ok fine.

And $\frac{k^2}{\lambda} \to 0$ even if $\lambda \to \infty$ b/c $k^2 \to 0$ faster.

I guess. $\Rightarrow$ So $0 < \mu_1 < 1 < \mu_2$. And this is important b/c $\mu_2 > 1$ is explosive, so TVC excludes it.

so one side-question that arises is Blanchard & Kahn conditions, which dictates as many explosive eigenvalues as fwd-looking vars. Why? And what's up w/ eigenvalues vs. roots of characteristic equations here?

→ ow: "characteristic roots are also known as eigenvalues." :-)

Alpha Chiang makes this connection explicit when he discusses dynamic stability of "simultaneous equations (ODE or difference eqs)"

→ Need to work thru but the intuition is thus. For an n-dimensional difference eq, transform it into a 1-dimensional matrix diff eq, and then proceed as usual.

So for Woodford, the complementary function is

$$\varphi_t = A_1 \mu_1^t + A_2 \mu_2^t$$

I guess TVC sets $A_2 = 0$.

So $\varphi_t = A_1 \mu_1^t$ is the complementary sol, for A_1 yet to be determined.

For the particular solution, guess $\varphi_t = b$

$$\beta k - \left(1 + \beta + \frac{\kappa^2}{\lambda}\right) k + k = \kappa x^*$$

$$k = \frac{\kappa x^*}{\cancel{1+\beta} \cancel{-1-\beta} - \frac{\kappa^2}{\lambda}} = \frac{x^*}{-\frac{\kappa}{\lambda}} = -\frac{\lambda x^*}{\kappa}$$

So the general sol is

$$\varphi_t = A_1 \mu_1^t - \frac{\lambda x^*}{\kappa}$$

Now use the initial condition $\varphi_{t-1} \Big|_{t=0} = 0$

(set $t=0$)

$$0 = A_1 \mu_1^0 - \frac{\lambda x^*}{\kappa} \Rightarrow A_1 = \frac{\lambda}{\kappa} x^*$$

So then the general sol is

$$\varphi_t = \frac{\lambda x^*}{\kappa} \mu_1^t - \frac{\lambda x^*}{\kappa} = -\frac{\lambda}{\kappa} x^* \left[1 - \mu_1^t\right] \qquad (1.12)$$

The only thing I don't know is why $t+1$, not t? Typo?

$$\pi_t = \varphi_{t-1} - \varphi_t$$

$$= -\frac{\lambda}{\kappa}x^*\left[1-\mu_1^{t-1}\right] - \left(-\frac{\lambda}{\kappa}x^*\left[1-\mu_1^{t}\right]\right)$$

$$= \cancel{-\frac{\lambda x^*}{\kappa}} + \frac{\lambda x^*}{\kappa}\mu_1^{t-1} + \cancel{\frac{\lambda x^*}{\kappa}} - \frac{\lambda x^*}{\kappa}\mu_1^{t}$$

$$\pi_t = \frac{\lambda x^*}{\kappa}\mu_1^{t-1}(1-\mu_1) \qquad (1.13)$$

↑except that again Woodford has t here.

Tomorrow: 1) ↑check on this

2) complete Alpha Chiang's eigenvalue & simultaneous eq. thing.

✓ 3) start reworking draft for midwest macro, using Ryan's abstract - ish...

2) Alpha Chiang's connection between matrices & difference equations (& differential eqs) 17 Feb 2020

Consider the 2nd order diff eq: (p. 602)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \qquad (17.21)$$

"concoct an artificial new variable $x_t = y_{t+1}$"

$$x_{t+1} + a_1 x_t + a_2 y_t = c$$

$$y_{t+1} = x_t$$

$$\Rightarrow \left.\begin{aligned} x_{t+1} + a_1 x_t + a_2 y_t &= c \\ y_{t+1} - x_t &= 0 \end{aligned}\right\} \qquad (17.21')$$

Suppose then that this looks like

$$x_{t+1} + 6x_t + 9y_t = 4 \qquad (17.24)$$

$$y_{t+1} - x_t = 0$$

1. Seek particular integrals (denote them by $\bar{x}$ & $\bar{y}$ since they're LR eqb values)

2. Seek complementary functions

① First try stationary sols. If that doesn't work, try moving sols of the form $x_t = k_1 t$, $y_t = k_2 t$ etc.

Supp. $x_{t+1} = x_t = \bar{x}$ and $y_{t+1} = y_t = \bar{y}$. Set

$$\left.\begin{aligned} 7\bar{x} + 9\bar{y} &= 4 \\ \bar{y} - \bar{x} &= 0 \end{aligned}\right\} \qquad \bar{x} = \bar{y} = \frac{4}{16} = \frac{1}{4}$$

② Complementary parts:

$x_t = m b^t \qquad y_t = n b^t \qquad$ Set.

$$\begin{aligned} x_{t+1} + 6x_t + 9y_t &= 0 \\ y_{t+1} - x_t &= 0 \end{aligned} \implies \begin{aligned} mb^{t+1} + 6mb^t + 9nb^t &= 4 \\ nb^{t+1} - mb^t &= 0 \end{aligned}$$

