# Shadow Price Learning and Expectationally Driven Business Cycles⋆

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**Abstract**

This paper explores the qualitative and quantitative effects of assuming economic agents are boundedly rational and employ an adaptive approach to understanding the economic environment called *shadow price learning* in which the current-period decisions of agents are informed by one-step ahead forecasts of both endogenous variables and the shadow prices which govern optimal behavior. The economic environment is the news-shock DSGE model of **?**, which is characterized by the ongoing arrival of "news shocks" which provide partial information about future economic fundamentals. This slight departure from the rational expectations hypothesis (REH) admits a restricted perceptions equilibrium (RPE) distinct from the model's corresponding rational expectations equilibrium (REE) for the same calibration. In some cases simulated moments from the RPE provide a better empirical match to than the REE.

*Keywords:* News shocks, bounded rationality, adaptive learning, business cycles, DSGE models

JEL Codes: C11, D83, D84, E13, E32

## 1. Introduction

Recently there has been great interest among structural macroeconomic modelers in exploring the potential role information and expectations might themselves play in generating the boom-bust cycles we refer to as the business cycle. While the notion that business cycles might be partially caused by swings in optimism and pessimism on the part of economic agents traces its lineage at least as far back as **?**, it was only after the asset-price bubbles during the late 1990s which culminated in the "dot-com bust" of 2001 that modelers of macroeconomic dynamic stochastic

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general equilibrium (DSGE) models began to appreciate the possible importance of constructing models which were capable of producing qualitatively realistic expectationally driven business cycles (EDBCs) in response to information about the future. The standard versions of both flexible and sticky-price models tend to struggle in generating "irrational exuberance" of the type witnessed at the turn of the century, and it is this gap which the original news-shock model of **?** attempted to fill. By modifying the structure of a RBC model to include a complementarity between durable and non-durable factors of in the production of output as well as a short-term substitutability constraint between consumption and investment, the authors were able to generate qualitatively realistic EDBCs resulting from households receiving erroneous news about the future. Yet this is just one way of solving the so-called "comovement problem" of neoclassical models, and it comes at the expense of making significant changes to the standard RBC model.

**?** on the other hand develop a news-shock DSGE model capable of solving the comovement problem while remaining quite close to the neoclassical framework established in **?**. In particular, by imposing a cost to adjusting investment between periods and allowing the utilization rate of capital to be variable, the model implies good news about future productivity causes investment to rise contemporaneously while capital is depreciated more quickly by increasing utilization. The latter effect implies the marginal product of labor, and hence the real wage, will rise. A novel preference specification nesting the well known specifications of **?** and **?** permits calibrations which ensure the wealth effect on labor supply is not dominated by the substitution effect and hence the higher level of investment is financed not by decreasing consumption but rather by increasing hours worked. The net result is to produce positive comovement between output, consumption, investment, and labor supply in response to good news about the future.

Other models featuring news as a primary driver of business cycles include the sticky-price formulations of **?** and **?** as well as the flexible price model of **?**, and each use different methods to solve the comovement problem. But the results in all previous cases are derived under the assumption that economic agents are endowed with accurate knowledge of the conditional distributions of all endogenous variables; that is, they are said to have rational expectations (RE). Given the focus on information as a cause of business cycle activity in these news-shock DSGE models, it seems natural to consider whether and how the predictions of such models might be affected by considering weaker assumptions for agent's expectation-formation, which is the express goal of the present study. Indeed, if the introduction of news-shocks constitutes a change to the usual input of agents' forecasting models, then it seems quite natural to consider how the output - and hence the behavior observed in the current period - is affected by different assumptions for the actual forecasting technology.

The paper proceeds as follows. Section 2 provides an overview of various alternatives to rational expectations commonly found in the learning literature. Section 3 describes the economic environment for the study, corresponding to the model developed including the calibration for the model, the equations governing a temporary equilibrium, and the way in which news shocks are modeled. Section 4 presents the results of calibrate-and-simulate exercises designed to compare US macroeconomic data to model predictions under various expectation-formation assumptions. Section 5 concludes by reviewing the main results and discussing avenues for future research.

## 2. Bounded Rationality and Adaptive Learning

Agents which use non-RE schemes to forecast the future are said to be "boundedly rational", and there are many different flavors. However, most have as their origin some version of either the *Euler-equation learning* approach of **?**, **?**, and others, or the *infinite-horizon learning* approach of **?**. Under the former, agents behave in accordance with their Euler equations which relate control decisions today with the forecasted values of decisions tomorrow. In contrast to the early work on *reduced-form learning* such as **?** which first solved agents' dynamic optimization problem under the RE assumption and then relaxed it, Euler-equation learning studies the evolution of agent's beliefs and the path of the economy when their expectations are boundedly rational even when solving their dynamic programming problem. One of the benefits of Euler-equation learning is its simplicity: agents need only make one-period ahead forecasts to determine their optimal control decisions.

One possible shortcoming to this approach noted in **?** is that agents must forecast the future value of endogenous control variables, a somewhat strange exercise in the usual context of a representative agent model where these are completely under the control of the agents themselves. Infinite-horizon learning avoids this by formally expressing decisions today as being determined by agent's expectations of their future lifetime budget constraint and transversality conditions. Behavior is thus based on the relationship between future wealth and the implied control decisions. One of the main benefits of the infinite-horizon approach is the strict adherence to microfoundations which implies behavior is truly optimal given beliefs.

In **?** a "hybrid" of the infinite-horizon and Euler-equation approaches called *N-step optimal learning* is developed. N-step optimal learning assumes agents explicitly take current and future expected values of wealth over a finite range into account when making their control decisions, which are themselves anchored to the Euler equation as a behavioral primitive. This learning mechanism has the infinite-horizon approach as a limiting case, and may be viewed as its finite-horizon version. N-step optimal learning therefore captures both the simplicity of Euler-equation learning and the rigorous adherence to fully optimal behavior imparted by infinite-horizon learning. However, it still requires considerable sophistication on the part of agents in the model.

A recent alternative developed in **?** and referred to as *shadow-price learning* (SP-learning), which is the one I choose to implement in the present paper, is similar in spirit but requires much less sophistication on the part of individual agents. Rather than taking the behavioral primitive to be the Euler equations, which often embody complicated nonlinear relations between contemporaneous control decisions and the evolution of the determinants of future income, SP-learning assumes agents base their decisions on the standard first-order necessary conditions (FONCs) derived from the intertemporal Lagrangian. These FONCs describe a set of conditions which must be satisfied by the agents control decisions in order to be considered optimal, and they are functions of the expected present value of shadow prices for endogenous state variables. SP-learners rely on a linear forecasting model to predict future shadow prices which is updated over time in response to forecast errors using the simple and familiar recursive least squares (RLS) algorithm. Because the shadow prices govern whether behavior is "optimal", and because inertia introduced by the backwards looking nature of adaptive learning can cause forecasts errors to be serially correlated, agent-level behavior under SP-learning can be described as "boundedly optimal": while it is (possibly) different than the behavior implied by RE, it is optimal in the sense that agents are still making choices rationally given expectations.

SP-learning is an attractive alternative to RE for many reasons. First, the informational assumptions are quite natural: agents need to know little more than their own preferences and budgetary constraint, taking as given many of the same variables that actual households do such as wages and interest rates. Second, the behavior of agents is quite intuitive: they simply contemplate how changing their behavior today would impact key variables tomorrow based on their beliefs and take action accordingly. Third, the updating of beliefs occurs via recursive least squares, an exercise which any student of introductory econometrics could themselves conduct. Finally, it lends itself to considering heterogeneity of agents along some dimension such as information or initial wealth, as these differences will cause different transition paths under learning for different households.

Much attention in the bounded rationality literature has focused on determining whether agents endowed with boundedly rational expectations may learn to behave rationally by updating their forecasting model until it is identical to that implied by the REH. REE which are "learnable" are said to be "Expectationally Stable (E-Stable)". E-stability results exist for a broad range of classic models including the textbook neoclassical and New Keynesian models, and recent work by **?** extends these results to include the news-shock DSGE models of **?** and **?**. There is also a somewhat smaller literature which examines the effect of relaxing RE on the quantitative predictions generated by a given DSGE model, either in calibrate-and-simulate exercises *a la* **?**, or in estimation approaches such as **?**, **?**, and **?**. Examples of these estimation exercises being performed in news-shock DSGE models can be found in **?** and **?**.

This paper proceeds in the spirit of **?** by trying to determine the quantitative effects on business cycle statistics generated by the **?** news-shock DSGE model when agents are assumed to be SP-learners. The central goal is to address what effect relaxing the assumption of RE in favor of SP-learning has on the empirical relevance of this particular news-shock model, and whether the behavior of SP-learning households will come to approximate that of their fully rational counterparts.

## 3. The Economic Environment

As in **?**, households earn income each period by renting their labor and capital to firms which pay wages and interest on the factors according to their marginal products; the households use their income to maximize their happiness via consumption and leisure while deciding the gross investment required to obtain the desired capital stock next period. The introduction of a cost to adjusting the level of gross investment implies net investment may differ; in addition, households may vary the utilization rate of their existing capital stock in any period. All markets are assumed to be competitive. Finally, the model features two sources of exogenous volatility: a shock to production technology and a shock to the marginal efficiency of investment i.e. the price of investment relative to consumption. Both shocks are stationary, and it is assumed that agents receive correct but incomplete information or "news" on the shocks in advance.

