I'm still here – trying to read up on 2 things 27 Feb 2020

1) Solutions to systems of diff eqs that are 1) non-autonom.
   2) have time-varying coeffs. i.e.

$$X_t = A_{t-1} X_{t-1} + M_t b_t$$

2) solutions to nonlinear diff eqs.

Reading Elaydi – Undergrad ... pdf

$k^{th}$ order nonhomogen diff eq:

$$y_{n+k} + p_{1,n} y_{n+k-1} + \dots + p_{k,n} y_n = g_n$$

forcing term

Erm... so far not a lot of useful stuff.

But on p. 115 (Mac), at least Elaydi says that in general, nonlinear diff. eqs cannot be solved explicitly. Some can, by transforming them to linear ones.

Type I: Riccati-style eq

$$x_{n+1} \cdot x_n + p_n x_{n+1} + q_n x_n = 0 \qquad (2.6.1)$$

$$x_{n+1}\cdot x_n + p_n x_{n+1} + q_n x_n = 0$$

Introduce $z_n = \frac{1}{x_n}$

$$\frac{1}{z_{n+1}}\frac{1}{z_n} + p_n \frac{1}{z_{n+1}} + q_n \frac{1}{z_n} = 0$$

$$\Rightarrow \quad 1 + p_n z_n + q_n z_{n+1} = 0 \qquad (2.6.2)$$

Holy! It works!

The nonhomogen Riccati:

$$y_{n+1}\cdot y_n + p_n y_{n+1} + q_n y_n = g_n \qquad (2.6.3)$$

Let $y_n = \frac{z_{n+1}}{z_n} - p_n$

$$\left(\frac{z_{n+2}}{z_{n+1}} - p_{n+1}\right)\left(\frac{z_{n+1}}{z_n} - p_n\right) + p_n\left(\frac{z_{n+2}}{z_{n+1}} - p_{n+1}\right) + q_n\left(\frac{z_{n+1}}{z_n} - p_n\right) = g_n$$

$$\frac{z_{n+2}\cancel{z_{n+1}}}{\cancel{z_{n+1}}z_n} - \cancel{p_n\frac{z_{n+2}}{z_{n+1}}} - p_{n+1}\frac{z_{n+1}}{z_n} + \cancel{p_{n+1}p_n} + \cancel{p_n\frac{z_{n+2}}{z_{n+1}}} - \cancel{p_n p_{n+1}}$$

$$+ q_n\frac{z_{n+1}}{z_n} - q_n p_n = g_n \qquad \Big| \cdot z_n$$

$$\Rightarrow \quad z_{n+2} - p_{n+1} z_{n+1} + q_n z_{n+1} - q_n p_n z_n - g_n z_n = 0$$

$$z_{n+2} + (q_n - p_{n+1}) z_{n+1} - (g_n + q_n p_n) z_n = 0$$ Yeah.

the logistic eq. (Verhulst-Pearl) or logistic map, is an example of this

$$x_{t+1} = x_t(a - bx_t)$$

<u>Type II General Riccati types</u>

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n} \qquad (2.6.6)$$

To solve this, let $c_n x_n + d_n = \frac{y_{n+1}}{y_n}$ (2.6.7)
(won't do)
and then group $a_n, b_n, c_n$ and $d_n$ in composite coeffs.

<u>Type III</u> Homogen. diff eq of type

$$f\left(\frac{x_{n+1}}{x_n}, n\right) = 0$$

Use the transformation $z_n = \frac{x_{n+1}}{x_n}$ to solve.

<u>Type IV</u> $y_{n+k}^{r_1} \cdot y_{n+k-1}^{r_2} \cdot \ldots \cdot y_n^{r_{k+1}} = g_n$ (2.6.13)

Let $z_n = \ln(y_n)$ to solve.

# Nonautonomous matrix system

Elaydi-Undergrad p. 143 Mac

$$x(n+1) = \underset{k\times k}{A(n)}\, x(n) \qquad (3.2.1)$$

$$x(n+1) = \underset{k\times k}{A(n)}\, x(n) + g(n) \qquad (3.2.2)$$

Sol is $\prod_{i=n_0}^{n-1} A(i) \cdot x_0$ (3.2.3)

In fact $\Phi(n) = \prod_{i=n_0}^{n-1} A(i)$ is known as the fundamental matrix

and satisfies $\Phi(n+1) = A(n)\Phi(n)$ (3.2.4)

where if $A(n) = A \ \forall n$, $\Phi(n) = A^n$.

**Thm 3.15.** Any sol of (3.2.2) can be written

$$y_n = \Phi_n \cdot c + y_p(n) \qquad (3.2.10)$$

**Lemma 3.16**

$$y_p(n) = \sum_{r=n_0}^{n-1} \Phi(n, r+1)\, g(r)$$

Rephrase: Thm 3.17 (Variation of Constants Formula)

The unique sol. to the initial value problem p.147 Mac

$$y(n+1) = A(n)\, y(n) + g(n) \qquad y(n_0) = y_0 \quad (3.2.11)$$

is

$$y(n, n_0, y_0) = \Phi(n, n_0)\, y_0 + \sum_{r=n_0}^{n-1} \Phi(n, r+1)\, g(r) \qquad (3.2.12)$$

Doesn't aesthetically look like what Tirelli has, but

<u>is</u> actually!

or, more explicitly,

$$y(n, n_0, y_0) = \left(\prod_{i=n_0}^{n-1} A(i)\right) y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i)\right) g(r) \quad (3.2.13)$$

Now I'm only confused b/c what is then the matrix $\Lambda$?