$| : b^t$

$$\left.\begin{aligned} mb + 6m + 9n &= 0 \\ nb - m &= 0 \end{aligned}\right\} \qquad (17.28)$$

Looking to solve this for (m, n).

$$\left.\begin{aligned} (b+6)m + 9n &= 0 \\ -1m + bn &= 0 \end{aligned}\right\} \text{ coefficient matrix} - \begin{pmatrix} b+6 & 9 \\ -1 & b \end{pmatrix}$$

In order to solve (17.28) w/o $m = n = 0$, need the determinant of the coeff matrix "to vanish" i.e.

$$\begin{vmatrix} b+6 & 9 \\ -1 & b \end{vmatrix} = 0$$

$$\Rightarrow (b+6)b + 9 = 0 \quad \Rightarrow \quad b^2 + 6b + 9 = 0 \qquad (17.29)$$

(17.29) is the characteristic equation, and its roots $b_{1,2}$ are the characteristic roots = eigenvalues of the simultaneous eq. system.

Having b, (17.28) gives us (m, n). (They usually are equations of the form $m_i = k_i n_i$ for each root b_i.)

Let's do the same thing w/ matrix notation

Express

$$x_{t+1} + 6x_t + 9y_t = 4 \qquad (17.24)$$

$$y_{t+1} - x_t \qquad = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=I} \underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}}_{z_{t+1}} + \underbrace{\begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_t \\ y_t \end{bmatrix}}_{z_t} = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{d} \qquad (17.24')$$

$$= I \quad z_{t+1} \quad + \quad K \ z_t = d$$

where $\underbrace{z_{t+1}}_{=u}$, $\underbrace{z_t}_{=v}$ in Alpha Chiang's notation

$$I z_{t+1} + K z_t = d$$

① Particular integral: try constant sols

$$z_{t+1} = z_t = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \quad \Rightarrow \quad (I + K)\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = d$$

$$\Rightarrow \quad \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = (I + K)^{-1} d \quad \text{if } (I+K)^{-1} \text{ exists.}$$

② Complementary functions

$$z_{t+1} = \begin{bmatrix} m b^{t+1} \\ n b^{t+1} \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} \qquad z_t = \begin{bmatrix} m \\ n \end{bmatrix} b^t$$

$$\Rightarrow \quad I \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} + K \begin{bmatrix} m \\ n \end{bmatrix} b^t = 0 \qquad \Big| : b^t$$

$$(bI + K)\begin{bmatrix} m \\ n \end{bmatrix} = 0 \qquad (17.28')$$

It is from this that we wanna find $\begin{bmatrix} m \\ n \end{bmatrix}$.

Again we wanna avoid $m = n = 0$ type solutions

so we demand $|bI + K| = 0$

which is again the characteristic eq, w/

eigenvalues / characteristic roots as roots.

Further comment on the characteristic equation (p.610)

1. Characteristic eq. of a matrix
2. of a single diff. eq / ODE
3. of system of diff eqs / ODEs.

What's the connection?

1. If I write an n-order single diff-eq as a 1st-order system of eqs, the characteristic eq. is <u>the same</u> (as we saw on the previous ex.)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

CE: $\underline{\underline{b^2 + a_1 b + a_2 = 0}}$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} a_1 & a_2 \\ -1 & 0 \end{bmatrix}}_{=K} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

CE: $|bI + K| = \begin{vmatrix} a_1 + b & a_2 \\ -1 & b \end{vmatrix} = (a_1 + b)b + a_2 = 0$

$$\underline{\underline{b^2 + a_1 b + a_2 = 0}}$$

2. Connection #2

There's a link between the characteristic eq. of a diff. eq (or an ODE) to that of a particular matrix, call it D.

$$|bI - D| = 0 \qquad (17.38')$$

makes it clear that $D = -K$. D, or $-K$, has a special meaning. Write $I z_{t+1} + K z_t = d$ as $I z_{t+1} = -K z_t$ (setting $d = 0$ for the reduced version of the system).

$\Rightarrow z_{t+1} = -K z_t$

$\Rightarrow -K$ is the matrix that can transform the system from z_t to z_{t+1}. (i.e. it's the transition matrix.)

$\Rightarrow$ The characteristic equation is the det(com-mat) = 0 relation!

NB. If $I = J$ (not identity) so $J z_{t+1} = -K z_t$ then we need to $z_{t+1} = J^{-1}(-K) z_t$ and look at $\det(-J^{-1}K) = 0$. (SVAR!)

Return to Woodford & try to obtain (1.12) again.

