Expectations and Monetary Policy

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Abstract

This paper uncovers a new channel of monetary policy transmission in a New-Keynesian model under Internal Rationality, which assumes that agents are rational but cannot forecast inflation and output perfectly. The departure from rational expectations is small; yet, empirical and policy implications differ significantly as the inflation-output trade-off responds to the evolution of agents' expectations. In particular, high inflation expectations lead to a persistent combination of low output and high inflation, which closely resemble the missing disinflation and the slow recovery associated with the Great Recession. The model matches several stylized facts on inflation, output and their expectations found both in the survey and aggregate data. On the methodological side, the paper provides a micro-foundation for the optimality of adaptive learning in a monetary economics model.

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1 Introduction

> *"In standard economic models, inflation expectations are an important determinant of actual inflation... Monetary policy presumably plays a key role in shaping these expectations... Even so, economists' understanding of exactly how and why inflation expectations change over time is limited."*
>
> Janet L. Yellen, the Chair of the Board of Governors of the FRS[1]

The behaviours of inflation and output after the Great Recession are a puzzle for workhorse macroeconomic models. Given the depth of the recession in 2008 and 2009, inflation was too high: the missing disinflation puzzle. In addition, the recovery from the Recession took longer than anticipated: the slow recovery puzzle. There is growing evidence that inflation expectations that are derived from survey data play an important role in understanding some of these dynamics. But, the existing models have difficulty explaining survey evidence for inflation expectations, even though they might be the key to uncovering the source of the puzzling behaviours of inflation and output. Some of the most important questions we are left with include: What are the driving forces of inflation expectations? How much do these forces actually matter for the real economy? And what is the role of monetary policy in shaping these expectations?

Figures 1 and 2 plot the dynamics of inflation, inflation expectations, and the output gap in the United States for two periods: the Great Moderation (JAN1985-NOV2007), and the Great Recession and recovery (DEC2007-MAY2016). Figure 1 focuses on the dynamics of inflation and inflation expectations. For the Great Moderation, inflation expectations, on average, have been in line with the actual inflation in the economy. Simultaneously, the output gap behaviour in Figure 2 fits well with the standard models based on the Rational Expectations Hypothesis (REH), which are presented by the red dashed line in Figure 2. So, for the Great Moderation, the rational expectations approach provides a reasonable explanation for the dynamic of the economy.

For the second period (the Great Recession and recovery), the behaviours of the variables in Figures 1 and 2 are puzzling and difficult to address with standard Rational Expectations Equilibrium (REE) models. First, inflation expectations depart from the actual inflation and remain persistently

[1]Speech, September 26, 2017, Inflation, Uncertainty, and Monetary Policy At the "Prospects for Growth: Reassessing the Fundamentals" 59th Annual Meeting of the National Association for Business Economics, Cleveland, Ohio

high for the entire period. This is difficult to rationalise via an REE model, which generates inflation expectations that are, on average, aligned with actual inflation. Additionally, inflation has decreased mildly since the financial crisis in 2008 and 2009 considering the depth of the recession. Because of these data, Ball and Mazumder (2011) and Hall (2011) question the overall dependence of inflation on slack variables, such as the output gap, during the post-Great Recession period. Simultaneously, the output gap exhibits a prolonged recession. The slow recovery from the Great Recession is viewed as one of the main challenges for the existing macro models according to Lindé et al. (2016).

[Figure 1 About Here]

[Figure 2 About Here]

Despite their limitations, the existing frameworks—in particular the New Keynesian Dynamic Stochastic General Equilibrium (NK DSGE) models—constitute the foundation for modern monetary policy analysis. How can we shape these models so that survey evidence can be directly incorporated into the analysis? This paper proposes a method to do just that. The standard approach for solving these models assumes that agents are not only rational decision-makers, but also have perfect knowledge of the economy. However, rational expectations is a strong assumption and the evidence from survey data demonstrates that economic agents may not be that knowledgeable. Therefore, a model that strives to match the behaviour of survey data must relax this assumption. Nevertheless, retaining the rationality of agents ensures internal consistency and allows the model to be used for projections and policy recommendations. Maintaining rational behaviour also allows for a clear comparison with the standard paradigm. The purpose of this paper is to provide a model that allows for minimal deviations from a standard workhorse model in monetary economics but at the same time deviates enough to be able to describe the beliefs in line with survey evidence.

This paper builds on the adaptive learning literature but maintains the rationality of the agents. Importantly, it is also specific about beliefs system that the agents have in the economy.[2] Both these modelling features are the hallmark of the Internal Rationality framework developed by Adam and Marcet (2011). This approach has not been applied to NK DSGE models before.[3] The building

[2]The adaptive learning literature does not specify what agents' views are on the evolution of macro-variables. They only equip them with a recursion, which tracks some moments of the variable. If beliefs are not fully specified in the model, then why, exactly, agents must form expectations according to a given recursion and how this relates to rational behaviour is unclear. Furthermore, if there are constraints in the economy, higher moments of the distribution of beliefs play a vital role. However, in these models they are not specified.

[3]Adam and Marcet (2011), Adam et al. (2017), and Adam et al. (2016) apply IR framework to asset pricing models.

principles of Internal Rationality are an explicit information structure and a belief system that ensure agents in the economy have expectations that are close to their experiences. Very importantly, unlike the existing application of this approach, this paper focuses on the formation of expectations about multiple variables, which is a key assumption for the main theoretical contribution of the paper. It shows how to determine the behavior of the economy in this case, and argues that it is likely to observe prolonged deviations from REE even when agents make undetectable mistakes.

The main difference of this belief system from alternative approaches is that agents form expectations about variables that are endogenous to their choices in general equilibrium. In this setting, the expectations affect the realised economic variables: for example, high inflation expectations translate into high inflation and then high observed inflation reinforces the agents' beliefs. This effect allows the model to generate sizeable deviations from the REE: expectations and outcomes depart from what would have been the dynamic under REH. The experience of the Great Recession suggests this type of modelling approach can help rationalise the dynamics of the economy. Importantly, monetary policy determined the exact extent of the output losses associated with high inflation expectations; while setting Taylor rule coefficients, monetary authorities shape the feedback from inflation expectations into output.

The modelling approach of Internal Rationality nests REE and thus can be seen as a generalisation of REE. Therefore, when beliefs are close to REE, the model's dynamics are also similar to the REE; and this feature allows the model to replicate dynamics of inflation and output during the Great Moderation. However, the learning setting also makes it possible to study the dynamics of an economy when inflation expectations temporarily deviate from the long-run equilibrium of the model. This helps to explain behavior of inflation and output during the Great Recession with the same model.

The model addresses two main puzzles for the Great Recession and its aftermath: missing disinflation and slow recovery. Inflation expectations were high at the beginning of the recession relative to the observed inflation. My main assumption is to initialise them at this level, that is, away from the long-run equilibrium at this point in time. The model transmits the expectations to the inflation series that drives actual inflation to a higher level than expected under the REE. Moreover, given this initial shock, the model can generate expectations that are persistently higher than the actual inflation for prolonged periods. This behaviour of inflation and inflation expectations is associated with persistently low output. This feature of the model helps to explain the slow recovery. Moreover, the dynamics delivered by the model for both periods of the Great Moderation and the Great Recession are consistent with various features of survey data on inflation expectations. In particular, tests for the REH indicate that the model is consistent with survey evidence for both periods, while REH is rejected by survey data for the Great Recession and the recovery.[4] Additionally, the model helps to rationalise the empirical evidence of a delayed response of the inflation expectations to monetary policy shocks.[5] The model also helps to address the forward guidance puzzle discussed in Del Negro et al. (2015a), McKay et al. (2016) and Gertler (2017). The application of the Internal Rationality approach to NK DSGE model dampens general equilibrium effects of monetary policy shocks because in this setting current or future monetary policy innovations affect expected path of interest rates but do not change agents' view about evolution of inflation and the output gap. Alternative ways of delivering a similar effect in the NK DSGE models include bounded rationality ("sparsity" developed in Gabaix (2016) and an interaction of k-level thinking with incomplete markets in Farhi and Werning (forthcoming)) and dispersed information (strategic complementarities and imperfect signals about interest rate changes in Angeletos and Lian (2018), also discussed in Angeletos and Lian (2017)). Similar to the dispersed information setting agents in the model are rational decision makers given their imperfect knowledge of the world. In line with the bounded rationality approaches agents do not fully understand how innovations to policy rate affect equilibrium inflation and output gap.

Finally, this paper provides a micro-foundation for adaptive learning models, focusing on the "Euler-equation learning" and connecting them to a competing approach of "expected utility" learning. This is a missing useful block in the adaptive learning literature because it helps to narrow the discussion about which method is preferable in any particular modelling case to a discussion regarding borrowing opportunities of the agents in the economy.

This paper builds on the literature that incorporates adaptive learning in a standard NK DSGE models. Bullard and Mitra (2002) study the stability of expectations under learning by using the methods developed in Marcet and Sargent (1989) and Evans and Honkapohja (2001). Evans and Honkapohja (2003) discuss optimal policy under learning according to the same criteria. Orphanides and Williams (2005) and Gaspar et al. (2006) study the same issues under the assumption of constant gain learning. Milani (2007) and Slobodyan and Wouters (2012) demonstrate that adaptive learning

[4] Hall (2016) reviews possible driving forces that could account for the persistent slump the US economy experienced after the Great Recession. Kozlowski et al. (2017) models the slow recovery within a Bayesian learning framework where a rare event induces agents to change their views about the distribution of exogenous shocks in the model. Del Negro et al. (2015b) show that the missing disinflation can be accounted for in the DSGE model with financial frictions under the REH. Nevertheless, they also incorporate in their approach survey-based inflation expectations (long term).

[5] Melosi (2016) addresses the same issue in an incomplete information model.

models provide a good fit with some aspects of the data. These papers do not address issues of rationality of agents away from the REE. Instead, these models are based on structural equations that are derived under the REH, and, therefore, these models can be inconsistent.

Preston (2005) applies the anticipated utility approach to learning in the NK DSGE model.[6] Given this assumption, Preston (2005) sets optimisation problems for households and firms and demonstrates that under learning, long-horizon expectations matter. This paper clarifies how to set up a fully rational model under learning, while specifying the beliefs about dividends, wages and interest rates and without relying on the anticipated utility approach. While Preston (2005) studies learnability of model equilibrium, I apply a different approach and focus on transition dynamics rather than asymptotic behaviour.

This paper is closely related to Coibion and Gorodnichenko (2015) and Coibion et al. (2018). They provide strong empirical evidence that the rise in households' inflation expectations explains the missing disinflation. They attribute the increase in expectations to rising oil prices. This paper provides a model which is able to generate patterns that can explain these empirical findings.

The remainder of the paper is organised as follows. Section 2 derives the standard New Keynesian DSGE model under an assumption of internal rationality. Section 3 describes the model's dynamics and stresses the role of monetary policy in shaping them. Section 4 compares the model to the data and shows that the model is capable of rationalising the behaviours of the output gap and inflation together with survey evidence on inflation expectations. Section 5 concludes.

2 Basic New Keynesian DSGE model under Internal Rationality

This section derives equilibrium conditions for the standard New Keynesian dynamic stochastic general equilibrium (DSGE) model of Woodford (2003) and Galí (2015) in an environment in which agents hold subjective beliefs about the evolution of the aggregate price level and aggregate output in the economy. Under the standard setting of REH, agents understand how fundamentals map to aggregate inflation and aggregate output. Instead, I assume the lack of common knowledge of general equilibrium linkages, and equip agents with a fully specified system of beliefs. I demonstrate that the subjective beliefs of internally rational agents are compatible with optimising behaviour, even if the beliefs differ from the probabilities delivered by the model in equilibrium. This provides a full set of microfoundations for a broad body of adaptive learning literature with application to DSGE models.

[6] See Kreps (1998), Sargent (2002) and Woodford (2013), also used in Eusepi and Preston (2011) for real business cycle setting.

There has been developed two competitive approaches to introducing adaptive learning into DSGE framework: "Euler-Equation" learning and "expected utility" learning. The first approach states that inflation and output gap today are defined by their one period ahead expectations; while the second one demonstrates that the entire future paths of expected output gap and inflation pin down realised values of those variables today. The qualitative results of this paper hold under both approaches. Moreover, I demonstrate that both approaches can be microfounded using internal rationality combined with two extreme assumptions about borrowing constraints of households in the economy and a very special belief system linking all prices to beliefs about inflation, output and some shocks. This section outlines full set of microfoundations of "expected utility" learning approach. Appendix C provides additional assumptions to rationalize the IS curve used in "Euler-equation" strand of learning literature.

2.1 Households

An economy is populated by a unit mass of identical infinitely lived consumers. A consumer i chooses his or her consumption, C_t^i, hours of work, N_t^i, and savings in the form of one-period nominal bond holdings, B_t^i, by maximising the expected utility, subject to a standard flow budget constraint:

$$\max_{\{C_t^i, N_t^i, B_t^i\}_{t=0}^{\infty}} E_0^{\mathcal{P}^h} \sum_{t=0}^{\infty} \delta^t \left[\frac{a_t (C_t^i)^{1-\sigma}}{1-\sigma} - \frac{(N_t^i)^{1+\phi}}{1+\phi} \right] \tag{2.1}$$

$$P_t C_t^i + \frac{1}{1+r_t} B_t^i \leq B_{t-1}^i + W_t N_t^i + D_t \tag{2.2}$$

$$\underline{B} \leq \frac{B_t^i}{P_t} \leq \overline{B}, \forall t \tag{2.3}$$

where P_t is the aggregate price level, a_t is the preference shock, r_t is the nominal interest rate, W_t is the nominal wage, and D_t are the dividends paid by firms. $\underline{B}$ and $\overline{B}$ are bounds on holding of real bonds. The problem defined above is standard to the DSGE literature; the only difference is the expectations operator $E^{\mathcal{P}^h}$.

The expectations of households are determined with a subjective probability measure $\mathcal{P}^h$, which assigns probabilities to all variables that agents take as given when solving their maximisation problems.[7] In order to introduce more structure to the agents' beliefs, and keep the model tractable, households are equipped with the following map:

$$\{W_t, r_t, D_t\} = F^h(Z_t, P_t, a_t) \tag{2.4}$$

beliefs of W,r,D are RE

where Z_t is the aggregate output, and $F^h(.)$ represents a relation which holds in the equilibrium of the model under learning. This is a crucial assumption for micro-foundations of consumers' optimality,

[7] Appendix A.4 provides further details of state spaces of each agent in the economy.

which was not made explicit previously in the adaptive learning literature. This assumption allows me to focus solely on the evolution of beliefs about inflation and output, and abstract from the problem of agents simultaneously learning about a large number of variables. Nevertheless, the mapping (2.4) is still not sufficient for agents to recover the REE.

I use a log-linear approximation to solve the model as in Preston (2005) and Woodford (2013).[8] Unlike Preston (2005) I allow for endogenous labour and find for large $\underline{B}$ and $\overline{B}$:[9]

$$c_t = (1-\delta)\widetilde{B}_{t-1} + E_t^{\mathcal{P}^h} \sum_{j=0}^{\infty} \delta^j \left((1-\delta) z_{t+j} - \delta \frac{1}{\sigma} (r_{t+j} - \pi_{t+j+1}) \right) + \epsilon_t^d \tag{2.5}$$

Optimal consumption today depends on the future expected path of aggregate output and inflation. The case of tight $\underline{B}$ and $\overline{B}$ is dealt with in Appendix C.

2.2 Firms

There is a continuum of firms indexed by j in the intermediate goods sector, $j \in [0,1]$. Each firm produces a differentiated good using the Cobb-Douglas production function:

$$Y_t(j) = N_t(j)^{1-\alpha} \tag{2.6}$$

where $N_t(j)$ is the labour input of firm j. Producers of intermediate goods operate in an environment with sticky prices. Following Calvo (1983), each firm can reset the price of its good in a given period with a probability $1-\theta$; and its price remains unchanged with probability θ, which is common across firms.

A firm that reoptimises in period t chooses the reset price, P_t^*, which maximises the current market value of the profits generated while the price remains effective:

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}^f} \left[Q_{t,t+k} (P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right] \tag{2.7}$$

subject to the demand schedule $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Z_t$[10] and taking the aggregate output of the final good, Z_t, and the aggregate price level in the economy, P_t, as given. The $Y_{t+k|t}$ is the output in period $t+k$ for a firm whose price was last reset in period t. The $Q_{t,t+k}$ is the stochastic discount factor for nominal payoffs and $\Psi_t(Y_{t+k|t}) = W_t \left[Y_{t+k|t}\right]^{\frac{1}{1-\alpha}} \chi_t$ is the cost function. $\chi_t = a_t \exp u_t$ is a shock to

[8] Appendix A.5 derives and linearises the solution of the household's problem and explains the difference of this approach to Preston (2005) in detail.

[9] In particular, equation (2.3) is substituted with a standard no-Ponzi-game condition $\lim_{j\to\infty} R_{t,t+j} B_{t+j} = 0$, where $R_{t,t+j} = \prod_{s=0}^{j-1} \left[\frac{1}{1+r_{t+s}}\right]$.

[10] This schedule follows from the optimality condition in the final good sector. The problem of the final goods producers is standard and is discussed in Appendix A.1

the firm's cost of production and $u_t \sim N(0, \sigma_u^2)$ is the cost push shock.[11] The problem defined above is standard; the only difference from the RE setting is that the expectations operator depends on firms' belief system $\mathcal{P}^f$.

Firms are equipped with a $Q_{t,t+k}$ that is a function of state variables:

$$Q_{t,t+k} = \prod_{j=1}^{k} \delta(Z_{t+j}/Z_{t+j-1})^{-\sigma}(P_{t+j-1}/P_{t+j}) \tag{2.8}$$

where δ and σ are known constants.[12]

Analogously to consumers, firms are assumed to know equilibrium maps from aggregate output and aggregate prices to wages; however, dividends are a choice variable for firms, so they cannot hold exogenous beliefs about their evolution. Therefore, firms internalise the following relation:

$$\{W_t\} = F^f(Z_t, P_t, a_t) \tag{2.9}$$

where $F^f(.)$ is the first element of F^h in (2.4).

The solution of the profit maximisation problem is presented in Appendix A.2. The intermediate goods producers have a belief system that is specified in detail in Appendix A.4 and Section 2.4 and it implies that they are setting the reset price according to:

$$p_t^* = (1 - \delta\theta) \sum_{k=0}^{\infty} (\theta\delta)^k E_t^{\mathcal{P}^f}(\Theta z_{t+k} + p_{t+k}) + u_t \tag{2.10}$$

The optimal reset price in period t is a function of the entire expected future path of aggregate output and the aggregate level of inflation.

2.3 General equilibrium and monetary policy

Monetary policy is conducted using a Taylor-type interest rate rule. Namely, the central bank sets the nominal interest rate in response to deviations in inflation and output from their targeted levels. The log-linearised Taylor rule takes the following form:[13]

$$r_t = \varphi_\pi \pi_t + \varphi_z z_t \tag{2.11}$$

Monetary authorities set an inflation target and the values for the Taylor rule coefficients, φ_π and φ_z in (2.11).

I now find equilibrium conditions in the model economy under Internal Rationality. Combining

[11] The cost push shock can be interpreted as systematic pricing error or it can reflect frictions in the wage-contracting process, as in Clarida et al. (1999).

[12] (2.8) deviates from RE, because $Q_{t,t+k}$ is not a subjective plan about the stream of future consumption. Under RE, the stochastic discount factor is a function of real marginal utilities of consumers. On the contrary, under IR, firms do not know details of households' problem, and vice versa.

[13] Appendix F.1 incorporate a zero-lower bound constraint for nominal interest rates.

the goods and bond market clearing conditions[14] with the household's Euler equation (2.5), leads to the IS curve:

$$z_t = E_t^{\mathcal{P}^h} \sum_{j=0}^{\infty} \delta^j \left((1-\delta) z_{t+j+1} - \frac{1}{\sigma}(r_{t+j} - \pi_{t+j+1}) \right) + \frac{\epsilon_t^d}{\delta} \tag{2.12}$$

and the output gap today depends on the whole future expected path of inflation and the output gap.

The combination of the labour and goods market clearing conditions with the intermediate firm's production function (2.6) and the household's optimal consumption and labour relation (A.10) leads to the following equilibrium relations between real wages and dividends and aggregate output:

$$\frac{W_t}{P_t} = Z_t^{\sigma + \frac{\phi}{1-\alpha}} a_t^{-1}; \frac{D_t}{P_t} = Z_t - a_t^{-1} Z_t^{\sigma + \frac{\phi+1}{1-\alpha}} \tag{2.13}$$

The combination of (2.11) and (2.13) defines F^h in (2.4) and is known to agents.

All producers of intermediate goods that are re-optimising in a given period choose the same reset price, and the fraction of those firms in the economy equals $1-\theta$. Thus inflation evolves according to the following equation:

$$\pi_t = (1-\theta)(p_t^* - p_{t-1}) \tag{2.14}$$

The relation (2.14) holds in equilibrium; however, it is not in the information set of firms. It would have been inconsistent for the firms to hold subjective beliefs about the aggregate inflation level and simultaneously understand how their choice variable maps into it.

The Phillips curve is derived by combining (2.10) and (2.14). Therefore, inflation today depends not only on inflation expectations tomorrow, but also on the entire future path of both agents' expectations about inflation and the output gap in this economy:

$$\pi_t = \kappa \sum_{k=0}^{\infty} (\theta\delta)^k E_t^{\mathcal{P}^f} z_{t+k} + (1-\theta)\delta \sum_{k=0}^{\infty} (\theta\delta)^k E_t^{\mathcal{P}^f} \pi_{t+k+1} + \epsilon_t^\pi \tag{2.15}$$

where $\kappa = \frac{1-\theta}{\theta}(1-\delta\theta)\Theta$. Equation (2.15) is the Phillips curve in an economy in which agents hold subjective beliefs about the evolution of inflation and the output gap.[15] It follows that the Phillips Curve in this economy under Internal Rationality does not have a recursive representation in the sense that inflation today does not only depend on its expectations one period ahead.[16]

[14] All market clearing conditions are formally stated in Appendix A.3

[15] Preston (2005) derives (2.15) and shows that long-horizon expectations matter in economy under learning.