Later, he says

matrix of eigenvectors

matrix of eigenvals

$$\Phi(n) = A^n P = P\Lambda \qquad (3.3.3)$$

(I think this is the non-time-varying A-case.)

so

$$\Phi(n) = [\lambda_1^n \xi_1, \lambda_2^n \xi_2, \ldots, \lambda_k^n \xi_k] \qquad (3.3.5)$$

eigenvector

So the general sol to (3.2.1) (= 3.3.2) is

$$x(n) = c_1 \lambda_1^n \xi_1 + c_2 \lambda_2^n \xi_2 + \ldots + c_k \lambda_k^n \xi_k$$

<u>Cont. w Ex. 3.21</u>

Intermezzo: I borrowed <u>28 Feb 2020</u>

Goldberg 1958 and Baumol 1975. (3rd ed.)

↳ This one doesn't look too useful b/c it doesn't have time-varying coeffs. But some useful notes from it.

For $Y(t) = M\,Y(t-1) + i$ Goldberg p. 237

if · elements of $M$ are $> 0$

· sum of elements in each column is less than 1

then eigs of $M$ are less than 1 in abs. value.

Baumol isn't great either :(

<u>Sturm criterion on the roots of $f(x)$</u> Baumol, p. 235

For a function $f(x)$ whose roots (eigs) we're trying to find, derive the Sturm functions $f_1(x)$, $f_2(x)$, etc. Each Sturm function is of order previous $-1$.

Then, define $V(x)$ = # changes in sign between $f(x), f_1(x), f_2(x) \dots f_k(x)$.

(Here $k$ is the order of $f(x)$)

Then: **Sturm's theorem**: if $f(x)$ is a polynomial w/ no multiple roots, and $a<b \in \mathbb{R}^2$, then the # of real roots of $f(x)$ which lie between $a$ & $b$ is equal to $V(a) - V(b)$.

How to obtain the Sturm functions?

$f_1(x) = \frac{\partial}{\partial x} f(x)$

$f_2(x) = $ (remainder of long division of $f_1(x)$ into $f(x)$)$(-1)$

$f_3(x)$ & onwards is analogous to $f_2(x)$.

I just don't remember how to do long division :S

Another thing Baumol talks about is:

nonlinear diff eqs

- initial conditions don't matter as much as for linear ones p. 256.
- you can rationalize cycles (vs. in a linear world, cycles will either explode or disappear unless the largest complex eig. is exactly unity)
  → independent of initial conditions
- seems to only treat its dynamics as a phase diagram
- so-called "limit cycles" := cycles of constant amplitude can be produced by a "closed phase line" i.e. one that is a circle, an elliptical, heart shaped, etc ...

<u>Ok so now continue w/ Elaydi - Underground</u>

I'm skipping the example. Instead, I note that the Jordan form, $J = P^{-1}AP$, is used when A is not diagonalizable, i.e. it has repeated roots.

$J = \mathrm{diag}(J_1, \ldots, J_r)$ where $J_i$ is called a **Jordan block**.

(Aka **Jordan canonical form**, p. 159 Mac)

The number of Jordan blocks corresponding to one eigenvalue is called the **geometric multiplicity** of $\lambda$, which also equals the number of lin. indep eigenvectors corresponding to $\lambda$.

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}$$

The **algebraic multiplicity** of $\lambda$ is the number of times it is repeated.

Notes on eigs of Block-diagonal matrices $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$
p. 166 Mac.

I've stopped my reading of Elaydi-Undergrad again. The reason is that there are two separate issues.

1) non-autonomous higher-order diff eqs w/ time-varying coeffs.

2) non-linear systems.

I want to do this step by step. B/c issue 1) also has a deeper issue: for a $2^{nd}$ order eq w/ constant coeffs but non-autonomous, how do I know what to do for the "modes"?

E.g. $y_t = \underset{2\times 2}{A} y_{t-1} + b_t$

Supp. $P^{-1}AP = \Lambda$ and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where $|\lambda_1| < 1$ and $|\lambda_2| > 1$ and $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Or, as in Woodford's (2.6), underlying this could be a $2^{nd}$ order diff eq

$$\beta \varphi_{t+1} - a \varphi_t + \varphi_{t-1} = \kappa x^* + u_t \qquad (2.6)$$

p. 488

where $a = \left(1 + \beta + \frac{\kappa^2}{\lambda}\right)$

We know that $\varphi_t^c = c_1 \lambda_1^t P_1 + c_2 \lambda_2^t P_2$

$$\varphi_t^p = \sum_{r=0}^{t-1} \left[\lambda_1^t \xi_1, \lambda_2^t \xi_2\right] u_r$$

$$= \sum_{r=0}^{t-1} \lambda_1^t \xi_1 u_r + \sum_{r=0}^{t-1} \lambda_2^t \xi_2 u_r$$ since $u$ scalar

Since $|\lambda_2| > 1$ it's clear that this won't do.

Try to iterate (2.6) fwd:

$$\beta \varphi_{t+1} - a\varphi_t + \varphi_{t-1} = kx^* + u_t \qquad (2.6)$$

$$\varphi_{t-1} = -\beta\varphi_{t+1} + a\varphi_t + kx^* + u_t$$

$$= -\beta\left[-\beta\varphi_{t+3} + a\varphi_{t+2} + kx^* + u_{t+2}\right]$$

$$+ a\left[-\beta\varphi_{t+2} + a\varphi_{t+1} + kx^* + u_{t+1}\right]$$

$$+ kx^* + u_t$$

Ignore the $kx^*$ terms and the vanishing $\varphi_{t+k}$.

$$= u_t - \beta u_{t+2} + a u_{t+1}$$

Sorry I can't think of what the pattern is. All I'm seeing is that $\beta$ looks like it was associated w/ the smaller eigenvalue and $a$ w/ the bigger one. So maybe stg like $\varphi_t^p = \sum_{r=0}^{t-1} \lambda_1^t \xi_1 u_r + \sum_{r=t}^{\infty} \lambda_2^{-(r-t)} \xi_2 u_r$

So maybe Woodford understood that $\xi_1$ blocks $u_t$ and $\lambda_2^t$ from materializing?

