

# The Discounted Euler Equation: A Note

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We present a simple model with income risk and borrowing constraints that yields a 'discounted Euler equation'. This feature of the model mutes the extent to which news about far future real interest rates (i.e. forward guidance) affects current outcomes. We show that this simple model approximates the outcomes of a rich model with uninsurable income risk and borrowing constraints in response to a forward guidance shock. The model is simple enough to be easily incorporated into simple New Keynesian models. We illustrate this with an application to the zero lower bound.

## INTRODUCTION

Recent work has highlighted that standard monetary business cycle models imply that far future forward guidance has implausibly large effects on current outcomes (Carlstrom *et al.* 2015; Del Negro *et al.* 2015).<sup>1</sup> In earlier research, we showed that this implausible behaviour of standard models results from the highly forward looking nature of the standard consumption Euler equation (McKay *et al.* 2016). To see this, consider the response of consumption to a forward guidance announcement by the central bank in a standard New Keynesian model. A key equation in this type of model is the standard consumption Euler equation, which once it has been log-linearized takes the form

$$\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n),$$

where  $\hat{C}_t$  denotes consumption,  $i_t$  denotes the nominal interest rate,  $\pi_t$  denotes inflation,  $r_t^n$  is the natural real rate of interest, i.e. the real interest rate that would prevail with flexible prices (all measured as percent deviations from steady state),  $\mathbb{E}_t$  denotes the expectations operator, and  $\sigma$  is the intertemporal elasticity of substitution. Solving this equation forward yields

$$\hat{C}_t = -\sigma \sum_{j=0}^{\infty} \mathbb{E}_t(i_{t+j} - \mathbb{E}_{t+j} \pi_{t+j+1} - r_{t+j}^n).$$

Notice that there is no discounting in the sum on the right-hand side of this equation. This implies that news about future real interest rates at any horizon—however far in the future—has the same effect on current consumption as an equally large change to the current interest rate.<sup>2</sup>

This prediction is sensitive to the presence of precautionary savings and liquidity constraints. In McKay *et al.* (2016), we present a heterogeneous agents New Keynesian model in which the response of current consumption to far future forward guidance is greatly muted relative to the standard representative agent New Keynesian model.<sup>3</sup> A drawback of the model presented in that paper is that it is considerably more complicated to solve than a standard New Keynesian model.

In this note, we present a simple model with income risk and borrowing constraints that yields a ‘discounted Euler equation’. The discounting in the Euler equation means that the effects of future real interest rates on current consumption are muted relative to the standard New Keynesian model.<sup>4</sup> We show that this simple model approximates the outcomes of our richer model with uninsurable income risk and borrowing constraints in response to a forward guidance shock. The model is simple enough to be easily incorporated into simple New Keynesian models.<sup>5</sup> We illustrate this with an application to the zero lower bound.

## I. DISCOUNTED EULER EQUATION MODEL

Consider a simplified version of the model presented in McKay *et al.* (2016), in which idiosyncratic income is i.i.d. and takes one of two values each period, which we refer to as high ( $z = 1$ ) and low ( $z = 0$ ). For simplicity, the income of those with low productivity is constant and equal to  $m$ , which could reflect a social welfare benefit. Agents are not allowed to borrow in the model. In this version of the model, we furthermore assume for tractability that the only assets available are risk-free bonds in zero net supply.

We show in the Appendix that under these assumptions we can derive the following log-linearized consumption Euler equation:

$$(1) \quad \hat{C}_t = \alpha \mathbb{E}_t \hat{C}_{t+1} - \sigma \zeta (i_t - \mathbb{E}_t \pi_{t+1} - r_t^n).$$

This equation differs from the standard consumption Euler equation because of the coefficients  $\alpha < 1$  and  $\zeta < 1$ . We also derive a New Keynesian Phillips curve for this model and show that it takes the same form as in the textbook representative agent model, the only difference being a slight increase in its slope for a given set of parameters.

