Materials 24 - Numerical implementation of the target criterion (TC) seems unstable

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Overview

1 Description of the three steps

Overall goal: find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and implements the target criterion in the simplified anchoring model, equation (B.1).

I proceed in 3 steps:

1) *find an exogenous sequence* $\{i_t\}_{t=1}^T$ *that fulfills the other model equations, including the Taylor-rule, w/o a target criterion;*

2) find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and fulfills the other model equations *including a simple target criterion from the RE model with discretion;*

3) find an exogenous sequence $\{i_t\}_{t=1}^T$ that replaces the Taylor rule as a DGP for i and fulfills the other model equations, *including the anchoring target criterion.*

- Implement using `fsolve`,
- implement using `fmincon`.

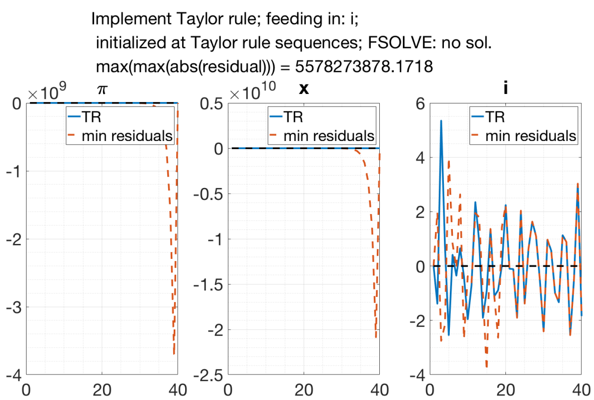
2 Questions/notes

1. I choose $\lambda_x = 0.5$ for these figures.
2. Numerical instability: `fmincon/fsolve` stops prematurely or finds no solution.
3. "Overparameterized?": for the anchoring TC, for $T + H$ periods, compute T residuals.
4. Solution conditional on sequence of innovations.
5. "Value function iteration-equivalent" solution method?
6. "Spline-equivalent" method of finding the optimal functional form that delivers the optimal sequence $\{i_t\}_{t=1}^T$? $\rightarrow$ a numerical approx to the optimal reaction function that replaces the TR.

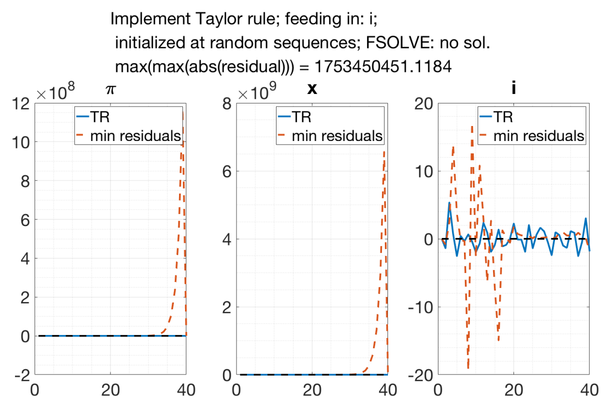
3 FSOLVE

3.1 FSOLVE: Implementing the Taylor-rule w/o using it

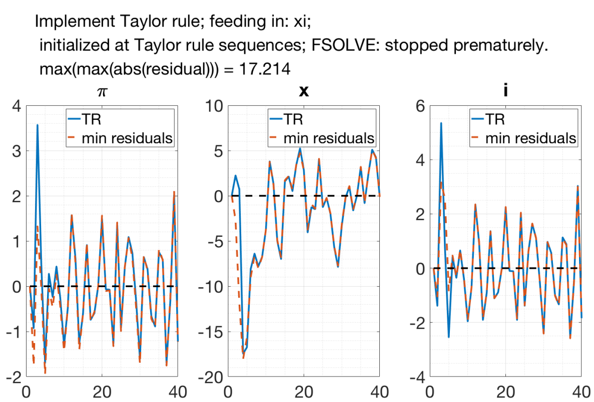
Figure 1: Simulation using Taylor rule against exogenous sequences that minimize equation residuals



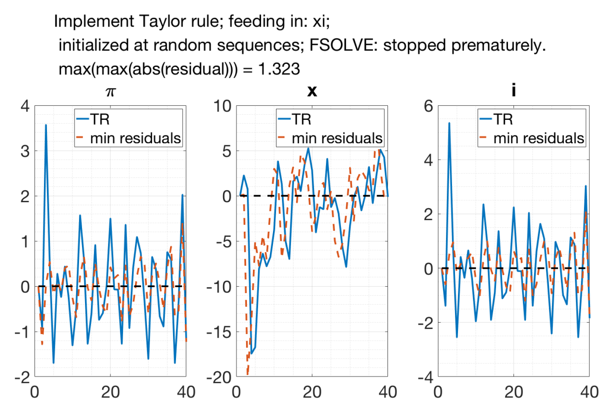
(a) Inputs: i, residual eq.: TR; initialized at TR