[16] Bullard and Mitra (2002) and many others study the behaviour of the economy under learning by using the recursive form of the Phillips curve, which holds in the RE version of the model and requires agents' knowledge of (2.14) for derivations. On a micro-level knowledge of (2.14) means that firms have two incompatible measures for the inflation process, which is inconsistent with Internal Rationality because (2.14) contradicts the beliefs that inflation is an exogenous process. This inconsistency means that firms in the "Euler Equation" learning models incorrectly use the law of iterated expectations at the moment of formulating the Phillips curve. Nevertheless, to be in line with the learning literature, the model's dynamics delivered by the recursive Phillips curve will be studied in Appendix C.

Equations (2.11) - (2.15) present the full set of equilibrium conditions that are required to close the model; these conditions describe the evolution of all endogenous variables as a function of fundamentals.

2.4 Beliefs' formation

The probability measure $\mathcal{P}$, which I assume is the same for both households and intermediate goods producers, specifies the joint distribution of aggregate inflation and the aggregate output gap, as well as the preference and cost push shocks in the economy.[17]

First of all, similar to RE setting, the agents are equipped with true data-generating processes for the preference and cost push shocks and know the exact values of all characteristics of these processes.

The beliefs system for inflation and output gap is introduced below as an additional building block of the model. This belief system differs from most of the literature on adaptive learning which equips agents with an updating recursion for beliefs without specifying those beliefs. It is important to understand what the agents' view is about the processes for the output gap and inflation to specify internally consistent model of rational agents.[18] The belief system of internally rational agents requires that agents do not make obvious mistakes while learning.

I assume agents believe that both inflation and the output gap follow an unobserved component model of the form:

replaces RLS w/ KF

$$y_t = \beta_t + \zeta_t \tag{2.16a}$$

$$\beta_{t+1} = (1-\rho)\bar{\beta} + \rho\beta_t + \xi_{t+1} \tag{2.16b}$$

where $y_t = [z_t, \pi_t]'$, $0 < \rho \leq 1$ is a known constant, $\bar{\beta}$ is REE and it is derived in Appendix B, and shocks $\zeta_t \sim N(0_{2,1}, \sigma_\zeta^2 * I_2)$, $\xi_t \sim N(0_{2,1}, \sigma_\xi^2 * I_2)$ are independent of each other. Note that the covariance matrices of ξ_t and ζ_t are diagonal, which implies that agents perceive innovations to inflation as independent to innovations to the output gap.[19] A possible simplification is to take $\rho = 1$, but for this paper, it is more reasonable to allow $\rho < 1$ in order to incorporate mean reversion in the beliefs. Consider the case where agents' prior beliefs are centred at the REE with the prior variance

[17]Please refer to Appendix A.4 for the details of $\mathcal{P}$. The Appendix explains that consumers and firms' state spaces have to differ but they are assumed to be mutually consistent. Therefore consumers and firms share the probability measure $\mathcal{P}$.

[18]This is also useful for further extensions of the model, including an incorporation of the zero-lower bound constraint for nominal rates where uncertainty about the future path of rate changes model behaviour even away from the bound. The details are in Appendix F.1.

[19]This is a simplifying assumption which is relaxed in Appendix G.3. This Appendix encompasses the scenario when agents perceive the correct correlation between inflation and output gap that occurs in model equilibrium. This assumption does not qualitatively change the model dynamics.

σ_0^2:

$$\beta_0 \sim N(\overline{\beta}, \sigma_0^2) \tag{2.17}$$

Note that agents' beliefs (2.16) encompass the REE of the model. If in addition to (2.17) the agents believe that $\sigma_\xi^2 = 0$, then the resulting equilibrium is indeed REE. Alternatively, if (2.17) is combined with a belief that σ_ξ^2 is small, even though the resulting dynamics of the economy are not going to be precisely given by REE, it will be close to REE.

The process (2.16) is a standard unobserved component model, and the learning problem arises because agents only observe realisations of y_t and not the permanent and transitory components of the process (2.16). The optimal filter for $E^{\mathcal{P}}(\beta_t|y^t) \equiv \hat{\beta}_t$ is a Kalman filter, which filters out the noise component ζ_t and simultaneously keeps track of permanent innovations ξ_t.

Agents' posterior beliefs are given by $\beta_t \sim N(\widehat{\beta}_t, \sigma_0^2)$, and optimal updating implies the following recursion, which explains the evolution of the beliefs about β_t:

$$\widehat{\beta}_t - \rho\widehat{\beta}_{t-1} = \lambda_t[y_t - \hat{\beta}_{t-1}] \tag{2.18}$$

where $\widehat{\beta}_t$ is an agent's estimate of β_t in (2.16). The λ_t defines a weight that agents assign to the most recent observation to update their forecasts from period $t-1$ to a period t.[20] If $\sigma_\xi^2 > 0$, then the optimal weight of the last observation in (2.18) is given by the steady state Kalman gain λ. The optimal filtering in this environment gives rise to constant gain learning.

2.5 Equilibrium dynamics under learning

The solution of the model under RE is presented in Appendix B. It shows that $\overline{\beta} = 0$ in this model. Under IR, the solution of the model is summarised by (2.11), (2.12), and (2.15) together with (2.18).

It follows from (2.16) and $\overline{\beta} = 0$ that beliefs about inflation and the output gap at k periods ahead are given by:

$$E_t^{\mathcal{P}} y_{t+k} = \rho^{k-1}\hat{\beta}_t \tag{2.19}$$

and these forecasts are updated each period with observed realisations of inflation and output according to (2.18). Since $\rho < 1$, agents internalise the knowledge that deviations from the long-run equilibrium of the output gap and inflation are temporary.

If expectations evolve according to (2.19), it follows that the actual dynamics of the output gap

[20] I fix $\sigma_0^2 = \left(\sigma_\xi^2 - (1-\rho^2)\sigma_\zeta^2 + \sqrt{(\sigma_\xi^2 - (1-\rho^2)\sigma_\zeta^2)^2 + 4\sigma_\zeta^2\sigma_\xi^2}\right)/2$ to be equal to the steady state Kalman filter uncertainty about β_t.

and inflation are a linear function of those expectations:

$$y_t = B^{LH}\beta_t + B_1\epsilon_t^y \tag{2.20}$$

where B^{LH} and B_1 are matrices of coefficients.[21]

I follow the method of Marcet and Sargent (1989) and Evans and Honkapohja (2001) and use the actual law of motion (2.20) and the perceived law of motion (2.16) to formulate the function $T(\beta)$ that maps the agents' expectations about parameters β to their realised values.

$$T(\beta_t) = E(y_t|\beta_t) = B^{LH}\beta_t \tag{2.21}$$

where E is true expectations given the model.

This function ($T(\beta)$) is recovered from the structural model (2.20) and is not known to agents. The fixed point in this mapping is a REE for the model. The T-mapping determines the evolution of beliefs in transition to the long-run equilibrium$-$REE. In other words, if agents believe that inflation is going to be high tomorrow, this expectation will transmit to actual realised inflation through (2.20); both inflation and the output gap respond to that belief according to (2.21). This is a key feature of self-referential learning models, and it is absent in Bayesian learning models. As a result, realised inflation tomorrow will indeed be high relative to RE; however, it will still be slightly lower than expected.[22]

Using the mapping (2.21), the scheme for sequential updating of beliefs (2.18) in the model economy can be rewritten in the following form:

$$\widehat{\beta}_t - \rho\widehat{\beta}_{t-1} = \lambda\left[T(\widehat{\beta}_{t-1}) - \widehat{\beta}_{t-1} + B_1\epsilon_{t-1}^y\right] \tag{2.22}$$

It follows that forecasting mistakes are also a function of agents' beliefs. As the literature highlights, learning generates high persistence in agents' beliefs by construction through setting λ to a small value. My focus is different. I show in Section 3 that when agents learn simultaneously about inflation and the output gap, learning generates conditions when agents make small forecasting mistakes even away from the REE and these mistakes create an additional endogenous source of persistence in beliefs. Therefore, persistent deviations in the beliefs from REE under learning, will transmit to persistent deviations in the actual inflation and the output gap from RE.

[21] For exact formulae for B^{LH} and B_1 and the derivations see Appendix A.6.

[22] When beliefs about inflation are higher than in REE, the realised inflation will be lower than expected because of the expectational-stability. And this is perfectly consistent with (2.16).

3 Model behaviour under IR and Monetary Policy

This section describes the model dynamic, given an initial condition for beliefs that is different from long-run model equilibrium. Unlike most of the literature on learning, the focus of this paper is on the transition dynamics of beliefs and their interaction with the actual economy to model equilibrium. The model makes predictions about the exact relation between the output gap and inflation that persists in the agents' beliefs, given an arbitrary initial condition for those beliefs.

3.1 Learning about inflation and the output gap - some technicalities

The model of the economy under learning is an example of a linear, two-dimensional T-map. A review of some general facts about two-dimensional dynamic systems is useful. Consider the following dynamic system describes an evolution of beliefs:

$$\beta_t = C\beta_{t-1} \tag{3.1}$$

where $\beta_t = [\beta_t^z, \beta_t^\pi]'$, and C is for now an arbitrary matrix with real eigenvalues: $-1 < h_2 < h_1 < 1$.[23] The v_1 and v_2 are the corresponding right eigenvectors, $v_i = [v_{i,1}, v_{i,2}]'$. I also define the eigenlines $V_i = \{(\pi, z) : \pi = \frac{v_{i,1}}{v_{i,2}} z\}$. The β_0 is an arbitrary initial condition for beliefs. The standard analysis under learning is focused on the largest eigenvalue of the matrix C (h_1) which is responsible for expectational stability and for the speed of convergence of beliefs to equilibrium under learning given the initial condition.[24]

However, both eigenvalues (h_1 and h_2) and the corresponding eigenvectors (v_1 and v_2) play a role in the understanding of the model transition to equilibrium. Along the direction of the eigenvector v_2 that corresponds to the lowest eigenvalue, the beliefs move at the speed of the smaller eigenvalue h_2 and, therefore, relatively fast. The beliefs travel fast towards the eigenline V_1 when the difference between the eigenvalues is sufficiently large.[25] Once the system reaches V_1, the movement of beliefs slows because from this moment the further evolution of beliefs to equilibrium along V_1 is mostly controlled by the largest eigenvalue h_1.[26]

[23] Note that (3.1) is a simplification of (2.22) for a particular C and in the absence of shocks.

[24] Section D in the Appendix presents the standard approach to monetary policy design under learning and examines the value of h_1.

[25] This result is formally stated in Proposition 3 in Appendix E.1.

[26] If the initial condition for beliefs is chosen on the eigenline V_1, the speed of the evolution of beliefs is solely described by the eigenvalue h_1, and movement of beliefs is slow because h_1 is close to one. The evolution of beliefs progresses to equilibrium on the eigenline V_1, with the speed h_1. However, even if beliefs lie close to the eigenline V_1, the evolution of beliefs is also controlled by h_1, and therefore slow, and the further path of beliefs takes place along V_1.

For the case of the learning scheme (2.19) C is connected to B^{LH} by:

$$C = \lambda B^{LH} + (\rho - \lambda) I_2 \tag{3.2}$$

where I_2 is the identity matrix. Then, the eigenvectors of matrix C are the same as the eigenvectors of matrix B^{LH}, and the eigenvalues relate in the following way:

$$h_i = \lambda h_i^{B^{LH}} + (\rho - \lambda), i = 1, 2 \tag{3.3}$$

Therefore, matrices B^{LH} and C share eigenvectors, and the eigenvector corresponding to the largest eigenvalue of matrix B^{LH} is the eigenvector vector v_1 of matrix C. If the economy happens to satisfy a condition in which h_1 is close to one and h_2 is not, the system finds itself on V_1 quickly for virtually any initial condition of beliefs.[27]

In other words, if $\beta \in V_1 \Rightarrow T(\beta) = B^{LH}\beta = h_1\beta \simeq \beta$, as $h_1 \simeq 1$. And $T(\beta)$ are actual realized inflation and the output gap given beliefs β according to (2.20) in absence of shocks. It follows, that actual inflation and the output gap are close to their expected values and also evolve around V_1. Therefore, V_1 of matrix B^{LH} defines the relation between actual inflation and the output gap that persists in the model under learning.

The fast movement of beliefs away from V_1 means that if agents are updating their beliefs according to (2.19), then their forecasting mistakes are large, while the evolution of beliefs is dominated by h_2. But, this particular combination of beliefs then will be quickly discarded by agents because the expectations are not matched by actual realisations of inflation and the output gap. On the contrary, V_1 describes the space of beliefs and realisations of inflation and the output gap which is associated with small forecasting mistakes. If h_1 is close to one, then agents' beliefs will be close to the actual realisations of variables and, therefore, will become stuck once they become sufficiently close to the eigenline V_1. Therefore, the slope of the eigenvector v_1 provides the key characteristics for the behaviour of both beliefs and actual economy under learning.

3.2 Expectational channel of Monetary Policy

In a standard DSGE model under IR, when agents simultaneously learn about inflation and the output gap, their expectations about inflation influence the path for the real economy. This environment creates an *expectational channel of monetary policy*, because the evolution of beliefs is highly dependent

[27]The only exception would be when the initial condition belongs to V_2. Given this set of conditions, the beliefs are independent of the eigenvalue h_1, and therefore, even if h_1 is close to one, the beliefs move quickly to REE because h_2 is low. However, in the stochastic environment this dynamic is irrelevant.

upon the values of the Taylor rule coefficients.[28]

Even in the baseline case described in section 2.4 when agents perceive inflation and the output gap as being independent, inflation expectations influence the future path of the output gap and its expectations through equilibrium linkages between those variables. For example, if inflation expectations are high this affects the consumption choices of households. Once actual consumption, and therefore aggregate output, respond to inflation expectations, the output gap expectations adjust as well. I demonstrate below that it is possible to quantify how much inflation expectations feed into evolution of the output gap and vice versa.

In order to quantify the evolution of beliefs and show how much inflation expectations matter for output dynamics, the model is calibrated using the estimates of parameters in Woodford (2003).[29] The Taylor rule coefficients are set to $\varphi_\pi = 1.5$ and $\varphi_z = 0.5$, as in Taylor(1993).

[Figure 3 About Here]

Figure 3 demonstrates the joint behaviour of beliefs about inflation and the output gap in the economy given different initial conditions. Each red curve that starts from the circle maps a trajectory for the beliefs according to (2.22). The blue lines represent the eigenlines V_1 and V_2 of the dynamic system, which were introduced in the previous subsection. For simplicity, we calibrate $\rho = 1$, this value is the benchmark in IR applications in Adam et al. (2016) and Adam et al. (2017). But this benchmark induces unrealistic persistence about the deviations in inflation and output in the minds of the agents. Therefore, considering robustness to $\rho < 1$ is of interest, because the expectational channel should be sensitive to this assumption.[30]

Figure 3 demonstrates that feedback from the output expectations to the inflation expectations is limited in this model. The trajectories with beliefs initialised on the vertical axis, $\beta_0^z \neq 0, \beta_0^\pi = 0$, show this limitation. Along these trajectories, beliefs about the output gap (β_0^z) do not transmit to the inflation expectations to any significant extent and the evolution of beliefs to REE happens close to the vertical axis. This happens because the eigenvector, corresponding to the lowest eigenvalue, V_2, is nearly aligned with the vertical axis.

[28]The expectational channel described in this section does not arise due to a multiplicity of equilibria in a standard DSGE model. In an environment with a unique equilibrium, if the agents' beliefs are not in REE, they bring about an abnormal dynamic in the actual series for inflation and the output gap when compared to their corresponding paths in the REE.

[29]The calibration is presented in Appendix D.

[30]Please refer to Appendix G.2 for robustness analysis.

At the same time, the second eigenvector (v_1) is negatively sloped; therefore it dictates that along V_1, high inflation expectations are accompanied by low output gap expectations. Given the initial condition for the inflation expectations (β_0^π), the output gap expectations quickly adjust so that beliefs arrive close to V_1. And, along this relation between the output gap and inflation expectations, beliefs are persistent. Therefore, high inflation expectations quickly transmit to a drop in expectations about output gap; once beliefs reach V_1, the economy converges slowly to REE along V_1 as the corresponding eigenvalue is close to one.[31]

Another observation is, that while the output gap expectations drop significantly and then go back up, the inflation expectations monotonically progress toward the REE. Once the output expectations reach their turning point, the inflation expectations are still high and remain close to their initial condition. Therefore, in this model, inflation is a slower variable for initial conditions away from V_1, while output is a faster variable and adjusts relatively quickly to innovations to the inflation expectations (when away from V_1).

The main conclusion from Figure 3, is that the inflation expectations are crucial for the evolution of the real economy. If inflation expectations are high, the actual production level drops. At the same time−the higher inflation expectations compared to the REE−a larger drop in the output gap is expected, because the slope of V_1 fixes the proportion of output costs to the inflation expectations. An important feature of the model dynamic under learning is that the dynamic of beliefs slows down not only when beliefs get close to REE, but also when in the neighbourhood of V_1.

To demonstrate why the economy stays away from the REE, Figure 4 plots the beliefs about the output gap and the actual realisations along trajectories produced by recursion (2.22). The recursion is initialised based on three different initial values for the inflation expectations. Further, the dynamic of the output gap is produced by the joint evolution of the expectations for the output gap and inflation along the trajectories.

[Figure 4 About Here]

The left plot on Figure 4 considers the Taylor rule coefficients as for Figure 3. The first observation is that the convergence of beliefs about the output gap to REE is non-monotonic. Initially, these beliefs overshoot the equilibrium in the opposite direction prior to returning to the REE. Near the turning point, the distance between the realised and expected levels of the output gap drops dramatically as

[31] Please refer to Appendix 3.3 for the details.

shown by the black line on Figure 4. Note that each trajectory intersects with the black line, which means that at some point on the way back to equilibrium, the expectations are exactly equal to the realised output gap.[32] Therefore, the movement of beliefs slows down, and the combination of output gap expectations and realisations stays at a similar level for many periods.

What matters in terms of the output and inflation dynamics under IR is that the drop of output associated with high inflation expectations is almost instantaneous. But the recovery of output takes a long time because agents' forecasts, reflecting their belief system, are very close to the actual dynamic of inflation and output, as can be seen in Figure 4. Therefore, there are no reasons for agents to revise those forecasts according to (2.19).

Along the recovery route, inflation, together with inflation expectations, are high. In particular, inflation expectations are higher than the realised inflation. Furthermore, the behavior of interest rates is as follows. If the central bank sets nominal rates according to the same Taylor rule; then along the recovery route, nominal interest rates decrease and come to equilibrium when inflation and output reach the REE.

3.3 Economic intuition for transition dynamic

The economic reasoning behind the transition dynamics presented in Figures 3 and 4 is as follows. For simplicity, in this paragraph I set $\varphi_z = 0$, so that the nominal rate does not respond to the output gap's dynamic. If in Period 1, $\beta_{z,0} = \bar{\beta}_z$ and $\beta_{\pi,0} = \bar{\beta}_\pi + \Delta$, $\Delta > 0$, then a price-setting decision made by firms in the economy, as reflected in the Phillips curve (2.15) leads to upward pressure for inflation in Period 1, and the inflation rate increases by exactly $\frac{(1-\theta)\delta}{1-\theta\delta\rho}\Delta$. At the same time, the central bank is setting a nominal interest rate according to the Taylor rule (2.11) and the described sequence of events dictates an increase in the nominal interest rate that is proportional to the increase in the inflation rate, $\varphi_\pi \frac{(1-\theta)\delta}{1-\theta\delta\rho}\Delta$.

The path of the real interest rate is highly dependent on the inflation expectations and Taylor rule coefficients. For the baseline calibration of the Taylor rule coefficient for inflation, this sequence of events leads to an increase in the real rate ($\varphi_\pi \frac{(1-\theta)\delta}{1-\theta\delta\rho}\Delta - \Delta > 0$) and, therefore, to a decrease in today's consumption and the output gap. This is in line with a dynamic of beliefs observed in the left panel in Figure 4.

It is also interesting to understand how the inflation-expectations-driven decrease in the output gap

[32] At the same time, the inflation expectations are different from realised inflation because this dynamic system has only one fixed point—REE.

accelerates and persists before returning to equilibrium. When the economy moves to Period 2, the inflation expectations are still going to be significantly higher than at the REE. This follows from the adaptive nature of beliefs' formation in 2.18. However, in this period, the output gap expectations are low. They are low because in Period 1 the realised output gap was lower than expected, given that the actions of the central bank in that period interfered with the intertemporal consumption-smoothing motives of rational economic agents. And rational agents will decrease their expectations of output gap for the next period according to (2.22). So, in Period 2, the downward pressure on realised output is coming not only from a policy-initiated increase in the real rate, but also by the expectations for the output gap that are lower than at the REE.

In this setting, the output gap expectations reinforce themselves in a downward spiral. However, when the expectations reach the minimum, the mechanism slowly unwinds: agents begin to be more optimistic about the future path of the economy, thereby correcting the path of actual consumption/output towards the equilibrium level. At the same time, the real rate returns to equilibrium.

3.4 Policy Implications

In this model's economy, monetary policy is responsible for the extent to which inflation expectations influence the path of the expected and actual output gaps. Taylor rule coefficients determine the inflation-output trade-off given by the slope of the V_1 in Figure 3. A desirable policy is one that targets the slope of V_1 and manipulates it according to policy objectives.[33] If V_1 is aligned with the horizontal axis, this alignment would make the output gap independent from innovations to the inflation expectations. The value of the Taylor rule coefficient, which gives a flat V_1, hence it eliminates feedback of high inflation expectations to the output gap, is as follows:[34]

$$\varphi_\pi^* = \frac{1 - \delta\theta\rho}{\delta(2 - \rho\delta - \theta)} \tag{3.4}$$

The middle panel in Figure 4 plots the output gap's trajectories for the Taylor rule that sets $\varphi_\pi = \varphi_\pi^*$. The difference between the path of the output gap is striking compared to the one produced by the baseline Taylor rule (left panel in the same figure). In particular, what used to be distinct trajectories for every initial condition for inflation expectations in the case of the baseline Taylor rule now transforms into a single blue curve.

For the standard calibration of deep parameters in the economy, the value of φ_π^* is greater than but close to one. The right panel of Figure 4 demonstrates that even if φ_π is not precisely set at the value

[33] Appendix G examines the change in the inflation target and robustness of the results to several model extensions.
[34] Appendix E.3 provides details of the slope of v_1 for different values of the Taylor rule coefficients.

stated in (3.4) but is fixed at a value close to one, then the output is much less sensitive to inflation expectations as compared to the baseline case presented in the left panel. The result, which is robust among different calibrations of the model's parameters is that the desirable policy sets φ_π close to one. This policy ensures that the output gap is less sensitive to inflation expectations that are not at the model's equilibrium.

