Work after

12 Feb 2020

COPY OF JOB MARKET TIMELINE

Time in Reverse

Feb/March 2021: accept job

Jan/Feb 2021: Interview Flyouts

Nov 2020: submit applications

Oct 2020: Job talk | (submit) GLMM ← Reading 2

Aug 31 2020: a very complete draft

Summer 2020: very hard work
writing → meaning math + text
i.e. not refining the question

May 2020: First draft ← Reading 1
sthg to circulate in Dept.

Branch's RPE (restricted perceptions eqb)

(an undercategory of which is a misspecification eqb, ME) is this model uncertainty / selection issue:

agents consider the set $\mathcal{F}$

$$\mathcal{F} = \{ x_t = a + b' \hat{z}_{t-1} + \varepsilon_t : \dim(\hat{z}) < \dim(z) \}$$

i.e. each element in $\mathcal{F}$ is a PLM that is misspecified

→ a testing model which omits a variable or a lag.

Agents then pick the best-performing model in an FEV/MSE or related concept-sense, like in Cho & Kasa.

→ and I'm also seeing that the Ball-effect is a negative feedback effect b/c then the int-rate, $E(\pi)$ & (π, x) move in opposite directions

↳ this is what Branch calls a third effect of exogenous disturbances: 1) direct effect on states x_t 2) indirect via expectations 3) policy feedback effect

⇒ joint determination of optimal policy & ME!

Woodford Chapter 8 - A General Linear-Quadratic Framework 13 Feb 2020

1.1. Optimal State-Contingent Paths

The model equations are summarized by

$$\hat{I}\begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = A\begin{bmatrix} Z_t \\ z_t \end{bmatrix} + B i_t + C s_t \qquad (1.1)$$

z_t := vector of "nonpredetermined endog vars" (jumps)

Z_t := vector of "predetermined endog vars" (endog states)

i_t := policy instrument

s_t := vector of exog disturbances

$$\hat{I} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = |\hat{I}| \begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} Z_t \\ z_t \end{bmatrix}$$

$$+ |\hat{I}|\begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} i_t + |\hat{I}|\begin{bmatrix} \tilde{E} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} s_t$$

Objectives of policy = $E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t$ (1.2)

where the period loss L_t = $L_t = \frac{1}{2}(\tau_t - \tau^*)' W (\tau_t - \tau^*)$ (1.3)

τ_t = vector of target variables, $\tau_t = T y_t$

where $y_t = \begin{bmatrix} Z_t \\ z_t \\ i_t \end{bmatrix}$ is the vector of endog. variables.

DEF. A policy rule which is optimal from a timeless perspective in this more general context is a rule such that

1.) jumps z_t can be expressed as a time-invariant function

$$z_t = f_0 + f_Z \bar{Z}_t + f_S \bar{S}_t \quad (1.5)$$

the bars reflect that the fct may involve additional endog. or exog states relative to the model

2) The eqb evolution of $\{y_t\}$ (i.e. all endog. vars) minimizes the loss (1.2) for all $t \geq t_0$, subject to the model (1.1) and s.t. $\tilde{E} z_{t_0} = \bar{e} \equiv \tilde{E}[f_0 + f_Z \bar{Z}_{t_0} + f_S \bar{S}_{t_0}]$ (1.6)

which restricts the initial values of the jumps. ("timelessness")

Indeed this method is the same as the one outlined in Chapter 7, just in a more general writing. The solution procedure is to set up the Lagrangian

$$\mathcal{L}_{t_0} = E_{t_0}\left\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left[L(y_t) + \varphi_{t+1}'\tilde{A}y_t - \beta^{-1}\varphi_t'\tilde{I}y_t\right]\right\} \quad (1.7)$$

where $\tilde{A} := [A \;\; B]$ $\quad \tilde{I} \equiv [\hat{I} \;\; 0]$

$\mathcal{L}$-multipliers on the model eqs.

$$\varphi_{t+1} = \begin{bmatrix} \xi_{t+1} \\ \Xi_t \end{bmatrix} \begin{array}{l} \rightarrow \text{pertains to endog states (predetermined at } t) \\ \rightarrow \text{pertains to jumps (set at } t) \end{array}$$

or vice versa :S

Shocks s_t have been suppressed in $\mathcal{L}$ b/c FOCs are independent of them anyway.

Note that the last term in $\mathcal{L}$, at t_0, gives the initial constraints

$$\varphi_{t_0}'\tilde{I}y_{t_0} = \xi_{t_0}'z_{t_0} + \Xi_{t_0-1}'\tilde{E}z_{t_0}$$

initial values of endog states

constraints (1.6) (the "timelessness")

$\boxed{\begin{array}{l}\text{Time-out: timelessly optimal vs. } t_0\text{-optimal:} \\ \text{time-consistent vs. time-inconsistent optimal commitment}\end{array}}$

Indeed:

A t_0-optimal plan minimizes (1.7) s.t. same constraints EXCEPT it replaces (1.6) ("timelessness") w/

$$\Xi_{t_0-1} = 0 \qquad (1.8)$$

i.e. the "bygones-be-bygones" assumption.

FOCs:

$$\tilde{A}' E_t \varphi_{t+1} + T' W(\tau_t - \tau^*) - \beta^{-1} \tilde{I}' \varphi_t' = 0 \qquad (1.9)$$

$\forall\, t \geq t_0$.

Restatement:

<u>t_0-optimal solution</u>: any bounded processes for $\{y_t\}$ & $\{\varphi_t\}$ that satisfy the FOCs (1.9) s.t. the time-inconsistency constraint (1.8)

<u>timelessly optimal sol</u>: any bounded processes for $\{y_t\}$ & $\{\varphi_t\}$ that satisfy the FOCs (1.9) s.t. the time-consistency constraint (1.6)

Yet again restate the timelessly optimal solution:

1. Eqb satisfies a time-invariant relation of the form (1.5)
2. $\{y_t\}$ $\forall t \geq t_0$ evolve according to the (by assumption bounded) unique solution to the FOCs (1.9) s.t. the model (1.1) w/ the given initial conditions and initial $\mathcal{L}$-multipliers Ξ_{t_0-1}.
3. The initial multipliers are given by a linear rule of the form

$$\Xi_{t_0-1} = g_0 + g_z \bar{z}_{t_0} + g_s \bar{s}_{t_0} \qquad (1.11)$$

w/ g_0, g_z, g_s satisfying

$$\tilde{E} z_t = e_0 + e_z z_t + e_s s_t + e_\Xi [g_0 + g_z \bar{z}_t + g_s \bar{s}_t] \qquad (1.12)$$

$\forall t \geq t_0$

(where what this is saying is that the rule for initial multipliers has to be timelessly optimal (i.e. compatible w/ 1.6) and at the same time conform to the time-invariant rule (1.5))

And note: these are just the optimal plans for $\{y_t\}$. Implementing this plan is another question.

I suggest now to revisit the simple example in Ch. 7 of deriving the to-/timelessly optimal plans, augmented w/ some review of dynamic systems from Alpha Chiang's wonderful book, before turning to the implementation of plans.

<u>Ch. 7 p. 472</u> The Ramsey Problem (shh... it's a secret)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}\left[\pi_t^2 + \lambda(x_t - x^*)^2\right] + \varphi_t\left[\pi_t - \kappa x_t - \beta \pi_{t+1}\right] \right\}$$

FOCs:

$$\pi_t: \quad \pi_t + \varphi_t - \varphi_{t-1} = 0 \qquad (1.7) \qquad \forall t \geq 0$$

$$x_t: \quad \lambda(x_t - x^*) - \kappa \varphi_t = 0 \qquad (1.8)$$

$$\text{and for } t = 0, \quad \varphi_{-1} = 0 \qquad (1.9)$$

This may in fact not be time-inconsistent b/c this is just saying that the model isn't a constraint in $t = -1$. But initial conditions may be, or stg.

Sub out π & x from

$$\pi_t = \kappa x_t + \beta \pi_{t+1} \qquad (1.1)$$

using (1.7) & (1.8)

$$\varphi_{t-1} - \varphi_t = \kappa \left[\frac{\kappa}{\lambda} \varphi_t + x^* \right] + \beta \left(\varphi_t - \varphi_{t+1} \right)$$

$\Leftrightarrow$

$$\varphi_{t-1} - \varphi_t - \beta \varphi_t + \beta \varphi_{t+1} - \frac{\kappa^2}{\lambda} \varphi_t = \kappa x^*$$

$$\Rightarrow \beta \varphi_{t+1} - \left(1 + \beta + \frac{\kappa^2}{\lambda} \right) \varphi_t + \varphi_{t-1} = \kappa x^* \qquad (1.10)$$

Ok, this is a difference eq. of order 2. Let's now revisit Alpha Chiang

General method of solving difference equations

(p. 554 book, pdf. p. 561 mac) Ch. 16 / Ch. 17

$$y_{t+1} + a y_t = c \qquad (16.6)$$

The general solution = $\underset{\substack{\uparrow \\ \text{particular} \\ \text{solution}}}{y_p} + \underset{\substack{\uparrow \\ \text{complementary} \\ \text{function}}}{y_c}$

y_p solves the nonhomogen. eq. (16.6), y_c the homogen. eq.:

$$y_{t+1} + a y_t = 0$$

y_p: represents intertemporal eqb level of y
(st. st. I guess)

y_c: signifies deviations of the time path from the equilibrium y_p. (business cycles I guess)

Focus on y_c first:

Try $y_t = Ab^t$

Then $y_{t+1} = Ab^{t+1}$

and the homogenous eq $y_{t+1} + a y_t = c$

implies $Ab^{t+1} + a Ab^t = 0 \quad | : Ab^t$

$$\rightarrow \quad b + a = 0 \rightarrow b = -a$$

then the complementary fct becomes

$$\underline{y_c = Ab^t = A(-a)^t}.$$

Then, the particular sol. Guess $y_t = k \rightarrow y_{t+1} = k$.

Sub into (16.6) $y_{t+1} + a y_t = c$

$$\rightarrow k + ak = c \Rightarrow \underline{\underline{k = \frac{c}{1+a} = y_p}}$$

Since this y_p is a constant, this is a stationary equilibrium.

Sometimes we want a moving equilibrium (a function of time, like a BGP st. st.), or $a = -1$, so the previous y_p is undefined. Then for the particular sol we can guess $y_t = kt$

$$\rightarrow \quad k(t+1) + akt = c$$

$$\rightarrow \quad k = \frac{c}{t+1+at}$$

which, if $a = -1$ implies $k = c$, so now our moving particular sol is $y_p = c \cdot t$.

Putting the sol together gives the general sol

$$(16.8) \quad y_t = A(-a)^t + \frac{c}{1+a} \qquad \text{if } a \neq -1$$

$$(16.9) \quad y_t = A(-a)^t + c \cdot t \qquad \text{if } a = -1$$

$$= A + c \cdot t$$

We still need to determine A.

To determine A, resort to initial conditions $y_t = y_0$

For (16.8), $t=0$ implies

$$y_0 = A + \frac{c}{1+a} \implies A = y_0 - \frac{c}{1+a}$$

Thus we obtain the "full-fledged" (lol) general sol

$$y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a} \qquad \text{for } a \neq -1 \qquad (16.8')$$

(analogous procedure for the $a=-1$ case)

<u>Dynamic stability of equilibrium</u> p. 558 / 564 Mac

Dynamic stability is the question of whether the general sol. $y_t = \underset{\text{"BC"}}{y_c} + \underset{\text{"st. st"}}{y_p}$ tends to y_p as $t \to \infty$.

$\Leftrightarrow$ In other words, whether $y_c \to 0$ as $t \to \infty$.

Dynamic stability depends then on $A(b)^t$, in fact, it depends fully on b!

$b > 1$	explosive
$b = 1$	constant, non-convergent
$b \in (0,1)$	convergent (stable)
$b = 0$	constant (stable)
$b \in (-1,0)$	damped oscillations
$b = -1$	uniform oscillations
$b < -1$	explosive oscillations

$\Rightarrow$ Nonoscillatory: $b > 0$ (else oscillatory)

Damped: $|b| < 1$ (else explosive)

Note: $|b| = 1$ must be ruled out b/c it's non-convergent.

A: - scales b

- sign of A can <u>mirror</u> b.

<u>Higher-order difference equations</u>

The sol for y_p, the particular integral, proceeds exactly the same way. So let's focus on y_c, the complementary function:

Here's a 2nd order diff. eq:

(17.1) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$

The homogen. version is

(17.3) $y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

Guess the same thing: $y_t = Ab^t$

$$A b^{t+2} + a_1 A b^{t+1} + a_2 A b^t = 0 \quad | : Ab^t$$

$$b^2 + a_1 b + a_2 = 0 \qquad (17.3')$$

(17.3)' is the characteristic equation of (17.3) or (17.1)

Let's repeat: the characteristic equation is the equation you obtain once you plug in the conjecture for the complementary function y_c of $y_t = Ab^t$ and simplify.

It follows that a diff. eq. of order p will have p roots to the characteristic eq, as (17.3') clearly has 2 "characteristic roots"*, b_1 and b_2.

Ok, but which is the solution?

*characteristic roots are also known as eigenvalues.

Quick side note: take Woodford's 2nd order diff eq:

$$\beta \varphi_{t+1} - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\varphi_t + \varphi_{t-1} = \kappa x^* \quad (1.10)$$

Take a sol. of the form $A\mu^t$ and plug

$$\beta A \mu^{t+1} - \left(1+\beta+\frac{\kappa^2}{\lambda}\right) A\mu^t + A\mu^{t-1} = \kappa x^*$$

Divide by $A\mu^{t-1}$ and consider the homogen case:

$$\beta\mu^2 - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\mu + 1 = 0$$

This is how you get eq. (1.11)

Ok, but back to Alpha Chiang and the question of what to do w/ the roots.

$$b_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Case 1. $a_1^2 - 4a_2 > 0 \Rightarrow$ 2 distinct, real roots

Case 2. $a_1^2 - 4a_2 = 0 \Rightarrow$ Repeated real roots

Case 3. $a_1^2 - 4a_2 < 0 \Rightarrow$ Complex roots

In all cases, all roots need to figure in the y_c b/c they are linearly independent.

Case 1. In this case the complementary function becomes

$$y_c = A_1 b_1^t + A_2 b_2^t$$

(a linear combination of two linearly independent parts)

Case 2. Since $b_1 = b_2$ here, $A_1 b_1^t + A_2 b_2^t$ becomes

$$(A_1 + A_2) b^t = A_3 b^t$$

In this case Alpha Chiang still insists that we need an additional linearly independent term, so he introduces $A_4 \cdot t \cdot b^t$, which is linearly $\perp$ b/c you can't obtain it by multiplying $A_3 b^t$ by some constant coefficient c. (unless $c = A_4 \cdot t$ I guess)

$\Rightarrow$ thus in this case the complementary sol becomes

$$y_c = A_3 b^t + A_4 \cdot t \cdot b^t$$

Case 3. $b_{1,2} = h \pm v i$ (conjugate complex roots)

$$h = -\frac{a_1}{2} \qquad v = \frac{\sqrt{4a - a_1^2}}{2} \qquad \text{(ahaa!)}$$

$$\Rightarrow y_c = A_5 (h + vi)^t + A_6 (h - vi)^t \qquad \text{(ahaa!!)}$$

thanks to DeMoivre's theorem (15.28')

$$(h \pm vi)^t = R^t(\cos\theta t \pm i \sin\theta t)$$

where $R = \sqrt{h^2+v^2} = \sqrt{a_2}$ (always >0) (radius)

the absolute value of the conjugate complex roots

$$\cos\theta = \frac{h}{R} = \frac{h}{\sqrt{a_2}} \quad \text{and} \quad \sin\theta = \frac{v}{R} = \frac{v}{\sqrt{a_2}}$$

$$\Rightarrow y_c = A_5(h+vi)^t + A_6(h-vi)^t$$

$$= A_5 R^t(\cos\theta t + i\sin\theta t) + A_6 R^t(\cos\theta t - i\sin\theta t)$$

$$= \underbrace{(A_5+A_6)}R^t\cos\theta t + \underbrace{(A_5-A_6)i}R^t\sin\theta t$$

$$y_c = R^t(A_7\cos\theta t + A_8\sin\theta t) \qquad (17.10)$$

<u>Convergence of the time path (dynamic stability) in the p-order difference equation case</u>

For concreteness, $p=2$ here.

A p-order diff eq converges to st.st if the absolute value of every characteristic root is less than 1.

<u>Schur THM</u>: this holds if all p determinants are positive (the first p subdeterminants or what) (p. 599/589 Mac)

Note that Viète's formulas are also an easy way to relate the roots to each other via their sums & products.

For a quadratic equation, the roots take the form

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Thus $x_1 + x_2 = \dfrac{-b+\sqrt{b^2-4ac} \; -b-\sqrt{b^2-4ac}}{2a} = -\dfrac{b}{a}$ $\square$

and $x_1 \cdot x_2 = \dfrac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$ $\quad$ $(a+b)(a-b) = a^2-b^2$

$$= \frac{b^2-(b^2-4ac)}{4a^2} = \frac{c}{a} \quad \square$$

$$\boxed{\begin{array}{l} \text{Thus, Viète:} \\ x_1 + x_2 = -\dfrac{b}{a} \\ x_1 \cdot x_2 = \dfrac{c}{a} \end{array}}$$

Back to Woodford then. The characteristic eq. (1.11)

$$\beta\mu^2 - \left(1+\beta+\frac{\kappa^2}{\lambda}\right)\mu + 1 = 0$$

has 2 solutions, μ_1 & μ_2.

By Viete,

$$\mu_1 + \mu_2 = -\frac{b}{a} = \frac{1+\beta+\frac{\kappa^2}{\lambda}}{\beta} > 0$$

$$\mu_1 \cdot \mu_2 = \frac{c}{a} = \frac{1}{\beta} > 0$$

So since $\mu_1\mu_2 > 0$, $\text{sign}(\mu_1) = \text{sign}(\mu_2)$.

Since $\mu_1 + \mu_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$.

How does Woodford know that $\mu_1 < 1 < \mu_2$?

(v1) $\mu_1 + \mu_2 = \frac{1}{\beta} + 1 + \frac{\kappa^2}{\lambda\beta} > 1$ (in fact > 2)

(v2) $\mu_1 \cdot \mu_2 = \frac{1}{\beta} > 1$ but close to 1. (around 1.1)

(v2) implies that at least 1 root > 1 b/c if $\mu_1 = \mu_2 \leq 1$,

(v2) ↯ So suppose $\mu_2 > 1$. Why must $\mu_1 < 1$? E.g.

Why can't $\mu_1 = 1$?

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

In Woodford's case this is

$$\mu_{1,2} = \frac{\left(1+\beta+\frac{\kappa^2}{\lambda}\right) \pm \sqrt{\left(1+\beta+\frac{\kappa^2}{\lambda}\right)^2 - 4\beta}}{2\beta}$$

Determinant is $\left(1+\beta+\frac{\kappa^2}{\lambda}\right)^2 - 4\beta \quad =: \Delta$

$= \left(1+\beta+\frac{\kappa^2}{\lambda}\right)^2 - \left(2\sqrt{\beta}\right)^2 \qquad a^2-b^2 = (a+b)(a-b)$

$= \left(1+\beta+\frac{\kappa^2}{\lambda} + 2\sqrt{\beta}\right)\underbrace{\left(1+\beta+\frac{\kappa^2}{\lambda} - 2\sqrt{\beta}\right)}_{\overset{!}{>}0}$

ok so for sure there is a value of λ that ensures that (close enough to 0).

Supp $\beta = 0.99$. Then $\sqrt{\beta} = 0.95 \to 2\sqrt{\beta} = 1.99$

which is miraculously $= 1+\beta = 1.99$!

$1+\beta - 2\sqrt{\beta} \Rightarrow 1 + x^2 - 2x \quad$ where $x = \sqrt{\beta}$

I can write this as $(1-x)^2 = (1-\sqrt{\beta})^2 > 0$ for $\beta < 1$

ok so at least I know I have 2 distinct real roots.

Ok but that means that $\Delta > 0$ so that $\sqrt{\Delta} > 0$

So the 2 roots are

$$\mu_{1,2} = \frac{\left(1 + \beta + \frac{k^2}{\lambda}\right) \pm \sqrt{\Delta}}{2\beta}$$

Is the idea that $\frac{1 + \beta + \frac{k^2}{\lambda}}{2\beta} \approx 1$?

$$= \frac{1}{2\beta} + \frac{1}{2} + \frac{k^2}{2\lambda\beta}$$

$$= \frac{1}{2}\left(\frac{1}{\beta} + 1 + \frac{k^2}{\lambda\beta}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\beta}\left(1 + \frac{k^2}{\lambda}\right)\right)$$

The problem is that it seems to me to be a quantitative question, but it should hold for all param values > 0.

So for $\beta \to 1$ and $k \to 0$ the above becomes

$\frac{1}{2}\left(1 + 1(1 + 0)\right) = \frac{1}{2}(1+1) = 2$ de fine.

And $\frac{k^2}{\lambda} \to 0$ even if $\lambda \to \infty$ b/c $k^2 \to 0$ faster.

I guess. $\Rightarrow$ So $0 < \mu_1 < 1 < \mu_2$. And this is important b/c $\mu_2 > 1$ is explosive, so TVC excludes it.

so one side-question that arises is Blanchard & Kahn conditions, which dictates as many explosive eigenvalues as fwd-looking vars. Why? And what's up w/ eigenvalues vs. roots of characteristic equations here?

→ ow: "characteristic roots are also known as eigenvalues." :>

Alpha Chiang makes this connection explicit when he discusses dynamic stability of "simultaneous equations (ODE or difference eqs)"

→ Need to work thru but the intuition is thus. For an n-dimensional difference eq, transform it into a 1-dimensional matrix diff eq, and then proceed as usual.

So for Woodford, the complementary function is

$\varphi_t = A_1 \mu_1^t + A_2 \mu_2^t$

I guess TVC sets $A_2 = 0$.

So $\varphi_t = A_1 \mu_1^t$ is the complementary sol, for A_1 yet to be determined.