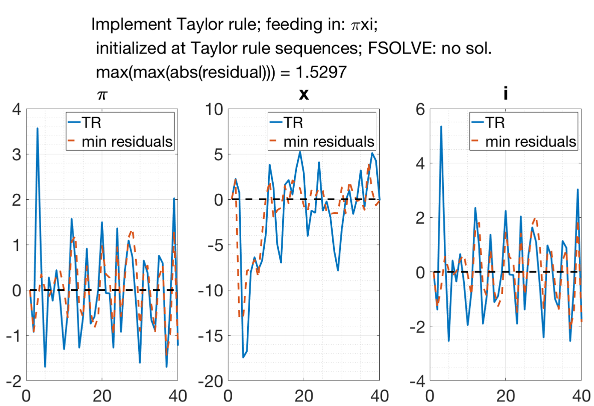
(b) Same as (a), initialized at random sequence(s)



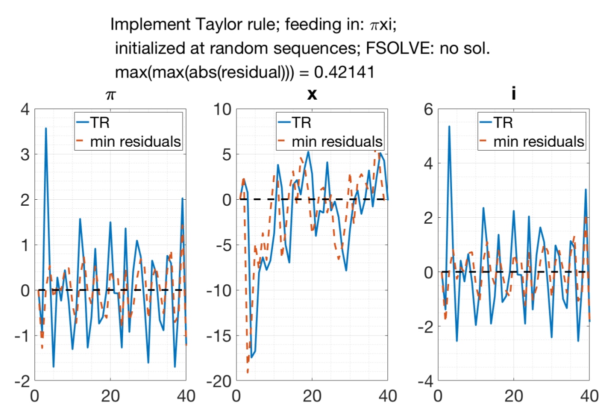
(c) Inputs: x, i, residual eq.: A9, TR; initialized at TR



(d) Same as (c), initialized at random sequence(s)



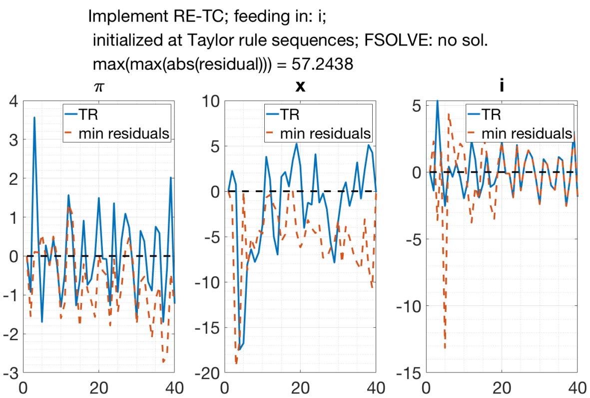
(e) Inputs: π, x, i, residual eq. A9, A10, TR; initialized at TR



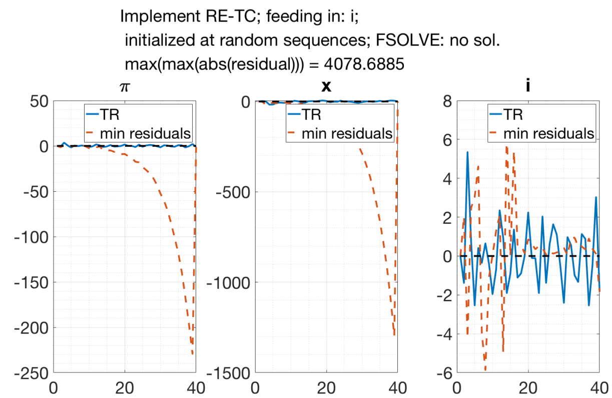
(f) Same as (e), initialized at random sequence(s)

3.2 FSOLVE: Implementing the RE-discretion target criterion

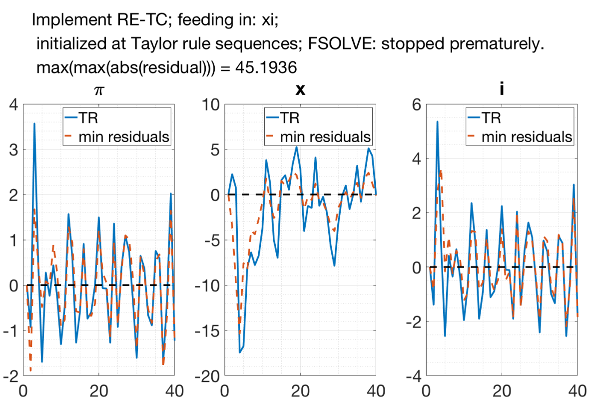
Figure 2: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including RE discretion target criterion



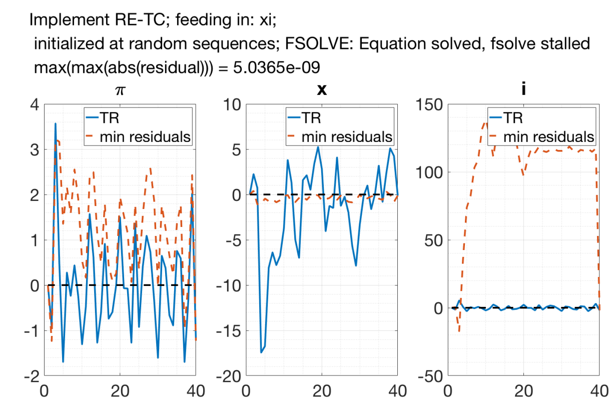
(a) Inputs: i, residual eq.: TC.