Formally, the representative household chooses consumption $C_t$ and hours worked $h_t$ to maximize the lifetime utility function

$$\hat{E}_0 \sum_{t=0}^{\infty} \beta^t U(V_t) \tag{1}$$

where $0 < \beta < 1$ characterizes the household's discount factor and $U$ is the period utility function which takes the CRRA form

$$U(V_t) = \frac{V_t^{1-\sigma} - 1}{1 - \sigma} \tag{2}$$

As usual, $\sigma > 0$ characterizes the (inverse) intertemporal elasticity of substitution. The argument $V_t$ is given by

$$V_t = C_t - \frac{\psi h_t^{1+\frac{1}{\theta}} S_t}{1 + \frac{1}{\theta}} \tag{3}$$

where $\psi > 0$ scales the disutility of labor supply and $\theta > 0$ governs the Frisch elasticity of labor supply. $S_t$ is a geometric average of current and past habit-adjusted consumption and takes the form

$$S_t = C_t^{\gamma} S_{t-1}^{1-\gamma} \tag{4}$$

where $0 \leq \gamma \leq 1$ governs the magnitude of the wealth elasticity of labor supply. This preference specification, often referred to as "JR preferences", allows the modeler to calibrate the wealth effect of labor supply to be "small" while permitting a balanced growth path. The comovement problem in typical RBC models is partially caused by the wealth effect of labor supply dominating the substitution effect, thereby causing consumption and hours worked to move in opposite directions upon receipt of good news, and hence this utility function makes it simple to develop qualitatively realistic expectationally driven business cycles.

Note this specification nests two well-known and important preference specifications. $\gamma = 0$ corresponds to the preferences of **?** in which labor supply depends only on current real wages and is independent of the marginal utility of income, while $\gamma = 1$ corresponds to the preferences of **?** which are compatible with a balanced growth path at the optimal steady state of the economy. Small values for $\gamma$ allow the economy to be consistent with a balanced growth path while also implying a very weak wealth effect of labor supply, both of which are important features for any

DSGE model to be capable of generating empirically relevant comovement in key macroeconomic variables in response to news shocks.[1]

Households are assumed to own a stock of physical capital $K_t$ and rent it to firms in a competitive factor market. Each household's stock of capital evolves according to the law of motion

$$K_t = (1 - \delta(u_t)) K_{t-1} + I_t \left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right] \tag{5}$$

where $I_t$ denotes gross private investment. Note the timing convention: subscripts correspond to the *end of period* realization of the variable.

The function $\Phi\left(\frac{I_t}{I_{t-1}}\right)$ imposes an increasing cost to adjusting investment from its previous level while $\delta(u_t)$ implies capital depreciation is an increasing function of its utilization rate $u_t$. I follow **?** in assuming the quadratic functional forms

$$\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2 \tag{6}$$

where $\delta_0 > 0$ is the steady-state depreciation rate, $\delta_1 > 0$ determines the steady-state value of $u_t$, and $\delta_2 > 0$ captures the rental rate elasticity of capacity utilization. Denoting the gross growth rate of private investment as $\mu_t^I \equiv \frac{I_t}{I_{t-1}}$, adjustment costs are given by

$$\Phi\left(\mu_t^I\right) = \frac{\kappa}{2}\left(\mu_t^I - \mu^I\right)^2 \tag{7}$$

where $\kappa > 0$ is a scaling parameter used in calibrating the model and $\mu^I$ is the steady-state growth rate of gross private investment. This specification implies there are no investment-adjustment costs on a balanced growth path i.e. $\Phi\left(\mu^I\right) = \Phi'\left(\mu^I\right) = 0$ and $\Phi''\left(\mu^I\right) > 0$.

Each period the household receives labor income from working $h_t$ hours at rate $W_t$, rental income from renting $u_t K_{t-1}$ units of effective capital at gross rental rate $R_t$, and lump sum firm-profits of $\Pi_t$. The household uses this income to purchase consumption and investment goods. The flow budget constraint is given by

$$C_t + A_t I_t \leq W_t h_t + R_t\left(u_t K_{t-1}\right) + \Pi_t \tag{8}$$

[1]Note that the case of $\gamma = 0$ is not consistent with a balanced growth path which implies the real interest rate (and hence the marginal utility of income) grows at a constant rate. A balanced growth path requires the wealth and substitution effects from the constantly growing real wage to cancel, but $\gamma = 0$ ensures that only the substitution effect exists, and hence labor supply will be changing at a rate inconsistent with the rest of the economy.

where $A_t$ is an exogenous process representing the current state of technology for producing investment goods from consumption goods with stationary growth rate $\mu_t^A \equiv \frac{A_t}{A_{t-1}}$ and steady-state value $\mu^A$. In a decentralized equilibrium $A_t$ may be interpreted as the relative price of investment goods in terms of consumption goods, that is $A_t$ units of consumption may be traded for a single unit of investment (equivalently, $1/A_t$ unites of the investment good are required to purchase a single unit of the consumption good).

Households are rational and hence make decisions in each period to maximize their expected lifetime utility. In other words, each period they make a labor-leisure and consumption-savings decision by solving a constrained optimization problem to maximize expected discounted utility. Formally, households choose a set of stochastic processes $\{C_t, h_t, u_t, I_t, K_t, S_t\}_{t=0}^{\infty}$ to maximize 1 subject to the constraints given by equations 2-8 and initial conditions for the endogenous state variables $I_{-1}, K_{-1}$, and $S_{-1}$. This can be written as a standard intertemporal constrained maximization problem:

$$
\begin{aligned}
\max_{C_t, u_t, h_t, I_t, K_t, S_t} \quad & \hat{E}_t \sum_{t=0}^{\infty} \beta^t U(V_t) \\
\text{subject to} \quad & U(V_t) = \frac{V_t^{1-\sigma} - 1}{1-\sigma} \\
& V_t = C_t - \frac{\psi h_t^{1+\frac{1}{\theta}} S_t}{1+\frac{1}{\theta}} \\
& S_t = C_t^{\gamma} S_{t-1}^{1-\gamma} \\
& C_t + A_t I_t \leq W_t h_t + R_t (u_t K_{t-1}) + \Pi_t \\
& K_t = (1-\delta(u_t))K_{t-1} + I_t \left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right]
\end{aligned}
$$

The household's time $t$ control decisions $\hat{u}_t = (C_t, u_t, h_t, I_t, K_t, S_t)'$ will depend on their expectations of the future shadow price of the endogenous state variables $x_t^1 = (I_{t-1}, K_{t-1}, S_{t-1})'$. These shadow prices have the familiar interpretation as the marginal value of these goods, and it is worth emphasizing their primitive role in intertemporal decisions making: adjustment costs, capital depreciation, and the habit formation-type indexing imply the value of future consumption and savings depend directly on their level today.[2] Put differently, without endogenous state variables (and their accompanying shadow prices) the decision maker would face an infinite sequence of unrelated

[2]While agents will also use their expectations of future realizations of the exogenous state variables in their

optimization decisions subject to stochastic processes.

Under RE all agents within the economy know the conditional distributions of all variables. In other words the laws of motion for the endogenous and exogenous state variables are known and the expected future value of these shadow prices is straightforward to compute. Standard solution techniques under RE involve deriving Euler equations relating today's control decisions to those of tomorrow to describe the behavior of households and combining these and other optimality conditions, resource constraints, and laws of motion to arrive a system of expectational difference equations. Typically this system is nonlinear, and hence the solution to the first-order approximation is considered.

While elegant and mathematically straightforward, rational expectations imparts an exceptional degree of sophistication on the forecasting and decision making behavior of the aggregate; furthermore, in the representative agent-type models so common in modern macroeconomics, the distinction between individual and aggregate decision making is blurred. In what follows I will replace RE with a version of bounded rationality based on shadow prices called SP-learning. Under SP-learning, households do not know the conditional distribution of all variables, and in particular they do not know the precise way in which shadow prices depend on their behavior. Instead they act as econometricians and estimate the value of these shadow prices using a linear forecasting rule which is updated each period in response to observed forecast errors.

The solution technique under SP-learning stands in stark contrast to that of RE. A major difference is that while agents are endowed with linear forecasting rules, their behavior is embedded within the nonlinear specification of the model. This implies the relevant equilibrium notion is that of a *restricted-perceptions equilibrium* (RPE) as described in **?**. A RPE can be thought of as an equilibrium arising from agents' optimally misspecified beliefs which are consistent with the stochastic processes realized in the economy; that is, given their forecasting model agents are unable to detect a misspecification. The first task of this paper is determining how similar this RPE is to the implied REE in the news-shock DSGE model presently considered.