For the particular solution, guess $\varphi_t = b$

$$\beta k - \left(1 + \beta + \frac{k^2}{\lambda}\right) k + k = k x^*$$

$$k = \frac{k x^*}{\cancel{1+\beta} \cancel{-1-\beta} - \frac{k^2}{\lambda}} = \frac{x^*}{-\frac{k}{\lambda}} = -\frac{\lambda x^*}{k}$$

So the general sol is

$$\varphi_t = A_1 \mu_1^t - \frac{\lambda x^*}{k}$$

Now use the initial condition $\varphi_{t-1} \Big|_{t=0} = 0$

(set $t=0$)

$$0 = A_1 \mu_1^0 - \frac{\lambda x^*}{k} \Rightarrow A_1 = \frac{\lambda x^*}{k}$$

So then the general sol is

$$\varphi_t = \frac{\lambda x^*}{k} \mu_1^t - \frac{\lambda x^*}{k} = -\frac{\lambda x^*}{k}\left[1 - \mu_1^t\right] \qquad (1.12)$$

The only thing I don't know is why $t+1$, not t? Typo?

$$\pi_t = \varphi_{t-1} - \varphi_t$$

$$= -\frac{\lambda}{\kappa} x^* \left[1 - \mu_1^{t-1} \right] - \left(-\frac{\lambda}{\kappa} x^* \left[1 - \mu_1^{t} \right] \right)$$

$$= \cancel{-\frac{\lambda x^*}{\kappa}} + \frac{\lambda x^*}{\kappa} \mu_1^{t-1} + \cancel{\frac{\lambda x^*}{\kappa}} - \frac{\lambda x^*}{\kappa} \mu_1^{t}$$

$$\pi_t = \frac{\lambda x^*}{\kappa} \mu_1^{t-1} (1 - \mu_1) \qquad (1.13)$$

↑ except that again Woodford has t here.

Tomorrow: 1) ↑ check on this

2) complete Alpha Chiang's eigenvalue & simultaneous eq. thing.

✓ 3) start reworking draft for midwest macro, using Ryan's abstract - ish ...

2) Alpha Chiang's connection between matrices & difference equations (& differential eqs) **17 Feb 2020**

Consider the 2nd order diff eq: (p. 602)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c \qquad (17.21)$$

"concoct an artificial new variable $x_t \equiv y_{t+1}$"

$$x_{t+1} + a_1 x_t + a_2 y_t = c$$

$$y_{t+1} = x_t$$

$$\Rightarrow \left.\begin{aligned} x_{t+1} + a_1 x_t + a_2 y_t &= c \\ y_{t+1} - x_t &= 0 \end{aligned}\right\} \qquad (17.21')$$

Suppose then that this looks like

$$\begin{aligned} x_{t+1} + 6x_t + 9y_t &= 4 \\ y_{t+1} - x_t &= 0 \end{aligned} \qquad (17.24)$$

1. Seek particular integrals (denote them by $\bar{x}$ & $\bar{y}$ since they're LR eqb values)
2. Seek complementary functions

① First try stationary sols. If that doesn't work, try moving sols of the form $x_t = k_1 t$, $y_t = k_2 t$ etc.

Supp. $x_{t+1} = x_t = \bar{x}$ and $y_{t+1} = y_t = \bar{y}$. Set

$$\left.\begin{aligned} 7\bar{x} + 9\bar{y} &= 4 \\ \bar{y} - \bar{x} &= 0 \end{aligned}\right\} \qquad \bar{x} = \bar{y} = \frac{4}{16} = \frac{1}{4}$$

② Complementary parts:

$x_t = m b^t \qquad y_t = n b^t \qquad$ Set.

$$\begin{aligned} x_{t+1} + 6x_t + 9y_t &= 0 \\ y_{t+1} - x_t &= 0 \end{aligned} \implies \begin{aligned} m b^{t+1} + 6 m b^t + 9 n b^t &= 4 \\ n b^{t+1} - m b^t &= 0 \end{aligned}$$

$| : b^t$

$$\left.\begin{aligned} mb + 6m + 9n &= 0 \\ nb - m &= 0 \end{aligned}\right\} \qquad (17.28)$$

Looking to solve this for (m, n).

$$\left.\begin{aligned} (b+6)m + 9n &= 0 \\ -1m + bn &= 0 \end{aligned}\right\} \text{ coefficient matrix } = \begin{pmatrix} b+6 & 9 \\ -1 & b \end{pmatrix}$$

In order to solve (17.28) w/o $m = n = 0$, need the determinant of the coeff matrix "to vanish". I.e.

$$\begin{vmatrix} b+6 & 9 \\ -1 & b \end{vmatrix} = 0$$

$$\Rightarrow (b+6)b + 9 = 0 \quad \Rightarrow \quad b^2 + 6b + 9 = 0 \qquad (17.29)$$

(17.29) is the characteristic equation, and its roots $b_{1,2}$ are the characteristic roots = eigenvalues of the simultaneous eq. system.

Having b, (17.28) gives us (m, n). (They usually are equations of the form $m_i = k_i n_i$ for each root b_i.)

Let's do the same thing w/ matrix notation

Express

$$\begin{aligned} x_{t+1} + 6x_t + 9y_t &= 4 \qquad (17.24) \\ y_{t+1} - x_t &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad (17.24')$$

$$= I \underbrace{z_{t+1}}_{=u} + K \underbrace{z_t}_{=v} = d$$

where $z_{t+1} = u$, $z_t = v$ in AlphaChiang's notation

$$I z_{t+1} + K z_t = d$$

① Particular integral: try constant sols

$$z_{t+1} = z_t = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \Rightarrow (I + K)\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = d$$

$$\Rightarrow \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = (I + K)^{-1} d \quad \text{if } (I+K)^{-1} \text{ exists.}$$

② Complementary functions

$$z_{t+1} = \begin{bmatrix} m b^{t+1} \\ n b^{t+1} \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} \qquad z_t = \begin{bmatrix} m \\ n \end{bmatrix} b^t$$

$$\Rightarrow I \begin{bmatrix} m \\ n \end{bmatrix} b^{t+1} + K \begin{bmatrix} m \\ n \end{bmatrix} b^t = 0 \qquad \Big| : b^t$$

$$(bI + K)\begin{bmatrix} m \\ n \end{bmatrix} = 0 \qquad (17.28')$$

It is from this that we wanna find $\begin{bmatrix} m \\ n \end{bmatrix}$.

Again we wanna avoid $m = n = 0$ type solutions

so we demand $|bI + K| = 0$

which is again the characteristic eq, w/

eigenvalues / characteristic roots as roots.

Further comment on the characteristic equation [p.610]

1. Characteristic eq. of a matrix
2. of a single diff. eq / ODE
3. of system of diff eqs / ODEs.

What's the connection?

1. If I write an n-order single diff-eq as a 1st-order system of eqs, the characteristic eq. is **the same** (as we saw on the previous ex.)

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

CE: $$b^2 + a_1 b + a_2 = 0$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} a_1 & a_2 \\ -1 & 0 \end{bmatrix}}_{=K} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

CE: $$|bI + K| = \begin{vmatrix} a_1 + b & a_2 \\ -1 & b \end{vmatrix} = (a_1 + b)b + a_2 = 0$$

$$b^2 + a_1 b + a_2 = 0$$

2. Connection #2

There's a link between the characteristic eq. of a diff. eq (or an ODE) to that of a particular matrix, call it D.

$$|bI - D| = 0 \qquad (17.38')$$

makes it clear that $D = -K$. D, or $-K$, has a special meaning. Write $I z_{t+1} + K z_t = d$ as $I z_{t+1} = -K z_t$ (setting $d = 0$ for the reduced version of the system).

$\Rightarrow z_{t+1} = -K z_t$

$\Rightarrow -K$ is the matrix that can transform the system from z_t to z_{t+1}. (i.e. it's the transition matrix.)

$\Rightarrow$ The characteristic equation is the $\det(\text{trans-mat}) = 0$ relation!

NB. If $I = J$ (not identity) so $J z_{t+1} = -K z_t$ then we need to $z_{t+1} = J^{-1}(-K) z_t$ and look at $\det(-J^{-1}K) = 0$. (SVAR!)

Return to Woodford & try to obtain (1.12) again.

Let me write (1.10) as

$$\beta \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = k x^*$$

$$\Rightarrow \beta \mu^2 - \alpha \mu + 1 = 0$$

$$\mu_{1/2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}$$

Ok, so $\alpha = 1 + \beta + \frac{k^2}{\lambda} \in [1+\beta, \infty]$

So for $\alpha = \min(\alpha) = 1+\beta$,

$$\mu_{1/2} = \frac{\alpha \pm \sqrt{(1+\beta)^2 - 4\beta}}{2\beta} = \frac{\alpha \pm \sqrt{\beta^2 + 2\beta + 1 - 4\beta}}{2\beta}$$

$$= \frac{\alpha \pm \sqrt{(\beta-1)^2}}{2\beta} = \frac{1+\beta \pm (\beta - 1)}{2} \begin{cases} \frac{1+\beta+\beta-1}{2} = \beta < 1 \\ \frac{1+\beta-\beta+1}{2} = 1 \end{cases}$$

If $\alpha \to \infty$

$$\mu_{1/2} = \frac{\infty \pm \sqrt{\infty^2}}{2\beta} \begin{cases} \frac{\infty + \infty}{2\beta} = \frac{1}{\beta} > 1 \\ \frac{\infty - \infty}{2\beta} = 0 < 1 \end{cases}$$ Ok!

Fine, Woodford, $0 \leq \mu_1 < 1 \leq \mu_2$

Ok so before turning a 100% to Woodford, let's think about solutions to DSGE models

In Lect 10, Basu looks for the solution of the 2-dimensional non-interacting system where the conj. sols are $z^1_{1t} = \alpha \omega_1^t$ and $z^1_{2t} = \beta \omega_2^t$

Since $\omega_1 > 1$ (and $\omega_2 \in (0,1)$), TVC sets $\alpha \stackrel{!}{=} 0$

(α, β) correspond here to Alpha Chiang's A_1, A_2 etc. which are usually determined by initial conditions. (what A.C. calls "definitize the constants A_i")

For bw-looking vars, TVCs exclude explosive sols.

So maybe the point is that when the economy is a bw-looking eq. system, this is what you do. But if the econ is a fw-looking system, then

$z_t = \omega_1 z_{t+1}$ $\quad \omega_1 < 1$, so $\quad z_t = \frac{1}{\omega_1} z_{t-1}$

$\frac{1}{\omega_1} > 1 \Rightarrow$ transforming the fw-looking one into a bw-looking one requires explosive eigs of the bw-looking system. (Blanchard & Kahn)

Finally back to Woodford, 100%

Alright, so I've solved the char. eq. to obtain μ_1.

So my conj must have been $\varphi_t = A\mu^t$

Or did Woodford conjecture $\varphi_t = A\mu^{t+1}$? So that

$$\beta A \mu^{t+2} - \alpha A \mu^{t+1} + A\mu = 0$$

$$\Leftrightarrow \quad \beta\mu^2 - \alpha\mu + 1 = 0$$

But technically it doesn't matter b/c also the

conjecture $\varphi_t = A\mu^t \rightarrow \beta A\mu^{t+1} - \alpha A\mu^t + A\mu^{t-1} = 0$

$\beta\mu - \alpha + \frac{1}{\mu} = 0$ leads to the same thing

if you multiply by μ,

$$\beta\mu^2 - \alpha\mu + 1 = 0$$

So taking Woodford's conjecture then, turn to the

particular integral:

A stationary sol would involve $\varphi_t = \bar{\varphi} \quad \forall t$

$$\beta\bar{\varphi} - \alpha\bar{\varphi} + \bar{\varphi} = kx^*$$

$$\bar{\varphi}(\beta - \alpha + 1) = kx^*$$

$$\bar{\varphi} = \frac{kx^*}{1+\beta-\alpha} = \frac{kx^*}{-\frac{k^2}{\lambda}} = -\frac{\lambda}{k}x^*$$

So w/o yet having definitized the constant, A,
the general sol is

$$y_t = A\mu_1^{t+1} + \left(-\frac{\lambda}{\kappa}x^*\right)$$

Definitize A using initial cond: when $t=-1$,

$y_{-1} = 0$, so $0 = A - \frac{\lambda}{\kappa}x^*$

$$\Rightarrow \quad A = \frac{\lambda}{\kappa}x^*$$

So the "full-fledged" general sol then is

$$y_t = \frac{\lambda}{\kappa}x^*\mu_1^{t+1} - \frac{\lambda}{\kappa}x^*$$

$$y_t = -\frac{\lambda}{\kappa}x^*\left(1-\mu_1^{t+1}\right) \qquad (1.12)$$

Alright!

And since $\pi_t = y_{t-1} - y_t$

$$= -\frac{\lambda}{\kappa}x^*\left(1-\mu_1^{t}\right) - \left(-\frac{\lambda}{\kappa}x^*\left(1-\mu_1^{t+1}\right)\right)$$

$$= \frac{\lambda}{\kappa}x^*\left[\cancel{-1}+\mu_1^{t}\cancel{+1}-\mu_1^{t+1}\right]$$

$$= \frac{\lambda}{\kappa}x^*\left(\mu_1^{t}\left(1-\mu_1\right)\right)$$

$$\Rightarrow \quad \pi_t = (1-\mu_1)\frac{\lambda}{\kappa}x^*\mu_1^{t} \qquad (1.13)$$

Consider the timeless optimal commitment policy. I think that the complementary solution is the same, but A will be determined differently b/c the initial condition (1.9) $\psi_{-1} = 0$ is replaced by $\pi_t = \bar{\pi} \ \forall t$ so that $\pi_0 = \bar{\pi}$.

(1.13) implies $\pi_t \stackrel{!}{=} \bar{\pi} = (1-\mu_1)\frac{\lambda}{k} x^* \mu_1^t$

let me recall that $A = \left(-\frac{\lambda}{k} x^*\right)$ so

$$\pi_t = (1-\mu_1)(-A)\mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$$(\mu_1 - 1) A \mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$$\mu_1 A \mu_1^t - A \mu_1^t \stackrel{!}{=} \bar{\pi} \quad \forall t$$

$\Rightarrow \quad A \stackrel{!}{=} 0$ and $\bar{\pi} \stackrel{!}{=} 0$ otherwise $\lightning$.

<u>Optimal responses to shocks</u> p. 488

Take the same loss (except in $E_{t_0}(\cdot)$) as before (eq (2.4)). Then clearly you obtain the same diff eq for the multiplier ψ_t (eq (1.10)) except there's an $E(\cdot)$ in front (eq (2.6)) and the cost-push shock u_t shows up on the RHS

$$\beta E_t \varphi_{t+1} - \alpha \varphi_t + \varphi_{t-1} = \kappa x^* + u_t \qquad (2.6)$$

The characteristic eq. is the same, eq. (1.11), which has 2 roots, $0 < \mu_1 < 1 < \mu_2$.

The question is: how the f* does Woodford obtain equation (2.7)?

We said that $\varphi_t = A_1 \mu_1^{t+1} + A_2 \mu_2^{t+1}$

but TVC set $A_2 = 0$.

So then the complementary sol is the same, $\varphi_t = A \mu_1^{t+1}$, except that A may be different.

What about the particular solution, $\bar{\varphi}$?

$$\bar{\varphi}(\beta - \alpha + 1) = \kappa x^* + u_t$$

$\uparrow$ how to treat this?

As long as u_t doesn't have a drift, its LR mean is 0. That would imply the same particular sol as in the non-stochastic case. But if $\bar{\varphi}$ is the same, then A is also the same.

Ok, so since the μ_1^{t+1} terms are gone, we must have gone somehow from (1.12):

$\varphi_t = -\frac{\lambda}{\kappa} x^* (1 - \mu_1^{t+1})$ to (2.7).

$= -\frac{\lambda}{\kappa} x^* (1 - \mu^t \cdot \mu_1)$

$= -\frac{\lambda}{\kappa} x^* (\mu_1 - \mu_1 \mu^t + 1 - \mu_1)$

$= \mu_1 \left[-\frac{\lambda}{\kappa} x^* (1 - \mu^t) \right] - \frac{\lambda}{\kappa} x^* (1 - \mu_1)$

$\varphi_t = \mu_1 \varphi_{t-1} - (1 - \mu_1) \frac{\lambda}{\kappa} x^*$ haha!

The question that remains is how / where do the u_{t+j} terms come from?

I'm wondering if from the perspective of the difference eq u_t isn't an "endog" variable?

Or is Woodford saying that u_t follows its own diff eq which can / is solved?

Wait. Maybe we guess a bit of a different complementary sol: $\varphi_t = A_1 \mu_1^{t+1} + A_2 u_t$

or, better yet: $\varphi_t = A_1 (\mu_1^{t+1} + u_t)$. But the problem even w/ this formulation is that it only explains why a single u_{t+j}-term would show up in y_{t+j}. But as long as the characteristic eq is the same and the particular sol too, I don't see how u would enter w/ a sum.

Bauer Lect 10 gives some hints: s.3 (p. 3 Mac)

$$z_t = a z_{t-1} + m_t \qquad |a| < 1$$

(Stable diff eq. w/ exog fct of time m_t)

Recursive sub

$$z_t = a\left[a z_{t-2} + m_{t-1}\right] + m_t$$

$$= a^2 z_{t-2} + a m_{t-1} + m_t$$

$$= a^k z_{t-k} + \sum_{s=k}^{t} a^{t-s} m_s$$

$\Rightarrow$

$$z_t = a^t z_0 + \sum_{s=1}^{t} a^{t-s} m_s$$

Let's reinterpret this a bit

$$z_t = z_0 \, a^t + \sum_{s=0}^{t} a^{t-s} m_s$$

$$z_t = A \cdot \mu^t + \sum_{s=0}^{t} \mu^{t-s} m_s$$

initial condition

dynamics from initial cond.

exog. function of time

Basu continues w/ unstable diff eqs w/ exog stuff:

$$z_t = a z_{t-1} + m_t \qquad |a| > 1$$

$$\Rightarrow \quad a z_{t-1} = z_t - m_t \quad \Rightarrow \quad z_{t-1} = a^{-1} z_t - a^{-1} m_t$$

$$\Rightarrow \quad z_{t-1} = a^{-1}\left[a^{-1} z_{t+1} - a^{-1} m_{t+1}\right] - a^{-1} m_t$$

$$= a^{-2} z_{t+1} - a^{-2} m_{t+1} - a^{-1} m_t$$

$$= a^{-k} z_{t+k} - \sum_{s=0}^{k} a^{-(s+1)} m_{t+s}$$

Susanto writes the index differently, but this way of writing it at least corresponds to Woodford.

Under uncertainty, just tack on "E_t" to the exog m_{t+s}

$$z_{t-1} = a^{-k} z_{t+k} - \underbrace{\sum_{s=0}^{k} a^{-(s+1)} E_t m_{t+s}}$$

so at least this is exactly where $\sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}$ comes from; and μ_2 is used b/c $\mu_2 > 1$

and β^{-1} will come from $\beta E_t \varphi_{t+1}$.

But this seems to indicate, like Basu writes,

$$z_t = a^t b_0 - \sum_{s=t+1}^{\infty} \left(\frac{1}{a} \right)^{s-t} m_s$$

Woodford, general solution for the multiplier (eq. 2.7?)

$$\text{or } \varphi_t = \underbrace{-\frac{\lambda}{k} x^* \left(1 - \mu_1^{t+1}\right)}_{\text{from (1.12)}} - \underbrace{\beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}}$$

effect of initial cond

effect of shocks for fwd-diff eq.

$$\underbrace{\text{general sol}}$$

$$\mu_1 \varphi_{t-1} - (1-\mu_1) \frac{\lambda}{k} x^*$$

So it seems like when there's a function of time on the RHS, you need to add a sum, fwd or bw-looking depending on the eq. or the roots.

But how do we know which sum (fw- or bw-looking) to add?

Supp: $\beta E_t \varphi_{t+1} = \alpha \varphi_t + u_t$

$$\alpha \varphi_t = \beta E_t \varphi_{t+1} - u_t$$

$$\varphi_t = \frac{\beta}{\alpha} E_t \varphi_{t+1} - \frac{1}{\alpha} u_t$$

$$y_t = \frac{\beta}{\alpha} E_t y_{t+1} - \frac{1}{\alpha} u_t$$

$$= \frac{\beta}{\alpha}\left[\frac{\beta}{\alpha} E_t y_{t+2} - \frac{1}{\alpha} E_t u_{t+1}\right] - \frac{1}{\alpha} u_t$$

$$= \left(\frac{\beta}{\alpha}\right)^2 E_t y_{t+2} - \frac{1}{\alpha} u_t - \frac{\beta}{\alpha^2} E_t u_{t+1} - \frac{\beta^2}{\alpha^3} E_t u_{t+2} + \dots$$

$$= \dots - \frac{1}{\alpha} \sum_{s=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^s u_{t+s}$$

ok, now this raises the issue of what exactly we're doing. In the Basu Lect. Notes, an AR(1) process's root is the AR-coeff. Maybe since we have a 2nd order process, we have a distinction between the AR-coeff & the roots.

⇒ Indeed: for a one-dim AR(1), $y_{t+1} = a y_t + c$ a is the root! [Alpha Chiang, p. 554)

So I'm pulling on Blume p. 585 ff (Mac, 559 ff) to study difference eqs. in general.

Note: Blume calls Basu's method of making the system noninteracting "decoupling" or "change of coordinates" (Mac, 564)

On p. 567 (Mac), Blume then says: the uncoupled system

is $z'_{n+1} = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_k \end{pmatrix} z'_n$

where r_i are the eigenvalues of of $\underset{k \times k}{A}$, when the

orig system was

$$z_{n+1} = A z_n$$

then the decoupled (non-interacting) z's are

$$z'_{1,n} = c_1 r_1^n$$
$$\vdots$$
$$z'_{k,n} = c_k r_k^n$$

So the original z-variables's solutions are

$$z_n = c_1 r_1^n v_1 + c_2 r_2^n v_2 + \ldots + c_k r_k^n v_k$$

where the v_i are eigenvectors.

i.e. a sum of terms of the form

$$c_i \cdot (\text{eigenvalue})^n \cdot \text{eigenvector}$$

where the c_i are constants determined from the

initial conditions.