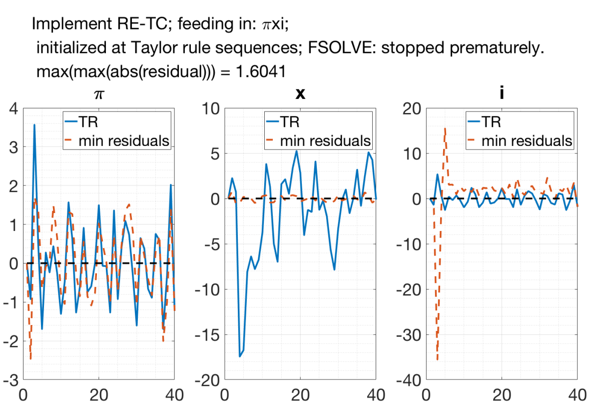
(b) Same as (a), initialized at random sequence(s)



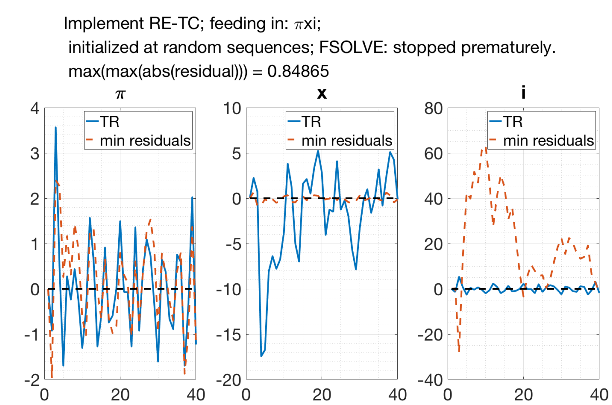
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



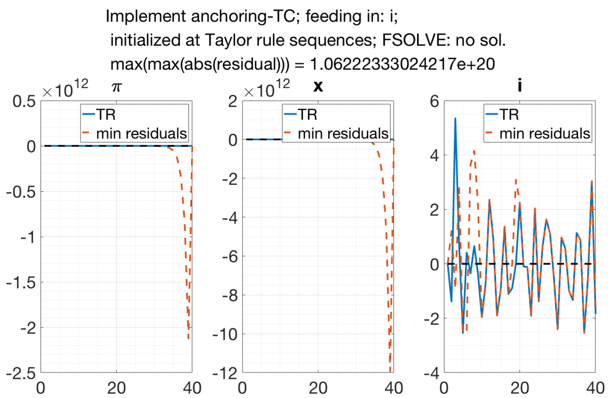
(e) Inputs: π, x, i, residual eq. A9, A10, TC.



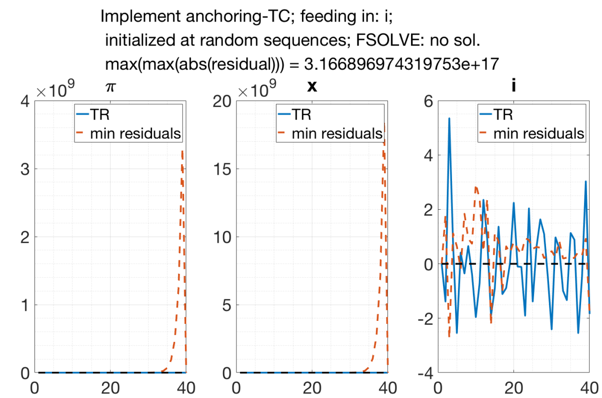
(f) Same as (e), initialized at random sequence(s)

3.3 FSOLVE: Implementing the simple anchoring target criterion

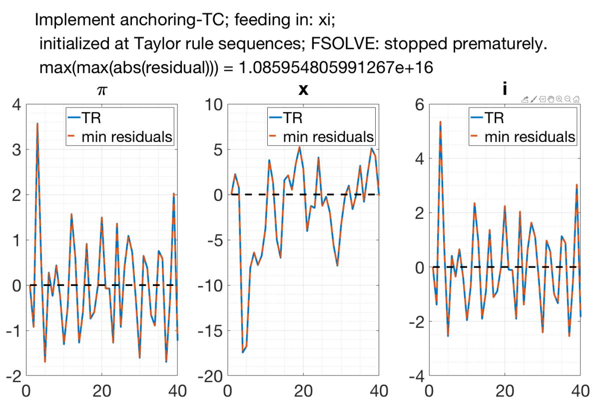
Figure 3: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including the simple anchoring target criterion