Let me write (1.10) as

$$\beta \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = k x^*$$

$$\Rightarrow \beta \mu^2 - \alpha \mu + 1 = 0$$

$$\mu_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}$$

Ok, so $\alpha = 1 + \beta + \frac{k^2}{\lambda} \in [1+\beta, \infty]$

So for $\alpha = \min(\alpha) = 1+\beta$,

$$\mu_{1,2} = \frac{\alpha \pm \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} = \frac{\alpha \pm \sqrt{\beta^2 + 2\beta + 1 - 4\beta}}{2\beta}$$

$$= \frac{\alpha \pm \sqrt{(\beta-1)^2}}{2\beta} = \frac{1+\beta \pm (\beta - 1)}{2} \begin{cases} \frac{1+\beta+\beta-1}{2} = \beta < 1 \\ \frac{1+\beta-\beta+1}{2} = 1 \end{cases}$$

If $\alpha \to \infty$

$$\mu_{1,2} = \frac{\infty \pm \sqrt{\infty^2}}{2\beta} \begin{cases} \frac{\infty + \infty}{2\beta} = \frac{1}{\beta} > 1 \\ \frac{\infty - \infty}{2\beta} = 0 < 1 \quad \text{Ok!} \end{cases}$$

Fine, Woodford, $0 \le \mu_1 < 1 \le \mu_2$

Ok so before turning a 100% to Woodford, let's think about solutions to DSGE models

In Lect 10, Basu looks for the solution of the 2-dimensional non-interacting system where the corr. sols are $z^1_{1t} = \alpha \omega_1^t$ and $z^1_{2t} = \beta \omega_2^t$

Since $\omega_1 > 1$ (and $\omega_2 \in (0,1)$), TVC sets $\alpha \overset{!}{=} 0$

(α, β) correspond here to Alpha Chiang's A_1, A_2 etc. which are usually determined by initial conditions. (what A.C. calls "definitize the constants A_i")

For bw-looking vars, TVCs exclude explosive sols.

So maybe the point is that when the economy is a bw-looking eq. system, this is what you do. But if the econ is a fw-looking system, then

$z_t = \omega_1 z_{t+1}$ $\quad \omega_1 < 1$, so $z_t = \frac{1}{\omega_1} z_{t-1}$

$\frac{1}{\omega_1} > 1 \Rightarrow$ transforming the fw-looking one into a bw-looking one requires explosive eigs of the bw-looking system. (Blanchard & Kahn)

Finally back to Woodford, 100%

Alright, so I've solved the char. eq. to obtain μ_1.

So my conj must have been $\varphi_t = A\mu^t$

Or did Woodford conjecture $\varphi_t = A\mu^{t+1}$? So that

$$\beta A\mu^{t+2} - \alpha A\mu^{t+1} + A\mu = 0$$

$$\Leftrightarrow \quad \beta\mu^2 - \alpha\mu + 1 = 0$$

But technically it doesn't matter b/c also the conjecture $\varphi_t = A\mu^t \rightarrow \beta A\mu^{t+1} - \alpha A\mu^t + A\mu^{t-1} = 0$

$\beta\mu - \alpha + \frac{1}{\mu} = 0$ leads to the same thing if you multiply by μ,

$$\beta\mu^2 - \alpha\mu + 1 = 0$$

So taking Woodford's conjecture then, turn to the particular integral:

A stationary sol would involve $\varphi_t = \bar{\varphi} \quad \forall t$

$$\beta\bar{\varphi} - \alpha\bar{\varphi} + \bar{\varphi} = kx^*$$

$$\bar{\varphi}(\beta - \alpha + 1) = kx^*$$

$$\bar{\varphi} = \frac{kx^*}{1+\beta-\alpha} = \frac{kx^*}{-\frac{k^2}{\lambda}} = -\frac{\lambda}{k}x^*$$

So w/o yet having definitized the constant, A, the general sol is

$$\varphi_t = A\,\mu_1^{t+1} + \left(-\frac{\partial}{k}x^*\right)$$

Definitize A using initial cond: when $t=-1$,

$\varphi_{-1} = 0$, so $\quad 0 = A - \frac{\partial}{k}x^*$

$$\Rightarrow \quad A = \frac{\partial}{k}x^*$$

So the "full-fledged" general sol then is

$$\varphi_t = \frac{\partial}{k}x^*\mu_1^{t+1} - \frac{\partial}{k}x^*$$

$$\varphi_t = -\frac{\partial}{k}x^*\left(1-\mu_1^{t+1}\right) \qquad (1.12)$$

Alright!

And since $\quad \pi_t = \varphi_{t-1} - \varphi_t$

$$= -\frac{\partial}{k}x^*\left(1-\mu_1^{t}\right) - \left(-\frac{\partial}{k}x^*\left(1-\mu_1^{t+1}\right)\right)$$

$$= \frac{\partial}{k}x^*\left[\cancel{-1} + \mu_1^{t} \cancel{+1} - \mu_1^{t+1}\right]$$

$$= \frac{\partial}{k}x^*\left(\mu_1^{t}\left(1-\mu_1\right)\right)$$

$$\Rightarrow \quad \pi_t = (1-\mu_1)\frac{\partial}{k}x^*\mu_1^{t} \qquad (1.13)$$

Consider the timelessly optimal commitment policy. I think that the complementary solution is the same, but A will be determined differently b/c the initial condition (1.9) $\psi_{-1} = 0$ is replaced by $\pi_t = \bar{\pi} \ \forall t$ so that $\pi_0 = \bar{\pi}$.

(1.13) implies $\pi_t \stackrel{!}{=} \bar{\pi} = (1-\mu_1)\frac{\lambda}{\kappa} x^* \mu_1^t$

let me recall that $A = \left(-\frac{\lambda}{\kappa} x^*\right)$ so

$$\pi_t = (1-\mu_1)(-A)\mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$$(\mu_1 - 1) A \mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$$\mu_1 A \mu_1^t - A \mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$\Rightarrow$ $A \stackrel{!}{=} 0$ and $\bar{\pi} \stackrel{!}{=} 0$ otherwise ↯.