Solving the Euler equation forward yields

$$\hat{C}_t = -\sigma \zeta \mathbb{E}_t \sum_{j=0}^{\infty} \alpha^j (i_{t+j} - \mathbb{E}_{t+j} \pi_{t+j+1} - r_{t+j}^n).$$

Notice that the effect of interest rates  $j$  periods in the future on current consumption is discounted by a factor  $\alpha^j$ . For this reason, we refer to equation (1) as the ‘discounted Euler equation’. The presence of  $\alpha < 1$  implies that far future interest rate changes have much smaller effects on current consumption than near term interest rate changes. The presence of  $\zeta$  is less consequential. It is equivalent to a change in the intertemporal elasticity of substitution  $\sigma$ .<sup>6</sup>

In this version of the model, the low-productivity households are liquidity constrained, while the high-productivity households are not. The presence of  $\alpha$  in equation (1) results from the fact that with some probability, the currently highly productive households will have low productivity next period, and next period’s expected marginal utility is therefore partly determined by the exogenous marginal utility in the low-productive state. For this reason, the currently highly productive households put some probability on the outcome that they will not benefit from an increase in next period’s consumption and therefore do not increase current consumption one-for-one with next period’s expected consumption.

The key feature of this stylized model is that the income and consumption of the high types is more cyclically sensitive than the income and consumption of the low types (Werning 2015).<sup>7</sup> Recent work by Guvenen *et al.* (2014) shows that top incomes rise substantially more than average incomes in expansions in the USA. Parker and Vissing-Jorgensen (2009) show that consumption growth of high-consumption households in the USA is much more exposed to aggregate fluctuations than consumption growth of the typical household. Our model is meant to capture this feature of the data in a stylized way.

Figure 1 shows that the discounted Euler equation—equation (1)—with  $\alpha = 0.97$  and  $\zeta = 3/4$  provides a good approximation to the response of output to a real interest rate shock 20 quarters in the future in the baseline incomplete markets model analysed in McKay *et al.* (2016). The approximation is nearly perfect until the time when the interest rate changes. What the discounted Euler equation misses is the fall in consumption after the interest rate shock. This fall is due to redistribution of wealth in the incomplete markets model (from households with high marginal propensities to consume to households with low marginal propensities to consume), which the discounted Euler equation does not capture.

The tractability of our model relies heavily on the simplifying assumption that households do not have any tradable wealth. We are following Krusell *et al.* (2011) and Ravn and Sterk (2013) in using this approach to formulate tractable incomplete markets models. This has the downside that our model cannot be embedded in medium-scale dynamic stochastic general equilibrium models that incorporate capital or government debt in positive net supply without substantial loss of tractability. However, the textbook New Keynesian model without such features is widely used to illustrate basic conceptual points about monetary economics. We therefore think that it is valuable to illustrate a

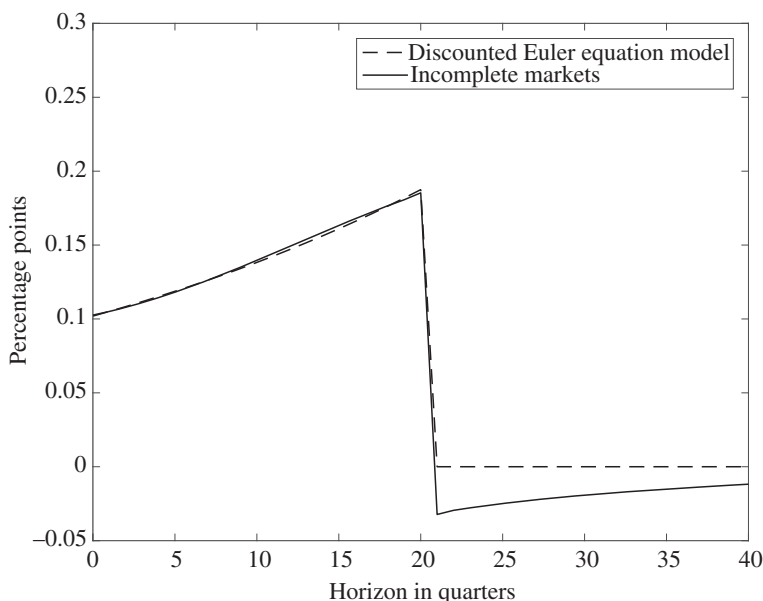


FIGURE 1. Response of output to a 50 basis point change in the expected real interest rate in quarter 20 in the incomplete markets model and in the discounted Euler equation model.

tractable way of incorporating the effects of precautionary savings into this benchmark model.