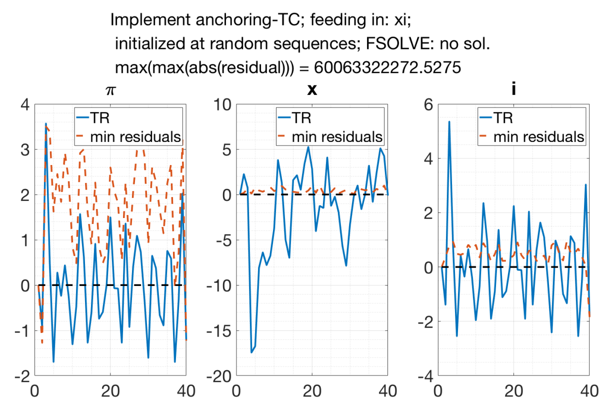
(a) Inputs: i, residual eq.: TC.



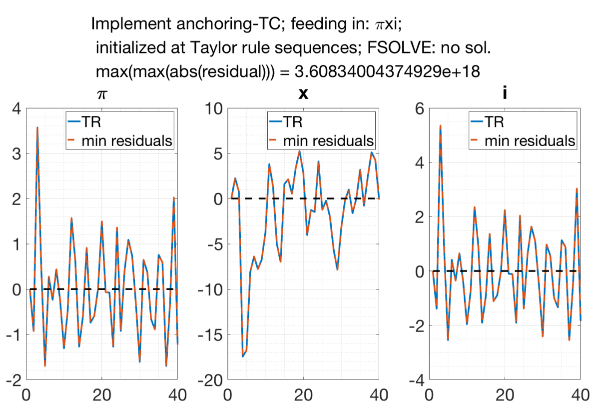
(b) Same as (a), initialized at random sequence(s)



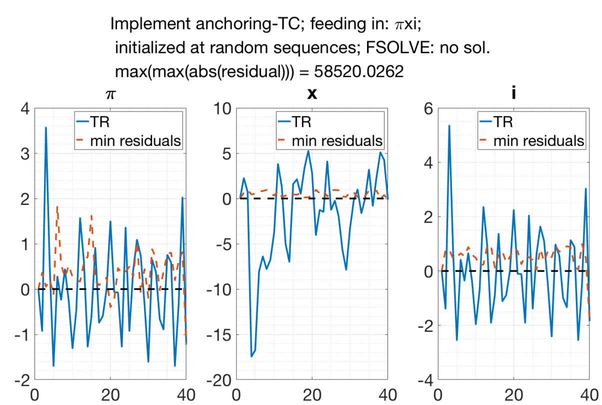
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



(e) Inputs: π, x, i, residual eq. A9, A10, TC.

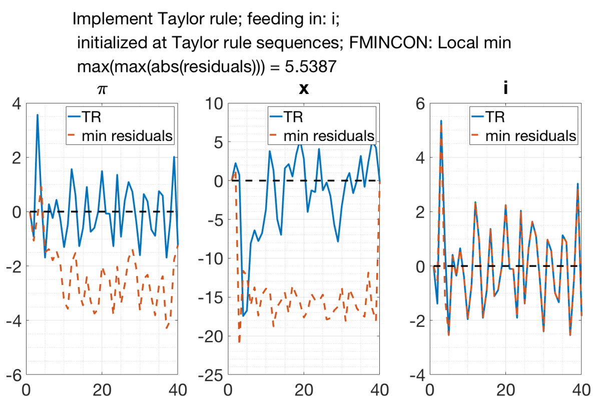


(f) Same as (e), initialized at random sequence(s)

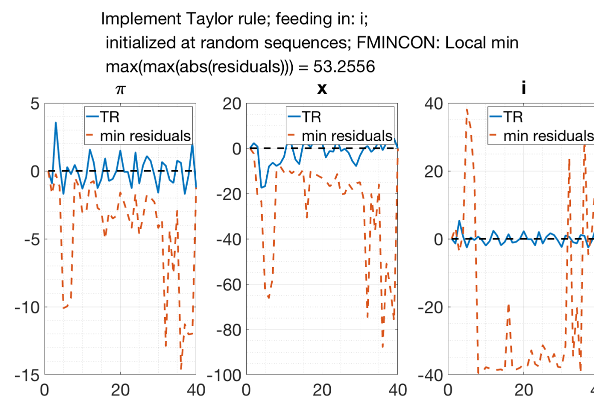
4 FMINCON

4.1 FMINCON: Implementing the Taylor-rule w/o using it

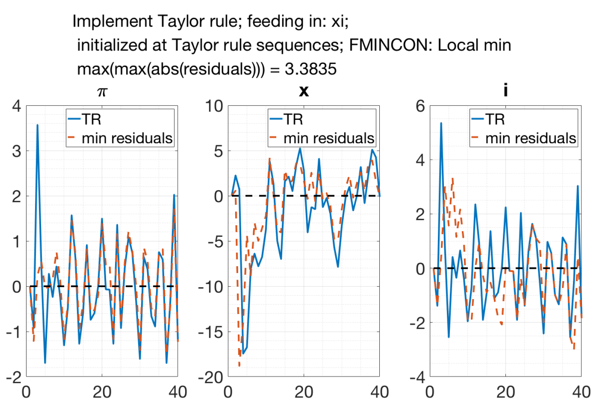
Figure 4: Simulation using Taylor rule against exogenous sequences that minimize equation residuals