<u>Optimal responses to shocks</u> p. 488.

Take the same loss (except in $E_{t_0}(\cdot)$) as before (eq (2.4)). Then clearly you obtain the same diff eq for the multiplier ψ_t (eq. (1.10)) except there's an $E(\cdot)$ in front (eq (2.6)) and the cost-push shock u_t shows up on the RHS

$$\beta E_t \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = \kappa x^* + u_t \qquad (2.6)$$

The characteristic eq. is the same, eq. (1.11), which has 2 roots, $0 < \mu_1 < 1 < \mu_2$.

The question is: how the f* does Woodford attain equation (2.7)?

We said that $\varphi_t = A_1 \mu_1^{t+1} + A_2 \mu_2^{t+1}$

but TVC set $A_2 = 0$.

So then the complementary sol is the same, $\varphi_t = A \mu_1^{t+1}$, except that A may be different.

What about the particular solution, $\bar{\varphi}$?

$$\bar{\varphi}(\beta - \alpha + 1) = \kappa x^* + u_t$$

how to treat this?

As long as u_t doesn't have a drift, its LR mean is 0. That would imply the same particular sol as in the non-stochastic case. But if $\bar{\varphi}$ is the same, then A is also the same.

Ok, so since the μ_1^{t+1} terms are gone, we must have gone somehow from (1.12):

$$\varphi_t = -\frac{\lambda}{\kappa} x^* (1 - \mu_1^{t+1}) \qquad \text{to } (2.7).$$

$$= -\frac{\lambda}{\kappa} x^* (1 - \mu^t \cdot \mu_1)$$

$$= -\frac{\lambda}{\kappa} x^* (\mu_1 - \mu_1 \mu^t + 1 - \mu_1)$$

$$= \mu_1 \left[-\frac{\lambda}{\kappa} x^* (1 - \mu^t) \right] - \frac{\lambda}{\kappa} x^* (1 - \mu_1)$$

$$\varphi_t = \mu_1 \varphi_{t-1} - (1 - \mu_1) \frac{\lambda}{\kappa} x^* \qquad \text{haha!}$$

The question that remains is how/where do the u_{t+j} terms come from?

I'm wondering if from the perspective of the difference eq u_t isn't an "endog" variable?

Or is Woodford saying that u_t follows its own diff eq which can/is solved?

Wait. Maybe we guess a bit of a different complementary sol: $\varphi_t = A_1 \mu_1^{t+1} + A_2 u_t$

or, better yet: $\varphi_t = A_1 (\mu_1^{t+1} + u_t)$. But the problem even w/ this formulation is that it only explains why a single u_{t+j}-term would show up in y_{t+j}. But as long as the characteristic eq is the same and the particular sol too, I don't see how u would enter w/ a sum.

Baum Lect 10 gives some hints: s. 3 (p. 3 Mac)

$$z_t = a z_{t-1} + m_t \qquad |a| < 1$$

(Stable diff eq w/ exog fct of time m_t)

Recursive sub

$$z_t = a\left[a z_{t-2} + m_{t-1}\right] + m_t$$

$$= a^2 z_{t-2} + a m_{t-1} + m_t$$

$$= a^k z_{t-k} + \sum_{s=k}^{t} a^{t-s} m_s$$

$\Rightarrow$

$$z_t = a^t z_0 + \sum_{s=1}^{t} a^{t-s} m_s$$

Let's reinterpret this a bit

$$z_t = z_0 \, a^t + \sum_{s=0}^{t} a^{t-s} m_s$$

$$z_t = A \cdot \mu^t + \sum_{s=0}^{t} \mu^{t-s} m_s$$

exog function of time

initial condition

dynamics from initial cond.

Basu continues w/ unstable diff eqs w/ exog stuff:

$$z_t = a z_{t-1} + m_t \qquad |a| > 1$$

$$\Rightarrow \quad a z_{t-1} = z_t - m_t \quad \Rightarrow \quad z_{t-1} = a^{-1} z_t - a^{-1} m_t$$

$$\Rightarrow \quad z_{t-1} = a^{-1}\left[a^{-1} z_{t+1} - a^{-1} m_{t+1}\right] - a^{-1} m_t$$

$$= a^{-2} z_{t+1} - a^{-2} m_{t+1} - a^{-1} m_t$$

$$= a^{-k} z_{t+k} - \sum_{s=0}^{k} a^{-(s+1)} m_{t+s}$$

Susanto writes the index differently, but this way of writing it at least corresponds to Woodford.

