Materials 15 - More on the CEMP vs. CUSUM criteria

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Overview

1 Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \tag{3}$$

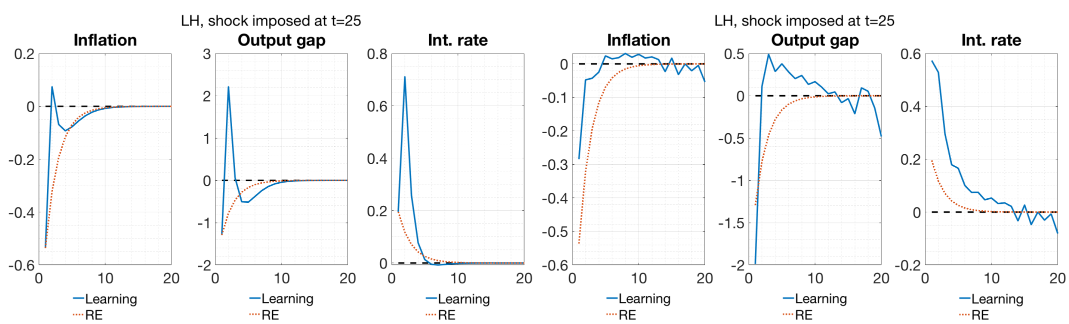
$$\hat{\mathbb{E}}_t z_{t+h} = \bar{z}_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \qquad \text{PLM} \tag{4}$$

$$\bar{z}_t = \bar{z}_{t-1} + k_t^{-1} \underbrace{\big(z_t - (\bar{z}_{t-1} + b s_{t-1})\big)}_{\text{fcst error using (4)}} \tag{5}$$

(Vector learning. For scalar learning, $\bar{z} = \begin{pmatrix} \bar{\pi} & 0 & 0 \end{pmatrix}'$. I'm also not writing the case where the slope b is also learned.)

$$k_t = \begin{cases} k_{t-1} + 1 & \text{for decreasing gain learning} \\ \bar{g}^{-1} & \text{for constant gain learning.} \end{cases} \tag{6}$$

Figure 1: Reference: baseline model



(a) Learning constant only

(b) Learning slope and constant

2 The CEMP vs. the CUSUM criterion

CEMP's criterion

$$\theta_t = |\hat{\mathbb{E}}_{t-1}\pi_t - \mathbb{E}_{t-1}\,\pi_t|/(\mathrm{Var(shocks)}) \tag{7}$$

$$\text{i.e.} \quad \text{PLM- E[ALM], scaled by shocks} \tag{8}$$

For my version of CEMP's criterion, I rewrite the ALM

$$z_t = A_a f_a + A_b f_b + A_s s_t \tag{9}$$

$$\text{as} \quad z_t = F + G s_t \tag{10}$$

$$\Leftrightarrow \quad z_t = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} 1 \\ s_t \end{bmatrix} \tag{11}$$

Then, since the PLM is $z_t = \phi \begin{bmatrix} 1 \\ s_t \end{bmatrix}$, the generalized CEMP criterion becomes

$$\theta_t = \max |\Sigma^{-1}(\phi - \begin{bmatrix} F & G \end{bmatrix})| \tag{12}$$

where Σ is the VC matrix of shocks. As for the CUSUM criterion, what I did in Materials 5 was

$$\omega_t = \omega_{t-1} + \kappa k_{t-1}^{-1}(FE_t^2 - \omega_{t-1}) \tag{13}$$

$$\theta_t = \theta_{t-1} + \kappa k_{t-1}^{-1}(FE_t^2/\omega_t - \theta_{t-1}) \tag{14}$$

where FE_t is the most recent short-run forecast error ($ny \times 1$), and ω_t is the agents' estimate of the forecast error variance ($ny \times ny$). To take into account that these are now matrices, I now write