Piergallini (2006) and Nistico (2012) provide an alternative micro-foundation for discounting in the Euler equation based on mortality shocks as in the perpetual-youth model of Blanchard (1985). This model has recently been applied to the power of forward guidance by Del Negro *et al.* (2015). In this model, discounting arises only from aggregation over different generations, and to generate a quantitatively important deviation from the standard Euler equation, these authors need to assume counterfactually high death rates. Our approach rationalizes why long-living consumers can have short planning horizons. Moreover, in the formulation of Piergallini and Nistico, the discounting in the Euler equation is larger, the larger is the amount of financial wealth in the economy, and disappears when financial wealth is in zero net supply. In contrast, in McKay *et al.* (2016), agents discount the future more when they have little liquid financial wealth to buffer shocks to income.<sup>8</sup>

## II. APPLICATION TO THE ZERO LOWER BOUND

We now illustrate the importance of discounting in the Euler equation using an application to the zero lower bound (ZLB). We follow the classic paper by Eggertsson and Woodford (2003) in how we model the ZLB, but show that our discounted Euler equation model yields substantially different conclusions.

First, using the aggregate resource constraint, we rewrite the Euler equation in terms of the output gap  $\hat{Y}_t$ <sup>9</sup>

$$(2) \quad \hat{Y}_t = \alpha \mathbb{E}_t \hat{Y}_{t+1} - \zeta \sigma (i_t - \mathbb{E}_t \pi_{t+1} - r_t^n).$$

The Calvo Phillips curve is

$$(3) \quad \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{Y}_t.$$

Finally, we assume that the central bank follows a ‘naive’ monetary policy,

$$i_t = \max(0, r_t^n + \phi \pi_t),$$

where  $\phi > 1$ . For comparability with Eggertsson and Woodford (2003), we assume that  $\beta = 0.99$ ,  $\sigma = 0.5$  and  $\kappa = 0.02$ .

Following Eggertsson and Woodford (2003), we assume that the ZLB binds due to a shock that lowers the natural rate below zero and persists at the same negative value with probability  $\lambda$  each quarter. With probability  $1 - \lambda$ , it reverts back to normal. For simplicity, we assume that once the natural rate reverts back to normal, the ZLB on nominal interest rates never binds again in the future.

We start by solving for the level of the output gap and inflation after the shock has dissipated. Since we have assumed that the natural rate will never go negative again, it is feasible for the monetary authority to set  $i_t = r_t^n$  at all times after the shock dissipates. This implies that both the output gap and inflation will be zero at all times after the shock dissipates. Given this, it is easy to solve for the output gap and inflation while the shock

persists. First, notice that all periods while the shock persists are identical since the probability of the shock reverting to normal does not change over time. This implies that output and inflation will be constant while the shock persists. We refer to the period during which the shock persists as the short run. Next, notice that in the short run,  $\mathbb{E}_t \hat{Y}_{t+1} = \lambda \hat{Y}_t$  and  $\mathbb{E}_t \pi_{t+1} = \lambda \pi_t$ , since with probability  $1 - \lambda$  the economy will revert to normal (in which case  $\hat{Y}_t = \pi_t = 0$ ). Using these facts and equations (3) and (2), a few steps of algebra (presented in the Appendix) yield

$$(4) \quad \pi_S = \frac{\kappa}{1 - \beta\lambda} \hat{Y}_S,$$

$$(5) \quad \hat{Y}_S = \frac{\zeta\sigma}{1 - \alpha\lambda - \zeta\sigma\lambda\kappa/(1 - \lambda\beta)} r_S^n,$$

where  $\pi_S$  and  $\hat{Y}_S$  denote inflation and the output gap in the short run, and  $r_S^n$  denotes the natural real rate of interest in the short run.

