You know, I don't know about that. <u>16 March 2020</u>

- I wanna pause omi for a sec b/c I can't seem to get it to work and I'm confused about how to get it time-varying anyway. So let's turn to estimation.

Estimation of the anchoring function

The issue is that we wanna estimate the anchoring fct together w/ the model. On the top of my head I can think of 3 ways of doing that:

1) IR-matching
2) Likelihood-based (either MLE or Bayesian)
3) VAR-representation ⇒ exist? est. that!
   ↳ it would be a time-varying one.

↳ I'm leaning toward #2 b/c 1) it's sexier
2) it's more general than conditional on shocks
3) a TV-VAR sounds challenging

The thing is: 17 March 2020

need to derive the log-likelihood of my model

Take the midsimple model w/ the TR.

In materials 21, this is eq. (5)-(10) + TR

$$\pi_t = \kappa x_t - (1-\alpha)\beta f_a(t) + \left[-\kappa\alpha\beta(I_3-\alpha\beta h x)^{-1} - e_3(I_3-\alpha\beta h x)^{-1}\right]s_t$$

$$x_t = -\sigma i_t - \sigma f_b(t) + \left[-(1-\beta)b_2(I_3-\beta h x)^{-1} + \sigma\beta b_3(I_3-\beta h x)^{-1} - \sigma e_1(I_3-\beta h x)^{-1}\right]s_t$$

$$f_a(t) = \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} - b_1(I_3-\alpha\beta h x)^{-1}s_t$$

$$f_b(t) = \frac{1}{1-\beta}\bar{\pi}_{t-1} - b_1(I_3-\beta h x)^{-1}s_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\right)$$

$$k_t^{-1} = k_{t-1}^{-1} + d\left(\pi_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})\right) + c$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t$$

This is a state-space model (believe it or not)

and I'm gonna eliminate some variables

$$\pi_t = \kappa x_t - \underline{\frac{(1-\alpha)\beta}{1-\alpha\beta}}\bar{\pi}_{t-1} - (1-\alpha)\beta b_1(I_3-\alpha\beta h x)^{-1}s_t$$
$$+ \left[-\kappa\alpha\beta(I_3-\alpha\beta h x)^{-1} - e_3(I_3-\alpha\beta h x)^{-1}\right]s_t$$

$$x_t = -\sigma\psi_\pi\pi_t - \sigma\psi_x x_t - \sigma\bar{i}_t - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1} - \sigma b_1(I_3-\beta h x)^{-1}s_t$$
$$+ \left[-(1-\beta)b_2(I_3-\beta h x)^{-1} + \sigma\beta b_3(I_3-\beta h x)^{-1} - \sigma e_1(I_3-\beta h x)^{-1}\right]s_t$$

$$\pi_t = \kappa x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h x)^{-1} S_1 - \kappa\alpha\beta (I_3 - \alpha\beta h x)^{-1} - e_3 (I_3 - \alpha\beta h x)^{-1}\right] S_t$$

$$x_t = -\sigma \psi_\pi \pi_t - \sigma\psi_x x_t - \sigma \bar{i}_t - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1} - \sigma b_1 (I_3 - \beta h x)^{-1} S_t$$

$$+ \left[-(1-\beta) b_2 (I_3 - \beta h x)^{-1} + \sigma\beta b_3 (I_3 - \beta h x)^{-1} - \sigma e_1 (I_3 - \beta h x)^{-1}\right] S_t$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t \qquad - \frac{\sigma}{1-\beta}\bar{\pi}_{t-1}$$

$$\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h x)^{-1} - (1-\beta) b_2 (I_3 - \beta h x)^{-1} + \sigma\beta b_3 (I_3 - \beta h x)^{-1} - \sigma e_1 (I_3 - \beta h x)^{-1}\right] S_t$$

$$\Rightarrow \quad x_t = -\frac{\sigma\psi_\pi}{1+\sigma\psi_x}\pi_t - \frac{1}{1+\sigma\psi_x}\frac{\sigma}{1-\beta}\bar{\pi}_{t-1}$$

$$+ \frac{1}{1+\sigma\psi_x}\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h x)^{-1} - (1-\beta) b_2 (I_3 - \beta h x)^{-1} + \sigma\beta b_3 (I_3 - \beta h x)^{-1} - \sigma e_1 (I_3 - \beta h x)^{-1}\right] S_t$$

Can even sub x_t out!

$$\pi_t = -\frac{\kappa\sigma\psi_\pi}{1+\sigma\psi_x}\pi_t - \frac{\kappa}{1+\sigma\psi_x}\frac{\sigma}{(1-\beta)}\bar{\pi}_{t-1}$$

$$+ \frac{\kappa}{1+\sigma\psi_x}\left[-\sigma e_2 - \sigma b_1 (I_3 - \beta h x)^{-1} - (1-\beta) b_2 (I_3 - \beta h x)^{-1} + \sigma\beta b_3 (I_3 - \beta h x)^{-1} - \sigma e_1 (I_3 - \beta h x)^{-1}\right] S_t$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h x)^{-1} S_1 - \kappa\alpha\beta (I_3 - \alpha\beta h x)^{-1} - e_3 (I_3 - \alpha\beta h x)^{-1}\right] S_t$$

$$\Leftrightarrow \left(1 + \frac{\kappa\sigma\psi_\pi}{1+\sigma\psi_x}\right)\pi_t = -\left(\frac{\kappa\sigma}{1+\sigma\psi_x(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right)\bar{\pi}_{t-1} + \text{stuff}\cdot S_t$$

$$\left(1+\frac{k\tau\psi_\pi}{1+\tau\psi_x}\right)\pi_t = -\left(\frac{k\tau}{1+\tau\psi_x(1-\beta)}+\frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right)\bar{\pi}_{t-1} +$$

$$\Bigg\{\frac{k}{1+\tau\psi_x}\left[-\tau e_2-\tau b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\tau\beta b_3(f_3-\beta h x)^{-1}-\tau e_1(f_3-\beta h x)^{-1}\right]$$

$$+\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Bigg\}S_t$$

$$\frac{1+\tau\psi_x+k\tau\psi_\pi}{1+\tau\psi_x}\pi_t = \text{same}$$

$$\Rightarrow \pi_t = -\frac{\cancel{1+\tau\psi_x}}{1+\tau\psi_x+k\tau\psi_\pi}\left(\frac{k\tau(1-\alpha\beta)+(1-\alpha)\beta(1-\beta)(1+\tau\psi_x)}{\cancel{(1+\tau\psi_x)}(1-\beta)(1-\alpha\beta)}\right)\bar{\pi}_{t-1}$$

$$+\Bigg\{\frac{k}{1+\tau\psi_x+k\tau\psi_\pi}\left[-\tau e_2-\tau b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\tau\beta b_3(f_3-\beta h x)^{-1}-\tau e_1(f_3-\beta h x)^{-1}\right]$$

$$+\frac{1+\tau\psi_x}{1+\tau\psi_x+k\tau\psi_\pi}\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Bigg\}$$

$$\cdot S_t$$

$$\Rightarrow \pi_t = -\frac{k\tau(1-\alpha\beta)+\beta(1-\alpha)(1-\beta)(1+\tau\psi_x)}{(1+\tau\psi_x+k\tau\psi_\pi)(1-\beta)(1-\alpha\beta)}\bar{\pi}_{t-1} +$$

$$\Bigg\{\frac{k}{1+\tau\psi_x+k\tau\psi_\pi}\left[-\tau e_2-\tau b_1(f_3-\beta h x)^{-1}-(1-\beta)b_2(f_3-\beta h x)^{-1}+\tau\beta b_3(f_3-\beta h x)^{-1}-\tau e_1(f_3-\beta h x)^{-1}\right]$$

$$+\frac{1+\tau\psi_x}{1+\tau\psi_x+k\tau\psi_\pi}\left[-(1-\alpha)\beta b_1(f_3-\alpha\beta h x)^{-1}s_1-k\alpha\beta(f_3-\alpha\beta h x)^{-1}-e_3(f_3-\alpha\beta h x)^{-1}\right]\Bigg\}S_t$$

Damn darn! So we have $\equiv A$

$$\pi_t = - \frac{\kappa\sigma(1-\alpha\beta) + \beta(1-\alpha)(1-\beta)(1+\sigma\psi_x)}{(1+\sigma\psi_x+\kappa\sigma\psi_\pi)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\Bigg\{ \frac{\kappa}{1+\sigma\psi_x+\kappa\sigma\psi_\pi} \Big[-\sigma e_2 - \sigma b_1 (f_3-\beta h x)^{-1} - (1-\beta) b_2 (f_3 - \beta h x)^{-1} + \sigma\beta b_3 (f_3-\beta h x)^{-1} - \sigma e_1 (f_3 - \beta h x)^{-1} \Big]$$

$$+ \frac{1+\sigma\psi_x}{1+\sigma\psi_x+\kappa\sigma\psi_\pi} \Big[-(1-\alpha)\beta b_1 (f_3 - \alpha\beta h x)^{-1} s_1 - \kappa\alpha\beta (f_3 - \alpha\beta h x)^{-1} - e_3 (f_3 - \alpha\beta h x)^{-1} \Big] \Bigg\} s_t$$

$\equiv B$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \right)$$

$$k_t^{-1} = k_{t-1}^{-1} + d \left(\pi_t - (\bar{\pi}_{t-1} - b_1 s_{t-1}) \right) + c$$

$\hookrightarrow$ 1 jump (π_t), 3 exog states $\left(s_t = \begin{bmatrix} r_t^n \\ \bar{i}_t \\ u_t \end{bmatrix} \right)$ and 2 endog states $\zeta_t = \begin{bmatrix} \bar{\pi}_t \\ k_t^{-1} \end{bmatrix}$ $\left(\text{or} \begin{bmatrix} \bar{\pi}_{t-1} \\ k_t^{-1} \end{bmatrix} \right)$ so $X_t = \begin{bmatrix} \zeta_t \\ s_t \end{bmatrix}$ (states)

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \zeta_t \\ s_t \end{bmatrix}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \right)$$

$$k_t^{-1} = k_{t-1}^{-1} + d \Big(\underbrace{\pi_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})}_{\text{call this the state(?) } fe_{t-1}} \Big) + c$$

$\pi_t = A\bar{\pi}_{t-1} + B s_t \quad = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$ $\quad Y_t = gx \cdot X_t$

$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} fe_{t-1}$ $\quad X_{t+1} = hx X_t + \eta \varepsilon_t$

$k_t^{-1} = k_{t-1}^{-1} + d \cdot fe_{t-1} + c$

$fe_{t-1} = \pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}$

Several issues y'all:

1) fe state or jump? depends on π_t (a jump)

→ there's gotta be some trick (like π_t in LEMP) to make it a pure state

2.) $\bar{\pi}_t$ nonlinear LOM!

ok – that's troubling. But let's pause it for a sec & let's read what Lütkepohl has to say about MLE & log-likelihoods.