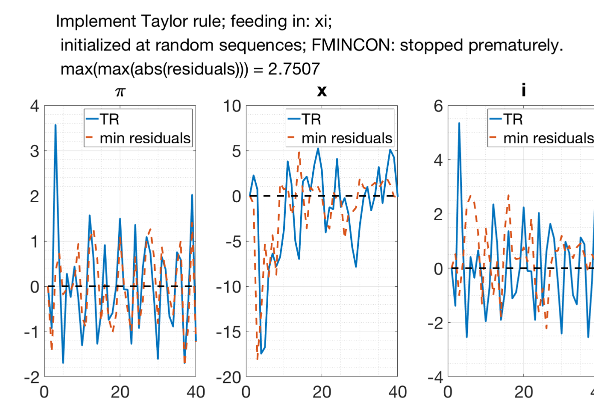
(a) Inputs: i, residual eq.: TR; initialized at TR



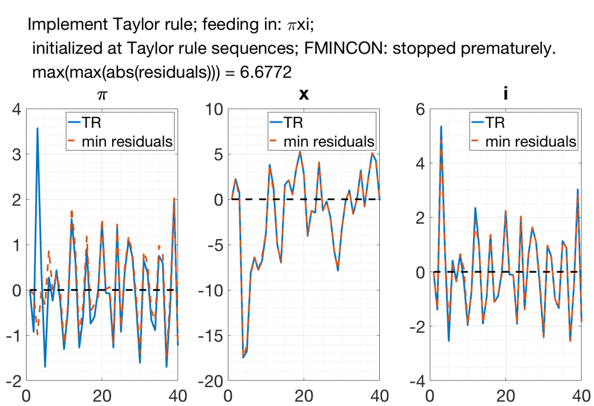
(b) Same as (a), initialized at random sequence(s)



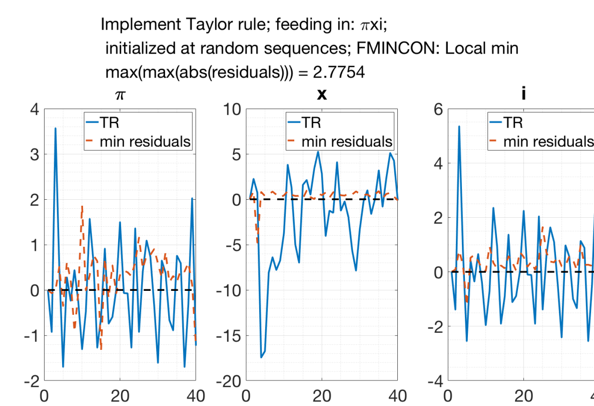
(c) Inputs: x, i, residual eq.: A9, TR; initialized at TR



(d) Same as (c), initialized at random sequence(s)



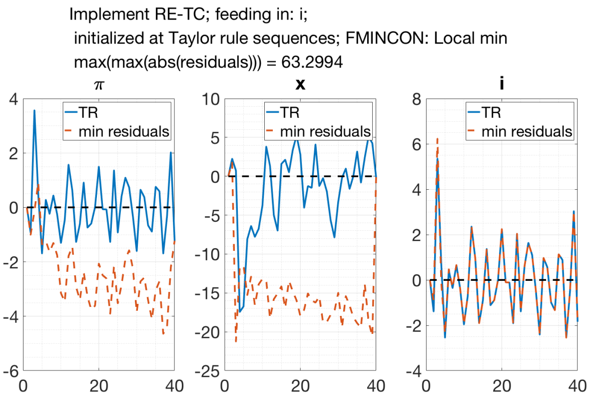
(e) Inputs: π, x, i, residual eq. A9, A10, TR; initialized at TR



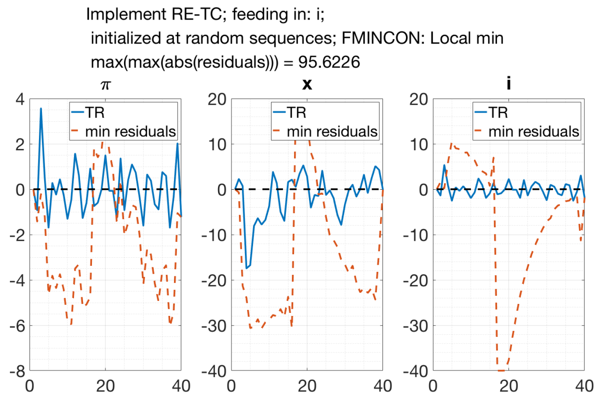
(f) Same as (e), initialized at random sequence(s)

