INTERNATIONAL ECONOMIC REVIEW
Vol. 49, No. 1, February 2008

ANTICIPATED UTILITY AND RATIONAL EXPECTATIONS AS APPROXIMATIONS OF BAYESIAN DECISION MAKING*

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We study a Markov decision problem with unknown transition probabilities. We compute the exact Bayesian decision rule and compare it with two approximations. The first is an infinite-history, rational-expectations approximation that assumes that the decision maker knows the transition probabilities. The second is a version of Kreps' anticipated-utility model in which decision makers update using Bayes' law but optimize in a way that is myopic with respect to their updating of probabilities. For several consumption-smoothing examples, the anticipated-utility approximation outperforms the rational expectations approximation. The rational expectations approximation misrepresents the market price of risk.

1. INTRODUCTION

A standard model of intertemporal choice assumes time-invariant beliefs about state transitions. Let s_t, x_t, and ε_t represent vectors of state variables, controls, and shocks, respectively, and suppose that an agent maximizes a discounted sum of utilities

$$E_0 \sum_{t=0}^{\infty} \beta^t r(s_t, x_t), \tag{1}$$

subject to a transition equation

$$s_{t+1} = g(s_t, x_t, \varepsilon_{t+1}). \tag{2}$$

The utility function is assumed to be concave, the constraint set is convex and compact, and the decision maker knows all the parameters of the model. With these assumptions, the maximization problem can be cast as a dynamic program. The Bellman equation is

$$V(s) = \max_x \{r(s, x) + \beta E(V[g(s, x, \varepsilon) \,|\, s])\} \tag{3}$$

and the optimal decision rule is $x = h(s)$.

* Manuscript received March 2004; revised November 2006.

[1] For comments and suggestions, we thank Lars Hansen, Narayana Kocherlakota, Frank Schorfheide, three referees, and seminar participants at Stanford and the CFS Summer School on "Learning and Macroeconomics." Sargent thanks the National Science Foundation for research support. Please address correspondence to: Thomas J. Sargent, Department of Economics, New York University, 19 W. 4th Street, 6th Floor, New York, NY 10012. Telephone: 212-998-3548. E-mail: *thomas.sargent@nyu.edu*.

The assumption that agents know the parameters of the model means that learning has been completed. Accordingly, the standard model is most useful for studying mature economies in which agents have already acquired so much information that new observations have no effects on beliefs about state transition probabilities. The no-learning assumption is a harmless and convenient simplification in cases like this. In other circumstances, however, learning can be a more prominent feature of the problem. Examples include transition economies in which equilibria are punctuated by big changes in economic or political institutions, and also economies in which government policy makers adapt their policy rules as their understanding of the structure evolves. Changing beliefs are likely to be more important in cases like these, and their effects on outcomes would be lost if learning were neglected.

One way to model dynamic choice with learning is to withdraw knowledge of the transition Equation (2) and replace it with an estimated transition equation

$$s_{t+1} = g(s_t, x_t, \varepsilon_{t+1}, \hat{\theta}_t), \tag{4}$$

where $\hat{\theta}_t$ represents estimates of the parameters governing (2) conditional on data through date t. At each date t, one could solve the dynamic program as before using the estimated transition Equation (4) instead of (2), still pretending that it is time-invariant when solving the time t Bellman equation. This delivers a time-varying decision rule

$$u_t = h(s_t, \hat{\theta}_t) \tag{5}$$

that depends on date-t beliefs about how the state vector evolves. As beliefs are updated, so too are decision rules. In related contexts, Kreps (1998) recommends such an approach and refers to it as an "anticipated-utility" model. This modeling strategy is used in much of the macroeconomics literature on learning.

This anticipated utility approach is of two minds in the way it treats $\hat{\theta}_t$. Agents treat parameters as random variables when they learn but as constants when they formulate decisions. Looking backward, agents can see how their beliefs have evolved, but looking forward they act as if beliefs will remain unchanged forever. Agents are eager to learn at the beginning of each period, but their decisions reflect a pretence that this is the last time they will update beliefs, a pretence that is falsified at the beginning of every subsequent period.

In contrast, a Bayesian would treat θ as a random variable both for learning and decision making and so would recognize that beliefs will continue to evolve going forward in time. He would recognize this source of uncertainty when formulating decision rules.

Often, this is easier said than done. In many interesting applications, especially those in macroeconomics, a full Bayesian procedure is too complicated to be implemented. Macroeconomists might justify anticipated-utility models as an approximation to a correctly formulated Bayesian decision problem. The computational cost of the full Bayesian calculation makes an anticipated utility approximation appealing, but the appeal would be more compelling if one could also show that anticipated-utility decisions approximate Bayesian decisions well. As far as we know, no one has assessed the quality of the approximation, mainly because one has to calculate Bayesian choices in order to make the comparison, and that is hard to do.

In this article, we develop a laboratory for exploring Bayesian and anticipated-utility choices. Our objectives are to examine how well anticipated-utility choices approximate Bayesian decisions and to compare that approximation with a no-learning or infinite-history rational expectations (IHRE) approximation.[2] We show how to find the exact Bayesian solution in a simple example rigged for maximum tractability, and then we compare Bayesian, anticipated-utility, and infinite-history choices. The Bayesian approach turns out to be simpler than we first imagined, making us hopeful about extensions to more realistic problems. We also find that the anticipated-utility model often provides an excellent approximation, which makes us more confident about its application as well.

Our key findings are: (1) in our permanent income economy, the anticipated utility solution does a good job of approximating the Bayesian solution, especially when the utility function is such that precautionary motives are muted; but (2) the full information rational expectations solution is a much worse approximation; (3) the contribution of learning to the market price of risk is well captured by an anticipated utility approximation, but completely missed by the full information rational expectations approximation.

There is a vast literature on learning in economics, of which Marimon (1996) provides a comprehensive overview. Kiefer and Nyarko (1995) and Blume and Easley (1995) survey Bayesian approaches, and Evans and Honkapohja (2001) and Sargent (1993, 1999) describe what we call anticipated-utility models. Papers that examine Bayesian dynamic programs with parameter uncertainty include Prescott (1972), Easley and Kiefer (1988), Kiefer and Nyarko (1989), and El-Gamal and Sundaram (1993). Relative to that literature, our contribution is to assess the quality of two approximations to Bayesian choices. Our analysis is closely related to that of Guidolin and Timmerman (2007), who study the properties of asset prices under alternative learning schemes.

The remainder of the discussion is organized as follows. Section 2 discusses a key step in the Bayesian treatment, namely, how to expand the state vector to encompass learning and how to derive the transition equation for the expanded state. It also describes our laboratory, a finite-horizon version of the permanent-income model for consumption. In our laboratory, agents know everything except parameters that govern the exogenous evolution of labor income. Because agents' actions do not influence the flow of income, they have no incentive to experiment. A pair of companion papers—by Cogley et al. (2005, 2007)—examine another laboratory in which incentives to experiment are active.

[2] We sometimes abbreviate "infinite-history rational expectations" simply as "rational expectations." This is an abuse of terminology because Bayesian expectations are rational when there is partial information. The distinction really concerns how much information agents have.

Section 3 characterizes Bayesian decisions and describes the two approximations, based on anticipated utility and infinite-history rational expectations, respectively. The former involves learning but neglects parameter uncertainty, whereas the latter attributes full information to agents and abstracts from learning altogether. Section 4 provides details about our simulations, and Sections 5–7 assess the quality of the approximations for three objects: consumption, asset prices, and the market price of risk. We conclude with an assessment of the relative merits of anticipated utility and infinite-history rational expectations as approximation strategies.

2. THE MODEL ENVIRONMENT

Agents know the functional form of $g(s_t, x_t, \epsilon_{t+1}\,; \theta)$, but they do not know θ. They update beliefs about θ via Bayes' theorem. To keep matters simple, we specialize the model by assuming that s_t is observable and exogenous and that there is no uncertainty about how controls affect future states. With these assumptions, the transition density simplifies to $f(s_{t+1}\,|\,s_t, \theta)$. In addition, the history of s_t—which we denote s^t—contains all the information relevant for learning. Conditional on s^t, all other variables are redundant for updating beliefs.

We also limit attention to probability models in which the information in s^t can be summarized by a finite-dimensional vector of sufficient statistics ζ_t.[3] These statistics encode the information relevant for learning. A big class of probability models satisfies this condition; for example, all members of the exponential family possess a finite-dimensional vector of sufficient statistics. We also limit attention to models in which the learning problem can be cast in terms of a conjugate prior and likelihood, so that the posterior distribution can be expressed analytically. This limitation is more restrictive, but many interesting models can be set up in this way.[4]

Our strategy is to append the vector of sufficient statistics ζ_t to the natural state vector s_t, $S_t = [s_t', \zeta_t']'$, and then to derive a transition density for the expanded state vector S_t

$$f(S_{t+1}\,|\,S_t). \tag{6}$$

If (6) can be expressed as a time-invariant function, then the Bayesian decision rule can be derived by dynamic programming. We first consider a general form for (6) and then describe a special case that we use in our simulations.

2.1. *The Transition Density for the Expanded State.*

To begin, factor (6) as

$$\begin{aligned} f(S_{t+1}\,|\,S_t) &= f(s_{t+1}, \zeta_{t+1}\,|\,s_t, \zeta_t), \\ &= f(\zeta_{t+1}\,|\,s_{t+1}, s_t, \zeta_t)\; f(s_{t+1}\,|\,s_t, \zeta_t). \end{aligned} \tag{7}$$

[3] For tractability, it is important that the dimension of ζ_t is constant.

[4] See Gelman et al. (2000, pp. 36–37), for an explanation of conjugacy. They also provide examples of nonconjugate priors.

The first term on the right side, $f(\zeta_{t+1} \mid s_{t+1}, s_t, \zeta_t)$, is the density for the statistics ζ_{t+1} conditional on their previous values along with realizations of the natural states. This term describes how the sufficient statistics are updated in light of observations on s_{t+1} and s_t. The updating rules are deterministic given the conditioning information, so we can express the updated value as a function $\zeta(s_{t+1}, S_t)$. That ζ_{t+1} is a deterministic function of (s_{t+1}, S_t) means that this term is a delta function that assigns unit probability mass to the updated value

$$f(\zeta_{t+1} \mid s_{t+1}, s_t, \zeta_t) = \delta(\zeta(s_{t+1}, S_t)). \tag{8}$$

The particular form of the updating rule $\zeta(s_{t+1}, S_t)$ follows from Bayes' theorem and depends on how the agent's priors and the conditional likelihood are specified. Analytical expressions for the updating formulas are available so long as we work within a conjugate family.

The second term on the right-hand side of (7) is a posterior predictive density. This term can be expanded as

$$\begin{aligned} f(s_{t+1} \mid s_t, \zeta_t) &= \int f(s_{t+1} \mid s_t, \zeta_t, \theta)\ f(\theta \mid s_t, \zeta_t)\, d\theta \\ &= \int f(s_{t+1} \mid s_t, \theta)\ f(\theta \mid s_t, \zeta_t)\, d\theta. \end{aligned} \tag{9}$$

On the second line, the first term in the integrand, $f(s_{t+1} \mid s_t, \theta)$, is the natural transition equation with which we started. Here it is conditioned on a particular value of θ, which makes ζ_t redundant as a conditioning variable. This is a conditional likelihood function for s_{t+1}.

Of course, θ is unknown, and the second term in the integrand summarizes beliefs about it. The term $f(\theta \mid s_t, \zeta_t)$ is the posterior for θ conditioned on information available through date t. Here s_t is redundant because ζ_t is a sufficient statistic for the history s^t. Hence this term can also be expressed as $f(\theta \mid \zeta_t) = f(\theta \mid s^t)$. Again, so long as we work within a conjugate family, expressions for $f(\theta \mid \zeta_t)$ will be available in closed form.

Equation (9) represents beliefs about s_{t+1} conditioned on information available through date t. Analytically solving this integral is often hard, but there are a number of cases in which it can be done. An example is given below. In other cases, the integral can be solved numerically, but we do not pursue such calculations here.

Equation (9) is the second piece needed for (6). Combining this with the deterministic updating relation (8) produces an expression for (6)

$$f(S_{t+1} \mid S_t) = \delta(\zeta(s_{t+1}, S_t)) \int f(s_{t+1} \mid s_t, \theta)\ f(\theta \mid \zeta_t)\, d\theta. \tag{10}$$

As promised, this is a time-invariant function of S_t, suitable for dynamic programs.