These policy suggestions are quite different from policy recommendations, if policy authorities are trying to anchor inflation expectations, that is ensure they move quickly to an equilibrium. The latter policy would dictate an aggressive reaction to deviations in inflation from the target that are accompanied by a minor reaction to deviations of the output gap.[35] In the case of this particular model this policy brings inflation expectations and actual inflation to equilibrium quickly while ignoring the output gap's dynamics along the way.

If policy satisfies (3.4), it implies that the largest eigenvalue (h_1) is high, therefore, convergence is slow. Moreover, it is higher than is the case for the baseline calibration of the Taylor rule. Therefore, inflation stays away from the REE for a longer period of time. However, output gap is not affected by such expectations. If the output gap's dynamics in the left and the middle/right panels in Figure 4 are compared, it becomes apparent that given an initial condition for beliefs that the output gap and its beliefs move much closer to the neighbourhood of the REE when compared to the baseline calibration in the same number of periods. This reasoning means that a desirable policy accepts high inflation if it is driven by high inflation expectations, because it does not lead to output costs.

4 Comparing the model with the data

This section describes how a standard NK DSGE model under the assumption of Internal Rationality can help to understand the co-movement of inflation, inflation expectations, and the output gap during both the Great Moderation and the Great Recession and the recovery in the US. Simultaneously, this section provides evidence that the model matches survey data better than the alternatives, based on the results of simple tests for the forecasting errors, which are derived from surveys of inflation expectations. Additionally, the model is capable of rationalising a delayed response of inflation expectations to monetary policy shocks, which is observed in the data.

The learning model can potentially be used to study both periods: the Great Moderation and the Great Recession. The model with beliefs that encompass agents being particularly sure about their

[35] Additional details about policy suggestions according to speed of convergence criteria can be found in Appendix D.

prior belief in (2.16) produces the REE itself. It is reasonable to expect that when beliefs are very close to REE, the generated model's behaviour is going to resemble the REE as well.[36] Therefore, the model with beliefs initialised at the REE, can be used to represent the behaviour of the US economy during the period of the Great Moderation. This approach ensures that the behaviour of the economy is close to the one produced by the REE model. However, it is also different enough so that it provides an opportunity for the model to match additional features of the data.

The same model is also capable of generating behaviour which is very different when compared to the REE counterpart: this behaviour requires expectations initialised away from the REE. Whenever agents expect inflation to be higher than the REE, the model predicts that output gap is going to be necessarily low,[37] and this behaviour persists in the model's dynamic. Therefore, it is reasonable to expect that this particular feature of the model dynamic has a chance of explaining the missing disinflation and the slow recovery of the Great Recession.

The only important initial condition that would make it possible for the model to rationalise the observed behaviour in the data is high inflation expectations. The path of inflation expectations presented in Figure 1 confirms that expectations were higher than actual inflation during the Great Recession and its recovery. One possible reason why expectations were high involves a structural break at the beginning of the crisis. At this initial juncture, the steady state moved to a new level with lower inflation and a lower output gap, but beliefs remained with at old equilibrium. Therefore, when compared to the new equilibrium, inflation expectations were high. An additional explanation involves an exogenous shock to inflation expectations. In particular, Coibion and Gorodnichenko (2015) and Coibion et al. (2018) attribute the increase of inflation expectations to oil prices that spiked during the initial phase of the Great Recession in 2009. This event inflated consumers' expectations about the future path of prices, but the effect of oil prices on actual inflation was not as strong. This initiated a departure of inflation expectations from actual inflation and consequently kept inflation high during the recovery period. This change in steady state combined with a shock to inflation expectations during the Great Recession is equivalent to the high initial inflation expectations considered in Section 3. The model is formulated in deviations from equilibrium, therefore a change in the deep equilibrium and a shock to inflation expectation or their combination lead to the same interpretation in terms of

[36] Moreover, Slobodyan and Wouters (2012) and, to some extent, Eusepi and Preston (2011) show that the adaptive learning model initialized at the REE produces behaviour which is very similar to the REE version of the same model.

[37] Bhandari et al. (2016) find that on average when consumer overpredict inflation they simultaneously underpredict output. They find strong co-movement in forecasting biases for inflation and the output gap. The model delivers the exact same sign of the dependence.

initial conditions for inflation expectations.

To summarise, the same model—but with different initial conditions for beliefs—is used to address facts about both the Great Moderation and the Great Recession. The model is simulated in the presence of demand and supply shocks, given the same characteristics for both periods, so that the initial condition for beliefs is the only difference between these two sets of simulations.[38]

4.1 Tests for belief system using survey data

If the survey about inflation expectations conveys information about the true expectations of future price movements ($E_t^{\mathcal{P}}\pi_{t+k}$), then it is possible to construct a test that verifies whether the assumption of REE holds in the data. Forecasting errors in an RE model must be orthogonal to all information that was available and relevant to the agents at the moment of making forecasts. The null hypothesis of the tests presented below encompasses not only the REE model, but also frameworks with limited information that retain the information structure about general equilibrium linkages in the economy.[39]

However, if agents form beliefs about inflation and the output gap according to (2.16), then forecasting errors are not supposed to be necessarily orthogonal to the variables agents observe when making the predictions. Actually, the model under IR suggests that if beliefs play a key role in the dynamics of the economy, then forecasting errors for the inflation process are positive when inflation is higher than at the REE or when the output gap is negative and vice versa.

The survey of inflation expectations ($s_{t,k}$) is considered to be a measure of subjective beliefs of consumers regarding future levels of inflation that are possibly subject to measurement error. Let $E_t^{\mathcal{P}}$ denote a subjective expectations operator using information up to the period t. This operator can potentially differ from the RE operator E_t. Therefore, $s_{t,t+k} = E_t^{\mathcal{P}}\pi_{t,t+k}$ represent an estimate of agents' subjective beliefs about the inflation rate between periods $t+k$ and t. Given the expectations horizon in the Michigan survey of consumers, k stands for 12 months.

$$\pi_{t,t+12} = \alpha_1 + \rho_1\pi_{t-12,t} + \epsilon_t \tag{4.1}$$

$$s_{t,t+12} = \alpha_2 + \rho_2\pi_{t-12,t} + \eta_t \tag{4.2}$$

Under the null hypothesis of the information structure of RE ($H_0 : E_t^{\mathcal{P}} = E_t$), $\hat{\rho}_1$ and $\hat{\rho}_2$ must be estimates of the same regression coefficient, because $\rho_1 = \rho_2$ under RE. This happens because

[38]The model's results for both periods are based on 1,000 independent simulations, and the model's estimates are those given for the median level of statistics.

[39]The exception would be behavioural models where agents' beliefs maintain persistent deviations from the true data generating process. I extend the proof of Proposition 1 in Adam et al. (2017) and show that agents are required to hold rational expectations given their information set that can be different from full information.

$s_{t,t+12} - \pi_{t,t+12}$ is a prediction error of the true data-generating process. If $\pi_{t-12,t}$ is in the information set for the time period t under RE, the prediction error must be orthogonal to $\pi_{t-12,t}$. Therefore, the null hypothesis of RE implies $H_0 : \rho_1 = \rho_2$.[40]

[Table 1 About Here]

Table 1 presents the results of the test. The last column presents the p-values of the test for both periods.[41]

The results presented in Table 1 demonstrate that the null hypothesis is not rejected for the Great Moderation period, which indicates that survey evidence is consistent with rational expectations. However, the data rejects the hypothesis for the Great Recession and the recovery, implying that rational expectations do not find empirical support for this period. Tables 2 and 3 in Appendix H confirm that these findings hold among different surveys of inflation expectations, as well as inflation constructed from different price indexes.

The same test can be conducted on the model-simulated data. If the learning model with beliefs at the REE is simulated, then there is not enough evidence to reject the null hypothesis at the 0.05 significance level 87% of the time. Therefore, the learning model initialized at REE is in line with the survey evidence on inflation expectations for the Great Moderation. However, when the same model is simulated given a different initial condition for beliefs−that is, in transition to the REE−the H_0 is rejected 99.9% of the time at the same significance level. This result is also in line with the results from Table 1. Therefore, the same model is capable of producing dynamics that are supported by survey evidence for both periods.

The model speaks to the relation between output gap and inflation expectations. In particular, the prediction error for inflation expectations in the model is negatively correlated with the output gap if beliefs are sufficiently remote from the long-run equilibrium. Therefore, I construct a test similar to (4.2) which checks how inflation expectations and actual inflation relate to the output or unemployment gaps and whether the forecasting errors of agents is orthogonal to these variables.

[40] Appendix A.8 presents a series of simpler tests based on the properties of the forecasting errors.

[41] The p-values of the test are constructed using a small sample correction procedure which addresses several estimation issues. First, it allows for dependencies among the error terms in (4.2) which are necessary because the error terms are not i.i.d. The variance-covariance matrix of the regressors in (4.2) is also subject to being poorly estimated because of the high persistence in the inflation series and a small sample size. These issues lead to an inflated rate of rejections of the null hypothesis. Therefore, to construct the p-values for the test I rely on Monte-Carlo simulations rather than on asymptotic results. Please refer to Section 2 and Appendix A.3 of Adam et al. (2017) for additional details of the test.

[Table 2 About Here]

The following relation is applied to the data:

$$\pi_{t,t+4} = \alpha_1 + \rho_1 x_{t-1} + \epsilon_t \tag{4.3}$$

$$s_{t,t+4} = \alpha_2 + \rho_2 x_{t-1} + \eta_t \tag{4.4}$$

where x_t is an output (or unemployment) gap in the period t.[42] Table 2 presents the results of the test. Table 2 reports the p-values of the test using the Michigan Survey of Consumers, and Table 4 in Appendix A.8 presents the robustness results using SPF expectations data. The middle panel on each table presents the results from using unemployment gap data, and the right panel utilises the output gap series.

The results of the test confirm the previous findings. The survey evidence indicates that the null hypothesis of REH is difficult to reject for the Great Moderation. However, for the Great Recession and the recovery, the data fails to support the hypothesis. The results are confirmed in simulations. The model-generated data for the Great Moderation rejects the null hypothesis only 6% of the time at 0.025 significance level. But in simulations of the model when the beliefs are initialised away from the long-run equilibrium, the data rejects the hypothesis 88% of the time.

4.2 Monetary policy shocks

This subsection explores whether the same model as above is also capable of addressing additional facts in the data for both periods. I study the response of inflation and inflation expectations to monetary policy both in the data and in model-simulated series.

I estimate a six-variable VAR, similar to Gertler and Karadi (2015), in order to measure responses of inflation and inflation expectations to monetary policy shocks. In particular, the set of variables consists of the monetary policy instrument, the Federal Funds rate and variables of interest; inflation and inflation expectations; and the industrial production growth and financial variables, the last of which includes excess bond premium and mortgage spread.[43] The monetary policy shock is identified using external instruments in a proxy SVAR.[44] I use movements of prices of the Fed Funds futures in

[42] The unemployment gap is proportional to the output gap in models without capital and without a difference between an extensive or intensive margin of labour (Blanchard and Galí (2010))

[43] The results are robust for different specifications—in particular, for the inclusion of VIX in the VAR instead of the mortgage spread as a measure of the financial cycle. The inclusion of financial variables is important for the identification procedure, as explained in Miranda-Agrippino (2016).

[44] see Stock and Watson (2012) and Mertens and Ravn (2013) for the description of the method, and, refer to Gertler and Karadi (2015) for its application in the context of US monetary policy or Gerko and Rey (2017) for application to

a tight window around monetary policy announcements as an external instrument in order to identify the monetary policy shock.

[Figure 5 About Here]

Figure 5 shows that the response of inflation to the shock is as expected: the inflation rate drops in response to the shock in the same period. However, inflation expectations react with a delay and to a lesser extent to monetary policy innovation, when compared to actual inflation.[45]

In order to address this issue, a monetary policy shock is incorporated into the Taylor rule (2.11). The Taylor rule becomes $r_t = \varphi_\pi \pi_t + \varphi_z z_t + \vartheta_t$, where ϑ_t is the monetary policy shock which is orthogonal to all other variables in the economy. The empirical evidence presented in Figure 5 shows that this shock should be modelled as persistent. Therefore, the persistence coefficient is set to 0.75, and the value of the coefficient is assumed to be known to agents under both RE and IR. Figure 6 presents the model-generated responses for inflation and inflation expectations to an increase in ϑ_t under both RE and IR assumptions.

The top panel of Figure 6 presents the response of the economy to a monetary policy shock in the REE. Under RE, when the shock ϑ_t appears, inflation expectations decrease in the same period because agents expect the monetary policy shock to persist and therefore lower inflation for the next period. In the RE setting, agents also know the true mapping of all shocks into aggregate variables. Further, at the same time this knowledge means that agents in the economy adjust their optimal choices, not only to the direct response of the shock to the path of interest rates but also through the adjustment of the output gap and inflation expectations to the shock. At the same time, this adjustment means that inflation expectations respond simultaneously to the shock but to a lesser extent than to the current inflation itself because of the calibration for the persistence of the shock. This evidence contradicts the empirical findings presented on Figure 5.

[Figure 6 About Here]

The same model under IR gives starkly different predictions about the response of inflation expectations to a monetary policy shock. Even though agents observe the shock when it appears and know its persistence, from this knowledge they can only conclude how this shock and its future path

UK monetary policy. Ramey (2016) discusses concerns associated with high frequency identification of monetary policy shocks

[45]Melosi (2016) finds similar results by applying sign restrictions to the identification of monetary policy shocks.

will affect the reset price of an individual firm or the optimal consumption choice of an individual household. Therefore, prices still drop in response to the shock as in the RE version of the model but now the downward pressure on inflation is coming only through the effects of future rates on the consumption choice in (2.5) and its general equilibrium effect on firms' pricing decisions. It follows that the actual inflation rate responds to the shock in a similar manner but through a single channel when compared to the case of the rational expectations solution for the same model.[46] However, even though individual consumers and producers fully understand how the shock and its future path affect their optimal decisions today, according to their belief system this shock is not useful for forecasting inflation. Therefore, the shock will propagate to inflation expectations only after it enters actual inflation and the agents learn about its effect from the inflation data.[47] The bottom panel of Figure 6 plots the dynamic of inflation and inflation expectations in the model. This dynamic is very similar to the empirical evidence.

4.3 How well the model does quantitatively

This subsection explores whether the model has the potential to quantitatively address two main puzzles associated with the Great Recession and the recovery.

4.3.1 Slow Recovery

Figure 7 plots inflation, inflation expectations, and the output gap together with the model's projections for recovery periods from early 1990s recession and from the Great Recession in the US. In each case, the beliefs about are initialized based on the survey evidence and output gap expectations are set in order to match the drop in the output gap.

For the first period, as before, inflation expectations are set to long-run equilibrium values in line with empirical evidence of inflation expectations being close to actual inflation during that period. Given this initial condition, the model's projections generate alignment in inflation and inflation expectations, while output recovers back to equilibrium levels. At the same time, output gap projection is very close to the projection from RE models presented in Figure 7 and is in line with the observed

[46] The same feature of the information structure in the model also addresses the forward guidance puzzle explained in Gertler (2017). Because of the limited response of expectations to innovations to the future path of interest rates, the effect of these innovations is different when compared to the RE version of the same model.

[47] The application of IR to NK DSGE model dampens general equilibrium effects of monetary policy shocks because in this setting current or future monetary policy innovations affect expected path of interest rates but do not change agents' view about evolution of inflation and the output gap. Alternative ways of delivering a similar effect in the NK DSGE models include bounded rationality (Gabaix (2016) and Farhi and Werning (forthcoming)) and dispersed information (Angeletos and Lian (2018)).

dynamic of the output gap. These findings support the evidence that the model under IR can generate behavior close to the REE model.

[Figure 7 About Here]

The projections for recovery from the Great Recession are very different compared to the first case. The model cannot account for the drop in the inflation rate at the beginning of the recovery. However, it matches the dynamic well for the remaining period. For the entire period, inflation expectations remain higher than actual inflation. However, now the projection for the output gap comes close to matching the phenomenon of a slow recovery from the Great Recession, that is particularly difficult to address in a standard DSGE model under RE. Further, the recovery is affected by policy: if the Taylor rule was in line with (3.4), the recovery would have had the same shape as for the first period.

4.3.2 Inflation and Output gap

In order to quantify the missing disinflation puzzle, the output gap is related to inflation for two subsamples by estimating of the following regression:

$$x_t = a + b\pi_t + \epsilon_t^x \tag{4.5}$$

Table 3 presents estimates of the regression coefficients in (4.5). The main difference in the results among the Great Moderation and the Great Recession is the drop in estimates of the intercepts in the latter period. This finding means that for the level of output gap observed in the economy, inflation was too high during the recovery from the Great Recession. If the same relation persisted between those variables as the one estimated during the Great Moderation, inflation should have been much lower. However, inflation remained high at this point in time and this behaviour is known as the missing disinflation puzzle.

[Table 3 About Here]

Table 10 in Appendix presents results of (4.5) in simulations for both periods. In line with Figure 7, the model-delivered dynamics for the recovery from the Great Recession also generates a drop in the estimates of constant terms and therefore can explain why inflation remains high during the recovery. The main reason for this dynamic lies with high inflation expectations, which keep inflation at higher levels when compared to what inflation should have been according to the REE model. This particular explanation is in line with the empirical evidence in Coibion and Gorodnichenko (2015) and Coibion et al. (2018).

4.3.3 Data Moments

Table 4 below presents summary statistics for inflation and inflation expectations data. The striking difference between the two periods is in the dependence between inflation expectations and current observed inflation. For the Great recession and the recovery expectations are, on average, higher than actual inflation. The estimates of coefficient $\hat{c}$, which are presented in the last row of each panel of Table 4, show that for the Great Moderation, consumers' expectations of future inflation are closely linked to the actual inflation they were experiencing at the moment they were surveyed. But for the Great Recession and the recovery, the estimate is greater than one, which means that consumers expected inflation to increase in the future compared to the level they were experiencing. Other findings are that inflation expectations are less volatile and slightly more persistent than actual inflation, even though the persistence of inflation is also high.[48]

Table 5 presents the statistics for the simulations of the learning model with different initial conditions for inflation expectations in two periods. Note that here I am focusing explicitly on the role of inflation expectations in model dynamics, in isolation from other structural differences between the two periods. In particular, the calibration pertains to all other parameters but the initial beliefs are the same.

The model-generated series under IR match several moments of distribution for the inflation and inflation expectations. For the first period, the average level of inflation is aligned with the mean level of inflation expectations. However, for the second period, the model generates a wedge between average levels of inflation and inflation expectations. At the same time, the expectations alone are not capable of accounting for the size of the difference in the average levels of inflation and inflation expectations, as found in the data. For example, the drop in the inflation rate observed during the Great Recession does not fit into the model's dynamics without being initiated by additional disturbances in the model. This dynamic within the model could be delivered by a combination of demand and cost push shocks, which are averaged out in calculations presented in Table 5. Nevertheless, the model delivers a significant wedge between inflation expectations and actual inflation.

[Table 4 About Here] [Table 5 About Here]

At the same time, the model comes close to rationalising the dependence of inflation expectations on observed inflation for both historical periods. The model-delivered estimates of $\hat{c}$ are broadly in

[48] The main reason for the high persistence in the inflation series is the particular method of measuring inflation, which is applied in order to match the forecasting horizon for inflation expectations.

line with those for the actual economy. The model is capable of matching these statistics because of the manner in which inflation expectations are formed in the model. For example, when expectations are in line with actual inflation, inflation is more volatile and therefore, on average, further apart from a long-run equilibrium than the agents' beliefs. This happens because all shocks first affect aggregate variables in the economy before they partially transmit to agents' expectations. This effect leads to an estimate of $\hat{c}$ being high, but still less than one for the first period.

In simulations for the Great Recession, this logic partially changes because of the different assumptions for the initial condition for inflation expectations. In this instance, the dynamic explained above interacts with the transition dynamic of beliefs, together with the actual economy on the way to the long-run equilibrium. Along the transition route, agents over predict inflation, and this channel outweighs the effect of other shocks for the entire period of time and keeps expectations, on average, higher than observed inflation. Therefore, the model delivers an estimate of c that is greater than one, which is in line with the empirical estimates presented in Table 4. The model also delivers inflation expectations that are less volatile than actual inflation. But, given that in simulation, the calibration of variances of shocks in the economy is the same for both periods, the model is not capable of generating an increase in the conditional variance of inflation for the Great Recession.

Finally, the model matches estimates of the persistence coefficients for both series in simulations: $\hat{\rho}$ for both inflation and inflation expectations series are high for all subsamples. For the period of the Great Recession, persistence in expected inflation is estimated to be slightly higher than persistence of actual inflation. The source of this additional persistence is the transition of beliefs to the long-run equilibrium.

5 Conclusions

This paper derives a standard DSGE model under the assumption of Internal Rationality and demonstrates how model dynamics change if inflation expectations depart from the long-run equilibrium. I show that a certain combination of beliefs about inflation and the output gap persists in model dynamics under learning. The model predicts that if inflation expectations and actual inflation are higher than the REE, then output expectations together with realized output are low and these dynamics can persist for years. This feature of the model helps to address the main puzzles associated with the Great Recession and its aftermath: namely, missing disinflation and the slow recovery.

The model is capable of generating sizable deviations in dynamics of inflation and output gap compared to the REE. Policy implications differ substantially in this case. For example, if the Central Banker's main objective is to stabilize inflation, output contracts severely in response to a positive shock to inflation expectations. At the same time, the sensitivity of output to inflation expectations can be eliminated by a mild reaction of interest rates to inflation innovations. These predictions also differ from policy recommendations if the policy is concerned about asymptotic properties of beliefs under learning.

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References

Adam, Klaus, Albert Marcet, and Johannes Beutel, "Stock Price Booms and Expected Capital Gains," *American Economic Review*, August 2017, *107* (8), 2352–2408.

_ , _ , and Juan Pablo Nicolini, "Stock market volatility and learning," *The Journal of Finance*, 2016, *71* (1), 33–82.