Supp. our VAR(p) looks like

$$y_t - \mu = A_1(y_{t-1} - \mu) + \ldots + A_p(y_{t-p} - \mu) + u_t \quad (3.3.1)$$

p. 75

Then if the VAR(p) is Gaussian, that is

$u \equiv \mathrm{vec}(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim N(0, I_T \otimes \Sigma_u)$, which equivalently means that the prob. density of u is

$$f_u(u) = \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp\left[-\frac{1}{2} u' (I_T \otimes \Sigma_u^{-1}) u \right]$$

then we can use the fact that $u = y - \mu^* - (X' \otimes I_K)\alpha$

where $\underset{K^2p \times 1}{\alpha := \mathrm{vec}(A)}$, $A := (A_1, \dots, A_p)$ $K \times Kp$

$Y^0 := (y_1 - \mu, \dots, y_T - \mu)$ $\quad K \times T$

$Y_t^0 := \begin{bmatrix} y_t - \mu \\ \vdots \\ y_{t-p+1} - \mu \end{bmatrix}$ $\quad (Kp \times 1)$

$X := (Y_0^0, \dots, Y_{T-1}^0)$,

to write

$$f_y(y) = \left| \frac{\partial u}{\partial y} \right| f_u(u)$$

$$= \frac{1}{(2\pi)^{KT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp\left[-\frac{1}{2} \left(y - \mu^* - (X' \otimes I_K)\alpha \right)' (I_T \otimes \Sigma_u^{-1}) \cdot \left(y - \mu^* - (X' \otimes I_K)\alpha \right) \right] \quad (3.4.4)$$

and

$$\ln l(\mu, \alpha, \Sigma_u) = -\frac{KT}{2} \ln(2\pi) - \frac{T}{2} \ln|\Sigma_u| - \frac{1}{2} \mathrm{tr}(Y^0 - AX)' \Sigma_u^{-1} (Y^0 - AX)$$

is the log-likelihood. (3.4.5)

Given that the updating eq. w/ endog. gain introduces non-linearities, I'm afraid that even a "simple" & quick estimation has to involve some form of particle filter. But let's see whether Lütkepohl has anything interesting to say about 1) state-space models 2) non-linearities

18 March 2020

Lütkepohl, State-space models (Ch. 13, p. 415 ff.)

$$\underset{N\times 1}{z_{t+1}} = \underset{N\times N}{B_t} z_t + \underset{N\times m}{F_t} x_t + \underset{N\times 1}{w_t} \quad \text{noise} \qquad t = 0, 1, 2, \ldots \qquad (13.2.1a)$$

$$\underset{k\times 1}{y_t} = \underset{k\times N}{H_t} z_t + \underset{k\times M}{G_t} \underset{M\times 1}{x_t} + \underset{k\times 1}{v_t} \qquad t = 1, 2, \ldots \qquad (13.2.2)$$

measurement error

inputs / instruments / policy vars

transition matrix

measurement matrix

input matrices

and $\begin{bmatrix} w_t \\ v_t \end{bmatrix} \sim WN\left[0, VC_t\right]$ $\qquad VC_t = \begin{bmatrix} \Sigma_{w_t} & \Sigma_{w_t v_t} \\ \Sigma_{v_t w_t} & \Sigma_{v_t} \end{bmatrix}$

Nonlinear state-space models

$$z_{t+1} = b_t(z_t, x_t, w_t, \delta_1)$$

$$y_t = h_t(z_t, x_t, v_t, \delta_2)$$

vectors of params

Example of nonlin state-space is the "bilinear" model:

$y_t = \alpha y_{t-1} + u_t + \beta y_{t-1} u_{t-1}$ p. 427 w/ Refs.

$\hookrightarrow$ $z_{t+1} = B z_t + w_t + C \operatorname{vec}(z_t z_t')$ (13.2.33)

$y_t = [I_K \; 0 \dots 0] \, z_t$ (13.2.34)

$\Rightarrow$ Bunch of refs on bilinear systems, univariate & multivariate on p. 427 bottom.

MLE of state-space models p. 434

Gather the time-invariant params from $B, F, H_t, G, \Sigma_w, \Sigma_v$ and Σ_0 & μ_0 in δ as

$$\delta = \begin{bmatrix} \operatorname{vec}[\nu, A_1, \dots, A_p] \\ \operatorname{vech}(\Sigma_u) \end{bmatrix}$$

where vech = "half-vectorization", $\operatorname{vech}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{bmatrix} a \\ b \\ d \end{bmatrix}$.

The log-likelihood for the Gaussian state-space model is:

$$\ln l(\delta | y_1, \dots, y_T) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln |\Sigma_y(t|t-1)|$$
$$- \frac{1}{2} \sum_{t=1}^{T} (y_t - y_{t|t-1})' \Sigma_y(t|t-1)^{-1} (y_t - y_{t|t-1}) \quad (13.4.1)$$

Denoting the fcst error $e_t(\delta) := y_t - y_{t|t-1}$

and $\Sigma_t(\delta) := \Sigma_y(t|t-1)$, we can rewrite this as

$$\ln l(\delta) = -\frac{KT}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln|\Sigma_t(\delta)| + e_t'(\delta)\Sigma_t(\delta)^{-1}e_t(\delta)\right] \quad (13.4.3)$$

which makes explicit

1) the dependence of $\ln l$ on δ,
2) that all quantities in the $\ln l$ all functions of δ and can (must) be computed using the Kalman filter.

locally identified: when in a subspace of the param space, δ is uniquely determined.

vs. **globally identified**: when δ is uniquely determined in the entire param space.

↳ identification: we need a min of $-\ln l$, so we need some sort of Hessian = pos. def. ⇒ the information matrix, = E[Hessian] = $E\left[\frac{\partial^2(-\ln l)}{\partial\delta\partial\delta}\Big|_{\delta_0}\right]$

A quick note on DSEMs (dynamic simultaneous equations models a.k.a. "linear systems") p. 323

essentially, these are linear VARMAX(p,s,q) models

$$A_0 y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + B_1 x_{t-1} + \dots + B_s x_{t-s} + w_t \quad (10.1.1)$$

- VARMAX(p,s,q) is w_t is MA(q)
- VARX(p,s) if $w_t \sim WN$.

VAR(p) models w/ time-varying coefficients p. 391ff.

periodic VARs → e.g. w/ seasonal dummies

intervention models → DGP_1 is replaced by DGP_2 at time τ.

$$y_t = v_t + A_{1t} y_{t-1} + \dots + A_{pt} y_{t-p} + u_t \quad (12.2.1)$$

↳ $WN(0, \Sigma_t)$

also time-varying

(not identically distrib!)

Rewrite the VAR(p) as a VAR(1)

$$Y_t = v_t + A_t Y_{t-1} + U_t \qquad (12.2.2)$$

$$\underset{kp \times 1}{Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}}, \quad \underset{kp \times 1}{v_t := \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \quad \underset{kp \times kp}{A_t := \begin{bmatrix} A_{1t} & \cdots & A_{p-1,t} & A_{pt} \\ I_k & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_k & 0 \end{bmatrix}}, \quad \underset{kp \times 1}{U_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

And by recursive sub we get

$$Y_t = \left[\prod_{j=0}^{h-1} A_{t-j}\right] Y_{t-h} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j}\right] V_{t-i} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j}\right] U_{t-i} \quad (12.2.3)$$

Defining $J := [I_k \ \ 0]$ such that $y_t = J Y_t$, we can premultiply (12.2.3) by J, define

$\Phi_{it} := J\left[\prod_{j=0}^{i-1} A_{t-j}\right] J'$ to get

$$y_t = \mu_t + \sum_{i=0}^{\infty} \Phi_{it} u_{t-i} \qquad (12.2.6)$$

where $\mu_t = E[y_t]$

$\Rightarrow$ Then the MSE (or FEV of the FE $y_{t+h} - y_t(h)$) is

$$\Sigma_t(h) := \sum_{i=0}^{h-1} \Phi_{i,t+h} \Sigma_{t+h-i} \Phi'_{i,t+h} \qquad (12.2.10)$$

MLE of TV-VAR p. 394

Write (12.2.1) as $y_t = B_t Z_{t-1} + u_t$ (12.2.11)

where $B_t := [\nu_t, A_{1t}, \dots, A_{pt}]$, $Z_{t-1} := (1, Y_{t-1}')'$

B_t depend on the vector $\underset{N\times 1}{\gamma}$ of time-invariant params.

Σ_t depend on $\underset{M\times 1}{\sigma}$ of fixed params.

If $u_t \sim N(0, \Sigma_t)$, the log-likelihood is

$$\ln l(\gamma, \sigma) = -\frac{KT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^{T} \ln |\Sigma_t| - \frac{1}{2} \sum_{t=1}^{T} u_t' \Sigma_t^{-1} u_t$$

(12.2.12)

where initial conditions have been ignored.

You can also derive an info-matrix.

So it seems like you just min $-\ln l(\gamma, \sigma)$!

It also seems like for certain special cases you can even derive the estimators in closed-form!

Ryan meeting 18 March 2020

Not yet convinced that the ~~procedure~~ perm is sensible

→ the only procedure that can work is

$f_{z,t}$ and more RHS-variables: k^{-1}, $\bar{\pi}$

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

things in $f_{z,t}$ will be 0 or simply described by the LOMs

It may be too restrictive.

A computationally intensive exercise is:

- Conj. path for the int. rate (exog.)
- solve the model for that int. rate
- check the target criterion, compute the resid
- fmincon to min that resid

⇒ find simulated optimal plan.

- Simulate the Ramsey model
- Simulate the model w/ Taylor-rule
  ⇒ see how close you can get

It might then be that most results won't be pencil & paper.

Comments:

- Concerning w/ linearness of (10)
  → "negative surprises cause me to be unanchored" shouldn't be "big mistakes" → so take the square
  ↳ like smooth one better than jumpy
- Estim: filter the data → App paper w/ Robert:
  match the params to moments cors of data
  ⇒ would give you results

They est an NK model: HP-filter both data and model, compute moments and try to match those.

Work after

It's the primal economy paper w/ Robert that Ryan meant. → chahrourWorkAt.pdf in "Literature."

⇒ They [1] take data on real per-capita output, inflation, nom interest rates & per-capita hours, [2] (BK) high-pass filter them, [3] Use GMM to estimate the process $\hat{\tau}_t$ that best-matches the autocov. structure of the data.

App. D p. 46 (Mac).

Wedges are an MA(14):

$$\tau_t = \Phi_\varepsilon(L)\,\varepsilon_t + \Phi_u(L)\,u_t$$

ε_t: $WN(0, \sigma_\varepsilon)$ $\quad$ u_t: $2\times 1\ WN(0, I_2)$

⇒ params to be estimated are

$$\gamma_{ma} \equiv (\Phi_\varepsilon, \Phi_u, \sigma_\varepsilon)$$

I think that the following is the target: $k=8$

$$\tilde{\Omega}_T = vech\left\{ Var\left[\tilde{q}_t^{data}, \ldots, \tilde{q}_{t-k}^{data}\right]\right\} \quad q: y_t, \pi_t, i_t, n_t$$

Filtering: Baxter-King filter w/ truncation horizon 32, lag-length 12

→ $\tilde{q}_t \equiv BK_{32}(q_t)$, so $\tilde{q}_t$ is filtered data.

Then, they do a trick in converting γ_{ma} into γ_{ar} to finally estimate γ as:

$$\hat{\gamma}_{ar} = \underset{\gamma_{ar}}{\operatorname{argmin}} \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{ar})\right)' W^{-1} \left(\tilde{\Omega}_T - \tilde{\Omega}(\gamma_{ar})\right)$$

- W is a diagonal matrix w/ the bootstrapped variances of $\tilde{\Omega}_T$ along the main diagonal.
- The model analogue $\tilde{\Omega}(\gamma_{ar})$ is computed after the model data has been similarly filtered as the data.

↳ Hold it there, now I know what I need to do.

↳ That and

- the numerical implementation of the target criterion ⇒ both are things to do once I have the big screen. So now give a last try to do the time-varying one.

