Materials 19 - Limits on analytical results?

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1 A simplified optimal policy with anchoring problem

Planner chooses $\{\pi_t, x_t, f_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$\mathcal{L} = \mathbb{E}_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Big\{ \pi_t^2 + \lambda x_t^2 + \varphi_{1,t}(\pi_t - \kappa x_t - \beta f_t + u_t)$$
$$+ \varphi_{2,t}(f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1})) + \varphi_{3,t}(k_t^{-1} - \mathbf{g}(\pi_t - f_{t-1})) \Big\}$$

where the IS-curve, $x_t = \mathbb{E}_t\, x_{t+1} + \sigma f_t - \sigma i_t + \sigma r_t^n$, is a non-binding constraint, and $\mathbb{E}_t\, x_{t+1}$ is rational. After some manipulation, FOCs reduce to:

$$2\pi_t + 2\frac{\lambda}{\kappa} x_t - \varphi_{2,t}(k_t^{-1} + \mathbf{g}_\pi(\pi_t - f_{t-1})) = 0 \tag{1}$$
$$-2\beta\frac{\lambda}{\kappa} x_t + \varphi_{2,t} - \varphi_{2,t+1}(1 - k_{t+1}^{-1} - \mathbf{g_f}(\pi_{t+1} - f_t)) = 0 \tag{2}$$

Combining FOCs with the three model equations, I obtain the following system in $\{\pi_t, x_t, f_t, k_t^{-1}, \varphi_t\}_{t=0}^{\infty}$ where I've relabeled $\varphi \equiv \varphi_2$ for simplicity:

$$2\pi_t + 2\frac{\lambda}{\kappa} x_t - \varphi_t(k_t^{-1} + \mathbf{g}_\pi(\pi_t - f_{t-1})) = 0 \tag{3}$$
$$-2\beta\frac{\lambda}{\kappa} x_t + \varphi_t - \varphi_{t+1}(1 - k_{t+1}^{-1} - \mathbf{g_f}(\pi_{t+1} - f_t)) = 0 \tag{4}$$
$$\pi_t - \kappa x_t - \beta f_t + u_t = 0 \tag{5}$$
$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) = 0 \tag{6}$$
$$k_t^{-1} - \mathbf{g}(\pi_t - f_{t-1}) = 0 \tag{7}$$

Unless I find some trick to convert this to a linear system, I can't solve it.

So I see 3 options:

1. Find trick to linearize.

2. Discuss optimal policy only up to target criterion, which can be derived even for the more complex (but simplified) model and is given by:

$$\pi_t = -\frac{\lambda}{\kappa}\left\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - bs_{t-1}))f_\pi(t)\right)\left(\sum_{i=1}^{\infty} x_{t+i}\prod_{j=1}^{i-1}(1 - k_{t+j}^{-1}(\pi_{t+1+j} - \bar{\pi}_{t+j} - bs_{t+j}))\right)\right\} \tag{8}$$

   One could also be "Woodfordian" and try to find simplifying expressions for this criterion that are approximately as good as this.

3. Since no distinction between discretion and commitment, could investigate a purely forward-looking policy rule (Taylor rule), i.e. reexamine the noninertial plan.

A Procedure to obtain the optimal interest rate rule: NKIS vs. target criterion

"NKIS approach" (Woodford 2003 and Handbook chapter, Molnár & Santoro)

1. Combine all FOCs to a single difference equation (system) in one variable (vector), solve it for the time path of the variable as a function of forcing terms.
2. Express optimal time paths for all endogenous variables, plug into NKIS.
3. Obtain optimal time path for the nominal interest rate as a function pf disturbances only.

"Target criterion approach" (Woodford 2003 and Handbook chapter)

1. From FOCs, express a target criterion as a relationship between endogenous variables that is optimal.
2. Derive the same (?) time paths for the endogenous variables, use NKIS and NKPC to express expectations for them.
3. Plug everything into the target criterion, rearrange to obtain the nominal interest rate path.

Why? The criterion is robust, but the rule isn't. Both result in a rule that delivers an indeterminate equilibrium. (In fact, they result in the same rule (Hbook, p. 19).)

B Details

Woodford:

1. Set up the above problem with RE. Get FOCs.
2. Combine FOCs to get a relationship between π, x; the target criterion.
3. Obtain the following 2nd order difference equation in the multiplier:

$$\beta\,\mathbb{E}_t\,\varphi_{t+1} - \big(1+\beta+\frac{\kappa^2}{\lambda}\big)\varphi_t + \varphi_{t-1} = \frac{\kappa}{\lambda}x^* + u_t \tag{9}$$

4. Obtain the optimal time path as the solution of the above equation:

$$\varphi_t = -\frac{\lambda}{\kappa}x^*(1-\mu_1^{t+1}) - \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}\,\mathbb{E}_t\,u_{t+j} \qquad 0<\mu_1<1<\mu_2 \tag{10}$$

(where I think β^{-j} is missing in the sum, and we know from the Viète formulas that $\mu_1 = \mu_2^{-1}\beta^{-1}$) Substitution into FOCs gives time path for π, x.

5. Sub time paths into NKPC and NKIS to solve for agents' expectations as functions of shocks only.

6. Evaluate target criterion at time t by substituting in time paths and expectations of π, x. Rearrange to express i_t.

Molnár & Santoro

1. Set up the above problem with learning. Get FOCs.

2. Express a target criterion, but don't think of it as such explicitly.

3. Sub out x from it using the NKIS to express $\mathbb{E}_t \pi_{t+1}$ as a function of current inflation, expectations and disturbances.

4. At this point they think of the economy as a trivariate system (inflation, inflation beliefs, output gap beliefs) where $t+1$ values depend on t ones and disturbances. Solve for the evolution of inflation as a function of inflation beliefs and disturbances only using a guess-and-verify method.

5. Once you have that for π, you can get the same for x.

6. Plug time paths for π, x into NKPC. Express i_t.