Recall that eigenvectors for a $2\times2$ are given by

$$\xi_i = \begin{bmatrix} \frac{\lambda_i - a_{22}}{a_{21}} \\ 1 \end{bmatrix}$$

But even if $\frac{\lambda_i - a_{22}}{a_{21}}$ would be zero,

$$\lambda_1^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t = \begin{bmatrix} 0 \\ \lambda_1^t u_t \end{bmatrix}$$

set $c_2 = 0$

$$\begin{bmatrix} \varphi_t \\ \varphi L_t \end{bmatrix} = P z_t$$ ... I don't know b/c I don't know how to translate the sol of a $2\times2$ system back to the $2^{nd}$ order one ... :(

Note: another way of analysing diff. eqs or ODE's is by transforming them to the frequency domain.

- ODE's: Laplace transform
- diff eqs: z-transform (DeMoivre, 1730s → "generating functions" in probability theory)

z-transform of a sequence $x(n)$ is defined as

$$\tilde{X}(n) = Z\left(x(n)\right) \equiv \sum_{j=0}^{\infty} x(j)\, z^{-j} \qquad \text{where } z \in \mathbb{C} \qquad (6.11)$$

For what $z$ does this series converge?

If $\lim_{j \to \infty} \left| \frac{x(j+1)}{x(j)} \right| = R$ (ratio test)

$R$ = "Radius of convergence". Then if $|z| > R$, $Z(x(n))$ converges, if $|z| < R$ it diverges.

A very interesting side note

A convolution of two sequences $x(n)$ & $y(n)$ is defined by

$$\underset{\text{symbol}}{x(n) * y(n)} = \sum_{j=0}^{\infty} x(n-j)\,y(j) = \sum_{j=0}^{\infty} x(n)\,y(n-j)$$

symbol

commutative property of convolution

The Z-transform of this convolution is

$$Z\big(x(n) * y(n)\big) = \tilde{x}(n)\,\tilde{y}(n) \qquad (6.1.12)$$

Is this helpful for me?

↳ on a similar note: check out Thm 6.27. p. 322 Mac.

## Oscillation theory

### Three-term difference equations

p. 328 Mac

$$x(n+1) - x(n) + p(n)\,x(n-k) = 0 \qquad (7.1.1)$$

This is a 3-term diff. eq. of order $k+1$.
$p(n)$ is a sequence.

A solution $x(n)$ is oscillatory if $x(n) - x^*$ fluctuates around 0 (where $x^* = 0$ here, st. st)

### Self-adjoint second order equations

p. 335 Mac

$$p(n)\,x(n+1) + p(n-1)\,x(n-1) = b(n)\,x(n) \qquad (7.2.2)$$

where $b(n) = p(n-1) + p(n) - q(n)$. $\qquad (7.2.3)$

Elaydi is able to find $p(n)$ and $q(n)$ here.

Control theory p. 442 Mac

Controlled system is the nonhomogeneous system:

$$x(n+1) = Ax(n) + u(n)$$

Or

$$x(n+1) = Ax(n) + Bu(n) \qquad (10.1.3)$$

Elaydi first investigates the question of whether this system is controllable: if it is, then a $u(n)$ sequence exists and a finite $N$ such that $x(N) = x_f$ ← desired value. In this case, we say $(A, B)$ is a controllable pair.

The system is controllable if a controllability matrix $W$ has rank$(W) = k$.

Observability: a state-space system is observable if from measuring the output only, you can infer the state.

Duality: a system is controllable if its counterpart is observable. (sic!)

Now I've returned to the "variations of constants" formula Elaydi describes on p. 147 Mac and which I've written earlier.

Maybe there he means that the system

$$y(n+1) = A(n)\,y(n) + g(n) \qquad y(n_0) = y_0 \quad (3.2.11)$$

is the already decoupled system, so that the sol,

$$y(n, n_0, y_0) = \left(\prod_{i=n_0}^{n-1} A(i)\right) y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i)\right) g(r) \quad (3.2.13)$$

is actually the sol of the transformed system so you still need to multiply by the eigenvectors $P$.

↳ maybe Woodford is focusing on the explosive eig for the particular sol b/c the stable one won't matter for the dynamics as $t \to \infty$?

I know I've already done this but let's make sure that (1.12) is really the complementary sol; i.e.

that $\varphi_t =$

$$-\frac{\lambda}{k} x^* (1 - \mu_1^{t+1}) = \mu_1 \varphi_{t-1} - (1-\mu_1)\frac{\lambda}{k} x^*$$

$$= \mu_1 \left(-\frac{\lambda}{k} x^* (1-\mu_1^t)\right) - (1-\mu_1)\frac{\lambda}{k} x^*$$

$$= -\frac{\lambda}{k} x^* \left[\mu_1 (1-\mu_1^t) + 1 - \mu_1\right]$$

$$= -\frac{\lambda}{k} x^* \left[\cancel{\mu_1} - \mu_1^{t+1} + 1 \cancel{-\mu_1}\right]$$

$$= -\frac{\lambda}{k} x^* \left[1 - \mu_1^{t+1}\right] \quad \text{ok!}$$

What's the eigvector of $A$?

$$\beta E \varphi_{t+1} - a \varphi_t + \varphi_{t-1} = 0$$

$$E \varphi_{t+1} - \frac{a}{\beta} \varphi_t + \frac{1}{\beta} \varphi_{t-1} = 0$$

Let $X_t \equiv \varphi_{t-1}$ so that $X_{t+1} = \varphi_t$

$$\varphi_{t+1} - \frac{a}{\beta}\varphi_t + \frac{1}{\beta} X_t = 0$$

$$X_{t+1} - \varphi_t = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{t+1} \\ X_{t+1} \end{bmatrix} + \begin{bmatrix} -\frac{a}{\beta} & \frac{1}{\beta} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_t \\ X_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\beta} & -\frac{1}{\beta} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors are $\xi_1 = \frac{\lambda_1 - a_{22}}{a_{21}} = \frac{\lambda_1 - 0}{1}$

so eigenvectors here are just $\xi_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $\xi_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

which might explain btw why Woodford gets

$y_t = -\frac{\lambda}{k} x^* (1 - \mu_1^{t+1})$.