Under uncertainty, just tack on "E_t" to the exog m_{t+s}

$$z_{t-1} = a^{-k} z_{t+k} - \underbrace{\sum_{s=0}^{k} a^{-(s+1)} E_t m_{t+s}}$$

so at least this is exactly where $\sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}$ comes from; and μ_2 is used <u>b/c</u> $\mu_2 > 1$.

and β^{-1} will come from $\beta E_t \varphi_{t+1}$

But this seems to indicate, like Baxon writes,

$$z_t = a^t b_0 - \sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{s-t} m_s$$

or $$\varphi_t = \underbrace{-\frac{\partial}{k} x^* \left(1 - \mu_1^{t+1}\right)}_{\text{from (1.12)}} - \underbrace{\beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}}$$

effect of initial cond — effect of shocks for

general sol — fwd-diff eq.

$$\underbrace{\mu_1 \varphi_{t-1} - (1-\mu_1)\frac{\partial}{k} x^*}_{\text{general sol}}$$

So it seems like when there's a function of time on the RHS, you need to add a sum, fwd or bw-looking depending on the eq. or the roots.

But how do we know which sum (fw- or bw-looking) to add?

Supp: $\beta E_t \varphi_{t+1} = \alpha \varphi_t + u_t$

$$\alpha \varphi_t = \beta E_t \varphi_{t+1} - u_t$$

$$\varphi_t = \frac{\beta}{\alpha} E_t \varphi_{t+1} - \frac{1}{\alpha} u_t$$

$$y_t = \frac{\beta}{\alpha} E_t y_{t+1} - \frac{1}{\alpha} u_t$$

$$= \frac{\beta}{\alpha}\left[\frac{\beta}{\alpha} E_t y_{t+2} - \frac{1}{\alpha} E_t u_{t+1}\right] - \frac{1}{\alpha} u_t$$

$$= \left(\frac{\beta}{\alpha}\right)^2 E_t y_{t+2} - \frac{1}{\alpha} u_t - \frac{\beta}{\alpha^2} E_t u_{t+1} - \frac{\beta^2}{\alpha^3} E_t u_{t+2} + \dots$$

$$= \dots - \frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^s u_{t+s}$$

ok, now this raises the issue of what exactly we're doing. In the Basu Lect. Notes, an AR(1) process's root is the AR-coeff. Maybe since we have a 2nd order process, we have a distinction between the AR-coeff & the roots.

$\Rightarrow$ Indeed: for a one-dim AR(1), $y_{t+1} = a y_t + c$ a is the root! [Alpha Chiang, p. 554)

So I'm pulling on Blume p. 585 ff (Mac, 559 ff) to study difference eqs. in general.

Note: Blume calls Basu's method of making the system noninteracting "decoupling" or "change of coordinates" (Mac, 564)

On p. 567 (Mac), Blume then says: the uncoupled system is $z'_{n+1} = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_k \end{pmatrix} z'_n$

where r_i are the eigenvalues of of $\underset{k \times k}{A}$, where the orig system was

$$z_{n+1} = A z_n$$

then the decoupled (non-interacting) z's are

$$z'_{1,n} = c_1 r_1^n$$
$$\vdots$$
$$z'_{k,n} = c_k r_k^n$$

So the original z-variables's solutions are

$$z_n = c_1 r_1^n v_1 + c_2 r_2^n v_2 + \dots + c_k r_k^n v_k$$

where the v_i are eigenvectors.

i.e. a sum of terms of the form

$$c_i \cdot (\text{eigenvalue})^n \cdot \text{eigenvector}$$

where the c_i are constants determined from the initial conditions.

I'm on econdse.org and find the remark: "DSE-Maths-... pdf" in literatur

"non-autonomous equations"

$$x_t = a x_{t-1} + b_t \qquad (2)$$

- The complementary function will be the same as for the autonomous eq. $x_t = a x_{t-1} + b$
- But the particular sol won't be the same b/c trying a constant value of x won't work, so no x_t and x_{t-1} can be the same. Good point!

There are 2 methods for arriving at the particular sol: backward sol & forward sol

See also "diffeqn.pdf" → this is a really good note!
In particular, this note shows that finding eigenvalues is really equal to factoring polynomials. E.g. p.17

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + b_t$ can be written as

$(1 - \phi_1 L - \phi_2 L^2)\, y_t = b_t$. Now, this we can write as

$$(1-\lambda_1 L)(1-\lambda_2 L)y_t = b_t$$

these are the eigs.

We know that since L behaves like z^{-1}, we can write

$1-\phi_1 L - \phi_2 L^2 = 0$ as $1-\phi_1 z^{-1} - \phi_2 z^{-2} = 0$

and multiply by z^2 to write the "associated polynomial"

$z^2 - \phi_1 z - \phi_2 = 0$ which we can solve for

$z_{1,2} = \lambda_{1,2}$

So $y_t = \dfrac{b_t}{(1-\lambda_1 L)(1-\lambda_2 L)\dots(1-\lambda_p L)}$

So if both eigenvalues are less than 1, then

$y_t = \hat{c}_1 \lambda_1^t + \hat{c}_2 \lambda_2^t + \dfrac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} b_t$ eq. (27)

(p. 14)

In terms of Woodford, what I still don't know is why he solves u forward (why not solve bw using $\mu_1 < 1$?)

and where β^{-1} is coming from?

Ok I don't know: the result of today's hard work is that in the case of a non-autonomous diff. eq, the sol. takes the form

$$y_t = \underbrace{y_c}_{\text{same as for autonomous eq}} + \underbrace{y_p}_{\text{not the same as for nonhom. eq. instead, this will be a forward or backward sum of } b_t \text{ (the exog. part), w/ the eigenvalues as coeffs.}}$$

What I don't know is how to determine which eigenvalue will be coefficient.