I'm on econdse.org and find the remark: "DSE-Maths-... pdf" in literatur

"non-autonomous equations"

$$x_t = a x_{t-1} + b_t \qquad (2)$$

- The complementary function will be the same as for the autonomous eq. $x_t = a x_{t-1} + b$
- But the particular sol won't be the same b/c trying a constant value of x won't work, so no x_t and x_{t-1} can be the same. Good point!

There are 2 methods for arriving at the particular sol: backward sol & forward sol

See also "diffeqn.pdf" → this is a really good note!

In particular, this note shows that finding eigenvalues is really equal to factoring polynomials. E.g. p.17

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + b_t$ can be written as

$(1 - \phi_1 L - \phi_2 L^2) y_t = b_t$. Now, this we can write as

$$(1-\lambda_1 L)(1-\lambda_2 L)\, y_t = b_t$$

these are the eigs.

We know that since L behaves like z^{-1}, we can write

$1-\phi_1 L-\phi_2 L^2 = 0$ as $1-\phi_1 z^{-1}-\phi_2 z^{-2} = 0$

and multiply by z^2 to write the "associated polynomial"

$z^2-\phi_1 z-\phi_2 = 0$ which we can solve for

$z_{1,2} = \lambda_{1,2}$

So $y_t = \dfrac{b_t}{(1-\lambda_1 L)(1-\lambda_2 L)\dots(1-\lambda_p L)}$

So if both eigenvalues are less than 1, then

$$y_t = \hat{c}_1 \lambda_1^t + \hat{c}_2 \lambda_2^t + \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} b_t$$ eq. (27)

(p. 14)

In terms of Woodford, what I still don't know is why he solves u forward (why not solve bw using $\mu_1 < 1$?) and where β^{-1} is coming from?

Ok I don't know: the result of today's hard work is that in the case of a non-autonomous diff. eq, the sol. takes the form

$$y_t = y_c + y_p$$

y_c: same as for autonomous eq

y_p: not the same as for nonhom. eq. instead, this will be a forward or backward sum of b_t (the exog. part), w/ the eigenvalues as coeffs.

What I don't know is how to determine which eigenvalue will be coefficient.

- If the system is univariate, there's only one eig λ and the fwd/bwd-nature of the sum depends on whether $|\lambda| \gtrless 1$
- If the system is p-variate, but all eigs $|\lambda_i| < 1$ then $y_p = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)\dots(1-\lambda_p L)}$ (guess... :S
- If all eigs $|\lambda_i| > 1$ then $y_p = \frac{1}{(1-\lambda_1^{-1}L)(1-\lambda_2^{-1}L)\dots(1-\lambda_p^{-1}L)}$

What I don't know is what when one eig $\mu_1 < 1$ the other $|\mu_2| > 1$.

The lecture notes by Tirelli "linear_...Tirelli.pdf" seem to provide the answer. Cont here.

Actually, they *almost* do <u>18 Feb 2020</u>

- but they also don't say what to do w/ "alternating" eigenvalues.

So I'll pause the issue of (2.7) here for a while.

I wanna go on to see that thing about implementation.

Note on nonintertial p. 497: purely fwd-looking policy, i.e. maximizes the same utility as under Ramsey but subject to the condition that endog. vars evolve according to $z_t = \bar{z} + f s_t$ (purely fwd-looking)

i.e. the optimal plan is purely fwd-looking.

It's mean b/c Woodford has actually introduced ωi here before he officially does starting p. 507.

What I said just now isn't quite right: splitting the same loss into $\mathcal{L}^{det}$ & $\mathcal{L}^{stab}$ serves the purpose of making ωi 1) time-consistent 2) still optimal. The point is that whatever path a restriction $z_t = \bar{z} + f s_t$ gives you is not time-consistent (b/c it is purely fwd-looking), but simply imposing the timelessly optimal plan from before is not optimal for $z_t = \bar{z} + f s_t$. So you split the problem in two, allowing the same t_0- or timelessly optimal policy to minimize $\mathcal{L}^{stab}$ (b/c responses to shocks optimally are the same) which is independent of the econ's state at any time t. And shit: the LR averages are exactly NOT those that min $\mathcal{L}^{det}$; instead you just adopt those from a timelessly optimal policy.

p. 516: Woodford also says that if a simple TR is found that is the optimal one, then a TR that involves fcsts does not improve upon the baseline TR. The reason is that already the baseline TR was purely fwd-looking, so you haven't changed that. In fact, as Giannoni & Woodford (2002b) show, for fcsts of horizon k large enough, the eqb becomes indeterminate.

By the way: given a timelessly optimal soln for the Ramsey problem, p. 518 calculates the opt. time paths for π & i (eqs. (4.1) & (4.2))

Then there's the problem of solving for optimal plans for (π, x, i) and using the plan for i as a reaction function. But that's indeterminate.

⇒ So the optimal time path isn't a good policy rule, but the purely forward-looking TR isn't fully optimal. ⇒

Are targeting rules the solution? p. 521

A targeting rule is a commitment of the CB to adjust its instrument such that a target criterion is projected to be fulfilled at all points in time.

One target criterion which is optimal from a timeless perspective is

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0 \qquad (5.1)$$

It is not just optimal, but it is also feasible (b/c an REE w/ it exists), moreover it's unique (not indeterminate)

Δx_t → reflects the history-dependance of optimal commitment (p. 525)

But you know, there are other robustly optimal target criteria. (p. 526-27)

But we still don't know how to implement the targeting rule, all we have is an optimal target criterion.

→ But you can use the target criterion together w/ model equations & LoMs of shocks to derive a reaction function for it.

As a first-pass, you obtain the fundamentals-based reaction function of E&H (5.4)

(You've assumed the private sector's expectations are RE)

The problem though is that for many param values, this again isn't determinate.

So instead of using expectations that "ought to be", plug in expectations of the private sector the CB actually observes: "expectations-based reaction fct" (w/o solving for 'em) (5.5)

This is determinate according to Prop 7.17.

→ so an interest rate rule (5.5) [= reaction function] is equivalent to committing to the π-target (5.1).

If implements it. This (5.5) is a forward-looking TR, however, it does have history-dependence thru

x_{t-1}. w/ $\frac{(\kappa^2+\lambda)-\lambda}{\sigma(\kappa^2+\lambda)} = \frac{\kappa^2}{\sigma(\kappa^2+\lambda)} = \phi_x$ in LR.

Looking at Gaspar et al 2011 & Preston 2008,
it's not quite clear to me what the Ramsey problem
should look like.

Ok: supposing that the indexation parameter $\gamma = 0$,
At least the period loss L_t takes the same form
in Gaspar et al 2011 & Preston 2008:

$$L_t = \pi_t^2 + \lambda x_t^2$$

$$W = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \quad \leftarrow \text{overall loss fct.}$$

And this is also the loss Molnár & Santoro (2016)
consider (except they call λ α for a change)
and they only claim to do discretionary policy (but
actually, they seem to be solving the commitment
problem.

And indeed: Preston takes Woodford's equations and
replaces the eigenvalue $\mu_2 > 1$ w/ $\mu_1 < 1$.
(see my comments in Preston 2008, p. 8 Mac)

Ok, brace yourselves, I'm approaching the "approach":

① $\min_{\pi_t, x_t, i_t, \phi_t} \mathcal{L}$ s.t. ALM, LOM (beliefs)
?

this will give you FOCs that, given eigs, can be solved for the evolution of $\mathcal{L}$-multipliers, which can be subbed back to obtain paths for $\{\pi_t, x_t, i_t, \phi_t\}$

② Sub out $\mathcal{L}$-multipliers from FOCs to obtain a target criterion: π_t in terms of x $(x_t, x_{t-1} \ldots)$.
(Analogue of (5.1))

③ Use the ALMs to figure out what $\hat{E}\pi$ & $\hat{E}x$ are and confront them w/ the criterion to obtain the i-rate rule / reaction function.

In materials17, I have written out 19 Feb 2020

both the big-Δ-ugly Lagrangian for the full system, as well as that of a simplified one (eq 20–25) (or 26–31)

Take FOCs of the simplified one.

π: $2\pi_t + \varphi_{1,t} - \varphi_{5,t}\,\bar{g}^{-1} = 0$ (1)

x: $2\lambda x_t - \varphi_{1,t} k + \varphi_{2,t} = 0$ (2)

i: $\varphi_{2,t}\, b = 0 \rightarrow \varphi_{2,t} = 0 \Rightarrow$ interesting: I'm also finding that IS-curve isn't binding.

fa: $-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0$ (3)

fb: $-\varphi_{2,t}\, b + \varphi_{4,t} = 0$ (4)

$\bar{\pi}_t$: $- E_t\beta\, \varphi_{3,t+1}\frac{1}{1-\alpha\beta} - E_t\beta\, \varphi_{4,t+1}\frac{1}{1-\beta}$

$+ \varphi_{5,t} + E_t\beta\, \varphi_{5,t+1}\left(-1 + \bar{g}^{-1}\right) = 0$ (5)

The good news is that these FOCs look similar to the ones Gaspar et al (2011) get.

In particular, using that $\varphi_{2,t} = 0$, I can combine (1) & (2)

Where I've rewritten the rational expectations:

If you can form $E(x)^{RE}$, then $E_t x_{t+1} = \underset{1\times nx}{g_2} E_t s_{t+1}$

$= g_2 h_x s_t$

$E_t x_{t+k} = g_2 h_x^k s_t \Rightarrow E_t x_T = g_2 h_x^{T-t} s_t$

$= b_2 h_x^{T-t-1} s_t$

$\rightarrow E_t x_{T+1} = g_2 h_x^{T-t+1} s_t = b_2 h_x^{T-t} s_t$

$\kappa \alpha \beta E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_{T+1} = \kappa \alpha \beta E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} b_2 h_x^{T-t} s_t$

$= \kappa \alpha \beta b_2 \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} h_x^{T-t} s_t$

$= \kappa \alpha \beta b_2 (I_3 - \alpha\beta h_x)^{-1} s_t$

$$2\pi_t + \varphi_{1,t} - \varphi_{5,t} \bar{g}^{-1} = 0 \quad (1)$$

$$2\lambda x_t - \varphi_{1,t} k = 0 \quad (2)$$

(2): $\varphi_{1,t} = 2\frac{\lambda}{k} x_t$

$\Rightarrow$ (1): $2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{5,t} \bar{g}^{-1} = 0$

$\Rightarrow \quad x_t = \frac{k}{\lambda}\left(\frac{\varphi_{5,t}}{2} \bar{g}^{-1} - \pi_t\right)$

$\Rightarrow \quad x_t = -\frac{k}{\lambda}\left(\pi_t - \frac{\varphi_{5,t}}{2} \bar{g}^{-1}\right)$ = Gaspar et al's (22)!

Now turn to (3) - (5): (using $\varphi_{2,t} = 0$)

$$-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0 \qquad (3)$$

$$\varphi_{4,t} = 0 \qquad (4)$$

$$- E_t\beta\, \varphi_{3,t+1} \frac{1}{1-\alpha\beta} - E_t\beta\, \varphi_{4,t+1} \frac{1}{1-\beta}$$

$$+ \varphi_{5,t} + E_t\beta\, \varphi_{5,t+1}\left(-1 + \bar{g}^{-1}\right) = 0 \qquad (5)$$

$\rightarrow$ $\varphi_{4,t} = 0$ b/c f_β only shows up in the NKIS, which isn't binding.

(3): $\varphi_{3,t} = (1-\alpha)\beta\, \varphi_{1,t}$

(5) $$- E_t\beta \frac{(1-\alpha)\beta\, \varphi_{1,t+1}}{(1-\alpha)\beta} + \varphi_{5,t} + E_t\beta\, \varphi_{5,t+1}\left(-1+\bar{g}^{-1}\right) = 0$$

But we know $\varphi_{1,t} = 2\frac{\lambda}{\kappa} x_t$ so (5) becomes

$$-E_t\beta\, 2\frac{\lambda}{\kappa} x_{t+1} + \varphi_{5,t} + E_t\beta\, \varphi_{5,t+1}\left(-1+\bar{g}^{-1}\right) = 0 \qquad (5')$$

Now let's solve (5') for $\varphi_{5,t}$ by iterating fwd

$$\varphi_{5,t} = 2\beta\frac{\lambda}{\kappa} E_t x_{t+1} + (1-\bar{g}^{-1})\beta E_t \varphi_{5,t+1}$$

$$= 2\beta\frac{\lambda}{\kappa} E_t x_{t+1} + (1-\bar{g}^{-1})\beta\left[2\beta\frac{\lambda}{\kappa} E_t x_{t+2} + (1-\bar{g}^{-1})\beta E_t \varphi_{5,t+1}\right]$$

$$= 2\beta\frac{\lambda}{\kappa} E_t x_{t+1} + (1-\bar{g}^{-1})\beta\, 2\beta\frac{\lambda}{\kappa} E_t x_{t+2} + \left((1-\bar{g}^{-1})\beta\right)^2 E_t \varphi_{5,t+1}$$

$\varphi_{5,t} = 2\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}$ except for the sign, it's the same as Gaspar et al (2011).

So now plug the sol for $\varphi_{s,t}$ into

$$x_t = -\frac{\kappa}{\lambda}\left(\pi_t - \frac{\varphi_{s,t}}{2}\bar{g}^{-1}\right)$$, expressed for π_t

$$-\frac{\lambda}{\kappa}x_t + \frac{\bar{g}^{-1}}{2}\varphi_{s,t} = \pi_t$$

$$\pi_t = -\frac{\lambda}{\kappa}x_t + \frac{\bar{g}^{-1}}{2}\left[2\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}\right]$$

$$\pi_t = -\frac{\lambda}{\kappa}x_t + \bar{g}^{-1}\beta\frac{\lambda}{\kappa}\sum_{i=0}^{\infty}(1-\bar{g}^{-1})^i E_t x_{t+1+i}$$

$$\pi_t = -\frac{\lambda}{\kappa}\left(x_t - \beta\bar{g}^{-1}E_t\sum_{i=0}^{\infty}(1-\bar{g}^{-1})x_{t+1+i}\right) \qquad (29)$$

Gaspar et al's

Allows you to get the nice interpretation of G et al 2011

When the gain is 0, (29) boils down to

$$\pi_t = -\frac{\lambda}{\kappa}x_t \quad (= \text{discretionary RE solution})$$

This is what Gaspar et al call an "intratemporal tradeoff"

(or actually, Molnár & Santoro do)

If the gain > 0, then there is an additional, inter-

temporal tradeoff between π & x b/c whatever π

you bring about today, it affects π-expectations tomorrow, presenting you w/ an intratemp. tradeoff tomorrow.

Molnár & Santoro derive an interest rate rule!

They say: plug ALM(π) & ALM(x) (eq 18 & 19) into the NKIS

Their ALMs take the following form

$$\pi_t = c_\pi a_t + d_\pi u_t \qquad \text{and} \qquad \hat{E}_t \pi_{t+1} = a_t$$

$$x_t = c_x a_t + d_x u_t \qquad \hat{E}_t x_{t+1} = b_t$$

NKIS: $x_t = \hat{E}_t x_{t+1} - \sigma^{-1}(r_t - \hat{E}_t \pi_{t+1})$

$$\Leftrightarrow \sigma x_t = \sigma \hat{E}_t x_{t+1} - r_t + \hat{E}_t \pi_{t+1}$$

$$r_t = \sigma \hat{E}_t x_{t+1} + \hat{E}_t \pi_{t+1} - \sigma x_t$$

$$= \sigma b_t + a_t - \sigma(c_x a_t + d_x u_t)$$

$$= \underbrace{(1-\sigma c_x)}_{} a_t + \underbrace{\sigma}_{} b_t \underbrace{- \sigma d_x}_{} u_t$$

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u u_t$$

which is what M&S obtain if you plug in c_x & d_x.

→ This is their expectation-based reaction function

$$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u a_t \qquad (20)$$

<u>Ryan meeting</u> <u>19 Feb 2020</u>

Mantra:

Question of the paper: "How does a concern for the anchoring of expectations affect the conduct of monetary policy?"

- 1) Take simple pencil & paper: how does it change (32)? An extra term ....
- 2.) Numerically solve the full-fledged model
- 3.) Estimate the gain function $k(\theta_t)$

"My est. suggests that $E(\cdot)$ are likely to be unanchored when people are surprised by 1%." → CB-ers would be interested to hear that. It'd be a great ending to the abstract.

"There should be a gentlemanly distance between assumptions and results." (Stephanie Schmitt-Grohe)

For me, this means that if I ask "does anchoring matter for policy?" then it's too easy to critique my work by saying "well, the way you set up the model, of course it does".

The mantra is mainly a marketing object: it doesn't assume that there is a concern for anchoring a priori, but given that I find that there is, how anchoring changes the policy problem, I avoid that kind of critique.

The reason Woodford might advocate an approach like Preston (2008) where the RE optimum should be implemented under learning is b/c he might say that "it is not obvious that a policymaker can credibly commit to a non-RE optimum."

Big issue: why is RE-discretion the alternative here? Did I not accidentally write down the discretion problem?

Ryan: discretion is when I choose a single i_t, commitment where I choose a whole plan of $\{i_t\}$.

↳ check discretion vs commitment in a very simple problem!

Work after 20 Feb 2020

So I think that discretion comes about here b/c there aren't really lagged multipliers.

Ryan made a remark that w/ commitment, you get lagged multipliers and it's intuitive that you do — when you choose sequences (today's and tomorrow's values) one multiplier will be date t, the other $t-1$. Which is in fact exactly why Woodford's (1.7) is

$$\pi_t + \varphi_t - \varphi_{t-1} = 0$$

Ok but this goes to the heart of my unease w/ the way I treat f_α & f_β.

Take a simplified version of Woodford's Ramsey problem (p. 472)

commitment

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + \varphi_t \left(\pi_t - \underline{\beta \pi_{t+1}} \right) \right\}$$

vs.

discretion

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \pi_t^2 + \varphi_t \left(\pi_t - \beta f_t \right) \right\}$$

where f_t is taken as given

FOCs:

$$\pi_t + \varphi_t - \varphi_{t-1} = 0 \qquad \text{commitment}$$

vs.

$$\pi_t + \varphi_t = 0 \qquad \text{discretion}$$

What I'm scared of is that in learning, f_t becomes a new variable that is bw-looking. So you can't commit?

peter

anticipated utility shouldn't be applied to the planner

Consider 2 models for CB

1. ant. u. would be like CB lives & rules for 1 period & doesn't care about W of future CB
2. an ∞-lived CB that could commit to forming a policy rule today → would be a diff.
   ↳ constrained at all periods

Can argue that p_{t+1}

CEMP isn't a model of reputation-building
   b/c it → discretionary RE; that would support eqb under commitment.

Expectations, Credibility ... Ireland, 2000, Macro Dynamics
   multiple eqba based on reputation, CB-its want

the one w/ credible → see if it's been cited
→ Cho & Matsui (1995) JEDC
E-forming scheme when all they ass
is that if the CB does the same thing,
sooner or later ppl get that that's how
it's gonna be
→ CB "if I do the right thing now, it will
have a cost, but it'll be worth it"
details of the transition to be worked out
↳ could do an extended learning scheme
which gives the CB incentives to tilt expectations
toward commitment sol.

Lucas & Sargent: PC w/ adaptive E(·)
→ need to be sensitive to that point; my
policy isn't to exploit E(·) → no one is
fooled all of the time in this model!

My work is less about the st. st., more about the transition like Ball (1994).

2 approaches to opt policy

Taylor → see what are coeffs of simple rules
→ I see an interesting tradeoff.

Woodford, Kribe & Schmitt-Grohe
opt. sequencial choice
I see interesting f/p btwn lag & future multipliers → mon. pol. shapes E( )
is similar in certain respects & diff in others
This isn't a model of building credibility for opt. commitment. In future work, one could think of alternative E-formation schemes.

Work after   21 Feb 2020

Reading Cho & Matsui (1995) & Peter Ireland (2000) kind of at the same time.

Cho & Matsui have a nice def of Ramsey policy and a nice sum-up of the problem:

Ramsey policy = the gov's scheme which maximizes its objective function if the gov behaves as a Stackelberg leader.

↳ leader moves first and the followers move sequentially

If the gov's objective fct = social welfare (diff to commit) fct, then the Ramsey policy max social welfare s.t. the private sector responding optimally.

The problem: once PS believes that gov follows Ramsey, the gov may have an incentive to deviate b/c Ramsey is (or may not be) its best-response. (Holy shit!)

→ Ramsey pol may be "time-inconsistent" (Kydland & Prescott 1977)

UNLESS the gov has a credible commitment strategy to the Ramsey policy.

Cho & Matsui say that one way to rationalize the Ramsey policy is by building reputation, meaning by repeating the same action many times in order to convince agents that that is what I'll do.

→ my confusion here is that adaptive learners are the same thing!

No, they are not: they are machines that take an action given a pre-programmed forecasting rule. Inductive forecasters, in Cho & Matsui's terms, however select the forecast optimally. I think this is a way to make them "strategic" in the way the gov is, and in the way they used to be w/ RE.

Peter's 2000 paper says that time-consistent policy features multiple eqlbm (is indeterminate), and he argues p This is due to RE. He says that RE is not good for modelling reputational eqb.

So what he does is:

1) $E(\pi)$ needs to move w/ π

2) $E(\pi) \xrightarrow{!} \pi$, that is

$E(\pi)$ needs to be inductive à la Cho & Matsui.

3) $E(\pi) = f(\pi_{t-1}, \pi_{t-2}, \dots)$ f continuous, diff-ble

(1) & (3) allow the CB to build credibility

This discussion seems to be related to the conclusion that learning can be a model selection criterion.

→ backward-lookingness ⇒ model selection

→ learning: too machine-like (not strategic)

⇒ time-inconsistent

It seems like what induces the multiplicity is that under RE, expectations are a jump variable... of course! stable eigs induce indeterminacy if ∀ is fw-lookin...