Ok but for the life of me I can't figure out the $\frac{1}{\beta}$.

"Difference Equations & Inequalities" (pdf)

An initial value problem is a special case of the boundary value problem.

difference_eqs_jensen.....pdf

For the inhomogen eq w/ variable coefficients

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = g(n) \qquad (6.30)$$

The variation of parameters method says that for $x_1$ & $x_2$ two linearly independent sols to the homogenous version of (6.30), the particular solution to (6.30) is

$$x_p(n) = u_1(n)x_1(n) + u_2(n)x_2(n)$$

where
$$u_1(n) = -\sum_{k=0}^{n-1} \frac{g(k)\,x_2(k+1)}{W(k+1)}$$

$$u_2(n) = \sum_{k=0}^{n-1} \frac{g(k)\,x_1(k+1)}{W(k+1)}$$

And I think $x_1$ & $x_2$ are the two eigenvalues and $W(k+1)$ is the Casoratian. WTF is that

## Def. Casoratian

$$W(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_N(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_N(n+1) \\ \vdots & & & \vdots \\ x_1(n+N-1) & & & x_N(n+N-1) \end{vmatrix}$$

The $N \times N$ determinant of a series of vectors (sequences).

i.e. if for vectors $x_1, \ldots, x_N$ the $0^{th}$ ($1^{st}$) element $n_0$ is s.t.

$$W(n_0) \neq 0$$

then the vectors are linearly independent.

I'm not sure what it's doing. Maybe that's where Woodford gets the $\frac{1}{\beta}$ from?

I'm confused about this goddamn Casoratian. In difference_gensys.pdf, we also have this interesting statement: for the $2^{nd}$ order diff eq, if we have two different real roots, $\lambda_1$ & $\lambda_2$,

1 March 2020

then $W(n_0) = W(0) = \begin{vmatrix} \lambda_1^0 & \lambda_2^0 \\ \lambda_1^1 & \lambda_2^1 \end{vmatrix}$

$= \lambda_1^0 \lambda_2^1 - \lambda_1^1 \lambda_2^0 = \lambda_2 - \lambda_1 \neq 0$

(so these seem to be to the powers 1 and 0 - that wasn't obvious from the def. at all!
Ah I get it! The sol $x_1(n) = \lambda_1^n$ so the sequence is $\lambda_1^0, \lambda_1^1, \lambda_2^2$ etc... Ok!)

So for Woodford $\beta E \varphi_{t+1} - a \varphi_t + \varphi_{t-1} = 0$

$\rightarrow$ so the characteristic is $\beta \mu^2 - a\mu + 1 = 0$

$\mu_{1,2} = \frac{a \pm \sqrt{a^2 - 4\beta}}{2}$ so $\mu_2 - \mu_1 = \frac{a + \sqrt{a^2-4\beta} - (a - \sqrt{\cdot})}{2}$

$= \sqrt{a^2 - 4\beta}$ hm. Well that's not zero at least

:D

But the thing is that if $W(0) \neq 0$, then $\forall n > 0$ $W(n) \neq 0$.

By the way, it seems like for systems of diff eqs,

$$Y_{t+1} = A_t Y_t + \text{stuff},$$

$A^t Y_0$ IS a sol, even if $A$ isn't diagonal. But, apparently, for non-diagonal $A$, $A^t$ is difficult to compute. And that's why we use

$$A^t = P \Lambda^t P^{-1} \quad \text{b/c } \Lambda^t \text{ is easy to get.}$$

I've tried to google "stochastic diff. eq" b/c the nonhomogeneous part should

- not be a constant
- not be a simple function of time, like $c \cdot t$
- ideally it should be a series, like $b_t$

→ so I've turned to time series b/c AR-processes usually have this feature.

The problem there is that for a scalar AR-process, you just check whether it's invertible and then you back out the shocks by the Wold decomp, which is

really just the particular solution to the equation...
But a scalar won't do b/c you just take the fwd or bwd-sol depending on whether the AR-coeff $\gtrless 1$.
$\Rightarrow$ So invertibility of VAR!

↳ Hamilton! Ryan Lect. 7 p. 7-8.

# HAMILTON

Note that Hamilton also has a treatment of difference equations. E.g. on p. 22 Mac,

$$\xi_t = F \xi_{t-1} + v_t \qquad [1.2.5]$$

F is called the companion matrix

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and Hamilton shows exactly this:

$$F^j = T \Lambda^j T^{-1} \qquad [1.2.19]$$

eigenvectors

An interesting result in Hamilton is that you can thus for the p-order system [1.2.5] use the eigenvalues & eigenvectors to write down a closed-form expression for the impulse response

$$\frac{\partial y_{t+k}}{\partial w_t} = F(1,1) = c_1 \lambda_1^k + c_2 \lambda_2^k + \ldots c_p \lambda_p^k$$

w/ $c_1 + c_2 + \ldots + c_p = 1$ and given by a particular transformation of eigenvectors t [1.2.25].

Ok finally a hint: Hamilton is talking about IRFs, which he refers to as dynamic multipliers. He says on p 28 (Mac) that if all eigs are real but 1 is > 1, then eventually that one dominates the dynamics. In that case the dynamic multiplier is eventually $\lim_{j \to \infty} \frac{\partial y_{t+j}}{\partial w_t} = c_1 \lambda_1^j$ So Woodford seems to consider that in the limit the bigger eig. dominates the system.

Oh lookee! Hamilton, p. 31 (Mac). For the 2nd order diff eq w/ real eigs $\lambda_1 > \lambda_2$, $\lambda_1 > 1$ whenever

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} > 1$$

or

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 - \phi_1$$

Not a big surprise tho if you think about it.