#### *How deep is a ZLB recession?*

Our first result is that the discounted Euler equation model implies that the same shocks that generate a huge recession at the ZLB in the standard model imply a much smaller recession in the discounted Euler equation model. Eggertsson and Woodford (2003) present results for a shock that lowers the natural rate to  $r_S^n = -0.02$  (annualized) and reverts to normal with probability  $1 - \lambda = 0.1$  (per quarter). They show that in the standard model, a shock of this size and persistence generates a very large recession—an output gap of  $-14.3\%$ —accompanied by a large amount of deflation ( $-10.5\%$ ). In Table 1, we show that in the discounted Euler equation model, this same shock leads to a much more modest recession. The output gap is a mere  $-2.9\%$ , and inflation falls by only  $2.1\%$ . Clearly, incorporating discounting of future interest rates radically alters the conclusions one comes to about the severity of the problem that the monetary authority faces with this type of shock.<sup>10</sup>

#### *When does the deflationary death spiral occur?*

Our second result is that the discounted Euler equation model implies that the economy is less prone to a ‘deflationary death spiral’ at the ZLB. The strength of the deflationary

TABLE 1  
HOW DEEP IS A ZLB RECESSION?

	Output	Inflation
Standard model ( $\alpha = 1, \zeta=1$ )	$-14.3\%$	$-10.5\%$
Discounted Euler equation model ( $\alpha=0.97, \zeta=0.75$ )	$-2.9\%$	$-2.1\%$

#### *Notes*

Response of output and inflation when the natural rate falls to  $-2\%$  (annualized) with a 10% per quarter probability of returning to normal.

forces in the standard model are due to a feedback loop that gets stronger the more persistent is the shock to the natural rate (Eggertsson 2010). The basic feedback loop results from the following chain of logic. The negative natural rate leads to a positive interest rate gap—a real rate that is higher than the natural rate—because the nominal rate cannot fall below zero. This leads output to fall, and, if the shock is persistent, then it leads expectations of future output to fall, which in turn leads expected inflation to fall, which causes the current real rate to rise further, and current output to fall further, and so on. The more persistent is the shock, the more it affects expected inflation and the stronger this feedback loop becomes.

It is well known that the strength of the deflationary forces associated with negative shocks to the natural real rate becomes infinitely strong—i.e. implying that the (log) output gap and inflation go to negative infinity—for even relatively modest levels of persistence. This can be seen by looking at the denominator of the expression for the short-run output gap in equation (5). As this denominator goes to zero, the short-run output gap goes to negative infinity. In the standard model (with  $\alpha = 1$  and  $\zeta = 1$ ), this occurs for a shock with an expected duration of 11 quarters.

In the discounted Euler equation model, the strength of the deflationary forces are muted, and consequently the persistence of the ZLB shock needs to be greater for this ‘deflationary death spiral’ to occur. This is depicted in Figure 2, which plots the drop in output for different levels of persistence of the ZLB shock. The solid line is the standard model, while the two broken lines are two calibrations of the discounted Euler equation model that match the baseline and high-risk calibrations, respectively, of the incomplete markets model in McKay *et al.* (2016).<sup>11</sup> The deflationary death spiral occurs only for shocks that are considerably more persistent in the discounted Euler equation model than in the standard model.

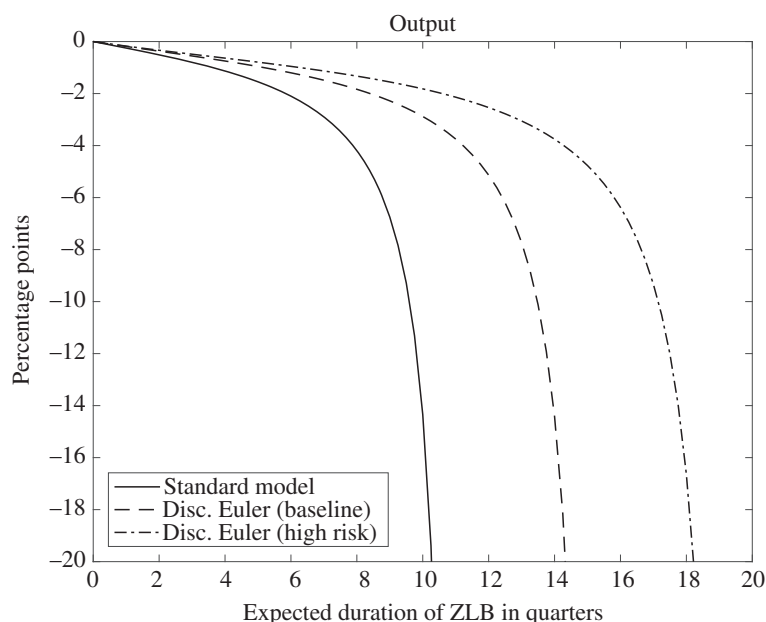


FIGURE 2. Response of output to shock that makes the natural real rate of interest  $-2\%$  (annualized) for different levels of persistence of the shock (different values of  $\lambda$ ).