_ and _ , "Internal rationality, imperfect market knowledge and asset prices," *Journal of Economic Theory*, 2011, *146* (3), 1224–1252.

Angeletos, George-Marios and Chen Lian, "Dampening General Equilibrium: From Micro to Macro," Working Paper No. 22785, National Bureau of Economic Research 2017.

_ and _ , "Forward guidance without common knowledge," *American Economic Review*, 2018, *108* (9), 2477–2512.

Ball, Laurence and Sandeep Mazumder, "Inflation Dynamics and the Great Recession.," *Brookings Papers on Economic Activity*, 2011, (1).

Bhandari, Anmol, Jaroslav Borovička, and Paul Ho, "Survey Data and Subjective Beliefs in Business Cycle Models," Working Paper No. 22225, National Bureau of Economic Research 2016.

Blanchard, Olivier and Jordi Galí, "Labor Markets and Monetary Policy: A New Keynesian Model with Unemployment," *American Economic Journal: Macroeconomics*, April 2010, *2* (2), 1–30.

Blanchard, Olivier Jean and Charles M Kahn, "The solution of linear difference models under rational expectations," *Econometrica: Journal of the Econometric Society*, 1980, pp. 1305–1311.

Bullard, James and Kaushik Mitra, "Learning about monetary policy rules," *Journal of Monetary Economics*, 2002, *49* (6), 1105 – 1129.

_ and _ , "Determinacy, Learnability, and Monetary Policy Inertia," *Journal of Money, Credit and Banking*, 2007, *39* (5), 1177–1212.

Calvo, Guillermo A, "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 1983, *12* (3), 383–398.

Christiano, Lawrence J, Martin Eichenbaum, and Charles L Evans, "Nominal rigidities and the dynamic effects of a shock to monetary policy," *Journal of Political Economy*, 2005, *113* (1), 1–45.

Clarida, Richard, Jordi Galí, and Mark Gertler, "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature*, 1999, *37*, 1661–1707.

Coibion, Olivier and Yuriy Gorodnichenko, "Is the Phillips curve alive and well after all? Inflation expectations and the missing disinflation," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 197–232.

_ , _ , and Rupal Kamdar, "The Formation of Expectations, Inflation and the Phillips Curve," *Journal of Economic Literature*, 2018, *56* (4), 1447–91.

Eusepi, Stefano and Bruce Preston, "Expectations, learning, and business cycle fluctuations," *The American Economic Review*, 2011, *101* (6), 2844–2872.

Evans, George W. and Seppo Honkapohja, *Learning and Expectations in Macroeconomics*, Princeton University Press, 2001.

Evans, George W and Seppo Honkapohja, "Expectations and the stability problem for optimal monetary policies," *The Review of Economic Studies*, 2003, *70* (4), 807–824.

_ , Eran Guse, and Seppo Honkapohja, "Liquidity traps, learning and stagnation," *European Economic Review*, 2008, *52* (8), 1438–1463.

Farhi, Emmanuel and Iván Werning, "Monetary Policy, Bounded Rationality, and Incomplete Markets," *American Economic Review*, forthcoming.

Ferrero, Giuseppe, "Monetary policy, learning and the speed of convergence," *Journal of Economic Dynamics and Control*, 2007, *31* (9), 3006 – 3041.

Gabaix, Xavier, "A behavioral New Keynesian model," Working Paper No. 22954, National Bureau of Economic Research 2016.

Galí, Jordi, *Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework and its applications*, Princeton University Press, 2015.

Gaspar, Vitor, Frank Smets, and David Vestin, "Adaptive learning, persistence, and optimal monetary policy," *Journal of the European Economic Association*, 2006, *4* (2-3), 376–385.

Gerko, Elena and Hélène Rey, "Monetary Policy in the Capitals of Capital," *Journal of the European Economic Association*, 2017, *15* (4), 721–745.

Gertler, Mark, "Rethinking the Power of Forward Guidance: Lessons from Japan," Working Paper No. 23707, National Bureau of Economic Research 2017.

_ **and Peter Karadi**, "Monetary policy surprises, credit costs, and economic activity," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 44–76.

Hall, Robert, "The Long Slump," *American Economic Review*, 2011, *101* (2), 431–69.

_ , "Macroeconomics of persistent slumps," *Handbook of Macroeconomics*, 2016, *2*, 2131–2181.

Kozlowski, Julian, Laura Veldkamp, and Venky Venkateswaran, "The tail that wags the economy: Belief-driven business cycles and persistent stagnation," Working Paper No. 21719, National Bureau of Economic Research 2017.

Kreps, David M, "Anticipated utility and dynamic choice," *Econometric Society Monographs*, 1998, *29*, 242–274.

Lindé, Jesper, Frank Smets, and Rafael Wouters, "Challenges for Central Banks' macro models," *Handbook of Macroeconomics*, 2016, *2*, 2185–2262.

Malmendier, Ulrike and Stefan Nagel, "Learning from inflation experiences," *The Quarterly Journal of Economics*, 2016, *131* (1), 53–87.

Marcet, Albert and Thomas J Sargent, "Convergence of least-squares learning in environments with hidden state variables and private information," *The Journal of Political Economy*, 1989, pp. 1306–1322.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The power of forward guidance revisited," *The American Economic Review*, 2016, *106* (10), 3133–3158.

Melosi, Leonardo, "Signalling Effects of Monetary Policy," *The Review of Economic Studies*, 2016, *84* (2), 853–884.

Mertens, Karel and Morten O Ravn, "The dynamic effects of personal and corporate income tax changes in the United States," *The American Economic Review*, 2013, *103* (4), 1212–1247.

Milani, Fabio, "Expectations, learning and macroeconomic persistence," *Journal of Monetary Economics*, 2007, *54* (7), 2065–2082.

Miranda-Agrippino, Silvia, "Unsurprising shocks: information, premia, and the monetary transmission," Working papers 626, Bank of England 2016.

Negro, Marco Del, Marc Giannoni, and Christina Patterson, "The forward guidance puzzle," Staff Report No. 574, Federal Reserve Bank of New York 2015.

_ , Marc P Giannoni, and Frank Schorfheide, "Inflation in the great recession and new keynesian models," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 168–196.

Orphanides, Athanasios and John Williams, "Imperfect knowledge, inflation expectations, and monetary policy," in "The inflation-targeting debate," University of Chicago Press, 2005, pp. 201–246.

Preston, Bruce, "Learning about Monetary Policy Rules when Long-Horizon Expectations Matter," *International Journal of Central Banking*, 2005, *1*.

Ramey, Valerie A, "Macroeconomic shocks and their propagation," *Handbook of Macroeconomics*, 2016, *2*, 71–162.

Sargent, Thomas J, *The conquest of American inflation*, Princeton University Press, 2002.

Slobodyan, Sergey and Raf Wouters, "Learning in a Medium-Scale DSGE Model with Expectations Based on Small Forecasting Models," *American Economic Journal: Macroeconomics*, April 2012, *4* (2), 65–101.

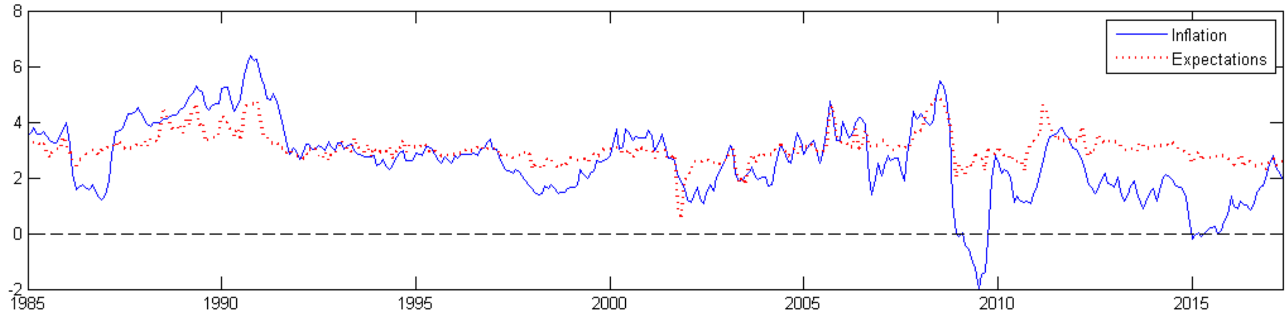
Smets, Frank and Raf Wouters, "An estimated dynamic stochastic general equilibrium model of the euro area," *Journal of the European Economic Association*, 2003, *1* (5), 1123–1175.

Stock, James H and Mark W Watson, "Disentangling the Channels of the 2007-2009 Recession," *Brookings Papers on Economic Activity: Spring 2012*, 2012, pp. 81–156.

Woodford, Michael, *Interest and prices: foundations of a theory of monetary policy*, Princeton university press, 2003.

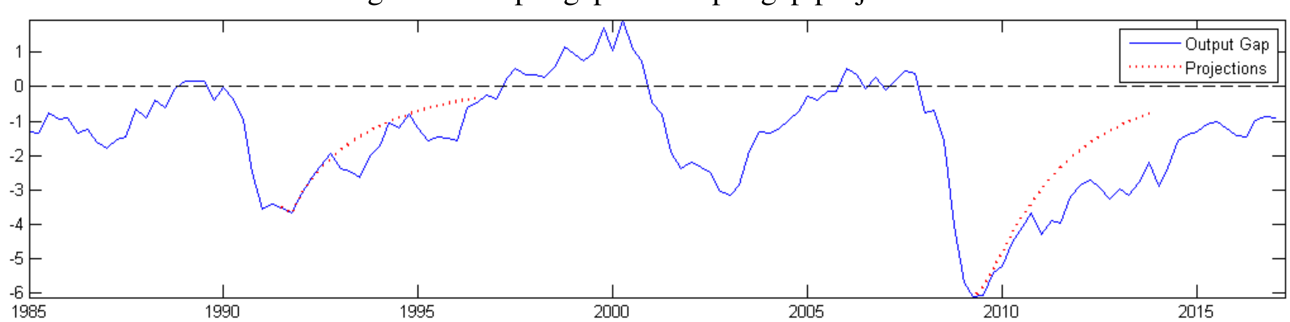
_ , "Macroeconomic analysis without the rational expectations hypothesis," *Annual Review of Economics*, 2013, *5* (1), 303–346.

Figure 1: Inflation and inflation expectations



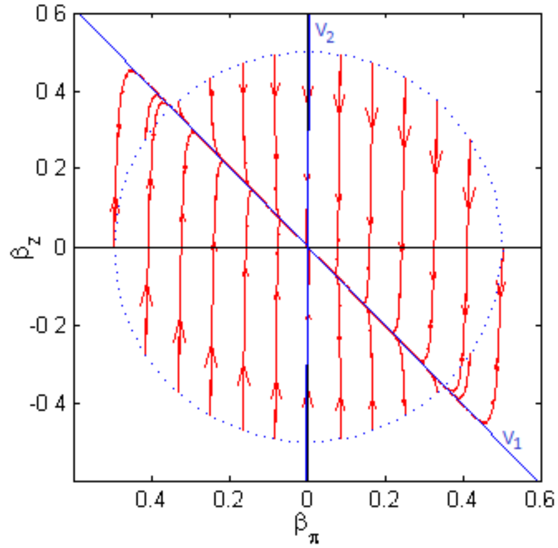
The solid line plots inflation series and the dashed line plots inflation expectations. Inflation is measured as a yearly percentage change in the CPI index. Inflation expectations are constructed from the Michigan Surveys of Consumers. Data sources are presented in Appendix H.

Figure 2: Output gap and output gap projections



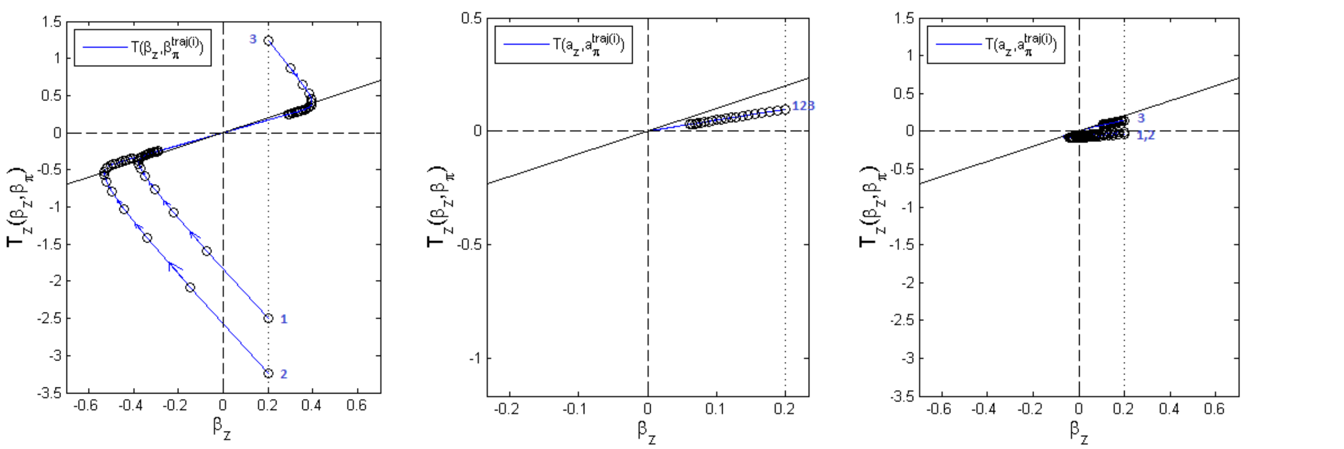
The solid line plots the output gap which is measured as a percentage deviation in the real gross domestic product from real potential GDP (CBO's estimates).. Data sources are presented in Appendix H. The red curves plot the projections for the output gap from an AR(2) process, which are roughly in line with those of sophisticated DSGE models based on the seminal models of Christiano et al. (2005) and Smets and Wouters (2003).

Figure 3: Trajectories of beliefs



The figure presents the trajectories for beliefs in the space of inflation expectations β_π and the output beliefs β_z with different initial conditions. The initial conditions for beliefs are fixed on a circle (dotted lines on each panel) so that the distance of the initial condition from the model's equilibrium, $(0, 0)$, is the same in each case. Each red curve that starts from the circle maps a trajectory for the beliefs according to (2.22). The arrows demonstrate the direction of the evolution according to (2.22). The blue lines represent the eigenlines V_1 and V_2 of the dynamic system.

Figure 4: Trajectories of T-map



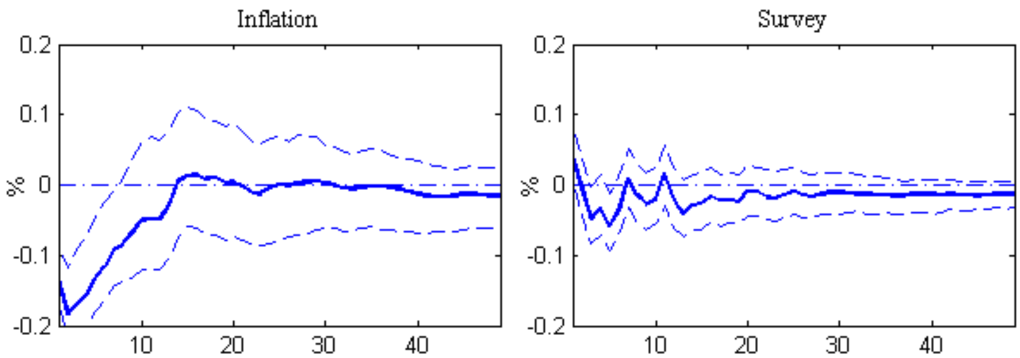
(a) Baseline calibration

(b) $\varphi_\pi = \varphi*$

(c) $\varphi_\pi = 1/\delta$, $\varphi_z = 0.02$

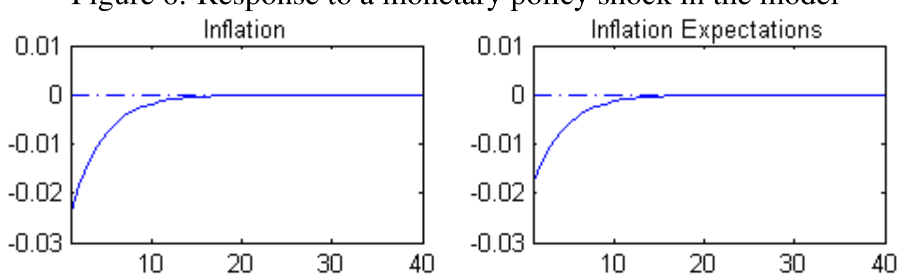
The figure plots the trajectories for beliefs about the output gap and its actual realisations given different initial conditions. Each panel considers different values of coefficients in a Taylor rule. Numbers 1-2-3 on each panel mark the location of the realised output gap, given the initial condition: **1.** $\beta_0^\pi = 0.5$; **2.** $\beta_0^\pi = 0.7$; **3.** $\beta_0^\pi = -0.5$ and $\beta_{z,0} = 0.2$ as marked by the vertical dotted line. Each blue curve represents the evolution of an actual output gap in time. The arrows define the direction of this dynamic given the initial conditions. The dots along each trajectory mark the location of the beliefs for the first 20 periods of $\hat{\beta}_t$ according to (2.18) after the economy begins evolving for each initial condition. A solid black line marks the exact area where the expectations are equal to the actual realised output gap. The distance between each point on the trajectory and the black line measures a forecasting mistake for a given combination of beliefs about the output gap and inflation.

Figure 5: Monetary Policy Shock

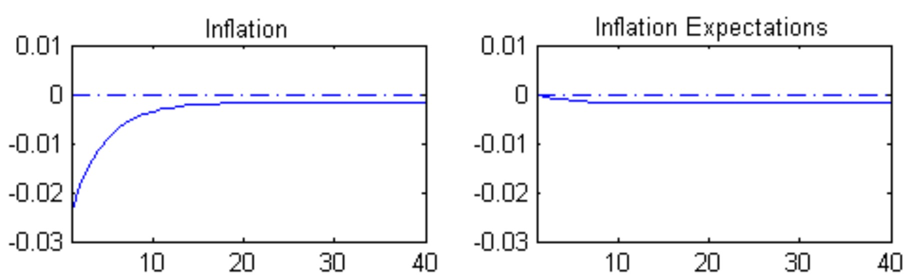


Response of variables to a 20bp increase in the Fed Funds rate. High frequency identification. The dashed lines mark 90% confidence intervals. F-stat:20.

Figure 6: Response to a monetary policy shock in the model

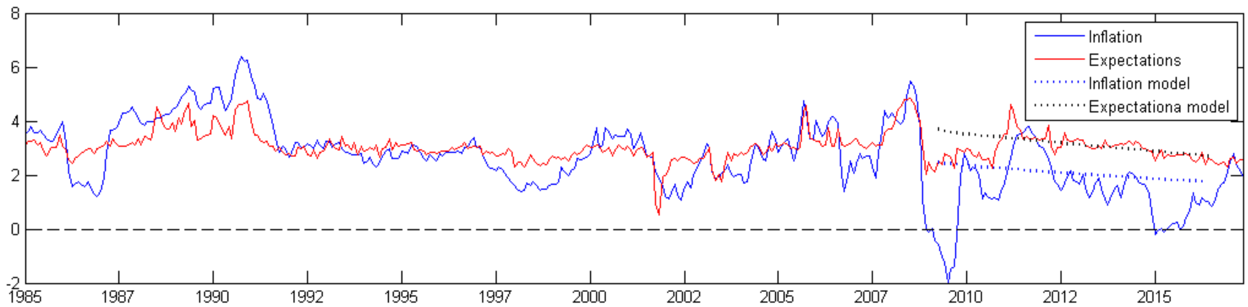


(a) RE



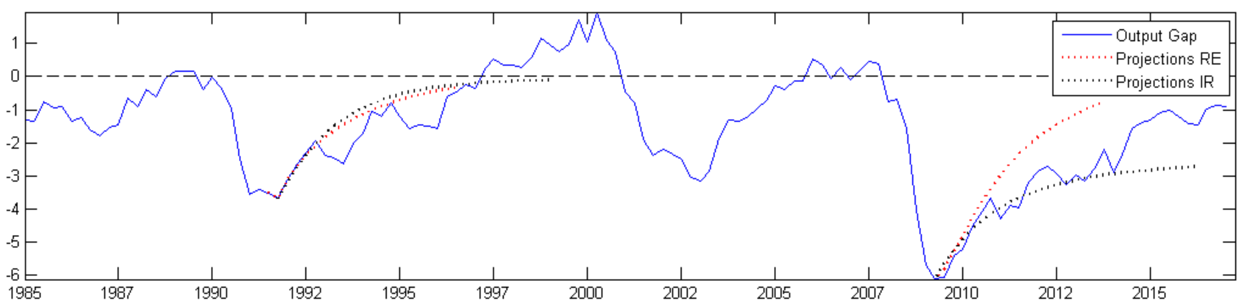
(b) IR

Figure 7: Data and model-delivered dynamic



(a) Inflation, inflation expectations and model projections

The solid line plots inflation series and the dashed line plots inflation expectations. Inflation is measured as a yearly percentage change in the CPI index. Inflation expectations are constructed from the Michigan Surveys of Consumers. Data sources are presented in Appendix H. The black curves plot the projections of the model under IR.



(b) Output gap and model projection

The solid line plots the output gap which is measured as a percentage deviation in the real gross domestic product from real potential GDP (CBO's estimates). Data sources are presented in Appendix H. The red curves plot the projections for the output gap from an AR(2) process, which are roughly in line with those of sophisticated DSGE models based on the seminal models of Christiano et al. (2005) and Smets and Wouters (2003). The black curves plot the projections of the model under IR.

Table 1: Test results for subsamples

Time period	ρ_1	ρ_2	P-value (ss correction) $H_0: \rho_1 \geq \rho_2$
JAN1985-NOV2007	0.37 (2.72)	0.48 (12.24)	0.37
DEC2007-MAY2016	-0.13 (-0.82)	0.42 (8.12)	0.03

Notes: *t* statistics in parentheses, HAC covariance estimator

The Table presents the results of the test (4.2)

Table 2: Test results with slack variables

Time period	ρ_1	ρ_2	P-value $H_0: \rho_1 \leq \rho_2$	ρ_1	ρ_2	P-value $H_0: \rho_1 \geq \rho_2$
	Unemployment Gap			Output Gap		
JAN1985-NOV2007	-0.35 (-1.43)	-0.36 (-4.08)	0.11	0.1 (0.88)	0.06 (0.99)	0.13
DEC2007-MAY2016	0.62 (3.06)	-0.30 (-2.18)	0.008	-0.51 (-2.60)	0.24 (1.95)	0.013

Notes: *t* statistics in parentheses, HAC covariance estimator
The Table presents the results of the test (4.4)

Table 3: Missing Disinflation

	$\hat{\mathbf{a}}$	$\hat{\mathbf{b}}$
JAN1985-NOV2007	-0.97 [-1.77; -0.16]	0.02 [-0.24; 0.27]
DEC2007-MAY2016	-3.47 [-4.3; -2.6]	0.39 [0.03; 0.75]

The Table presents the estimates of coefficients in (4.5).