The ori – a first shot

Ryan wrote:

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

I think he meant $z_t = \bar{z}_t + h_{z,t} z_{t-1} + f_{z,t} u_t$

Let me ignore r_t^n b/c it just blows things up and I just wanna see if it works. Ignore $\bar{z}_t$ too.

$$h_{\pi,t} \pi_{t-1} + f_{\pi,t} u_t - \kappa (h_{x,t} x_{t-1} + f_{x,t} u_t)$$
$$- (1-\alpha)\beta \left(h_{fa,t} fa(t-1) + f_{fa,t} u_t \right) + exog_1 \cdot u_t = 0 \quad (9)$$
$$h_{x,t} x_{t-1} + f_{x,t} u_t + \sigma(h_{i,t} i_{t-1} + f_{i,t} u_t)$$
$$- \sigma \left(h_{fb,t} fb(t-1) + f_{fb,t} u_t \right) + exog_2 \cdot u_t = 0 \quad (10)$$

$\Leftrightarrow$

$$h_{\pi,t} \pi_{t-1} - \kappa h_{x,t} x_{t-1} - (1-\alpha)\beta h_{fa,t} fa(t-1)$$
$$+ \underbrace{\left(f_{\pi,t} - \kappa f_{x,t} - (1-\alpha)\beta f_{fa,t} + exog_1 \right)}_{\text{this is exactly what I have in materials 21}} u_t = 0 \quad (10)$$

$$h_{x,t} x_{t-1} + \sigma h_{i,t} i_{t-1} - \sigma h_{fb,t} fb(t-1)$$
$$+ \left(f_{x,t} + \sigma f_{i,t} - \sigma f_{fb,t} + exog_2 \right) u_t = 0$$

$$z_t = \bar{z}_t + h_{z,t}\, z_{t-1} + f_{z,t}\, u_t$$

$$h_{fa,t}\, f_a(t-1) + f_{fa,t}\, u_t - \frac{1}{1-\alpha\beta}\left(h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + f_{\bar{\pi},t-1}\, u_{t-1}\right) + exog_3\, u_t$$

$$h_{fa,t}\, f_a(t-1) - \frac{1}{1-\alpha\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + \underbrace{\left(f_{fa,t} + exog_3\right) u_t - \frac{1}{1-\alpha\beta} f_{\bar{\pi},t-1}\, u_{t-1} = 0}_{\text{same as in materials 21}} \quad (13)$$

$$h_{fb,t}\, f_b(t-1) - \frac{1}{1-\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2} + \overbrace{\left(f_{fb,t} + exog_3\right) u_t - \frac{1}{1-\beta} f_{\bar{\pi},t-1}\, u_{t-1} = 0} \quad (14)$$

Honestly, I don't even think I will complete this b/c we can see that if, in say (14), $h_{fb,t} = 0$ and $h_{\bar{\pi},t-1} = 0$ then we're back to exactly the system I had in mat21.

So suppose $h_{fb,t} \neq 0$. But then in each equation we have an earlier iteration of itself. Now I might argue $h_\pi = h_x = h_i = 0$, but if I ass that for $h_{fa,t} = h_{fb,t} = 0$ then again the link between (9)-(10) and (15)-(16) is broken. Then (14) would give an additional constraint

$$h_{fb,t}\, f_b(t-1) = \frac{1}{1-\beta} h_{\bar{\pi},t-1}\, \bar{\pi}_{t-2}$$

In two unknowns. Mm... You'd get a proliferation of unknowns...! I'll stop here, this wasn't for endog states, merde!

① GMM of midsimple 19 March 2020

② Numerical implementation of target criterion

① GMM of midsimple

sim_learnLH.m w/ PLM = constant_only

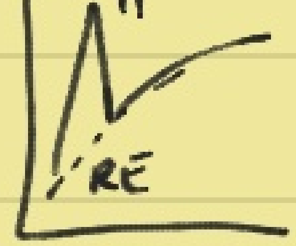
→ Need to:

1) add smooth function as a third criterion

2) To have "midsimple", we need that x (and i) aren't learned, so input $a = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$, $b = b^{RE}$

↳ Ok, I've learned one thing (comparing w/ IRFs of materials 9)

⇒ when only π is learned, IRFs are less oscillatory in the sense that they overshoot only once:



vs.

when $(\pi, x, i)'$ learned

→ did I ever conclude this? I thought so, but now it doesn't seem like it. In materials12 I just state that vector vs. scalar learning is pretty much the same

...And it almost is, just not quite.

⇒ so mental note!

Learning only LOM(π) dampens the oscillations b/c E(·) in NKIS & NKPC are moving less, and therefore E(i_{t+k}) is moving less. I think more importantly since the bulk of action is in x, the fact that FE(x) is now cut out gets rid of the FE that was most oscillatory.

Ok I just wasted an hour trying to customize keys in Matlab...

Back to 1) the anchoring function:

$$k_t^{-1} - k_{t-1}^{-1} = c + d(FE)$$

has the interpretation that the gain decreases when $FE < 0$. Ryan is right: the easiest thing that makes Δk_t^{-1} big when FE is big in absolute value is FE^2

$$\to \quad k_t^{-1} - k_{t-1}^{-1} = c + d\, FE_t^2$$

does this even make sense?

> Not really: it's saying that the gain always changes, even if the $FE^2 = 0$.

↳ So then gather data and implement

- HP
- BK
- Hamilton filters

But there is a problem: for this current form

$$k_t^{-1} - k_{t-1}^{-1} = c + d \cdot FE^2,$$

y always explodes, even more so if $c=0$.

And it's b/c $k \to 0$, so I guess k^{-1} explodes.

↳ Need to be smart about this!

Normally, a dgain would be <u>20 March 2020</u>

$k_{t+1} = k_t + 1$

So I could just do $k_t = k_{t-1} + \frac{1}{FE^2}$

so that if in the limit $FE^2 \to \infty$, $k_t = k_{t-1}$ (cgain)

Then one could have

$$k_t = k_{t-1} + d \frac{1}{FE^2}$$ in this case I guess d small $d<1$

or $$k_t = k_{t-1} + \frac{1}{d\, FE^2}$$ in this case I guess d big $d>1$

↳ both of them work and exactly the way I thought

or even $k_t = k_{t-1} + \frac{1}{(d FE)^2} = k_{t-1} + (d FE)^{-2}$

works.

HP-filter

A time series y_t (in logs) is

$$y_t = \underset{\substack{\uparrow \\ \text{growth} \\ \text{component}}}{g_t} + \underset{\substack{\uparrow \\ \text{cyclical component}}}{c_t}$$

Obtain g_t as

$$\min_{\{g_t\}_{t=-1}^{T}} \left\{ \sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=1}^{T} \left[(g_t - g_{t-1}) - (g_{t-1} - g_{t-2}) \right]^2 \right\}$$

where $\lambda = 1600$.

Hamilton provides a closed-form solution to this problem as $g^* = (H'H + \lambda Q'Q)^{-1} H' y$ (2)

where $\tilde{T} := T + 2$

$$\underset{T \times 1}{y} := (y_T, y_{T-1}, \ldots, y_1)' \qquad \underset{\tilde{T} \times 1}{g} = (g_T, g_{T-1}, \ldots, g_0, g_{-1})'$$

and $\underset{T \times \tilde{T}}{H} = \left[\underset{T \times T}{I_T} \quad \underset{T \times 2}{0} \right]$ and $\underset{T \times \tilde{T}}{Q} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{bmatrix}$

Hamilton filter

21 March 2020

Idea: $c = y_{t+h} - E[y_{t+h} | y_t]$

choose $h=8$ for quarterly data.

How to calc $E[y_{t+h} | y_t]$?

just regress y_{t+h} on $y_t, y_{t-1}, y_{t-2}, y_{t-3}$

and take the fitted value from that reg.

In fact, Hamilton's eq (21) gives us c directly:

$$c = \hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

So: $Y = y_{t+h}$, $X = \overset{4\times1}{\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}}$ $\Rightarrow \underset{1\times4}{\beta} = \underset{1\times1}{(\underset{1\times4}{X'}\underset{4\times1}{X})^{-1}}\underset{1\times4}{X'}\underset{1\times1}{Y}$

and $\hat{v}_{t+h} = y_{t+h} - \beta X$

Baxter & King filter (BK filter, bandpass filter)

Def: business cycle = cyclical components of a ts that last between 6 and 32 quarters in duration

Notation: $BP_k(p,q)$ = filter that passes cycles between $q=32$ and $p=6$ cycle length, and is truncated at leads/lags k (→ recommend $k=12$)

↳ Such a filter is a MA(k).

So we're interested in constructing the filtered series

$$y_t^* = \sum_{k=-K}^{K} a_k y_{t-k} \qquad (1)$$

and the question is how to obtain a_k?

(where the MA is symmetric, i.e. $a_k = a_{-k}$)

Eq. (10) gives the answer:

$$a_h^{BP} = (\bar{b}_h - \underline{b}_h) + (\bar{\theta} - \underline{\theta})$$

where the bars denote things pertaining to low-pass filters that only allow stuff below the frequency $\bar{\omega}$ & $\underline{\omega}$,

i.e periodicities $\bar{p}$ & $\underline{p}$,
where $p = \frac{2\pi}{\omega}$

θ is a normalizing constant (coming from truncation)

$$\theta = \left(1 - \sum_{h=-k}^{k} b_h\right) \frac{1}{2k+1}$$

and $b_0 = \frac{\omega}{\pi}$ and $b_h = \frac{\sin(h\underline{\omega})}{h\pi}$ for $h = 1, 2, \ldots$ (7)

$q = 32 \qquad p = 6$

Ok, so we have $\underline{p} = 32$, $\bar{p} = 6$

$\Rightarrow \underline{\omega} = \frac{2\pi}{\underline{p}} = \frac{2 \cdot \pi}{32} = \frac{\pi}{16}$, $\bar{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}$

Autocovariances of VARs (Lütkepohl, p. 21)

For a stationary VAR(1)

$$y_t = v + A_1 y_{t-1} + \underbrace{u_t}_{\hookrightarrow N(0, \Sigma_u)}$$

define $\Gamma_y(h) \equiv E[(y_t - \mu)(y_{t-h} - \mu)']$

where $\mu := E[y_t]$. Then if you know $\Gamma_y(0)$, you can compute

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) \quad \text{recursively!} \quad (2.1.31)$$

If you know A_1 and Σ_u, you can compute

$$\mathrm{vec}\,\Gamma_y(0) = (I_{k^2} - A_1 \otimes A_1)^{-1} \mathrm{vec}\,\Sigma_u \qquad (2.1.32)$$

But I don't want to est a VAR... I just want the empirical auto-covariance matrix of the data and of the simulated data.

So, the GMM of the anchoring fit works 23 March 2020

so far. Issues.

1.) I'm not sure if I should model the raw data w/ a time series. So e.g. if they were a VAR(1) then I could estimate a RF-VAR, I could bootstrap using the residuals and I could estimate the autocov structure as in (2.1.32) & (2.1.31) in Lütkepohl.

This issue shows up for $W = \begin{bmatrix} \hat{\sigma}^2_{act_1} & & \\ & \ddots & \\ & & \hat{\sigma}^2_{act_{15}} \end{bmatrix}$

b/c $\hat{\sigma}^2_{act,i}$ are super small, thus W^{-1} is huge, and therefore the estimation does not move.

2) Filters may not be working?

→ I think now they are!

↳ check bootstrap! ✓ It's fine.

Write up materials22_peter. ✓

Read "Stochastic". On it!

I'm not sure what I'm learning from 24 March 2020

the GMM so far, so until I talk to Ryan, I'll postpone it and turn to questions of implementation of the target criterion.

Peter meeting — 24 March 2020

- Ask Clough

- Intro: CBs always talk about anchoring, eq (2) formalizes that!

- Where to go #1:

- Suggestions

  (1): A common theme: does it make sense to have $h_t = k_t^{-1}$ and just track the evolution of that.

(2): Eq. (2)

If you work w/ nonlinear diff eqs, they are harder but not imposs to solve

→ mit-shock business is all about that w/ perfect foresight

- take FOCs
- keep 'em in nonlinear form
- ass. the econ is hit by 1-time shock
- trace out transition to new st. st.

A precise way of putting it:

any dynamic econ: all vars → X_t

$$F(X_t, X_{t-1}, \varepsilon_t) = 0$$

If F has a lin form, $A X_t + B X_{t+1} + C \varepsilon_t = 0$

then we can solve it numerically quite easily.

If F is nonlinear, but nonstochastic:

$$F(x_t, x_{t-1}) = 0$$

is harder, but doable.

↳ Sect 2. point (1)

"can use analytic methods to describe eqs in opt policy & key variables"

& simulate vars under those policies.

⇒ allows you to answer

1) How close does a TR come to opt. policy

2) Opt TR will try to do the same thing as opt policy and you'll see what they both are trying to do.

Suggestion overall:

- simulate numerically the model under some kind of optimal pol that's more flex than a TR.