4.2 FMINCON: Implementing the RE-discretion target criterion

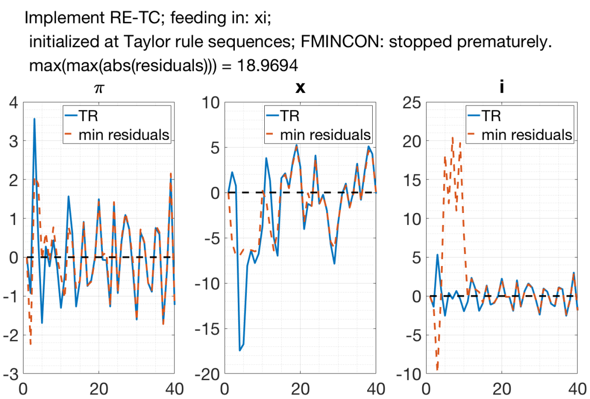
Figure 5: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including RE discretion target criterion



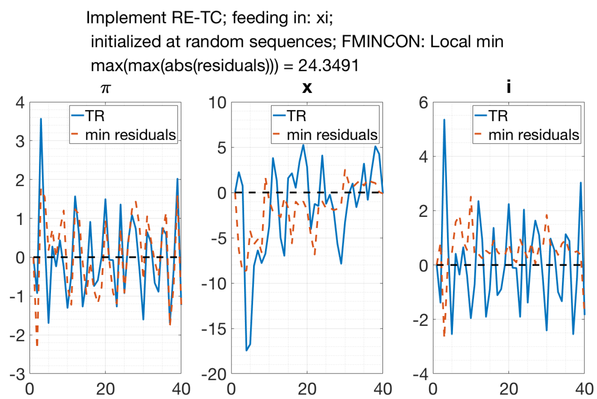
(a) Inputs: i, residual eq.: TC.



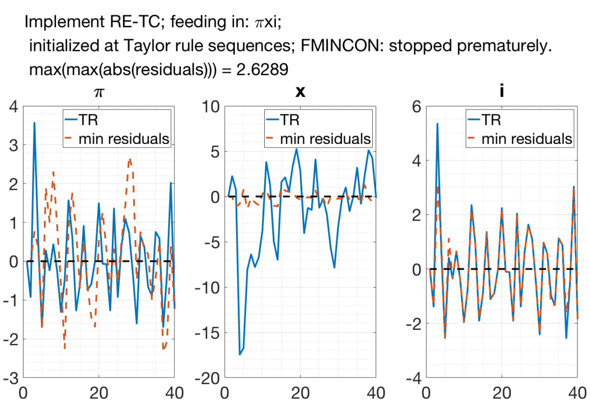
(b) Same as (a), initialized at random sequence(s)



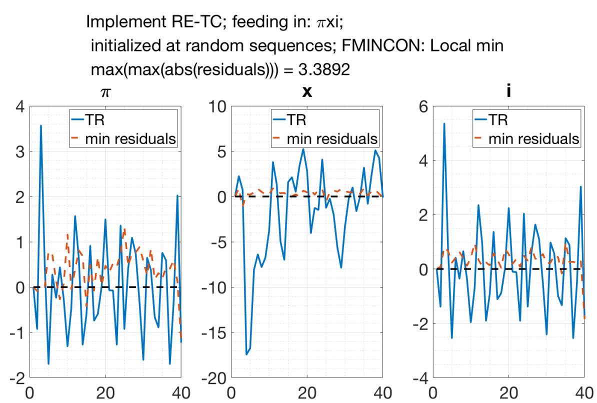
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



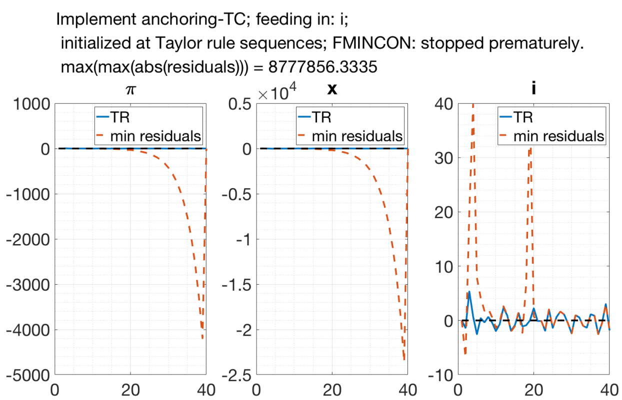
(e) Inputs: π, x, i, residual eq. A9, A10, TC.