2.2. *A Beta-Binomial Model for a Two-State Markov Process.* In the remainder of the article, we specialize to a two-state Markov process with unknown transition probabilities.[5] Suppose that state s_t takes two values and that its evolution is governed by the following Markov transition matrix:

$$\Pi = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \tag{11}$$

where $\Pi_{ij} = pr(s_{t+1} = j \mid s_t = i)$. The states are observable, but the transition probabilities are unknown. Our agent learns about them using Bayes' theorem.

Because the states are discrete and take on two values, we adopt a beta-binomial probability model. We assume the agent has independent beta priors over (p, q) and that the likelihood function for a batch of data s^t is proportional to the product of binomial densities. It follows that the posteriors for p and q are independent and have the beta form (Gelman et al., 2000, pp. 481–483)

$$f(p \mid s^t) = beta\left(n_t^{11}, n_t^{12}\right), \quad f(q \mid s^t) = beta\left(n_t^{22}, n_t^{21}\right). \tag{12}$$

The variable n_t^{ij} is a counter that records the number of transitions from state i to j experienced through date t. The parameter n_0^{ij} represents prior beliefs about the frequency of transitions, which may, for example, come from a training sample, and $(n_t^{ij} - n_0^{ij})$ is the number of transitions observed in the sample.

2.2.1. *The augmented state space.* In the notation of the last subsection, Equation (11) is the natural transition density, and the transition probabilities (p, q) are the unknown parameters. The vector of counters $n_t = [n_t^{11}, n_t^{12}, n_t^{21}, n_t^{22}]$ is a sufficient statistic for the transition probabilities. Thus, the expanded state S_t consists of the Markov variable s_t along with the counters n_t, $S_t = [s_t, n_t]$.

For finite-horizon economies, S_t is a finite-state Markov random variable. At any date there are only two possible values for s_t plus a finite number of permutations of n_t. The nodes for S_t consist of each value of s_t combined with every possible permutation of n_t that can be attained through period t. By appending all the permutations of n_t to the two values of s_t, we transform the two-state process for s_t into a large multistate process for S_t. Also notice that the number of possible values for S_t grows with t because the number of permutations of n_t increases with t.

Our goal is to derive a matrix of transition probabilities that maps the probability of moving from any element of S_t to any element of S_{t+1}

$$P_{t,t+1}^{lm} = pr[S_{t+1} = m \mid S_t = l]. \tag{13}$$

The expanded state S_t is a Markov random variable because the probability that $S_{t+1} = m$ conditional on the past history S^t depends on a single lag S_t. The process

[5] This can be extended to a multistate-Markov process by adopting a Dirichlet-multinomial probability model.

is not homogenous, however, because of the expansion of the number of nodes. The matrix $P_{t,t+1}$ is rectangular, not square, and its dimensions increase with t. It follows that the transition probabilities cannot be independent of t. The elements of $P_{t,t+1}$ are, however, time-invariant functions of S_t.

To derive $P^{lm}_{t,t+1}$, we first deduce the conditional predictive density, $f(s_{t+1} \mid s_t, n_t)$, and then incorporate how the counters are updated conditional on the passage from s_t to s_{t+1}. For a given s_t and n_t, there are two possible outcomes for s_{t+1}, and the posterior predictive density assigns probabilities to them. The first of these is $pr(s_{t+1} = 1 \mid s_t, n_t)$, which can be calculated as

$$pr(s_{t+1} = 1 \mid s_t, n_t) = \begin{cases} \int pf(p|s_t, n_t)\, dp & \text{if } s_t = 1, \\ \int (1-q) f(q|s_t, n_t)\, dq & \text{if } s_t = 2. \end{cases} \tag{14}$$

This is the posterior mean of p if $s_t = 1$ and one minus the posterior mean of q if $s_t = 2$. After integrating with respect to the beta density, we find the intuitive result that the posterior mean of p is the fraction of times the system stays in state 1 when it begins there, counting both prior and observed transitions:

$$E_t p = n_t^{11} \big/ \big(n_t^{11} + n_t^{12}\big) \equiv \hat{p}_t. \tag{15}$$

Similarly, the posterior mean of q is $\hat{q}_t = n_t^{22}/(n_t^{21} + n_t^{22})$. The second element of the posterior predictive density is $pr(s_{t+1} = 2 \mid s_t, n_t)$, which can be found by following the same steps. This is equal to

$$pr(s_{t+1} = 2 \mid s_t, n_t) = \begin{cases} (1 - \hat{p}_t) & \text{if } s_t = 1, \\ \hat{q}_t & \text{if } s_t = 2. \end{cases} \tag{16}$$

To represent the joint transition density for s_{t+1} and n_{t+1}, we combine these marginal predictive probabilities with the deterministic conditional relationship that governs how the counters are updated in light of the news in s_{t+1}. For each of the four possible s-transitions, one element of n_t is incremented by 1, and the others remain constant. The first element n^{11}_{t+1} increases by 1 when passing from $s_t = 1$ to $s_{t+1} = 1$, the second element n^{12}_{t+1} increases by 1 when passing from $s_t =$ 1 to $s_{t+1} = 2$, and so on. The probabilities associated with these joint transitions are

$$\begin{aligned} &pr(s_{t+1} = 1, n_{t+1} = n_t + [1\ 0\ 0\ 0]' \mid s_t = 1, n_t) = \hat{p}_t, \\ &pr(s_{t+1} = 1, n_{t+1} = n_t + [0\ 1\ 0\ 0]' \mid s_t = 2, n_t) = 1 - \hat{q}_t, \\ &pr(s_{t+1} = 2, n_{t+1} = n_t + [0\ 0\ 1\ 0]' \mid s_t = 1, n_t) = 1 - \hat{p}_t, \\ &pr(s_{t+1} = 2, n_{t+1} = n_t + [0\ 0\ 0\ 1]' \mid s_t = 2, n_t) = \hat{q}_t. \end{aligned} \tag{17}$$

To form the matrix $P_{t,t+1}$, imagine iterating through the rows of S_t and S_{t+1}, matching each element of the former with each element of the latter. There are two kinds of matches, admissible and inadmissible. A match is admissible if the updating of n_{t+1} is consistent with the movement from s_t to s_{t+1}; in that case

$\delta(\zeta(s_{t+1}, S_t)) = 1$ and the joint probability can be read from those in Equation (17). All other matches correspond to pairs in which the change in s_{t+1} is inconsistent with the change in n_{t+1}. Accordingly, $\delta(\zeta(s_{t+1}, S_t)) = 0$, and the joint transition probability is zero.

This delivers one-step transition matrices for S_t. As promised, the transition probabilities depend on a single lag of S_t, and the elements of $P_{t,t+1}$ are time-invariant functions of the state. Armed with $P_{t,t+1}$, one can solve finite-horizon dynamic programs in the usual way. The primary challenge is coping with the curse of dimensionality.[6]

Infinite-horizon problems are more difficult because the counters are unbounded, violating the assumption that the state space is compact. In this example, however, the Bayesian consistency theorem holds, and $\hat{p}_t$ and $\hat{q}_t$ converge in probability to p and q. Since beliefs settle down, the effects of learning eventually become negligible. Therefore, one way to approximate the solution of an infinite-horizon model would be to choose a horizon large enough that further learning can be neglected. Then adopt the finite-horizon approach to approximate outcomes until that date and a no-learning model for outcomes thereafter. Terminal conditions for the finite-horizon calculation can be derived from the value function and decision rules for the no-learning model. Thus, at least in principle, one could approximate an infinite-horizon problem by decomposing it into two segments, a finite-horizon problem in which learning matters plus an infinite-horizon remainder in which beliefs have settled down.

We do not pursue this idea further. All the calculations reported below involve finite-horizon economies.

2.3. *A Finite-Horizon Permanent-Income Model.* Having cast the Bayesian problem in a form suitable for dynamic programming, we can compare Bayesian outcomes with the two approximations. As a laboratory for this comparison, we adopt a finite-horizon general equilibrium version of Hall's (1978) permanent income model. We chose the permanent income model because it is a canonical example of dynamic choice. We assume a finite horizon because this keeps the state space compact.

Imagine an economy that operates for T periods. Aggregate income is exogenous and follows a two-state Markov process. A representative agent knows the value of income in each state but not the transition probabilities. The representative agent updates beliefs about the transition probabilities via Bayes' law.

The representative agent formulates a consumption plan to maximize lifetime utility

$$V_0 = E_0 \sum_{t=0}^{T} \beta^t u(c_t). \tag{18}$$

[6] Appendix A.1 describes an algorithm for constructing the state space. Its chief virtue is that it finds a parsimonious representation for $\{S_t\}_{t=1}^{T}$.

Three assets are available for smoothing consumption. One is a linear storage technology

$$A_{t+1} = R(A_t + i_t), \tag{19}$$

where A_t represents the amount in storage at the beginning of period t and i_t is the net amount added to it. There are no shocks here, so storage is a risk-free investment paying a constant gross return R, which we assume is the inverse of the subjective discount factor, $R = \beta^{-1}$.

The consumer also trades two Arrow securities at each date. Each pays one unit of consumption at $t+1$ in the event that a particular income state is realized. They are purely inside assets, however, and are in zero net supply. After substituting $i_t = y_t - c_t$, we can rewrite the aggregate resource constraint as

$$A_{t+1} = R(A_t + y_t - c_t). \tag{20}$$

To find the equilibrium, we solve a planning problem for the consumption allocation and then calculate the Arrow prices implied by that allocation. Assuming absence of arbitrage, returns on the Arrow securities must satisfy the Euler equation

$$E_t \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{j,t+1} = 1, \tag{21}$$

where $R_{j,t+1}$ is the gross return on the Arrow security that pays off in income state j. Because $R_{j,t+1}$ is the inverse of the price of that security, the Euler Equation pins down Arrow prices as a function of consumption

$$Q_{jt} = \beta \, \frac{u'[c_{t+1}(s_{t+1} = j)]}{u'[c_t(s_t = i)]} \, pr(s_{t+1} = j \mid s_t = i). \tag{22}$$

A portfolio consisting of one share of every Arrow security replicates the risk-free investment, so Arrow prices also satisfy $\sum_j Q_{jt} = R^{-1}$.

We examine two versions of the PIH to explore how the quality of the approximation depends on the strength of precautionary motives. In one version, period utility is quadratic[7]

$$u(c_t) = -1/2(c_t - c_0)^2. \tag{23}$$

In this case, certainty equivalence holds and precautionary motives are absent. This is an important benchmark for macroeconomics because many macroeconomic models exploit certainty equivalence, either by specifying quadratic preferences or by adopting an approximation that imposes certainty equivalence.

[7] In our simulations, the bliss point c_0 is high enough that consumption never approaches it.

Departures from certainty equivalence can be important for asset pricing and welfare comparisons. Accordingly, we also consider a version of the PIH in which period utility is isoelastic

$$u(c_t) = c_t^{1-\alpha}/(1-\alpha). \tag{24}$$

The parameter α is the coefficient of relative risk aversion; in the simulations reported below, we set α equal to 2, 5, 10, or 20. Values around 2 are probably most relevant for macroeconomics, but higher values are sometimes adopted in asset-pricing models. This specification activates precautionary motives.[8]

Three remarks about our laboratory are in order. First, there is no incentive for experimentation because the consumer understands perfectly how controls affect states. The consumer just learns about parameters that govern the evolution of exogenous states, and his decisions have no bearing on the flow of information. The difference between Bayesian and approximate decision rules is likely to be more pronounced when agents have an incentive to experiment. For quantitative studies of optimal experimentation, see Prescott (1972), Wieland (2000a, 2000b), Beck and Wieland (2002), and Cogley et al. (2005, 2007), among others.[9]

Second, information and priors are homogenous and agents know it, and prices convey no information about hidden parameters over and above that contained in realizations of labor income. Thus, there is no need to solve a signal-extraction problem or forecast the forecasts of others. These complications could be important in other settings.

Third, because this is a finite-state economy, expected utility exists even though period utility is unbounded. Existence is not guaranteed in continuous-state economies such as that studied by Geweke (2001). The anticipated-utility approximation is a tactic for trimming the tails of Bayesian predictive densities, and that could matter more for decision rules in economies like his.

3. OPTIMAL DECISIONS AND TWO APPROXIMATIONS

For quadratic preferences, the consumption decision rule can be found by backward induction and expressed analytically. As in the infinite-horizon version of the PIH, consumption depends on wealth plus the expected present value of future labor income

$$c_{T-h} = \left[\sum_{m=0}^{h} R^{-m}\right]^{-1} \left[A_{T-h} + E_{T-h} \sum_{m=0}^{h} R^{-m} y_{T-h+m}\right]. \tag{25}$$

[8] The chief difference between quadratic and *CRRA* utility is not that one involves risk aversion and the other does not, for consumers with quadratic preferences are also risk averse. The main difference is that *CRRA* preferences involve precautionary motives (which requires nonzero third derivatives of utility functions) whereas quadratic utility does not.