2^{nd} suggestion w/ nonlinearity:

- Analogy: stochastic Ramsey model (neoclassical growth model) no closed-form sol once depreciation isn't $=1$ and not log u. So to solve this:
  - loglin
  - discretize the state-space
    - TFP follows not an AR1 but one of 3 values
    - then decision rules are also 3-form

<u>1 option</u> Continue working w/ full-blown nonlinear model but replace the stoch. disturbances

$$u = \begin{matrix} u^{high} \\ u^{low} \end{matrix} \qquad r^n = \begin{matrix} r^{high} \\ r^{low} \end{matrix}$$

$\Rightarrow$ 4 states of the world

$\hookrightarrow$ calculating $E(\cdot)$ of ∞ future terms in (2) may be easier

2 option: Take (1)

$$f_t = f_{t-1} + h_t (\pi_t - f_{t-1})$$

and loglin

Another way to solve a DSGE model

loglin ass's that decision rules are lin functions of the states

-> well you can say that they are quadratic function of the states

↳ so instead of Taylor-approx, maybe the right thing is a cubic or spline

In eq (3) I try to approx something of unknown form w/ a class

Simulate the model under some optimal targeting rule & compare w/ behavior under an optimized TR economy.

Draft for end of semester
→ she'll read it then
and afterwards too.

↳ Talk to Ryan now:
What is the most promising avenue?

Tomorrow noon → send Peter email for macro seminar

Work after

Takeaways:

1) In my introduction, I talk about how CB-ers often talk about anchoring. The target criterion (eq. (2) in materials22peter) formalizes this!

2) Small detail: replace $h_t \equiv k_t^{-1}$

3.) The main point: need to identify the set of results (and avenues) that can be achieved and then written up. For him it seems to be that this set of results is the target criterion and its numerical implementation so we can make statements about the optimal plan and how it compares to the optimal Taylor rule.

↳ he seemed adamant that I settle this w/ Ryan.

4) Taking the analogy of the neoclassical growth model, no closed-form sol exists for the nonlinear model unless log u and $\beta = 1$. So if we wanted a tighter analytical result, we could 1) either loglin, or 2) discretize the stochastic disturbance

e.g. $u_t = \begin{cases} u^{high} \\ u^{low} \end{cases}$

↳ this way we might more fully describe the target criterion.

5) As for the loglin, which is a Taylor-Approx, maybe a higher order approx (cubic or spline) would be sensible.

Here's the deal: I don't think I wanna pursue further analytical work. Not w/ the discretization, not w/ a higher-order approx, although both might work. I think that building on the existing

work, I want to go on to simulating the optimal Ramsey policy and comparing it to the opt. TR.
⇒ I think that that is 1) feasible 2) makes sense in terms of storytelling & also is likely to provide additional inputs to that storytelling.

I will continue reading Stochastic Opt. in Continuous Time, not b/c I really hope for inputs for more analytical work, more for my intellectual completion – as diff. eqs contributed too.

50: numerical implementation of the target criterion <u>25 March 2020</u>

- I wonder if I couldn't do it via value function iteration → can I not solve the nonlinear system numerically and obtain the optimal paths like that?
→ which would allow me to back out i^{Ramsey} from NKIS?

g_π :

$y \quad k_t = k_{t-1} + \overbrace{(d \cdot fe_{t-1})}^{=g}{}^{-2}$

$$k_t = k_{t-1} + d^{-2}\,(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-2}$$

Then $g_\pi(t) = d^{-2}(-2)\,(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-3}$

$$= -2\,(d\, fe_{t-1})^{-2}\, fe_{t-1}^{-1}$$

Now what I need to solve is the fact that when $\{i_t\}$ is exog, the model sol

y = function (matrices, f_a, f_b) changes.

In particular, if we go back to the model summary in materialsA in (1) – (3) we know that (3) doesn't hold (TR), and $\{i_t\}$ influences the econ via the NKIS curve (1).

$$x_t = -\sigma i_t + [\sigma, (1-\beta), -\sigma\beta] f_b + \sigma[1,0,0](I_{nx} - \beta h_x)^{-1} s_t$$

$$\pi_t = \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_a + [0,0,1](I_{nx} - \beta h_x)^{-1} s_t$$

$$\begin{array}{c} \begin{bmatrix} 0 & 1 \\ -\kappa & 1 \end{bmatrix} \\ 1 \quad -\kappa \end{array} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} -\sigma i_t + [\sigma, (1-\beta), -\sigma\beta] f_b + \sigma[1,0,0](I_{nx} - \beta h_x)^{-1} s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_a + [0,0,1](I_{nx} - \beta h_x)^{-1} s_t \end{bmatrix}$$

So 27 March

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\sigma i_t + [\sigma, (1-\beta), -\sigma\beta] f_\beta + \sigma[1,0,0](I_{nx} - \beta h_x)^{-1} s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_a + [0,0,1](I_{nx} - \beta h_x)^{-1} s_t \end{bmatrix}$$

$1 \quad -\kappa$

$$\begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where f_β, f_a unchanged.

$$= \begin{bmatrix} -\kappa & -1 \\ -1 & 0 \end{bmatrix}\frac{1}{-1} = \begin{bmatrix} \kappa & 1 \\ 1 & 0 \end{bmatrix}$$

Ryan meeting

25 March 2020

$$k_t = k_{t-1} + \frac{1}{(d\,fe)^2} - c$$

so that when fe large, the gain grows

Calibration: $E(fe)^{RE}$ would be the size of c.

- check if drawing w/ replacement!
  - take raw data: run a VAR
  - bootstrap from residuals

} What Ryan did in the paper w/ Robert

- put sim data in filter

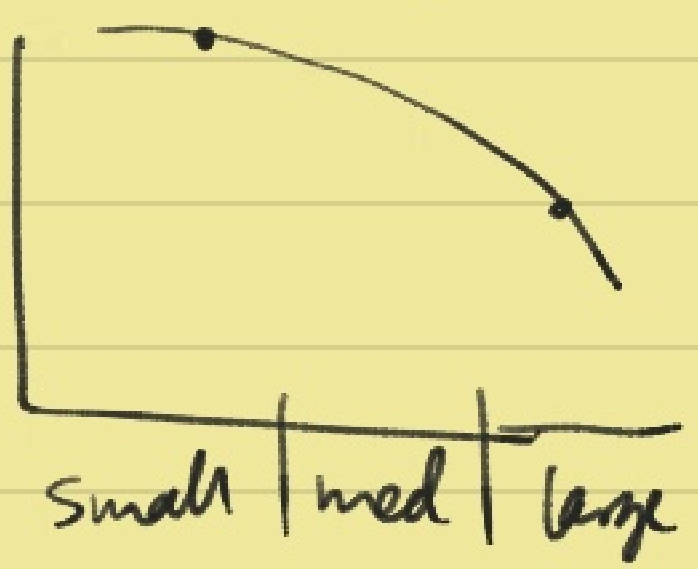
- g should
  - tend to go down
  - but when it goes up, it goes up a lot
    (i.e. for very large FE, it ↑ a lot.

- check if you're close to the target moments

$k_t = k_{t-1} + \hat{g}(fe^c)$

$\hat{g}$ is a spline w/ x nodes

piecewise linear w/ x nodes



Each node comes w/ 2 params
→ so can est a 4-param family

↳ here you just est 3 slopes

→ eventually you could even est. what a small, large, FE is.

shape-preserving splines: are either always convex or always concave

Judd, the textbook, Numerical Methods for Econ

piece-wise linear interpolation

spline / quadratic spline

basis functions

Econ librarian: Sonia Ensins

Makes sense:

Hamilton $\hat{d}$ lower b/c Var ↑, so FEs lower.

4 states, so VFI could be done.

Work after 26 March 2020

EBSCO host: lauragati
empirical_77 methods

GMM

1) close to target?

No, not at all! loss is huge (5.4e+08)!

2) w/ replacement

randperm really wasn't w/ replacement

randi(maxvalue, a x b vector) is w/ replacement

but it's still blowing up. $\overset{\cdot\cdot}{\frown}$

Maybe I do need to estimate a VAR?

I'm surprised b/c Ryan said that it sounds like my procedure is valid.

3) An anchoring function w/ the feature that

a) it generally decreases b) grows a lot when FE large

$k_t = k_{t-1} + d - c\,fe_{t-1}^2$

If $cfe^2 < d$, dgain → a) → $g_\pi = -2c\,fe_{t-1}$

For $cfe^2 > d$, cgain → b) Dam. Real bad.

Try

$$k_t = k_{t-1} + \frac{1}{(\ell f_t)^2} - c$$

Not good either.

27 March 2020

$$x_t = -\gamma i_t + \text{shuff}_1 f_b + \text{shuff}_2 s_t$$

$$\pi_t = \kappa x_t + \text{shuff}_3 f_a + \text{shuff}_4 s_t$$

$$1 x_t + 0 \pi_t = -\gamma i_t + \text{shuff}_1 f_b + \text{shuff}_2 s_t$$

$$-\kappa x_t + 1 \pi_t = \text{shuff}_3 f_a + \text{shuff}_4 s_t$$

$$\underset{2\times 1}{\begin{bmatrix} 0 \cdot \pi_t + 1 x_t \\ 1 \pi_t - \kappa x_t \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} -\gamma i_t + \text{shuff}_1 f_b + \text{shuff}_2 s_t \\ \text{shuff}_3 f_a + \text{shuff}_4 s_t \end{bmatrix}}$$

$$\underbrace{\underset{2\times 2}{\begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} \pi_t \\ x_t \end{bmatrix}}}_{2\times 1}$$

I actually think they are the same - 28 March 2020

it's just that since π is exploding, small numerical

errors become considerable.
Strange that x doesn't explode.
→ Like I thought yesterday, pi_x_given_i.m should be correct. It's the target criterion that may not be.

Or actually: fa(3) and fb(3) may not be correct.

29 March 2020

I think I know what the problem is. fafb_... m just computes $\hat{E}_t \sum_T (\alpha)\beta^{T-t} z_{T+1}$
and now in pi_x_given_i there may be a discrepancy in that $f_b(3)$ is used, and yet the A-matrices that specified that $f_b(3)$ wasn't used before are no longer valid.
→ Need to make an assumption on how $f_b(3)$ is formed. The most natural is to say that the CB announces the future path of i, i.e. it's like an

exog sequence. On the 24 March, I had:

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & -k\end{bmatrix}^{-1} \begin{bmatrix} -\sigma i_t + [\sigma, (1-\beta), -\sigma\beta] f_\beta + \sigma[1,0,0](I_{nx} - \beta h x)^{-1} s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0] f_a + [0,0,1](I_{nx} - \beta h x)^{-1} s_t \end{bmatrix}$$

$$-\sigma\beta f_\beta = -\sigma\beta E_t \sum_{\tau=0}^{\infty} \beta^{\tau-t} i_{\tau+1}$$ Together with i_t, this gives

$$-\sigma\left[i_t + \beta E_t \sum_{\tau=0}^{\infty} \beta^{\tau-t} i_{\tau+1}\right]$$
$$= -\sigma\left[i_t + E_t \sum_{\tau=0}^{\infty} \beta^{\tau-t+1} i_{\tau+1}\right]$$
$$= -\sigma\left[E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} i_\tau\right]$$

Since the i-sequence may not be an AR(1), it may be completely arbitrary, I should simply write $\text{discsum\_i} = \sum_{\tau=t}^{T+H} \beta^{\tau-t} i_\tau =: dsi$

So pi_x_given_i.m modifies to

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & -k\end{bmatrix}^{-1} \begin{bmatrix} -\sigma dsi + [\sigma, (1-\beta), 0] f_\beta + \sigma[1,0,0](I_{nx} - \beta h x)^{-1} s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0] f_a + [0,0,1](I_{nx} - \beta h x)^{-1} s_t \end{bmatrix}$$

and stuff1 modifies to $[\sigma, 1-\beta, 0]$

Still exploding ...

I'm wondering if (A.1) – (A.3) yield an i-sequence, and I input that i-sequence into pi-x-given-i.m, should I get the same $\{\pi_t, x_t\}_t^T$?