Ok so the vicious circle we're having is this:

- Ramsey sol under RE is cool but may be indeterminate (in fact is <u>indeterminate</u> - unless you do Woodford's target criterion thing. That one is determinate, but it the target criterion (and thus the rule) isn't unique)
- You can intro learning to make it bw-looking and make it determinate :)
- But learning takes away the commitment tech. b/c agents are machines (not strategic) and so you get the discretionary sol :(
- Cho & Matsui (1995) try to get the best of both worlds by having <u>inductive expectations</u> which are both <u>bw-looking</u> and <u>optimal</u> (i.e. strategic).

And this is what Peter does too. Holy shit.

Materials 18: very simple problem

$$\mathcal{L} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \pi_t^2 + \varphi_{1,t}(\pi_t - \beta f_t)$$
$$+ \varphi_{2,t}(f_t - f_{t-1} - k^{-1}(\pi_t - f_{t-1})) \}$$

FOC

$$\pi: \quad 2\pi_t + \varphi_{1,t} - \varphi_{2,t} k^{-1} = 0 \qquad (1)$$

$$f_t: \quad -\beta \varphi_{1,t} + \varphi_{2,t} + E_t \varphi_{2,t+1}(-1 + k^{-1}) = 0 \qquad (2)$$

$$(1) \quad \varphi_{1,t} = \varphi_{2,t} k^{-1} - 2\pi_t$$

$$(2) \quad 2\beta\pi_t - \beta \varphi_{2,t} k^{-1} + \varphi_{2,t} + E_t \varphi_{2,t+1}(-1 + k^{-1}) = 0$$

$$2\beta\pi_t + \varphi_{2,t}(1 - \beta k^{-1}) - E_t \varphi_{2,t+1}(1 - k^{-1}) = 0$$

$$\varphi_{2,t} = E_t \varphi_{2,t+1} \frac{1-k^{-1}}{1-\beta k^{-1}} - \frac{2\beta}{1-\beta k^{-1}} \pi_t$$

$$= \frac{1-k^{-1}}{1-\beta k^{-1}} \left(E_t \varphi_{2,t+2} \frac{1-k^{-1}}{1-\beta k^{-1}} - \frac{2\beta}{1-\beta k^{-1}} E_t \pi_{t+1} \right) - \frac{2\beta}{1-\beta k^{-1}} \pi_t$$

$$= \underbrace{\left(\frac{1-k^{-1}}{1-\beta k^{-1}} \right)^n}_{\to 0 \text{ as } n \to \infty} E_t E \varphi_{2,t+2} - \sum_{s=0}^{n-1} \left(\frac{2\beta}{1-\beta k^{-1}} \right)^s E_t \pi_{t+s}$$

$$\cancel{1} - k^{-1} \overset{<}{\gtrless} \cancel{1} - \beta k^{-1} \to \beta k^{-1} \overset{<}{\gtrless} k^{-1} \to \beta \overset{<}{\gtrless} 1$$

Ok, so should turn to optimal policy in my simplified model w/ smooth anchoring.

So return to system (26)-(31) of materials 17.

Need to replace $\bar{g}^{-1}$ in (31) w/ k_t^{-1}

and add an equation like $k_t = f(\theta_t)$

and one for $f(\theta_t)$.

Suppose for simplicity that there's only the cost-push shock, so $\Sigma = \eta\eta'$ is reduced to σ_u^2.

Then the CUSUM-criterion is much simpler:

$$\omega_t = \omega_{t-1} + \tilde{\kappa} k_{t-1}^{-1} (f_t^2 - \omega_{t-1})$$

↳ scalar b/c we're only learning π now

→ ω_t is a scalar

$$\theta_t = \theta_{t-1} + \tilde{\kappa} k_{t-1}^{-1} \left(\frac{f_t^2}{\omega_t} - \theta_{t-1} \right)$$

(was also previously a scalar)

But now, as time goes by, $\omega_t \to \sigma_u^2$
and as agents are learning, the period FE f_t^2
is on avg. going to be $\to \sigma_u^2$ as well.
$\Rightarrow \theta_t \to 1$ from above on average.

So suppose then the gain is

$$\underline{k_t = f(\theta_t)} = c + d \cdot \theta_t$$

actually I'm stuck w/ this

since $\theta_t \to 1$, $c \overset{!}{=} -d$, so we could also write this as $-d + d\theta_t = d(-1+\theta_t)$
and simply set $d=1$ and write $f(\theta_t) = \theta_t - 1$.
My only concern is that occasionally it may happen that $f_t^2 < \omega_t$ so $\theta_t < 1$. So maybe therefore we could write $f(\theta_t) = \underline{\theta_t - \gamma}$ where $\gamma < 1$ but close.

So in this case, the system to solve for optimal policy is eq. (26)-(31) and the gain, so:

Materials 18 sets up the Lagrangian 22 Feb 2020

FOCs: (ignoring E_t's for simplicity)

π_t: $2\pi_t + \varphi_{1,t} - \varphi_{5,t} k_t^{-1} - \varphi_{9,t} = 0$ (1)

x_t: $-\varphi_{1,t}\kappa + 2\lambda x_t = 0 \Rightarrow \varphi_{1,t} = 2\frac{\lambda}{\kappa} x_t$ (2)

Let me not take FOCs wrt i_t & f_β b/c $\varphi_{2,t} = \varphi_{4,t} = 0$ anyway.

$f_{a,t}$: $-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0 \Rightarrow \varphi_{3,t} = (1-\alpha)\beta 2\frac{\lambda}{\kappa} x_t$ (3)

$\bar{\pi}_t$: $-\varphi_{3,t+1}\frac{1}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{9,t} = 0$ (4)

k_t: $-\varphi_{5,t}(-1)k_t^{-2}(\pi_t - (\bar{\pi}_{t-1} + bs_{t-1})) + \varphi_{6,t}$

$-\varphi_{7,t+1}\tilde{\kappa}(-1)k_t^{-2}(f_{t+1}^2/\omega_{t+1} - \theta_t)$

$-\varphi_{8,t+1}\tilde{\kappa}(-1)k_t^{-2}(f_{t+1}^2 - \omega_t) = 0$ (5)

θ_t: $-\varphi_{6,t} d + \varphi_{7,t} - \varphi_{7,t+1}(1-\tilde{\kappa}k_t^{-1}) = 0$ (6)

ω_t: $-\varphi_{7,t}\tilde{\kappa}k_{t-1}^{-1}\frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1-\tilde{\kappa}k_t^{-1}) = 0$ (7)

f_t: $-\varphi_{7,t}\tilde{\kappa}k_{t-1}^{-1}\cdot 2\frac{f_t}{\omega_t} - \varphi_{8,t}\tilde{\kappa}k_{t-1}^{-1} 2f_t + \varphi_{9,t} = 0$ (8)

Let me rewrite in a simplified manner

$$\pi_t: \quad 2\pi_t + 2\frac{\lambda}{\kappa} x_t - \varphi_{5,t} k_t^{-1} - \varphi_{9,t} = 0 \qquad (1)$$

$$\bar{\pi}_t: \quad -\frac{(1-\alpha)}{1-\alpha\beta}\beta\, 2\frac{\lambda}{\kappa} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1 - k_t^{-1}) + \varphi_{9,t} = 0 \qquad (4)$$

$$k_t: \quad -\varphi_{5,t}(-1) k_t^{-2}\left(\pi_t - (\bar{\pi}_{t-1} + b s_{t-1})\right) + \varphi_{6,t}$$
$$- \varphi_{7,t+1}\tilde{\kappa}(-1) k_t^{-2}\left(f_{t+1}^2/\omega_{t+1} - \theta_t\right)$$
$$- \varphi_{8,t+1}\tilde{\kappa}(-1) k_t^{-2}\left(f_{t+1}^2 - \omega_t\right) = 0 \qquad (5)$$

$$\theta_t: \quad -\varphi_{6,t} d + \varphi_{7,t} - \varphi_{7,t+1}(1 - \tilde{\kappa} k_t^{-1}) = 0 \qquad (6)$$

$$\omega_t: \quad -\varphi_{7,t}\tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1 - \tilde{\kappa} k_t^{-1}) = 0 \qquad (7)$$

$$f_t: \quad -\varphi_{7,t}\tilde{\kappa} k_{t-1}^{-1} \cdot 2\frac{f_t}{\omega_t} - \varphi_{8,t}\tilde{\kappa} k_{t-1}^{-1} 2 f_t + \varphi_{9,t} = 0 \qquad (8)$$

The approach will be sthg like this: Solve (1) & (4) for $\varphi_{5,t} \to \varphi_{5,t}$

Solve (6), (7) for $\varphi_{7,t}$ & $\varphi_{8,t}$ as a function of $\varphi_{6,t}$

Do sthg w/ (5) & (8).

$$(1) \quad \varphi_{5,t} = 2\pi_t + 2\frac{\lambda}{\kappa} x_t - \varphi_{5,t} k_t^{-1}$$

$$\Rightarrow (4) \quad -2\frac{(1-\alpha)}{1-\alpha\beta}\frac{\lambda}{\kappa} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1 - k_t^{-1}) + 2\pi_t + 2\frac{\lambda}{\kappa} x_t - \varphi_{5,t} k_t^{-1} = 0$$

$$\Leftrightarrow -2\pi_t - 2\frac{\lambda}{\kappa}\left(x_t - \frac{1-\alpha}{1-\alpha\beta} x_{t+1}\right) + (1 - k_t^{-1})\varphi_{5,t+1} = (1 - k_t^{-1})\varphi_{5,t}$$

$$\Rightarrow \varphi_{5,t} = \frac{1}{1-k_t^{-1}}\left(-2\pi_t - 2\frac{\lambda}{\kappa}\left(x_t - \frac{1-\alpha}{1-\alpha\beta} x_{t+1}\right) + \varphi_{5,t+1}\right.$$

Damn. Neither fwd nor bw-looking. It's b/c the gain is endog.

Parse that: Focus on (6) & (7)

$$-\varphi_{6,t} d + \varphi_{7,t} - \varphi_{7,t+1}(1 - \tilde{\kappa} k_t^{-1}) = 0 \qquad (6)$$

$$\varphi_{7,t} = \varphi_{7,t+1}(\underbrace{1 - \tilde{\kappa} k_t^{-1}}_{<1}) + \varphi_{6,t} \cdot d$$

so solve fwd:

$$\varphi_{7,t} = (1 - \tilde{\kappa} k_t^{-1}) \left[(1 - \tilde{\kappa} k_t^{-1}) \varphi_{7,t+2} + \varphi_{6,t+1} \cdot d \right] + \varphi_{6,t} \cdot d$$

$$= (1 - \tilde{\kappa} k_t^{-1})^j \varphi_{7,t+j} + d \sum_{i=0}^{j-1} (1 - \tilde{\kappa} k_t^{-1})^i \varphi_{6,t+i}$$

$$\underline{\underline{\varphi_{7,t} = d \sum_{i=0}^{\infty} (1 - \tilde{\kappa} k_t^{-1})^i \varphi_{6,t+i}}}$$

Analogously: (7)

$$-\varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1 - \tilde{\kappa} k_t^{-1}) = 0 \qquad (7)$$

$$\varphi_{8,t} = (1 - \tilde{\kappa} k_t^{-1}) \varphi_{8,t+1} - \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2} \varphi_{7,t}$$

$$\Rightarrow \underline{\underline{\varphi_{8,t} = -\tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2} \sum_{i=0}^{\infty} (1 - \tilde{\kappa} k_t^{-1}) \varphi_{7,t+i}}}$$

How to rewrite this in terms of φ_6?

Try stmy different: plug in all multipliers that only show up w/ a t-subscript:

(8) in (1) & (4):

$$2\pi_t + 2\frac{\partial}{\kappa} x_t - \varphi_{5,t} k_t^{-1} - \varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \cdot 2\frac{f_t}{\omega_t} - \varphi_{8,t} \tilde{\kappa} k_{t-1}^{-1} 2 f_t = 0 \quad (1)$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta} 2\frac{\partial}{\kappa} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1 - k_t^{-1})$$

$$+ \varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \cdot 2\frac{f_t}{\omega_t} + \varphi_{8,t} \tilde{\kappa} k_{t-1}^{-1} 2 f_t = 0 \quad (4)$$

(6) in (5):

$$-\varphi_{5,t}(-1) k_t^{-2} \left(\pi_t - (\bar{\pi}_{t-1} + b s_{t-1})\right) + \frac{1}{\lambda}\varphi_{7,t} - \frac{(1 - \tilde{\kappa} k_t^{-1})}{\lambda} \varphi_{7,t+1}$$

$$- \varphi_{7,t+1} \tilde{\kappa} (-1) k_t^{-2} \left(f_{t+1}^2 / \omega_{t+1} - \theta_t \right)$$

$$- \varphi_{8,t+1} \tilde{\kappa} (-1) k_t^{-2} \left(f_{t+1}^2 - \omega_t \right) = 0 \quad (5)$$

And we still have (7)

$$-\varphi_{7,t} \tilde{\kappa} k_{t-1}^{-1} \frac{f_t^2}{\omega_t^2}(-1) + \varphi_{8,t} - \varphi_{8,t+1}(1 - \tilde{\kappa} k_t^{-1}) = 0 \quad (7)$$

Having a hard time w/ this system, I don't really know why.

What if we said for a sec that all stays the way it was w/ an exog dgain, except that $k_t = f(\cdot)$ some function. Then eqs (15)-(17) are gone and (14) is just $\varphi_{6,t}(k_t - f(\cdot))$. Suppose $f(\cdot) = f(\pi_t, \bar{\pi}_{t-1}, k_{t-1})$ and the derivatives of f are f_π, $f_{\bar\pi}$ and f_k.

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FOCs: $\quad + \varphi_{6,t}\left(k_t - f(\pi_t, \bar{\pi}_{t-1}, k_{t-1})\right)$

π_t: $2\pi_t + \varphi_{1,t} - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_\pi = 0$ (1)

x_t: $2\lambda x_t - \varphi_{1,t} k = 0 \qquad \underline{\varphi_{1,t} = 2\frac{\lambda}{k} x_t}$ (2)

$fa_{,t}$: $-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0 \qquad \underline{\varphi_{3,t} = 2(1-\alpha)\beta\frac{\lambda}{k} x_t}$ (3)

$\bar\pi_t$: $-\varphi_{3,t+1}\frac{1}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{6,t} \cdot f_{\bar\pi}$ (4)

k_t: $\varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \underline{\varphi_{6,t+1}} f_k = 0$ (5)

$\varphi_{6,t+1}$ also here the same prob

k_{t-1} in the endog gain is intro-ing the difficulty.

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varphi_{5,t}k_t^{-1} + \varphi_{6,t} f_\pi = 0 \qquad (\mathrm{I}) \quad \text{Gaspar's (22)}$$

$$\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \underline{\varphi_{6,t}} \cdot f_{\bar\pi} = 0 \qquad (\mathrm{II}) \quad \text{Gaspar's (21)}$$

$\varphi_{6,t+1}$

$$\varphi_{5,t}k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1} f_k = 0 \qquad (\mathrm{III})$$

Gaspar solves (II) for & plugs into (I). Maybe I can solve (III) for $\varphi_{6,t}$, plug into (II), solve that for $\varphi_{5,t}$, and plug that into (I).

(III) $\varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1} f_k = 0$

$$\varphi_{6,k} = f_k \varphi_{6,t+1} - k_t^{-2} \varphi_{5,t}$$

↑ have to assume $f_k < 1$

$$\varphi_{6,t} = k_t^{-2} \sum_{i=0}^{\infty} f_k^{\,i} \; \varphi_{5,t+i}$$

a problem...

(II) $\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{6,t} \cdot f_{\bar{\pi}} = 0$

$$\varphi_{5,t} = (1-k_t^{-1})\varphi_{5,t+1} - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} x_{t+1} - f_{\bar{\pi}} \varphi_{6,t}$$

$$\varphi_{5,t} = -\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} \sum_{j=0}^{\infty} (1-k_t^{-1})^j x_{t+1} - f_{\bar{\pi}} \sum_{j=0}^{\infty} (1-k_t^{-1})^j \varphi_{6,t+j}$$

(I): $2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_\pi = 0$

$$\pi_t = -\frac{\lambda}{k} x_t + \frac{k_t^{-1}}{2} \varphi_{5,t} - \frac{f_\pi}{2} \varphi_{6,t}$$

If $\varphi_{6,t} = -\varphi_{5,t} k_t^{-2}$, then (I) is $\pi_t = -\frac{\lambda}{k} x_t + \frac{k_t^{-1}}{2} \varphi_{5,t} + \frac{k_t^{-2} f_\pi}{2} \varphi_{5,t}$

$$\Rightarrow \pi_t = -\frac{\lambda}{k} x_t + k_t^{-1} \frac{(1 + k_t^{-1} f_\pi)}{2} \varphi_{5,t} \qquad (\mathrm{I}')$$

If $f(\cdot)$ didn't depend on k_{t-1} then (III) would be $\varphi_{5,t} k_t^{-2} + \varphi_{6,t} = 0$

$\rightarrow \varphi_{6,t} = -\varphi_{5,t} k_t^{-2}$ Even this isn't right b/c there's a $\varphi_{6,t+1}$ still...

(II) $\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{\kappa} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) - \varphi_{5,t} \cdot k_t^{-2} \cdot f_{\bar{\pi}} = 0$

$$\rightarrow \varphi_{5,t}\left(1 - k_t^{-2} f_{\bar{\pi}}\right) = (1-k_t^{-1})\varphi_{5,t+1} - \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{\kappa} x_{t+1}$$

$$\rightarrow \varphi_{5,t} = \frac{1-k_t^{-1}}{1-k_t^{-2} f_{\bar{\pi}}} \varphi_{5,t+1} - \frac{2(1-\alpha)\beta}{(1-k_t^{-2} f_{\bar{\pi}})(1-\alpha\beta)} \frac{\lambda}{\kappa} x_{t+1} \qquad (\text{II}')$$

$k_t^{-2} < k_t^{-1}$ since $\frac{1}{k_t^2} < \frac{1}{k_t} \Leftrightarrow \frac{1}{k_t} < 1$.

So whether $\frac{1-k_t^{-1}}{1-k_t^{-2} f_{\bar{\pi}}} \gtrless 1$ depends on $f_{\bar{\pi}}$. If $f_{\bar{\pi}} = k_t$

then $\frac{1-k_t^{-1}}{1-k_t^{-2} f_{\bar{\pi}}} = 1$. So I'll assume $f_{\bar{\pi}} < k_t$. A stronger ass. that also works is $f_{\bar{\pi}} < 1$.

Rewrite

$$\varphi_{5,t} = \frac{1-k_t^{-1}}{1-k_t^{-2} f_{\bar{\pi}}} \varphi_{5,t+1} - \frac{2(1-\alpha)\beta}{(1-k_t^{-2} f_{\bar{\pi}})(1-\alpha\beta)} \frac{\lambda}{\kappa} x_{t+1} \qquad (\text{II}')$$

as

$$\varphi_{5,t} = \underbrace{\frac{1-k_t^{-1}}{1-k_t^{-2} f_{\bar{\pi}}}}_{=: a_t} \varphi_{5,t+1} - \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{\kappa}}_{c} \underbrace{\frac{1}{1-k_t^{-2} f_{\bar{\pi}}}}_{b_t} x_{t+1} \qquad (\text{II}'')$$

$$y_{s,t} = a_t\, y_{s,t+1} - c \cdot b_t\, x_{t+1}$$

$$= a_t \left[a_{t+1}\, y_{s,t+2} - c\, b_{t+1}\, x_{t+2} \right] - c\, b_t\, x_{t+1}$$

$$= a_t \cdot a_{t+1}\, y_{s,t+2} - a_t\, c\, b_{t+1}\, x_{t+2} - c\, b_t\, x_{t+1}$$

$$= a_t\, a_{t+1} \left[a_{t+2}\, y_{s,t+3} - c\, b_{t+2}\, x_{t+3} \right] - a_t\, c\, b_{t+1}\, x_{t+2} - c\, b_t\, x_{t+1}$$

$$= a_t\, a_{t+1}\, a_{t+2}\, y_{s,t+3} - a_t\, a_{t+1}\, c\, b_{t+2}\, x_{t+3} - a_t\, c\, b_{t+1}\, x_{t+2} - c\, b_t\, x_{t+1}$$

$$= \prod_{i=0}^{k-1} a_{t+i}\, y_{s,t+k} - c \sum_{j=0}^{k} \prod_{l=0}^{j-1} a_{t+l}\, b_{t+j}\, x_{t+1+j}$$

Omg - even this is ugly...

Can we make a case for $\prod_{i=0}^{\infty} a_{t+i} \rightarrow 0$?

We've assumed $f_{\pi} < 1$ so $a_t < 1$. The problem is that as $t \rightarrow \infty$, $k_t^{-1} \rightarrow 0$ so $a_t \rightarrow 1$. So $\prod_{i=0}^{\infty} a_{t+i}$ is sthg like $0.5 \cdot 0.6 \cdot 0.7 \cdot 0.8 \ldots$ I'm not entirely sure what the limit of this is (1?) but it sure ain't 0.

Or is it? So this is a decreasing series ... hm.

I'm wondering if this shows up when you do this for dgain learning as in Molnár & Santoro. There, it doesn't seem to show up.

Let's see Molnár & Santoro: FOCs

$$2\pi_t - \lambda_{2,t} + \overset{\text{gain}}{\gamma_{t+1}} \lambda_{3t} = 0 \quad (11)$$

$$2\alpha x_t + k\lambda_{2,t} = 0 \quad \rightarrow \lambda_{2,t} = -\frac{2\alpha}{k} x_t \quad (12)$$

$$\beta^2 \lambda_{2t+1} + \beta(1-\gamma_{t+2})\lambda_{3t+1} = \lambda_{3t} \quad (13)$$

(10) & (12)

$$2\pi_t + \frac{2\alpha}{k} x_t + \gamma_{t+1}\lambda_{3,t} = 0 \quad (26)$$

Molnár & Santoro already use this to conclude that once beliefs have converged, you're in discretion.

Let's look at my analogue of this equation:

$$2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{5,t} k_t^{-1} + \varphi_{6,t} f_\pi = 0 \quad (\pm) \quad \overset{\text{Gaspar's}}{(22)}$$

As $t \to \infty$ $k_t^{-1} \to 0$ and $f_\pi \to 0$ as the gain changes less and less in response to π. I guess. In that case I also get the same result that in the LR, you have discretion.