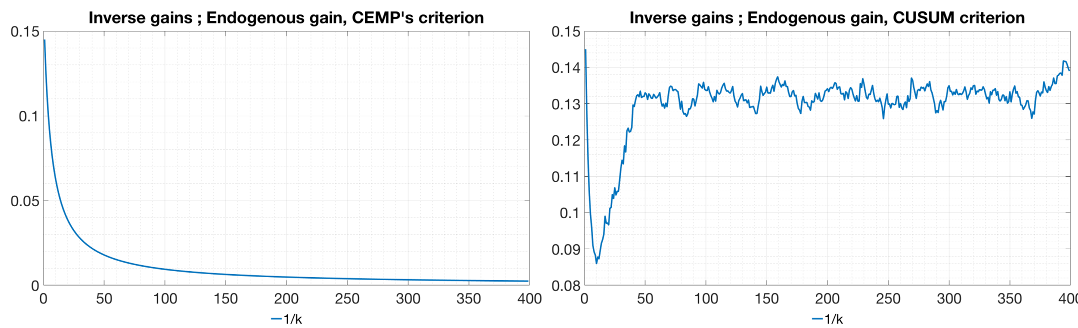
$$\omega_t = \omega_{t-1} + \kappa k_{t-1}^{-1}(FE_t FE_t' - \omega_{t-1}) \tag{15}$$

$$\theta_t = \theta_{t-1} + \kappa k_{t-1}^{-1}\mathrm{mean}((\omega_t^{-1} FE_t FE_t' - \theta_{t-1})) \tag{16}$$

3 Investigating the behavior of CEMP and CUSUM criteria

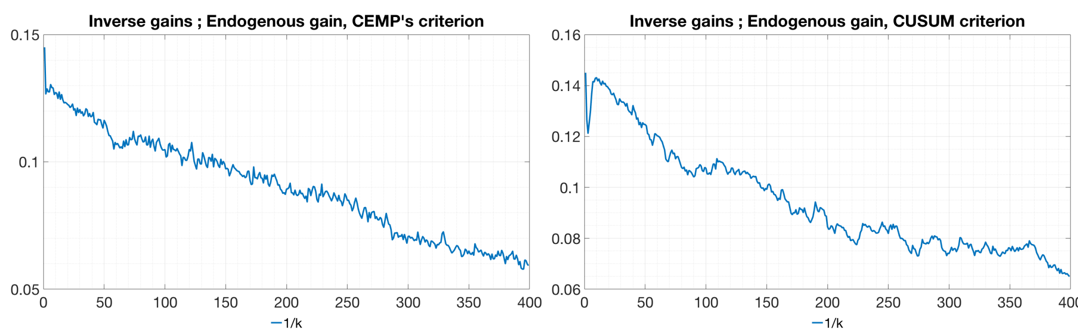
3.1 Anchoring as a function of ψ_π, fixing $\psi_x = 0, \bar{\theta} = 4, \tilde{\theta} = 0.2$

Figure 2: Inverse gains, $\psi_\pi = 1.01$



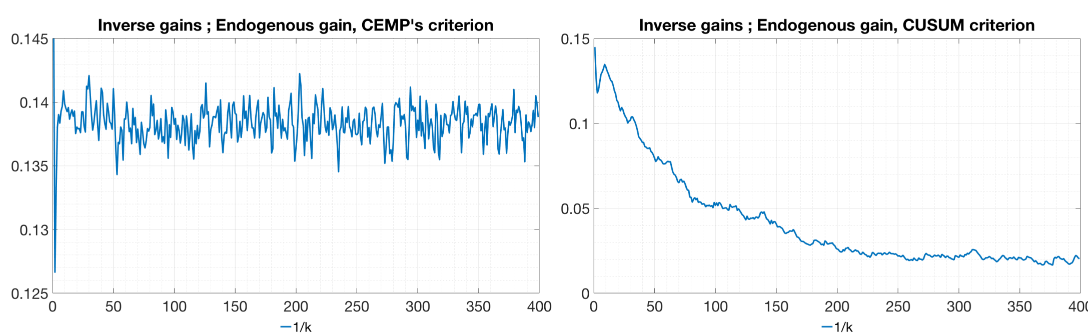
(a) CEMP's criterion (b) CUSUM criterion