### III. CONCLUSION

In this note, we develop simple micro-foundations for a ‘discounted Euler equation’ that is easy to incorporate into simple New Keynesian models. This discounted Euler equation implies that the effects of far future changes in real interest rates as well as far future changes in the natural real rate have much smaller effects on contemporaneous outcomes than in the standard New Keynesian model. We show that this has important implications for policy experiments when nominal interest rates are constrained by their zero lower bound.

### APPENDIX

#### *A simple incomplete markets model*

The discounted Euler equation (1) can be micro-founded with a simplified version of the model analysed in McKay *et al.* (2016). The simplified model deviates from the full model in the following ways:

- The idiosyncratic productivity shock takes just two values, which we will call high ( $z = 1$ ) and low ( $z = 0$ ).
- Idiosyncratic productivity is i.i.d. across time:  $\Pr(z'|z) = \Pr(z')$ .
- The supply of government debt is zero:  $B = 0$ .
- The tax system pays a social benefit  $m$  to low productivity households financed by lump-sum taxes on high productivity households.
- Firm dividends  $D_t$  are distributed only to the high-productivity households.

This version of our model is analytically tractable because there is no wealth in the economy and there is a strict borrowing constraint,  $b \geq 0$ . As the gross supply of assets is zero, there is no possibility of saving in equilibrium, so the distribution of wealth remains degenerate at zero.

*The environment* Here we briefly summarize the model in McKay *et al.* (2016), including the simplifications above. The economy is populated by a unit continuum of *ex ante* identical households with preferences given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{q_t} \left[ \frac{c_{h,t}^{1-\gamma}}{1-\gamma} - \frac{\ell_{h,t}^{1+\psi}}{1+\psi} \right],$$

where  $c_{h,t}$  is consumption of household  $h$  at time  $t$ , and  $\ell_{h,t}$  is labour supply of household  $h$  at time  $t$ . The random variable  $q_t$  is an aggregate patience shock that determines the natural rate of interest and is given by  $q_t = q_{t-1} + r_{t-1}'$  with  $q_0 = 0$ . This shock is an addition to the model in McKay *et al.* (2016). Households are endowed with stochastic idiosyncratic employment status  $z_{h,t} \in \{0, 1\}$ , which is drawn i.i.d. with probabilities  $\Pr(z_{h,t} = 0) = \rho$  and  $\Pr(z_{h,t} = 1) = 1 - \rho$ . High-productivity households earn labour income  $W_t \ell_{h,t}$ , where  $W_t$  is the aggregate real wage. Low-productivity households receive  $m$  units of the consumption good as a transfer, and high-productivity households pay a tax of  $\rho m / (1 - \rho)$  to finance the transfer.

In this economy, a final good is produced from intermediate inputs according to the production function

$$Y_t = \left( \int_0^1 y_{j,t}^{1/\mu} dj \right)^\mu,$$

where  $Y_t$  denotes the quantity of the final good produced at time  $t$ , and  $y_{j,t}$  denotes the quantity of the intermediate good produced by firm  $j$  in period  $t$ . The intermediate goods are produced using labour as an input according to the production function



$$y_{j,t} = n_{j,t},$$

where  $n_{j,t}$  denotes the amount of labour hired by firm  $j$  in period  $t$ .

While the final good is produced by a representative competitive firm, the intermediate goods are produced by monopolistically competitive firms. The intermediate goods firms face frictions in adjusting their prices that imply that they can update their prices only with probability  $\theta$  per period as in Calvo (1983). These firms are controlled by a risk-neutral manager who discounts future profits at rate  $\beta$ . Whatever profits are produced are paid out immediately to the high-productivity households, with each of them receiving an equal share.