Denote the shadow prices of investment, capital, and habit-adjustment by $\lambda_t^I, \lambda_t^K$ and $\lambda_t^S$, respectively. These have the simple interpretation as the time $t$ value of an additional unit of their

decision making process, their actions do not affect the marginal values of these variables; hence, they need not concern themselves with the expected future shadow prices of these exogenous states.

corresponding endogenous state in time $t$. For example, $\lambda_t^K$ is exactly how much a household would be willing to pay in time $t$ for an additional unit of preinstalled capital $K_{t-1}$. With this interpretation the FONCs describing optimal household choices of $C_t, h_t$, and $u_t$ given beliefs about the future values of these shadow prices can be obtained via simple variational arguments.[3] The details of this derivation are presented in the Appendix. The end result is the following three FONCs in the controls

$$\begin{aligned}
U_{C_t}(V_t) + \frac{\partial S_t}{\partial C_t}\beta\hat{E}_t\lambda_{t+1}^S &= \frac{\partial I_t}{\partial C_t}\left(\beta\hat{E}_t\lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta\hat{E}_t\lambda_{t+1}^K\right) \\
-U_{h_t}(V_t) &= \frac{\partial I_t}{\partial h_t}\left(\beta\hat{E}_t\lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta\hat{E}_t\lambda_{t+1}^K\right) \\
\frac{\partial K_t}{\partial u_t}\beta\hat{E}_t\lambda_{t+1}^K &= \frac{\partial I_t}{\partial u_t}\left(\beta\hat{E}_t\lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta\hat{E}_t\lambda_{t+1}^K\right)
\end{aligned}$$

and the following three FONCs in the endogenous state variables

$$\begin{aligned}
\lambda_t^I &= \frac{\partial K_t}{\partial I_{t-1}}\beta\hat{E}_t\lambda_{t+1}^K \\
\lambda_t^K &= \frac{\partial I_t}{\partial K_{t-1}}\left(\beta\hat{E}_t\lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta\hat{E}_t\lambda_{t+1}^K\right) + \frac{\partial K_t}{\partial K_{t-1}}\beta\hat{E}_t\lambda_{t+1}^K \\
\lambda_t^S &= U_{S_{t-1}}(V_t) + \frac{\partial S_t}{\partial S_{t-1}}\beta\hat{E}_t\lambda_{t+1}^S
\end{aligned}$$

It can be shown that the endogenous shadow prices can be expressed in terms of the contemporaneous controls and states, and hence their actual value is determined as a function of the contemporaneous decisions of the household. However, the household makes these decisions based on their expectations of the value of future shadow prices without knowing the nonlinear way in which the actual values are determined. Given a set of beliefs and a linear forecasting model (called a *perceived law of motion (PLM)*) for the evolution of these shadow prices, SP-learners take optimal actions given their forecasts of future shadow prices. This behavior results in the *actual law of motion (ALM)* for the shadow prices: beliefs as described in the PLM give way to actions which have consequences described by the ALM. Deviations between the ALM and the PLM - that is, forecast

[3]Note that given optimal decisions for consumption, labor supply, and capacity utilization the choices for the geometric average of habit-adjusted consumption $S_t$, gross investment $I_t$, and hence next period's capital stock $K_{t+1}$ are pinned down by equations (4), (8), and (5) respectively.

errors - are observed and incorporated into the household's forecasting model via recursive least squares. A formal description of this process is given in section 3.2. A crucial question addressed by studying household behavior under this less stringent specification for expectation formation is whether the household's beliefs will (approximately) converge to those of a rational agent.

The model is closed by describing the production side of the economy. The representative firm pays for $h_t$ worker-hours and rents $u_t K_{t-1}$ units of effective capital to produce output $Y_t$ using constant returns to scale (CRTS) technology according to the production function

$$Y_t = z_t \left(u_t K_{t-1}\right)^{1-\alpha} h_t^{\alpha} \tag{9}$$

where $\alpha \in (0,1)$ governs the labor share of output in steady state. As is typical in RBC-type models, the supply side of the economy is subjected to exogenous stochastic shocks, given here by a transitory shock $z_t$. Factor markets are competitive and hence the gross rental rate equals the value of the marginal product of effective capital

$$R_t = (1-\alpha)\frac{Y_t}{u_t K_{t-1}} \tag{10}$$

and the wage paid is equal to the value of the marginal product of labor

$$W_t = \alpha \frac{Y_t}{h_t} \tag{11}$$

Output is fungible and may be used for private consumption or gross investment. Total demand is thus given by

$$Y_t = C_t + A_t I_t \tag{12}$$

*3.1. Equilibrium*

A temporary equilibrium for all periods $t \geq 0$ is described collectively by the optimal behavior of households given beliefs:

$$U_{C_t}(V_t) + \frac{\partial S_t}{\partial C_t}\beta \hat{E}_t \lambda_{t+1}^S = \frac{\partial I_t}{\partial C_t}\left(\beta \hat{E}_t \lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda_{t+1}^K\right) \tag{13a}$$

$$-U_{h_t}(V_t) = \frac{\partial I_t}{\partial h_t}\left(\beta \hat{E}_t \lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda_{t+1}^K\right) \tag{13b}$$

$$\frac{\partial K_t}{\partial u_t}\beta \hat{E}_t \lambda_{t+1}^K = \frac{\partial I_t}{\partial u_t}\left(\beta \hat{E}_t \lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda_{t+1}^K\right) \tag{13c}$$

$$K_t = \left(1-\delta\left(u_t\right)\right) K_{t-1} + I_t \left[1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right] \tag{13d}$$

$$C_t + A_t I_t \leq W_t h_t + R_t \left(u_t K_{t-1}\right) \tag{13e}$$

$$S_t = C_t^{\gamma} S_{t-1}^{1-\gamma} \tag{13f}$$

$$I_{-1}, K_{-1}, S_{-1} \text{ given} \tag{13g}$$

where the utility function, its argument, and the functional forms for $\delta(u_t)$ and $\Phi\left(\frac{I_t}{I_{t-1}}\right)$ are given by equations (2), (3), (6), and (7), respectively. The Appendix expresses these equations in terms of the underlying variables as opposed to derivatives, which I present here for the sake of expositional clarity. The market clearing conditions, aggregate resource constraint, production function, and factor-market prices are given by

$$Y_t = C_t + A_t I_t \tag{14a}$$

$$Y_t = z_t \left(u_t K_{t-1}\right)^{1-\alpha} \left(h_t\right)^{\alpha} \tag{14b}$$

$$W_t = \alpha \frac{Y_t}{h_t} \tag{14c}$$

$$R_t = (1-\alpha) \frac{Y_t}{u_t K_{t-1}} \tag{14d}$$

The laws of motion for the exogenous stochastic processes $A_t$ and $z_t$ along with the precise way in which news shocks are included in the model are described below.

*3.2. News Shocks and Expectations Formation*

It remains to describe the particular way in which shadow price learning agents form their expectations. The variables relevant to household decision making but outside of their control in time $t$ - that is, the predetermined control variables, factor prices, and the exogenous state variables (plus a constant) - are collected into a vector $x_t = \left(1, I_{t-1}, K_{t-1}, S_{t-1}, W_t, R_t, A_t, z_t\right)'$, and households are assumed to have a PLM for the endogenous-state shadow prices given by $\lambda_t = H_t' x_t$, where $\lambda_t = \left(\lambda_t^I, \lambda_t^K, \lambda_t^S\right)'$ and $H_t$ a conformable matrix denoting the household's time $t$ estimates of the coefficient-matrix relating the state variables to the value of the shadow price.

There is one issue that must be addressed: $W_t$ and $R_t$ are exact functions of labor $h_t$ and effective physical capital $u_t K_{t-1}$, as well as the exogenous processes $z_t$ and $A_t$, and hence including all variables in a regression will lead to perfect multicollinearity. To avoid this I drop $W_t$ and $R_t$ from the set of regressors utilized by the household and denote this subset of state variables as

$\tilde{x}_t = (1, I_{t-1}, K_{t-1}, S_{t-1}, A_t, z_t)'$.[4]

The household's PLM for the shadow prices is thus given by the linear model

$$\lambda_t = \tilde{H}_t' \tilde{x}_t \tag{15}$$

where $\tilde{H}_t$ is updated via recursive least squares according to the dynamic system

$$R_{H,t} = R_{H,t-1} + g_t \left( \tilde{x}_{t-1} \tilde{x}_{t-1}' - R_{H,t-1} \right) \tag{16}$$

$$\tilde{H}_t = \tilde{H}_{t-1} + g_t R_{H,t}^{-1} \tilde{x}_{t-1} \left( \lambda_{t-1} - \tilde{H}_{t-1}' \tilde{x}_{t-1} \right)' \tag{17}$$

$R_{H,t}$ is the household's time $t$ estimate of the second-moment matrix for the regressors while the gain parameter $g_t$ controls how much weight households put on new information. Equation 17 describes the process for forecast-revision as being one of gradual adjustment of existing beliefs (captured by $\tilde{H}_{t-1}$) in response to forecast errors arising from differences between the PLM and ALM (captured by $\lambda_{t-1} - \tilde{H}_{t-1}' \tilde{x}_{t-1}$), weighted by the gain parameter $g_t$ and the uncertainty induced by volatility in the underlying data (captured by $R_{H,t}^{-1} \tilde{x}_{t-1}$).

Much of the theoretical work exploring whether agents can learn the coefficient-matrix of an economy's REE, such as **?** assumes a *decreasing gain* such as $g_t = t^{-1}$ , which implies the household response to forecast errors vanishes asymptotically. Alternatively, studies focused on simulation or estimation of DSGE models under learning, such as **?** and **?**, employ a *constant gain* where $g_t = \bar{g}$, which implies the household is a "lifelong learner" and continually revises its coefficient estimates placing the most weight on the most recent observations. I will simulate the model under both assumptions. Under constant-gain learning I calibrate the gain to be 0.0152, which is in line with recent estimates from the empirical adaptive learning literature.