Initially I thought yes, but now I think no b/c A_a, A_b were derived w/ the assumption that $f_b(3) = \gamma_\pi f_b(1) + \gamma_2 f_b(2)$ + expected i-shock

→ I turned off i-shock, but that actually makes things explode faster!

Well, it depends. B/c if you set all $\bar{i}_t = 0$, $\forall t$, then to make $\hat{E}_t i_{t+k}$ consistent even under the TR-regime, you need to set $\rho_i = \sigma_i = 0$.

But at least it makes sense that x explodes more than π, b/c $\pi = \underbrace{\kappa}_{\text{small}} x + (\text{stuff} \perp i)$

Hm, if T+H small, then things don't have time to explode and therefore fmincon works. Otherwise somehow loss = 0 already initially. why?

Initial loss decreases in H

so when $H \geq 36$, loss = 0 (when $T = 10$)

But this seems to be extremely robust across T.

Initial loss is ∩-shaped in T

so when $T = 2$, loss = 0 (when $H = 35$)

$T = 10$, loss = 49.479

$T = 100$, loss = 1.4761

→ after that, it's 1.48 for $\forall T$.

But this is not robust to H.

E.g. if $H = 5$, loss ↑ in T monotonically.

→ maybe I should take the vector of results as moments, not the biggest one.

That doesn't help w/ loss-explosions either, but I wonder if makes the loss function well-behaved?

So I'm back to thinking that $\{\bar{a}_t, x_t \mid i_t\}$ shouldn't explode - I can accept that they aren't the same as $\{a_t, x_t \mid i_t^{TR}\}$ b/c $E(i)$ isn't the same, but I guess they shouldn't always explode.

→ they're exploding b/c fa & fb are exploding. of course it's only fa(1) & fb(1), the others are numerically 0 (e-16), which makes sense b/c $a_2 = a_3 = 0$ a $b = gx\,hx$ = zeros now that I set all elements of $hx = 0$.

Somehow the learning isn't E-stable (indeed $a = \bar{a}$ explodes) when you just feed in a i-sequence b/c it's as if the TR was just specified as $i_t = \bar{I}_t$.

→ If there was some way of letting the private sector know that i isn't chosen randomly, that it follows a rule...

Need to find a way to have the 30 March 2020

private sector internalize the target criterion.

→ the TC needs to be a model equation

But if that is so, then the expectation of the RHS needs to be the public's expectation!

But before I do this, this is the time to verify that the TC of (B.1) in materials23 is correct!

I solve for TC in materials18. (Notes 6, p. 95 ff.)

$$\mathcal{L} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Big\{ (\pi_t^2 + \lambda x_t^2)$$

$$+ \varphi_{1,t} \left(\pi_t - \kappa x_t + (1-\alpha)\beta f_a(t) + \text{exog} \right)$$

$$+ \varphi_{2,t} \left(x_t + \sigma i_t + \sigma f_b(t) + \text{exog} \right)$$

$$+ \varphi_{3,t} \left(f_a(t) - \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + \text{exog} \right)$$

$$+ \varphi_{4,t} \left(f_b(t) - \frac{1}{1-\beta} \bar{\pi}_{t-1} + \text{exog} \right)$$

$$+ \varphi_{5,t} \left(\bar{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1} \left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \right) \right)$$

$$+ \varphi_{6,t} \left(k_t^{-1} - \underline{k_{t-1}^{-1}} + g \left(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1} \right) \right)$$

This is a change compared to materials18.

$$\mathcal{L} = E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Big\{\left(\pi_t^2 + \lambda x_t^2\right)$$

$$+\varphi_{1,t}\left(\pi_t - \kappa x_t - (1-\alpha)\beta f_a(t) + exog\right)$$

$$+\varphi_{2,t}\left(x_t + \sigma i_t - \sigma f_b(t) + exog\right)$$

$$+\varphi_{3,t}\left(f_a(t) - \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} + exog\right)$$

$$+\varphi_{4,t}\left(f_b(t) - \frac{1}{1-\beta}\bar{\pi}_{t-1} + exog\right)$$

$$+\varphi_{5,t}\left(\bar{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1}\left(\pi_t - \left(\bar{\pi}_{t-1} + b_1 s_{t-1}\right)\right)\right)$$

$$+\varphi_{6,t}\left(k_t^{-1} \underline{- k_{t-1}^{-1}} - g\left(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}\right)\right)$$

FOCs:

π: $$2\pi_t + \varphi_{1,t} - k_t^{-1}\varphi_{5,t} - \varphi_{6,t}\, g_\pi(t) = 0 \qquad (1)$$

x: $$2\lambda x_t - \kappa\varphi_{1,t} + \varphi_{2,t} = 0 \rightarrow \underline{\varphi_{1,t} = 2\frac{\lambda}{\kappa}x_t} \qquad (2)$$

i: $$\underline{\sigma\varphi_{2,t} = 0} \qquad (3)$$

f_a: $$-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0 \rightarrow \varphi_{3,t} = \underline{(1-\alpha)\beta\, 2\frac{\lambda}{\kappa}x_t} \qquad (4)$$

f_b: $$\varphi_{4,t} + \sigma\varphi_{2,t} = 0 \rightarrow \underline{\varphi_{4,t} = 0} \qquad (5)$$

$\bar{\pi}$: $$\varphi_{3,t+1}\left(-\frac{1}{1-\alpha\beta}\right) + \varphi_{5,t} + \varphi_{5,t+1}\left[-1 + k_{t+1}^{-1}\right]$$
$$+\varphi_{6,t+1}\, g_{\bar{\pi}}(t) = 0 \qquad (6)$$

k^{-1}: $$-\varphi_{5,t}\left(\pi_t - \left(\bar{\pi}_{t-1} + b_1 s_{t-1}\right)\right) + \varphi_{6,t} - \varphi_{6,t+1} = 0 \qquad (7)$$

$$2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t^{-1}\varphi_{3,t} - \varphi_{6,t}\, g_\pi(t) = 0 \qquad (1)$$

Gaspar's 22

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}x_{t+1} + \varphi_{3,t} - \varphi_{5,t+1}\left(1 - k_{t+1}^{-1}\right)$$

Gaspar's 21

$$+\varphi_{6,t+1}\, g_{\bar{\pi}}(t) = 0 \qquad (6)$$

$$-\varphi_{5,t}\left(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\right) + \varphi_{6,t} - \varphi_{6,t+1} = 0 \qquad (7)$$

new

Note: Eq. (1) is exactly Eq. (I), p. 95 of Notes 6.

Eq (6) is exactly Eq (II).

Eq (7) is (III) + an additional $\varphi_{6,t+1}$ term that comes from redefining the anchoring function as $k_t = \underline{k_{t-1}} + g(fe)$

NEW $\rightarrow$ $\underline{\varphi_{6,t+1}}$

The issue that now comes up is the one I anticipated in materials 18, Notes 6 before 95: if the anchoring function has k_{t-1}^{-1} in it, then $\varphi_{6,t+1}$ shows up and causes trouble. Let me simplify the system:

Define $c \equiv -\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}$ and $fe_t \equiv \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$,

Then: (where I'm annoyingly using a different notation than usual...)

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1}\varphi_{5,t} - g_\pi(t)\varphi_{6,t} = 0 \quad (1)$$

$$c\, x_{t+1} + \varphi_{5,t} - (1-k_{t+1}^{-1})\varphi_{5,t+1} + g_{\bar{\pi}}(t)\varphi_{6,t+1} = 0 \quad (6)$$

$$-fe_t \cdot \varphi_{5,t} + \varphi_{6,t} - \varphi_{6,t+1} = 0 \quad (7)$$

From (7), $\varphi_{5,t} = \dfrac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t}$.

Plug into (1)

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1}\left(\frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t}\right) - g_\pi(t)\varphi_{6,t} = 0$$

$$\Leftrightarrow \quad 2\pi_t + \frac{2\lambda}{\kappa} x_t - \left(\frac{k_t^{-1}}{fe_t} + g_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} = 0 \quad (1')$$

---

and plug into (6)

$$c x_{t+1} + \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t} - (1-k_{t+1}^{-1})\left(\frac{\varphi_{6,t+1} - \varphi_{6,t+2}}{fe_{t+1}}\right) + g_{\bar{\pi}}(t)\varphi_{6,t+1} = 0$$

This I should be able to solve for $\varphi_{6,t}$.

$$c x_{t+1} + \frac{\varphi_{G,t} - \varphi_{G,t+1}}{fe_t} - (1-k_{t+1}^{-1})\left(\frac{\varphi_{G,t+1} - \varphi_{G,t+2}}{fe_{t+1}}\right) + g_{\bar{\pi}}(t)\varphi_{G,t+1} = 0$$

$$c x_{t+1} + \frac{1}{fe_t}\varphi_{G,t} + \left[-\frac{1}{fe_t} - \frac{(1-k_{t+1}^{-1})}{fe_{t+1}} + g_{\bar{\pi}}(t)\right]\varphi_{G,t+1} + \frac{(1-k_{t+1}^{-1})}{fe_{t+1}}\varphi_{G,t+2}$$

6'

$$c x_{t+1} + \alpha_t \varphi_{G,t} + \beta_t \varphi_{G,t+1} + \gamma_{t+1}\varphi_{G,t+2} = 0$$

Now the question is: solve forward or backward?

$1-k^{-1}$ is always < 1, and in the limit it $\to 1$.

fe doesn't have a limit, it's Normal and in the limit its variance is Σ ($=$ var(shocks)), mean zero.

For the most part, then, and close to the limit, if Σ's elements are $\sigma_i^2 \leq 1$, then fe (in the limit) will be mainly (or always?) below 1. So $\frac{1}{fe} > 1$

$$c fe_t x_{t+1} + \varphi_{G,t} + \left[-1 - \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) + fe_t g_{\bar{\pi}}(t)\right]\varphi_{G,t+1} + \underbrace{\frac{fe_t}{fe_{t+1}}}_{<1}(1-k_{t+1}^{-1})\varphi_{t+2}$$

Here, $\frac{fe_t}{fe_{t+1}} \simeq 1$ in expectation, so $\frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) < 1$

$$\varphi_{6,t} = -c\, fe_t\, x_{t+1} + \underbrace{\left[1 + \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) - \underbrace{fe_t\, g_{\bar{\pi}}(t)}\right]}_{=:\alpha_t} \varphi_{6,t+1} \underbrace{- \underbrace{\frac{fe_t}{fe_{t+1}}}_{<1}(1-k_{t+1}^{-1})}_{=:\beta_t} \varphi_{6,t+2}$$

in the limit, this needs to be >1 in abs. value. But it's not.

$$\lim \left[1 + \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) - fe_t\, g_{\bar{\pi}}(t)\right] \rightarrow 1 + 1 - \Sigma \cdot$$ (some small number, might even be zero)

>1

Can I factor this using lag polynomials then? I need to solve both backward & forward.

$$\varphi_{6,t} = -c\, fe_t\, x_{t+1} + \alpha_t\, \varphi_{6,t+1} + \beta_t\, \varphi_{6,t+2} \qquad |: \alpha_t$$

$$|\alpha_t| > 1 \quad |\beta_t| < 1$$

$$\frac{1}{\alpha_t}\varphi_{6,t} = -\frac{c\, fe_t}{\alpha_t} x_{t+1} + \varphi_{6,t+1} + \frac{\beta_t}{\alpha_t}\varphi_{6,t+2}$$

$$\varphi_{6,t+1} = \frac{c\, fe_t}{\alpha_t} x_{t+1} + \underbrace{\frac{1}{\alpha_t}}_{<1}\varphi_{6,t} - \underbrace{\frac{\beta_t}{\alpha_t}}_{<1}\varphi_{6,t+2} \qquad (6'')$$

Before we go there, let me just check the target criterion as it now is, i.e. if $\varphi_{6,t+1} = 0$.