- If the system is univariate, there's only one eig λ and the fwd/bwd-nature of the sum depends on whether $|\lambda| \gtrless 1$
- If the system is p-variate, but all eigs $|\lambda_i| < 1$ then $y_p = \dfrac{1}{(1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_p L)}$ (guess... :S
- If all eigs $|\lambda_i| > 1$ then $y_p = \dfrac{1}{(1-\lambda_1^{-1}L)(1-\lambda_2^{-1}L)\ldots(1-\lambda_p^{-1}L)}$

What I don't know is what when one eig $\mu_1 < 1$ the other $|\mu_2| > 1$.

The lecture notes by Tirelli "linear_...Tirelli.pdf" seem to provide the answer. Cont here.

Actually, they *almost* do — 18 Feb 2020

- but they also don't say what to do w/ "alternating" eigenvalues.

So I'll pause the issue of (2.7) here for a while.
I wanna go on to see that thing about implementation.

Note on noninertial p. 497: purely fwd-looking policy, i.e. maximizes the same utility as under Ramsey but subject to the condition that endog. vars evolve according to $z_t = \bar{z} + f s_t$ (purely fwd-looking)
i.e. the optimal plan is purely fwd-looking.

It's mean b/c Woodford has actually introduced OMI here before he officially does starting p. 507.

What I said just now isn't quite right: splitting the same loss into $\mathcal{L}^{det}$ & $\mathcal{L}^{stab}$ serves the purpose of making OMI 1) time-consistent 2) still optimal. The point is that whatever path a restriction $z_t = \bar{z} + f s_t$ gives you is not time-consistent (b/c it is purely fwd-looking), but simply imposing the timelessly optimal plan from before is not optimal for $z_t = \bar{z} + f s_t$. So you split the problem in two, allowing the same t_0- or timelessly optimal policy to minimize $\mathcal{L}^{stab}$ (b/c responses to shocks optimally are the same) which is independent of the econ's state at any time t. And shit: the LR averages are exactly NOT those that min $\mathcal{L}^{det}$; instead you just adopt those from a timelessly optimal policy.

p. 516: Woodford also says that if a simple TR is found that is the optimal one, then a TR that involves fcsts does not improve upon the baseline TR. The reason is that already the baseline TR was purely fwd-looking, so you haven't changed that. In fact, as Giannoni & Woodford (2002b) show, for fcsts of horizon k large enough, the eqb becomes indeterminate.

> By the way: given a timelessly optimal soln for the Ramsey problem, p. 518 calculates the opt. time paths for π & i (eqs. (4.1) & (4.2))

Then there's the problem of solving for optimal plans for (π, x, i) and using the plan for i as a reaction function. But that's indeterminate.

$\Rightarrow$ So the optimal time path isn't a good policy rule, but the purely forward-looking TR isn't fully optimal. $\Rightarrow$

Are targeting rules the solution? p. 521

A targeting rule is a commitment of the CB to adjust its instrument such that a target criterion is projected to be fulfilled at all points in time.

One target criterion which is optimal from a timeless perspective is

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0 \qquad (5.1)$$

It is not just optimal, but it is also feasible (b/c an REE w/ it exists), moreover it's unique (not indeterminate)

Δx_t → reflects the history-dependence of optimal commitment (p. 525)

But you know, there are other robustly optimal target criteria. (p. 526-27)

But we still don't know how to implement the targeting rule, all we have is an optimal target criterion.

→ But you can use the target criterion together w/ model equations & LOMs of shocks to derive a reaction function for it.

As a first-pass, you obtain the fundamentals-based reaction function of E&H (5.4)

(You've assumed the private sector's expectations are RE)

The problem though is that for many param values, this again isn't determinate.

So instead of using expectations that "ought to be", plug in expectations of the private sector the CB actually observes: "expectations-based reaction fct" (w/o solving for 'em) (5.5)

This is determinate according to Prop 7.17.

→ so an interest rate rule (5.5) [= reaction function] is equivalent to committing to the π-target (5.1) & implements it. This (5.5) is a forward-looking TR, however, it does have history-dependence thru

x_{t-1} w/ $\frac{(\kappa^2+\lambda)-\lambda}{\sigma(\kappa^2+\lambda)} = \frac{\kappa^2}{\sigma(\kappa^2+\lambda)} = \phi_x$ in LR.

Looking at Gaspar et al 2011 & Preston 2008,
it's not quite clear to me what the Ramsey problem should look like.

Ok: supposing that the indexation parameter $\gamma = 0$,
At least the period loss L_t takes the same form in Gaspar et al 2011 & Preston 2008:

$$L_t = \pi_t^2 + \lambda x_t^2$$

$$W = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \leftarrow \text{ overall loss fct.}$$

And this is also the loss Molnár & Santoro (2016) consider (except they call λ α for a change) and they only claim to do discretionary policy (but actually, they seem to be solving the commitment problem.