The value of shadow prices in time $t$ is not known *ex-ante* to the household, and is determined as a result of their control choices. Hence, time $t$ beliefs are updated using information available through time $t-1$. This clarifies the central idea of shadow price learning: households make optimal decisions *given* their (misspecified) beliefs and the information available to them in the

[4] One could also consider assuming the household does not observe some of the exogenous processes and instead uses wages and the gross rental rate in their forecasts. This would require specifying a PLM and dynamic system for updating beliefs for these price variables. It can also lead to serious issue with asymptotic multicollinearity as household's beliefs converge to those of rational agents, because as the residuals in the dynamic system for updating beliefs go to zero the actions of the household become perfectly collinear with market prices.

moment. That beliefs may be misspecified highlights a particularly salient feature of SP-learning in particular and bounded rationality in general by embracing the notion that agents in the economy may certainly fail to fully understand the dynamics governing its evolution and yet an equilibrium may still exist.

Taking expectations of equation (15) we have $\hat{E}_t \lambda_{t+1} = H_t \hat{E}_t \tilde{x}_{t+1}$, and hence households must forecast the future values of regressors.[5]. The values of the time $t+1$ endogenous state variables $I_t, K_t$, and $S_t$ are pinned down by the time $t$ flow budget constraint, capital accumulation equation, and geometric-average identity respectively, and it is natural to assume the household knows these values. In addition the transition equations for the exogenous processes $z_t$ and $A_t$ are assumed to be known to the household.[6]

These expectations are augmented by the inclusion of news shocks to the information set. A news shock can be thought of as information which arrives exogenously to the household about the value of some future innovation to an exogenous process, but (importantly) does not impact any economic fundamentals contemporaneously. The action generated by news shocks is therefore entirely in the response of agents to this information about the future. News is modeled as an anticipated shock to the exogenous processes and hence has the interpretation of imparting incomplete (but accurate) information to the household about future economic fundamentals.

Let the law of motion for the exogenous processes, indexed by $w = (A, z)$, take the form

$$
\begin{aligned}
\ln(w_t) &= \rho_w \ln(w_{t-1}) + \varepsilon^0_{w,t} \\
\varepsilon^0_{w,t} &= \varepsilon^1_{w,t-1} + \nu^0_{w,t} \left[\nu^0_{w,t} \in \mathcal{I}_t\right] \\
\varepsilon^1_{w,t} &= \varepsilon^2_{w,t-1} + \nu^1_{w,t} \left[\nu^1_{w,t-1} \in \mathcal{I}_t\right]
\end{aligned}
$$

[5] I have assumed certainty equivalence on the parts of households; that is, they believe their time $t$ estimate of the coefficient matrix is the correct estimate now and in the future, and hence behave accordingly. Thus household's do not need to attempt to forecast the way in which their estimates will change in the future.

[6] Since there is no feedback between the household's decisions and the evolution of these processes this is a very simple estimation exercise from an econometric standpoint if the households actually observe all of this data. An alternative approach would carefully consider the variables a household seems likely to observe. Exogenous variables that directly influence the household, such as the technology available for converting consumption goods into investment goods $A_t$ in the flow budget constraint seem quite natural; however it is less obvious that households would directly observe productivity $z_t$. As mentioned previously, one could drop these as regressors in favor of e.g. wages and/or the rental rate of capital, but this may cause the coefficient estimation routine to suffer from asymptotic multicollinearity and/or bias from sampling error present in using estimated data as an explanatory variable.

$$\vdots \;=\; \vdots$$

$$\varepsilon_{w,t}^{n}=\varepsilon_{w,t-1}^{n-1}+\nu_{w,t}^{n}\left[\nu_{w,t-n}^{n}\in\mathcal{I}_t\right]$$

where $\mathcal{I}_t$ the time $t$ information set of the representative household, $\nu_{w,t-k}^{k}$ the $k$-period ahead anticipated shock, and $\left[\nu_{w,t-k}^{k}\in\mathcal{I}_t\right]=1$ if $\nu_{w,t-k}^{k}\in\mathcal{I}_t$ and 0 otherwise. This specification allows the modeler to easily and parsimoniously consider a variety of assumptions regarding the precise details of informational acquisition by households.[7]

The system above makes clear the sense in which news shocks here are being modeled as partial information about the total value of the innovation. This permits a compact autoregressive representation

$$\ln(w_t)=\rho_w\ln(w_{t-1})+\varepsilon_{w,t} \tag{18}$$

$$\varepsilon_{w,t}=\varphi_w\varepsilon_{w,t-1}+M_w\nu_{w,t} \tag{19}$$

where $\nu_{w,t}=\left(\nu_{w,t}^{0},\nu_{w,t}^{1},...,\nu_{w,t}^{n}\right)'$ distributes i.i.d normal with mean zero and variance-covariance matrix equal to the identity matrix so that the non-zero elements of $M_w$ characterize the standard deviation of the exogenous processes. The matrix $\varphi_w$ is a nilpotent matrix which characterizes the timing and flow of information to households (including "surprise" shocks). More details are presented in the Appendix.

To see how introducing news as anticipated shocks can have real effects, suppose households were to receive information four and eight periods in advance about the time $t+1$ shock, and that the exogenous process is subjected to an unanticipated shock which arrives in time $t+1$. Then the time $t$ expectations about the evolution of process $w$ in time $t+1$ would be

$$\hat{E}_t\ln(w_{t+1})=\rho_w\ln(w_t)+\nu_{w,t+1-4}^{4}+\nu_{w,t+1-8}^{8}$$

while the actual value realized in time $t+1$ would be

$$\ln(w_{t+1})=\rho_w\ln(w_t)+\nu_{w,t+1}^{0}+\nu_{w,t+1-4}^{4}+\nu_{w,t+1-8}^{8}$$

[7] In fact, this requires the creation of only as many auxiliary state variables as the lengthiest forecasting horizon. For example, if a household receives information four and eight periods in advance the longest forecasting horizon is eight periods, and thus eight auxiliary state variables must be generated. An alternative but equivalent specification would generate a set of $k$ state variables for each signal $v_{w,t-k}^{k}$ so that e.g. if a household received information four and eight periods in advance one would need to generate 12 additional state variables.

This enables the consideration of a variety of interesting behavior as households revise their expectations in response to new information. For example, the expected value for the innovation prior to time $t+1$ could fail to materialize, that is, $\nu^0_{w,t+1} = -\left(\nu^4_{w,t+1-4} + \nu^8_{w,t+1-8}\right)$, or could materialize exactly as expected i.e. $\nu^0_{w,t+1} = 0$. More generally one may think of the household continually adjusting expectations throughout time in response to the receipt of new information, and this information being descriptive of the innovation's final value to a varying degree in any given period.

Recall the household's PLM for shadow prices is a linear function of the time $t+1$ endogenous state variables and exogenous processes. Since households are assumed to know the law of motion for all exogenous processes they will incorporate the anticipated shocks directly into their decision rules for time $t$ controls via their forecast of future shadow prices. That is, the time $t$ expectation of future shadow prices based on the PLM is

$$\hat{E}_t \lambda_{t+1} = H_t' \hat{E}_t \tilde{x}_{t+1}$$

$$\text{where } \hat{E}_t \tilde{x}_{t+1} = \begin{pmatrix} 1 \\ \hat{E}_t I_t \\ \hat{E}_t K_t \\ \hat{E}_t S_t \\ \hat{E}_t A_{t+1} \\ \hat{E}_t z_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ I_t \\ K_t \\ S_t \\ \rho_A A_t + \hat{E}_t \varepsilon^0_{A,t+1} \\ \rho_z z_t + \hat{E}_t \varepsilon^0_{z,t+1} \end{pmatrix}$$

where the household's expectation of the future shock conditional on all information received up to time $t$, $\hat{E}_t \varepsilon^0_{w,t+1}$, is described above.

*3.3. Calibration*

The model is calibrated using a combination of commonly used values in the literature, estimates obtained from **?**, and steady-state targets for some endogenous variables. Periods correspond to quarters. $\sigma = 1$ which corresponds to logarithmic utility, $\theta = 1.4$ so that the wage elasticity of labor supply is 2.5 when the wealth effect of labor supply is shut off, and $\gamma$ is set to 0.001 which is simultaneously consistent with an extremely small wealth effect of labor supply and a balanced growth path. $\beta$ is assumed to be 0.985 so that the steady-state gross real interest rate is 1.5 percent, and $\alpha$ is set to 0.64 so that labor's share of output in steady-state is 64 percent. Steady state quarterly depreciation $\delta_0$ is set to 2.5 percent and $\delta_2$ is chosen so that the elasticity of $\delta(u_t)$ is

0.15. $\delta_1$ is calibrated to ensure steady-state capacity utilization equals 1, while the disutility-scale parameter $\psi$ is chosen so that household's spent 20% of their time working. The second derivative of the investment adjustment cost function $\kappa$ is set to 1.3, though this is subjected to robustness checks since the literature has little to say about this parameter. The autoregressive parameters for growth rates of $z_t$ and $A_t$ are set to 0.5 and 0.9, respectively, which are consistent with the estimated values obtained in **?**. This is summarized in Table 1 below.

| Parameter | Value | Description |
|---|---|---|
| $\sigma$ | 1 | Intertemporal Elasticity of Substitution |
| $\theta$ | 1.4 | Frisch-labor Supply Elasticity |
| $\gamma$ | 0.001 | Wealth Elasticity of Labor Supply |
| $\beta$ | 0.985 | Subjective Discount Factor |
| $\alpha$ | 0.64 | Steady-state Labor Share |
| $\delta_0$ | 0.025 | Steady-state Depreciation Rate |
| $u$ | 1 | Steady-state Capacity Utilization Rate |
| $h$ | 0.2 | Steady-state Labor Supply |
| $\kappa$ | 1.3 | Adjustment Cost Acceleration |
| $\rho_z$ | 0.5 | Persistence of Investment-specific TFP Growth |
| $\rho_A$ | 0.9 | Persistence of TFP Growth |

Table 1: Calibrated Parameters for One-sector Model

The relative importance of anticipated vs surprise shocks for each exogenous process is set consistent with estimates obtained from **?** which estimates an augmented version of the model presently considered. In particular, the standard deviations for the surprise components of $z_t$ and $A_t$ are set to 0.21 and 0.65, respectively, while the corresponding standard deviations for the (cumulative) anticipated components are 0.32 and 0.2. This implies the majority (60%) of variation in the growth rate of investment-specific technology is anticipated, while just under 25% of variation in TFP is anticipated.