For Woodford:

$\mu_2 > \mu_1$

$$\beta \varphi_{t+1} - a\varphi_t + \varphi_{t-1} = 0$$

$$\mu^2 - \frac{a}{\beta}\mu + \frac{1}{\beta} = 0$$

$$a = \left(1 + \beta + \frac{k^2}{\lambda}\right)$$

or

$$\varphi_{t+1} = \underbrace{\frac{a}{\beta}}_{\phi_1}\varphi_t - \underbrace{\frac{1}{\beta}}_{\phi_2}\varphi_{t-1}$$

$$\sqrt{\frac{a^2}{\beta^2} - 4\frac{1}{\beta}} > 2 - \frac{a}{\beta} \quad ?$$

Is $\frac{a}{\beta} > 2$ ? $\quad \frac{a}{\beta} = \frac{1 + \beta + \frac{k^2}{\lambda}}{\beta}$

Even if $\frac{k^2}{\lambda} = \min = 0$, then $\frac{a}{\beta} > 2$ b/c $\beta < 1$

Ok! Both $|\phi_1| > 1$ and $|\phi_2| > 1$.

But so Hamilton reasons that if $\lambda_1, \lambda_2 \in \mathbb{R}^2$

$$\sqrt{\phi_1^2 + 4\phi_2} > 2 - \phi_1$$

will hold for any $\phi_1 > 2$. If $\phi_1 < 2$, then if $(\phi_1, \phi_2)$ lies northeast of the line $\phi_2 = 1 - \phi_1$ we're still good

There's also a discussion of the "effect on the present value of $y$ of a transitory increase in $w$".
First consider the vector system and note the effect on the present value of $\xi$ of a transitory increase in $v$:

$$\frac{\partial \overbrace{\sum_{j=0}^{\infty} \beta^j \xi_{t+j}}^{\text{PV of } \xi}}{\partial v_t} = \sum_{j=0}^{\infty} \beta^j F^j = (I_p - \beta F)^{-1} \quad [1.2.46]$$

(a discounted sum of IRFs, like a discounted cumsum)
For this not to explode we need all $|\lambda_i| < \beta^{-1}$.
The effect on $y_t$-PV is just the (1,1)-element of [1.2.46]. Is Woodford using this?

→ exactly! As Hamilton says, a cumsum is a special case of this w/o discounting ($\beta = 1$)

Prop. 1.3. (p. 34 Ham)

If all eigs of the $p \times p$ matrix $F$ of [1.2.3] are less than $\beta^{-1}$ in modulus, the matrix $(I_p - \beta F)^{-1}$ exists and the effect of $w$ on the present value of $y$ is given by its first element, which is

$$\frac{1}{1 - \phi_1 \beta - \phi_2 \beta^2 - \dots - \phi_p \beta^p}$$

So if $F$ is $2 \times 2$ (as for Woodford), and eigs were all $< 1$, then this would be $\dfrac{1}{1 - \phi_1 \beta - \phi_2 \beta^2}$

Hamilton says that this can also be interpreted as the long-run effect on $y$ of a permanent change in $w$. Can it be that for an eig $> 1$, you'd write sthg like

$1 / (1 - \phi_1^{-1} \beta - \phi_2^{-1} \beta^2)$ ?

Cont tomorrow w/ lag operators in Hamilton.

<u>Lag operators</u> <u>2 March 2020</u>

A time series operator transforms one time series into a new time series. Has as an input $\{x_t\}_{t=-\infty}^{\infty}$ and as an output $\{y_t\}_{t=-\infty}^{\infty}$

Transform a series $\{x_t\}$ into $\{y_t\} = \{x_{t-1}\}$ as

$$Lx_t = x_{t-1}$$

Applying it twice is

$$L^2 x_t = x_{t-2}$$

Note that L is commutative, associative & distributive.

From this follows that

$$\begin{aligned} y_t &= (a + bL)Lx_t \\ &= aLx_t + bL^2 x_t \\ &= (aL + bL^2)x_t \end{aligned}$$

That's where the factoring of polynomials comes in.

Such an expression is called a <u>lag polynomial</u>.

Another example for a polynomial in the lag operator

is $(1-\lambda_1 L)(1-\lambda_2 L)x_t = (1-\lambda_1 L-\lambda_2 L+\lambda_1\lambda_2 L^2)x_t$

$$= (1-(\lambda_1+\lambda_2)L+\lambda_1\lambda_2 L^2)x_t$$

$$= x_t-(\lambda_1+\lambda_2)x_{t-1}+\lambda_1\lambda_2 x_{t-2}$$

Thus the first-order difference eq

$$y_t = \phi y_{t-1} + w_t$$

can be written as

$$y_t = \phi L y_t + w_t$$

$$(1-\phi L)y_t = w_t$$

We also saw previously that we can recursively sub backward to write the series as

$$y_t = \phi^{t+1} y_{-1} + w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \dots + \phi^t w_0$$

$$= \phi^{t+1} L^{t+1} y_t + (1+\phi L+\phi L^2+\dots+\phi^t L^t)w_t$$

$$(1-\underbrace{\phi^{t+1}L^{t+1}})y_t = (1+\phi L+\phi L^2+\dots+\phi^t L^t)w_t$$

As $t\to\infty$, this term vanishes. That's nothing new.