Table 4: Statistics in the data

	Inflation Expectations ($s_{t,t+12}$)	CPI inflation ($\pi_{t-12,t}$)
JAN1985-NOV2007		
mean	3.04	3.05
$\hat{\rho}$	0.996	0.996
$v\hat{a}r(x_t \vert x_{t-1})$	0.09	0.11
$\hat{c}$	0.93	
DEC2007-MAY2016		
mean	3.14	1.69
$\hat{\rho}$	0.99	0.96
$v\hat{a}r(x_t \vert x_{t-1})$	0.09	0.25
$\hat{c}$	1.13	

$x_t = \rho x_{t-1} + \epsilon_t^x$; $s_{t,t+12} = c\pi_{t-12,t} + \epsilon_t^\pi$

Table 5: Statistics in model simulations

	Inflation Expectations ($s_{t,t+12}$)	Inflation ($\pi_{t-12,t}$)
JAN1985-NOV2007		
mean	3.00	3.01
$\hat{\rho}$	0.999	0.999
$v\hat{a}r(x_t \vert x_{t-1})$	0.00	0.02
$\hat{c}$	0.92	
DEC2007-MAY2016		
mean	2.1	1.75
$\hat{\rho}$	0.997	0.994
$v\hat{a}r(x_t \vert x_{t-1})$	0.00	0.02
$\hat{c}$	1.15	

$x_t = \rho x_{t-1} + \epsilon_t^x$; $s_{t,t+12} = c\pi_{t-12,t} + \epsilon_t^\pi$

Online Appendix to
Expectations and Monetary Policy

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June 27, 2019

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A Additional model details

A.1 Final goods producers

The single final good in the economy is produced from intermediate goods by perfectly competitive firms, subject to a standard CES production function:

$$Z_t = \left(\int_0^1 Y_t(j)^{1-\frac{1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \tag{A.1}$$

where Z_t is a quantity of the final good and $Y_t(j)$ is a quantity of the intermediate good j used in the production of the final good.

The optimality condition in the final good sector leads to the following demand function for the intermediate good j:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Z_t \tag{A.2}$$

where P_t is a price of the final good (aggregate price index) and $P_t(j)$ stands for the price of the intermediate good j. The aggregate price index is given by $P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$.

The final goods producers are the only agents in the model who are observing the whole distribution of the intermediate goods' prices and know how the price of the final consumption good is formulated. It follows, that their state space is very different from other agents' in the economy. However, in this model economy, final producers' role is limited to transformation of intermediate goods inputs into the consumption good, plus aggregation of the distribution of the intermediate goods prices into a single price for the final good. Therefore, they do not communicate their knowledge to other agents in the economy.

A.2 Intermediate goods producers

The solution of the profit maximisation problem 2.7 leads to the following expression for the optimal reset price:

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}^f} \left[\delta^k \left(Z_{t+k} \right)^{1-\sigma} P_{t+k}^{\epsilon} MC_{t+k|t} \right]}{\sum_{k=0}^{\infty} \theta^k E_t^{\mathcal{P}^f} \left[\delta^k \left(Z_{t+k} \right)^{1-\sigma} P_{t+k}^{\epsilon-1} \right]} \tag{A.3}$$

where the real marginal costs are a known function:

$$MC_{t+k,t} = \frac{1}{1-\alpha} \frac{W_{t+k}}{P_{t+k}} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\frac{\epsilon\alpha}{1-\alpha}} Z_{t+k}^{\frac{\alpha}{1-\alpha}} \chi_t \tag{A.4}$$

The standard log-linearisation of an optimal decision on price setting (A.3) together with (A.4) and (2.9) leads to a well-known relation.

$$p_t^* = (1 - \delta\theta) \sum_{k=0}^{\infty} (\theta\delta)^k E_t^{\mathcal{P}^f} \left(\Theta z_{t+k} + p_{t+k} \right) + u_t \tag{A.5}$$

A.3 Market clearing conditions and Aggregate price level

In equilibrium the goods market clears:

$$Z_t = \int_0^1 C_t^i di = C_t \tag{A.6}$$

at all t. Bond holdings are assumed to be in zero net supply:

$$B_t = 0 \tag{A.7}$$

Market clearing in the labour market requires that at all t:

$$\int_0^1 N_t(j)dj = \int_0^1 N_t^i di \tag{A.8}$$

Intermediate goods producers are using the same production technology (2.6) and are facing the same probability of resetting prices for their goods, $1-\theta$. Therefore, all firms that are re-optimizing in a given period are going to choose the same reset price, and the fraction of those firms in the economy equals $1-\theta$. Combining this fact with the definition of the aggregate price level results in the following mapping between inflation and optimal reset price:

$$P_t = \left[\theta(P_{t-1})^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{A.9}$$

A.4 Further details on agents' state spaces

This subsection gives the details about the state spaces of agents in the economy. I aim for simplicity, so in principle it is desirable to introduce the same beliefs for all agents in the economy. However, because state spaces differ for households and firms, this is not possible. For example, firms choose dividends, but households take dividends as given while solving for optimal consumption plans. This subsection explains how the original state spaces of each agent are reduced to a set of variables for which they share the probability measure. Importantly, I assume that all agents hold the same beliefs about common variables (z_t, π_t) and true maps of (z_t, π_t, a_t, u_t) for other variables.

Households A household i solves its utility maximisation problem (2.1)-(2.3) by taking aggregate price level, preference shocks, wages, dividends and the nominal interest rate as given. It follows that the underlying state space Ω^h consists of the space of realisations of those variables. More precisely, an element $\omega^h \in \Omega^h$ is given by $\omega^h = \{P_t, W_t, D_t, r_t, a_t\}_{t=0}^{\infty}$ because they enter the feasibility constraints of households. The household's subjective probability measure $\mathcal{P}^h$ specifies the joint distribution of ω^h.

Additionally, I assume that the household is equipped with the knowledge of several mappings (2.4), which reduces its information set to a minimum number of variables in this setting. This is done in order to minimise the extent of deviations from the RE version of the model and to keep the model tractable. In particular, the household realises how the central bank sets the nominal interest rate (2.11), including the values of the Taylor rule coefficients and the inflation target. Furthermore, households are fully aware of the equilibrium mappings of the aggregate output and preference shock to real wages and dividends (2.13). Therefore, (2.4) is actually given by the joint knowledge of (2.11) and (2.13). Given that the households recognise and take into account the equilibrium maps of the remaining set of exogenous variables into wages, dividends, and the nominal interest rates, these variables carry only redundant information and without loss of generality can be substituted in the underlying household's state space with aggregate output Z_t.

Is it rational for the household to hold subjective beliefs about aggregate output and inflation?

Given that the household takes prices as given, it seems natural that it holds subjective beliefs about the aggregate price level. A less obvious assumption is that household i treats the aggregate output as an exogenous variable and holds subjective beliefs about its evolution. Aggregate output is consumed by a continuum of households, and from the stance of a single household i its consumption or savings decision does not affect the aggregate level of production. In other words, analogous to the RE setting of the same model, household i does not internalise the reality that it is a representative agent in the economy. Under the standard assumption of lack of common knowledge, it is impossible for this household to realise that the economy is populated by identical agents. Therefore, a household holds subjective beliefs about aggregate output even though in equilibrium, the level of aggregate output will be dictated by a household's choice of consumption level.[2]

Firms Producer j of intermediate goods chooses the optimal reset price for his or her good by maximising the expected stream of profits (2.7) and takes aggregate production, aggregate price level, wages, stochastic discount factor, cost push, and preference shocks as given. It follows that the typical element $\omega^f \in \Omega^f$ of the underlying firm's state space Ω^f is given by $\omega^f = \{P_t, Z_t, W_t, Q_t, a_t, u_t\}_{t=0}^{\infty}$, and the firm's subjective probability measure $\mathcal{P}^f$ specifies the joint distribution of ω^f. Similar to that of households, the firm's state space is reduced by equipping it with the exact equilibrium mapping of the remaining set of exogenous variables into real wages (2.13) and into the stochastic discount factor (2.8). Therefore, without loss of generality, these variables can be excluded from the firm's state space.

The producers of intermediate goods are price setters in the economy. However, they do not possess sufficient knowledge to conjecture on the relation between the aggregate price level and their decision on the reset price in every period. As with households, the lack of common knowledge prevents an individual firm from discovering how its reset price maps into the aggregate price level, and therefore it is consistent that it holds subjective beliefs about the aggregate price level.

Therefore, $\mathcal{P}^h$ and $\mathcal{P}^f$ can be simplified to a probability measure that specifies the joint distribution of the histories of aggregate output, aggregate price level, and exogenous disturbances. Specifically, the probability space is given by $(\Omega, \mathcal{P})$ with a typical element $\omega \in \Omega$, $\omega = \{P_t, Z_t, a_t, u_t\}_{t=0}^{\infty}$. In addition, it is further assumed that households and firms share this probability measure.[3] However, this does not restrict their views about other variables. For example, the firm chooses the dividend, but the household takes the dividend as a given function of fundamentals while solving its maximisation problem.

[2] Another concern could be that given that all agents are identical, bond trades never happen in equilibrium, but the household's savings plans are never updated when given this information. Whether household i is not borrowing today, even though it might have expected to borrow, is an irrelevant signal for the behaviour of aggregates in the economy under lack of common knowledge. Heterogeneity among households can potentially be introduced: it would generate bond trading, thereby eliminating this paradox.

[3] The probability measure for inflation and output is assumed to be given by a specific stochastic process, which is close to the actual processes for aggregate inflation and aggregate output, delivered by the model in equilibrium.

The main difference from the standard RE setting of the same model is that under RE agents' state space consists only of the shocks. Under RE, agents understand the equilibrium mapping of shocks into aggregate inflation and output, and the information contained in those series turns out to be redundant.[4]

A.5 Derivation of LH problem

The agent's optimal plan is characterised by the standard first order conditions:

$$\frac{(N_t^i)^{\phi}}{a_t(C_t^i)^{-\sigma}} = \frac{W_t}{P_t} \tag{A.10}$$

$$\frac{1}{1+r_t} = \delta E_t^{\mathcal{P}^h}\left[\frac{a_{t+1}}{a_t}\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{A.11}$$

Equation (A.10) defines the consumption and labour choice, and equation (A.11) is the Euler equation which pins down the intertemporal consumption decision for household i.

The main difference from the analysis of Preston (2005) is that internally rational households that are solving the problem (2.1), (2.2) and (2.3) cannot take their income today or in future periods as given. These households realise that their income is endogenous and depends on their choice of hours in every period. Therefore, it would be internally inconsistent for them to substitute aggregate output in the economy in their budget constraint. Instead, they use the first order conditions for utility maximisation, (A.10) and (A.11), together with their beliefs (2.4) to find an optimal consumption plan.

The flow budget constraint, given (2.3), can be written in the following form:

$$B_{t-1} = \sum_{j=0}^{\infty} R_{t,t+j}P_{t+j}C_{t+j} - \sum_{j=0}^{\infty} R_{t,t+j}\left(W_{t+j}N_{t+j} + D_{t+j}\right) \tag{A.12}$$

The household needs to take into account long-horizon expectations about the income stream in order to choose a consumption level today. Substituting first order condition (A.10) together with (2.4) into budget constraint (A.12) leads to the following expression:

$$\begin{aligned} B_{t-1} = \sum_{j=0}^{\infty} R_{t,t+j}P_{t+j}C_{t+j} - &\\ \sum_{j=0}^{\infty} R_{t,t+j}P_{t+j}&\left(a_{t+j}^{-1}Z_{t+j}^{\sigma+\frac{\phi}{1-\alpha}}\left[Z_{t+j}^{\sigma+\frac{\phi}{1-\alpha}}C_{t+j}^{-\sigma}\right]^{\frac{1}{\phi}} + Z_{t+j} - a_{t+j}^{-1}Z_{t+j}^{\sigma+\frac{\phi+1}{1-\alpha}}\right) \end{aligned} \tag{A.13}$$

Loglinearizing A.13 around zero-inflation steady state and applying a standard approximation $\exp x \approx 1 + x$

[4] Even though under IR, the agents' perceived probabilities about ω are not going to be equal to the equilibrium distribution of inflation and output, agents' perception is defined in such a way so that the agents are not making obvious mistakes while learning. In particular, the beliefs are going to be close to the observed behaviour, because agents will be optimally updating their views about inflation and the output gap with actual data.

one can arrive at:

$$B_{t-1} = \hat{P}\hat{Z}\sum_{j=0}^{\infty}\delta^j(c_{t+j} - z_{t+j}) + \frac{\sigma}{\phi}\hat{P}\hat{Z}^{\sigma+\frac{\phi+1}{1-\alpha}}\sum_{j=0}^{\infty}\delta^j(c_{t+j} - z_{t+j}) \tag{A.14}$$

or, alternatively:

$$\tilde{B}_{t-1} = \sum_{j=0}^{\infty}\delta^j(c_{t+j} - z_{t+j}) \tag{A.15}$$

where $\tilde{B}_{t-1} = \frac{B_{t-1}}{\hat{P}\hat{Z}+\frac{\sigma}{\phi}\hat{P}\hat{Z}^{\sigma+\frac{\phi+1}{1-\alpha}}}$ and this is different from Preston (2005). Taking expectations from both sides of A.15 results in an expression which means that the expected discounted difference between consumption and income streams must be equal to initial savings:

$$\widetilde{B}_{t-1} = E_t^{\mathcal{P}^h}\sum_{j=0}^{\infty}\delta^j(c_{t+j} - z_{t+j}) \tag{A.16}$$

Log-linearization of A.11 around the steady state leads to a well-known optimality condition:

$$c_t = E_t^{\mathcal{P}^h}c_{t+1} - \frac{1}{\sigma}\left(r_t - E_t^{\mathcal{P}^h}\pi_{t+1}\right) + \epsilon_t^d \tag{A.17}$$

This condition can be expressed in the following form:

$$E_t^{\mathcal{P}^h}c_{t+k} = c_t + \frac{1}{\sigma}E_t^{\mathcal{P}^h}\sum_{j=0}^{k-1}(r_{t+j} - \pi_{t+j+1}) - \epsilon_t^d \tag{A.18}$$

Substituting A.18 into intertemporal budget constraint A.15:

$$\widetilde{B}_{t-1} = E_t^{\mathcal{P}^h}\sum_{j=0}^{\infty}\delta^j\left(c_t + \frac{1}{\sigma}E_t^{\mathcal{P}^h}\sum_{k=0}^{j-1}(r_{t+1} - \pi_{t+k+1}) - \epsilon_t^d\right) - E_t^{\mathcal{P}^h}\sum_{j=0}^{\infty}\delta^j z_{t+j} \tag{A.19}$$

From A.19 it follows that consumption today is pinned down by the weighted sum of savings and expected discounted future earnings corrected by future movements in interest rates and inflation:

$$c_t = (1-\delta)\widetilde{B}_{t-1} + E_t^{\mathcal{P}^h}\sum_{j=0}^{\infty}\delta^j\left((1-\delta)z_{t+j} - \delta\frac{1}{\sigma}(r_{t+j} - \pi_{t+j+1})\right) + \epsilon_t^d \tag{A.20}$$

A.6 Model equilibrium under learning and E-stability

Combining the system of structural equations (2.11), (2.12) and (2.15) with the forecasting model (2.19) results in the following system of linear equations:

$$z_t = \frac{1-\delta}{1-\rho\delta}\beta^z - \sigma^{-1}(\varphi_\pi\pi_t + \varphi_z z_t - \beta^\pi) - \frac{\delta}{\sigma(1-\rho\delta)}\left((\varphi_\pi - \rho)\beta^\pi + \varphi_z\beta^z\right) \tag{A.21}$$

$$\pi_t = \kappa z_t + \frac{\kappa\theta\delta}{1-\theta\delta\rho}\beta^z + \frac{(1-\theta)\delta}{1-\theta\delta\rho}\beta^\pi \tag{A.22}$$

where β^z is a forecast of output gap for the next period and β^π is a forecast of inflation. After rearranging terms:

$$\text{(A.23)}\quad \pi_t = \left(\frac{(\sigma(1-\delta)-\varphi_z\delta)\kappa(1-\theta\delta\rho)+(\sigma+\varphi_z)\kappa\theta\delta(1-\rho\delta)}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}\right)\beta^z + \left(\frac{\kappa(1-\theta\delta\rho)((1-\rho\delta)+(\varphi_\pi-\rho)\delta)+(1-\rho\delta)(\sigma+\varphi_z)(1-\theta)\delta}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}\right)\beta^\pi$$

$$\text{(A.24)}\quad z_t = \left(\frac{\sigma(1-\delta)-\varphi_z\delta}{(1-\rho\delta)(\sigma+\varphi_z)} - \frac{\varphi_\pi}{\sigma+\varphi_z}\left(\frac{\kappa(\sigma+\delta(\varphi_z(\theta-1)+\sigma(\theta-1-\rho\theta)))}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}\right)\right)\beta^z + \left(\frac{(1-\rho\delta)+(\varphi_\pi-\rho)\delta}{(1-\rho\delta)(\sigma+\varphi_z)} - \frac{\varphi_\pi}{\sigma+\varphi_z}\frac{\kappa(1-\theta\delta\rho)(1+\delta(\varphi_\pi-2\rho))+(1-\rho\delta)(\sigma+\varphi_z)(1-\theta)\delta}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}\right)\beta^\pi$$

Therefore, the dynamic system can be presented in the following manner:

$$\text{(A.25)}\qquad y_t = B^{LH}\beta_t + B_1\epsilon_t^y$$

where B_1 is the same as in (B.4) and $B^{LH} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and its elements are as follows:

$$\text{(A.26)}\quad b_{11} = \frac{\sigma(1-\delta)-\varphi_z\delta}{(1-\rho\delta)(\sigma+\varphi_z)} - \frac{\varphi_\pi}{\sigma+\varphi_z}\left(\frac{\kappa(\sigma+\delta(\varphi_z(\theta-1)+\sigma(\theta-1-\rho\theta)))}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}\right)$$

$$\text{(A.27)}\quad b_{12} = \frac{(1-\rho\delta)+(\varphi_\pi-\rho)\delta}{(1-\rho\delta)(\sigma+\varphi_z)} - \frac{\varphi_\pi}{\sigma+\varphi_z}\frac{\kappa(1-\theta\delta\rho)(1+\delta(\varphi_\pi-2\rho))+(1-\rho\delta)(\sigma+\varphi_z)(1-\theta)\delta}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}$$

$$\text{(A.28)}\quad b_{21} = \frac{\kappa(\sigma+\delta(\varphi_z(\theta-1)+\sigma(\theta-1-\rho\theta)))}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}$$

$$\text{(A.29)}\quad b_{22} = \frac{\kappa(1-\theta\delta\rho)(1+\delta(\varphi_\pi-2\rho))+(1-\rho\delta)(\sigma+\varphi_z)(1-\theta)\delta}{(1-\rho\delta)(\sigma+\varphi_z+\kappa\varphi_\pi)(1-\theta\delta\rho)}$$

Preston (2005) studies e-stability of this system for the case of $\rho = 1$ and the results are not very different from those of Bullard and Mitra (2002) for a plausible set of policy coefficients.[5]

B Rational Expectations Equilibrium

Under RE, both the IS curve and the Phillips curve have recursive representation, and the model is presented by the following well-known system of equations:

$$\text{(B.1)}\qquad z_t = E_t z_{t+1} - \sigma^{-1}(r_t - E_t\pi_{t+1}) + \epsilon_t^d$$

$$\text{(B.2)}\qquad \pi_t = \kappa z_t + \delta E_t \pi_{t+1} + \epsilon_t^\pi$$

$$\text{(B.3)}\qquad r_t = \varphi_\pi \pi_t + \varphi_z z_t$$

[5]Please refer to Proposition 2 and its proof in Preston (2005)

where E_t is the expectations operator with regard to objective distributions of all variables.

The system of equations (B.1)-(B.3) can be further summarised in a way similar to (2.20):

$$y_t = BE_t y_{t+1} + B_1 \epsilon_t^y \tag{B.4}$$

where $y_t = \begin{bmatrix} z_t \\ \pi_t \end{bmatrix}$, $\epsilon_t^y = \begin{bmatrix} \epsilon_t^z \\ \epsilon_t^\pi \end{bmatrix}$, $B = \frac{1}{\sigma+\varphi_z+\kappa\phi_\pi} \begin{bmatrix} \sigma & 1-\delta\varphi_\pi \\ \kappa\sigma & \kappa+\delta(\sigma+\varphi_z) \end{bmatrix}$ and

$B_1 = \frac{1}{\sigma+\varphi_z+\kappa\varphi_\pi} \begin{bmatrix} \sigma & -\varphi_\pi \\ \kappa\sigma & \sigma+\varphi_z \end{bmatrix}$.

Under the concept of *rational expectations*, the agents have full knowledge of the structure and all interdependencies in a given economy. The model (B.4) can be solved by applying forward iterations, and the resulting rational expectations equilibrium (REE) is given by:

$$y_t = \overline{\beta} + B_1 \epsilon_t^y \tag{B.5}$$

where $\beta' = (\beta^z, \beta^\pi)$ denotes a generic vector of coefficients and in the REE $\overline{\beta}$: $(\bar{\beta^z}, \bar{\beta^\pi})' = 0$. Therefore, under rational expectations the evolution of the economy is fully described by (B.5).

Under rational expectations the monetary policy rule should be designed in such a way that the resulting equilibrium is unique. For the case of the standard DSGE model outlined above, the determinacy of the REE (B.5) is guaranteed if all eigenvalues of matrix B in (B.4) lie inside the unit circle.[6] This happens when the following condition for model parameters is met:

$$\kappa(\varphi_\pi - 1) + (1-\delta)\varphi_z > 0 \tag{B.6}$$

Therefore, a Taylor Rule which actively offsets deviations of inflation from its targeted level with interest rate ($\varphi_\pi > 1$) is sufficient to guarantee determinacy of REE.