To develop a benchmark against which to compare the results from SP-learning, I linearize the temporary equilibrium around the non-stochastic steady state. The resulting system of first order expectational difference equations can be easily solved under rational expectations, and the resulting equilibrium is the REE.

*3.4. The Response to News*

With the model solved and calibrated, I turn to an exploration of the response to news shocks by key macroeconomic variables in the model. Figure 1 shows the responses by consumption, investment, hours worked, and output when a fully rational household learns at time $t = 0$ that there will be a 1 unit increase in the value of investment-specific or total factor productivity in time $t = 3$, and the expected innovation arrives as expected. In both cases the model generates positive comovement amongst all variables at the time the news is received: good news about the future causes all aggregate variables to rise.

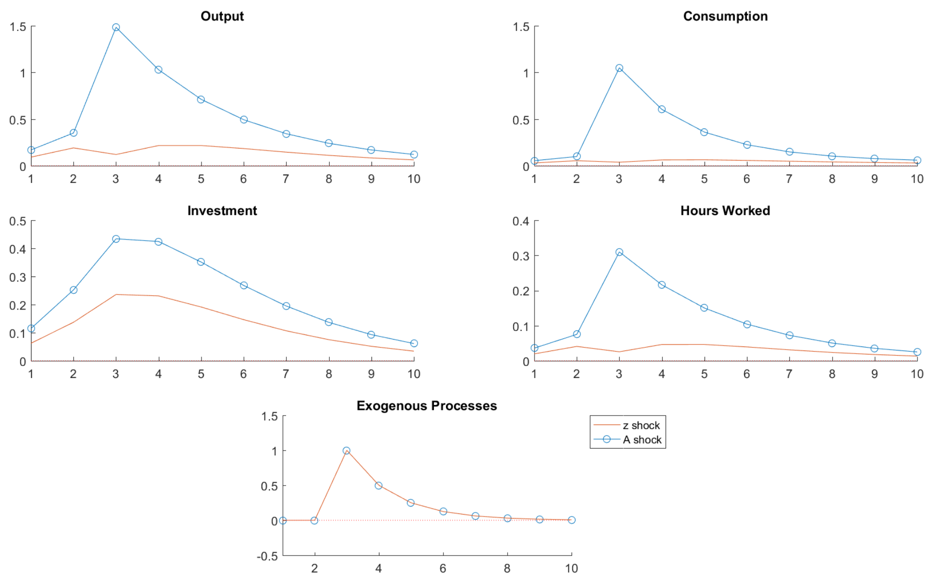


Figure 1: IRF for Accurate News

The total response to news about a shock to investment-specific technology $z_t$ is more subdued than that regarding a shock to TFP $A_t$, because the price of investment affects output only through its effect on capital accumulation while an increase in TFP directly increases output and factor prices everything else held constant. Interestingly this gives rise to a discrepancy in the relative impact of news on the overall response: most of the movement in key variables stemming from news about $z_t$

occurs in the period the news is received, while most of the movement from news about $A_t$ occurs in the period the shock occurs. Clearly including additional exogenous processes will lead to a richer set of possible reactions to news of each; indeed **?** include five additional exogenous disturbances and allow agents to receive information of their future values, in which case one should no longer speak generically of "news".

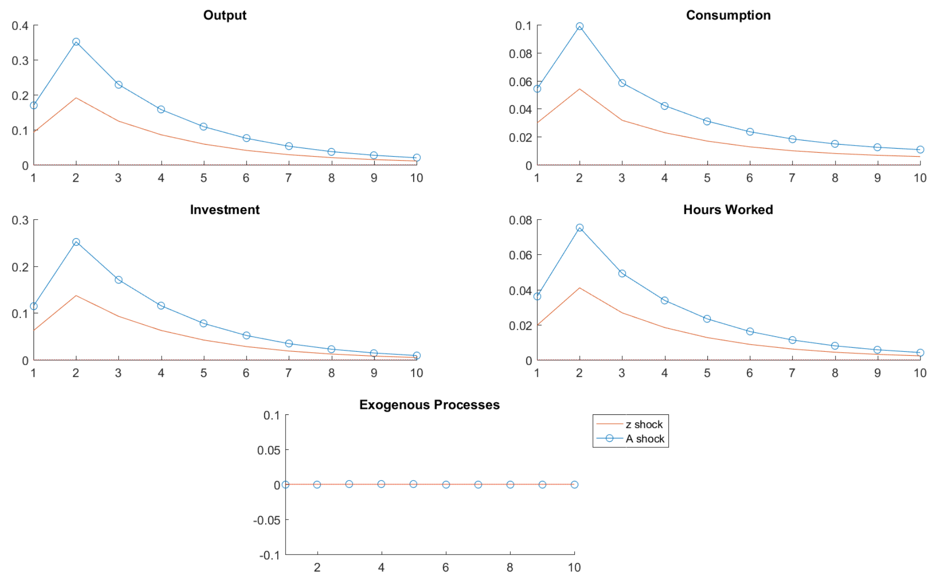


Figure 2: IRF for Inaccurate News

Now suppose the news received by households turns out to be completely false. That is, at $t = 0$ households come to expect a 1 unit increase in the exogenous processes will occur in $t = 3$ which does not materialize. Figure 2 plots the IRF resulting from this thought experiment on rational households. Again, the receipt of news generates positive comovement in the key macroeconomic variables. However, once the news is shown to have been erroneously optimistic all variables tend back towards their initial steady state values. Contrary to the prediction of **?**, in this model all variables remain above their steady-state levels for an extended period of time. Thus even in the face of incorrect news the positive comovement amongst variables remains is preserved, which is a particular strength of this news-shock model.

## 4. Simulation and Model Performance

Having demonstrated the ability of the model to generate qualitatively realistic expectationally driven business cycles in response to news about the future I now turn to an evaluation of the model's ability to generate quantitatively realistic empirical moments under SP-learning. I begin by determining whether the behavior of SP-learners will cause the economy to converge (approximately) to the REE. Figure 3 shows a simulation in which the initial beliefs of agents are perturbations of the linearized RE solution, the economy begins in steady state, and agents update their beliefs with a constant gain. For each variable the solid red line is the RE steady-state value, while the dashed line is the average of the actual variable realizations.

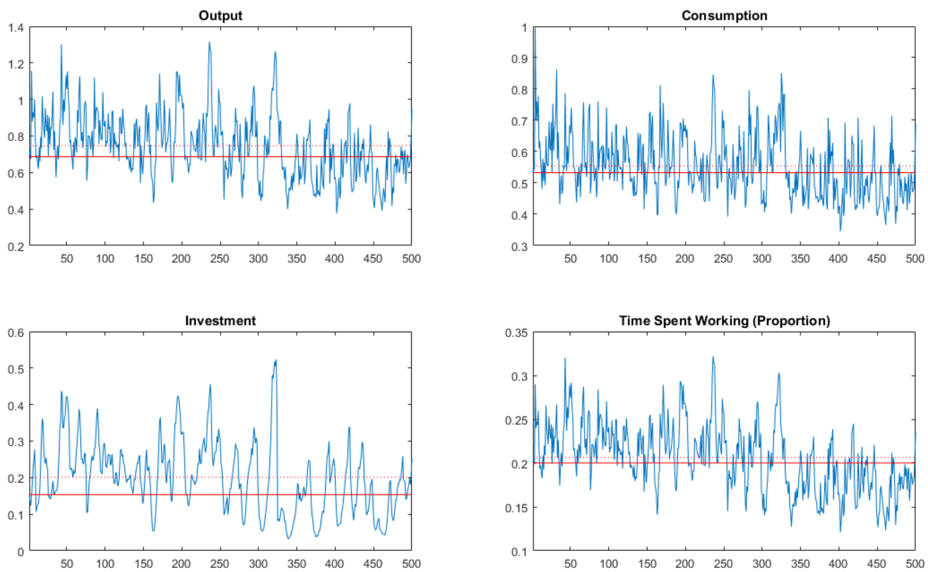


Figure 3: Model Simulation Under SP-learning

Since households employ a linear forecasting model in a nonlinear environment, convergence to the rational expectations equilibrium is not to be expected. Indeed, Figure 3 provides strong evidence that the economy converges to some RPE - the key macroeconomic variables all fluctuate around some stationary value - but the implied RPE seems quite far from the REE as judged by the respective central tendencies of the variables.

| | RE Mean | SPL Mean | pval | 95% Low | 95% High |
|---|---|---|---|---|---|
| Consumption | 0.533 | 0.568 | 0.000 | -0.036 | -0.032 |
| Labor Supply | 0.200 | 0.212 | 0.000 | -0.013 | -0.012 |
| Investment | 0.154 | 0.172 | 0.000 | -0.019 | -0.018 |
| Output | 0.687 | 0.736 | 0.000 | -0.052 | -0.046 |
| Investment SP | 0.000 | 0.060 | 0.000 | -0.061 | -0.059 |
| Capital SP | 4.250 | 4.043 | 0.000 | 0.193 | 0.219 |
| Habit-adjustment SP | -154.107 | -155.120 | 0.000 | 0.704 | 1.321 |

Table 2: t-tests for Data Generating Process, News, 230 Periods

Tables 2 and 3 perform tests for equality of means for key model variables from simulating the model 1000 times each for 230 periods under varying assumptions regarding households expectations formation mechanism and their access to news. Table 2 displays the cross-sectional means of consumption, investment, hours worked, output, and the endogenous shadow prices under RE and SP-learning when agents receive news. Generally the cross-sectional means are higher under SP-learning than under RE, and the difference is highly statistically significant: the p-value for the t-test of mean equality across expectations formation mechanisms imply rejection of the null hypothesis that the means are equal at any reasonable level of significance. Put another way, I am able to reject the null hypothesis that the data comes from the same data generating process, which of course is true. Similar findings are found for the case where households do not receive news in Table 3. Again the null hypothesis that the simulated data came from the same data generating process can be rejected at any meaningful level of significance.