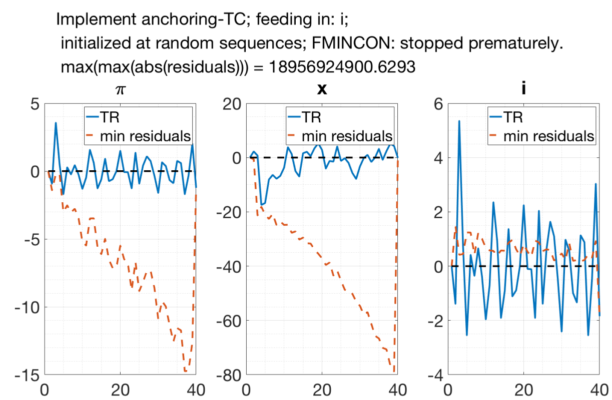
(f) Same as (e), initialized at random sequence(s)

4.3 FMINCON: Implementing the simple anchoring target criterion

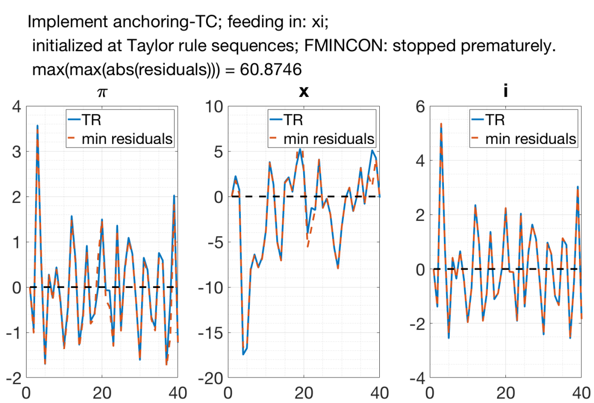
Figure 6: Simulation using Taylor rule against exogenous sequences that minimize equation residuals including the simple anchoring target criterion



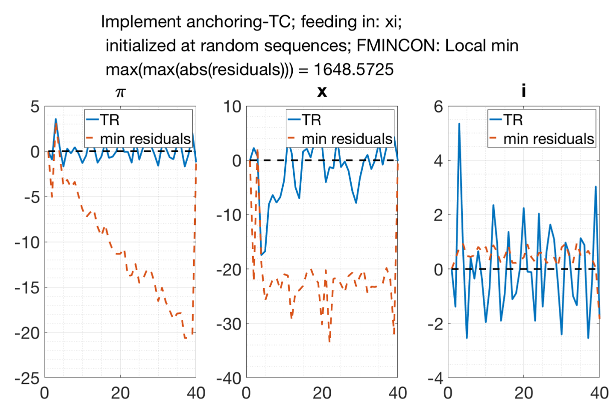
(a) Inputs: i, residual eq.: TC.



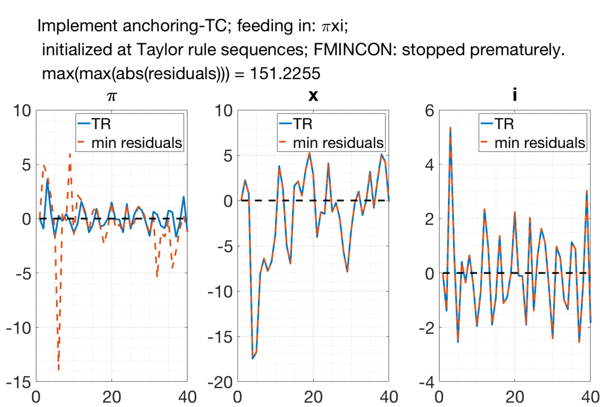
(b) Same as (a), initialized at random sequence(s)



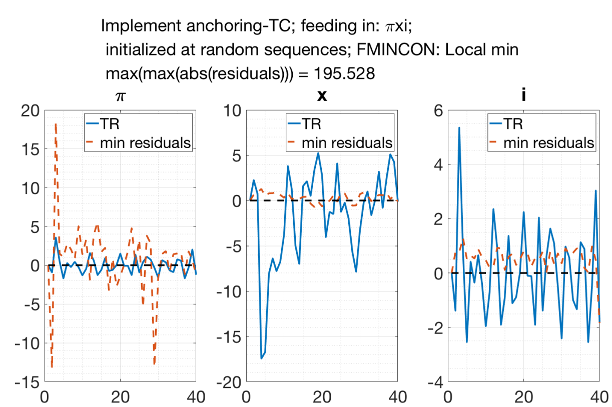
(c) Inputs: x, i, residual eq.: A9, TC.



(d) Same as (c), initialized at random sequence(s)



(e) Inputs: π, x, i, residual eq. A9, A10, TC.