[9] A nonexperimental setting is a useful benchmark for thinking about atomistic agents who want to learn about aggregate relationships. Because an atomistic agent in a competitive market cannot influence aggregate outcomes, he has no incentive to experiment in order to learn about aggregate relations.

The primary difference from the infinite-horizon model is that the marginal propensity to consume out of wealth depends on age. In the infinite-horizon model, consumption equals the annuity value of wealth. In the finite-horizon version, consumption equals the annuity value plus a fraction of the principal. That fraction increases as the consumer ages.

To solve for consumption, we just need to calculate the present value of labor income. How that is done depends on how expectations are modeled. Bayesian expectations exploit the conditional expectations operator, whereas the approximations each take a shortcut.

Consumption decisions involve multistep forecasts, so we want the m-step transition matrix $P^{ij}_{t,t+m} = pr[S_{t+m} = j \mid S_t = i]$. A Bayesian exploits the Chapman–Kolmogorov Equation to express the m-step transition matrix $P_{t,t+m}$ as the product of one-step transition matrices (Feller, 1968, p. 421)

$$P_{t,t+m} = P_{t,t+1} P_{t+1,t+2} \ldots P_{t+m-1,t+m}. \tag{26}$$

Armed with this formula, multistep forecasts of labor income are calculated as

$$E^B(y_{t+m} \mid S_t = i) = \sum_j P_{t,t+m}(i,\, j) S_{t+m}(j, 1), \tag{27}$$

with the convention that labor income is recorded in the first column of S_{t+m}.[10]

For the two approximations, the consumption decision rule still takes the form of (25), but with different expectations operators. For the anticipated-utility approximation, we update estimates of the transition probabilities in each period by plugging in the current counters, arriving at an estimate Π_t of the transition matrix Π. Looking forward in time, we simplify by disregarding how future realizations of labor income alter the counters and future one-step transition probabilities. In other words, we pretend the chain is homogenous, with constant transition probabilities going forward in time. If the chain were homogenous, the m-step transition density would simplify to

$$\Pi_{t,t+m} = \Pi_t \Pi_t \ldots \Pi_t = \Pi_t^m. \tag{28}$$

Accordingly, anticipated-utility forecasts of labor income are given by

$$E^{AU}(y_{t+m} \mid s_t = i) = \sum_j \left(\Pi_t^m\right)(i,\, j)\, s_{t+m}(j, 1). \tag{29}$$

To construct an infinite-history rational-expectations approximation, we simplify further by assuming that the consumer knows the true transition probabilities, Π. With this assumption, the multistep transition matrix simplifies to

$$\Pi_{t,t+m} = \Pi^m, \tag{30}$$

[10] The other columns store the counters.

and expected labor income becomes

$$E^{RE}(y_{t+m} \mid s_t = i) = \sum_j (\Pi^m)(i, j)\, s_{t+m}(j, 1). \tag{31}$$

Thus, in this setting, differences between Bayesian choices and the two approximations boil down to how multistep forecasts are formed. A Bayesian recognizes that the Markov chain is not homogenous and alters the transition matrix according to date and horizon. An anticipated-utility modeler updates beliefs each period, but then uses a homogenous-chain approximation for long-horizon forecasts. An infinite-history modeler abstracts from learning altogether and uses a single homogenous-chain approximation at all dates.

Matters are more complicated when preferences are isoelastic because the Bayesian decision rule can no longer be expressed analytically. Instead, we resort to numerical dynamic programming. Since it is optimal to consume all of one's resources at date T, the terminal decision rule and value function are $A_{T+1} = 0$ and $V_{T+1} = 0$, respectively. For $t \leq T$, the Bayesian Bellman equation takes the form

$$V_t^B(y_t, A_t, n_t) = \max_{A_{t+1}} \left[u(A_t + y_t - R^{-1} A_{t+1}) + \beta E_t^B V_{t+1}^B(y_{t+1}, A_{t+1}, n_{t+1}) \right], \tag{32}$$

where E_t^B is the mathematical expectation with respect to the distribution implied by Bayes' Law. This still involves the transition matrix $P_{t,t+1}$, but now it applies to continuation values instead of future labor income. We solve this once and for all by backward induction on the expanded state space.

The anticipated-utility allocation is also found by backward induction, but the problem and solution algorithm differ in two respects. First, because an anticipated-utility planner disregards how future counters influence continuation values, the state vector reduces to (y_t, A_t). Second, the anticipated-utility expectations operator depends on Π_t instead of $P_{t,t+1}$. After taking these shortcuts, the anticipated-utility Bellman equations become

$$V_t^{AU}(y_t, A_t) = \max_{A_{t+1}} \left[u(A_t + y_t - R^{-1} A_{t+1}) + \beta E_t^{AU} V_{t+1}^{AU}(y_{t+1}, A_{t+1}) \right]. \tag{33}$$

Although an anticipated-utility planner neglects the influence of n_t, he reoptimizes every period, revising value functions and decision rules in accordance with updates of Π_t. The problem is still solved by backward induction, but the solution is recalculated every period on the reduced state space.

Within an infinite-history model, the consumer knows the true transition probabilities and need not keep track of the counters. Thus, the state vector is (y_t, A_t), and the Bellman equation is

$$V_t^{RE}(y_t, A_t) = \max_{A_{t+1}} \left[u(A_t + y_t - R^{-1} A_{t+1}) + \beta E_t^{RE} V_{t+1}^{RE}(y_{t+1}, A_{t+1}) \right]. \tag{34}$$

The IHRE expectations operator depends on the true transition probabilities Π. Because there is no news about transition probabilities, there is no need to update decision rules, so an infinite-history planner solves the dynamic program only once.

4. DESCRIPTION OF THE EXPERIMENT

For computational reasons, we study a consumer who lives 20 periods. We assume that each period lasts 3 years, and we set $R = \beta^{-1} = 1.04^3$. To minimize the influence of initial and terminal conditions, we focus on outcomes in interior periods.

Precautionary behavior will be important for the quality of the approximation. Thus, we calibrate the labor-income process to give consumers ample reason to take precautions, assuming that labor income switches between a "normal" and a "crash" state,[11]

$$y_h = 1, \quad y_l = 0.75, \tag{35}$$

with transition probabilities,

$$\Pi = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}. \tag{36}$$

The high-income state, in which income is normalized to 1, occurs 83% of the time and is persistent. Conditional on being in the high-income state, the mean switching time is 10 periods, which means that a consumer might go 30 years without experiencing a crash. The low-income state represents a sharp but infrequent drop in income, and it is not persistent. Conditional on being in the low-income state, the mean switching time is 2 periods, so a young consumer can look forward to a recovery. Nevertheless, with *CRRA* preferences, the presence of a low-income state activates a motive for precautionary saving.

For Bayesian and anticipated-utility consumers, we simulate the model for four sets of priors that represent various kinds and degrees of disagreement with the true transition probabilities. Table 1 displays the prior counters for each case.

The first example is designed to illustrate what happens when initial beliefs are correctly centered but not dogmatic. In case 1, the prior means coincide with p and q, but there are few enough degrees of freedom that the consumer is open to alternatives.[12] In this case, experience will confirm prior beliefs on average, but learning is still relevant because the prior variance is not zero. The IHRE approximation makes the prior variance zero as well. The example elicits how much this matters.

[11] Earlier versions of our article studied several other calibrations, including ones designed to mimic business cycles and persistent shifts in increments to income. Although the details of the experiments differ, the lessons we draw are the same as those recorded below.

[12] The sum of prior counters is 12, so that by mid-life the consumer will assign approximately equal weight to priors and experience.

TABLE 1
PRIOR COUNTERS

	n_0^{hh}	n_0^{hl}	n_0^{lh}	n_0^{ll}
Case 1: Correctly Centered Priors	9	1	1	1
Case 2: Contraction Optimism	9	1	3	1
Case 3: Contraction Pessimism	9	1	1	3
Case 4: Global Pessimism	3	1	1	3

The other three cases represent examples in which consumers are initially mistaken about the frequency and persistence of income fluctuations. These experiments illustrate what happens when experience does not merely tighten priors, but also alters mean beliefs. We chose to focus on learning about transition probabilities because they are important for consumption smoothing and precautionary saving. In the model with quadratic preferences, beliefs about the persistence of labor income determine how much of income is consumed and how much is saved. In the model with *CRRA* preferences, beliefs about the likelihood of a crash also influence the strength of precautionary motives.

In cases 2 and 3, consumers have correctly centered priors about p but mistaken beliefs about q. In case 2, the consumer initially underestimates the persistence of the crash state. Not only does this make him overoptimistic about the prospects for recovering from a crash, it also causes him to underestimate its unconditional probability. This is an example of "irrational exuberance."

Case 3 represents the opposite scenario, one of excessive pessimism about the low-income state. Here the consumer overestimates the persistence of the crash state. He believes that crashes occur more often and last longer than they actually do.

Consumer 4 is even more pessimistic. He shares the prior of consumer 3 about the crash state, and he also underestimates the persistence of the high-income state. This further increases the unconditional probability of the low-income state. Because consumer 4 begins life with such a grim prior, we think of him as a member of the "depression generation." This example is especially relevant for the *CRRA* model because this degree of pessimism magnifies interest in precautionary saving.

Each of our simulations starts with 1000 draws of $\{y_t\}_{t=1}^{20}$ from the exogenous income process. We initialize wealth at $A_0 = 0$ and allow households to borrow or lend as much as they want at interest rate R, subject to paying off their debts before they die. Then we compare the consumption choices of Bayesian, anticipated-utility, and infinite-history rational-expectations consumers who face identical income paths, as well as the Arrow security prices that decentralize those allocations.

For the quadratic-preference model, consumption is computed recursively from Equation (25). At each date t, we update beliefs in light of the income realization, then use the appropriate expectations operator to forecast labor income over horizons $h = t + 1, \ldots, T$. Those forecasts are discounted at rate $R^{-(h-t)}$ and summed to find the expected present value of labor income. Adding this to the predetermined value of assets A_t gives total wealth. To determine consumption, we multiply by the period-t marginal propensity to consume out of wealth. Finally, we update next period's assets A_{t+1} by substituting consumption into the aggregate resource constraint.

For the model with *CRRA* preferences, consumption is calculated by solving Bellman Equations (32)–(34). The state variables y_t and n_t are discrete, and it is convenient to approximate the solution by discretizing A_t as well. Accordingly, we specify a grid for A_t in increments of 0.01 and set a range that is broad enough so that the end points are never reached in the simulations. Then value functions and decision rules are calculated by solving the finite-horizon, finite-state dynamic program, using backward-induction methods described in Judd (1998).

In this case, our simulations work as follows. First, we solve for the Bayesian and IHRE decisions rules and store them. Then we simulate 1000 paths for the exogenous income process. Having stored the decision rules, we simply read off the associated Bayesian and RE consumption paths for each income path. Solving for anticipated-utility consumption involves an extra step because consumers must re-optimize at each date. So at each date on each path, they re-solve the AU dynamic program using updated transition probabilities, then read off the consumption choice from the revised decision rule.

The final step is to calculate the Arrow security prices that decentralize the consumption allocation. With consumption and updated probabilities in hand, Arrow prices can be calculated directly from Equation (22).

5. APPROXIMATING CONSUMPTION

The first set of results compares Bayesian and anticipated-utility outcomes for consumption. Figures 1 and 2 portray consumption in period 10 of each simulation. The middle period was chosen to avoid end-point problems; apart from that, the results are typical of those in other periods. Simulations of the quadratic-preference model are shown in Figure 1, with one panel for each prior. Figure 2 depicts outcomes for *CRRA* preferences. Here we concentrate on prior number 4 because it generates the most precautionary savings. The four panels illustrate how the results depend on α, the coefficient of relative risk aversion.

Each panel represents a cross section of consumption choices over 1000 draws for labor income, with the Bayesian choice shown on the horizontal axis and the anticipated-utility choice on the vertical. Because the two decision makers face identical income streams within each experiment, a good approximation should result in a tight scatterplot along the 45° line. The figures confirm that Bayesian and anticipated-utility consumption choices are highly correlated and tightly arrayed along the 45° line.