While the discussion above suggests the rejection of mean equality should not be surprising, it is is worth noting each simulation is run for a relatively short amount of time. In Tables 4 and 5 I increase the simulation length to 1000 periods in order to allow the learning algorithm more time to converge. No significant changes arise: the RPE is statistically quite distinct from the REE.

A deeper inspection of the exact mechanisms driving this wedge between the RPE and REE may proceed from several points of observation. First, the data-generating process under SP-learning is inherently nonlinear. This contrasts with the REE which is obtained via a first-order approximation to the equilibrium dynamics of the model. Exploring higher-order approximations may shed light on precisely how important the nonlinearities are to the data-generating processes. Second, comparing

| | RE Mean | SPL Mean | pval | 95% Low | 95% High |
|---|---|---|---|---|---|
| Consumption | 0.534 | 0.563 | 0.000 | -0.031 | -0.027 |
| Labor Supply | 0.200 | 0.210 | 0.000 | -0.011 | -0.010 |
| Investment | 0.154 | 0.169 | 0.000 | -0.016 | -0.014 |
| Output | 0.688 | 0.728 | 0.000 | -0.043 | -0.038 |
| Investment SP | -0.000 | 0.034 | 0.000 | -0.035 | -0.034 |
| Capital SP | 4.248 | 4.068 | 0.000 | 0.167 | 0.192 |
| Habit-adjustment SP | -154.085 | -154.949 | 0.000 | 0.558 | 1.170 |

Table 3: t-tests for Data Generating Process, No News, 230 Periods

tables 2 and 3 suggest that including anticipated shocks in agents' information sets increases the size of the wedge between the RPE and REE. While the cross-sectional mean values under REE are the same whether news is included or not, those obtained under SP-learning are closer to that of REE when there is no news. This suggest something about the news pushes SP-learning household behavior away from that of their ration counterparts, which is especially interesting given the results of the previous chapter showing news shocks should not matter to the learnability of an REE.

Finally, it appears that SP-learning agents systematically behave in such a way as to cause the shadow price of investment to be positive. This will occur if household behavior is such that the gross level of investment is increasing ($I_t > I_{t-1}$) and capital is being utilized at a rate above the rational expectations steady state ($u_t > 1$), or if the opposite cases are true. It is telling that this is exactly the behavior prescribed by the model in the event that the household receives news about future productivity, and likely explains why news drives a wedge between RE and NRE: under NRE households may go through periods in which they respond too strongly (relative to RE) to news of the future.

In contrast to **?**, where reduced-form learning converged to the REE and simulated data did not differ meaningfully whether agents were rational or boundedly rational, these results suggest that the convergence to the RPE should cause simulated business cycle statistics to differ from those obtained under RE. To explore this further, Figure 6 compares simulated data from the model with and without news under the assumption of RE and SP-learning against that of actual quarterly U.S. data. The data is from **?** and covers the range 1944:Q1-2004:Q4. The logarithm of the simulated data is detrended by the Hodrick-Prescott (HP) filter using a smoothing parameter of 1600, con-

| | RE Mean | SPL Mean | pval | 95% Low | 95% High |
|---|---|---|---|---|---|
| Consumption | 0.533 | 0.635 | 0.000 | -0.103 | -0.101 |
| Labor Supply | 0.200 | 0.238 | 0.000 | -0.038 | -0.037 |
| Investment | 0.154 | 0.192 | 0.000 | -0.039 | -0.039 |
| Output | 0.687 | 0.824 | 0.000 | -0.138 | -0.136 |
| Investment SP | -0.000 | 0.117 | 0.000 | -0.118 | -0.116 |
| Capital SP | 4.251 | 3.620 | 0.000 | 0.625 | 0.638 |
| Habit-adjustment SP | -154.125 | -162.404 | 0.000 | 8.017 | 8.542 |

Table 4: t-tests for Data Generating Process, News, 1000 Periods

| | RE Mean | SPL Mean | pval | 95% Low | 95% High |
|---|---|---|---|---|---|
| Consumption | 0.533 | 0.600 | 0.000 | -0.068 | -0.066 |
| Labor Supply | 0.200 | 0.223 | 0.000 | -0.024 | -0.023 |
| Investment | 0.153 | 0.177 | 0.000 | -0.023 | -0.023 |
| Output | 0.686 | 0.774 | 0.000 | -0.088 | -0.087 |
| Investment SP | -0.000 | 0.038 | 0.000 | -0.038 | -0.038 |
| Capital SP | 4.251 | 3.797 | 0.000 | 0.449 | 0.460 |
| Habit-adjustment SP | -154.070 | -158.385 | 0.000 | 4.056 | 4.573 |

Table 5: t-tests for Data Generating Process, No News, 1000 Periods

| | US Data | RE (News) | SPL (News) | RE (No News) | SPL (No News) |
|---|---|---|---|---|---|
| $\sigma_h/\sigma_Y$ | 0.968 | 0.714 | 0.716 | 0.714 | 0.715 |
| $\sigma_I/\sigma_Y$ | 3.103 | 2.386 | 2.167 | 2.282 | 2.023 |
| $\sigma_C/\sigma_Y$ | 0.712 | 0.737 | 0.706 | 0.746 | 0.725 |
| $corr(Y,h)$ | 0.860 | 1.000 | 1.000 | 1.000 | 1.000 |
| $corr(Y,I)$ | 0.890 | 0.850 | 0.879 | 0.922 | 0.912 |
| $corr(Y,C)$ | 0.770 | 0.969 | 0.969 | 0.977 | 0.986 |

Table 6: Predicted Business Cycle Statistics

sistent with quarterly data. I conducted 1000 simulations of 230 periods each, which is the length of the US data sample. I report the statistics for the cross-sectional average standard deviations of hours worked, investment, and consumption relative to that of output across all simulations along with the correlations between output and hours worked, investment, and consumption.

US data suggests investment is the most volatile, while hours worked tends to be roughly as volatile as output and consumption is much smoother. Significant differences exist in the volatility of simulated data between RE and SP-learning, whether news shocks are included or not. All model simulations except SP-learning with news shocks suggest hours worked are much too smooth, and relative volatility of consumption under SP-learning is extremely close to the data. Only the specification with SP-learning and news shocks matches the stylized facts pertaining to the ordering of relative volatilities of investment, hours worked, and consumption. The simulated correlation between hours worked and output is unit, which does not match the data, but is expected: output and hours worked are perfectly correlated by assumption in the RBC model. The correlation between output and investment under SP-learning is closer to that of the data than under RE regardless of whether news is included or not, while both expectations formations mechanisms do a poor job capturing the correlation between output and consumption.

## 5. Conclusion

The analysis of this paper has shown the behavior implied by SP-learning households is quite distinct from that of their rational counterparts. While agents in this news-shock model can not converge to the exact REE, the process of making decisions based on linear forecasting rules of shadow prices and updating these forecasts via adaptive least squares leads agents to converge to an RPE. The difference between the implied RPE and the REE seem to be exacerbated by the inclusion of anticipated shocks in agents' information sets. Furthermore, this behavior causes the model to generate simulated moments for key business cycle statistics which are quite distinct from those obtained under RE. In some cases the model under SP-learning appears to better approximate US data, while in others the RE approach is more accurate.

One interesting extension would be to consider informational heterogeneity in this environment. **?** considers the qualitative and quantitative effects of informational heterogeneity in a Lucas-Phelps "island" model where households on a continuum of islands receive public and private signals about the productivity of aggregate and island-specific productivity. They are somewhat surprised to find that information heterogeneity does not have any serious effect on welfare or quantitative features of the model. But **?** observes this goes back to an insight going back to Hayek (1945) suggesting that markets parsimoniously convey all relevant information through prices, and thus heterogeneity by itself is not likely to represent a significant source of amplification or persistence in simulated data. However, SP-learning introduces a behavioral friction in the way agents utilize information which is likely to be exacerbated by informational frictions stemming from heterogeneity. While it is purely speculative, I suspect SP-learning (and potentially other forms of bounded rationality) may provide a simple means of making informational heterogeneity matter in DSGE models.