Figure 3: Inverse gains, $\psi_\pi = 1.5$



(a) CEMP's criterion (b) CUSUM criterion

Figure 4: Inverse gains, $\psi_\pi = 2$



(a) CEMP's criterion (b) CUSUM criterion

3.2 Why do the two criteria behave opposite ways?

A rough restatement of the two criteria: you get unanchored expectations if:

$$\theta_t^{CEMP} = |(\phi - \begin{bmatrix} F & G \end{bmatrix})| > \bar{\theta} \qquad \text{vs.} \qquad \theta_t^{CUSUM} = f'\omega^{-1}f > \tilde{\theta} \tag{17}$$

where ϕ is the agents' estimated matrix, F, G are the ALM matrices that incorporate long-horizon expectations, f is the one-period ahead forecast error and ω is the estimated forecast error variance matrix. (Note: I'm using Lütkepohl's *Introduction to Multiple Time Series Analysis*, p. 160 to reformulate the CUSUM criterion as a statistic that has a χ^2 distribution.)

Here's the key difference between the two criteria:

- F, G incorporate LH expectations. Thus when ψ_π is large, F, G move a lot, opening up the gap between ϕ and itself, leading to unanchored expectations.
- f doesn't incorporate long-horizon expectations and thus doesn't move as much. In fact, when ψ_π is large, current inflation responds less, and thus one-period ahead forecast errors are *smaller*; you get more anchoring.

4 Analytical expressions for optimal Taylor rule coefficients

Following Woodford's *Interest and Prices*, here's a procedure to obtain optimal Taylor rule coefficients. In Woodford's terminology, this consists of solving for the *optimal noninertial plan* (oni) for the endogenous variables, and then doing coefficient comparison between the Taylor rule and the *oni*.

4.1 In-a-nutshell algorithm for optimal Taylor rule coefficients

1. Postulate conjectures $z_t = \bar{z} + f_j u_t + g_j \hat{r}_t^n, \quad j = \pi, x, i \quad z = (\pi, x, i)'$ for the model consisting of an NKPC and NKIS relation and the AR(1) shocks $u, \hat{r}^n$, where $\hat{r}_t^n = r_t^n - \bar{r}$ (so that the natural rate has a drift, but $\hat{r}^n$ is detrended).

2. Plug the conjectures into the two model equations to derive 2 constraints on the 3 deterministic components $(\bar{\pi}, \bar{x}, \bar{i})$ and 4 constraints on the 6 coefficients on disturbances, $f_j, g_j, \quad j = \pi, x, i$. (Use the known LOMs of shocks to write everything in terms of time t shocks.)

3. Solve 3 sets of optimizations

   (a) $(\bar{\pi}, \bar{x}, \bar{i}) = \arg\min L^{det} \quad s.t.$ the 2 constraints on deterministic components

   (b) $f_j = \arg\min L^{stab,u} \quad s.t.$ 2 out of 4 constraints on shock-coefficients, $j = \pi, x, i$.

   (c) $g_j = \arg\min L^{stab,r} \quad s.t.$ the last 2 out of 4 constraints on shock-coefficients, $j = \pi, x, i$.

4. Compare coefficients of Taylor rule to *oni*-solution of i_t.

4.2 Details

1. The optimal noninertial plan (oni)

   A purely forward-looking set of optimal policies that specifies a LOM for each endogenous variable as the sum of a deterministic component (a long-run average, denoted above by "bar") and a state-contingent component with an optimal response to state t disturbances (the f_j and g_j above). Moreover, the deterministic components are optimal from a timeless perspective (i.e. they minimize L^{det}), and the state-contingent components minimize fluctuations coming from shocks ($L^{stab,u}, L^{stab,r}$).