The next table provides more detail about the quality of the approximation, reporting the relative mean-square approximation error (RMSAE) for various periods. If we regard the anticipated-utility model as an approximation to the more complex Bayesian decision problem, the approximation error is $\varepsilon = c^B - c^{AU}$. The RMSAE is defined as the ratio of mean-square error of ε to the variance of c^B, and it is analogous to $1 - R^2$ in a regression. Thus, a small number signifies a good approximation.

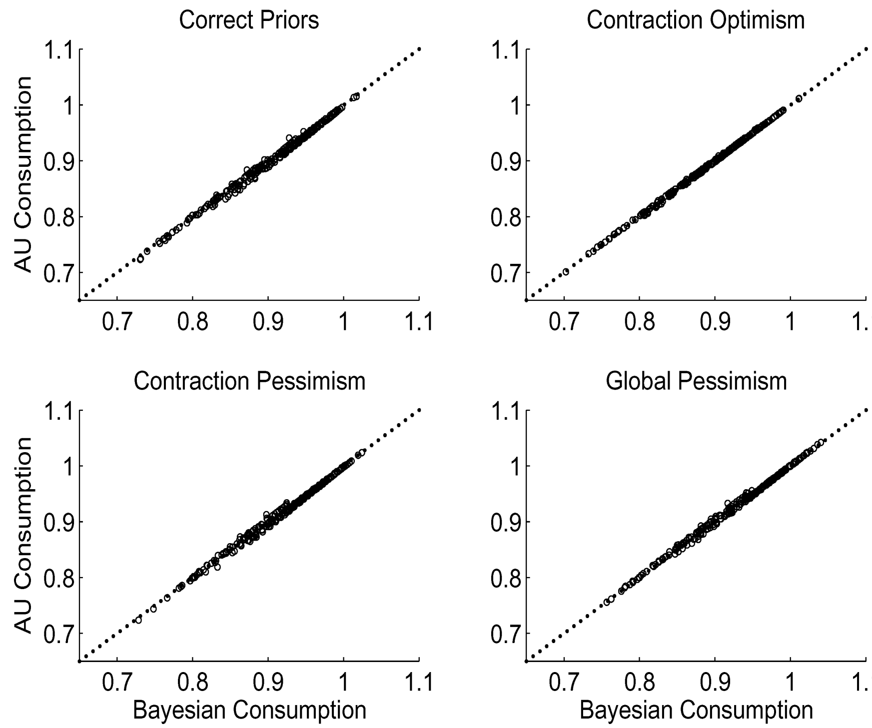

FIGURE 1

AU VERSUS BAYESIAN CONSUMPTION, QUADRATIC PREFERENCES, PERIOD 10

Measured on this scale, the approximation errors are often quite small, confirming the visual impression made by Figures 1 and 2. In the quadratic preference simulations, the entries are no larger than 1%, and many are less than 0.5%. In the *CRRA* example, the quality of the approximation is also quite good when α is small, but it deteriorates as α increases. For α equal to 2 or 5, the RMSAE is less than 3%, but it increases to 5 or 10% when $\alpha = 10$ or 20. These numbers suggest that an anticipated-utility approximation for consumption is excellent for typical calibrations in macroeconomics but that approximation errors can be larger when there is more than the usual degree of curvature.

Results for the IHRE approximation are summarized in Figures 3 and 4 and in Table 3. In many cases, the quality of the approximation is still quite good, although not as good as the anticipated-utility approximation. IHRE consumption is still highly correlated with outcomes in the Bayesian economies, but the figures reveal two problems. The scatterplots are less tightly concentrated, and sometimes they are not centered on the 45° line.

Notice that the quality of the approximation deteriorates even when prior beliefs are correctly centered (see the upper left-hand panel of Figure 3). This example should dispel the notion that learning is important to model only when initial

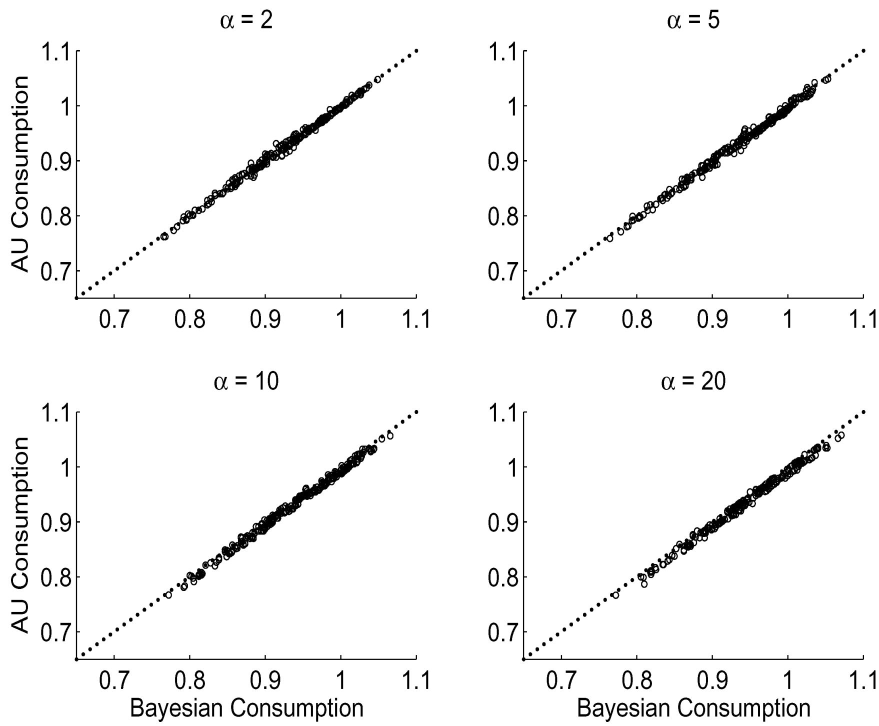

FIGURE 2

AU VERSUS BAYESIAN CONSUMPTION, CRRA PREFERENCES, PERIOD 10

TABLE 2

RMSAE FOR THE AU APPROXIMATION TO CONSUMPTION

Period	5	10	15
Quadratic Preferences			
Correct Priors	0.0078	0.0066	0.0100
Contraction Optimism	0.0007	0.0005	0.0005
Contraction Pessimism	0.0071	0.0064	0.0046
Global Pessimism	0.0045	0.0037	0.0027
CRRA Preferences			
$\alpha = 2$	0.0235	0.0054	0.0102
$\alpha = 5$	0.0137	0.0079	0.0277
$\alpha = 10$	0.0068	0.0163	0.0580
$\alpha = 20$	0.0447	0.0316	0.0954

NOTE: RMSAE $= \mathrm{MSE}(c^B - c^{AU})/\mathrm{var}(c^B)$.

beliefs are centered far from true parameter values. Bayesian outcomes depend on the prior variance as well as the prior mean. The IHRE approximation goes astray here not because of distorted mean beliefs relative to a Bayesian decision maker, but because it is dogmatic.

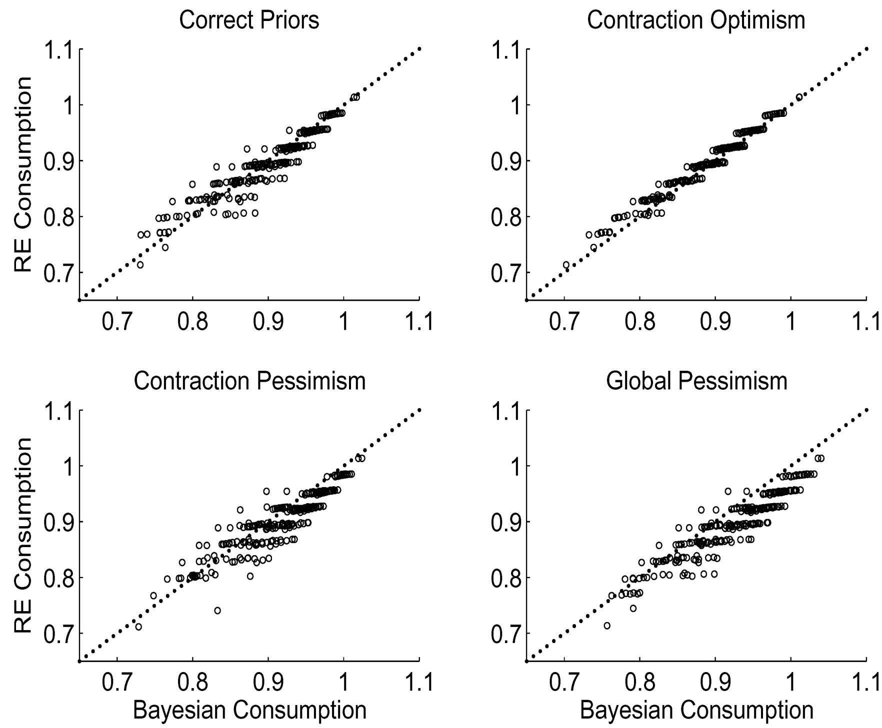

FIGURE 3

IHRE VERSUS BAYESIAN CONSUMPTION, QUADRATIC PREFERENCES, PERIOD 10

TABLE 3

RMSAE FOR THE IHRE APPROXIMATION TO CONSUMPTION

Period	5	10	15
Quadratic Preferences			
Correct Priors	0.1027	0.0813	0.1503
Contraction Optimism	0.0375	0.0238	0.0761
Contraction Pessimism	0.1886	0.1927	0.3106
Global Pessimism	0.1782	0.3004	1.0019
CRRA Preferences			
$\alpha = 2$	0.2075	0.3859	1.2980
$\alpha = 5$	0.1997	0.3918	1.5126
$\alpha = 10$	0.1755	0.4533	1.7660
$\alpha = 20$	0.2565	0.4363	1.6738

NOTE: RMSAE $= \mathrm{MSE}(c^{RE} - c^{B})/\mathrm{var}(c^{B})$.

RMSAEs for the IHRE approximation are shown in Table 3. In many instances, they are not excessively large. For example, the RMSAEs are less than 10% in a number of the certainty-equivalent cases, and they range between 15 and 25% in the early periods of the *CRRA* simulations. But the RMSAEs are uniformly higher

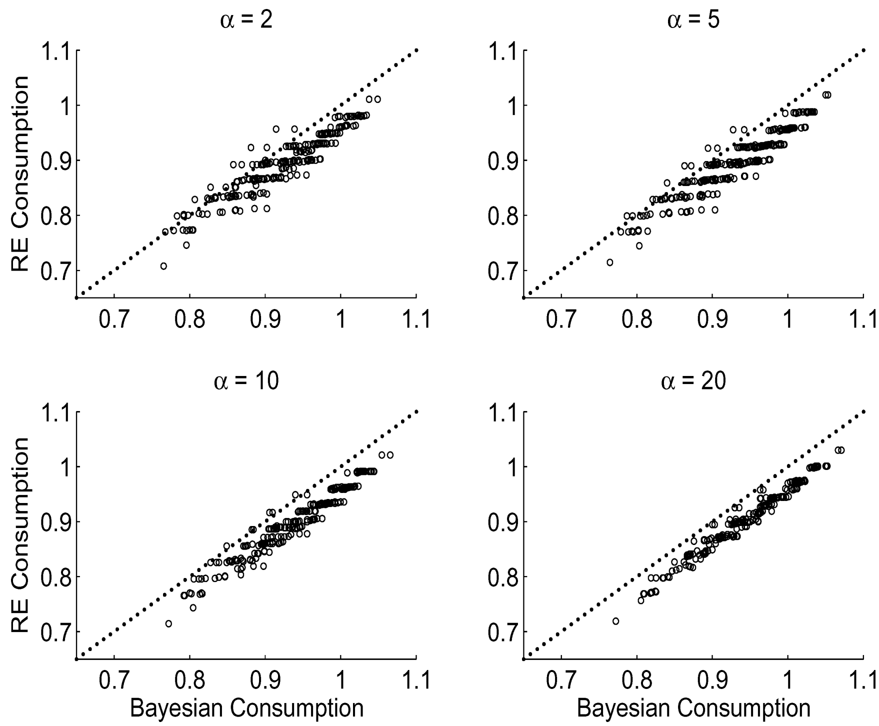

FIGURE 4

IHRE VERSUS BAYESIAN CONSUMPTION, CRRA PREFERENCES, PERIOD 10

than those for the AU approximation. In addition, the IHRE approximation is not uniformly reliable. In some cases, the mean-square approximation error is larger than the variance of consumption itself!