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## Appendix A. Solving the Model

### *Appendix A.1. Variational Arguments*

Suppose the household was acting optimally and contemplated reallocating a small amount of consumption for investment. The marginal cost of the reduction in consumption is the direct loss of utility today as well as the expected discounted value of changing the geometric average of habit-adjusted consumption. The marginal benefit is the expected discounted value of the additional investment, which affects future investment adjustment costs and increases tomorrow's capital stock. Hence the household's first-order necessary condition for $C_t$ is

$$U_{C_t}(V_t) + \frac{\partial S_t}{\partial C_t}\beta \hat{E}_t \lambda^S_{t+1} = \frac{\partial I_t}{\partial C_t}\left(\beta \hat{E}_t \lambda^I_{t+1} + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda^K_{t+1}\right) \tag{A.1}$$

Now suppose the household considered a small increase in their labor supply. The additional labor income could be used to increase investment, and hence the marginal benefit is the expected discounted value of the additional investment they can afford, which again affects future investment adjustment costs and increases tomorrow's capital stock. The marginal cost is the disutility associated with higher labor supply, and hence the household's first-order necessary condition for $h_t$ is

$$U_{h_t}(V_t) = \frac{\partial I_t}{\partial h_t}\left(\beta \hat{E}_t \lambda^I_{t+1} + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda^K_{t+1}\right) \tag{A.2}$$

If the household were to utilize more of their predetermined capital stock they would receive rental income which could be used to finance additional investment, again affecting tomorrow's capital stock and investment adjustment costs. However working capital more intensely also increase the rate of depreciation, and thus the first-order necessary condition for $u_t$ is

$$\frac{\partial K_t}{\partial u_t}\beta \hat{E}_t \lambda^K_{t+1} = \frac{\partial I_t}{\partial u_t}\left(\beta \hat{E}_t \lambda^I_{t+1} + \frac{\partial K_t}{\partial I_t}\beta \hat{E}_t \lambda^K_{t+1}\right) \tag{A.3}$$

The optimal choices of $C_t$, $h_t$, and $u_t$ pin down the value for $I_t$ through the flow budget constraint, which pins down the value for $K_t$ through the capital accumulation equation. Likewise, the optimal choice of $C_t$ generically determines the value of $S_t$. Thus equations (A.1), (A.2), and (A.3) fully describe the optimal behavior of the household as a function of their beliefs about the future shadow prices.

The expected value of these shadow prices may also be derived using variational arguments. The time $t$ value of additional $I_{t-1}$ in time $t$ holding everything else constant is the resulting change to investment adjustment costs which changes the size of tomorrow's capital stock, that is

$$\lambda_t^I = \frac{\partial K_t}{\partial I_{t-1}} \beta \hat{E}_t \lambda_{t+1}^K \tag{A.4}$$

The time $t$ value of additional preinstalled capital in time $t$ is the expected discounted value of the additional rental income (which could be invested) plus the value of directly increasing tomorrow's capital stock, that is

$$\lambda_t^K = \frac{\partial I_t}{\partial K_{t-1}} \left( \beta \hat{E}_t \lambda_{t+1}^I + \frac{\partial K_t}{\partial I_t} \beta \hat{E}_t \lambda_{t+1}^K \right) + \frac{\partial K_t}{\partial K_{t-1}} \beta \hat{E}_t \lambda_{t+1}^K \tag{A.5}$$

Finally, the time $t$ value of a small change to the predetermined level of habit-adjusted consumption in time $t$ is the direct change to utility today, as well as the expected discounted value of the resulting change in the level of habit-adjusted consumption, that is

$$\lambda_t^S = U_{S_{t-1}}(V_t) + \frac{\partial S_t}{\partial S_{t-1}} \beta \hat{E}_t \lambda_{t+1}^S \tag{A.6}$$

The solution to the household's optimization problem satisfies the FONCs for the controls given by

$$V_t^{-\sigma} + \frac{\gamma S_t}{C_t} \beta \hat{E}_t \lambda_{t+1}^S = z_t \beta \hat{E}_t \left( \lambda_{t+1}^I + \left( 1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) - \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right) \right) \lambda_{t+1}^K \right) \tag{A.7a}$$

$$V_t^{-\sigma} \left( \frac{-\theta \psi h_t^\theta S_t}{W_t h_t} \right) = z_t \beta \hat{E}_t \left( \lambda_{t+1}^I + \left( 1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) - \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right) \right) \lambda_{t+1}^K \right) \tag{A.7b}$$

$$\frac{\delta'(u_t)}{R_t} \beta \hat{E}_t \lambda_{t+1}^K = z_t \beta \hat{E}_t \left( \lambda_{t+1}^I + \left( 1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) - \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right) \right) \lambda_{t+1}^K \right) \tag{A.7c}$$

$$\lambda_t^I = \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)^2 \beta \hat{E}_t \lambda_{t+1}^K \tag{A.7d}$$

$$\lambda_t^K = (R_t u_t)\, z_t \beta \hat{E}_t \left( \lambda_{t+1}^I + \left( 1 - \Phi\left(\frac{I_t}{I_{t-1}}\right) - \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right) \right) \lambda_{t+1}^K \right) + (1 - \delta(u_t)) \beta \hat{E}_t \lambda_{t+1}^K \tag{A.7e}$$

$$\lambda_t^S = V_t^{-\sigma} \left( -\frac{(1-\gamma)\psi h_t^\theta S_t}{S_{t-1}} \right) + \beta \frac{(1-\gamma) S_t}{S_{t-1}} \hat{E}_t \lambda_{t+1}^S \tag{A.7f}$$

It can be shown that the shadow prices can be calculated as

$$\lambda_t^I = \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)^2 \left(\frac{R_t}{\delta'(u_t)}\right) V_t^{-\sigma} \left(\frac{\theta \psi h_t^\theta S_t}{W_t h_t}\right) \tag{A.8a}$$

$$\lambda_t^K = (1 + u_t \delta'(u_t) - \delta(u_t)) \left(\frac{R_t}{\delta'(u_t)}\right) V_t^{-\sigma} \left(\frac{\theta \psi h_t^\theta S_t}{W_t h_t}\right) \tag{A.8b}$$

$$\lambda_t^S = \left[ V_t^{-\sigma} \left( \frac{(\gamma - 1) S_t}{S_{t-1}} \right) \right] \left[ \left(\frac{1}{\gamma}\right) \left(\frac{C_t}{S_t}\right) \left( 1 - \frac{\gamma \psi h_t^\theta S_t}{C_t} - \frac{\theta \psi h_t^\theta S_t}{W_t} h_t \right) + \psi h_t^\theta \right] \tag{A.8c}$$

The full dynamic system under SP-learning is described by the following equations, comprised of the optimality conditions for households and firms, resource constraints, and identities:

$$V_t^{-\sigma}+\frac{\gamma S_t}{C_t}\beta\hat{E}_t\lambda_{t+1}^S = z_t\beta\hat{E}_t\left(\lambda_{t+1}^I+\left(1-\Phi\left(\frac{I_t}{I_{t-1}}\right)-\Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)\right)\lambda_{t+1}^K\right) \tag{A.9}$$

$$V_t^{-\sigma}\left(-\frac{\theta\psi h_t^\theta S_t}{W_t h_t}\right) = z_t\beta\hat{E}_t\left(\lambda_{t+1}^I+\left(1-\Phi\left(\frac{I_t}{I_{t-1}}\right)-\Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)\right)\lambda_{t+1}^K\right) \tag{A.10}$$

$$\frac{\delta'(u_t)}{R_t}\beta\hat{E}_t\lambda_{t+1}^K = z_t\beta\hat{E}_t\left(\lambda_{t+1}^I+\left(1-\Phi\left(\frac{I_t}{I_{t-1}}\right)-\Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)\right)\lambda_{t+1}^K\right) \tag{A.11}$$

$$\lambda_t^I = \Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)^2\left(\frac{R_t}{\delta'(u_t)}\right)V_t^{-\sigma}\left(\frac{\theta\psi h_t^\theta S_t}{W_t h_t}\right) \tag{A.12}$$

$$\lambda_t^K = \left(1+u_t\delta'(u_t)-\delta(u_t)\right)\left(\frac{R_t}{\delta'(u_t)}\right)V_t^{-\sigma}\left(\frac{\theta\psi h_t^\theta S_t}{W_t h_t}\right) \tag{A.13}$$

$$\lambda_t^S = V_t^{-\sigma}\left(\frac{(\gamma-1)S_t}{S_{t-1}}\right)\left[\left(\frac{C_t}{\gamma S_t}\right)\left(1-\frac{\gamma\psi h_t^\theta S_t}{C_t}-\frac{\theta\psi h_t^\theta S_t}{W_t h_t}\right)+\psi h_t^\theta\right] \tag{A.14}$$

$$K_t = \left(1-\delta\left(u_t\right)\right)K_{t-1}+I_t\left[1-\Phi\left(\frac{I_t}{I_{t-1}}\right)\right] \tag{A.15}$$

$$C_t + A_t I_t = W_t h_t + R_t\left(u_t K_{t-1}\right) \tag{A.16}$$

$$S_t = C_t^\gamma S_{t-1}^{1-\gamma} \tag{A.17}$$

$$V_t = C_t - \psi h_t^\theta S_t \tag{A.18}$$

$$Y_t = C_t + \left(\frac{1}{z_t}\right)I_t \tag{A.19}$$

$$Y_t = A_t\left(u_t K_{t-1}\right)^{\alpha_k}\left(h_t\right)^{\alpha_h}F^{1-\alpha_k-\alpha_h} \tag{A.20}$$

$$W_t = \alpha_h\frac{Y_t}{h_t} \tag{A.21}$$

$$R_t = \alpha_k\frac{Y_t}{u_t K_{t-1}} \tag{A.22}$$

and where under SPL expectations are assumed to follow the recursive least squares algorithm i.e.