2. The loss function of the monetary authority and its decomposition into deterministic and shock-contingent parts

Following Woodford, I augment my loss function with some concern for interest rate stabilization:

$$L^{CB} = \mathbb{E}_t \sum_{T=t}^{\infty} \{\pi_T^2 + \lambda_x (x_T - x^*)^2 + \lambda_i (i_T - i^*)\} \tag{18}$$

Woodford decomposes this as $L^{CB} = L^{det} + L^{stab}$ where the former only depends on the "deterministic component of the equilibrium paths of the target variables," while the latter "depends only on the equilibrium responses to unexpected shocks" (p. 509). In particular:

$$L^{det} = \sum_{T=t}^{\infty} \beta^{T-t} \{\mathbb{E}_t\, \pi_T^2 + \lambda_x (\mathbb{E}_t\, x_T - x^*)^2 + \lambda_i (\mathbb{E}_t\, i_T - i^*)^2\} \tag{19}$$

$$L^{stab} = \sum_{T=t}^{\infty} \beta^{T-t} \{\mathrm{var}_t(\pi_T) + \lambda_x \mathrm{var}_t(x_T) + \lambda_i \mathrm{var}_t(i_T)\} \tag{20}$$

Woodford then further decomposes L^{stab} into an element conditional on each shock, but this is only for algebraic convenience.

3. The Taylor rule

   Woodford postulates a Taylor rule of the form

$$i_t = \bar{i} + \phi_\pi (\pi_t - \bar{\pi}) + \phi_x (x_t - \bar{x})/4 \tag{21}$$

(He divides by 4 to make the output gap quarterly.) Substituting in the conjectured and solved for *oni*-solutions for the endogenous variables, one obtains:

$$i_t = \bar{i} + \phi_\pi (f_\pi u_t + g_\pi \hat{r}_t^n) + \phi_x (f_x u_t + g_x \hat{r}_t^n)/4 \tag{22}$$

$$i_t = \bar{i} + f_i u_t + g_i \hat{r}_t^n \tag{23}$$

allowing one to solve for (ϕ_π^*, ϕ_x^*) as the solution to

$$f_i = \phi_\pi f_\pi + \phi_x f_x \tag{24}$$

$$g_i = \phi_\pi g_\pi + \phi_x g_x \tag{25}$$

4.3 Optimal Taylor rule coefficients for RE model, with the simplifying assumption $\rho_u = \rho_r = \rho$

$$\phi_\pi^* = \frac{\kappa\sigma}{\lambda_i(\rho-1)(\beta\rho-1) - \kappa\lambda_i\rho\sigma} \tag{26}$$

$$\phi_x^* = \frac{\lambda_x\sigma(1-\beta\rho)}{\lambda_i(\rho-1)(\beta\rho-1) - \kappa\lambda_x\rho\sigma} \tag{27}$$

which is - fabulously enough - exactly what Woodford obtains.

4.4 Optimal Taylor rule coefficients for the learning model, with the simplifying assumption $\rho_u = \rho_r = \rho$

$$\psi_\pi^* = \frac{\kappa\sigma(\beta(\rho-1)-1)(\alpha\beta(\rho-1)-1)}{\kappa\lambda_i\sigma(\alpha\beta(\rho-1)-1) + \beta\lambda_i(\rho-1)(\alpha\beta(\rho-1)+\beta-1)} \tag{28}$$

$$\psi_x^* = \frac{\lambda_x\sigma(\beta(\rho-1)-1)(\alpha\beta(\rho-1)+\beta-1)}{\kappa\lambda_i\sigma(\alpha\beta(\rho-1)-1) + \beta\lambda_i(\rho-1)(\alpha\beta(\rho-1)+\beta-1)} \tag{29}$$

For a simple calibration like my model of $\lambda_i = 1$ and $\lambda_x = 0$ and $\{\beta \to 0.99, \sigma \to 1, \kappa \to 0.16, \rho \to 0.3, \alpha \to 0.5\}$, I get $\phi_\pi^* = 0.360279$ and $\psi_\pi^* = 11.5371$.