If $\psi_{6,t+1} = 0$, the system is

$$2\pi_t + \frac{2\lambda}{k} x_t - k_t^{-1} \psi_{5,t} - g_{\bar{\pi}}(t)\psi_{6,t} = 0 \qquad (1)$$

$$c\, x_{t+1} + \psi_{5,t} - (1 - k_{t+1}^{-1})\psi_{5,t+1} + g_{\bar{\pi}}(t)\psi_{6,t+1} = 0 \qquad (6)$$

$$-fe_t \cdot \psi_{5,t} + \psi_{6,t} - \cancel{\psi_{6,t+1}} = 0 \qquad (7)$$

and eq (7) corresponds to Notes 6, p. 95, eq. III.

$$\Rightarrow \psi_{6,t} = fe_t \psi_{5,t}$$

$$\Rightarrow (1): \quad 2\pi_t + \frac{2\lambda}{k} x_t - (k_t^{-1} + g_{\bar{\pi}}(t) fe_t)\psi_{5,t} = 0 \qquad (1')$$

$$\Rightarrow (6) \quad c\, x_{t+1} + \psi_{5,t} + [-(1-k_{t+1}^{-1}) + g_{\bar{\pi}}(t) fe_{t+1}]\psi_{5,t+1} = 0 \quad (6')$$

$$\psi_{5,t} = -c\, x_{t+1} + [1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) fe_{t+1}]\psi_{5,t+1}$$

To be compatible w/ Notes 6, p. 96, let me redefine

$$c := -c \quad \text{and} \quad \alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) fe_{t+1}$$

and assume that $\alpha_t < 1$ for most t. Which is likely actually b/c $\alpha_{t+1} \to 1$ as $t \to \infty$ if $k_t^{-1} \to 0$ and $g_{\bar{\pi}}(t) \to 0$. For any $k_t^{-1}, g_{\bar{\pi}}(t) > 0$, $\alpha_{t+1} < 1$. It can be < -1 though! Suppose $k_{t+1}^{-1} \approx 1$. Then $1 - k_{t+1}^{-1}$ is close to 0. If $g_{\bar{\pi}}(t) fe_{t+1} > 1$ by enough, then this could happen. But it's not likely b/c 1) $fe < 1$ if $\Sigma \leq 1$; 2)

$g_{\bar{\pi}}(t)$ is likely to be not just < 1, but quite small since the overall level of the gain is < 1, so its change is likely to be an order of magnitude smaller.

$\Rightarrow$ So I think $\alpha_t < 1 \ \forall t$ is safe to assume.

$$\varphi_{s,t} = c x_{t+1} + \alpha_{t+1} \varphi_{s,t+1} \qquad | \text{Iter. fwd.}$$

$$= c x_{t+1} + \alpha_{t+1} \left[c x_{t+2} + \alpha_{t+2} \varphi_{s,t+2} \right]$$

$$= c x_{t+1} + \alpha_{t+1} \left[c x_{t+2} + \alpha_{t+2} \left(c x_{t+3} + \alpha_{t+3} \varphi_{s,t+3} \right) \right]$$

$$= c \left[x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \right.$$

$$\left. + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \varphi_{s,t+3} \right.$$

$$= c \left[x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \right]$$

$$+ \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \left[c x_{t+4} + \alpha_{t+4} \varphi_{s,t+4} \right]$$

$$= c \left[x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} x_{t+4} \right]$$

$$+ \underbrace{\alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \alpha_{t+4} \varphi_{s,t+4}}_{\to 0 \text{ as } t \to \infty}$$

$$\varphi_{s,t} = c x_{t+1} + c \sum_{i=2}^{\infty} x_{t+1} \prod_{j=1}^{i-1} \alpha_{t+j}$$

So that (1') becomes:

$$2\pi_t + \frac{2\lambda}{k}x_t - (k_t^{-1} + g_\pi(t)fe_t)\left(c\,x_{t+1} + c\sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\alpha_{t+j}\right) = 0$$

$$\Rightarrow 2\pi_t = -\frac{2\lambda}{k}x_t + c\left(k_t^{-1} + g_\pi(t)fe_t\right)\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\alpha_{t+j}\right)$$

Since $c = \frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}$, we have

$$\pi_t = -\frac{\lambda}{k}x_t + \frac{(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}\left(k_t^{-1} + g_\pi(t)fe_t\right)\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\alpha_{t+j}\right)$$

Recalling that $\alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t)fe_{t+1}$, this is:

$$\pi_t = -\frac{\lambda}{k}\left\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe_t\, g_\pi(t)\right)\right.$$

$$\left.\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{\underline{\underline{j=0}}}^{i-1}\left(1 - k_{\underline{\underline{t+1+j}}}^{-1} - g_{\bar{\pi}}(t+j)fe_{t+1+j}\right)\right)\right\}$$

Note this is not 1, as in p.97, Notes 6.

And lastly I defined $fe_t \equiv \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$, so

$$\pi_t = -\frac{\lambda}{k}\left\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} \underline{\underline{-}} b_1 s_{t-1})g_\pi(t)\right)\right.$$

$$\left.\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{\underline{\underline{j=0}}}^{i-1}\left(1 - k_{\underline{\underline{t+1+j}}}^{-1} - g_{\bar{\pi}}(t+j)\left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}\right)\right)\right)\right\}$$

was t+j

= minus, not plus!

so I write

$$\pi_t = -\frac{\lambda}{\kappa}\Big\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\big(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - \underline{\underline{b_1}} s_{t-1}) g_\pi(t)\big)$$

$$\Big(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{\underline{\underline{j=0}}}^{i-1}\big(1 - \underline{\underline{k_{t+1+j}^{-1}}}\ \underbrace{-}_{=\text{minus, not plus!}} g_{\bar{\pi}}(t+j)\big)\big(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}\big)\Big)\Big\}$$

changing the i-index and using the notation $\prod_{j=0}^{0} := 1$,

as

$$\pi_t = -\frac{\lambda}{\kappa}\Big\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\big(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - \underline{\underline{b_1}} s_{t-1}) g_\pi(t)\big)$$

$$\Big(\quad \sum_{i=1}^{\infty} x_{t+i} \prod_{\underline{\underline{j=0}}}^{i-1}\big(1 - \underline{\underline{k_{t+1+j}^{-1}}}\ \underbrace{-}_{=\text{minus, not plus!}} g_{\bar{\pi}}(t+j)\big)\big(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}\big)\Big)\Big\}$$

where I'll highlight the 5 changes vis-à-vis p. 97 of Notes 6, and I'll adopt these changes to materials 23, (B.1)

and note also that $-g_{\bar{\pi}}(t+j)$ got lost in materials 18 and thus the paper too! Need to correct!

$\checkmark$ = corrected in materials 23

Let me write it in a way good to interpret:

$$\pi_t = -\frac{\lambda}{\kappa}\left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_\pi(t)\right)\right.$$

$$\left.\left(E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+1+j}^{-1} - \left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}\right) g_{\bar{\pi}}(t+j)\right)\right)\right\}$$

Which allows the even neater formulation

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \underbrace{\frac{\lambda}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe^{eve}_{t|t-1}\, g_\pi(t)\right)}\cdot$$

$$\underbrace{E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+1+j}^{-1} - fe^{eve}_{t+1+j|t+j}\, g_{\bar{\pi}}(t+j)\right)}$$

"discretion tradeoff + tradeoff coming from how the gain today responded to latest FE + tradeoff coming from how the gain is expected to change respond to future FE"

Rewrite once more

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \overbrace{\frac{\lambda}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe^{eve}_{t|t-1}\, g_\pi(t)\right) E_t \sum_{i=1}^{\infty} x_{t+i}}^{\text{effect of current level and change of gain on future tradeoffs}}$$

$$\underbrace{-\frac{\lambda}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left[k_t^{-1} + fe^{eve}_{t|t-1}\, g_\pi(t)\right) E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(k_{t+1+j}^{-1} + fe^{eve}_{t+1+j|t+j}\, g_{\bar{\pi}}(t+j)\right)}_{\text{effect of future expected levels and changes of the gain on future tradeoffs, given current level \& change}}$$

Taking the 2nd expression again to consider how one could simplify it,

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \underline{\frac{\lambda}{\kappa} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + fe^{eve}_{t+1|t-1}\, g_\pi(t)\right)} \cdot$$

$$\underline{E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k^{-1}_{t+1+j} - fe^{eve}_{t+1+j|t+j}\, g_{\bar{\pi}}(t+j)\right)}$$

reveals that only the orange part is not available at time t to the CB. Evaluating even $E_t x_{t+k}$ $k>0$ is difficult b/c it requires evaluating $E_t k^{-1}_{t+k}$

But: if I use a rule that the CB relies on agents' expectations (otherwise it couldn't implement the TC).

But then anticipated utility would bind:

$\hat{E}_t k^{-1}_{t+k} = k_t^{-1}$ (is that true?)

$\hat{E}_t fe^{eve}_{t+1+k|t+k} = 0$

$\hat{E}_t g_{\bar{\pi}}(t+k) = 0$ (is that true?)

So the "expectations-based TC" would be

$$\pi_t = -\frac{\lambda}{\kappa} x_t + \underline{\frac{\lambda}{\kappa} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + fe^{eve}_{t+1|t-1}\, g_\pi(t)\right)} \cdot \underbrace{\sum_{i=1}^{\infty} b_x h x^{i-1} s_t \left(1 - k_t^{-1}\right)^{i-1}}_{\text{check later.}}$$

Let's go back to the question of solving (6''):

$$\psi_{6,t+1} = \frac{c\, fe_t}{\alpha_t} x_{t+1} + \underbrace{\frac{1}{\alpha_t}}_{<1} \psi_{6,t} - \underbrace{\frac{\beta_t}{\alpha_t}}_{<1} \psi_{6,t+2} \qquad \boxed{|\alpha_t| > 1 \quad |\beta_t| < 1}$$

Time-out: write this (following Hamilton) as a std 2nd order diff eq:

$$\beta_t \psi_{6,t+2} + \alpha_t \psi_{6,t+1} - \psi_{6,t} = c\, fe_t\, x_{t+1}$$

$$(\beta_t + \alpha_t L - L^2)\, \psi_{6,t+2} = c\, fe_t\, x_{t+1} \qquad |:\beta_t$$

$$\Big(1 + \underbrace{\frac{\alpha_t}{\beta_t}}_{>1} L - \underbrace{\frac{1}{\beta_t}}_{>1} L^2\Big)\, \psi_{6,t+2} = \frac{c\, fe_t}{\beta_t} x_{t+1}$$

$$\Big(1 - \Big(-\frac{\alpha_t}{\beta_t}\Big) L - \frac{1}{\beta_t} L^2\Big)\, \psi_{6,t+2} = \frac{c\, fe_t}{\beta_t} x_{t+1}$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L)\, \psi_{6,t+2} = \frac{c\, fe_t}{\beta_t} x_{t+1}$$

Let me drop the 6 and roll backward <u>31 March 2020</u>

$$\beta_{t-2} \psi_t + \alpha_{t-2} \psi_{t-1} - \psi_{t-2} = c\, fe_{t-2}\, x_{t-1}$$

$$\psi_t = -\underbrace{\frac{\alpha_{t-2}}{\beta_{t-2}}}_{>1} \psi_{t-1} + \psi_{t-2} + c\, fe_{t-2}\, x_{t-1}$$

can't solve bw. But can't solve fw either!