Households trade a risk-free real bond with real interest  $r_t$  between periods  $t$  and  $t + 1$ . Borrowing constraints prevent these households from taking negative bond positions. There is no external supply of bonds. Finally, a monetary authority determines the nominal interest rate.

*Discounted Euler equation* As individual assets are constant at zero, it follows that all households of a given productivity status must choose the same consumption. Let  $c_{H,t}$  be the consumption of the high-productivity agents, and let  $c_{L,t}$  be the consumption of the low-productivity agents. Moreover, the absence of opportunities to borrow and save implies that consumption must be equal to income for all individuals. The low-productivity group will therefore consume  $m$ . The Euler equation for high-productivity agents is

$$c_{H,t}^{-\gamma} \geq \beta e^{r_t} (1 + r_t) \mathbb{E}_t [(1 - \rho) c_{H,t+1}^{-\gamma} + \rho m^{-\gamma}].$$

The Euler equation for a low-productivity agent is

$$m^{-\gamma} \geq \beta e^{r_t} (1 + r_t) \mathbb{E}_t [(1 - \rho) c_{H,t+1}^{-\gamma} + \rho m^{-\gamma}].$$

Notice that the right-hand side of the Euler equation is independent of current productivity as productivity is i.i.d. across time. Therefore if  $m < c_{H,t}$ , then the low-productivity households must be constrained and their Euler equation will not hold with equality. We assume that  $m$  is low enough that  $m < c_{H,t}$  for all  $t$ .<sup>12</sup>

Following Krusell *et al.* (2011) and Ravn and Sterk (2013), we will focus on the equilibrium of this economy in which the Euler equation of the high-productivity households holds with equality in all periods. In this equilibrium, the high-productivity households choose zero savings and are therefore up against their constraint. But the constraint does not bind in the sense that they would not strictly prefer to borrow more if allowed. There are other equilibria of this economy in which the Euler equation for the high-productivity household holds with inequality (implying that the high-productivity households would strictly prefer to borrow if allowed). We focus on the equilibrium in which the high-productivity households' Euler equation holds with equality for the following reason. In the more realistic case in which the gross supply of assets (either internal or external) is positive, there is a unique equilibrium in which the high-productivity households' Euler equation holds with equality. Consider a sequence of such economies with smaller and smaller gross supplies of assets. The equilibrium on which we focus is the unique equilibrium of the limiting economy for which the supply goes to zero.<sup>13</sup>

Aggregate consumption is

$$(A1) \quad C_t = \rho m + (1 - \rho) c_{H,t}.$$

Substituting into the high-productivity Euler equation, we have

$$\left( \frac{C_t}{1 - \rho} - \frac{\rho m}{1 - \rho} \right)^{-\gamma} = \beta e^{r_t} R_t \mathbb{E}_t \left[ (1 - \rho) \left( \frac{C_{t+1}}{1 - \rho} - \frac{\rho m}{1 - \rho} \right)^{-\gamma} + \rho m^{-\gamma} \right].$$



Log-linearizing this equation and using the Fisher equation yields

$$\hat{C}_t = -\sigma\zeta[i_t - \mathbb{E}\pi_{t+1} - r_t^n] + \alpha\mathbb{E}[\hat{C}_{t+1}],$$

where

$$\begin{aligned}\alpha &\equiv \left(1 + \frac{\rho}{1-\rho} \left(\frac{\bar{c}_H}{m}\right)^\gamma\right)^{-1}, \\ \zeta &\equiv 1 - \frac{\rho m}{\bar{C}}, \\ \sigma &\equiv \frac{1}{\gamma},\end{aligned}$$

and bars denote steady-state values and hats denote log deviations from steady state. Notice that  $\alpha$  is decreasing in the probability  $\rho$  of having low productivity, and the consumption differential  $\bar{c}_H/m$ .