But what Hamilton shows is the following:

You can derive the Wold decomp by taking

$$(1-\phi L)y_t = w_t$$

and simply multiplying both sides by $(1+\phi L+\phi L^2+\dots+\phi^t L^t)$

During the calculation, you discover that

$$(1+\phi L+\phi L^2+\dots+\phi^t L^t)(1-\phi L)y_t = (1-\phi^{t+1}L^{t+1})y_t$$

(You can verify this by multiplying out the lag polynomial on the LHS.) Since $\phi^{t+1}L^{t+1} \to 0$ as $t\to\infty$

$$(1+\phi L+\phi L^2+\dots+\phi^t L^t)(1-\phi L) \approx 1 \quad \text{when } t\to\infty$$

$$\Leftrightarrow (1+\phi L+\phi L^2+\dots+\phi^t L^t) \approx \frac{1}{1-\phi L} \quad \text{when } t\to\infty \text{ (sic!)}$$

(when $|\phi|<1$)

Formally, when $|\phi|<1$ and the sequence $\{y\}$ is bounded,

$$\lim_{j\to\infty}(1+\phi L+\phi^2L^2+\dots+\phi^j L^j) = (1-\phi L)^{-1}$$

Which is why $y_t = (1-\phi L)^{-1}w_t$

$$= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \dots$$

## Second-order difference equations

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t \qquad [2.3.1]$$
$$= (\phi_1 L + \phi_2 L^2) y_t + w_t$$

$$\underbrace{(1 - \phi_1 L - \phi_2 L^2)}_{\text{factor this as}} y_t = w_t$$

factor this as $(1-\lambda_1 L)(1-\lambda_2 L) = 1 - \lambda_1 L - \lambda_2 L + \lambda_1\lambda_2 L^2$
$$= 1 - (\lambda_1 + \lambda_2) L + \lambda_1 \lambda_2 L^2$$

$\Rightarrow$ it is clear that $(\lambda_1, \lambda_2)$ need to satisfy:

$$\lambda_1 + \lambda_2 \overset{!}{=} \phi_1$$
$$\lambda_1 \lambda_2 \overset{!}{=} -\phi_2$$

scalar

You can also do this the following way: replace $L$ by $z$:

$$1 - \phi_1 z - \phi_2 z^2 \overset{!}{=} (1 - \lambda_1 z)(1 - \lambda_2 z)$$

What values of $z$ make this hold? If $z = \lambda_1^{-1}$ or $\lambda_2^{-1}$

**Prop. 2.1.**

Factoring the polynomial $(1 - \phi_1 L - \phi_2 L^2)$ as

$1 - \phi_1 L - \phi_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L)$ is the same thing as finding the eigenvalues of $F$, in fact, you obtain the same eigs $\lambda_1, \lambda_2$.

A possible confusion in language:

We just stated the equivalence between solving the characteristic equation

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0 \qquad [2.3.19]$$

for its roots, the eigenvalues $\lambda_{1,2}$ and we said they should lie inside the unit circle for stability, and the factored polynomial

$$1 - \phi_1 z - \phi_2 z^2 = 0 \qquad [2.3.20]$$

whose roots are $z_{1,2} = \lambda_{1,2}^{-1}$, and who should lie outside the unit circle for stability (so that $|\lambda_{1,2}| < 1$).

$\Rightarrow$ Therefore, to avoid confusion about which roots we're talking about, it's better to talk of eigenvalues.

Nice clarification!

Supp. $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $\lambda_1 \neq \lambda_2$.

then $(1-\lambda_1 L)^{-1} = 1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \dots$

$(1-\lambda_2 L)^{-1} = 1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \dots$

So then the 2nd order diff eq, which was

$(1-\lambda_1 L)(1-\lambda_2 L) y_t = w_t$

$\Rightarrow \qquad y_t = (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1} w_t$

From this it is clear that

$$y_t = w_t + \lambda_1 w_{t-1} + \lambda_1^2 w_{t-2} + \dots$$

$$+ w_t + \lambda_2 w_{t-1} + \lambda_2^2 w_{t-2} + \dots$$

$$= \sum_{k=0}^{\infty} \lambda_1^k w_{t-k} + \sum_{k=0}^{\infty} \lambda_2^k w_{t-k} \quad , \text{or,}$$

$$= \sum_{k=0}^{\infty} (\lambda_1^k + \lambda_2^k) w_{t-k}$$

## 2.5. Initial conditions and unbounded sequences

For a diff eq $(1-\phi L) y_t = w_t$,

Sargent's (1987) advice was to solve backward when $|\phi| < 1$, multiplying by $(1-\phi L)^{-1}$ to get

$$y_t = (1-\phi L)^{-1} w_t$$

and solve forward when $|\phi| > 1$, multiplying by $(1-\phi L)^{-1}$ still, but factoring it differently as

$$(1-\phi L)^{-1} = \frac{-\phi^{-1} L^{-1}}{1-\phi^{-1} L^{-1}} \qquad \text{where } L^{-1} w_t = w_{t+1}$$

$$\Leftrightarrow 1-\phi^{-1}L^{-1} = -\phi^{-1}L^{-1}(1-\phi L)$$

$$1-\phi^{-1}L^{-1} = -\phi^{-1}L^{-1} + 1$$ ok so they are the same indeed!

So then $y_t = \dfrac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}} w_t$

$$= (-\phi^{-1}L^{-1})\left[1+\phi^{-1}L^{-1} + \phi^{-2}L^{-2} + \ldots\right] w_t$$

$$= -\left[\phi^{-1}L^{-1} + \phi^{-2}L^{-2} + \phi^{-3}L^{-3} + \ldots\right] w_t$$

$$= -\sum_{s=1}^{\infty} \left(\frac{1}{\phi}\right)^s w_{t+s} \quad \text{or} \quad = -\frac{1}{\phi}\sum_{s=0}^{\infty}\left(\frac{1}{\phi}\right)^s w_{t+1+s}$$

which is EXACTLY what Tirelli has

Now I don't know the following: if we can write a scalar diff eq so simply as a function of the AR-coeffs, why doesn't Woodford do that?

I think I know. Sorry: it's b/c technically that's no longer a scalar process: it's a 2nd order diff eq.

So maybe in a p-order diff eq case, you follow these steps (more or less) for the particular sol.