Let's now iterate their (13) find

$$-\frac{2\alpha\beta^2}{k}x_{t+1} + \beta(1-\gamma_{t+2})\lambda_{3t+1} = \lambda_{3t} \qquad (13)$$

$$\Rightarrow \lambda_{3t} = \beta(1-\gamma_{t+2})\left[\beta(1-\gamma_{t+3})\lambda_{3t+2} - \frac{2\alpha\beta^2}{k}x_{t+2}\right] - \frac{2\alpha\beta^2}{k}x_{t+1}$$

$$= \beta^2\underbrace{(1-\gamma_{t+2})(1-\gamma_{t+3})}_{\text{they get the same feature as I do}}\lambda_{3t+2} - \beta\underline{(1-\gamma_{t+2})}\frac{2\alpha\beta^2}{k}x_{t+2} - \frac{2\alpha\beta^2}{k}x_{t+1}$$

Hmm. In the appendix, where they let $\gamma_t = \frac{1}{t}$, somehow only very neat things show up ($\frac{1}{t+1}$ only) although here I get $\left(1-\frac{1}{t+2}\right)\left(1-\frac{1}{t+3}\right)\dots$

$$= \left(\frac{t+2-1}{t+2}\right)\left(\frac{t+3-1}{t+3}\right)\dots = \left(\frac{t+1}{\cancel{t+2}}\right)\left(\frac{\cancel{t+2}}{t+3}\right)\dots \text{ hoppé!}$$

so we're left w/ $\frac{t+1}{t+\infty}$ so that way λ_{3t+2} does disappear over time. As for the x_t elements, there we have (ignoring the constants $-\frac{2\alpha\beta^2}{k}$)

$$x_{t+1} + \left(1-\frac{1}{t+2}\right)x_{t+2} + \left(1-\frac{1}{t+2}\right)\left(1-\frac{1}{t+3}\right)x_{t+3} + \dots$$

$$= x_{t+1} + \left(\frac{t+2-1}{t+2}\right)x_{t+2} + \left(\frac{t+2-1}{t+2}\right)\left(\frac{t+3-1}{t+3}\right)x_{t+3} + \dots$$

$$= x_{t+1} + \left(\frac{t+1}{t+2}\right)x_{t+2} + \left(\frac{t+1}{t+3}\right)x_{t+3} + \left(\frac{t+1}{t+4}\right)x_{t+4} + \dots$$

$$= (t+1)\left[\frac{1}{t+1} x_{t+1} + \frac{1}{t+2} x_{t+2} + \frac{1}{t+3} x_{t+3} + \dots\right]$$

$$= (t+1) \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i}$$

so then $\lambda_{3,t} = -\frac{2\alpha\beta^2}{k}(t+1) \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i}$

which for their (16) gives

$$2\pi_t + \frac{2\alpha}{k} x_t - \frac{2\alpha\beta^2}{k} \sum_{i=1}^{\infty} \frac{1}{t+i} x_{t+i} = 0 \qquad (16)$$

Nice!

The issue is that in my case, even the   24 Feb 2020

dgain won't be this simple b/c if $k_{t+1} = k_t + 1$

then the series $k_{t+2} k_{t+1} k_t = (k_{t+1} + 1)(k_t + 1) k_t$

$= (k_t + 1 + 1)(k_t + 1) k_t$ oh that's not too bad

$= (k_t + 2)(k_t + 1) k_t$

$= \prod_{i=0}^{\tau} (k_t + i)$ ok, that's not too bad. The problem is

that the endog. gain is gonna break this.

Let's go back to (II').

$$\varphi_{5,t} = \underbrace{\frac{1-k_t^{-1}}{1-k_t^{-2}f_{\bar{\pi}}}}_{=:a_t} \varphi_{5,t+1} - \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}}_{c} \underbrace{\frac{1}{1-k_t^{-2}f_{\bar{\pi}}}}_{b_t} x_{t+1} \quad (\text{II}'')$$

$$a_t = \frac{1-\frac{1}{k_t}}{1-\frac{f_{\bar{\pi}}}{k_t^2}} = \frac{\frac{k_t-1}{k_t}}{\frac{k_t^2-f_{\bar{\pi}}}{k_t^2}} = \frac{k_t-1}{\frac{k_t^2-f_{\bar{\pi}}}{k_t}} = \frac{(k_t-1)k_t}{k_t^2-f_{\bar{\pi}}}$$

Not useful.

So let me instead rewrite it like this

$$\varphi_{5,t} = \frac{1-k_t^{-1}}{d_t}\varphi_{5,t+1} - c\,\frac{1}{d_t}x_{t+1} \quad \text{with } d_t = 1-k_t^{-2}f_{\bar{\pi}}$$

The problem is still: what to do w/ $k_{t+2}\,k_{t+1}\,k_t$

$f(\pi_t$ ...

ok, I just noticed that this is not right either b/c due to $\bar{\pi}_{t-1}$ we have a $\varphi_{6,t+1}$ floating around...

→ ok so what I first need to check is how to solve these darn systems of nonautonomous diff. eqs.

So the system is:

$$2\pi_t + 2\frac{\lambda}{\kappa}x_t - \varphi_{5,t}k_t^{-1} + \varphi_{6,t}f_\pi = 0 \qquad (\mathrm{I}) \quad \overset{\text{Gaspar's}}{(22)}$$

$$\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{6,t+1}\cdot f_{\bar{\pi}} = 0 \qquad (\mathrm{II}) \quad \overset{\text{Gaspar's}}{(21)}$$

$$\varphi_{5,t}k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1}f_k = 0 \qquad (\mathrm{III})$$

Is the following dumb? Express $\varphi_{6,t}$ from (III) plug in (I).
Then: express $\varphi_{6,t+1}$ from (I), plug in (II). Solve for $\varphi_{5,t}$.
Yeah it's dumb.
What if instead we express $\varphi_{5,t}$ from (III) and plug in (I) & (II)

$$\varphi_{5,t} = \frac{f_k\varphi_{6,t+1}}{k_t^{-2}} - \frac{1}{k_t^{-2}}\varphi_{6,t}$$

$\Rightarrow$ (I): $2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t f_k\varphi_{6,t+1} - k_t\varphi_{6,t} + f_\pi\varphi_{6,t} = 0$

(II) $\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}x_{t+1} + \frac{f_k}{k_t^{-2}}\varphi_{6,t+1} - \frac{1}{k_t^{-2}}\varphi_{6,t}$

$$-(1-k_t^{-1})\left[\frac{f_k}{k_{t+1}^{-2}}\varphi_{6,t+2} - \frac{1}{k_{t+1}^{-2}}\varphi_{6,t+1}\right] + f_{\bar{\pi}}\varphi_{6,t+1} = 0$$

(I): $2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t f_k \varphi_{6,t+1} + (f_\pi - k_t)\varphi_{6,t} = 0$

(II) $\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}x_{t+1} - k_t^2\varphi_{6,t} + \left(k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}\right)\varphi_{6,t+1}$
$- k_{t+1}^2(1-k_t^{-1}) f_k \varphi_{6,t+2} = 0$

I think I can solve any of (I) or (II) for $\varphi_{6,t}$ (and $\varphi_{6,t+1}$) and plug in the other. I could also have done it the other way: expressed for $\varphi_{6,t}$ and have these two eqs for $\varphi_{5,t}$. The economic interpretation is interesting in either case: in the system w/ x and φ_6, we'll relate the π-x tradeoff to the shadow value of the anchoring constraint; in the system w/ x and φ_5, we relate the π-x tradeoff to the shadow value of the expectation of LR inflation, $\bar{\pi}$.

But before we can solve any of those, we need to deal w/ these k_t's. For that reason, I suggest to focus on (III) for a sec, and b/c it has an interesting interpretation: it says that φ_5, the shadow value of the LR π-expectation, $\bar{\pi}$, depends on all future ones of

the shadow value of the anchoring constraint:
The other way around!

$$\varphi_{5,t} k_t^{-2} + \varphi_{6,t} - \varphi_{6,t+1} f_k = 0 \qquad \text{(III)}$$

$$k_t^{-2} \varphi_{5,t} = f_k \varphi_{6,t+1} - \varphi_{6,t}$$

$$\varphi_{5,t} = k_t^2 f_k \varphi_{6,t+1} - k_t^2 \varphi_{6,t}$$

This is saying that that the extent of today's LR-expectation constraint depends on the change in how the anchoring constraint binds. In fact,

$$\varphi_{5,t} = k_t^2 \left(f_k \varphi_{6,t+1} - \varphi_{6,t} \right)$$

$\varphi_{5,t} = 0$ when $\varphi_{6,t} = \varphi_{6,t+1} = 0$, i.e. it's not enough that there's a single period where the anchoring doesn't bind. This makes intuitive sense b/c you want the learning to have converged. Another way to do it is to have

$\varphi_{6,t} = f_k \varphi_{6,t+1} \Rightarrow \varphi_6$ is <u>increasing</u> $(f_k < 1)$?!

or $k_t = 0$. But I guess φ_6 can't increase if $f_k \downarrow$.

I kinda feel that the anchoring constraint binds as long as $(k_t, \bar{\pi}_t, f_t(\cdot))$ is non-zero. That is, as long as $\varphi_5 \neq 0$.

time-varying!

One thing this points to which I've ignored so far is that $f(\pi_t, \bar{\pi}_{t-1}, k_{t-1})$ is clearly time varying, so that I should call it $f(t)$, and its derivatives are likely also time-varying $f_K(t)$.

(If $f(t)$ isn't linear in its arguments, then the derivatives will indeed be time-varying.)

So $\varphi_{5,t} = k_t^2 \left(f_K(t)\, \varphi_{6,t+1} - \varphi_{6,t} \right)$

That is one way $\varphi_{5,t} = 0$ is is

$f_K(t) > 1$ but $\varphi_{6,t+1} < \varphi_{6,t} \Rightarrow \varphi_6 \downarrow$ Case 1

the other: $f_K(t) < 1$ but $\varphi_{6,t+1} > \varphi_{6,t} \rightarrow \varphi_6 \uparrow$ Case 2

Take Case 1: tomorrow's anchoring constraint binds less than today's, but the anchoring function is changing a lot in response to k_{t-1}

→ I'm not sure I understand what this means!

You can rewrite as $\varphi_{5,t} = k_t^2 f_K(t) \left(\varphi_{6,t+1} - \frac{\varphi_{6,t}}{f_K(t)} \right)$

· If $f_K(t) = 0$ the gain isn't changing, $k_t = 0$.

Case 1: $f_K(t)$ large, $\varphi_{6,t} > \varphi_{6,t+1} \longrightarrow f_K(t)$ large means/implies that $\varphi_{0,t}$ is large b/c when the gain changes a lot, that's when the constraint binds more.

So if $\varphi_{0,t+1} < \varphi_{0,t}$, this implies that $E\, f_K(t+1) < f_K(t)$ $\Rightarrow$ so when anchoring is expected to happen, then learning may not be a binding constraint

Case 2: $f_K(t)$ small, $\varphi_{0,t+1} > \varphi_{0,t}$

It seems like if anchoring or unanchoring is expected to happen is a particular ratio, then $\varphi_{5,t} = 0$.

I'm still unsure about this interpretation and don't know if it's just a mathematical awkwardness or if it really carries a deep, economic meaning.

Now let's finally turn to solving this:

$$k_t^{-2} \varphi_{5,t} = f_k(t) \varphi_{6,t+1} - \varphi_{6,t} \qquad (\text{III})$$

$$\varphi_{6,t} = f_k(t) \varphi_{6,t+1} - k_t^{-2} \varphi_{5,t}$$

Let me call $\gamma_t = k_t^{-1}$

$$\varphi_{6,t} = f_k(t) \varphi_{6,t+1} - \gamma_t^2 \varphi_{5,t}$$

Wait one sec: this is a non-autonomous diff eq. w/ time-varying coeffs $f_k(t)$ and γ_t^2. But for γ_t^2, we can map it to a non-autonomous diff. eq. by introducing $b_t = -\gamma_t^2 \varphi_{5,t}$. Calling $f_k(t) = a_t$, we can write like Tirelli (p. 5)

$$\varphi_{6,t} = a_t \varphi_{6,t+1} + b_t$$ or redefining

dubiously: $x_{t+1} = \alpha_{t+1} x_t + b_t$

Fuck! Tirelli just states that

$$x_t^{pa} = \begin{cases} \sum_{k=1}^{t-1} \prod_{s=0}^{k-1} \alpha_{t-s} b_{t-k} + b_t & \text{if } |\alpha| < 1 \\ -\sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} \left(\frac{1}{\alpha_{t+1+s}} \right) b_{t+1+k} \right. & \text{if } |\alpha| > 1 \end{cases}$$

which is what I've found but that doesn't help a lot!

$$\varphi_{6,t} = f_k(t)\,\varphi_{6,t+1} + b_t$$

$$b_t \equiv -\gamma_t^2 \varphi_{5,t}$$

$$\varphi_{6,t} = f_k(t)\left[f_k(t+1)\,\varphi_{6,t+2} + b_{t+1}\right] + b_t$$

$$= f_k(t) f_k(t+1)\,\varphi_{6,t+2} + f_k(t)\, b_{t+1} + b_t$$

$$= f_k(t) f_k(t+1)\left[f_k(t+2)\,\varphi_{6,t+3} + b_{t+2}\right] + f_k(t) b_{t+1} + b_t$$

$$= f_k(t) f_k(t+1) f_k(t+2)\,\varphi_{6,t+3} + f_k(t) f_k(t+1)\, b_{t+2} + f_k(t)\, b_{t+1} + b_t$$

I assume that all $f_k(t) < 1$ $\forall t$, so the stuff $\varphi_{6,t+k} \to 0$ as $k \to \infty$. At least.

$$\varphi_{6,t} = b_t + f_k(t)\, b_{t+1} + f_k(t) f_k(t+1)\, b_{t+2} + f_k(t) f_k(t+1) f_k(t+2)\, b_{t+3}$$

$$= b_t + \sum_{i=1}^{\infty} b_{t+i} \left(f_k(t) \cdots f_k(t+i-1)\right) + \ldots$$

$$= b_t + \sum_{i=1}^{\infty} b_{t+i} \prod_{j=0}^{i-1} f_k(t+j)$$

$$\Rightarrow \varphi_{6,t} = -\gamma_t^2 \varphi_{5,t} - E_t \sum_{i=1}^{\infty} \gamma_{t+i}^2\, \varphi_{5,t+i} \prod_{j=0}^{i-1} f_k(t+j)$$

The anchoring constraint binds today to the extent that the learning constraint binds today and is expected to bind in the future.

Ok, now go back to the 2 equations:

(I): $2\pi_t + 2\frac{\lambda}{k}x_t - k_t f_k \varphi_{6,t+1} + (f_\pi - k_t)\varphi_{6,t} = 0$

(II) $\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}x_{t+1} - k_t^2\varphi_{6,t} + \left(k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}\right)\varphi_{6,t+1}$

$- k_{t+1}^2(1-k_t^{-1}) f_k \varphi_{6,t+2} = 0$

Solve (II) for $\varphi_{6,t}$ to then plug back $\varphi_{6,t}$ & $\varphi_{6,t+1}$ in (I)

$$\varphi_{6,t} = \frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}\frac{1}{k_t^2}x_{t+1} + \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2}\varphi_{6,t+1}$$

$$- \frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k \varphi_{6,t+2} = 0$$

Sigh. Have to solve a 2nd order non-autonomous eq.

Call $\varphi_{6,t+1} \equiv y_t$

$$\varphi_{6,t} = \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}\frac{1}{k_t^2}}_{(*)}x_{t+1} + \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2}\varphi_{6,t+1}$$

$$- \frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k y_{t+1} = 0$$

$y_t = \varphi_{6,t+1}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varphi_{6,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2} ; & -\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi_{6,t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} (*) \\ 0 \end{bmatrix} x_{t+1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varphi_{G,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2} \; ; & -\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi_{G,t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} (*) \\ 0 \end{bmatrix} x_{t+1}$$

$$\begin{bmatrix} \varphi_{G,t} \\ y_t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2} \; ; & -\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k \\ 1 & 0 \end{bmatrix}}_{=:A}\begin{bmatrix} \varphi_{G,t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} (*) & 0 \\ 0 & 0 \end{bmatrix} X_{t+1}$$

So to solve this, we need to make this non-interacting.

→ need to diagonalize A !

I'm using Tirelli here, but could use Susanto's lecture notes too.

The characteristic polynomial here is

$$\det \begin{bmatrix} \frac{k_t^2 f_k + (1-k_t^{-1})k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2} - \lambda \; ; & -\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k \\ 1 & -\lambda \end{bmatrix} = 0$$

Take an extra step just b/c it gives nice intuition:

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = 0 \implies (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=0$$

$$\Rightarrow \quad a_{11}a_{22}-\lambda a_{11}-\lambda a_{22}+\lambda^2-a_{12}a_{21}=0$$

$$\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0$$

$$\Rightarrow \quad \lambda^2-\operatorname{tr}(A)\lambda+\det(A)=0$$

$$\Rightarrow \quad \lambda_{1,2}=\frac{\operatorname{tr}(A)\pm\sqrt{\operatorname{tr}(A)^2-4\det(A)}}{2}$$

Useful!

$$\operatorname{tr}(A)=a_{11}+a_{22}=\frac{k_t^2 f_k+(1-k_t^{-1})k_{t+1}^2+f_\pi}{k_t^2}$$

$$\det(A)=a_{11}a_{22}-a_{12}a_{21}=0-\left(-\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1})f_k\right)$$

So the eigenvalues are

$$\lambda_{1,2}=\frac{\frac{k_t^2 f_k+(1-k_t^{-1})k_{t+1}^2+f_\pi}{k_t^2}\pm\sqrt{\left(\frac{k_t^2 f_k+(1-k_t^{-1})k_{t+1}^2+f_\pi}{k_t^2}\right)^2-4\frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1})f_k}}{2}$$

Under the $\sqrt{\cdot}$, $a_t:=\frac{(1-k_t^{-1})k_{t+1}^2}{k_t^2}$, then

$$\left(f_k+a_t+\frac{f_\pi}{k_t^2}\right)^2-4f_k a_t = f_k^2+a_t^2+\frac{f_\pi^2}{k_t^4}+\underline{2f_k a_t}+2f_k\frac{f_\pi}{k_t^2}$$

$$+2a_t\frac{f_\pi}{k_t^2}-4f_k a_t$$

$$=f_k^2+a_t^2+\frac{f_\pi^2}{k_t^4}-2f_k a_t+2f_k\frac{f_\pi}{k_t^2}+2a_t\frac{f_\pi}{k_t^2}$$

Damn!

ok so at least establish stg about signs & sizes of the λ.

$$\lambda_1 \cdot \lambda_2 = \det(A)$$
$$\lambda_1 + \lambda_2 = \text{trace}(A)$$

$$\lambda_1 \cdot \lambda_2 = \frac{k_{t+1}^2}{k_t^2}(1-k_t^{-1}) f_k$$
$$\lambda_1 + \lambda_2 = \frac{k_t^2 f_k + (1-k_t^{-1}) k_{t+1}^2 + f_{\bar{\pi}}}{k_t^2}$$

Sorry guys, even this is unwieldy. Step back a little:

assume a little more structure on f:

$$k_1 = f = f(\pi_t - \bar{\pi}_{t-1} - bS_{t-1})$$
$$\Rightarrow \frac{\partial f}{\partial \pi_t} = f_\pi < 1 \quad \text{but } f_\pi > 0.$$

$$\frac{\partial f}{\partial \bar{\pi}} = -f_{\bar{\pi}} \quad \text{where } f_{\bar{\pi}} \in (0,1) \text{ also.}$$

In fact, since $FE = \pi_t - \bar{\pi}_{t-1} - bS_{t-1}$, and $f = f(FE)$, it's likely that $f_\pi = f_{\bar{\pi}}$. Can impose if necessary.

In that case the FOCs are: was wrong previously anyway

Also I'm now treating k_t^{-1} as a variable.

FOCs: $+ \varphi_{6,t}\left(k_t^{-1} - f(\pi_t - \bar{\pi}_{t-1} - bS_{t-1})\right)$

π_t: $2\pi_t + \varphi_{1,t} - \varphi_{5,t} k_t^{-1} - \varphi_{6,t} f_\pi(t) = 0$ (1)

x_t: $2\lambda x_t - \varphi_{1,t} k = 0$ $\quad \underline{\varphi_{1,t} = 2\frac{\lambda}{k} x_t}$ (2)

$fa_{1,t}$: $-\varphi_{1,t}(1-\alpha)\beta + \varphi_{3,t} = 0$ $\quad \underline{\varphi_{3,t} = 2(1-\alpha)\beta \frac{\lambda}{k} x_t}$ (3)

$\bar{\pi}_t$: $-\varphi_{3,t+1}\frac{1}{1-\alpha\beta} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{6,t+1} f_{\bar{\pi}}(t)$ (4)

k_t^{-1}: $-\varphi_{5,t}\underline{(\pi_t - \bar{\pi}_{t-1} - bS_{t-1})} + \varphi_{6,t} = 0$ (5)

that was missing too!

$\Rightarrow$ This becomes the following 3 equations:

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varphi_{5,t}k_t^{-1} - \varphi_{6,t} f_\pi(t) = 0 \quad (I)$$

$$-\frac{2(1-\alpha)\beta\frac{\lambda}{k}}{1-\alpha\beta} x_{t+1} + \varphi_{5,t} - \varphi_{5,t+1}(1-k_t^{-1}) + \varphi_{6,t+1} f_{\bar{\pi}}(t) = 0 \quad (II)$$

$$\varphi_{6,t} = (\pi_t - \bar{\pi}_{t-1} - bS_{t-1})\varphi_{5,t} \quad (III)$$

The nice thing is that this makes the relationship between φ_6 & φ_5 much more clear: one is FE · the other.