In Section I we argue that  $\alpha = 0.97$  and  $\zeta = 0.75$  matches the dynamics produced by the model in McKay *et al.* (2016) shown in Figure 1. It is hard to get these two values for reasonable values of the underlying parameters  $\rho$ ,  $\bar{c}_H$ ,  $m$  and  $\gamma$ . Values of these parameters that yield  $\zeta = 0.75$  tend to yield values of  $\alpha$  below 0.97. This reflects the fact that the larger model that we work with in McKay *et al.* (2016) generates a somewhat different initial drop relative to the subsequent discounting than the model analysed here. However, what really matters is not  $\zeta$  itself but the product  $\zeta\sigma$ . We can interpret our calibration as a combination of underlying parameters that generate  $\alpha = 0.97$ , a  $\zeta$  slightly below 1, and a value of  $\sigma$  that is somewhat smaller than 0.5.

*Phillips curve* Market incompleteness does not change the form of the New Keynesian Phillips curve, but slightly changes the relationship between the slope of the Phillips curve and the structural parameters. To see this, note that the labour supply condition for the high-productivity workers is

$$c_{H,t}^{-\gamma} W_t = \ell_t^\psi.$$

Using the fact that aggregate labour supply is  $L_t = (1-\rho)\ell_t$  and aggregate consumption is given by equation (A1), this last expression can be rewritten as

$$\left(\frac{C_t}{1-\rho} - \frac{\rho m}{1-\rho}\right)^{-\gamma} W_t = \left(\frac{L_t}{1-\rho}\right)^\psi.$$

Log-linearizing this equation and using the aggregate resource constraint  $C_t = Y_t$  and a first-order approximation to the aggregate production function,<sup>14</sup>  $\hat{L}_t = \hat{Y}_t$ , yields

$$\hat{W}_t = \left(\psi + \frac{\gamma}{1 + \rho m/\bar{Y}}\right) \hat{Y}_t.$$

If we set  $\rho = 0$ , then we arrive at the standard, complete-markets case. The effect of market incompleteness is to strengthen the wealth effect on labour supply because a subset of the workers are bearing the full effects of aggregate income movements.

Market incompleteness does not change the relationship between inflation and marginal cost. To a first-order approximation, this relationship is

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\beta(1-\theta))\theta}{1-\theta} \hat{M}C_t.$$

As the real marginal cost is the wage, we have

$$(A2) \quad \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1 - \beta(1 - \theta))\theta}{1 - \theta} \left( \psi + \frac{\gamma}{1 + \rho m / \bar{Y}} \right) \hat{Y}_t.$$

As the flexible price equilibrium is constant in our model,  $\hat{Y}_t$  is the output gap and (A2) is the standard New Keynesian Phillips curve when  $\rho = 0$  and differs in the strength of the wealth effect when  $\rho > 0$ .

### *Algebra behind equations (4) and (5)*

Consider first the Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{Y}_t.$$

Since the output gap and inflation are constant at  $\hat{Y}_S$  and  $\pi_S$ , respectively, and  $\mathbb{E}_t \pi_{t+1} = \lambda \pi_S$  in the short run, we have that

$$\pi_S = \beta \lambda \pi_S + \kappa \hat{Y}_S,$$

which implies

$$(A3) \quad \pi_S = \frac{\kappa}{1 - \beta \lambda} \hat{Y}_S$$

as long as  $\hat{Y}_S$  and  $\pi_S$  are finite.

Consider next the discounted Euler equation

$$\hat{Y}_t = \alpha \mathbb{E}_t \hat{Y}_{t+1} - \zeta \sigma (i_t - \mathbb{E}_t \pi_{t+1} - r_t^n).$$

Again, since the output gap and inflation are constant at  $\hat{Y}_S$  and  $\pi_S$ , respectively, and  $\mathbb{E}_t \pi_{t+1} = \lambda \pi_S$  and  $\mathbb{E}_t \hat{Y}_{t+1} = \lambda \hat{Y}_S$  in the short run, and, in addition, since the natural real rate is  $r_S^n$  in the short run, we have that

$$\hat{Y}_S = \alpha \lambda \hat{Y}_S + \zeta \sigma (\lambda \pi_S + r_S^n).$$

If we now use equation (A3) to eliminate  $\pi_S$  from this equation, then we obtain

$$\hat{Y}_S = \alpha \lambda \hat{Y}_S + \zeta \sigma \left( \lambda \frac{\kappa}{1 - \beta \lambda} \hat{Y}_S + r_S^n \right),$$

which implies that

$$\hat{Y}_S = \frac{\zeta \sigma}{1 - \alpha \lambda - \zeta \sigma \lambda \kappa / (1 - \lambda \beta)} r_S^n$$

as long as  $\hat{Y}_S$  is finite.