Let me step back: can we simplify this somehow?

$$\varphi_{6,t} = -c\,fe_t\, x_{t+1} + \left[1 + \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) - fe_t\, g_{\bar{\pi}}(t)\right]\varphi_{6,t+1} - \underbrace{\frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1})}_{<1}\varphi_{6,t+2}$$

Obs:

1. If $fe_t = 0$, $\varphi_{6,t} = \varphi_{6,t+1}$. I guess π_t is $\downarrow$ then.
2. If $fe_{t+1} = 0$, $\varphi_{6,t} = \infty\,(\varphi_{6,t+1} - \varphi_{6,t+2}) = \infty \cdot 0$?
3. If $fe_t = \infty$, $\varphi_{6,t} = \infty$
4. If $fe_{t+1} = \infty$, $\varphi_{6,t} = -c\,fe_t\, x_{t+1} + (1 - fe_t\, g_{\bar{\pi}}(t))\,\varphi_{6,t+1}$

An unorthodox thing to do: what if we solved (1')

$$2\pi_t + \frac{2\lambda}{\kappa}x_t - \left(\frac{k_t^{-1}}{fe_t} + g_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} = 0 \quad (1')$$

for $\varphi_{6,t}$, and plugged that in (6')?

$$\varphi_{6,t} = \underbrace{\frac{\frac{k_t^{-1}}{fe_t}}{\frac{k_t^{-1}}{fe_t} + g_\pi(t)}}_{<1} \varphi_{6,t+1} - \gamma_t \qquad \gamma_t := 2\left(\pi_t + \frac{\lambda}{\kappa}x_t\right)$$

$=: \delta_t$

$$\varphi_{6,t} = \delta_t\,\varphi_{6,t+1} - \gamma_t$$

$$\psi_{6,t} = \delta_t \psi_{6,t+1} - \gamma_t$$

$$\psi_{6,t} = \delta_t \left[\delta_{t+1} \psi_{6,t+2} - \gamma_{t+1}\right] - \gamma_t$$

$$= \delta_t \left[\delta_{t+1} \left[\delta_{t+2} \psi_{6,t+3} - \gamma_{t+2}\right] - \gamma_{t+1}\right] - \gamma_t$$

$$= \delta_t \delta_{t+1} \delta_{t+2} \psi_{6,t+3} - \left[\delta_t \delta_{t+1} \gamma_{t+2} + \delta_t \gamma_{t+1} + \gamma_t\right]$$

$$= \delta_t \delta_{t+1} \delta_{t+2} \delta_{t+3} \psi_{6,t+4}$$

$$- \delta_t \delta_{t+1} \delta_{t+2} \gamma_{t+3}$$

$$- \left[\delta_t \delta_{t+1} \gamma_{t+2} + \delta_t \gamma_{t+1} + \gamma_t\right]$$

$$= - \left[\delta_t \delta_{t+1} \delta_{t+2} \gamma_{t+3} + \delta_t \delta_{t+1} \gamma_{t+2} + \delta_t \gamma_{t+1} + \gamma_t\right]$$

$$= - \gamma_t - \sum_{i=1}^{\infty} \gamma_{t+i} \prod_{j=0}^{i-1} \delta_{t+j}$$

$$\psi_{6,t} = - \sum_{i=0}^{\infty} \gamma_{t+i} \prod_{j=0}^{i-1} \delta_{t+j}$$ (again using the $\prod_{j=0}^{0} = 1$ notation)

$$\psi_{6,t} = -2 E_t \sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i}\right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{fe_{t+j}}}{\frac{k_{t+j}^{-1}}{fe_{t+j}} + g_\pi(t+j)}$$

Sol of (1') for $\psi_{6,t}$.

This implies (plugging into (6')):

$$E_t \sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{k} x_{t+i}\right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{fe_{t+j}}}{\frac{k_{t+j}^{-1}}{fe_{t+j}} + g_{\pi}(t+j)}$$

$$= E_t \, c \, fe_t \, x_{t+1}$$

$$+ E_t \left[1 + \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1}) - fe_t \, g_{\bar{\pi}}(t)\right] \left(\sum_{i=0}^{\infty} \left(\pi_{t+1+i} + \frac{\lambda}{k} x_{t+1+i}\right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+1+j}^{-1}}{fe_{t+1+j}}}{\frac{k_{t+1+j}^{-1}}{fe_{t+1+j}} + g_{\pi}(t+1+j)} \right)$$

$$- E_t \frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) \left(\sum_{i=0}^{\infty} \left(\pi_{t+2+i} + \frac{\lambda}{k} x_{t+2+i}\right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+2+j}^{-1}}{fe_{t+2+j}}}{\frac{k_{t+2+j}^{-1}}{fe_{t+2+j}} + g_{\pi}(t+2+j)} \right)$$

the target criterion if the anchoring function is specified in terms of changes in the gain.

(I divided terms by -2.)

(Bigugly)

Back to (6')

$$\varphi_{6,t+1} = \frac{c\,fe_t}{\alpha_t} x_{t+1} + \underbrace{\frac{1}{\alpha_t}}_{<1} \varphi_{6,t} - \underbrace{\frac{\beta_t}{\alpha_T}}_{<1} \varphi_{6,\underline{t+2}}$$

Let's do a really crazy thing: sub fwds & backwards

$$\varphi_t = c_{t-1} x_t + \underbrace{a_{t-1}}_{<1} \varphi_{t-1} + \underbrace{b_{t-1}}_{<1} \varphi_{t+1}$$

let me call $b_{t-1} = -\frac{\beta_t}{\alpha_t}$

Iter 1

(Mathematica)

$$\varphi_t = c_{t-1} x_t$$
$$+ a_{t-1} (c_{t-2} x_{t-1} + a_{t-2} \varphi_{t-2} + b_{t-2} \varphi_t)$$
$$+ b_{t-1} (c_t x_{t+1} + a_t \varphi_t + b_t \varphi_{t+2})$$
$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$
✓ $+ a_{t-1} a_{t-2} \varphi_{t-2}$

✓ $- a_{t-1} b_{t-2} \varphi_t + b_{t-1} a_t \varphi_t$

✓ $+ b_{t-1} b_t \varphi_{t+2}$

I shouldn't have plugged in here. I would have saved myself the bold-circled terms.

Iter 2

$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$

↳ LHS $1 - a_{t-1} b_{t-2} - b_{t-1} a_t$

$$+ a_{t-1} a_{t-2} (c_{t-3} x_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1})$$
$$- (a_{t-1} b_{t-2} + b_{t-1} a_t)(c_{t-1} x_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1})$$
$$+ b_{t-1} b_t (c_{t+1} x_{t+2} + a_{t+1} \varphi_{t+1} + b_{t+1} \varphi_{t+3})$$

$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$

$$+ a_{t-1} a_{t-2} \left(c_{t-3} x_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1} \right)$$

$$+ \left(a_{t-1} b_{t-2} + b_{t-1} a_t \right) \left(c_{t-1} x_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1} \right)$$

$$+ b_{t-1} b_t \left(c_{t+1} x_{t+2} + a_{t+1} \varphi_{t-1} + b_{t+1} \varphi_{t+3} \right)$$

$$= \underline{c_{t-1} x_t} + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$

$$a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$\underline{+ \left(a_{t-1} b_{t-2} + b_{t-1} a_t \right) c_{t-1} x_t}$$

$$+ b_{t-1} b_t c_{t+1} x_{t+2}$$

$$\left. \begin{aligned} & + a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3} \\ & + a_{t-1} a_{t-2} b_{t-3} \varphi_{t-1} \\ & + \left(a_{t-1} b_{t-2} + b_{t-1} a_t \right) a_{t-1} \varphi_{t-1} \\ & + \left(a_{t-1} b_{t-2} + b_{t-1} a_t \right) b_{t-1} \varphi_{t+1} \\ & + b_{t-1} b_t a_{t+1} \varphi_{t+1} \\ & + b_{t-1} b_t b_{t+1} \varphi_{t+3} \end{aligned} \right\} \text{these are all 0 in the limit}$$

the x parts: $a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$\left(1 + a_{t-1} b_{t-2} + b_{t-1} a_t \right) c_{t-1} x_t$$

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3}$$

$$+ \left(a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1} \right) \varphi_{t-1}$$

$$+ \left((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1} \right) \varphi_{t+1}$$

$$+ b_{t-1} b_t b_{t+1} \varphi_{t+3}$$

|| Now plug only the x-term. 1 ker 3

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} (c_{t-4} x_{t-3} + \dots)$$

$$+ \left(a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1} \right) (c_{t-2} x_{t-1} + \dots)$$

$$+ \left((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1} \right) (c_t x_{t+1} + \dots)$$

$$+ b_{t-1} b_t b_{t+1} (c_{t+2} x_{t+3} + \dots)$$

So only the x parts:

$$= a_{t-1}c_{t-2}x_{t-1} + a_{t-1}a_{t-2}c_{t-3}x_{t-2}$$
$$+ b_{t-1}c_t x_{t+1} + b_{t-1}b_t c_{t+1}x_{t+2}$$
$$(1 + a_{t-1}b_{t-2} + b_{t-1}a_t)c_{t-1}x_t$$
$$+ a_{t-1}a_{t-2}a_{t-3}(c_{t-4}x_{t-3} + \dots)$$
$$+ (a_{t-1}a_{t-2}b_{t-3} + (a_{t-1}b_{t-2} + b_{t-1}a_t)a_{t-1})(c_{t-2}x_{t-1} + \dots)$$
$$+ ((a_{t-1}b_{t-2} + b_{t-1}a_t)b_{t-1} + b_{t-1}b_t a_{t+1})(c_t x_{t+1} + \dots)$$
$$+ b_{t-1}b_t b_{t+1}(c_{t+2}x_{t+3} + \dots)$$

$$= a_{t-1}c_{t-2}x_{t-1} + a_{t-1}a_{t-2}c_{t-3}x_{t-2} + a_{t-1}a_{t-2}a_{t-3}c_{t-4}x_{t-3}$$
$$+ b_{t-1}c_t x_{t+1} + b_{t-1}b_t c_{t+1}x_{t+2} + b_{t-1}b_t b_{t+1}c_{t+2}x_{t+3}$$
$$+ (1 + a_{t-1}b_{t-2} + b_{t-1}a_t)c_{t-1}x_t$$
$$+ (a_{t-1}a_{t-2}b_{t-3} + (a_{t-1}b_{t-2} + b_{t-1}a_t)a_{t-1})\,c_{t-2}x_{t-1}$$
$$+ ((a_{t-1}b_{t-2} + b_{t-1}a_t)b_{t-1} + b_{t-1}b_t a_{t+1})\,c_t x_{t+1}$$

→ there will be a backward sum & a forward sum

$$BS := \sum_{i=1}^{\infty} x_{t-i}\, c_{t-1-i} \prod_{j=1}^{i} a_{t-j}$$

$$FS := \sum_{i=1}^{\infty} x_{t+i}\, c_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j}$$

and one or more cross-sums. Let's look at those.

$$+ (1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

Let's multiply out partly

$$(1 + a_{t-1} b_{t-2} + a_t b_{t-1}) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1}^2 b_{t-2} + a_{t-1} a_t b_{t-1})) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} b_{t-1} + a_t b_{t-1}^2) + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ a_{t-1} (a_{t-2} b_{t-3} + (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}})) c_{t-2} x_{t-1}$$

$$+ b_{t-1} ((\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) + b_t a_{t+1}) c_t x_{t+1}$$

$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ \underline{(a_{t-2} b_{t-3} + (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}})) a_{t-1} c_{t-2} x_{t-1}}$$

$$+ \underline{((\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) + b_t a_{t+1}) b_{t-1} c_t x_{t+1}}$$

I think (and hope) that more and more powers and multiplications can be pulled out, so that the bw- & fw- cross-terms go to 0. In that case we'd be left with

$$x_t c_{t-1} (1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \ldots \text{ expanding sum})$$

$\longrightarrow$ in fact,

I might even be able to neglect some or all of this sum. Or it may not be expanding. Try to verify w/ Mathemidica. If it doesn't expand, I'd get:

$$\varphi_{6,t} = BS + FS + CS$$

$$= \sum_{i=1}^{\infty} X_{t-i}\, C_{t-1-i} \prod_{j=1}^{i} a_{t-j}$$

$$+ \sum_{i=1}^{\infty} X_{t+i}\, C_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j}$$

$$+ X_t C_{t-1} \left(1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \ldots \cancel{\text{expanding sum}}\right)$$

where $C_{t-1} = \dfrac{c\, fe_t}{\alpha_t}$ $\qquad c = -\dfrac{2(1-\alpha)\beta}{1-\alpha\beta}$

$a_{t-1} = \dfrac{1}{\alpha_t}$ $\qquad \alpha_t = \left[1 + \dfrac{fe_t}{fe_{t-1}}\left(1 - k_{t-1}^{-1}\right) - fe_t\, g_{\bar{\pi}}(t)\right]$

$b_{t-1} = -\dfrac{\beta_t}{\alpha_t}$ $\qquad \beta_t = \dfrac{fe_t}{fe_{t-1}}\left(1 - k_{t+1}^{-1}\right)$

But now the cross-terms simplify if we hadn't subbed in $\varphi_{6,t}$ on RHS, but on LHS we'd have

$$(1 - a_{t-1} b_{t-2} - b_{t-1} a_t)\, \varphi_{6,t} = BS + FS + CS$$

and CS would the previous minus the bold stuff; i.e.

$$CS = a_{t-2}\, a_{t-1}\, b_{t-3}\, C_{t-2}\, X_{t-1} + a_{t+1}\, b_t\, b_{t+1}\, C_t\, X_{t+1}$$

Now the hope is that the CS-term will vanish.