The largest errors occur later in the consumer's life. In the quadratic preference model, this reflects the consequences for wealth of expectations errors made earlier in life by Bayesian agents. For example, those who were irrationally exuberant saved too little early on and had to cut back later when their hopes were not realized. In the IHRE model, consumers were more realistic and saved more early in life, so they could consume more later on. If we regard the Bayesian model as true and the IHRE model as an approximation to it, this means the IHRE model systematically overstates consumption later in life. Rational-expectations outcomes are still highly correlated with Bayesian outcomes, but the mean approximation error contributes to a higher RMSAE. The same phenomenon occurs with priors 3 and 4, except there Bayesian agents are initially too pessimistic, and the mean approximation error has the opposite sign.

In the *CRRA* model, a second force is also at work. The bias is even more pronounced in the *CRRA* simulation because of how model uncertainty and pessimism affect the slope of the life-cycle consumption path. In the Bayesian economy, three factors contribute to precautionary saving, namely, *CRRA* preferences,

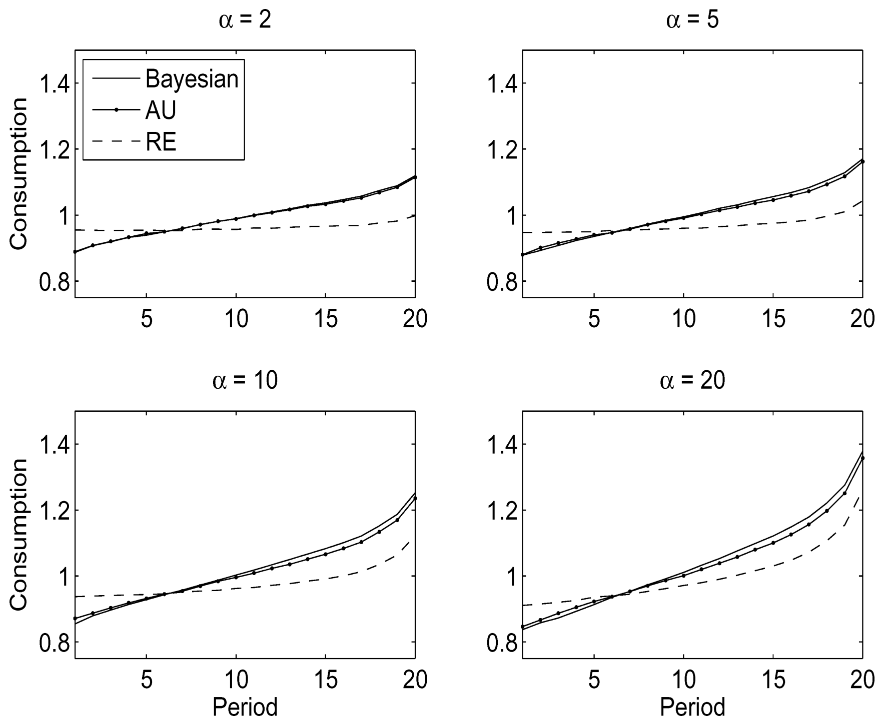

FIGURE 5

AVERAGE LIFE-CYCLE CONSUMPTION WITH CRRA PREFERENCES

model uncertainty, and initial pessimism. Consumers in the IHRE approximating model have the same preferences, but they are neither pessimistic nor uncertain about the transition probabilities, so they engage in less precautionary saving.

Figure 5 illustrates the magnitude of the problem. Solid lines portray Bayesian outcomes, and dashed lines illustrate the infinite-history approximation. Because the IHRE approximation abstracts from model uncertainty and pessimism, it understates the amount of precautionary savings early in life and therefore also understates the amount of consumption later on. This flattens the life-cycle consumption profile relative to that of Bayesian consumers and biases the approximation to consumption, first upward and then downward. These biases contribute to the high RMSAEs recorded in Table 3. The anticipated-utility approximation—depicted by a solid line with dots—does a better job fitting this feature of the Bayesian economy.

6. APPROXIMATING SECURITY PRICES

Next we turn to evidence on Arrow security prices. Figures 6 and 7 depict outcomes for the Arrow security that pays off in the high-income state. As before,

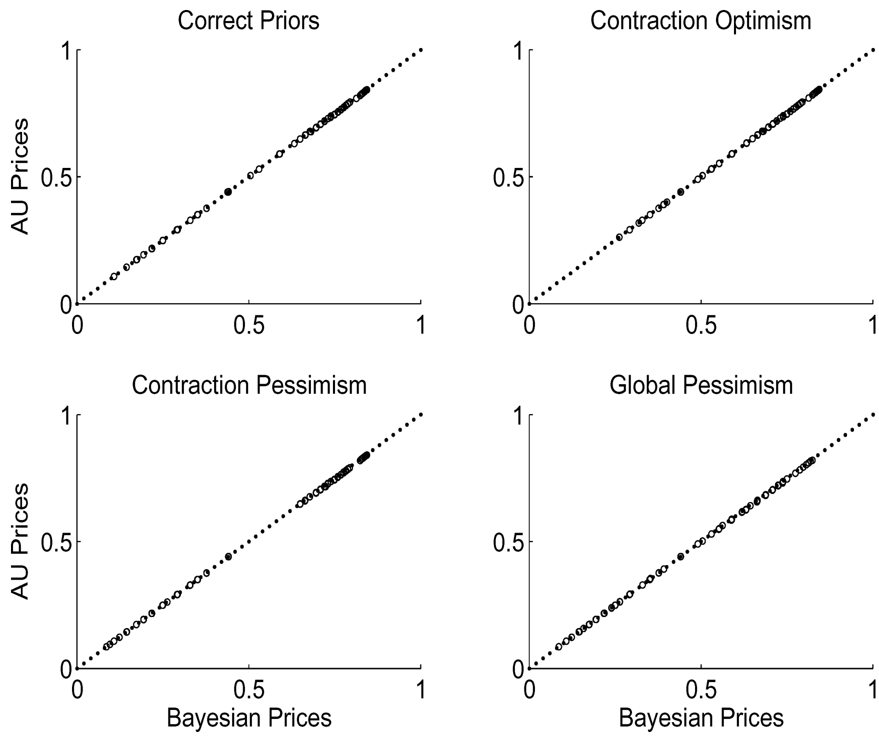

FIGURE 6

AU VERSUS BAYESIAN PRICES, QUADRATIC PREFERENCES, PERIOD 10

Bayesian prices are plotted on the horizontal axis, and the anticipated-utility approximation is shown on the vertical.

Not only is Bayesian consumption well approximated by anticipated-utility models, so too are security prices, provided that consumers are not too risk averse. In the quadratic preference simulations, the scatterplots indicate that Bayesian and anticipated-utility prices are tightly arrayed along the 45° line. Pricing errors are also quite small in the *CRRA* example when α is 2 or 5, but they grow in magnitude as α increases. Notice in particular that as α increases the scatterplot becomes steeper than the 45° line, which means that the mean pricing error is negative. For high α, an anticipated-utility model systematically overstates the price of an Arrow security in a Bayesian economy.

Table 4 records the RMSAE for prices in various years. As before, the quality of the approximation is excellent for the quadratic preference simulations; indeed the RMSAEs for security prices are even lower than those for consumption. In these examples, the mean-square approximation error amounts to less than one-twentieth of 1% of the variance of Bayesian prices. Pricing errors are also quite small in *CRRA* simulations with small α. When α is 2, the RMSAEs are still less than 1%, but the quality of the approximation deteriorates as α increases. The

TABLE 4

RMSAE FOR THE AU APPROXIMATION TO ARROW PRICES

Period	5	10	15
Quadratic Preferences			
Correct Priors	0.0001	0.0001	0.0000
Contraction Optimism	0.0000	0.0000	0.0000
Contraction Pessimism	0.0002	0.0002	0.0001
Global Pessimism	0.0004	0.0002	0.0001
CRRA Preferences			
$\alpha = 2$	0.0015	0.0021	0.0032
$\alpha = 5$	0.0104	0.0204	0.0373
$\alpha = 10$	0.0489	0.1339	0.3071
$\alpha = 20$	0.1790	0.7143	1.9584

NOTE: RMSAE $= \text{MSE}(Q^{AU} - Q^B)/\text{var}(Q^B)$.

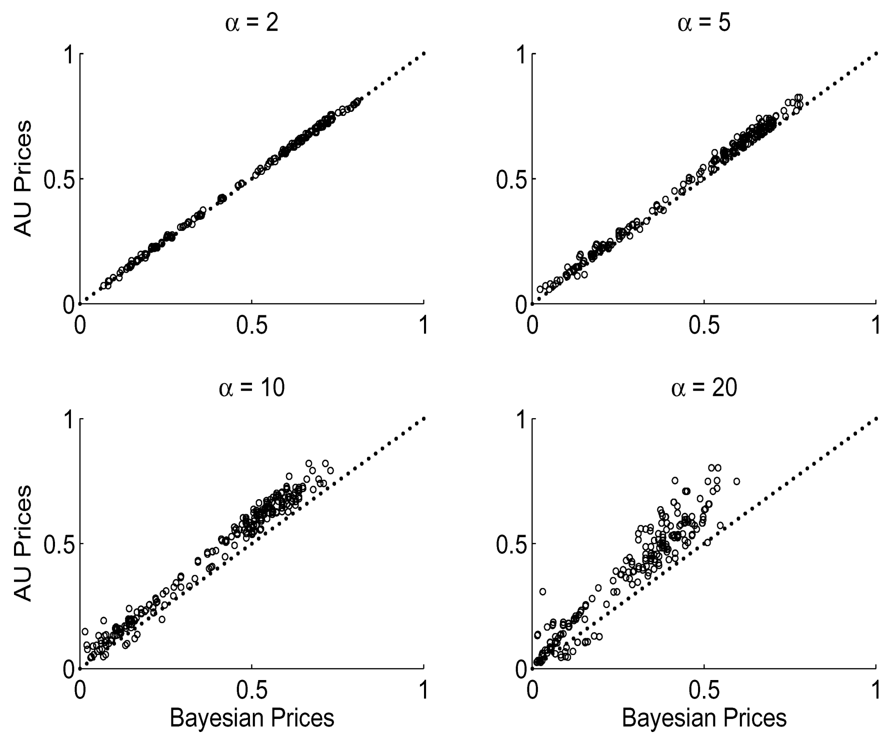

FIGURE 7

AU VERSUS BAYESIAN PRICES, CRRA PREFERENCES, PERIOD 10

approximation is still very good for $\alpha = 5$, but the pricing errors are larger when α increases to 10, and they become very large—sometimes more than 100%—when $\alpha = 20$. The large RMSAEs that occur in this case reflect the upward bias in AU prices that was mentioned above.

The close correspondence of anticipated-utility and Bayesian prices in the quadratic preference and small-α CRRA cases follows from two facts, that consumption allocations are very similar across models and that one-step-ahead transition probabilities are identical. In each case, there are two possible outcomes for s_{t+1} given (s_t, n_t), and along every sample path the one-step-ahead transition probabilities to these outcomes depend on the same counters n_t in the same way. Thus, the one-step-ahead transition probabilities always agree. Multistep transition probabilities differ because Bayesian consumers update counters across potential future paths, whereas anticipated-utility consumers do not, but multistep transition probabilities matter only indirectly for Arrow prices, affecting $Q(s_{t+1}, s_t)$ only through c_{t+1}. In the quadratic preference and small-α CRRA examples, the disparities in consumption are negligible, so Arrow prices are also in close agreement.

The consumption disparities are a bit larger in the large-α simulations, however, and they are magnified because inverse-consumption growth is raised to a large exponent when calculating security prices. Thus, seemingly minor discrepancies in approximating consumption can matter a lot for asset prices when consumers are highly risk averse.

Once again, the lesson seems to be that an anticipated-utility approximation is fine as long as consumers are not too risk averse. But it can be problematic for modeling security prices when consumers are highly risk averse.

Figures 8 and 9 illustrate the infinite-history approximation to Bayesian security prices. Bayesian and IHRE outcomes are highly correlated (i.e., IHRE prices are systematically above average when Bayesian prices are), but notice how the IHRE models predict essentially only two values for prices, whereas the Bayesian economy predicts many. This feature is especially sharp in the quadratic preference and small-α CRRA simulations, though somewhat more diffuse in the large-α case.