C "Euler-Equation" version of the model and relation to IR

The adaptive learning literature usually studies model's dynamics, which are produced by equations that hold for the general equilibrium of the RE version of the same model—but at the same time, the objective probabilities in these equations are mechanically substituted with agents' subjective updating recursion for beliefs. Under this setting, to what extent, on a micro-level, agents remain optimal decision-makers is unclear. Further, if they are, what is the mechanism that prevents them from discovering the REE of the model and discarding their subjective beliefs because they are not consistent with the model? The current setup shows that under some modifications the adaptive learning model is consistent with rational behaviour.[7]

[6]Blanchard and Kahn (1980)

[7]The adaptive learning literature does not specify what agents' views are on the evolution of macro-variables. They only equip them with a recursion, which tracks some moments of the variable. If beliefs are not fully specified in the model, then why, exactly, agents must form expectations according to a given recursion and how this relates to rational behaviour is unclear. Furthermore, if there are constraints in the economy, higher moments of the distribution of beliefs play a vital role. However, in these models they are not specified.

C.1 Additional condition and derivation of "myopic" model

Assume that borrowing constraints are tight and the constraint (2.3) is substituted with a standard no-Ponzi-game condition:

$$\underline{B} \leq \frac{B_t^i}{P_t} \leq \overline{B}, \forall t \tag{C.1}$$

where $\underline{B}$ and $\overline{B}$ are finite bounds on holding of real bonds.

Assume that $|\underline{B}|$ and $|\overline{B}|$ in (C.1) are very small. Household i understands that in a given period its consumption is not restricted by the aggregate production level because it is free to choose any level of savings within defined bounds, $[\underline{B}, \overline{B}]$. However, if the bounds for the bond holdings are small enough compared to labour income and dividends, the household also internalises that the expected consumption level for the next period cannot differ much from the aggregate output of the economy.[8] Therefore, if the household combines (2.4) with the assumption of tight borrowing constraints, then it can approximate its expected consumption level for the next period using the expected aggregate output. This result is formally stated as:

Proposition 1. $\forall \widehat{\delta} > 0 \; \exists \widehat{\epsilon} > 0$: if $-\underline{B} < \widehat{\epsilon}$ and $\overline{B} < \widehat{\epsilon} \Rightarrow Prob^{\mathcal{P}^h}(|(C_{t+1}^i - Z_{t+1})| < \widehat{\delta}) = 1$.

Assumption 1 shows the conditions for underlying parameters of the model which guarantee that the household approximates its consumption for the next period with the expected aggregate output in the Euler equation (A.11).

Assumption 1. The bounds for real bond holdings are small, $-\underline{B} < \widehat{\epsilon}$ and $\overline{B} < \widehat{\epsilon}$, so that the result stated in Proposition 1 and the approximation (C.2) hold with sufficient accuracy (for a given small $\widehat{\delta}$).

Given Assumption 1, a household i can rely on the following approximation:

$$E_t^{\mathcal{P}^h}\left[\frac{a_{t+1}}{a_t}\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \simeq E_t^{\mathcal{P}^h}\left[\frac{a_{t+1}}{a_t}\left(\frac{Z_{t+1}}{C_t^i}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right] \tag{C.2}$$

Under an Approximation (C.2) the household's Euler equation (A.11) takes the following form:

$$\frac{a_t(C_t^i)^{-\sigma}}{1+r_t} = \delta E_t^{\mathcal{P}^h}\left[\frac{a_{t+1}(Z_{t+1})^{-\sigma}}{\Pi_{t+1}}\right] \tag{C.3}$$

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the aggregate inflation level. Therefore, the optimal consumption choice is pinned down by the agent's expectations of aggregate output and inflation for the next period.[9]

The standard log-linearisation of (C.3) around a zero-inflation steady state results in the following equation:

$$c_t^i = E_t^{\mathcal{P}^h} z_{t+1} - \sigma^{-1}(r_t - E_t^{\mathcal{P}^h}\pi_{t+1}) + \sigma^{-1}\epsilon_t^a \tag{C.4}$$

The approximation (C.2) rationalises the "myopic" view of the household about its consumption decision: to choose the consumption level today it needs to know the expected value of aggregate output solely for the next

[8] Expected consumption being close to the expected aggregate output follows from the first order condition (A.10) and the household's budget constraint (2.2) combined with (2.4).

[9] Note that under Assumption 1, a labour decision is also close to the equilibrium mapping of fundamentals into labour choice which holds in the REE. This result comes from the household's first order condition (A.10) and the budget constraint (2.2).

period. Under Assumption 1, the approach of Bullard and Mitra (2002), Bullard and Mitra (2007) of combining subjective probabilities about inflation and output in the IS curve is compatible with rational behaviour.

Proof of Proposition 1

The optimal consumption/labor choice is given by the solution of the following problem

$$\max_{\{C_t,N_t,B_t\}_{t=0}^{\infty}} E_0^{\mathcal{P}^h} \sum_{t=0}^{\infty} \delta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{C.5}$$

subject to

$$C_t + \frac{B_t}{(1+r_t)P_t} - \frac{B_{t-1}}{P_t} \leq \frac{W_t}{P_t} N_t + \frac{D_t}{P_t} \tag{C.6}$$

$$\underline{B} \leq B_t \leq \overline{B} \tag{C.7}$$

taking r_t, P_t, B_t, W_t and D_t as given. Denote $\Delta B_t = \frac{B_t}{(1+r_t)P_t} - \frac{B_{t-1}}{P_t}$.

To prove Statement 1 it is necessary to follow the steps, outlined below:

1. Maximization of a concave function given a convex constraint leads to a unique solution $C^*(\Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and $N^*(\Delta B, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and, by implication, in optimum the budget constraint (C.6) binds. Therefore, the unique solution satisfies the following system of equations:

$$\frac{N_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \tag{C.8}$$

$$C_t + \Delta B = \frac{W_t}{P_t} N_t + \frac{D_t}{P_t} \tag{C.9}$$

2. Given that $r_t \gg -1$ and P_t is bounded away from zero, from the definition of ΔB it follows that as the bounds for bond holdings $[\underline{B}, \overline{B}]$ converge to zero, $\Delta B- > 0$.

Formally: $\underline{B} = -\epsilon$ and $\overline{B} = \epsilon \Rightarrow \lim_{\epsilon \to 0}(\Delta B) = 0$.

3. Solving (C.8) for labor:

$$N_t = \left[\frac{W_t}{P_t} C_t^{-\sigma} \right]^{\frac{1}{\phi}} \tag{C.10}$$

and substituting for the budget constraint allows us to define the following function:

$$f(C_t, \Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) \equiv C_t + \Delta B - \left(\frac{W_t}{P_t} \right)^{1+\frac{1}{\phi}} C_t^{\frac{-\sigma}{\phi}} + \frac{D_t}{P_t} \tag{C.11}$$

which in turn defines the optimal consumption choice:

$$C^*(\Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) : f(C_t^*, \Delta B_t, \frac{W_t}{P_t}, \frac{D_t}{P_t}) = 0 \tag{C.12}$$

4. By the Implicit Function Theorem

$$\frac{\partial C}{\partial(\Delta B)}|_{\Delta B=0} = -\frac{\partial f/\partial(\Delta B)}{\partial f/\partial C} = -\frac{1}{\partial f/\partial C} \tag{C.13}$$

Therefore, to finish the proof it is necessary to demonstrate that $\partial f/\partial C|_{\Delta B=0}$ exists and it is invertible.

Taking the derivative of (C.11) with respect to consumption, and evaluating at $\Delta B = 0$ leads to the following expression:

$$\partial f/\partial C|_{\Delta B=0} = 1 + \frac{\sigma}{\phi}\left(\frac{W_t}{P_t}\right)^{1+\frac{1}{\phi}} C_t^{\frac{-\sigma}{\phi}-1} > 0 \tag{C.14}$$

which is strictly positive given the non-negativity of consumption and a real wage.

Therefore, it follows that when $\Delta B \to 0$, the consumption choice converges to the REE, $C^* \to C^*(0, \frac{W_t}{P_t}, \frac{D_t}{P_t})$ and, from (C.10), it also follows that the labor choice converges to REE, $N^* \to N^*(0, \frac{W_t}{P_t}, \frac{D_t}{P_t})$.

Given that household understands the mapping of fundamentals into wages and dividends, the equilibrium consumption equals to aggregate output. This can be easily seen by substituting (2.13) into (C.11), which leads to $C_t^* = Z_t$.

C.2 Model equilibrium, E-stability and Asymptotic speed of Convergence

If expectations evolve according to (2.16), then the actual dynamics of the output gap and inflation are linear functions of expectations:

$$y_t = B\beta_t + B_1\epsilon_t^y \tag{C.15}$$

where B and B_1 are matrices of coefficients defined in B.4. Details of derivations can be found in Bullard and Mitra (2002). Therefore, the T-map is given by:

$$T(\beta_t) = B\beta_t \tag{C.16}$$

A general concern for monetary models studied under learning is whether learning makes the model equilibrium unstable. A standard analysis of monetary policy under learning would set the Taylor rule coefficients in such a way that they ensure that if agents' beliefs are away from REE, those beliefs return to equilibrium. The equilibrium is defined as expectationally-stable (e-stable) if, once initialized away from REE, beliefs converge back to their REE values. This happens if the conditions of stability of (2.22) are met: the real part of all eigenvalues of the first derivatives of $T(\beta)$ with respect to β are smaller than one.

The e-stability of the learning scheme in this model economy is guaranteed if the following condition for model parameters is met:[10]

$$\kappa(\varphi_\pi - 1) + (1-\delta)\varphi_z > 0 \tag{B.6}$$

And the rate at which the successive iterates of (2.22) approach the fixed point $-$ rate of asymptotic conver-

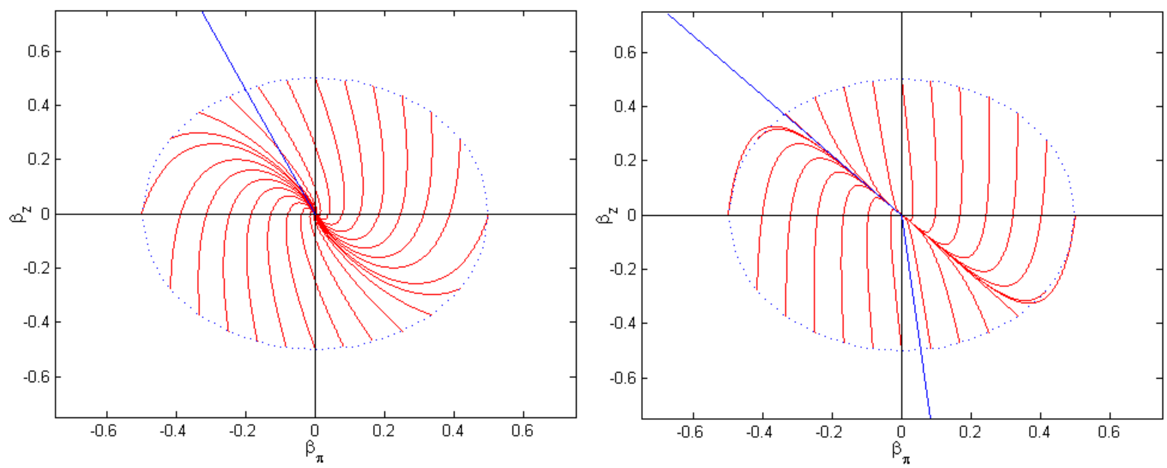
[10]please refer to Appendices A and B in Bullard and Mitra (2002) for the proof

gence, – is defined by the largest eigenvalue of matrix B.[11] Figure 2a visualizes condition (B.6) in the space of Taylor rule coefficients for the same calibration as it was used in Bullard and Mitra (2002). A desirable policy that guarantees determinacy under REE, expectational stability and faster speed of convergence, is an aggressive reaction to deviations of inflation from the target, accompanied by a minor reaction to output gap deviations.

C.3 Model dynamic under myopic beliefs

This section studies model dynamic in the "Euler equation" learning model and highlights that results presented for the model with loose borrowing constraints apply in this case.

Figure 1: Trajectories of T-map, "Myopic" model



(a) $\sigma = 6$ (b) $\sigma = 2$

Figure 1 plots trajectories of beliefs for the case of the "myopic" model. The dynamic is very similar to Figure 3. Namely, high inflation expectations lead to output losses.

However, the policy (3.4) which eliminates feedback from inflation expectations into output gap is different because the T-map (C.16) differs for the case of this model. The policy that creates an evolution of output gap which is independent of a future path of inflation requires setting the coefficient on inflation in the Taylor rule to an inverse of the discount factor. These results are formally stated in Proposition 2:

Proposition 2.

$\forall \varphi_z$ if $\varphi_\pi = 1/\delta \Rightarrow \partial a_z / \partial a_\pi = 0$ and $h_1 = \delta$.

Proof of Proposition 2. Follows from $\partial a_z / \partial a_\pi = 1 - \delta \varphi_\pi$ and (C.17) below.

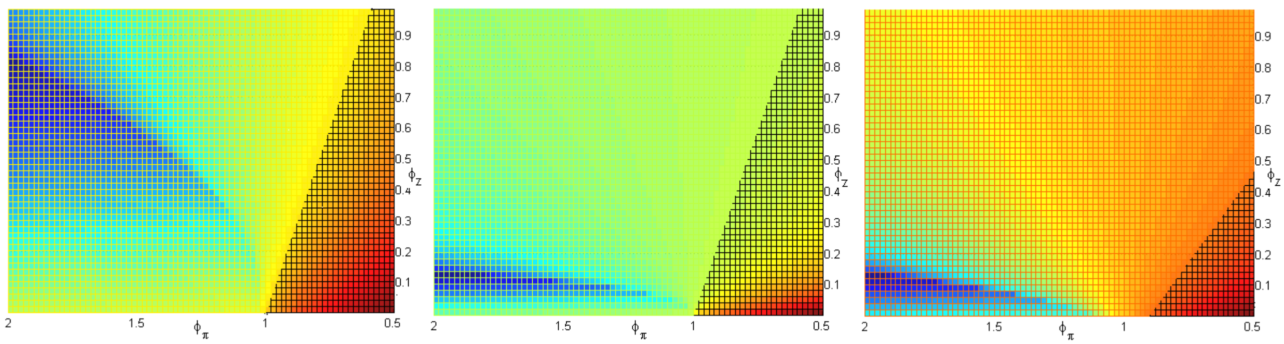
The analytical expression for both eigenvalues and eigenvectors of matrix B, which describes the dynamic of beliefs about inflation and output gap, can be easily derived from the expression for B in (2.20). Their values are defined by the following equations:

(C.17) $$h_{1,2} = \frac{\sigma + \delta\sigma + \kappa + \delta\varphi_z \pm \sqrt{-4\delta\sigma(\sigma + \kappa\varphi_\pi + \varphi_z) + (\sigma + \delta\sigma + \kappa + \delta\varphi_z)^2}}{2(\sigma + \kappa\varphi_\pi + \varphi_z)}$$

[11]Ferrero (2007) studies speed of convergence of beliefs in a similar model.

$$
v_{1,2} = \left[\frac{\sigma - \delta\sigma - \kappa - \delta\varphi_z \pm \sqrt{-4\delta\sigma(\sigma + \kappa\varphi_\pi + \varphi_z) + (\sigma + \delta\sigma + \kappa + \delta\varphi_z)^2}}{2\sigma\kappa}, 1 \right] \tag{C.18}
$$

D Policy analysis according to e-stability and speed of convergence criteria



(a) "Myopic" model (b) LH beliefs model (c) LH beliefs with mean reversion

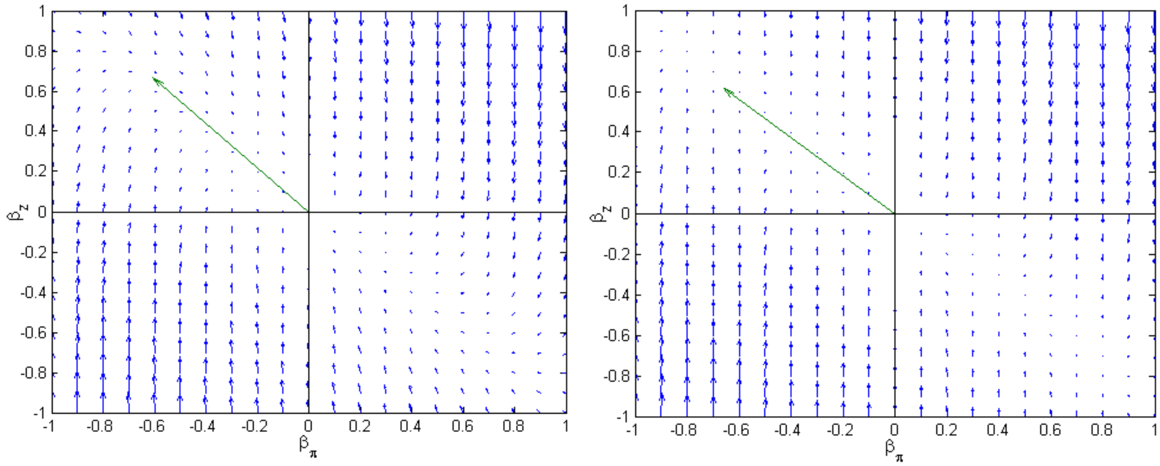
Figure 2: Regions of e-stability

In order to visualize these results and define a preferable set of Taylor rule coefficients according to the criteria of the value of the largest eigenvalue, the calibration from Woodford (2003) for the US economy is applied to all variations of models outlined above. It is assumed that $\sigma^{-1} = 0.157$, $\kappa = 0.024$ and $\delta = 0.99$ − are the values of the parameters that are considered plausible in business cycle literature. For the model with mean-reversion in beliefs the persistence coefficient is set to $\rho = 0.985$. Taylor rule coefficients for a baseline case are set to standard values, $\varphi_\pi = 1.5$ and $\varphi_z = 0.5$, as proposed by Taylor (1993). Bullard and Mitra (2002) and Preston (2005) also use the same calibration to study asymptotic properties of the model. The gain parameter is set in line with estimates in Malmendier and Nagel (2016).

Figure 2 gives a sense of which set of Taylor rule coefficients leads to e-stability of the learning scheme. Areas marked with black check pattern are characterised by the largest eigenvalue of the B matrix being larger than 1. Figure2a also demonstrates the region of determinacy of REE, which is the same among all three models. Interestingly, the presence of mean-reversion in beliefs enlarged the set of policy coefficients that guarantee expectational stability of model equilibrium. It is easy to verify that the baseline Taylor rule for any modification of the learning model produces equilibrium that is both determinate under RE and e-stable under IR.

The blue area on each subplot indicates a set of policy coefficients, which ensures not only e-stability, but also a higher speed of convergence of beliefs to REE. This policy dictates an aggressive reaction to deviations of inflation from the target, accompanied by a minor reaction to output gap deviations. This policy ensures that even if beliefs happen to disagree with their REE values, they would return to equilibrium as fast as possible given the constraints for policy coefficients.

Figure 3: Phase planes



(a) BM

(b) Preston

E Supplementary material for Section 3

E.1 Proposition 3

Let $V = [v_1, v_2]$ be a matrix of right eigenvectors of matrix C and H is a diagonal matrix whose diagonal elements are the corresponding eigenvalues. Therefore, C can be factorized in the following way:

$$C = VHV^{-1} \tag{E.1}$$

Combining (E.1) with (3.1) leads to:

$$V^{-1}\beta_t = HV^{-1}\beta_{t-1} \tag{E.2}$$

It follows, that

$$\frac{1}{det(H)}\begin{pmatrix} v_2^2 & -v_2^1 \\ -v_1^2 & v_1^1 \end{pmatrix}\beta_t = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\frac{1}{det(H)}\begin{pmatrix} v_2^2 & -v_2^1 \\ -v_1^2 & v_1^1 \end{pmatrix}\beta_{t-1} \tag{E.3}$$

which results in the following system of equations given an initial condition β_0:

$$\beta_t^1 - \frac{v_2^1}{v_2^2}\beta_t^2 = (h_1)^t(\beta_0^1 - \frac{v_2^1}{v_2^2}\beta_0^2) \tag{E.4}$$

$$\beta_t^1 - \frac{v_1^1}{v_1^2}\beta_t^2 = (h_2)^t(\beta_0^1 - \frac{v_1^1}{v_1^2}\beta_0^2) \tag{E.5}$$

Proposition 3. If $|h_2| < |h_1| < 1 \Rightarrow \lim_{t\to\infty} \frac{\beta_t^1 - \frac{v_1^1}{v_1^2}\beta_t^2}{\beta_t^1 - \frac{v_2^1}{v_2^2}\beta_t^2} \to 0$

Proof of Proposition 3. $\lim_{t\to\infty} \frac{\beta_t^1 - \frac{v_1^1}{v_1^2}\beta_t^2}{\beta_t^1 - \frac{v_2^1}{v_2^2}\beta_t^2} = \lim_{t\to\infty} \left(\frac{h_2}{h_1}\right)^2 \left[\frac{\beta_0^1 - \frac{v_1^1}{v_1^2}\beta_0^2}{\beta_0^1 - \frac{v_2^1}{v_2^2}\beta_0^2}\right] = 0$ given that $|h_2| < |h_1| < 1$.

E.2 Proposition 4

Proposition 3 states that along V_1, when inflation expectations are high, agents overpredict inflation and underpredict the output gap. The latter behaviour can be observed in Figure 4: when the output gap expectations move in the direction of equilibrium, the corresponding T-map (initial condition 1 and 2) is above the black line. This map means that agents expect output to be lower than its actual realisations.

Proposition 4. $\forall \beta_t \in V_1$ if $\beta_t^\pi > \bar{\beta} \Rightarrow \pi_t < \beta_t^\pi, z_t < \beta_t^z$.

Proof of Proposition 4. Follows from $h_1 < 1$, which is guaranteed by $\varphi_\pi > 1$.