Mathematica: Iter 3: new t-term shows up:

$(a_{t-2}\, a_{t-1}\, b_{t-3}\, b_{t-2} + a_t\, a_{t+1}\, b_{t-1}\, b_t)\, \psi_{6,t}$

$\rightarrow$ have to subtract from LHS!

And in Iter 5 it will appear again.

So instead of subbing them always from the LHS, I'll assume that already 4 multiplications of any a or b combos is close enough to 0 so I don't need to worry (i.e. from Iter 3 onward.)

Ok, so I'm letting all $\psi_{6,t\pm k}$ $k \in \mathbb{R}$ $\rightarrow 0$

Let me write down what I get after 5 iters in Mathematica, where I took $\psi_{6,t}$ on the LHS only in Iter 3: Coefficients:

x_{t-5}: $a_{t-5}\, a_{t-4}\, a_{t-3}\, a_{t-2}\, a_{t-1}\, c_{t-6}$

x_{t-4}: $a_{t-4}\, a_{t-3}\, a_{t-2}\, a_{t-1}\, c_{t-5}$

Good up to x_{t-3}:

X_{t-3}: $a_{t-3}\, a_{t-2}\, a_{t-1}\, c_{t-4}$

$+ a_{t-4}\, a_{t-3}\, a_{t-2}\, a_{t-1}\, b_{t-5}\, c_{t-4}$

$+ a_{t-3}^2\, a_{t-2}\, a_{t-1}\, b_{t-4}\, c_{t-4}$

$+ a_{t-3}\, a_{t-2}^2\, a_{t-1}\, b_{t-3}\, c_{t-4}$

X_{t-2}: $a_{t-2}\, a_{t-1}\, c_{t-3}$

$+ a_{t-3}\, a_{t-2}\, a_{t-1}\, b_{t-4}\, c_{t-3}$

$a_{t-2}^2\, a_{t-1}\, b_{t-3}\, c_{t-3}$

X_{t-1}: $a_{t-1}\, c_{t-2}$

$+ a_{t-2}\, a_{t-1}\, b_{t-3}\, c_{t-2}$

$+ a_{t-3}\, a_{t-2}\, a_{t-1}\, b_{t-4}\, b_{t-3}\, c_{t-2}$

$+ a_{t-2}^2\, a_{t-1}\, b_{t-3}\, c_{t-2}$

It's impossible to see any pattern!

My only chance is to make restrictions on the coefficients. In particular I'm interested in doing so for a & b, not so much for c.

$$c_{t-1} = \frac{c\, fe_t}{\alpha_t} \qquad c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}$$

$$a_{t-1} = \frac{1}{\alpha_t} \qquad a_T = \left[1 + \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1}) - fe_t\, g_{\bar{\pi}}(t)\right]$$

$$b_{t-1} = -\frac{\beta_t}{\alpha_t} \qquad \beta_t = \frac{fe_t}{fe_{t+1}}(1-k_{t+1}^{-1})$$

But $c_{t-1} = c\, fe_t\, a_{t-1}$, so if I have any restriction that says that $a \cdot b = 0$, then $b \cdot c = 0$ too.

(which will kill the forward sum FS).

So $a_{t-1}\, b_{t-1} = -\frac{1}{\alpha_t} \frac{\beta_t}{\alpha_t}$

Can't make the case that $\alpha_t^2 \to \infty$ b/c then

$a_{t-1}\, c_{t-1} = 0$ too, kill everything.

What one can do is to make a time-dependent argument.

$\lim_{t\to\infty} \beta_t \approx 1 \quad \lim_{t\to 0} \beta_t = 0 \quad \longrightarrow \lim_{t\to\infty} b_t = -\frac{1}{2}, \lim_{t\to 0} b_t = \frac{0}{0}$

$\lim_{t\to\infty} \alpha_t \approx 2 \quad \lim_{t\to 0} \alpha_t \approx 0 \quad \longrightarrow \lim_{t\to\infty} a_t = \frac{1}{2}, \lim_{t\to 0} a_t = \infty$

$\lim_{t\to\infty} c_t = \frac{c\,\Sigma}{2} \quad \lim_{t\to 0} c_t = \infty$

Ok, so one can't even solve that. All I can do is interpret it: the target criterion under anchoring function specified in gain changes, not levels, takes the form of two equations:

$$\varphi_{6,t} = -c\, fe_t x_{t+1} + \left[1 + \frac{fe_t}{fe_{t-1}}(1-k_{t-1}^{-1}) - fe_t g_{\bar{\pi}}(t)\right]\varphi_{6,t+1} - \frac{fe_t}{fe_{t-1}}(1-k_{t+1}^{-1})\varphi_{6,t+2} \quad (6')$$

$$2\pi_t + \frac{2\lambda}{\kappa}x_t - \left(\frac{k_t^{-1}}{fe_t} + g_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} = 0 \quad (1')$$

One way to represent this is (bigugly) on p. 65. The other is in the 3-eq-form:

$c = 2\frac{(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}$ (+ or −?) (from p. 53)

Define $c \equiv -\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}$ and $fe_t \equiv \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$, (where I'm annoyingly using a different notation than usual...

Then:

$$2\pi_t + \frac{2\lambda}{\kappa}x_t - k_t^{-1}\varphi_{5,t} - g_\pi(t)\varphi_{6,t} = 0 \quad (1)$$

$$c\, x_{t+1} + \varphi_{5,t} - (1-k_{t+1}^{-1})\varphi_{5,t+1} + g_{\bar{\pi}}(t)\varphi_{6,t+1} = 0 \quad (6)$$

$$fe_t \cdot \varphi_{5,t} = \varphi_{6,t} - \varphi_{6,t+1} \quad (7)$$

(1) says "discretion + learning ($\varphi_{5,t}$) × effect of stance of anchoring + "effect of how anchoring changes" where (7) says that if learning

is a strong constraint now, then at least anchoring constraint will be relaxed strongly tomorrow.

Interpreting (bigingly) is an alternative way: the CB can follow a target criterion as computing the anchoring multiplier $\varphi_{6,t}$ for t, $t+1$ and $t+2$ and then evaluating (6'). Compute the multiplier as the solution to (1') as:

$$\varphi_{6,t} = -2E_t \sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{k} x_{t+i}\right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{f_{e_{t+j}}}}{\frac{k_{t+j}^{-1}}{f_{e_{t+j}}} + g_{\pi}(t+j)} \qquad (\text{sol } 1')$$

Interpretation:

$\varphi_{6,t} = 0$ if $\pi_{t+i} + \frac{\lambda}{k} x_{t+i} = 0 \quad \forall i$ or

$k_{t+j}^{-1} = 0 \quad \forall j$

i.e. if you hit the target or expectations are anchored. But exp. can't be anchored if you don't hit the target (vice versa can happen - you hit the target but there was a shock that got expectations unanchored).

I think (sol 1') suggests that

$$\varphi_{6,t} = -2E_t\left(\pi_t + \frac{\lambda}{\kappa} x_t\right) + \varphi_{6,t+1}$$

So if $\varphi_{6,t} = 0$ b/c $\left(\pi_t + \frac{\lambda}{\kappa} x_t\right) = 0$ $\forall t$, then $\varphi_{6,t+1} = 0$ too.

But $\varphi_{6,t} = 0 \not\Rightarrow \varphi_{6,t+1} = 0$.

Then $\varphi_{6,t} = -2\left(\pi_t + \frac{\lambda}{\kappa} x_t\right) - 2E_t\left(\pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1}\right) + \varphi_{6,t+2}$

So $\varphi_{6,t+2} = \varphi_{6,t} + 2\left(\pi_t + \frac{\lambda}{\kappa} x_t\right) + 2E_t\left(\pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1}\right)$

$\varphi_{6,t+1} = \varphi_{6,t} + 2\left(\pi_t + \frac{\lambda}{\kappa} x_t\right)$

Plugging this in (6')

$$\varphi_{6,t} = -c\, fe_t\, x_{t+1} + \left[1 + \frac{fe_t}{fe_{t+1}}\left(1 - k_{t+1}^{-1}\right) - fe_t\, g_{\bar{\pi}}(t)\right]\varphi_{6,t+1} - \frac{fe_t}{fe_{t+1}}\left(1 - k_{t+1}^{-1}\right)\varphi_{6,t+2}$$

$$\varphi_{6,t} = -c\, fe_t\, x_{t+1} + \alpha_t\left(\varphi_{6,t} + 2\left(\pi_t + \frac{\lambda}{\kappa} x_t\right)\right)$$
$$-\beta_t\left(\varphi_{6,t} + 2\left(\pi_t + \frac{\lambda}{\kappa} x_t\right) + 2E_t\left(\pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1}\right)\right)$$

$$(1 - \alpha_t + \beta_t)\varphi_{6,t} = -c\, fe_t\, x_{t+1} + 2(\alpha_t - \beta_t)\left(\pi_t + \frac{\lambda}{\kappa} x_t\right)$$
$$-2\beta_t E_t\left(\pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1}\right)$$

$$\varphi_{6,t} = \frac{-c\, fe_t x_{t+1} + 2(\alpha_t - \beta_t)(\pi_t + \frac{\lambda}{\kappa} x_t) - 2\beta_t E_t(\pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1})}{1 - \alpha_t + \beta_t}$$

$$\alpha_t - \beta_t = \left[1 + \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1}) - fe_t g_{\bar{\pi}}(t)\right] - \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1})$$

$$= 1 - fe_t g_{\bar{\pi}}(t)$$

$$\Rightarrow \varphi_{6,t} = \frac{-c\, fe_t x_{t+1} + 2(1 - fe_t g_{\bar{\pi}}(t))(\pi_t + \frac{\lambda}{\kappa} x_t) - 2\frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1})(E_t \pi_{t+1} + \frac{\lambda}{\kappa} x_{t+1})}{fe_t g_{\bar{\pi}}(t)}$$

(finite)

Which is neat I guess b/c you don't need to evaluate all the future periods.

Can we plug this in (1')? I don't really think so b/c we used knowledge of the sol. to (1') to derive this.

Hmm. Let it rest.

← stuff potentially to be used for interpretation in the paper. Otherwise for the implementation of the target criterion, I'll stick w/ anchoring function in levels.

Ryan meeting 1 April 2020

Conjecture a sequence of $\{x, i\}$ s.t. A.1. & A.3

fmincon to find the sequence of $\{x, i\}$ for which A.1. & A.3 hold

Think of the $\hat{E}$ you'd compute over period 50.

think of A.1. & A.3 as residual eq.

and (A.4) holding exactly

You can also do the opposite: solve for expectations that satisfy all the processes.

Ryan thinks

A.1. & A.3. need to be residual

A.2 we don't know

A.5, 6, 7 can be used exactly

Start w/ sim you know works → check resids of 1-7 → should be zero. Then perturb your sequence & see.

Work after

My comments/ interpretation:

It seems like there is a language for these kinds of problems:

residual equations vs. those that are fulfilled exactly

And it also seems (Ryan was suggesting) that there is a right/wrong decision about which equation to treat as residual or exactly fulfilled equation.

It seems to me that

1) initially I treated all equations as exact (or something)

2) Need to figure out which to treat how.