In the quadratic preference and small-α CRRA cases, the existence of essentially two prices under full information (i.e., infinite history) follows from three facts, that labor income evolves as a two-state process, that the true transition probabilities are known, and that the consumer's intertemporal marginal rate of substitution does not vary much. Consider the Arrow security that pays off in the high-income state. The upper and lower branches in the scatterplots represent $Q(s_{t+1} = s_h, s_t = s_h)$ and $Q(s_{t+1} = s_h, s_t = s_l)$, respectively. From the household's first-order condition, these prices are given by

$$Q(s_{t+1} = s_h, s_t = i) = \beta \, \frac{u'[c_{t+1}(s_{t+1} = s_h)]}{u'[c_t(s_{t+1} = i)]} \, pr(s_{t+1} = s_h \,|\, s_t = i). \tag{37}$$

The linear storage technology provides a powerful tool for consumption smoothing, so the IMRS does not vary much across states, always staying fairly close to $1/R = \beta$ in the quadratic preference models and not straying too far from there in the small-α CRRA case. It follows that Arrow prices are well approximated by

$$Q(s_{t+1} = s_h, s_t = i) \doteq \beta pr(s_{t+1} = s_h \,|\, s_t = i). \tag{38}$$

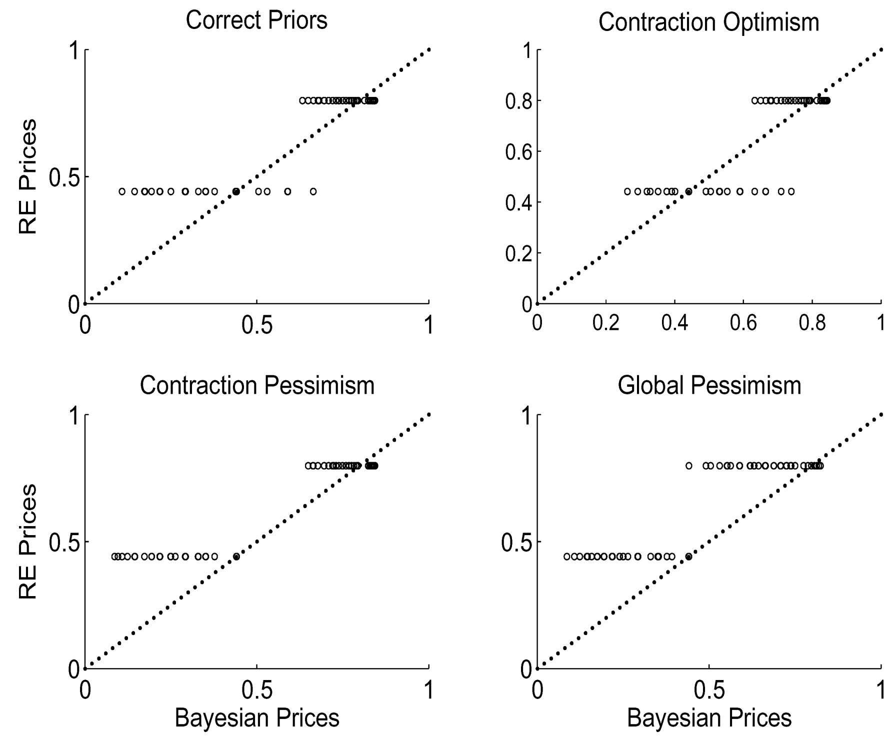

FIGURE 8

IHRE VERSUS BAYESIAN PRICES, QUADRATIC PREFERENCES, PERIOD 10

Because the two transition probabilities are constant under full information, it follows that there are essentially just two prices. The price is higher when $s_t = s_h$, reflecting a high probability that $s_{t+1} = s_h$ when it starts there, and the price is lower when $s_t = s_l$, reflecting a low probability of making a transition to the high-income state.

This two-value representation mirrors in a rough way what goes on in a Bayesian economy. There it is also the case that $Q(s_{t+1} = s_h, s_t)$ tends to be higher when $s_t = s_h$, again reflecting that prices are higher for securities promising a payoff with higher probability. Much of the variation in prices in the Bayesian economy represents movements across branches, and the IHRE model captures this feature of the data. Cross-branch movements in prices account for the correlation between the Bayesian and rational-expectations economies.

But there is an additional source of variation in the Bayesian economy that the IHRE model neglects. In addition to variation across branches, there is also price variation within branches arising from the updating of transition probabilities. A version of Equation (38) holds for the Bayesian economy as well,[13] but there are

[13] The IMRS varies more in the Bayesian economies, but not a lot more.

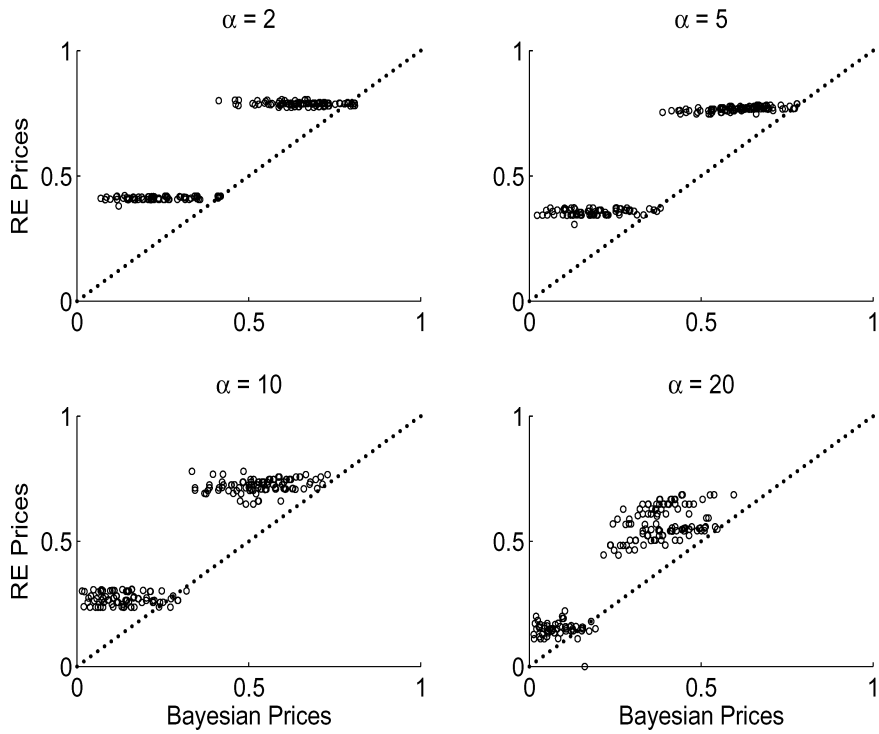

FIGURE 9

IHRE VERSUS BAYESIAN PRICES, CRRA PREFERENCES, PERIOD 10

many more nodes in the expanded Bayesian state space, hence many more possible values for the transition probabilities. Accordingly, there is price uncertainty in the Bayesian model even conditional on knowing the state of income today and tomorrow; prices depend on the counters as well. Because the IHRE approximation abstracts from this source of uncertainty, the RMSAE for within-branch variation exceeds 100%.[14]

This two-value characterization of prices begins to break down as α increases because higher α magnifies variation in the IMRS. In that case, we get many values for prices under full information as well, and two clouds emerge in the scatterplots instead of two branches. But it remains true that within each cloud there is more variation in Bayesian prices than in IHRE prices, again reflecting that variation in transitions probabilities in Bayesian economies magnifies asset price volatility.

Whether abstracting from learning is critical for modeling security prices depends on the relative importance of variation within and across branches in the Bayesian economy. In many instances, the overall quality of the IHRE

[14] The RMSAE equals 1 if mean Bayesian prices within each branch are the same as the rational expectations value, and it exceeds 1 if the mean error is nonzero.

TABLE 5
RMSAE FOR THE IHRE APPROXIMATION TO ARROW PRICES

Period	5	10	15
Quadratic Preferences			
Correct Priors	0.1624	0.1716	0.1992
Contraction Optimism	0.5121	0.4410	0.5825
Contraction Pessimism	0.2088	0.1918	0.1876
Global Pessimism	0.3848	0.2954	0.2625
CRRA Preferences			
$\alpha = 2$	0.4230	0.3044	0.2586
$\alpha = 5$	0.5008	0.3441	0.2524
$\alpha = 10$	0.7623	0.4780	0.2444
$\alpha = 20$	2.7515	1.0014	0.3787

NOTE: RMSAE = $\text{MSE}(Q^{RE} - Q^B)/\text{var}(Q^B)$.

approximation is not too bad. For instance several of the RMSAEs reported in Table 5 are 20% or less. But in some cases the IHRE approximation is very wide of the mark, and in a few cases the RMSAE exceeds 100%.

7. THE MARKET PRICE OF RISK

Viewed from another perspective, IHRE calculations are even more off the mark. That metric involves the market price of risk, which Hansen and Jagannathan (1991, 1997) define as

$$\rho_t(m_{t+1}) = \sigma_t(m_{t+1})/\mu_t(m_{t+1}). \tag{39}$$

Here m_{t+1} represents a stochastic discount factor, $\mu_t(\cdot)$ is a conditional mean, and $\sigma_t(\cdot)$ is a conditional standard deviation.

Hansen and Jagannathan characterize asset pricing puzzles by contrasting two measures of $\rho_t(m_{t+1})$. Within an infinite-history rational-expectations model, one can directly estimate the conditional mean and standard deviation of m_{t+1} and form an estimate by taking the ratio. When stochastic discount factor models are calibrated to represent "plausible" levels of risk aversion, the implied price of risk is typically small.[15] They also show how to estimate a lower bound on $\rho_t(m_{t+1})$ from security market data, without any reference to a model discount factor. Estimates reported by Hansen and Jagannathan and Cochrane and Hansen (1992) suggest a price of risk that is so high that it can be attained in conventional models only if agents are very risk averse.

Thus, although many economists have a strong prior that investors are risk tolerant, their behavior in securities markets suggests a high degree of risk aversion.

[15] According to Kocherlakota (1996), "a vast majority of economists believe that values for [the coefficient of relative risk aversion] above ten (or, for that matter, above five) imply highly implausible behavior on the part of individuals."

Here we examine whether the conflict can be resolved by understanding what happens when an IHRE modeler calculates prices of risk based on outcomes from a Bayesian economy.

At least three prices of risk are relevant for our economies. If one were to survey the agents in our model and ask them about the price of risk, they would report calculations based on their own preferences and beliefs. Within a Bayesian equilibrium, asset returns and consumption satisfy

$$E_t^B(m_{t+1}R_{t+1}) = 1, \tag{40}$$

where $E_t^B(\cdot)$ represents an expectation taken with respect to probabilities coming from Bayes' Law, $pr_B(s_{t+1} = j \mid S_t)$, and $m_{t+1} = \beta u'(c_{t+1})/u'(c_t)$ is the consumers' intertemporal marginal rate of substitution. Their subjective price of risk is accordingly

$$\rho_t^B(m_{t+1}) = \sigma_t^B(m_{t+1})/\mu_t^B(m_{t+1}), \tag{41}$$

where a superscript B indicates that conditional moments are evaluated with respect to Bayesian transition probabilities. In a Markov economy such as ours, the subjective price of risk can be expressed in terms of Arrow security prices and Bayesian probabilities. Appendix A.2 shows how to do this and displays a formula for $\rho_t^B(m_{t+1})$.

Now imagine an IHRE modeler confronting data on prices and consumption from the Bayesian equilibrium. He has a model discount factor m_{t+1}, which in this case happens to be correctly specified, but he follows the precept of equating the perceived and actual laws of motion, so he tries to make sense of observed prices in terms of the actual transition probabilities $pr_A(s_{t+1} = j \mid s_t)$.[16] This combination will not correctly price securities, for his model implies an Euler equation

$$E_t^A(m_{t+1}R_{t+1}) = 1 \tag{42}$$

that involves expectations with respect to different probabilities than those used by consumers. Since $E_t^A(\cdot) \neq E_t^B(\cdot)$, it follows that equilibrium consumption and asset returns will not conform to (42).

Following Hansen and Jagannathan, an IHRE modeler might summarize the discrepancy by calculating two other prices of risk, one associated with his model discount factor and another that characterizes the properties a stochastic discount factor must have in order to rationalize observed security prices with his probability model. We label the first $\rho_t^A(m_{t+1})$ and the second $\rho_t^A(\tilde{m}_{t+1})$. Both are calculated with respect to the actual infinite-history probabilities $pr_A(s_{t+1} = j \mid s_t)$, hence the A superscript.

The IHRE model price of risk, $\rho_t^A(m_{t+1})$, is easy to calculate. One just substitutes observations on consumption into the model discount factor and then computes

[16] A rational expectations modeler can consistently estimate these transition probabilities ex post.

conditional moments under the assumed probability law. Appendix A.2 displays an alternative calculation that is less straightforward but easier to implement using our simulation output.