$$\begin{aligned}
\hat{E}_t\lambda_t^I &= H_t^I\tilde{x}_t\\
\hat{E}_t\lambda_t^K &= H_t^K\tilde{x}_t\\
\hat{E}_t\lambda_t^S &= H_t^S\tilde{x}_t\\
R_{H,t} &= R_{H,t-1}+g_t\left(\tilde{x}_{t-1}\tilde{x}_{t-1}'-R_{H,t-1}\right)\\
H_t &= H_{t-1}+g_t R_{H,t}^{-1}\tilde{x}_{t-1}\left(\lambda_{t-1}-H_{t-1}'\tilde{x}_{t-1}\right)'
\end{aligned}$$

*Appendix A.2. Steady State*

These equations can be used to solve for a non-stochastic rational expectations steady state. I normalize the steady-state values of the driving processes to be unit i.e. impose $A = 1$ and $z = 1$, and assume households spend 20% of their total available time engaged in labor i.e. $h = 0.2$. The parameter $\delta_1$ of the depreciation function $\delta(u_t) = \delta_0 + \delta_1(u_t - u) + \delta_2/2(u_t - u)^2$ can be used to calibrate the steady-state value of capacity utilization which I normalize to be unit i.e. $u = 1$. To see how, first note that in steady state households will by definition have no value for additional gross investment i.e. from equation A.7d we have that $\lambda^I = 0$. Then optimal decision making in capacity utilization depends only on the value of pre-installed capital; this is described by equations A.7c and A.7e and can be shown to imply

$$\delta_1 = \frac{\frac{1}{\beta} - (1 - \delta_0)}{u}$$

This can be used in A.7c to show $R = \delta_1/z$; intuitively, the rental rate is the marginal benefit of effective capital and $\delta_1/z$ is the marginal cost (in terms of consumption goods) in steady state. Furthermore, in steady state the rate of net investment must exactly replace depreciated capital; the law of motion for capital in A.7e implies

$$\begin{aligned} I &= \delta_0 K \\ \Rightarrow \frac{I}{K} &= \delta_0 \end{aligned}$$

while the factor market determination of the rental rate in A.22 implies effective capital's share of income in steady-state is $\alpha_k$ i.e.

$$\begin{aligned} \frac{R(uK)}{Y} &= \alpha_k \\ \Rightarrow = \frac{Y}{K} &= \left(\frac{1}{\alpha_k}\right) Ru \end{aligned}$$

The production function A.20 can then be solved for the steady-state level of capital

$$\begin{aligned} Y &= A\,(uK)^{\alpha_k}\,(h)^{\alpha_h}\,(F)^{1-\alpha_k-\alpha_h} \\ \Rightarrow K &= \left[\left(A\,(u)^{\alpha_k}\,(h)^{\alpha_h}\,(F)^{1-\alpha_k-\alpha_h}\right)^{-1}\left(\frac{Y}{K}\right)\right]^{\frac{1}{1-\alpha_k}} \end{aligned}$$

which can be used with the previous expressions for $I/K$ and $Y/K$ to solve for the steady-state levels of gross investment and output i.e.

$$\begin{aligned} I &= \left(\frac{I}{K}\right) K \\ Y &= \left(\frac{Y}{K}\right) K \end{aligned}$$

Armed with expressions for the levels of output and gross investment, the aggregate resource constraint A.19 pins down the level of consumption[8]

$$Y = C + \left(\frac{1}{z}\right) I$$
$$\Rightarrow C = Y - \left(\frac{1}{z}\right) I$$

furthermore, the steady-state wage rate is given directly by A.21 is

$$W = \alpha_h \frac{Y}{h}$$

Past habit-adjusted consumption stock from equation A.17 implies

$$S = C^{\gamma} S^{1-\gamma}$$
$$\Rightarrow S = C$$

It remains to derive the steady state values for the shadow price of capital, the shadow price of the habit-adjusted consumption stock, the argument of the utility function, and to ensure that the disutility of labor parameter is calibrated appropriately to imply labor supply is indeed equal to 0.2. To accomplish this, first combine the FONCs for consumption and labor supply given by equations A.9 and A.10 by substituting out the shadow price of household wealth (i.e. the right-hand side of each equation); then use the resulting expression along with that describing the shadow price of the habit-adjusted consumption stock from equation A.14 to solve for the disutility of labor parameter $\psi$, which is given by

$$\psi = \left[\left(\frac{\left(\beta\left(\gamma-1\right)-1\right)\left(\frac{\theta S}{Wh}\right)-\gamma\beta\left(\gamma-1\right)}{1+\beta\left(\gamma-1\right)}+\gamma\right)h^{\theta}\right]^{-1}$$

[8]Of course one can also use the household's flow budget constraint described in equation A.16. This, together with equations A.21 and equation A.22 implying that labor and effective capital's income in steady-state is $\alpha_h Y$ and $\alpha_k Y$, respectively, implies

$$C + \left(\frac{1}{z}\right) I = Wh + R(uK)$$
$$\Rightarrow C = Wh + R(uK) - \left(\frac{1}{z}\right) I$$
$$= \alpha_h Y + \alpha_k Y - \left(\frac{1}{z}\right) I$$
$$= Y - \left(\frac{1}{z}\right) I$$

This can then be used to solve for the argument of the period utility function directly from equation A.18

$$V = C - \psi h^{\theta} S$$

which can then be used to calculate the shadow prices of habit-adjusted consumption stock and capital directly from equations A.14 and A.13

$$\lambda^S = V^{-\sigma}(\gamma - 1)\left[\left(\frac{1}{\gamma}\right)\left(1 - \gamma\psi h^{\theta} - \frac{\theta\psi h^{\theta} S}{Wh}\right) + \psi h^{\theta}\right]$$
$$\lambda^K = (1 + u\delta_1 - \delta_0)\left(\frac{R}{\delta_1}\right)V^{-\sigma}\left(\frac{\theta\psi h^{\theta} S}{Wh}\right)$$

The entire collection of steady-state values and implied structural parameters in nested order is thus

$$z = 1 \tag{A.23a}$$
$$A = 1 \tag{A.23b}$$
$$h = 0.2 \tag{A.23c}$$
$$l = 1 - h \tag{A.23d}$$
$$\lambda^I = 0 \tag{A.23e}$$
$$u = 1 \tag{A.23f}$$
$$\delta_1 = \frac{\frac{1}{\beta} - (1 - \delta_0)}{u} \tag{A.23g}$$
$$R = \delta_1 / z \tag{A.23h}$$
$$\frac{I}{K} = \delta_0 \tag{A.23i}$$
$$\frac{Y}{K} = \left(\frac{1}{\alpha_k}\right) Ru \tag{A.23j}$$
$$K = \left[\left(A\,(u)^{\alpha_k}\,(h)^{\alpha_h}\,(F)^{1-\alpha_k-\alpha_h}\right)^{-1}\left(\frac{Y}{K}\right)\right]^{\frac{1}{1-\alpha_k}} \tag{A.23k}$$
$$I = \left(\frac{I}{K}\right) K \tag{A.23l}$$
$$Y = \left(\frac{Y}{K}\right) K \tag{A.23m}$$
$$W = \alpha_h \frac{Y}{h} \tag{A.23n}$$

$$C = Y - \left(\frac{1}{z}\right) I \tag{A.23o}$$

$$S = C \tag{A.23p}$$

$$\psi = \left[\left(\frac{\left(\beta\left(\gamma-1\right)-1\right)\left(\frac{\theta S}{Wh}\right)-\gamma\beta\left(\gamma-1\right)}{1+\beta\left(\gamma-1\right)}+\gamma\right)h^{\theta}\right]^{-1} \tag{A.23q}$$

$$V = C - \psi h^{\theta} S \tag{A.23r}$$

$$\lambda^{S} = V^{-\sigma}\left(\gamma-1\right)\left[\left(\frac{1}{\gamma}\right)\left(1-\gamma\psi h^{\theta}-\frac{\theta\psi h^{\theta}S}{Wh}\right)+\psi h^{\theta}\right] \tag{A.23s}$$

$$\lambda^{K} = \left(1+u\delta_1-\delta_0\right)\left(\frac{R}{\delta_1}\right)V^{-\sigma}\left(\frac{\theta\psi h^{\theta}S}{Wh}\right) \tag{A.23t}$$

## Appendix B. Recursive Representation of News-Shocks

We can write the evolution of any exogenous variable indexed by $w$ as

$$\ln(w_t) = \rho_w \ln(w_{t-1}) + \varepsilon^0_{w,t} \tag{B.1}$$

$$\varepsilon_{w,t} = \varphi_w \varepsilon_{w,t-1} + M_w \nu_{w,t} \tag{B.2}$$

where $\varepsilon_{w,t} = \left(\varepsilon^0_{w,t}, \varepsilon^1_{w,t}, .., \varepsilon^n_{w,t}\right)$ a vector of auxiliary state variables which carry the news shocks through time, $\varphi$ a lower-shift matrix with 1's on the super-diagonal and zeros elsewhere, $\nu_{w,t} = \left(\nu^0_{w,t}, \nu^1_{w,t}, ..., \nu^n_{w,t}\right)'$ a vector of (anticipated and unanticipated) shocks, and $M_w = \left(\left[\nu^0_{w,t} \in \mathcal{I}_t\right], \left[\nu^1_{w,t-1} \in \mathcal{I}_t\right], ..., \left[\nu^n_{w,t-n}\right.\right.$ a row-vector of 1's and 0's respecting the specific assumptions regarding the information obtained by households.