1. Decide which $\lambda_k$ is the dominant one (largest in modulus
2. Think of the system as a scalar in that and define your lag polynomial accordingly.
3. Factor as you would normally, as fwd- or backward-looking

$$(1-\lambda_k)^{-1} = \begin{cases} (1+\lambda_k L + \lambda_k^2 L^2 + \dots) & \text{(bw.)} \\ -(1+\lambda_k^{-1}L^{-1} + \lambda_k^{-2}L^{-2} + \dots)(\lambda_k^{-1}L^{-1}) & \text{(fw)} \end{cases}$$

So for Woodford's eq we have

$$y_{t+1} = \underbrace{\frac{a}{\beta}}_{\phi_1} y_t \underbrace{- \frac{1}{\beta}}_{\phi_2} y_{t-1} + w_t$$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

$$(1 - \phi_1 L - \phi_2 L^2) y_t = w_t$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L) y_t = w_t$$

$$\underbrace{\underset{<1}{\downarrow}}_{\text{ignore this}} \qquad \underset{>1}{\downarrow}$$

$$y_t = (1 - \lambda_2 L)^{-1} w_t$$

where I use the forward-def of $(1 - \lambda_2 L)^{-1}$ as

$$\frac{-\lambda_2^{-1} L^{-1}}{1 - \lambda_2^{-1} L^{-1}} = -\lambda_2^{-1} L^{-1} \left(1 + \lambda_2^{-1} L^{-1} + \lambda_2^{-2} L^{-2} + \ldots\right) w_t$$

$$= -\lambda_2^{-1} \left(1 + \lambda_2^{-1} L^{-1} + \lambda_2^{-2} L^{-2} + \ldots\right) E_t w_{t+1}$$

$$= -\lambda_2^{-1} \sum_{s=0}^{\infty} (\lambda_2)^{-s} E_t w_{t+1+s}$$

or in Woodford's notation

$$= -\mu_2^{-1} \sum_{j=0}^{\infty} (\mu_2)^{-j} E_t w_{t+1+s}$$

(the $\frac{1}{\beta}$ I still don't know but whatevs...)

$$y^P_{me} = -\mu_2^{-1} \sum_{j=0}^{\infty} (\mu_2)^{-j} \; E_t w_{t+(1+s)}$$

ok so we don't quite agree here

$$y^P_{woodford} = -\beta^{-1} \sum_{j=0}^{\infty} (\mu_2)^{-j-1} \; E_t u_{t+j}$$

(Notes 6, p. 40)

and here

but I think it's ok. I know now the essence of what Woodford's solution is made up out of and I also know that I can write my 2nd or 3rd order system as

$$(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L)\, y_t = w_t$$

choose the dominant eigenvalue $\lambda_k$ and write $(1-\lambda_k L)^{-1}$ as a fwd- or bw-looking polynomial depending on whether $|\lambda_k| > 1$ or not.

So I'm still not a 100% sure I see & know what Woodford is doing with that 2nd order eq. 3 March 2020

But since I might not get to some problem like it unless I can simplify my non-linear equations, I turn to the issue of non-linearity.

In materials 19, take eq (6), which is the root of the non-linearity:

$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) = 0 \qquad (6)$$

$$\Leftrightarrow f_t - f_{t-1} - k_t^{-1}\pi_t + k_t^{-1}f_{t-1} = 0$$

Try $z_t = \frac{1}{k_t^{-1}}$

$$f_t - f_{t-1} - \frac{\pi_t}{z_t} + \frac{f_{t-1}}{z_t} = 0$$

Somehow what seems problematic is that the interaction terms (or $k_t^{-1}$) aren't everywhere, wherefore a guess like $z_t = \frac{1}{k_t^{-1}}$ fails. Try to reduce the system: (ignore the inputs to $g(\cdot)$)

(5) in (3): $\pi_t = \kappa x_t + \beta f_t + u_t$

$$2\kappa x_t + 2\beta f_t + 2u_t + 2\frac{\lambda}{\kappa} x_t - \varphi_t k_t^{-1} - \varphi_t g_\pi = 0$$

$$\Rightarrow 2\kappa\left(1 + \frac{\lambda}{\kappa^2}\right) x_t = \varphi_t k_t^{-1} + \varphi_t g_\pi - 2\beta f_t - 2u_t$$

$$x_t = \frac{1}{2\kappa\left(1+\frac{\lambda}{\kappa^2}\right)}\left(\varphi_t k_t^{-1} + \varphi_t g_\pi - 2\beta f_t - 2u_t\right)$$

in (4):

$$\underbrace{\frac{-\cancel{2}\beta\frac{\lambda}{\kappa}}{\cancel{2}\kappa\left(1+\frac{\lambda}{\kappa^2}\right)}}\left(\varphi_t k_t^{-1} + \varphi_t g_\pi - 2\beta f_t - 2u_t\right) + \varphi_t - \varphi_{t+1} + \varphi_{t+1} k_t^{-1} - \varphi_{t+1} g_f = 0$$

$$-\beta\frac{\frac{\lambda}{\kappa^2}}{1+\frac{\lambda}{\kappa^2}} = -\beta\frac{\lambda}{\kappa^2+\lambda}$$

$$-\beta\frac{\lambda}{\kappa^2+\lambda}\varphi_t k_t^{-1} - \beta\frac{\lambda}{\kappa^2+\lambda}\varphi_t g_\pi + 2\beta^2\frac{\lambda}{\kappa^2+\lambda} f_t + 2\beta\frac{\lambda}{\kappa^2+\lambda} u_t$$
$$+ \varphi_t - \varphi_{t-1} + \varphi_{t+1} k_t^{-1} - \varphi_{t+1} g_f = 0$$

Try $\Delta\varphi_t = \varphi_t - \varphi_{t-1}$

$$2\beta^2\frac{\lambda}{\kappa^2+\lambda} f_t + 2\beta\frac{\lambda}{\kappa^2+\lambda} u_t + \Delta\varphi_t + \varphi_{t+1}(k_t^{-1} - g_f) - \varphi_t\beta\frac{\lambda}{\kappa^2+\lambda}(k_t^{-1} - g_\pi) = 0$$

$$2\beta^2\frac{\lambda}{\kappa^2+\lambda} f_t + 2\beta\frac{\lambda}{\kappa^2+\lambda} u_t + \Delta\varphi_t + \Delta\varphi_{t+1}(k_t^{-1} - g_f)$$
$$- \varphi_t\left(\beta\frac{\lambda}{\kappa^2+\lambda}(k_t^{-1} - g_\pi) + (k_t^{-1} - g_f)\right) = 0$$

The form is the same as for (6).