The IHRE required price of risk, $\rho_t^A(\tilde{m}_{t+1})$, is calculated from securities market data alone, without reference to consumption data or a model discount factor. Assuming absence of arbitrage opportunities, in a Markov economy with transition probabilities $pr_A(s_{t+1} = j \mid s_t)$, Arrow security prices must satisfy

$$Q_t(s_{t+1} = j \mid s_t) = \tilde{m}_{t+1}(s_{t+1} = j \mid s_t)\ pr_A(s_{t+1} = j \mid s_t), \tag{43}$$

for some discount factor $\tilde{m}_{t+1}$. Conditional moments for the unknown discount factor $\tilde{m}_{t+1}$ can therefore be calculated from deflated security prices $Q_t(\cdot)/pr_A(\cdot)$. The prices $Q_t(\cdot)$ are observed in our economies, and the modeler specifies the probabilities $pr_A(s_{t+1} = j \mid s_t)$, so the properties of $\tilde{m}_{t+1}$ can indeed be inferred from security market data. Once again, see Appendix A.2 for the details and a formula for $\rho_t^A(\tilde{m}_{t+1})$.[17]

The literature typically finds that the model price of risk $\rho_t^A\ (m_{t+1})$ is small if consumers are not too risk averse and that the required price of risk $\rho_t^A(\tilde{m}_{t+1})$ is larger. This is also what we find in our Bayesian economies. For example, Figures 10 and 11 compare the required price of risk $\rho_t^A(\tilde{m}_{t+1})$—shown on the horizontal axis—with the model price of risk $\rho_t^A\ (m_{t+1})$—plotted on the vertical.

The scatterplots are always far from the 45 degree line, and the required price of risk is almost always larger than the model price of risk. Differences between the two are especially pronounced in the quadratic preference and small-α CRRA simulations. In those cases, the required price of risk is often one or two orders of magnitude larger than the model price of risk. In the *CRRA* example, the model price of risk grows as α increases, but most of the points remain below the 45° line, so the model price of risk still falls short of the required price of risk.

This is usually interpreted as a sign that the model discount factor is misspecified, but it could also signify that the transition probabilities are off the mark. Indeed, in this instance, we know that the model discount factor is correctly specified, so the discrepancies must be due to how the transition probabilities are modeled.

Figures 12 and 13 compare the required price of risk with the subjective measure $\rho_t^B\ (m_{t+1})$ that appraises the price of risk from the vantage of Bayesian consumers. The required price of risk $\rho_t^A(\tilde{m}_{t+1})$ is again plotted on the horizontal axis, and the consumers' price of risk $\rho_t^B\ (m_{t+1})$ is now shown on the vertical.

Once again, the required price of risk is often higher by one or two orders of magnitude. If one regards the consumers' price of risk as the true value and the

[17] Hansen and Jagannathan estimate a lower bound on the required price of risk, but in our economies we can calculate $\rho_t^A(\tilde{m}_{t+1})$ itself. This is just a detail. Following their example, we still compare a model price of risk with a required price of risk estimated from security market data alone.

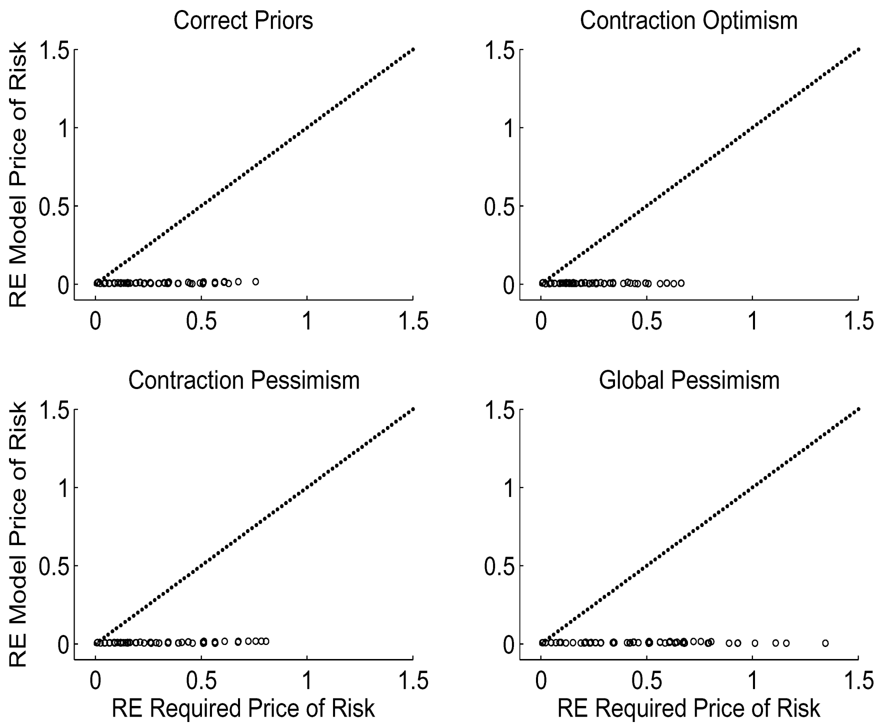

FIGURE 10

MODEL AND REQUIRED MPR, QUADRATIC PREFERENCES, PERIOD 10

IHRE calculation as an estimate of it, then the figures can be interpreted as an assessment of the IHRE approximation. Many of the RMSAE statistics for this comparison are off the chart, on the order of 1.0E+02 to 1.0E+06. Notice also that the IHRE model price of risk, shown in the previous figures, is actually closer to subjective evaluations. The irony in this example is that the IHRE model price of risk, which is usually judged to be too low, is nearer to the truth. It is the required price of risk that is exaggerated.

The reason that the IHRE estimate $\rho_t^A(\tilde{m}_{t+1})$ is so much larger is that it encompasses both risk aversion per se and model uncertainty. To see why, rewrite the Bayesian first-order condition (40) in terms of infinite-history probabilities:

(44)

$$1 = \sum_j m_{t+1}(s_{t+1} = j \mid S_t)\; R_{t+1}(s_{t+1} = j \mid S_t)\; pr_B(s_{t+1} = j \mid S_t),$$

$$= \sum_j \left[m_{t+1}(s_{t+1} = j \mid S_t)\; \frac{pr_B(s_{t+1} = j \mid S_t)}{pr_A(s_{t+1} = j \mid s_t)} \right] R_{t+1}(s_{t+1} = j \mid S_t)\; pr_A(s_{t+1} = j \mid s_t).$$

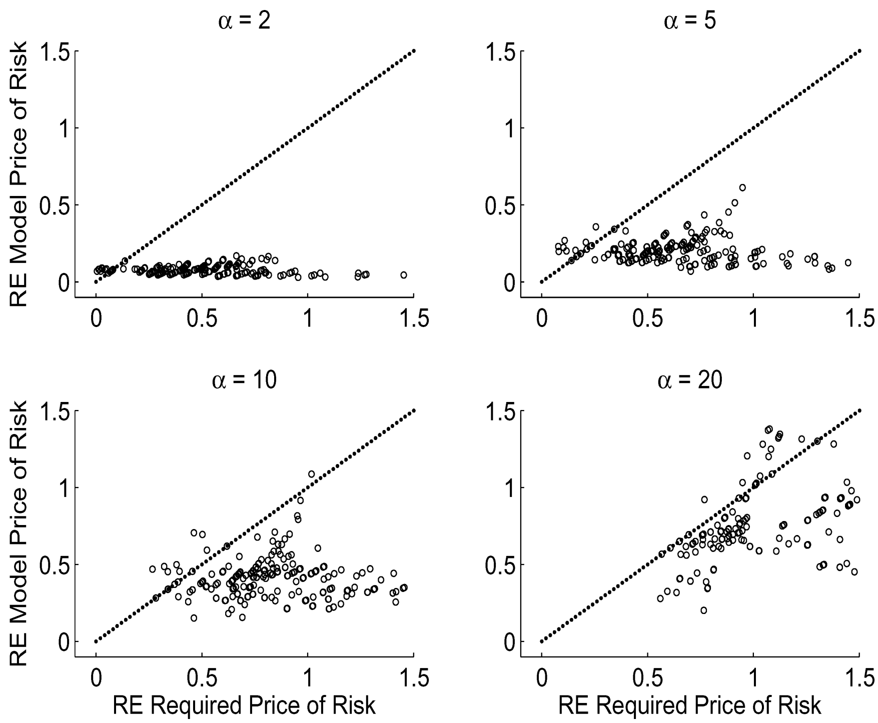

FIGURE 11

MODEL AND REQUIRED MPR, *CRRA* preferences, period 10

The term in brackets is the modified discount factor $\tilde{m}_{t+1}$ that reconciles Bayesian prices with infinite-history transition probabilities. It involves the consumers' IMRS along with the Radon–Nikodým derivative of the Bayesian transition density with respect to the actual transition density. Variation in the IMRS reflects how averse consumers are to risk, whereas variation in the Radon–Nikodým derivative reflects their uncertainty about the law of motion for income. If the perceived and actual laws of motion were the same, the latter term would always equal 1, and its variance would be zero. But because consumers are uncertain about the law of motion and try to learn about it, their beliefs change over time, giving rise to variation in this probability ratio. Hence variation in $pr_B(\cdot)/pr_A(\cdot)$ reflects consumers' uncertainty about the right model for income.

Risk aversion and model uncertainty are both in play in the Bayesian economy, but our consumers are actually very risk tolerant. Their IMRS varies, but unless α is large it varies only a little. It follows that most of the variation in $\tilde{m}_{t+1}$ recorded in $\rho_t^A(\tilde{m}_{t+1})$ arises from variation in the Radon–Nikodým derivative. Thus, the high price of risk required under full information mostly reflects model uncertainty and changing beliefs; risk aversion makes only a small contribution.

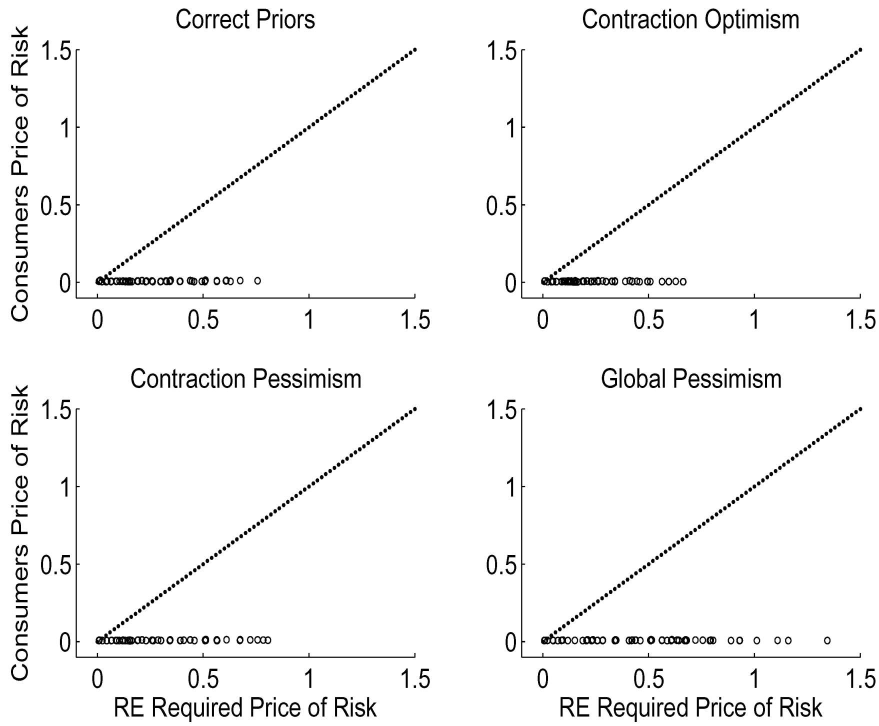

FIGURE 12

SUBJECTIVE AND REQUIRED MPR, QUADRATIC PREFERENCES, PERIOD 10

Finally, notice that an anticipated-utility modeler trying to rationalize prices and quantities would draw the same conclusions about the conditional price of risk as a Bayesian modeler. This follows from the fact that one-step ahead transition probabilities in the anticipated-utility model coincide with those of a Bayesian model. Transition probabilities disagree over longer horizons but are the same for one-period forecasts. Since the one-step transition probabilities agree, the implicit stochastic discount factor $\tilde{m}_{t+1}^{AU}$ is the same as the Bayesian discount factor m_{t+1} . In addition, conditional moments evaluated with respect to one set of probabilities are equivalent to those evaluated with respect to the other, so $\rho_t^B(\cdot) = \tilde{\rho}_t^{AU}(\cdot)$. It follows that the AU price of risk is identical to Bayesian price of risk:[18]

$$\rho_t^B(m_{t+1}) = \rho_t^{AU}\big(\tilde{m}_{t+1}^{AU}\big). \tag{45}$$

The AU approximation is a shortcut for calculating multistep transition probabilities. Since multistep transitions are not in play for calculating the

[18] One can confirm this by inspecting the formulas in Appendix A.2. Reinterpret $pr_A(\cdot)$ as an anticipated-utility transition probability and then equate $pr_B(\cdot) = pr_A(\cdot)$. The equality of the prices of risk follows directly.