And indeed: Preston takes Woodford's equations and replaces the eigenvalue $\mu_2 > 1$ w/ $\mu_1 < 1$.
(see my comments in Preston 2008, p. 8 Mac)

Ok, brace yourselves, I'm approaching the "approach":

① $\min\limits_{\pi_t, x_t, i_t, \phi_t} \mathcal{L}$ s.t. ALM, LOM (beliefs)

$\quad\quad\quad\quad ?$

this will give you FOCs that, given eigs, can be solved for the evolution of $\mathcal{L}$-multipliers, which can be subbed back to obtain paths for $\{\pi_t, x_t, i_t, \phi_t\}$

② Sub out $\mathcal{L}$-multipliers from FOCs to obtain a target criterion: π_t in terms of x $(x_t, x_{t-1} \dots)$.
(Analogue of (5.1))

③ Use the ALMs to figure out what $\hat{E}\pi$ & $\hat{E}x$ are and confront them w/ the criterion to obtain the i-rate rule / reaction function.

In materials17, I have written out both the big-Δ-ugly Lagrangian for the full system, as well as that of a simplified one (eq 20–25) (or 26–31)

19 Feb 2020

Take FOCs of the simplified one.

π: $\quad 2\pi_t + \varphi_{1,t} - \varphi_{5,t}\,\bar{g}^{-1} = 0$ (1)

x: $\quad 2\lambda x_t - \varphi_{1,t} k + \varphi_{2,t} = 0$ (2)

i: $\quad \varphi_{2,t}\, b = 0 \rightarrow \varphi_{2,t} = 0 \Rightarrow$ interesting: I'm also finding that IS-curve isn't binding.

fa: $\quad -\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0$ (3)

fb: $\quad -\varphi_{2,t}\, b + \varphi_{4,t} = 0$ (4)

$\bar{\pi}_t$: $\quad - E_t\beta\, \varphi_{3,t+1} \frac{1}{1-\alpha\beta} - E_t\beta\, \varphi_{4,t+1}\frac{1}{1-\beta}$

$\quad + \varphi_{5,t} + E_t\beta\, \varphi_{5,t+1}\left(-1 + \bar{g}^{-1}\right) = 0$ (5)

The good news is that these FOCs look similar to the ones Gaspar et al (2011) get.

In particular, using that $\varphi_{2,t} = 0$, I can combine (1) & (2)

Where I've rewritten the rational expectations:

If you can form $E(x)^{RE}$, then $E_t x_{t+1} = \underset{1 \times nx}{g_2} E_t s_{t+1}$

$= g_2 h_x s_t$

$E_t x_{t+k} = g_2 h_x^k s_t \Rightarrow E_t x_T = g_2 h_x^{T-t} s_t$

$= b_2 h_x^{T-t-1} s_t$

$\rightarrow E_t x_{T+1} = g_2 h_x^{T-t+1} s_t = b_2 h_x^{T-t} s_t$

$\kappa \alpha \beta E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_{T+1} = \kappa \alpha \beta E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} b_2 h_x^{T-t} s_t$

$= \kappa \alpha \beta b_2 \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} h_x^{T-t} s_t$

$= \kappa \alpha \beta b_2 (I_3 - \alpha\beta h_x)^{-1} s_t$

$$2\pi_t + \varphi_{1,t} - \varphi_{5,t} \bar{g}^{-1} = 0 \quad (1)$$

$$2\lambda x_t - \varphi_{1,t} k = 0 \quad (2)$$

(2): $\varphi_{1,t} = 2 \frac{\lambda}{k} x_t$

$\Rightarrow (1)$: $2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{5,t} \bar{g}^{-1} = 0$

$\Rightarrow \quad x_t = \frac{k}{\lambda}\left(\frac{\varphi_{5,t}}{2} \bar{g}^{-1} - \pi_t\right)$

$\Rightarrow \quad x_t = -\frac{k}{\lambda}\left(\pi_t - \frac{\varphi_{5,t}}{2} \bar{g}^{-1}\right)$ = Gaspar et al's (22)!

Now turn to (3)-(5): (using $\varphi_{2,t}=0$)

$$-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0 \qquad (3)$$

$$\varphi_{4,t} = 0 \qquad (4)$$

$$-E_t\beta\,\varphi_{3,t+1}\frac{1}{1-\alpha\beta} - E_t\beta\,\varphi_{4,t+1}\frac{1}{1-\beta}$$
$$+\varphi_{5,t} + E_t\beta\,\varphi_{5,t+1}\left(-1+\bar{g}^{-1}\right) = 0 \qquad (5)$$

$\rightarrow$ $\varphi_{4,t}=0$ b/c f_β only shows up in the NKIS, which isn't binding.