Plug $\varphi_{6,t}$ into (I) & (II)

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varphi_{5,t}k_t^{-1} - \left((\pi_t - \bar{\pi}_{t-1} - bS_{t-1})\varphi_{5,t}\right) f_\pi(t) = 0$$

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varphi_{5,t}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - bS_{t-1}) f_\pi(t)\right) = 0$$

$$2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{S,t}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - bS_{t-1}) f_{\pi}(t)\right) = 0 \qquad (I')$$

And (II):

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k} x_{t+1} + \varphi_{S,t} - \varphi_{S,t+1}(1-k_t^{-1})$$

$$+ f_{\bar{\pi}}(t)\,(\pi_{t+1} - \bar{\pi}_t - bS_t)\varphi_{S,t+1} = 0$$

$$\varphi_{S,t} = \frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k} x_{t+1} + \varphi_{S,t+1}\left[(1-k_t^{-1}) - f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - bS_t)\right]$$

$$(II')$$

So we're down to 2 equations, (I') & (II') Gaspar et al's 22 821

$$\boxed{\begin{aligned} &2\pi_t + 2\frac{\lambda}{k} x_t - \varphi_{S,t}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - bS_{t-1}) f_{\pi}(t)\right) = 0 && (I') \\ &\varphi_{S,t} = \underbrace{\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}}_{c} x_{t+1} + \varphi_{S,t+1}\underbrace{\left[1 - k_t^{-1} - f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - bS_t)\right]}_{\alpha_{t+1}} && (II') \end{aligned}}$$

$$\varphi_{S,t} = c\,x_{t+1} + \varphi_{S,t+1}\cdot\alpha_{t+1}$$ (assuming $\alpha_t < 1$ at least for most t ... :S)

$$= c\,x_{t+1} + \alpha_{t+1}\left[c\,x_{t+2} + \varphi_{S,t+2}\cdot\alpha_{t+2}\right]$$

$$= c\,x_{t+1} + c\,\alpha_{t+1}x_{t+2} + \alpha_{t+1}\alpha_{t+2}\varphi_{S,t+2}$$

$$= c(x_{t+1} + \alpha_{t+1}x_{t+2}) + \alpha_{t+1}\alpha_{t+2}\left[c\,x_{t+3} + \varphi_{S,t+3}\,\alpha_{t+3}\right]$$

$$= c\left(x_{t+1} + \alpha_{t+1}x_{t+2} + \alpha_{t+1}\alpha_{t+2}x_{t+3} + \dots\right)$$

$$\varphi_{S,t} = c\,x_{t+1} + c\sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\alpha_{t+j}$$

Which means that (†') becomes:

$$2\pi_t = -2\frac{\lambda}{k}x_t + \left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_\pi(t)\right)\varphi_{s,t}$$

$$\pi_t = -\frac{\lambda}{k}x_t + \frac{1}{2}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_\pi(t)\right)\left(c\, x_{t+1} + c\sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1} a_{t+j}\right)$$

$$\pi_t = -\frac{\lambda}{k}x_t + \left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_\pi(t)\right)\frac{(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1} a_{t+j}\right)$$

$$\boxed{\pi_t = -\frac{\lambda}{k}\left\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_\pi(t)\right)\cdot\left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\left(1 - k_{t+j}^{-1} + f_{\bar{\pi}}(t+j)\left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b\, s_{t+j}\right)\right)\right)\right\}}$$

b1

- (minus, not plus) in front of f_pibar

j=0

Gaspar et al's (24).

In materials18, I wrote this using $\prod_{j=1}^{0} \equiv 1$ as

$$\pi_t = -\frac{\lambda}{k}\left\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + (\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) f_\pi(t)\right)\cdot\left(\sum_{i=1}^{\infty} x_{t+i}\prod_{j=1}^{i-1}\left(1 - k_{t+j}^{-1} + f_{\bar{\pi}}(t+j)\left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b\, s_{t+j}\right)\right)\right)\right\}$$

Interpretation: we have

$\pi_t = -\frac{\lambda}{k} x_t + \frac{\lambda}{k} \cdot \text{stuff}$.

"No grim trigger result" is here too.

Clearly, stuff $\to 0$ if k_t^{-1} & $f_\pi(t) \to 0$.

that stuff can be decomposed as:

$$\underbrace{k_t^{-1}}_{\text{learning}} \left(\Sigma \pi \right) + \underbrace{fe_t \cdot f_\pi(t)}_{\text{anchoring}} \left(\Sigma \pi \right)$$

As you learn things, $k_t^{-1} \to 0$.

The more anchored you are, the more $f_\pi(t) \to 0$.

Note that the FE doesn't go to 0 b/c exog shocks.
The FE only goes to 0 in expectation.
So in theory a sitn may happen that $k_t^{-1} \simeq 0$ but fe_t is large and you become unanchored.
So the "no grim trigger" result is here, is however qualified. Why? B/C in a sitn where $f = f(FE)$ (so θ^{CEMP} doesn't work, θ^{cusum} does), you can at any time in history be hit by a shock large enough such that $f_\pi(t)$ is big! This is "as if" the grim trigger

was back, but not really in the sense that PS is still a machine and not strategic: has no recourse to threats. It's more like nature can discipline the CB by spitting out a shock large enough to restart the learning.

Hoping to simplify prod:

$(1-a)(1-b)(1-c)$

$(1-b)(1-c) - a(1-b)(1-c)$

$1-c-b+bc \ - a(1-c-b+bc)$

$1-c-b+bc-a+ac+ab-abc$

$1-a-b-c+ac+ab+bc-abc$

Unfortunately that's not much simpler.

The next step is to solve for the interest rate rule.

As a first step, let's work thru how Molnár & Santoro do it (Proof of Prop. 2, in App B., p. 19 (Mac))

Eqs (10) - (13) and (15) are: $\left((10)\ \lambda_{1t} = 0 \quad (15)\ \lambda_{4t} = 0 \right)$

$$2\pi_t - \lambda_{2,t} + \gamma_{t+1}\lambda_{3,t} = 0 \qquad (11)$$

$$2\alpha x_t + k\lambda_{2,t} = 0 \rightarrow \underline{\lambda_{2,t} = -\frac{2\alpha}{k} x_t} \qquad (12)$$

$$E_t \beta^2 \lambda_{2t+1} + \beta(1-\gamma_{t+2})\lambda_{3t+1} = \lambda_{3t} \qquad (13)$$

$$\left.\begin{aligned} &2\pi_t + \frac{2\alpha}{k} x_t + \gamma_{t+1}\lambda_{3t} = 0 && (11') \\ &-E_t\beta^2 \frac{2\alpha}{k} x_{t+1} + \beta(1-\gamma_{t+2})\lambda_{3,t+1} = \lambda_{3t} && (13') \end{aligned}\right\}$$

Now impose $\gamma_t = \frac{1}{t}$

$$2\pi_t + \frac{2\alpha}{k} x_t + \frac{1}{t+1}\lambda_{3,t} = 0 \qquad (11'')$$

$$-E_t\beta^2 \frac{2\alpha}{k} x_{t+1} + \beta\underbrace{\left(1 - \frac{1}{t+2}\right)}_{<1}\lambda_{3,t+1} = \lambda_{3,t} \qquad (13'')$$

So $\lambda_{3,t} = \beta\left(\frac{t+1}{t+2}\right)\left[\beta\left(\frac{t+2}{t+3}\right)\lambda_{3,t+1} - E_t\beta^2\frac{2\alpha}{k}x_{t+2}\right] - E_t\beta^2\frac{2\alpha}{k}x_{t+1}$

$$\lambda_{3,t} = -E_t\beta^2\frac{2\alpha}{k}\left[\beta\left(\frac{t+1}{t+2}\right)x_{t+2} + x_{t+1}\right]$$

$$= -E_t\beta^2\frac{2\alpha}{k}\left[\beta^2\left(\frac{t+2}{t+3}\right)\left(\frac{t+1}{t+2}\right)x_{t+3} + \beta\left(\frac{t+1}{t+2}\right)x_{t+2} + x_{t+1}\right]$$

$$= -E_t \beta^2 \frac{2\alpha}{\kappa} \left[\beta^2 \left(\frac{t+1}{t+3}\right) x_{t+3} + \beta \left(\frac{t+1}{t+2}\right) x_{t+2} + x_{t+1} \right]$$

$$\lambda_{3,t} = -E_t \beta^2 \frac{2\alpha}{\kappa} (t+1) \sum_{i=0}^{\infty} \beta^i \left(\frac{1}{t+1+i}\right) x_{t+1+i}$$

(11'') was $2\pi_t + \frac{2\alpha}{\kappa} x_t + \frac{1}{t+1} \lambda_{3,t} = 0$, so it becomes

$$2\pi_t + \frac{2\alpha}{\kappa} x_t = \frac{2\alpha}{\kappa} \beta^2 E_t \sum_{i=0}^{\infty} \beta^i \left(\frac{1}{t+1+i}\right) x_{t+1+i}$$

which is nowhere near Molnár & Santoro's expression!
(p. 19)

Suppose for a sec that their expression is right, i.e.
↳ for the target criterion
given by

$$\frac{\kappa}{\alpha} \pi_t + x_t = \beta E_t \left[\beta \frac{1}{t+1} x_{t+1} + \frac{\kappa}{\alpha} \pi_{t+1} + x_{t+1} \right]$$

Use (2): $x_t = \frac{1}{\kappa} \pi_t - \frac{\beta}{\kappa} E_t^* \pi_{t+1} - \frac{1}{\kappa} u_t$ (NKPC)

to sub out x:

$$\frac{\kappa}{\alpha} \pi_t + \frac{1}{\kappa} \pi_t - \frac{\beta}{\kappa} E_t^* \pi_{t+1} - \frac{1}{\kappa} u_t = \beta E_t \frac{\kappa}{\alpha} \pi_{t+1}$$

$$+ \beta E_t \left[\left(\beta \frac{1}{t+1} + 1\right) \left(\frac{1}{\kappa} \pi_{t+1} - \frac{\beta}{\kappa} E_{t+1}^* \pi_{t+2} - \frac{1}{\kappa} u_{t+1} \right) \right]$$

Note: they ass. $u_t \sim iid$, so $E_t u_{t+k} = 0 \quad \forall k > 0$

$$\frac{k}{\alpha}\pi_t + \frac{1}{k}\pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \beta E_t \frac{k}{\alpha} \pi_{t+1}$$

$$+ \beta E_t \left[\left(\beta \frac{1}{t+1} + 1\right) \left(\frac{1}{k}\pi_{t+1} - \frac{\beta}{k} E_{t+1}^* \pi_{t+2} - \frac{1}{k} u_{t+1} \right) \right]$$

$$\Rightarrow \left(\frac{k}{\alpha} + \frac{1}{k}\right)\pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \beta E_t \frac{k}{\alpha} \pi_{t+1} + \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{1}{k} E_t \pi_{t+1}$$

$$- \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{\beta}{k} E_t E_{t+1}^* \pi_{t+2}$$

$$\Leftrightarrow \left(\frac{k}{\alpha} + \frac{1}{k}\right)\pi_t - \frac{\beta}{k} E_t^* \pi_{t+1} - \frac{1}{k} u_t = \left(\beta \frac{k}{\alpha} + \frac{\beta}{k}\left(\beta \frac{1}{t+1} + 1\right)\right) E_t \pi_{t+1}$$

$$- \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{\beta}{k} E_t E_{t+1}^* \pi_{t+2}$$

Now sub $\pi - E(\cdot)$: $E_t^* \pi_{t+1} = a_t = a_{t-1} + t^{-1}(\pi_{t-1} - a_{t-1})$

$$\Rightarrow \left(\frac{k}{\alpha} + \frac{1}{k}\right)\pi_t - \frac{\beta}{k} a_t - \frac{1}{k} u_t = \left(\beta \frac{k}{\alpha} + \frac{\beta}{k}\left(\beta \frac{1}{t+1} + 1\right)\right) E_t \pi_{t+1}$$

$$- \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{\beta}{k} \left(a_t + \frac{1}{t+1} (\pi_t - a_t) \right)$$

$$\Leftrightarrow \left(\frac{k}{\alpha} + \frac{1}{k}\right)\pi_t - \frac{\beta}{k} a_t - \frac{1}{k} u_t = \left(\beta \frac{k}{\alpha} + \frac{\beta}{k}\left(\beta \frac{1}{t+1} + 1\right)\right) E_t \pi_{t+1}$$

$$- \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{\beta}{k} \left(1 - \frac{1}{t+1} \right) a_t - \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{1}{t+1} \pi_t$$

$$\Leftrightarrow \left(\frac{k}{\alpha} + \frac{1}{k} + \beta\left(\beta \frac{1}{t+1} + 1\right) \frac{1}{t+1} \right) \pi_t + \left(\beta\left(\beta \frac{1}{t+1} + 1\right) \frac{\beta}{k} \left(1 - \frac{1}{t+1}\right) - \frac{\beta}{k} \right) a_t - \frac{1}{k} u_t$$

$$= \left(\beta \frac{k}{\alpha} + \frac{\beta}{k}\left(\beta \frac{1}{t+1} + 1\right)\right) E_t \pi_{t+1}$$

So $E_t \pi_{t+1} = A_{11,t} \pi_t + A_{12,t} a_t + P_{1,t} u_t$

where

$$A_{11,t} = \frac{\left(\frac{k}{\alpha} + \frac{1}{k} + \beta\left(\beta\frac{1}{t+1}+1\right)\frac{1}{t+1}\right)}{\left(\beta\frac{k}{\alpha} + \frac{\beta}{k}\left(\beta\frac{1}{t+1}+1\right)\right)}$$

$$A_{12,t} = \frac{\left(\beta\left(\beta\frac{1}{t+1}+1\right)\frac{\beta}{k}\left(1-\frac{1}{t+1}\right) - \frac{\beta}{k}\right)}{\left(\beta\frac{k}{\alpha} + \frac{\beta}{k}\left(\beta\frac{1}{t+1}+1\right)\right)}$$

and $$P_{1,t} = \frac{-\frac{1}{k}}{\left(\beta\frac{k}{\alpha} + \frac{\beta}{k}\left(\beta\frac{1}{t+1}+1\right)\right)}$$

Let's rewrite the denom: $\cdot\, \alpha k \rightarrow \frac{1}{\alpha k}\Big(\underbrace{\beta k^2 + \alpha\beta\left(\beta\frac{1}{t+1}+1\right)}_{\text{this is what they have.}}\Big)$

→ So multiply numer's by αk:

→ $\text{num}(P_{1,t}) = -\alpha$ ✓ Got it!

→ $\text{num}(A_{11,t}) = k^2 + \alpha + \underline{\alpha k \beta}\left(\beta\frac{1}{t+1}+1\right)\frac{1}{t+1}$

↑ This is diff, they have β^2 here – may be a typo.

→ $\text{num}(A_{12,t}) = \alpha k\, \frac{\beta}{k}\left[k\left(\beta\frac{1}{t+1}+1\right)\left(1-\frac{1}{t+1}\right) - 1\right]$

$$\Rightarrow \quad -\alpha\beta\left[-\kappa\left(\beta\frac{1}{t+1}+1\right)\left(1-\frac{1}{t+1}\right)+1\right]$$

$$= -\alpha\beta\left[1-\left(1-\frac{1}{t+1}\right)\left(1+\beta\frac{1}{t+1}\right)\cdot\kappa\right]$$

this guy again turned into β for them, otherwise we're good.

Ok so we have (more or less) their eq. (31).

So having $E_t \pi_{t+1} = A_{11,t}\pi_t + A_{12,t} a_t + P_{1,t} u_t$

they write a big system using the LOMs of beliefs

$$y_t = \begin{bmatrix} \pi_t \\ a_t \\ b_t \end{bmatrix}$$

and they need to solve that somehow

for $\pi_t = c_{\pi,t} a_t + d_{\pi,t} u_t$ (23) (Prop. 2)

Then (maybe from the NKPC somehow?)

they get $x_t = \frac{c_{\pi,t}-\beta}{\kappa} a_t + \frac{d_{\pi,t}-1}{\kappa} u_t$ (24)

Then from the NKIS they back out the int. rate rule

$r_t = \delta_\pi a_t + \delta_x b_t + \delta_u \cdot u_t$.

Ok - what I'm trying to understand is what this method does exactly and how that's related to what Woodford does. So in words: Molnár & Santoro:

- Take FOCs, get target criterion, solve for $E_t \pi_{t+1}$
- derive π_t as a fct of a_t, u_t (optimal path)
- derive x_t as a fct of a_t, u_t (optimal path) (from NKPC)
- use NKIS to get i_t as a fct of a_t, b_t, u_t

Woodford:

The econ is characterized by:

(2.1): $\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \Rightarrow x_t = \frac{1}{\kappa}\pi_t - \frac{\beta}{\kappa} E_t \pi_{t+1}$

(2.23): $x_t = E_t x_{t+1} - \sigma\left[i_t - E_t \pi_{t+1} - r_t^n\right]$

(5.1): $\pi_t = -\frac{\lambda}{\kappa}(x_t - x_{t-1})$

Woodford says that you calc $E_t x_{t+1}$ & $E_t \pi_{t+1}$ and plug into (5.1).

Can you simply plug NKPC & NKIS into (5.1)?

→ No b/c it's recursive! It looks as if Woodford also derived the optimal plans for π_t, x_t. He does so to eval the $E(\cdot)$.

It looks like Woodford does sthg like this:
if we've solved for $E_t\pi_{t+1} = a_x x_t + a_u \cdot u_t$
and $E_t x_{t+1} = b_x x_t + b_u \cdot u_t$

and sub these into NKIS & NKPC:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t$$

$$x_t = E_t x_{t+1} + \sigma E_t \pi_{t+1} - \sigma(i_t - r_t^n)$$

$$\Rightarrow \pi_t = \beta a_x x_t + \beta a_u u_t + \kappa x_t$$

$$x_t = b_x x_t + b_u \cdot u_t + \sigma a_x x_t + \sigma a_u u_t - \sigma(i_t - r_t^n)$$

$$\Rightarrow \pi_t = (\beta a_x + \kappa) x_t + \beta a_u u_t$$

$$x_t = (b_x + \sigma a_x) x_t + (b_u + \sigma a_u) u_t - \sigma(i_t - r_t^n)$$

$$x_t = \frac{b_u + \sigma a_u}{1 - b_x - \sigma a_x} u_t - \frac{\sigma}{1 - b_x - \sigma a_x}(i_t - r_t^n)$$

$$\Rightarrow \pi_t = \frac{(\beta a_x + \kappa)(b_u + \sigma a_u)}{1 - b_x - \sigma a_x} u_t - \frac{\sigma(\beta a_x + \kappa)}{1 - b_x - \sigma a_x}(i_t - r_t^n) + \beta a_u u_t$$

$$\pi_t = \left(\frac{(\beta a_x + \kappa)(b_u + \sigma a_u)}{1 - b_x - \sigma a_x} + \beta a_u\right) u_t - \frac{\sigma(\beta a_x + \kappa)}{1 - b_x - \sigma a_x}(i_t - r_t^n)$$

So we get x_t & π_t as fcts of u_t & $(i_t - r_t^n)$:

$$\pi_t = \left(\frac{(\beta a_x + \kappa)(b_u + \sigma a_u)}{1 - b_x - \sigma a_x} + \beta a_u \right) u_t - \frac{\sigma(\beta a_x + \kappa)}{1 - b_x - \sigma a_x} \left(i_t - r_t^n \right)$$

$$x_t = \frac{b_u + \sigma a_u}{1 - b_x - \sigma a_x} u_t - \frac{\sigma}{1 - b_x - \sigma a_x} (i_t - r_t^n)$$

Then do we plug these into the target criterion?

$$\pi_t + \frac{\lambda}{\kappa} x_t = \frac{\lambda}{\kappa} x_{t-1}$$

given?

$$\alpha_u u_t + \alpha_i (i_t - r_t^n) + \frac{\lambda}{\kappa} \beta_u u_t + \frac{\lambda}{\kappa} \beta_i (i_t - r_t^n) = \frac{\lambda}{\kappa} x_{t-1}$$

$$(\alpha_i + \frac{\lambda}{\kappa} \beta_i) i_t + \alpha_u u_t - \alpha_i r_t^n + \frac{\lambda}{\kappa} \beta_u u_t - \frac{\lambda}{\kappa} \beta_i r_t^n = \frac{\lambda}{\kappa} x_{t-1}$$

$$(\alpha_i + \frac{\lambda}{\kappa} \beta_i) i_t = -(\alpha_u + \frac{\lambda}{\kappa} \beta_u) u_t + (\alpha_i + \frac{\lambda}{\kappa} \beta_i) r_t^n + \frac{\lambda}{\kappa} x_{t-1}$$

$$i_t = \frac{-(\alpha_u + \frac{\lambda}{\kappa} \beta_u) u_t}{\alpha_i + \frac{\lambda}{\kappa} \beta_i} + \frac{(\alpha_i + \frac{\lambda}{\kappa} \beta_i) r_t^n}{\alpha_i + \frac{\lambda}{\kappa} \beta_i} + \frac{\frac{\lambda}{\kappa} x_{t-1}}{\alpha_i + \frac{\lambda}{\kappa} \beta_i}$$

Oh yeah!

Alright! That looks ok, in particular b/c it's indeterminate ;)

So Woodford: 1) Solve for optimal plan as a fct of x_t & shocks

2) Use these plans to eval $E(\cdot)$ in NKIS & NKPC. Solve for π_t & x_t as a function of shocks only.

3) Plug the latter functions into the target criterion. Solve for i.

→ But for learning, we don't need to eval these $E(\cdot)$ b/c we have the LOMs of $E(\cdot)$. The problem is that those aren't in functions of shocks only. Which is why Molnár & Santoro solve for $\begin{bmatrix} E_t \pi_{t+1} \\ a_t \\ b_t \end{bmatrix}$ as a function of shocks only.

Most likely whether you eval NKIS first or target criterion, doesn't matter.

→ Tomorrow: try to understand Molnár & Santoro's target criterion & their solution procedure.