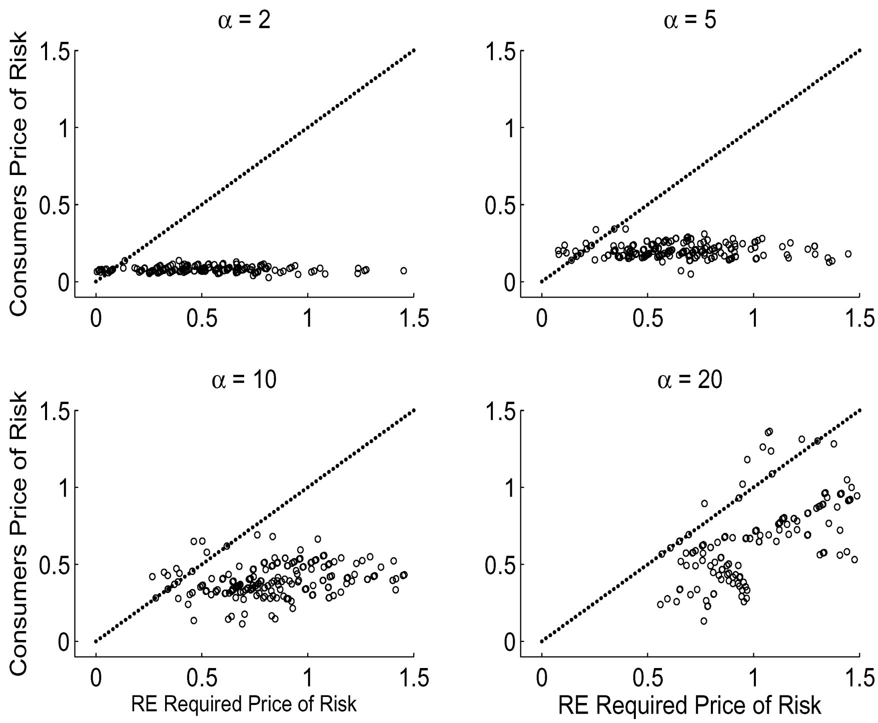

FIGURE 13

SUBJECTIVE AND REQUIRED MPR, *CRRA* preferences, period 10

conditional market price of risk, the AU approximation coincides with Bayesian calculations.

8. CONCLUSION

This article is a progress report on our research on anticipated utility. Our results are encouraging in three respects. First, at least for the nonexperimental setting considered here, the anticipated-utility approximation is excellent provided that precautionary motives are not too strong. This can be understood in terms of how an anticipated-utility model approximates a Bayesian predictive density. In our examples, the anticipated-utility model well approximates the mean of the predictive density, but it neglects parameter uncertainty and therefore has tails that are too thin. Whether the quality of the approximation in the tails matters for decisions depends on the strength of precautionary motives. When decisions depend mostly on the mean, as in quadratic preference or small-α CRRA models, errors in approximating the tails matter hardly at all for decisions. The range of α over which we obtain good results covers values typically used in macroeconomics. The quality of the approximation deteriorates as α increases, however, so anticipated-utility modeling may be problematic for applications in finance when high risk aversion is assumed. But the full Bayesian analysis also turns out to be more tractable than we expected, making us optimistic that the methods can be generalized to more realistic applications when one has doubts about the anticipated-utility approach.

The results about market prices of risk are tantalizing. In Cogley and Sargent (2005), we explore further how learning drives a wedge between subjective and infinite-history prices of risk. That paper builds on Cecchetti et al. (2000) and Abel (2002) by studying how pessimism and learning interact in a version of the Mehra–Prescott (1985) model. Our analysis predicts a declining price of risk and equity premium. Weitzman (2005) develops a closely related Bayesian model that simultaneously explains a number of asset pricing puzzles. In his model, parameter uncertainty is a permanent feature of the equilibrium, and the equity premium and price of risk remain high forever. Thus, at least in principle, the two approaches could be distinguished by examining empirical evidence on whether the equity premium declines over time.

APPENDIX

A.1. *How the State Space Is Constructed.* For a two-state model, there are four counters, one for each possible transition of s_t. The state space consists of all possible permutations of counters as well as the two values for the natural states. To construct this state space, we start with a large $k \times 4$ matrix in which each row represents a potential value for S_t. For the counters, we initially include all combinations of integers from 1 to T, the time horizon of the model. In T periods, the economy can remain in the same state the entire time, so the counters for s_h to s_h and s_l to s_l transitions can in principle reach T. But the state cannot switch from s_h to s_l or s_l to s_h that many times. The maximum number of switches in T periods is $(T+1)/2$, so we prune the original matrix by eliminating rows in which the counters for s_h to s_l or s_l to s_h exceed this limit. Call this matrix N_0.

Next, we rearrange the rows of N_0 so that counters are grouped by date. Admissible counters for date t must sum to t, and the elements representing switches should not exceed $(t+1)/2$. Once again, we prune elements of N_0 that violate this constraint. We label the resulting matrices N_{1t}.

Next we add the prior counters to N_{1t} to get a new set of counters N_{2t}. We also append s_h and s_l to N_{2t}, obtaining a state array

$$\mathcal{S}_{0t} = \begin{bmatrix} s_h \cdot \iota & N_{2t} \\ s_l \cdot \iota & N_{2t} \end{bmatrix}, \tag{A.1}$$

where ι is a column vector of ones conformable with N_{2t}. Thus, $\mathcal{S}_{0t}$ consists of every permutation of the counters and levels of income.

The matrix $\mathcal{S}_{0t}$ is a profligate representation of the state, however, containing many elements that the Markov process cannot actually reach. To identify redundant elements, we compute transition probabilities

$$P^0_{t,t+1}(i, j) = pr(\mathcal{S}_{0t+1} = x_j \mid \mathcal{S}_{0t} = x_i). \tag{A.2}$$

This is done by iterating through the rows and columns of $\mathcal{S}_{0t}$ and $\mathcal{S}_{0t+1}$ and looking for admissible matches. For $i = 1, 2$ and $j = 1, 2$, a match is admissible if $s_t = s_i$, $y_{t+1} = s_j$, the counter for i to j transitions increases by one, and the other counters remain unchanged. In that case, the transition probability is $n_t^{ij}/(n_t^{ii} + n_t^{ij})$. All other matches between $\mathcal{S}_{0t}$ and $\mathcal{S}_{0t+1}$ correspond to pairs in which the change in s is inconsistent with the change in n, and their transition probabilities are zero.

After cycling through the rows and columns of $\mathcal{S}_{0t}$ and $\mathcal{S}_{0t+1}$, we have a transition matrix $P^0_{t,t+1}$. We identify redundant elements of $\mathcal{S}_{0t+1}$ by inspecting the columns of this matrix. If an element of $\mathcal{S}_{0t+1}$ cannot be reached from $\mathcal{S}_{0t}$, the corresponding column of $P^0_{t,t+1}$ is zero. Accordingly, we eliminate rows of $\mathcal{S}_{0t+1}$ that correspond to columns of $P^0_{t,t+1}$ that sum to zero. The reduced state array is denoted $\mathcal{S}_{t+1}$, and the trimmed transition matrix connecting the reduced states $\mathcal{S}_t$ and $\mathcal{S}_{t+1}$ is denoted $P_{t,t+1}$. These tree-trimming operations substantially reduce the number of nodes on the state space and are helpful for coping with the curse of dimensionality.

A.2. *Arrow Prices and the Market Price of Risk in a Finite-State Markov Economy.* Here we show how to express prices of risk in terms of Arrow security prices. Consider first the subjective price of risk, $\rho_t^B\,(m_{t+1})$. In our Markov economies, the conditional mean is

$$\begin{aligned} \mu_t^B(m_{t+1}) &= \sum_{j=1}^{k} m_{t+1}(s_{t+1} = j \mid S_t)\ pr_B(s_{t+1} = j \mid S_t), \\ &= \sum_{j=1}^{k} Q_t(s_{t+1} = j \mid S_t) = \beta. \end{aligned} \tag{A.3}$$

Similarly, the conditional second moment is

$$\begin{aligned} E_t^B\big(m_{t+1}^2\big) &= \sum_{j=1}^{k} m_{t+1}^2(s_{t+1} = j \mid S_t)\ pr_B(s_{t+1} = j \mid S_t), \\ &= \sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{pr_B(s_{t+1} = j \mid S_t)}. \end{aligned} \tag{A.4}$$

It follows that the subjective price of risk is

$$\begin{aligned} \rho_t^B(m_{t+1}) &= \frac{[E_t^B\big(m_{t+1}^2\big) - \mu_t^B(m_{t+1})^2]^{1/2}}{\mu_t^B(m_{t+1})}, \\ &= \left[\sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{\beta^2 pr_B(s_{t+1} = j \mid S_t)} - 1\right]^{1/2}. \end{aligned} \tag{A.5}$$

Next, consider the model price of risk under full information, $\rho_t^A\ (m_{t+1})$. Under the actual transition law, the conditional first and second moments are

$$\text{(A.6)}\quad \mu_t^A(m_{t+1}) = \sum_{j=1}^{k} m_{t+1}(s_{t+1} = j \mid S_t)\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)}\, pr_B(s_{t+1} = j \mid S_t)$$
$$= \sum_{j=1}^{k} Q_t(s_{t+1} = j \mid S_t)\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)},$$

and

$$\text{(A.7)}\ E_t^A(m_{t+1}^2) = \sum_{j=1}^{k} m_{t+1}^2(s_{t+1} = j \mid S_t)\, \left(\frac{pr_B(s_{t+1} = j \mid S_t)}{pr_B(s_{t+1} = j \mid S_t)}\right)^2\, pr_A(s_{t+1} = j \mid s_t)$$
$$= \sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{pr_B(s_{t+1} = j \mid S_t)}\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)},$$

respectively. It follows that the RE model price of risk is

$$\text{(A.8)}\quad \rho_t^A(m_{t+1}) = \frac{\left[\sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{pr_B(s_{t+1} = j \mid S_t)}\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)} - \left(\sum_{j=1}^{k} Q_t(s_{t+1} = j \mid S_t)\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)}\right)^2\right]^{1/2}}{\sum_{j=1}^{k} Q_t(s_{t+1} = j \mid S_t)\, \frac{pr_A(s_{t+1} = j \mid s_t)}{pr_B(s_{t+1} = j \mid S_t)}}.$$

This is not the most straightforward way to calculate $\rho_t^A(m_{t+1})$, and an IHRE modeler would not adopt it because he would not know $pr_B(s_{t+1} = j \mid S_t)$. But we do know the Bayesian probabilities, and this formula simplifies our computations given what is available from our simulation output.

Finally, consider the infinite-history required price of risk, $\rho_t^A(\tilde{m}_{t+1})$. The conditional mean is

$$\text{(A.9)}\qquad \mu_t^A(\tilde{m}_{t+1}) = \sum_{j=1}^{k} \frac{Q_t(s_{t+1} = j \mid S_t)}{pr_A(s_{t+1} = j \mid s_t)}\, pr_A(s_{t+1} = j \mid s_t),$$
$$= \sum_{j=1}^{k} Q_t(s_{t+1} = j \mid S_t) = \beta.$$

Similarly, the conditional second moment is

$$
\text{(A.10)} \qquad E_t^A(\tilde{m}_{t+1}^2) = \sum_{j=1}^{k} \left[\frac{Q_t(s_{t+1} = j \mid S_t)}{pr_A(s_{t+1} = j \mid s_t)} \right]^2 \; pr_A(s_{t+1} = j \mid s_t),
$$

$$
= \sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{pr_A(s_{t+1} = j \mid s_t)},
$$

It follows that the RE required price of risk is

$$
\text{(A.11)} \qquad \rho_t^A(\tilde{m}_{t+1}) = \left[\sum_{j=1}^{k} \frac{Q_t^2(s_{t+1} = j \mid S_t)}{\beta^2 pr_A(s_{t+1} = j \mid s_t)} - 1 \right]^{1/2}.
$$

This depends on Arrow security prices, which are observed, and on actual transition probabilities, which an IHRE modeler can consistently estimate.

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