Go back to (6)

$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) = 0 \qquad (6)$$

$$f_t - f_{t-1} - k_t^{-1}\pi_t + k_t^{-1} f_{t-1} = 0$$

Now this is big! !  4 March 2020

Woodford "OMP_Hbook" – the Opt. Monpol Handbook chapter in the lit-folder of info inflation!

Solves the simple optimal policy problem, & on p. 11 (Mac) it arrives at the 2nd order diff eq.

$$E_t\left[\beta \varphi_{t+1} - \left(1+\beta+\frac{k^2}{\lambda}\right)\varphi_t + \varphi_{t-1}\right] = kx^* + u_t \qquad (1.12)$$

which gives the characteristic eq

$$\beta\mu^2 - \left(1+\beta+\frac{k^2}{\lambda}\right)\mu + 1 = 0 \qquad (1.13)$$

Using Viète, we have

$$\mu_1 + \mu_2 = -\frac{b}{a} = \frac{1+\beta+\frac{k^2}{\lambda}}{\beta} = \frac{1}{\beta} + 1 + \frac{k^2}{\beta\lambda} > 1$$

and

$$\mu_1\mu_2 = \frac{c}{a} = \frac{1}{\beta} > 1 \Rightarrow \underline{\mu_1 = \beta^{-1}\mu_2^{-1}}$$

Hold on to this.

Then we factor (1.12)

$$E_t[\beta(1-\mu_1 L)(1-\mu_2 L)\varphi_{t+1}] = kx^* + u_t$$

(don't know wtf happened to $(1+\beta+\frac{k^2}{a})$ ...)

Multiply by $(1-\mu_2 L)^{-1}\beta^{-1}$

$$(1-\mu_1 L)\varphi_{t+1} = E_t \beta^{-1}(1-\mu_2^{-1}L^{-1})(-\mu_2^{-1}L^{-1})(kx^*+u_t)$$

$$(1-\mu_1 L)\varphi_{t+1} = -\beta^{-1}\mu_2^{-1} E_t (1-\mu_2^{-1}L^{-1})L^{-1}(kx^*+u_t)$$

$$\Leftrightarrow (1-\mu_1 L)L\varphi_{t+1} = -\beta^{-1}\mu_2^{-1} E_t(1-\mu_2^{-1}L^{-1})(kx^*+u_t)$$

$$\varphi_t - \mu_1\varphi_{t-1} = -\mu_1 E_t \sum_{j=0}^{\infty}\mu_2^{-j}(kx^*+u_{t+j})$$

$$\varphi_t = \mu_1\varphi_{t-1} - \mu_1 E_t \sum_{j=0}^{\infty}\mu_2^{-j}(kx^*+u_{t+j})$$

Now I miss a $\beta^j$ here.

Like in Hamilton, the eigs replace the AR-coeffs.

$\beta$ seems to stay in here b/c the CB is discounting future stuff, here it's discounting $\beta^j\varphi_{t+j}$.

→ so that's why $\beta$ stays in the eq at the top

– and is reintroduced in the sum at the bottom.

So Woodford wasn't ignoring the smaller root after all! He was factoring smoothly and taking present value. So that issue is solved!

!

Now the other thing is

- optimal path into NKPC, done
- optimal path into target criterion, done.

- Woodford says in "OMP" that you can back out the optimal paths of $\pi$ & $x$ from the FOCs, and subbing those into the NKPC you get it for the int rate.

- He also says that a policy using the target criterion will do exactly the same thing as the optimal Ramsey policy – the diff. seems to be the communication of the policy. OMP seems not to be feasible to communicate and to implement, whereas a target criterion is.

Now: back to (6)

$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) = 0 \qquad (6)$$

This comes from an "optimal least squares" algorithm along the lines of

$$f_t = f_{t-1} + k_T^{-1}(\pi_t - f_{t-1})$$

But supp. that was something like

$$f_t = f_{t-1}\, k_T^{-1}\left(\frac{\pi_t}{f_{t-1}}\right)$$

then calling things $\ln(x)$

$$\ln(f_t) = \ln(f_{t-1}) + \ln(k_T^{-1}) + \ln \pi_t - \ln(f_{t-1})$$

That's not quite cool but: I tried to see if I can find some help on Mathematica. The issue is that a part is linear, a part isn't, and also they're interacting: an unknown is not just a polynomial in itself.

Ryan meeting 4 March 2020

- Nice insight that TR might be optimal now altho it wasn't before!
- guess a sol! & verify
- prob. not going to work
- some way to specify $g(\cdot)$ such that it's slope is exactly right (e.g. $k_{+}^{-1}$) so that it all works
- less freedom w/ (6) than w/ $g(\cdot)$
  b/c lots of precedents for the learning
- continuous time could solve things or make things easier: there are only a few theorems you need to know (Ito's Lemma and a couple of others) and then you're already good to go. ("Functionals".) Wouldn't spend too much time on it tho – would instead write up the paper

so that you can 1) benefit from the insights you already have 2) find out while writing what is important, where you need to go deeper etc. And then for one hour a day, find a math textbook and as a hobby, develop your continuous time math. (← goal there is to see whether a "stab at it" using those tools has a chance of success or not.)

Ricardo Reis: "Each equation has an economic interpretation, et vice versa."