E.3 Slope of v_1

The discounted present value of the entire future path of the real rate given the shock to the inflation expectation in (2.12) is $\varphi_\pi \frac{(1-\theta)\delta}{1-\theta\delta\rho}\Delta + (\frac{\varphi_\pi\delta-1}{1-\delta\rho})\Delta$ and φ_π^* is a solution to $\varphi_\pi^* \frac{(1-\theta)\delta}{1-\theta\delta\rho} + \frac{\varphi_\pi^*\delta-1}{1-\delta\rho} = 0$.

Figure 4: Slope of eigenvector v_1

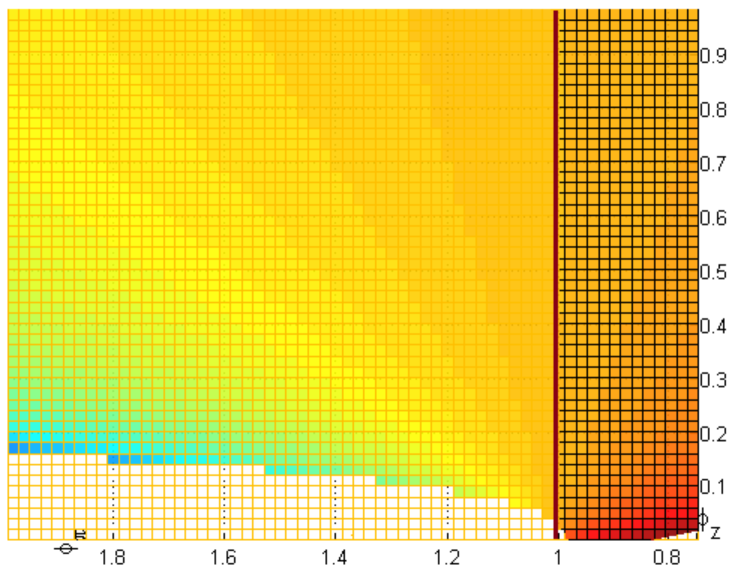


Figure 4 plots values of the inflation output trade-off (slope of the eigenvector v_1) in the space of Taylor rule coefficients for standard calibration of remaining parameters. The horizontal axis measures values of Taylor rule coefficients on inflation, and the vertical axis sets the coefficient on output gap in the policy rule. The red vertical line marks the value of the $\varphi_\pi = \varphi_\pi^*$ and therefore eliminates the response of output gap to inflation expectations. The black checked area to the right (from the red line) marks the space of policy parameters, which make the slope positive. The area to the left (from the red line) presents combinations of policy parameters that result in negative slope and therefore lead to output costs of inflation expectations being high. The color of the plot represents the idea of the size of the slope: the blue area marks the space of policy coefficients that result in a larger contraction in output gap, and the dark orange area indicates for which combination of policy coefficients the feedback from inflation expectations into output is lower. The figure confirms that the desirable policy would be to take the value of $\varphi_\pi = \varphi_\pi^*$; and if $\varphi_\pi > \varphi_\pi^*$, then the relatively large φ_z will guarantee that the output costs of high inflation expectations are minimized.

Proposition 5. $\forall \delta < 1, \forall \theta \; \varphi_\pi^* > 1$.

Proof of Proposition 5. Follows from $\varphi_\pi^* \frac{1-\delta\theta}{\delta(2-\delta-\theta)}$ and $1-\delta > \delta(2-\delta), \forall \delta < 1$.

If $\varphi_\pi = \varphi_\pi^*$ then both the slope of the eigenvector v_1 and the eigenvalue h_1 are independent of φ_z. This is formally stated in Proposition 4 below.

Proposition 6. $\forall \varphi_z$ if $\varphi_\pi = \varphi^* \Rightarrow \partial a_z / \partial a_\pi = 0$ and $\partial h_1 / \partial \varphi_z = 0$.

Proof of Proposition 6. Follows from $\partial a_z / \partial a_\pi = b_{2,1}$, where B^{LH} is defined in (A.26). $\varphi_\pi = \varphi^*$ guarantees $b_{2,1} = 0$. If $b_{2,1} = 0 \Rightarrow h_1 = b_{2,2}$. $b_{2,2}(\varphi^*) = \frac{\delta(1-\theta)}{1-\delta\rho\theta}$, therefore $\partial h_1 / \partial \varphi_z = 0$.

However, the second eigenvalue is decreasing in φ_z; therefore, it is still beneficial for policy to set φ_z to higher levels. This would minimize the effect of output expectations on actual path of economy. If the policy manages to fix the Taylor rule coefficient for inflation in the following region, $[1, \varphi_\pi^*]$, the eigenvector becomes slightly upward sloping; therefore, higher-than-in-REE inflation expectations result in small output gains. Simultaneously, this policy design preserves the expectational stability of the model under learning.

F Zero-Lower Bound constaint

F.1 Extension: Zero-lower bound for nominal interest rates

This subsection introduces the zero-lower bound (ZLB) for nominal interest rates into the model under IR. This is an important extension of the model in light of the interest rates behavior during the Great Recession and its aftermath. Given the ZLB constraint, the Taylor rule (2.11) is modified and takes the following form:

$$r_t^{ZLB} = \max[\underline{r}, \varphi_\pi \pi_t + \varphi_z z_t] \tag{F.1}$$

where $\underline{r}$ is the lower bound for r_t^{ZLB}, because the model is formulated in deviations from the steady state. In the spirit of keeping agents' beliefs as a good approximation of the actual behaviour of the economy, agents' perception of the Taylor rule is changed to (F.1). But, this perception does not change the beliefs about the evolutions of dividends, wages, and the stochastic discount factor.

Under IR, the model solution combines (F.1) with the true probability distribution, as it would be under REE,[12] but also with agents' perceptions of inflation and the output gap (2.16). Thus, the forecasts of future interest rates are calculated with respect to these perceptions. However, these perceptions differ compared to the adaptive learning literature as this literature starts out at (2.18), so it is unsuitable to address the ZLB constraint when long-horizon expectations matter. Under IR, it is clear that first moments are not sufficient to extrapolate the future behaviour of rates in the presence of the ZLB constraint; the whole distribution of beliefs matters for the model's dynamics.

[12] In general, under REE, the ZLB constraint significantly alters the model solution; this is linked to the fact that agents possess knowledge of the true probability distributions of all variables in the economy. This knowledge means that in order to solve the model, the agents need to calculate the exact periods in the future when the ZLB constraint is binding in general equilibrium. This is a complicated problem.

The forecasting model for interest rates differs from the one before, and internally rational households forecast nominal interest rates at k periods ahead according to:

$$E_t^{\mathcal{P}} r_{t+k}^{ZLB} = E_t^{\mathcal{P}} \max[\underline{r}, \varphi_\pi \pi_{t+k} + \varphi_z z_{t+k}] \tag{F.2}$$

Combining (F.2) with the perception that inflation and the output gap are independently and normally distributed, it follows then that the expectations of nominal interest rates in the presence of the ZLB constraint are simply calculated as the first moment for truncated normal distribution:

$$E_t^{\mathcal{P}} r_{t+k}^{ZLB} = \mu_{t+k} + \sigma_{t+k} \frac{\phi\left(\frac{\underline{r}-\mu_{t+k}}{\sigma_{r_{t+k}}}\right)}{1-\Phi\left(\frac{\underline{r}-\mu_{t+k}}{\sigma_{t+k}}\right)} \tag{F.3}$$

where ϕ and Φ are correspondingly pdf and cdf of the standard normal distribution, and the moments are given by:

$$\mu_{t+k} = \varphi_\pi \rho^k \hat{\beta}_t^\pi + \varphi_z \rho^k \hat{\beta}_t^z \tag{F.4}$$

$$\begin{aligned} \sigma_{t+k}^2 = {} & \varphi_\pi^2 (\rho^{2k} \sigma_0^{\pi,2} + \sum_{i=1}^{k} \rho^{2(i-1)} \sigma_{\xi_t^\pi}^2 + \sigma_{\zeta^\pi}^2) + \varphi_z^2 (\rho^{2k} \sigma_0^{z,2} \\ & + \sum_{i=1}^{k} \rho^{2(i-1)} \sigma_{\xi_t^z}^2 + \sigma_{\zeta^z}^2) \end{aligned} \tag{F.5}$$

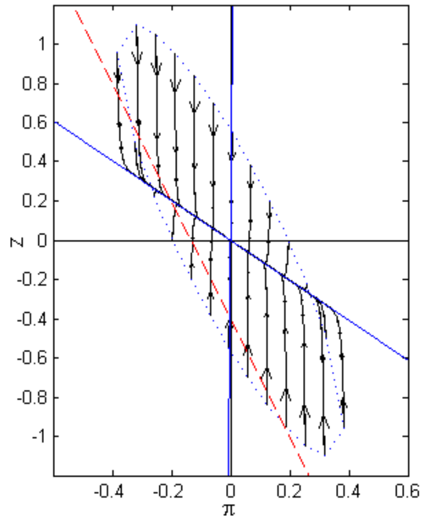
To summarise, the ZLB constraint under IR changes the agents' expectations about the future dynamic of the nominal interest rate because agents incorporate in their forecasts of inflation and the output gap the possibility of revising beliefs in the future. Even away from the ZLB constraint, agents expect future rates to be higher compared to the case of the economy without the constraint. When the economy is already at the ZLB, agents assess the probability of exiting the constraint according to the (F.3), and this probability increases with the forecasting horizon because the uncertainty about the future path of the economy accumulates in time ($\sigma_{r_{t+k}}^2$ increases in k). This affects consumption plans according to (2.12).

F.2 Dynamics in the presence of the ZLB constraint

This subsection conveys that the main result of the high inflation expectations relative to the REE that leads to persistently low output is robust to the incorporation of the ZLB constraint. Simultaneously, it highlights that the Internal Rationality approach is useful for analyzing the economy's dynamics under learning in the presence of constraints.

Figure 5 plots the dynamics of the output gap and inflation in the economy without the ZLB, which is associated with the beliefs in Figure 3. The blue line is the eigenline V_1, and it is the same on both figures. The ZLB constraint $r^0 = \{(\pi, z) : \pi = -\bar{r} - \frac{\varphi_z}{\varphi_\pi} z\}$ for a baseline calibration of the Taylor rule coefficients is presented by a dashed line in the space for the output gap and inflation. Therefore, for any combination below the dashed line, interest rates are negative if the rates follow the Taylor rule (2.11). This negativity means that in the presence of the ZLB constraint and when the economy is below r^0, interest rates do not respond to innovations in the output gap and inflation.

Figure 5: Model dynamic and the ZLB constraint



Therefore, as it was pointed out in Evans et al. (2008) for the case of the "Euler equation" learning model, there are different mappings of beliefs into the actual inflation and output gap when the economy is above or below r^0. In the latter case the mapping is not e-stable, and economy enters the deflationary spiral. This spiral happens because according to the results presented in Figure 2, $\varphi_\pi = \varphi_z = 0$ does not retain e-stability of the model's equilibrium under learning. When ZLB does not bind, then the REE is locally stable and agents do not incorporate the possibility of ZLB binding in future periods in their forecasts. This is different in a long-horizon model under the IR approach.

Under IR, the ZLB changes the agents' view about the future path of interest rates even away from the bound.[13] Agents understand that now rates cannot be negative and therefore they are forecasting them according to (F.3) and (F.4). Away from the ZLB, this forecasting means that for the same beliefs about the output gap and inflation, the future rates are expected to be higher compared to the case when rates are not restricted by a zero bound. In this setting, the uncertainty about forecasts of inflation and the output gap (σ_ζ and σ_ξ in (2.16)) have an explicit role in the model's dynamics, because it increases the probability of the ZLB binding for any given level of beliefs.

Some of the results in Evans et al. (2008) remain under IR. Mainly, if the economy is at the ZLB and beliefs are such that ZLB is expected to bind for several periods, the economy dynamics resembles the deflationary trap. Figure 5 demonstrates that this region is likely to be characterised by inflation with expectations being lower than at the REE. Therefore, the ZLB dramatically changes the model's dynamics when inflation expectations are low. These changes suggest that output gains that are associated with the inflation expectations being lower than the REE are limited in the presence of the ZLB constraint.

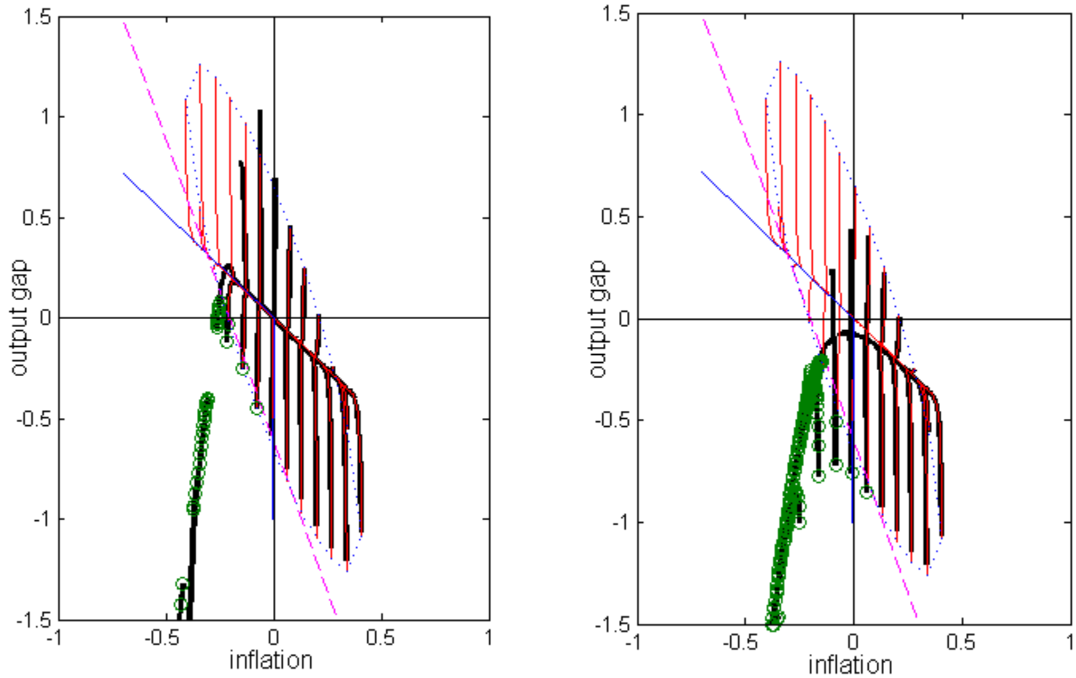
It is also the case that the model's dynamics are robust to the ZLB constraint if beliefs are sufficiently away from the bound, and the bound is not expected to bind in the future. According to Figure 5, this region is

[13] Note that how much model dynamic changes for a given level of inflation and output gap compared to dynamics on 5 depends on the distance of the economy from r^0 and the agents' views about the volatilities of inflation and the output gap (σ_ζ and σ_ξ in (2.16)).

characterised by high inflation expectations and it is the main focus of this paper.[14] Moreover, if the economy is subjected to the ZLB by a shock, but agents expect ZLB not to bind in the future (high inflation expectations), then those beliefs help the economy to move away from the ZLB and back to V_1.

The most interesting dynamics under IR arises when the beliefs are in the region where the economy is close to the ZLB. If beliefs are such that in future periods the constraint has a high probability to bind, this binding changes the feedback from the beliefs to the actual inflation and output gap. Those beliefs drive the economy to the ZLB and lead to persistently low output accompanied by minor deflation. Moreover, when σ_ζ and σ_ξ are high, then the REE is not locally stable, which is different than in Evans et al. (2008).

Figure 6: Model dynamic with and without the ZLB constraint



(a) Low uncertainty in beliefs

(b) High uncertainty in beliefs

Figure 6 plots dynamics of inflation and output gap in the economy with and without the ZLB constraint for different levels of uncertainty in agents' beliefs in (2.16). The left panel of the figure considers a low level of uncertainty in beliefs. The red curves plot the dynamics of actual inflation and the output gap associated with beliefs in Figure 3 in absence of ZLB constraint as on Figure 5. The ZLB constraint for a baseline calibration of the Taylor rule coefficients is presented by a dashed line in the space for inflation and output gap. Therefore, for any combination of inflation and output gap below the dashed line, interest rates are negative if rates follow the Taylor rule.

Black curves on the left panel in Figure 6 present model dynamics in the presence of the ZLB constraint, given the same initial conditions for beliefs as in Figure 3. Green dots mark periods when the ZLB is binding, giving one an idea how much time the economy spends at the ZLB for any given trajectory. The left panel of

[14]Note that for any $\varphi_\pi > 0, \varphi_z > 0$ slope of V_1 is smaller than of r^0 which means that when the beliefs and the economy are close to V_1 and $\beta_\pi > \bar{\beta}_\pi$, ZLB does not bind. Proof in a companion paper.

Figure 6 outlines an important result, namely, that even if the economy arrives at the ZLB at some period this does not imply that it necessarily stays at the bound. If agents believe that the bound is not going to bind in future periods this changes model dynamic significantly. Even though the underlining T-map below r_0 pushes the economy away from REE, beliefs about a future path for the economy and the associated path of interest rates outweigh this effect. This is demonstrated by the trajectories in Figure (6), when the economy initially is at the ZLB but consequently avoids the trap; this happens only because of agents' expectations. The same logic applies if the economy is taken to the ZLB by an exogenous shock, but this shock does not affect agents' expectations. If beliefs remain on a trajectory that is away from the ZLB being binding, this dynamic does not lead to the trap.

The dynamic presented by black curves, if compared to red ones, suggests that output gains associated with inflation expectations being lower than REE are limited in the presence of the ZLB. At the same time, low inflation expectations are dangerous because they are capable of pushing the economy to the deflationary trap because of the proximity to the ZLB constraint. Therefore, it is impossible for policymakers to achieve output gains associated with low inflation expectations in the presence of the ZLB constraint. However, and importantly, ZLB does not alter the model dynamic when inflation expectations are high. When inflation expectations are high, even if the economy is taken to the ZLB by a shock to output, the model dynamic suggests that beliefs about inflation help it to recover and move away from the constraint. The figure also demonstrates that the economy is attracted to the same ratio between inflation and output gap as before$-$the blue line representing v_1. Therefore, the result that inflation expectations lead to persistently low output is maintained in the presence of the ZLB constraint. Along the recovery route, interest rates are low, but ZLB does not bind according to agents' beliefs.

The right panel of Figure 6 studies the case of higher uncertainty in agents' expectations about inflation and output gap, as compared to the left panel. This means that variances in both disturbances in (2.16) are increased proportionally, and therefore this increase does not change the gain value. But according to (F.4), this will alter agents' views about future rates. If we compare the black curves on the right and left panels in Figure 6, it becomes apparent that higher uncertainty about inflation and output gap significantly affects the model dynamic. Interestingly, the economy is now driven to the trap, even if it is initialized away from the ZLB. Moreover, the REE of the model is no longer a stable solution if agents perceive inflation and output gap as volatile. The logic is as it follows: consider that inflation and output gap expectations are at the REE; when rates are not restricted by a lower bound according to agents' belief systems, rates remain at zero for all future horizons. Therefore, the resulting output gap and inflation are also at the REE. However, if there is a ZLB constraint present in the economy, simultaneously with high uncertainty in agents beliefs about inflation and output gap, this increases the present discounted value for a future path of interest rates in (2.12). This in turn lowers output gap, which brings the economy closer to the ZLB constraint in the next period, which in turn increases expected rates and eventually affects inflation and inflation expectations. Nevertheless, the dynamic associated with high inflation expectations remains similar to the case of the absence of the ZLB constraint.

G Robustness

G.1 Change in the inflation target

The previous analysis focuses on the model's dynamic, given that the inflation target is fixed at a zero level and beliefs happen to disagree with it. Therefore it follows that in this particular case, inflationary expectations are costly for the output. It is important to note that the model is formulated in deviations from the steady state levels, and a change in the inflation target affects the steady state. And what is important for the mechanism outlined above is whether agents expect inflation to be lower or higher compared to the central bank's targeted level and not in absolute terms. For the policy to be successful the current state of inflation expectations should be taken into account.

If monetary authorities increase the inflation target, then, according to the information structure under internal rationality, agents understand the change in the central bank's inflation target. However, they are doubtful about how successful monetary authorities will be with respect to reaching the new target. Therefore agents' expectations remain in the neighbourhood of the old equilibrium, this position means that inflation expectations are low compared to the new equilibrium.[15] Further, the transition dynamic to a new equilibrium operates as if inflation expectations are initialised to the left of equilibrium (the negative region for inflation expectations) in Figure 3. Therefore, inflation expectations that are lower than the targeted level of inflation increase the output gap away from the zero-lower bound.

On the other hand, a small decrease in the inflation target will create output losses when compared to a new equilibrium, because in this scenario the expected level of inflation$|-$which agrees with the old REE$|-$is higher than the new target. So the model predicts that along the transition route to a new equilibrium, the central bank, acting to lower inflation forces output to contract. However, if the central bank lowers the inflation target in an environment in with low inflation expectations, then the costs of these actions are minimised.

G.2 Robustness to changes in ρ

If the persistence of inflation expectations is low the expectational channel of monetary policy is less strong. However, the persistence of the output gap is less crucial because output expectations are expected to adjust quickly given that the output gap is a faster variable under learning. The results are robust to low levels of persistence in the beliefs about the output gap, but for simplicity the persistence of these beliefs is set to the same level as the persistence of the inflation expectations.

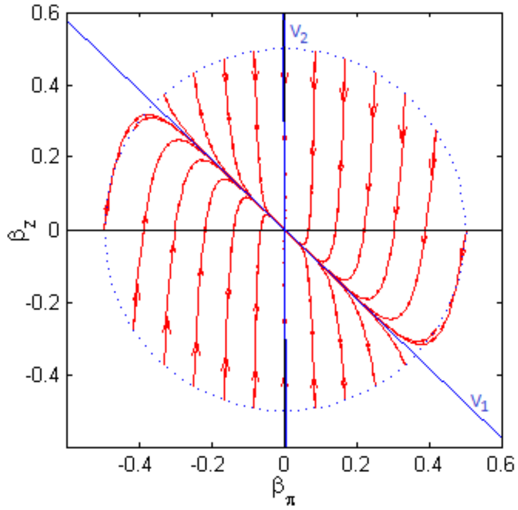
I calibrate ρ by recovering it from estimates of the mean reversion coefficient in the survey of inflation expectations data from Michigan Surveys of Consumers.