One thing I wanted to check is what if we iterate materials 18, eq (16) backwards? No, there's no way we can do that b/c $(1-k_t^{-1}) \leq 1$

25 Feb 2020

Molnár & Santoro have solved for

$$\pi_t = c_{\pi,t} a_t + d_{\pi,t} u_t$$

$$x_t = c_{x,t} a_t + d_{x,t} u_t$$

and NKIS $=$ $x_t = \underbrace{E_t^* x_{t+1}}_{= b_t} - \sigma^{-1} r_t + \sigma^{-1} \underbrace{E_t^* \pi_{t+1}}_{= a_t}$

$$\Rightarrow \quad r_t = -\sigma x_t + \sigma b_t + a_t$$

$$r_t = -\sigma (c_{x,t} a_t + d_{x,t} u_t) + \sigma b_t + a_t$$

$$r_t = (1 - \sigma c_{x,t}) a_t + \sigma b_t - \sigma d_{x,t} u_t$$

which is what they get.

So the procedure is: solve for optimal plans (and expectations if necessary) and plug into the NKIS.

So in either case you solve for the optimal plans. What I'm a little confused about is the following: if you have optimal plans for (x_t, π_t) and you've solved for expressions for $E_t \pi_{t+1}$ & $E_t x_{t+1}$ then the NKIS should give you the interest rate you wanna set; the only one consistent w/ this path. So why need the target?

Stop for a sec – investigate diff. eqs w/ Mathematica
(Mathematica calls them recurrence equations)

materials18.nb

1st order homogenous diff. eq.

$y_{t+1} + a y_t = 0$

If $|a| < 1$, $\quad y_{t+1} = -a y_t = -a(-a) y_{t-1} \quad \dots$

$\Rightarrow \quad y_t = (-a)^k y_{t-k} \quad$ or $\quad y_t = (-a)^t y_0$

1st order nonhomogenous diff eq.

$y_{t+1} + a y_t = c$

$\rightarrow$ Alpha Chiang p. 556, (16.8') $\quad y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a}$

is exactly what Mathematica gets. Neat!

1st order nonhomogenous, nonautonomous diff eq.

$y_{t+1} + a y_t = x_{t+1}$

It seems like there is no way to tell Mma that $|a| < 1$,

so assume $|a| > 1$.

$$y_t = -\frac{1}{a} y_{t+1} + \frac{1}{a} x_{t+1}$$

$$\to y_t = -\frac{1}{a}\left[\left(-\frac{1}{a}\right) y_{t+2} + \frac{1}{a} x_{t+2}\right] + \frac{1}{a} x_{t+1}$$

$$= \underbrace{\left(-\frac{1}{a}\right)^2}_{\to 0} y_{t+2} + \frac{1}{a}\left[\left(-\frac{1}{a}\right) x_{t+2} + x_{t+1}\right]$$

$\vdots$

$$y_t = \underbrace{(-a)^t y_0}_{\substack{\text{complementary} \\ \text{sol is the same}}} + \underbrace{\frac{1}{a} \sum_{j=0}^{t-1} \left(-\frac{1}{a}\right)^j x_{t+1+j}}_{\text{particular}}$$

Mma gets $\ominus$ here

and $(-a)^t$ is multiplying everything for Mma. Why?

In diffeqn.pdf, we have this exact example:

$$x_t = a x_{t-1} + b_t$$

If $|a| < 1$ $\quad x_t = a\left[a x_{t-2} + b_{t-1}\right] + b_t$

Basso, Lec 10 $\sum_{s=1}^{t} a^{t-s} b_s$

$$= a^2 x_{t-2} + a b_t + b_t$$

$$= a^k x_{t-k} + \sum_{j=0}^{k-1} a^j b_{t-k}$$

$\vdots$

Tirelli too! in diffeqn.

$$= \sum_{j=0}^{\infty} a^j b_{t-j} = \sum_{s=-\infty}^{t} a^{t-s} b_s \quad (2)$$

If $|a| > 1$ $\quad a x_{t-1} = x_t - b_t \Rightarrow x_{t-1} = \frac{1}{a} x_t - \frac{1}{a} b_t$

$$x_{t-1} = \frac{1}{a} x_t - \frac{1}{a} b_t$$

$$= \frac{1}{a}\left(\frac{1}{a} x_{t+1} - \frac{1}{a} b_{t+1}\right) - \frac{1}{a} b_t$$

$$x_{t-1} = \left(\frac{1}{a}\right)^2 x_{t+1} - \frac{1}{a}\left(\frac{1}{a} b_{t+1} + b_t\right)$$

$$x_t = \underbrace{\left(\frac{1}{a}\right)^2 x_{t+2}}_{\to 0} - \frac{1}{a}\left(\frac{1}{a} b_{t+2} + b_{t+1}\right)$$

$$\vdots$$

$$= -\frac{1}{a} \sum_{i=0}^{k-1} \left(\frac{1}{a}\right)^i b_{t+1+i}$$

Tirelli gets this yeah!

$$= -\frac{1}{a} \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i b_{t+1+i} = \underbrace{-\frac{1}{a} \sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{t+1-s} b_s}_{\text{not quite what diffeqn. says.}}$$

$$= -\sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^{i+1} b_{t+1+i}$$

$$= -\sum_{j=1}^{\infty} \left(\frac{1}{a}\right)^{j} b_{t+j}$$

Basu, Lect 10:

$$z_t = a z_{t-1} + m_t \qquad |a| > 1$$

$$a z_{t-1} = z_t - m_t$$

$$z_{t-1} = \frac{1}{a} z_t - \frac{1}{a} m_t$$

He writes

$$z_t = -\sum_{s=t+1}^{\infty} \left(\frac{1}{a}\right)^{t-s} m_s$$

this doesn't quite look right to me

Some messy work, but I think I'm fine.

So for the diff. eq. $y_{t+1} = a\, y_t + b_{t+1}$

I get: $y^p_t = -\frac{1}{a} \sum_{i=0}^{\infty} \left(\frac{1}{a}\right)^i b_{t+1+i}$

Mma gets: $y_t = a^{t-1}\left(a\, y0 + \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1} - a\, b_0\right)$

$$\Rightarrow y_t = \underbrace{\underbrace{a^t y_0}_{\text{this is fine, ignore.}} - \underbrace{a^t b_0}_{\text{this is also fine, ignore.}}}_{y^c} + a^{t-1} \underbrace{\sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1}}_{y^p}$$

Mma is saying that this system has evolved since $t=0$ for t periods only (not ∞): $a^t\left(y_0 + a^{t-1} \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^{k+1} a\, b_{k+1} - b_0\right)$

$y_t = a^t\left[y_0 - b_0 + \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^{k+1} b_{k+1}\right]$. Focus on y^p & let $s = k+1$

$$y^p_t = -b_0 + \sum_{s=0}^{t} \left(\frac{1}{a}\right)^s b_s = -b_0 + \left(\frac{1}{a}\right) \sum_{s=0}^{t} \left(\frac{1}{a}\right)^{s-1} b_s$$

$$= -b_0 + b_0 + \left(\frac{1}{a}\right) \sum_{s=1}^{t} \left(\frac{1}{a}\right)^{s-1} b_s$$

$$= \frac{1}{a} \sum_{j=0} \left(\frac{1}{a}\right)^j b_{j+1}$$ Yeah baby! Up to the sign ":)"

So when Mma outputs

$$y^p = \sum_{k=-1}^{t-1} \left(\frac{1}{a}\right)^k b_{k+1} \quad , \quad \text{you can convert that to}$$

$$y^p = \cdot(-1)\, \frac{1}{a} \sum_{j=0}^{\infty} \left(\frac{1}{a}\right)^j b_{t+1+j}$$

and multiply by (-1). It seems like Mma isn't getting that right.

<u>1st order nonhomogenous nonautonomous diff. eq.</u>

<u>w/ variable coefficients</u>

$$x_{t+1} = \alpha_{t+1} x_t + b_{t+1} \qquad \text{Tirelli, p. 5.}$$

Supp $|\alpha| < 1$

$$\begin{aligned} x_{t+1} &= \alpha_{t+1} \left(\alpha_t x_{t-1} + b_t \right) + b_{t+1} \\ &= \alpha_{t+1} \alpha_t x_{t-1} + \alpha_{t+1} b_t + b_{t+1} \\ &= \alpha_{t+1} \alpha_t \left(\alpha_{t-1} x_{t-2} + b_{t-1} \right) + \alpha_{t+1} b_t + b_{t+1} \\ &= \alpha_{t+1} \alpha_t \alpha_{t-1} x_{t-2} + \alpha_{t+1} \alpha_t b_{t-1} + \alpha_{t+1} b_t + b_{t+1} \end{aligned}$$

$$y^c_t = \prod_{s=0}^{t} \alpha_{s+1} x_0 \quad \text{as Tirelli notes}$$

y^p: $\quad \alpha_{t+1}\alpha_t b_{t-1} + a_{t+1} b_t + b_{t+1} \quad + \dots$

$$= \alpha_t \alpha_{t-1} b_{t-2} + a_t b_{t-1} + b_t \quad + \dots$$

$$= \alpha_t \alpha_{t-1} \alpha_{t-2} b_{t-3} + \alpha_t \alpha_{t-1} b_{t-2} + \alpha_t b_{t-1} + b_t$$

$$= \sum_{k=0}^{\infty} b_{t-k} \left(\alpha_{t-0} \cdots \alpha_{t-k} \right)$$

$$= b_t + \sum_{k=1}^{\infty} b_{t-k} \left(\alpha_{t-0} \cdots \underset{\alpha_{t-(k-1)}}{\alpha_{t-k+1}} \right)$$

$$= b_t + \sum_{k=1}^{\infty} b_{t-k} \prod_{s=0}^{k-1} \alpha_{t-s}$$

ok. good. This is what Tirelli gets.

Now if $|\alpha| > 1$

$$x_{t+1} = \alpha_{t+1} x_t + b_{t+1}$$

$$\Leftrightarrow \quad x_t = \frac{1}{\alpha_{t+1}} x_{t+1} - \frac{1}{\alpha_{t+1}} b_{t+1}$$

$$\Rightarrow x_t = \frac{1}{\alpha_{t+1}} \left[\frac{1}{\alpha_{t+2}} x_{t+2} - \frac{1}{\alpha_{t+2}} b_{t+2} \right] - \frac{1}{\alpha_{t+1}} b_{t+1}$$

$$= \frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} x_{t+2} - \left(\frac{1}{\alpha_{t+1}} \frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1} \right)$$

$$y^c = \prod_{s=1}^{k} \left(\frac{1}{\alpha_{t+s}} \right) x_{t+k}$$

$$y^P = -\left(\frac{1}{\alpha_{t+1}}\frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1}\right)$$

$$= -\left(\frac{1}{\alpha_{t+1}}\frac{1}{\alpha_{t+2}}\frac{1}{\alpha_{t+3}} b_{t+3} + \frac{1}{\alpha_{t+1}}\frac{1}{\alpha_{t+2}} b_{t+2} + \frac{1}{\alpha_{t+1}} b_{t+1}\right)$$

$$\vdots$$

$$= -\sum_{k=1}^{\infty} b_{t+k} \prod_{s=1}^{k}\left(\frac{1}{\alpha_{t+s}}\right)$$

$$= -\sum_{k=0}^{\infty} b_{t+1+k} \prod_{s=0}^{k} \frac{1}{\alpha_{t+1+s}}$$

Good. Torelli gets this, eq. 7, part 2.

Mumma gets

$\Pi(y_0 - b_0)$ and then the following term:

$$y^P (\text{I hope}) = \sum_{k_2=-1}^{t-1} b_{1+k_2} \prod_{k_1=0}^{k_2}\left(\frac{1}{a_{1+k_1}}\right)$$

so again b_0 got lost, but otherwise adding it, and t we get the correct thing up to the sign:

$$y^P = (-1)\sum_{k_2=0}^{\infty} b_{t+1+k_2} \prod_{k_1=0}^{k_2}\left(\frac{1}{\alpha_{t+1+k_1}}\right)$$

1st order simultaneous nonhomogeneous diff. eq.

Finally! Diagonalizing to solve diff. eqs!

$$x_t = A x_{t-1} + b \qquad (3)$$

in DSE Notes.

Particular sol: Try $x_t = x \quad \forall t$

$$x = Ax + b$$

$$x - Ax = b$$

$$\Leftrightarrow (I - A)x = b$$

$$x = (I - A)^{-1} b$$ → the unique st. st! (if it exists)

Complementary sol: sol to $x_t = A x_{t-1}$

We need to diagonalize A (decouple the equations)

⇒ We're looking for $M^{-1} A M = \Lambda$, where Λ is a diagonal matrix w/ λ_i on the diags (eigenvals of A) (and actually, M contains the eigenvectors)

$M^{-1} A M = \Lambda \Rightarrow AM = M\Lambda \Rightarrow A = M \Lambda M^{-1}$ | Sub.

$x_t = M \Lambda M^{-1} x_{t-1}$ | M^{-1} from left $\Rightarrow \underbrace{M^{-1} x_t}_{y_t} = \Lambda \underbrace{M^{-1} x_{t-1}}_{y_{t-1}}$

$$y_t = \Lambda y_{t-1}$$

$\rightarrow$ decoupled!

Then the sol of the orig system is just

$$x_t = M y_t + (I - A)^{-1} b$$

Suppose the st. st. is just 0 (homogeneous eq.)

$$x_t = M \Lambda^t$$

eigenvectors

e.g. $x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$ $M = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$ $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$x_{1,t} = c_1 e_{11} \lambda_1^t + c_2 e_{12} \lambda_2^t$$

$$x_{2,t} = c_1 e_{21} \lambda_1^t + c_2 e_{22} \lambda_2^t$$

(w/ constant c to be pinned down)

Let's work thru the example in diffeqn.pdf (3.2) p. 7 Mac.

$$x_t = A x_{t-1} + b$$

where $\begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x^1_{t-1} \\ x^2_{t-1} \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}$, so we're aiming

for $\begin{pmatrix} \hat{x}_t^1 \\ \hat{x}_t^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \hat{x}^1_{t-1} \\ \hat{x}^2_{t-1} \end{pmatrix} + \begin{pmatrix} \hat{b}^1 \\ \hat{b}^2 \end{pmatrix}$.

<u>Step 1.</u> Find eigenvals:

$$\det\left[A - \lambda \cdot I_n \right] = 0$$

In the 2×2 case: $\lambda_{1,2} = \frac{tr(A)}{2} \pm \frac{1}{2}\sqrt{(tr(A))^2 - 4\det(A)}$

<u>Step 2.</u> Find eigenvectors:

Def. Eigenvector i is defined as

$$A e_i = e_i \lambda_i \qquad (13)$$

(Note that (13) doesn't pin down a unique e_i. But this is actually cool b/c we can choose our favorite one.)

So we pick the eigenvector with $e_{i2} = 1$ (y-coordinate...)

$$A \begin{pmatrix} e_i \\ 1 \end{pmatrix} = \begin{pmatrix} e_i \\ 1 \end{pmatrix} \lambda_i$$

[some abuse of notation...)

So we only need to find the x-coordinate (hmm...)

Write this out:

$$a_{11} e_i + a_{12} = e_i \lambda_i \rightarrow e_i (a_{11} - \lambda_i) = -a_{12} \rightarrow e_i = -\frac{a_{12}}{a_{11} - \lambda_i}$$

$$a_{21} e_i + a_{22} = \lambda_i \rightarrow e_i = \frac{\lambda_i - a_{22}}{a_{21}}$$

These are apparently identical equations...

If that's true then the two sols should equal:

$$e_i = -\frac{a_{12}}{a_{11}-\lambda_i} = \frac{\lambda_i - a_{22}}{a_{21}}$$

$$\Leftrightarrow \quad -a_{12}a_{21} = (\lambda_i - a_{22})(a_{11}-\lambda_i)$$

$$(a_{11}-\lambda_i)(a_{22}-\lambda_i) - a_{12}a_{21} = 0$$ Ok. this is the condition $\det(A - \lambda I_n) = 0$. ✓

So then $e_i = \begin{pmatrix} \frac{\lambda_i - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$

Step 3. Put things together

Thus: $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad P = \begin{pmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 & 1 \end{pmatrix} = (e_1, e_2)$

If both eigenvals are distinct, P isn't singular, and thus P^{-1} exists. $P^{-1}\Lambda P = \Lambda \Rightarrow A = P\Lambda P^{-1}$

$$x_t = A x_{t-1} \Rightarrow x_t = P\Lambda P^{-1} x_{t-1}$$

$$\hat{x}_t = \Lambda \hat{x}_{t-1} \quad \text{(decoupled)}$$

And the thing is WE KNOW the sol to this: it's "$c\Lambda^t \hat{x}_0$"

Thus $\hat{x}_t = \Lambda \hat{x}_{t-1}$ has the complementary sol

$$\begin{pmatrix} \hat{x}^1_t \\ \hat{x}^2_t \end{pmatrix} = \begin{bmatrix} c_1 \lambda_1^t \hat{x}^1_0 \\ c_2 \lambda_2^t \hat{x}_0 \end{bmatrix}$$

And since $x_t = P \hat{x}_t$, we have

$$\begin{pmatrix} x^1_t \\ x^2_t \end{pmatrix} = \begin{bmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} & \frac{\lambda_2 - a_{22}}{a_{22}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \lambda_1^t \hat{x}^1_0 \\ c_2 \lambda_2^t \hat{x}^2_0 \end{bmatrix} + \bar{x}$$

By the superposition principle, the general sol just adds $\bar{x}$ here, where $\bar{x} = (I_2 - A)^{-1} b$

$$\rightarrow \left.\begin{aligned} x^1_t &= \left(\frac{\lambda_1 - a_{22}}{a_{22}}\right) c_1 \lambda_1^t \hat{x}^1_0 + \left(\frac{\lambda_2 - a_{22}}{a_{22}}\right) c_2 \lambda_2^t \hat{x}^2_0 \\ x^2_t &= c_1 \lambda_1^t \hat{x}^1_0 + c_2 \lambda_2^t \hat{x}^2_0 \end{aligned}\right\}$$

Voilà!

Now I'm wondering whether we can get the plans for π_t, x_t using this method from (15)-(17) in materials18.

Woodford subs the FOCs into the NKPC. I'm not sure I can do that though (+ into the other model equations) b/c everything is so darn backward-looking and GE...

I think what I need to do is simplify the system as much as possible, and then solve the system of diff eqs. As it seems to me, it may be 5-dimensional, easily...

We have. 26 Feb 2020

- NKPC, LOM(fa), LOM($\bar{\pi}$), LOM(k), eq. (15), (16), (17) (7)
- π_t, x_t, fa, $\bar{\pi}_{t-1}$, k_t^{-1}, φ_5, φ_6 (7)

Ok we should be good.

The equations are: m_i $(i=1,\dots)$ are coeffs on exog s_t.

$$\pi_t - kx_t - (1-\alpha)\beta fa(t) = m_1 \cdot s_t \qquad (9)$$

$$fa(t) = \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} + m_2 \cdot s_t \qquad (11)$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}(\pi_t - \bar{\pi}_{t-1} - bs_{t-1}) \qquad (13)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - bs_{t-1}) \qquad (14)$$

$$2\pi_t + 2\frac{\lambda}{k}x_t - \varphi_{5,t}k_t^{-1} - f_\pi(t)(\pi_t - \bar{\pi}_{t-1} - bs_{t-1})\varphi_{5,t} = 0 \qquad (15+17)$$

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}x_{t+1} + \varphi_{5,t} - (1-k_t^{-1})\varphi_{5,t+1} + f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - bs_t)\varphi_{5,t+1}$$
$$= 0 \qquad (16+17)$$

6 in 6. Sub (11) into (9)

$$\pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} - (1-\alpha)\beta m_2 \cdot s_t = m_1 \cdot s_t$$

$$\Leftrightarrow \pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} = \underbrace{(m_1 + (1-\alpha)\beta m_2)}_{=:m_3} s_t$$

$\pi_t, x_t, \bar{\pi}_{t-1}, k_t^{-1}, \varphi_{5,t}$

$$\pi_t - kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} = m_3 s_t \qquad (1)$$

$$\bar{\pi}_t - (1-k_t^{-1})\bar{\pi}_{t-1} - k_t^{-1}\pi_t + bs_{t-1}k_t^{-1} = 0 \qquad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - bs_{t-1}) \qquad (3)$$

$$2\pi_t + 2\frac{\lambda}{k}x_t - (k_t^{-1} + f_\pi(t)(\pi_t - \bar{\pi}_{t-1} - bs_{t-1}))\varphi_{5,t} = 0 \qquad (4)$$

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{k}x_{t+1} + \varphi_{5,t} - \left(1 - k_t^{-1} - f_{\bar{\pi}}(t)(\pi_{t+1} - \bar{\pi}_t - bs_t)\right)\varphi_{5,t+1} = 0 \qquad (5)$$

$\pi_t, x_t, \bar{\pi}_{t-1}, k_t^{-1}, \varphi_{5,t}$

$$\pi_t - \kappa x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} = m_3 s_t \quad (1)$$

$$\bar{\pi}_t - (1-k_t^{-1})\bar{\pi}_{t-1} - k_t^{-1}\pi_t + b s_{t-1} k_t^{-1} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\pi_t + 2\frac{\lambda}{\kappa} x_t - (k_t^{-1} + f_\pi(t)(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}))\varphi_{5,t} = 0 \quad (4)$$

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa} x_{t+1} + \varphi_{5,t} - \left(1 - k_t^{-1} - f_{\bar{\pi}}(t)\left(\pi_{t+1} - \bar{\pi}_t - b s_t\right)\right)\varphi_{5,t+1} = 0 \quad (5)$$

Express x_t from (1) & sub into (4) & (5)

$$x_t = \frac{1}{\kappa}\pi_t - \frac{1}{\kappa}\frac{(1-\alpha)\beta}{(1-\alpha\beta)}\bar{\pi}_{t-1} - \frac{m_3}{\kappa} s_t$$

$$\bar{\pi}_t - (1-k_t^{-1})\bar{\pi}_{t-1} - k_t^{-1}\pi_t + b s_{t-1} k_t^{-1} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}) \quad (3)$$

$$2\pi_t + 2\frac{\lambda}{\kappa}\left(\frac{1}{\kappa}\pi_t - \frac{1}{\kappa}\frac{(1-\alpha)\beta}{(1-\alpha\beta)}\bar{\pi}_{t-1} - \frac{m_3}{\kappa} s_t\right) - (k_t^{-1} + f_\pi(t)(\pi_t - \bar{\pi}_{t-1} - b s_{t-1}))\varphi_{5,t} = 0 \quad (4)$$

$$-\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa}\left(\frac{1}{\kappa}\pi_{t+1} - \frac{1}{\kappa}\frac{(1-\alpha)\beta}{(1-\alpha\beta)}\bar{\pi}_t - \frac{m_3}{\kappa} s_{t+1}\right) + \varphi_{5,t} - \left(1 - k_t^{-1} - f_{\bar{\pi}}(t)\left(\pi_{t+1} - \bar{\pi}_t - b s_t\right)\right)\varphi_{5,t+1} = 0 \quad (5)$$

Now we're down to 4 eqs in $\pi_t, \bar{\pi}_t, k_t^{-1}, \varphi_{5,t}$

$$\bar{\pi}_t - (1-k_t^{-1})\bar{\pi}_{t-1} - \underline{k_t^{-1}\pi_t} + b\,\underline{S_{t-1}k_t^{-1}} = 0 \quad (2)$$

$$k_t^{-1} = f(\pi_t - \bar{\pi}_{t-1} - bS_{t-1}) \quad (3)$$

$$2\left(1+\frac{\lambda}{k^2}\right)\pi_t - 2\frac{\lambda}{k^2}\frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_{t-1} - 2\frac{\lambda}{k^2}m_3 S_t - k_t^{-1}$$

$$- f_\pi(t)\,\underline{\pi_t \varphi_{5,t}} + f_\pi(t)\,\underline{\bar{\pi}_{t-1}\varphi_{5,t}} + f_\pi(t)\,b\,\underline{S_{t-1}\varphi_{5,t}} = 0 \quad (4)$$

$$-2\frac{\lambda}{k^2}\frac{(1-\alpha)\beta}{1-\alpha\beta}\pi_{t+1} + 2\frac{\lambda}{k^2}\frac{(1-\alpha)\beta}{1-\alpha\beta}\bar{\pi}_t + 2\frac{\lambda}{k^2}\frac{(1-\alpha)\beta}{1-\alpha\beta}m_3 S_{t+1} + \varphi_{5,t}$$

$$-\varphi_{5,t+1} + \underline{k_t^{-1}\varphi_{5,t+1}} + f_{\bar{\pi}}(t)\,\underline{\pi_{t+1}\varphi_{5,t+1}} - f_{\bar{\pi}}(t)\,\underline{\bar{\pi}_t\varphi_{5,t+1}} - f_{\bar{\pi}}(t)\,b\,\underline{S_t\varphi_{5,t+1}} = 0$$

$$(5)$$

The problem is the presence of these interactions.