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## NOTES

1. Since this ‘puzzle’ is based on a thought experiment rather than hard empirical evidence, not everyone agrees that it is a puzzle. Haberis *et al.* (2014) point out that far future forward guidance may be time-inconsistent and therefore imperfectly credible, which would reduce its effect.
2. Here it is important to remember that consumption will return to its pre-shock steady state in the long run. In other words, temporary monetary policy actions have no long-run effects in this model.
3. Kaplan *et al.* (2016) present similar results in a richer model. Werning (2015) presents conditions under which the response of consumption to interest rates in a heterogeneous agents model will be the same as in a representative agents model.
4. Interestingly, such a formulation of the consumption Euler equation has actually been used in policy calculations by the Central Bank of Norway, to combat the forward guidance puzzle. We thank Oistein Roisland for pointing this out to us.
5. The two main limitations of our model are (1) that assets are in zero net supply (which rules out models with capital), and (2) that heterogeneity is acyclical (which rules out search models with cyclical unemployment).
6. Our model is quite different from one with a fraction of hand-to-mouth agents. Such a model does not give rise to an  $\alpha < 1$ . Hand-to-mouth agents give rise to a  $\zeta < 1$  but also yield an additional income term in the consumption Euler equation (Gali *et al.* 2007).
7. Werning (2015) shows that if the income, liquidity and borrowing constraints of all agents scale proportionately with aggregate income, then the presence of uninsurable income risk and borrowing constraints will not affect the response of output to current or future interest rates. The reason for this is that stronger general equilibrium multiplier effects of the higher income that results from the change in interest rates makes up for the more muted partial equilibrium effects of interest rate change. If, however, high-income agents (with low marginal propensities to spend) get a disproportionate share of the additional income when aggregate income rises, then this general equilibrium feedback is not strong enough to make up for the weaker partial equilibrium effect, and the overall effects of interest rates are muted.
8. Discounting of this kind does not arise in the spender–saver setup (see, for example, Bilbiie 2008).
9. In our model the level of output under flexible prices is constant, so  $\hat{Y}_t$  is both the log-deviation of output from steady state and the output gap.
10. We set  $\alpha = 0.97$  and  $\zeta = 0.75$  as in Figure 1. We compare this to the standard case of  $\alpha = 1$  and  $\zeta = 1$ .
11. As we discuss in the text,  $\alpha = 0.97$  and  $\zeta = 0.75$  matches the baseline calibration of the incomplete markets model in McKay *et al.* (2016). We use the same approach to match the high-risk calibration with the discounted Euler equation model. This yields  $\alpha = 0.94$  and  $\zeta = 0.7$ .
12. It is easy to relax the assumption that idiosyncratic productivity is i.i.d.; however, with i.i.d. productivity it is especially easy to see that the low-productivity households will be constrained if  $m$  is low enough.
13. Suppose that there is an outstanding supply of government debt in amount  $\varepsilon > 0$  financed by a lump sum tax. In this case, bond market clearing requires some household to hold this debt at the prevailing interest rate (which is set by the monetary authority). This household must be indifferent at the margin regarding how much they hold of the debt (i.e. their Euler equation must hold with equality). If instead the Euler equation of all households held with strict inequality, then no household would want to hold the outstanding bonds. There would be excess supply of bonds and excess demand for consumption goods. Since firms must meet demand at posted prices, they would produce more, which would increase income and consumption, and lead marginal utility of consumption to fall. This process would continue until the agents with the lowest marginal utility of consumption—the high-productivity agents—were willing to hold the bonds. Thus for any positive  $\varepsilon$ , the Euler equation of the high-productivity households will hold with equality in equilibrium, and the equilibrium on which we focus is the limit of these equilibria as  $\varepsilon$  goes to zero.
14. Here we abstract from the efficiency loss due to price dispersion, which is zero in a first-order approximation around a zero-inflation steady state.

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