G.3 Robustness to changes in the belief system

The Taylor rule coefficients are set in response to a broad range of policy concerns and shocks to the economy. The policy rule, which is described by (3.4), can be suboptimal if other types of disturbances are considered. One of the appealing features of the Internal Rationality approach is that the agents' belief system (2.16) constitutes a well-defined building block for the model. This subsetion examines robustness of the results to

[15] A change in the inflation target results in a change in the origin of Figure 3

Figure 7: Trajectories of beliefs when $\rho = 0.985$



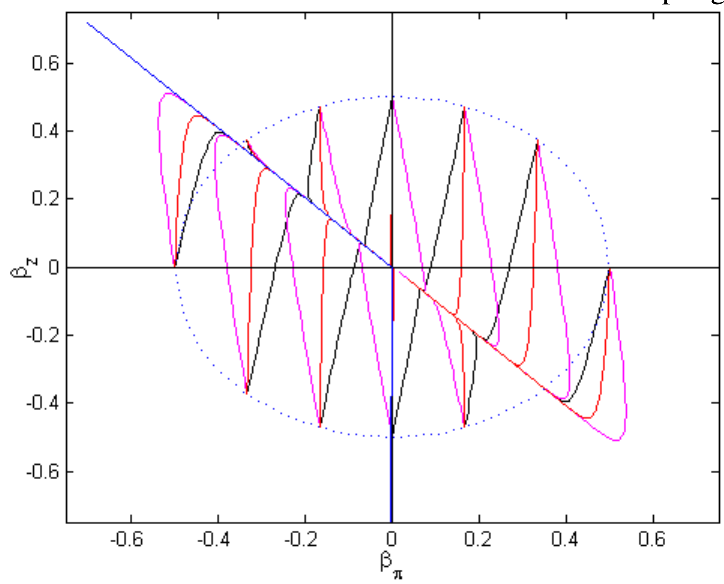
modifications of the system of beliefs.

It is interesting to compare the dynamics of the model if the central bank manages to convince the agents that some inflation is either positive or negative news for the future output. Appendix G.2 provides details of this robustness exercise. An important finding is that if inflation expectations are high but monetary authorities manage to signal to households and firms that some inflation is associated with an increase in future output, this signal improves the output gap's path to equilibrium. Appendix G.2 also describes the robustness for the asymmetry in gain parameters in (2.18), which stresses that if the gain's parameter for beliefs about the output gap in (2.22) is higher than for inflation, the output losses associated with high inflation expectations are even higher when compared to the baseline case.

In this section we study model dynamics given a change in the covariance matrix of ξ_t in (2.16), in particular we set its off-diagonal elements to either positive or negative values. This change in the covariance matrix results in a modification of the optimal filter (2.18) and (2.22) by transforming gain into a symmetric matrix with off-diagonal elements being different from zero. This change in gain matrix will affect both eigenvectors and eigenvalues of the dynamic system according to (3.2). It is interesting that the eigenvector corresponding to the largest eigenvalue does not change significantly, given this type of generalisation about beliefs system. However, the other eigenvector is very different when compared to the baseline case, so we can expect the transition seen to V_1 in Figure 3 to be most affected by this change.

Consider that agents internalize that the notion that permanent innovations to inflation and output are positively correlated in (2.16). The resulting dynamic is presented by the black lines in Figure 8. Red lines present trajectories from Figure 3, the baseline calibration when innovations to inflation and output gap are perceived by agents as independent. It is apparent that if the initial condition for beliefs is above V_1, then the dynamic presented by the black lines is preferable to the case of the baseline calibration in terms of output dynamic. However, the consequent dynamic of beliefs along the blue line also changes. A positive correlation slows down movements in beliefs along the blue line compared to the baseline case because the largest eigenvalue is now slightly larger compared to the case of no correlation in agents' beliefs. The pink curves in Figure 8

Figure 8: Trajectories of beliefs when beliefs about inflation and output gap are correlated



The figure presents trajectories for beliefs in the economy given three assumptions about the correlation between beliefs about inflation and the output gap in agents' perceived behaviour of economy (2.16). Red curves present trajectories for the case when inflation and output gap are autonomous variables in agents' minds, as represented in Figure 3. Black curves plot the case for a positive correlation between inflation and output gap in agents' belief systems. Pink curves plot the opposite case—when expectations of high inflation are connected to low output in agents' minds.

demonstrate the dynamics of the economy if agents perceive inflation and output gap as being negatively correlated. This can be interpreted as an increase in inflation being bad news for the dynamic of output in agents' minds. The pink curves in Figure 8 show that this modification of a belief system can be preferable in terms of the resulting dynamic of output, if beliefs happen to fall below V_1. Given $\beta < V_1$, the transition of the economy to the model equilibrium is accompanied by higher levels of output gap compared to the baseline calibration. At the same time, along V_1 the movement of beliefs in the direction of equilibrium is slightly faster compared to the case of independent innovations to inflation and output gap in agents' beliefs systems. This happens because the largest eigenvalue is slightly larger in the case of negative correlation when compared to the baseline case.

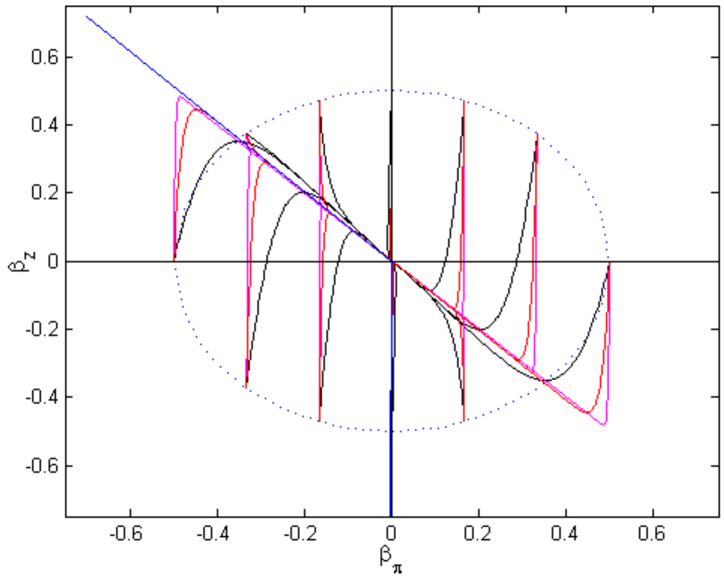
Therefore, if the monetary authorities have an accurate conception of the initial condition for the agents' beliefs and can persuade agents with regard to the sign and the extent of correlation between the dynamic of inflation and the output gap, it is possible to tailor the dynamic of the economy in such a way that along the transition route to equilibrium, the actual output costs are minimised.

G.4 Gain Asymmetry

In this section we study model dynamics given a change in the covariance matrix of ξ_t in (2.16), in particular we relax the assumption that $\sigma^2_{\xi,\pi} = \sigma^2_{\xi,z}$. However, shocks to inflation and output gap agents perceive as uncorrelated as before. This modification of agents' belief system changes gain parameters for inflation and output gap in (2.22).

The results of this robustness exercise are presented in Figure 9. Red curves present trajectories from Figure

Figure 9: Trajectories of beliefs given different assumptions about gain parameter



The figure presents trajectories for beliefs in the economy given three assumptions about the gain parameters in (2.19). Red curves present trajectories for the case when $\lambda_\pi = \lambda_z = 0.02$, as represented in Figure 3. Black curves plot the case $\lambda_z = 0.01$ and $\lambda_\pi = 0,05$. Pink curves plot the opposite case—$\lambda_z = 0.05$ and $\lambda_\pi = 0,01$.

3, which assumes that agents perceive $\sigma^2_{\xi,\pi} = \sigma^2_{\xi,z}$. The black curves present economy dynamics when gain parameter for inflation process is higher than for output gap. These dynamics demonstrate that in this case expectational channel is not as strong, even though high inflation expectations still result in output losses. The pink curves in Figure 8 demonstrate the dynamic of the economy when gain parameter for output gap is higher than for inflation. Figure 9 demonstrates that in this case high inflation expectation can lead to even bigger drop in output gap.

H Empirical facts

H.1 Data Sources

The *unemployment gap* is calculated as the difference between actual realised unemployment and the CBO's estimate of the short-run natural level of unemployment

The *output gap* which is measured as a percentage deviation in the real gross domestic product from real potential GDP (CBO's estimates).

Inflation is measured as Percent Change in price index from Year Ago.

H.2 Additional tests and discussion of results

The model's dynamics under IR, derived in Section 4, offer a prediction that when inflation expectations are high, then the agents' forecasting errors for inflation are positively related to the level of inflation in the economy. It is possible to directly check the sign of this dependence to see whether the model manages to match

Table 1: Data Sources

Variable	Frequency	Description	Source
RGDP	Quarterly	Real Gross Domestic Product, Billions of Chained 2012 Dollars, (SA)	U.S. Bureau of Economic Analysis*
RPGDP	Quarterly	Real Potential Gross Domestic Product, Billions of Chained 2009 Dollars, (NSA)	U.S. Congressional Budget Office*
UNRATE	Quarterly	Civilian Unemployment Rate, Percent, (SA)	U.S. Bureau of Labor Statistics*
NROU	Quarterly	Natural Rate of Unemployment (Short-Term), Percent (NSA)	U.S. Congressional Budget Office*
CPI	Monthly	Consumer Price Index for All Urban Consumers: All Items, (SA)	U.S. Bureau of Labor Statistics*
PCE	Monthly	Personal Consumption Expenditures, Billions of Dollars, (SA)	U.S. Bureau of Economic Analysis*
Deflator	Quarterly	Gross Domestic Product: Implicit Price Deflator, Index 2012=100, (SA)	U.S. Bureau of Economic Analysis*
MP data and surprises		Gertler and Karadi (2015) dataset	
	Inflation expectations		
MSC	Monthly	Surveys of Consumers, University of Michigan, Median expected price change next 12 months, Surveys of Consumers, percent	University of Michigan
SPF	Quarterly	Survey of Professional Forecasters, Livingston Survey, are one-year-ahead inflation forecasts	The Federal Reserve Bank of Philadelphia

*retrieved from FRED, Federal Reserve Bank of St. Louis

Table 2: Test results with GDP deflator and SPF forecasts of GDP deflator, quarterly data

Time period	ρ_1	ρ_2	P-value (ss correction) $H_0 : \rho_1 = \rho_2$	P-value (ss correction) $H_0 : \rho_1 \geq \rho_2$
JAN1978-MAY2016	0.8698 (5.8604)	0.8826 (20.7953)	0.64	0.06
JAN1985-NOV2007	0.6333 (4.6126)	0.7661 (5.5950)	0.28	0.05
DEC2007-MAY2016	-0.2951 (-1.8038)	0.2727 (5.4053)	0.00	0.035

Table 3: Robustness to different price level series (P-value of the test with small sample correction)

Time period	CPI	PCE	core CPI	core PCE
Surveys of Consumers - University of Michigan				
JAN1978-MAY2016	0.5	0.42	0.71	0.64
JAN1985-NOV2007	0.37	0.51	0.92	0.95
DEC2007-MAY2016	0.03	0.02	0.047	0.05
OECD Consumer Opinion Survey				
JAN1978-MAY2016	0.4175	0.247	0.63	0.4735
JAN1985-NOV2007	0.1740	0.276	0.84	0.7610
DEC2007-MAY2016	0.025	0.02	0.11	0.079

Table 4: Test results with slack variables

Time period	ρ_1	ρ_2	P-value $H_0 : \rho_1 \leq \rho_2$	ρ_1	ρ_2	P-value $H_0 : \rho_1 \geq \rho_2$
	Unemployment Gap			Output Gap		
JAN1985-NOV2007	-0.29 (-1.84)	-0.47 (-2.23)	0.07	0.03 (0.36)	-0.09 (-0.78)	0.14
DEC2007-MAY2016	0.11 (1.77)	-0.29 (-5.13)	0.004	-0.01 (-0.17)	0.18 (4.60)	0.01

it.[16] For the Great Moderation, the survey evidence indicates that forecasting errors are not related to current inflation; however, for the Great Recession period forecasting errors increase with current inflation.[17]

What the model does not match is the negative value of the point estimate for ρ_1 for the Great Recession, which is not an issue because the estimate is insignificant in the data. The estimates of $\hat{\rho_1}$ and $\hat{\rho_2}$ in simulated data are positive for both subsamples. However, the source of the negative sign of the $\hat{\rho_1}$ is deflation, which took place during the recession period. At the same time, the model is designed to match the recovery from recession (this is what simulations present) and not the recession itself. If ρ_1 is estimated solely for the recovery after the Great Recession, then the estimate of ρ_1 becomes positive($\hat{\rho_1} = 0.21$ with t-stat of 5). Therefore, the model matches $\hat{\rho}_1$ and $\hat{\rho}_2$ for the recovery after the Great Recession.

Table 5: Test results with slack variables

Time period	ρ_1	ρ_2	P-value $H_0 : \rho_1 \leq \rho_2$	ρ_1	ρ_2	P-value $H_0 : \rho_1 \geq \rho_2$
	Unemployment Gap			Output Gap		
JAN1985-NOV2007	-0.35 (-1.43)	-0.36 (-4.08)	0.11	0.1 (0.88)	0.06 (0.99)	0.13
DEC2007-MAY2016	0.62 (3.06)	-0.30 (-2.18)	0.008	-0.51 (-2.60)	0.24 (1.95)	0.013
JUL2009-MAY2016	0.83 (5.23)	0.03 (0.25)	0.01	-1.01 (-6.38)	-0.44 (-3.30)	0.03

Notes: *t* statistics in parentheses, HAC covariance estimator

The point estimates of the slope coefficients in (4.4) for the output gap in simulations are negative for both periods. The model matches the fact that estimates of ρ_1 and ρ_2 for the Great Moderation are smaller in absolute value than for the Great Recession. At the same time, the estimates for the first period are not significant, which is in line with results from Table 5. For the Great Recession, the mean estimates of coefficients in simulations are $\rho_1^{m,GR} = -0.39$ and $\rho_2^{m,GR} = -1.04$, and these estimates are close to those for the recovery period, which are in the last row in Table 5.

According to Figures 3, the model provides predictions that positive forecasting errors for inflation are linked to a negative output gap. If I estimate a relationship between the forecasting error and the same period's output gap (in the test above, the output gap is lagged one more quarter to ensure that it was in the information set at t) for the Great Recession, the prediction error is significantly negatively correlated with the contemporaneous output gap.[18] This empirical evidence supports the predictions of the model.

[16]The results presented in Table1 check the relation between forecasting error and lagged inflation. The results presented in the table show that the forecasting error and lagged inflation rate are positively related, which is also consistent with the model-generated dynamic.

[17]The regression model is $e_t = \alpha + \beta\pi_t + \epsilon_t^{\pi}$ and the point estimates for the two sub-periods with t-stats in brackets are $\beta^{GM} = 0.03(0.17)$ and $\beta^{GR} = 0.64(3.01)$.

[18]The $e_t = \beta x_t + \epsilon_t^e$, $\beta^{\hat{G}R} = -0.31, (-2.53)$ for the Great Recession and $\beta^{\hat{G}M} = -0.05, (-0.83)$ for the Great Moderation.

H.3 Simple additional tests using the forecasting error properties

Under REH prediction errors must be i.i.d. and orthogonal to all information observed at the moment of forecasting. This section presents several simple tests that are built around those properties of forecasting error, which enriches the findings documented in the main text.

The first test checks bias in the forecasting errors in the data for the periods of Great Moderation and Great Recession. The following model is estimated with the forecasting errors (e_t) that are calculated as the difference between surveyed ($s_{t,t+12}$) and actually-realised inflation ($\pi_{t,t+12}$):

$$\underbrace{s_{t,t+12} - \pi_{t,t+12}}_{e_t} = a + \epsilon_t \tag{H.1}$$

Under RE there must be no statistically significant bias in H.1 ($a = 0$) because $s_{t,t+12} - \pi_{t,t+12}$ is a prediction error from the true data-generating process. The measurement error associated with surveys would not bias the results if its properties did not change dramatically between subsamples, and this is unlikely.

The table below reports estimates of a using the Michigan Survey of Consumers and two measures of price indices:

Table 6: Bias in prediction error

Time period		CPI	Core CPI
	$\hat{a}$	0.1386	0.0746
JAN1978-MAY2016	t-stat	(0.4216)	(0.2226)
	$\hat{a}$	0.0022	-0.0070
JAN1985-NOV2007	t-stat	(0.0123)	(-0.0270)
	$\hat{a}$	1.8383^{***}	1.4853^{***}
DEC2007-MAY2016	t-stat	(4.1163)	(10.2798)

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$
Newey-West HAC standard errors

The estimates demonstrate a significant bias in forecasting errors for the period of the Great Recession and the test fails to reject the null hypothesis for the Great Moderation period. This evidence once again suggests that at least for the Great Recession period, the data fails to support REH and expectations of inflation were persistently higher than realised inflation. These findings are confirmed in simulations, and the learning model initialized at the REE does not generate a persistent bias in forecasting errors, but at the same simulations with beliefs initialized away from long-run equilibrium generate a persistent positive bias in forecasting error. This confirms not only the significance of estimates of the bias in simulations (across all simulations t-stats are higher than 10) but also its sign.

The next test checks whether forecasting error for inflation next year is orthogonal to the inflation experienced in the previous period (π_{t-1}).[19] This inflation rate was in the information set of the consumers when they were surveyed, therefore, under RE the prediction errors must be orthogonal to it.

[19]This test similar to the one presented in Section 4, but it uses less information.

$$e_t = \beta\pi_{t-1} + \epsilon_t^2 \tag{H.2}$$

In other words, under RE $H_0 : \beta = 0$. Table 7 below presents results for estimates of β in H.2. The results are confirmed with simulated data and for the simulations corresponding to the Great Recession and recovery period the model delivers high significant estimates of β.

The research has shown that forecasting errors for inflation have high persistence in the data, and this finding holds for both subsamples. An adaptive learning model with learning about inflation dynamics, which the model presented in this paper nests, can generate high persistence in forecasting errors. This result follows from the adaptive nature of the expectations' formation itself, when agents only partially update their beliefs with each new observation. Therefore, inflation expectations can be persistently lower or higher than actual inflation when agents are learning about the evolution of prices.

Forecasting errors for inflation exhibit high persistence in the data, and this finding holds for both subsamples. In particular, we estimate the following relationship for two subsequent periods:

$$e_t = \rho_e e_{t-1} + \eta_t^e \tag{H.3}$$

where e_t is the forecasting error, calculated as the difference between the survey measure of inflation expectations and realised inflation, $e_t = s_{t,t+12} - \pi_{t,t+12}$.[20] The estimates of persistence coefficients in the data are $\hat{\rho}_e^{GM} = 0.86$ and $\hat{\rho}_e^{GR} = 0.96$ for periods of the Great Moderation and the Great Recession, respectively. This fact cannot be easily addressed by the RE model because under rational expectations forecasting errors are supposed to be i.i.d, as was explained in the previous section. Nevertheless, the data delivers high values of estimates of autocorrelation coefficient for both periods.

An adaptive learning model with learning about inflation dynamics, which the model presented in this paper nests, can generate persistence in forecasting errors. This result follows from the adaptive nature of expectations formation itself, when agents only partially update their beliefs with each new observation. Therefore, inflation expectations can be persistently lower or higher than actual inflation when agents are learning about the evolution of prices. If the autocorrelation coefficient for forecasting errors is calculated using simulated data, the point estimate is highly independent of the initial condition for beliefs. The persistence coefficient in (H.3) is estimated at $\hat{\rho}_e = 0.93$ with 95% confidence bands [0.9;0.95], which is in line with empirical estimates.[21]

[20]the Michigan Surveys of Consumers inflation expectation data and CPI inflation.

[21]The point estimate of the persistence coefficient is calculated as a median level among 1000 estimates of ρ^e, obtained from independent simulations. The confidence interval is constructed using the quantiles of the same empirical distribution.

Table 7: Estimates of β in H.2

Time period	CPI	Core CPI	PCE	Core PCE
	Surveys of Consumers - University of Michigan			
JAN1978-MAY2016	-0.0439	-0.0776	0.0937	0.0890*
	(-0.5405)	(-1.5646)	(1.3748)	(2.0811)
JAN1985-NOV2007	0.0057	-0.0588	0.1787*	0.1390
	(0.0887)	(-0.8566)	(2.1210)	(1.3894)
DEC2007-MAY2016	0.8138***	0.8311***	1.0171***	1.0867***
	(4.6118)	(11.7229)	(5.8364)	(14.1358)
	OECD Consumer Opinion Survey			
JAN1978-MAY2016	0.1323	0.1090	0.3084***	0.3210***
	(1.3269)	(1.8531)	(3.6390)	(5.4959)
JAN1985-NOV2007	0.2656***	0.2005**	0.4915***	0.4544***
	(3.8587)	(3.1369)	(5.3829)	(4.5607)
DEC2007-MAY2016	1.0555***	1.2033***	1.3289***	1.5129***
	(6.2569)	(15.5470)	(8.1824)	(19.8479)

t statistics in parentheses, HAC covariance estimator
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 8: SPF and deflator

	SPF $(s_{t,t+12})$	Deflator $(\pi_{t-12,t})$	FE (e_t)
GM			
mean	2.77	2.48	0.35
variance	0.72	0.59	0.72
ρ	0.99	0.99	0.92
$var(x_t \vert x_{t-1})$	0.05	0.05	0.1
b		1.08	
GR			
mean	1.82	1.57	0.32
variance	0.07	0.30	0.4
ρ	0.99	0.96	0.88
$var(x_t \vert x_{t-1})$	0.027	0.093	0.12
b		1.07	

$e_t = s_{t,t+12} - \pi_{t,t+12}$
$x_t = \rho x_{t-1} + \epsilon_t$; $E\pi_t = b\pi_t + \epsilon_t$
quarterly data

Table 9: Missing Disinflation

	$\hat{\mathbf{a}}$	$\hat{\mathbf{b}}$
JAN1985-NOV2007	-0.97 [-1.77; -0.16]	0.02 [-0.24; 0.27]
DEC2007-MAY2016	-3.47 [-4.3; -2.6]	0.39 [0.03; 0.75]
JUL2009-MAY2016	-2.91 [-3.8; -2.01]	-0.11 [-0.59; 0.37]

Table 10: Missing Disinflation (Model)

	$\hat{\mathbf{a}}$	$\hat{\mathbf{b}}$
JAN1985-NOV2007	-0.08	0.03
DEC2007-MAY2016	-2.34	-0.11