They seem to be coming from φ_6. But not only. Also from k_t^{-1}?

$\Rightarrow$ are they breaking the linearity ...?

The other issue is eq. (3) b/c I can't separate time t & $t-1$ elements. One could "chop it up" as

$$f(\pi_t - \bar{\pi}_{t-1} - bS_{t-1}) = g(\pi_t) - g(\bar{\pi}_{t-1}) - g(bS_{t-1})$$ in which case we have an eq. system in π_t, $\bar{\pi}_t$ & $\varphi_{5,t}$. Ignore the interactions for a sec. Define $y_t = \begin{bmatrix}\pi_t \\ \bar{\pi}_t \\ \varphi_{5,t}\end{bmatrix}$

Then we have $A E_t y_{t+1} + B y_t + C y_{t-1} = D E_t S_{t+1} + E S_t + F S_{t-1}$

Let $z_t \equiv E_t y_{t+1}$, $e_t \equiv E_t S_{t+1}$, $\varepsilon_t \equiv S_{t-1}$

$$\Rightarrow A z_t + B y_t + C y_{t-1} = D e_t + E s_t + F \varepsilon_t$$

Call $x_t \equiv \begin{bmatrix} y_t \\ z_t \end{bmatrix}$ and $m_t = \begin{bmatrix} s_t \\ e_t \\ \varepsilon_t \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} B & A \end{bmatrix} x_t = -\begin{bmatrix} C \\ 1 \end{bmatrix}_{t-1} + \begin{bmatrix} E & D & F \end{bmatrix} m_t$$

$$x_t = \underbrace{-\begin{bmatrix} B & A \end{bmatrix}^{-1} C}_{=:A} x_{t-1} + \underbrace{\begin{bmatrix} E & D & F \end{bmatrix}}_{=:M} m_t$$

$$x_t = A x_{t-1} + M m_t$$

Ryan meeting 26 Feb 2020

- no credible threats → Mele, Molnár & Santoro, 2019
- eq (6)-(7) if k_t endog, does that break linearity?
- If optimal plan, doesn't NKIS suffice to get i? Why target out?
- Commitment = discretion under learning
- $\Delta k = f(FE)$ ← It's not obvious in (14) that level of k should be a fct of the fcst error.

  ↑ change in k
- The interactions have to be there. Maybe one can't get anything closed form but eq. (18) is good to describe the tradeoff. Try the simple model!

Work after

The simple model is (5)-(7), but of course we'd need to add a constraint governing k_t^{-1}. I'm rephrasing the problem in Materials19 so that x_t shows up too, so that we can get an interest rate rule.

FOCs:

π: $2\pi_t + \varphi_{1,t} - \varphi_{2,t} k_t^{-1} - \varphi_{3,t}\, g_\pi = 0$ (1)

x_t: $2\lambda x_t - K\varphi_{1,t} = 0 \rightarrow \varphi_{1,t} = 2\frac{\lambda}{K} x_t$ (2)

f_t: $-\beta\varphi_{1,t} + \varphi_{2,t} - (1 - k_{t+1}^{-1})\varphi_{2,t+1} + \varphi_{3,t+1}\, g_f = 0$ (3)

k_t^{-1}: $-\varphi_{2,t}(\pi_t - f_{t-1}) + \varphi_{3,t} = 0$ (4)

$\rightarrow \varphi_{3,t} = (\pi_t - f_{t-1})\varphi_{2,t}$

$\Rightarrow$ $2\pi_t + 2\frac{\lambda}{K} x_t - \varphi_{2,t} k_t^{-1} - g_\pi(\pi_t - f_{t-1})\varphi_{2,t} = 0$ (1)

$-2\beta\frac{\lambda}{K} x_t + \varphi_{2,t} - (1 - k_{t+1}^{-1})\varphi_{2,t+1} + g_f(\pi_{t+1} - f_t)\varphi_{2,t+1} = 0$ (2)

$\pi_t = K x_t + \beta f_t + u_t$ (3)

$f_t = f_{t-1} - k_t^{-1}(\pi_t - f_{t-1})$ (4)

$k_t^{-1} = g(\pi_t - f_{t-1})$ (5)

in $\pi_t, x_t, f_t, k_t^{-1}$ and $\varphi_{2,t}$

This guy, eq. (4), is what is causing the trouble.

$$f_t = f_{t-1} + k_t^{-1}(\pi_t - f_{t-1})$$

What if we "loglinearized" this?

$$\ln f_t = \ln\left(f_{t-1} - k_t^{-1}(\pi_t - f_{t-1})\right) \quad \Big| \text{Total diff.}$$

$$\frac{1}{f} \cdot df_t = \frac{1}{\underbrace{f - k^{-1}(\pi - f)}_{=0 \text{ in st.st.}}}\left[(1-k^{-1})\, df_{t-1} + (\pi - f)\, dk_t + k^{-1}\, d\pi_t\right]$$

$$\hat{f}_t = \frac{1}{f}\left[f \cdot \hat{f}_{t-1} + (\pi - f)\, \underbrace{\hat{k}_t^{-1}}_{\text{just call } dk_t \ \hat{k}_t}\right]$$

$$\rightarrow \hat{f}_t = \hat{f}_{t-1} + \underbrace{\frac{(\pi - f)}{f}}_{=: \varepsilon \ = \text{ st.st. } \frac{FE}{\text{Expectation}}} \hat{k}_t^{-1}$$

Supp we replace

$$f_t - f_{t-1} - k_t^{-1}(\pi_t - f_{t-1}) \quad \text{w/}$$

$$\hat{f}_t - \hat{f}_{t-1} - \varepsilon\, \hat{k}_t^{-1}$$

in the opti problem. Then FOCs become:

π_t: $2\pi_t + \varphi_{1,t} - \varphi_{3,t}\, g_\pi = 0$ (1)

x_t: $2\lambda x_t - k\varphi_{1,t} = 0 \quad \rightarrow \quad \varphi_{1,t} = 2\frac{\lambda}{k} x_t$ (2)

f_t: $\varphi_{2,t} - \varphi_{2,t+1} + \varphi_{3,t}\, g_f = 0$ (3)

k_t^{-1}: $-\varphi_{2,t}\,\varepsilon + \varphi_{3,t} = 0 \quad \rightarrow \quad \varphi_{3,t} = \varepsilon\varphi_{2,t}$ (4)

$\Rightarrow$

$$2\pi_t + 2\frac{\lambda}{k} x_t - \varepsilon g_\pi \varphi_{2,t} = 0 \qquad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \qquad (2)$$

$$\varphi_{2,t+1} = \underbrace{(1+\varepsilon g_f)}_{>1}\varphi_{2,t} \quad \rightarrow \quad \varphi_{2,t} = \frac{1}{1+\varepsilon g_f}\varphi_{2,t+1}$$

Problem: this is explosive $\quad \varphi_{2,t+1} = (1+\varepsilon g_f)^t \varphi_0$ explodes.

But there is some hope of at least deriving the optimal paths

$$2\pi_t + 2\frac{\lambda}{k} x_t - \varepsilon g_\pi \varphi_{2,t} = 0 \qquad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \qquad (2)$$

$$\pi_t = k x_t + \beta f_t + u_t \qquad (3)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon \hat{k}_t^{-1} \qquad (4)$$

$$k_t^{-1} = g(\pi_t - f_{t-1}) \qquad (5)$$

$$2\pi_t + 2\frac{\lambda}{k} x_t - \varepsilon g_\pi \varphi_{2,t} = 0 \qquad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \qquad (2)$$

$$\pi_t = k x_t + \beta f_t + u_t \quad \longrightarrow \quad x_t = \frac{1}{k}\pi_t - \frac{\beta}{k} f_t - \frac{1}{k} u_t \qquad (3)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon g(\pi_t - f_{t-1}) \qquad (4)$$

$$2\left(1 + \frac{\lambda}{k^2}\right)\pi_t - 2\beta\frac{\lambda}{k^2} f_t - 2\frac{\lambda}{k^2} u_t - \varepsilon g_\pi \varphi_{2,t} = 0 \qquad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \qquad (2)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon g(\pi_t - f_{t-1}) \qquad (4)$$

A system in $\pi_t, f_t, \varphi_{2,t}$. Call $\varphi_2 = \varphi$, $g = g(\pi_t) + g(-f_{t-1})$

$$\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}^{=a} \begin{bmatrix} \pi_{t+1} \\ f_{t+1} \\ \varphi_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 2\left(1+\frac{\lambda}{k^2}\right) & -2\beta\frac{\lambda}{k^2} & -\varepsilon g_\pi \\ 0 & 0 & 1+\varepsilon g_f \\ g(\pi_t) & 0 & 0 \end{bmatrix}}_{=b} \begin{bmatrix} \pi_t \\ f_t \\ \varphi_t \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1+g(-f_{t-1}) & 0 \end{bmatrix}}_{=c} \begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ \varphi_{t-1} \end{bmatrix} = \begin{bmatrix} 2\frac{\lambda}{k^2} \\ 0 \\ 0 \end{bmatrix} u_t$$

Scalar analogy:

$a x_{t+1} + b x_t + c x_{t-1} = u_t$ Intro: $xP_t = x_{t+1}$

$\Rightarrow a\, xP_t + b x_t + c x_{t-1} = u_t$

$xP_{t-1} = x_t$

$$\begin{bmatrix} b & a \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ xP_t \end{bmatrix} + \begin{bmatrix} c & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ xP_{t-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t$$

So:

$$\underbrace{\begin{bmatrix} \overbrace{\begin{matrix} 2\left(1+\frac{\lambda}{\kappa^2}\right) & -2\beta\frac{\lambda}{\kappa^2} & -\varepsilon g_\pi \\ 0 & 0 & 1+\varepsilon g_f \\ g(\pi_t) & 0 & 0 \end{matrix}}^{=b} & \overbrace{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{matrix}}^{=a} \\ \begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{matrix} & 0 \end{bmatrix}}_{=:D} \underbrace{\begin{bmatrix} \pi_t \\ f_t \\ \varphi_t \\ \pi P_t \\ fP_t \\ \varphi P_t \end{bmatrix}}_{=:g_t}$$

$$+ \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1+g(-f_{t-1}) & 0 \end{matrix}}_{=c} & 0 \\ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & 0 \end{bmatrix}}_{=:E} \begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ y_{t-1} \\ \pi P_{t-1} \\ fP_{t-1} \\ \varphi P_{t-1} \end{bmatrix} = \begin{bmatrix} 2\frac{\lambda}{\kappa^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t$$

$$D y_t + E y_{t-1} = [\text{stuff}]\, u_t$$

$$y_t = -D^{-1} E\, y_{t-1} + D^{-1} \underset{6\times 1}{[\text{stuff}]}\, u_t$$

$$\underset{6\times 1}{y_t} = \underset{6\times 6}{A}\, \underset{6\times 1}{y_{t-1}} + \underset{6\times 1}{M}\, \underset{1\times 1}{u_t}$$

The question is whether D^{-1} exists. Of course not: the 4^{th} & 5^{th} column are not lin. independent.

Yo bigmouth! What if you roll eq. (2) bw?

$$2\left(1+\frac{\lambda}{\kappa^2}\right)\pi_t - 2\beta\frac{\lambda}{\kappa^2} f_t - 2\frac{\lambda}{\kappa^2} u_t - \varepsilon g_\pi \varphi_{2,t} = 0 \qquad (1)$$

$$\varphi_{2,t} - \varphi_{2,t+1} + \varepsilon g_f \varphi_{2,t} = 0 \rightarrow \varphi_{2,t} = (1+\varepsilon g_f)\varphi_{2,t-1} \qquad (2)$$

$$\hat{f}_t = \hat{f}_{t-1} + \varepsilon g(\pi_t - f_{t-1}) \qquad (4)$$

$$\Rightarrow 2\left(1+\frac{\lambda}{\kappa^2}\right)\pi_t - 2\beta\frac{\lambda}{\kappa^2} f_t - \varepsilon g_\pi \varphi_{2,t} = 2\frac{\lambda}{\kappa^2} u_t \qquad (1)$$

$$\varphi_{2,t} = (1+\varepsilon g_f)\varphi_{2,t-1} \qquad (2)$$

$$-\varepsilon \cdot g(\pi_t) + \hat{f}_t = \hat{f}_{t-1} + \varepsilon g(-f_{t-1}) \qquad (3)$$

$$\underbrace{\begin{bmatrix} 2\left(1+\frac{\lambda}{\kappa^2}\right) & -2\beta\frac{\lambda}{\kappa^2} & -\varepsilon g_\pi \\ 0 & 0 & 1 \\ -\varepsilon g(\pi_t) & 1 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \pi_t \\ f_t \\ \varphi_t \end{bmatrix}}_{y_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1+\varepsilon g_f \\ 0 & 1+\varepsilon g(-f_{t-1}) & 0 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \pi_{t-1} \\ f_{t-1} \\ \varphi_{t-1} \end{bmatrix}}_{y_{t-1}} + \begin{bmatrix} 2\frac{\lambda}{\kappa^2} \\ 0 \\ 0 \end{bmatrix} u_t$$

$y_t = A_t\, y_{t-1} + M_t\, u_t$ (see Mathematica, materials19.nb)

Then $y_t^c = P\, \hat{y}_t$ (P = matrix of eigenvectors)

is the complementary solution where

$$\hat{y}_t = c_i\, \lambda_i^t\, \hat{y}_0^i \qquad i = 1, 2, 3$$

The eigenvalues are (see Mma)

$$\underset{<1}{\lambda_1 = 0}, \quad \lambda_2 = \frac{(\kappa^2 + \lambda)(1 + \varepsilon g(-f_{t-1}))}{\kappa^2 + \lambda - \beta\varepsilon\lambda g(\pi_t)}, \quad \underset{>1}{\lambda_3 = 1 + \varepsilon g_f}$$

$$\lambda_2 = \frac{1 + \varepsilon g(-f_{t-1})}{1 - \frac{\beta\varepsilon\lambda g(\pi_t)}{\kappa^2 + \lambda}}$$ looks like this is >1 as well.

But since we have 2 bw-looking vars, we might have 2 TVCs which set $c_2 = c_3 = 0$ so we're left w/ the only stable eig. I don't like it that it's 0. But allow the system to evolve for long enough such that the initial condition disappears.

And turn to the particular sol:

$$y_t = A_t \left[A_{t-1}\, y_{t-2} + M_{t-1}\, u_{t-1} \right] + M_t\, u_t$$

$$y_t = A_t \; A_{t-1} y_{t-2} + A_t M_{t-1} u_{t-1} + M_t u_t$$

$$= A_t A_{t-1} \left[A_{t-2} y_{t-3} + M_{t-2} u_{t-2} \right] + A_t M_{t-1} u_{t-1} + M_t u_t$$

$$= \underbrace{A_t A_{t-1} A_{t-2} y_{t-3}} + A_t A_{t-1} M_{t-2} u_{t-2} + A_t M_{t-1} u_{t-1} + M_t u_t$$

I wonder how on earth this is s'posed to vanish

But A^{-1} doesn't exist so can do nothing else.

Or were we s'posed to do this for the diagonal system

$\hat{y}_t = \Lambda y_{t-1}$ ... No b/c that's just the homogen. part.

So

$$y_t^P = M_t u_t + \sum_{i=1}^{\infty} M_{t-i} u_{t-i} \left(A_{t-0} A_{t-1} \dots A_{t-i+1} \right)$$

$$y_t^P = M_t u_t + \sum_{i=1}^{\infty} M_{t-i} u_{t-i} \prod_{k=0}^{i-1} A_{t-k}$$

Ok – I think we had this confusion before:

diffeqn.pdf says that for an n-dim system:

$$y_t^P = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)\dots(1-\lambda_n L)}$$

where $\dfrac{1}{1-\lambda_1 L} = \sum_{s=-\infty}^{t} \lambda_1^{t-s} b_s$ if $|\lambda_1| < 1$

and $\dfrac{1}{1-\lambda_1 L} =$ sum of future b_t if $|\lambda_1| > 1$.

But many questions. First of all, does that mean that

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \sum_{s=-\infty}^{t} \lambda_1^{t-s} b_s \cdot \sum_{s=-\infty}^{t} \lambda_2^{t-s} b_s \quad ?$$

(if both roots > 1?) Second, if $|\lambda_1| < 1$ but $|\lambda_2| > 1$, then

$$= \sum_{s=-\infty}^{t} \lambda_1^{t-s} b_s \sum_{j=t}^{\infty} \left(\frac{1}{\lambda_2}\right)^{j-1} b_j \quad ?$$

Third, how does this relate to iterating fwd/bw?

→ I think I know. In the scalar case, $eig(\underset{1\times 1}{A}) = A$. So the sum is a sum of eigs. In the matrix case, we go directly to the eigs.

Fourth, A is time-varying. If it wasn't, then the particular sol would have been:

$$y_t^p = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L)} M u_t \longrightarrow$$

With A time-varying, it is (maybe)

$$y_t^p = \frac{1}{(1-\lambda_{1,t} L)(1-\lambda_{2,t} L)(1-\lambda_{3,t} L)} M u_t$$

Fifth, M_t is time-varying too.

→ this may not be correct. Look at Tirelli p. 6 Mac

$$x_t^{pa} = \begin{cases} \sum_{k=1}^{t} \left(\prod_{s=0}^{k-1} \alpha_{t-s} \right) b_{t-k} + b_t & \text{if } |\alpha| < 1 \\ -\sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} \left(\frac{1}{\alpha_{t+1+s}} \right) \right) b_{t+1+k} & \text{if } |\alpha| > 1 \end{cases}$$

So for a 2-dim system, this should be something like:

$$\left(\sum_{k=1}^{t} \left(\prod_{s=0}^{k-1} \lambda_{1,t-s} \right) b_{t-k} + b_t \right) \left(-\sum_{k=0}^{\infty} \left(\prod_{s=0}^{k} \left(\frac{1}{\lambda_{2,t+1+s}} \right) \right) b_{t+1+k} + b_t \right)$$

where $|\lambda_1| < 1$, $|\lambda_2| > 1$.

Tirelli also says on p. 13 Mac

$$z_t^{pa} = \sum_{s=0}^{t} \Lambda^{t-s} c_s$$

where $z_t = P x_t$ and the transformed system is

$$z_t = \Lambda z_{t-1} + C_t \qquad w/ \; C_t = P^{-1} b_t$$

$\Lambda^t = \text{diag}(\lambda_1^n, \lambda_2^n)$ which implies that you simply sum separately as $\sum \lambda_1^t + \sum \lambda_2^t$

$$x_t = A x_{t-1} + b_t$$

$$x_t = P^{-1} \Lambda P x_{t-1} + b_t$$

$$P x_t = \Lambda P x_{t-1} + P b_t$$

$$z_t = \Lambda z_{t-1} + c_t$$ so c_t should be Pb_t.

So $y_t^P = \underbrace{\sum \Pi \lambda_1}_{\substack{\text{but since} \\ \lambda_1 = 0 \\ \text{that drops}}} + \overbrace{\sum \Pi \lambda_2}^{\text{forward}} + \overbrace{\sum \Pi \lambda_3}^{\text{forward.}}$

- To do tomorrow: search online for particular sol. of non-autonomous diff eq. w/ time-varying coeffs.
- Only that way can you characterize the sol. for the simple model w/ "loglin-ed" Lom (f_t).
- If that doesn't work, drop the "loglin" Lom(f_t), and return to trying to deal w/ the interacting system.