(3): $\varphi_{3,t} = (1-\alpha)\beta\,\varphi_{1,t}$

(5) $$-E_t\beta\frac{(1-\alpha)\beta\,\varphi_{1,t+1}}{(1-\alpha)\beta} + \varphi_{5,t} + E_t\beta\,\varphi_{5,t+1}(-1+\bar{g}^{-1}) = 0$$

But we know $\varphi_{1,t} = 2\frac{\lambda}{\kappa}x_t$ so (5) becomes

$$-E_t\beta\, 2\frac{\lambda}{\kappa}x_{t+1} + \varphi_{5,t} + E_t\beta\,\varphi_{5,t+1}(-1+\bar{g}^{-1}) = 0 \qquad (5')$$

Now let's solve (5') for $\varphi_{5,t}$ by iterating fwd

$$\varphi_{5,t} = 2\beta\frac{\lambda}{\kappa}E_t x_{t+1} + (1-\bar{g}^{-1})\beta E_t\varphi_{5,t+1}$$
$$= 2\beta\frac{\lambda}{\kappa}E_t x_{t+1} + (1-\bar{g}^{-1})\beta\left[2\beta\frac{\lambda}{\kappa}E_t x_{t+2} + (1-\bar{g}^{-1})\beta E_t\varphi_{5,t+1}\right]$$
$$= 2\beta\frac{\lambda}{\kappa}E_t x_{t+1} + (1-\bar{g}^{-1})\beta\, 2\beta\frac{\lambda}{\kappa}E_t x_{t+2} + \left((1-\bar{g}^{-1})\beta\right)^2 E_t\varphi_{5,t+1}$$

$\varphi_{5,t} = 2\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}$ except for the sign, it's the same as Gaspar et al (2011).

So now plug the sol for $\varphi_{5,t}$ into

$$x_t = -\frac{\kappa}{\lambda}\left(\pi_t - \frac{\varphi_{5,t}}{2}\bar{g}^{-1}\right)$$, expressed for π_t

$$-\frac{\lambda}{\kappa}x_t + \frac{\bar{g}^{-1}}{2}\varphi_{5,t} = \pi_t$$

$$\pi_t = -\frac{\lambda}{\kappa}x_t + \frac{\bar{g}^{-1}}{2}\left[2\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}\right]$$

$$\pi_t = -\frac{\lambda}{\kappa}x_t + \bar{g}^{-1}\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}$$

$$\pi_t = -\frac{\lambda}{\kappa}\left(x_t - \beta\bar{g}^{-1}E_t\sum_{i=0}^{\infty}(1-\bar{g}^{-1})x_{t+1+i}\right) \quad (29)$$

Gaspar et al's

Allows you to get the nice interpretation of G et al 2011

When the gain is 0, (29) boils down to

$\pi_t = -\frac{\lambda}{\kappa}x_t$ (= discretionary RE solution)

This is what Gaspar et al call an "intratemporal tradeoff" (or actually, Molnár & Santoro do)

If the gain > 0, then there is an additional, inter-temporal tradeoff between π & x b/c whatever π

you bring about today, it affects π-expectations tomorrow, presenting you w/ an intratemp. tradeoff tomorrow.

Molnár & Santoro derive an interest rate rule!

They say: plug ALM(π) & ALM(x) (eq 18 & 19) into the NKIS

Their ALMs take the following form

$$\pi_t = c_\pi a_t + d_\pi u_t \qquad \text{and} \quad \hat{E}_t \pi_{t+1} = a_t$$

$$x_t = c_x a_t + d_x u_t \qquad \hat{E}_t x_{t+1} = b_t$$

NKIS: $x_t = \hat{E}_t x_{t+1} - \sigma^{-1}(r_t - \hat{E}_t \pi_{t+1})$

$$\Leftrightarrow \sigma x_t = \sigma \hat{E}_t x_{t+1} - r_t + \hat{E}_t \pi_{t+1}$$

$$r_t = \sigma \hat{E}_t x_{t+1} + \hat{E}_t \pi_{t+1} - \sigma x_t$$

$$= \sigma b_t + a_t - \sigma(c_x a_t + d_x u_t)$$

$$= \underbrace{(1-\sigma c_x)}_{} a_t + \underbrace{\sigma}_{} b_t \underbrace{- \sigma d_x}_{} u_t$$

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u u_t$$

which is what M&S obtain if you plug in c_x & d_x.

→ This is their expectation-based reaction function

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u a_t \qquad (20)$$

Ryan meeting 19 Feb 2020

Mantra:

Question of the paper: "How does a concern for the anchoring of expectations affect the conduct of monetary policy?"

- 1) Take simple pencil & paper: how does it change (32)? An extra term....
- 2.) Numerically solve the full-fledged model

3.) Estimate the gain function $k(\theta_t)$

"My est. suggests that $E(\cdot)$ are likely to be unanchored when people are surprised by 1%." → CB-ers would be interested to hear that. It'd be a great ending to the abstract.

"There should be a gentlemanly distance between assumptions and results." (Stephanie Schmitt-Grohe)

For me, this means that if I ask "does anchoring matter for policy?" then it's too easy to critique my work by saying "well, the way you set up the model, of course it does".

The mantra is mainly a marketing object: it doesn't assume that there is a concern for anchoring a priori, but given that I find that there is, how anchoring changes the policy problem, I avoid that kind of critique.

The reason Woodford might advocate an approach like Preston (2008) where the RE optimum should be implemented under learning is b/c he might say that "it is not obvious that a policymaker can credibly commit to a non-RE optimum."

Big issue: why is RE-discretion the alternative here? Did I not accidentally write down the discretion problem?

Ryan: discretion is when I choose a single i_t, commitment where I choose a whole plan of $\{i_t\}$.

↳ check discretion vs commitment in a very simple problem!