(f) Same as (e), initialized at random sequence(s)

5 A value function iteration attempt at finding the optimal interest-rate-sequence

The planner chooses $\{\pi_t, x_t, i_t, f_{a,t}, f_{b,t}, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$V(\mathbf{x}_t, t) = \max -\left\{(\pi_t^2 + \lambda_x x_t^2) + \beta\, \mathbb{E}_t\, V(\mathbf{x}_{t+1}, t+1)\right\} \tag{1}$$

$$\text{s.t. to model equations} \tag{2}$$

Model equations are:

$$\pi_t = \kappa x_t + (1-\alpha)\beta f_a(t) + \kappa\alpha\beta b_2(I_3 - \alpha\beta h_x)^{-1} s_t + e_3(I_3 - \alpha\beta h_x)^{-1} s_t \tag{3}$$

$$x_t = -\sigma i_t + \sigma f_b(t) + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} s_t - \sigma\beta b_3 (I_3 - \beta h_x)^{-1} s_t + \sigma e_1 (I_3 - \beta h_x)^{-1} s_t) \tag{4}$$

$$f_a(t) = \frac{1}{1-\alpha\beta}\bar{\pi}_{t-1} + b_1 (I_3 - \alpha\beta h_x)^{-1} s_t \tag{5}$$

$$f_b(t) = \frac{1}{1-\beta}\bar{\pi}_{t-1} + b_1 (I_3 - \beta h_x)^{-1} s_t \tag{6}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}\big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})\big) \tag{7}$$

$$k_t^{-1} = k_{t-1}^{-1} + \mathbf{g}(\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \tag{8}$$

Let's substitute out $x_t, f_{a,t}$ and $f_{b,t}$, so that the state vector is simply $\mathbf{x}_t = (\bar{\pi}_t, k_t^{-1}, r_t^n, u_t)'$.

The problem becomes to choose $\{\pi_t, i_t, \bar{\pi}_t, k_t^{-1}\}_{t=t_0}^{\infty}$ to minimize

$$
\begin{aligned}
V(\mathbf{x}_t, t) = \max -\Bigg\{ & \pi_t^2 + \lambda_x \sigma^2 i_t^2 + \lambda_x \frac{\sigma^2}{(1-\beta)^2} \bar{\pi}_{t-1}^2 - \lambda_x \frac{\sigma^2}{1-\beta} i_t \bar{\pi}_{t-1} \\
& - \lambda_x \sigma \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) i_t s_t \\
& + \lambda_x \frac{\sigma}{1-\beta} \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big) \bar{\pi}_{t-1} s_t \\
& + \lambda_x \Big(\sigma b_1 (I_3 - \beta h_x)^{-1} + (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \sigma e_1 (I_3 - \beta h_x)^{-1} \Big)^2 s_t \\
& + \beta \, \mathbb{E}_t \, V(\mathbf{x}_{t+1}, t+1) \Bigg\}
\end{aligned} \tag{9}
$$

s.t. to model equations

$$
\begin{aligned}
\pi_t = & -\kappa \sigma i_t + \left(\kappa \sigma \frac{1}{1-\beta} + \frac{(1-\alpha)\beta}{1-\alpha\beta} \right) \bar{\pi}_{t-1} \\
& + \Big(\kappa \sigma b_1 (I_3 - \beta h_x)^{-1} + \kappa (1-\beta) b_2 (I_3 - \beta h_x)^{-1} - \kappa \sigma \beta b_3 (I_3 - \beta h_x)^{-1} + \kappa \sigma e_1 (I_3 - \beta h_x)^{-1} \\
& + (1-\alpha) \beta b_1 (I_3 - \alpha \beta h_x)^{-1} + \kappa \alpha \beta b_2 (I_3 - \alpha \beta h_x)^{-1} + e_3 (I_3 - \alpha \beta h_x)^{-1} \Big) s_t
\end{aligned}
$$

$$
\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} \big(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}) \big) \tag{10}
$$

$$
k_t^{-1} = k_{t-1}^{-1} + \mathbf{g} (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) \tag{11}
$$

A Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \qquad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \qquad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + b s_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \qquad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \qquad fe_{t-1} = z_t - (a_{t-1} + b s_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \qquad f_a(t) = \frac{1}{1-\alpha\beta} a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta} a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1} s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \begin{pmatrix}\bar{\pi}_t & 0 & 0\end{pmatrix}'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix}\sigma & 1-\beta & -\sigma\beta\end{bmatrix} f_b + \sigma \begin{bmatrix}1 & 0 & 0\end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1} s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix}(1-\alpha)\beta & \kappa\alpha\beta & 0\end{bmatrix} f_a + \begin{bmatrix}0 & 0 & 1\end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1} s_t \tag{A.10}$$

B Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\Bigg\{x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t)\Big)$$
$$\Bigg(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j))\Bigg)\Bigg\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^{0} \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i}$$
$$-\frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\Bigg(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j)\Bigg) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs

C A target criterion system for an anchoring function specified for gain changes

$$k_t = k_{t-1} + \mathbf{g}(fe_{t|t-1}) \tag{C.1}$$

Turns out the k_{t-1} adds one $\varphi_{6,t+1}$ too many which makes the target criterion unwieldy. The FOCs of the Ramsey problem are

$$2\pi_t + 2\frac{\lambda}{\kappa}x_t - k_t^{-1}\varphi_{5,t} - \mathbf{g}_\pi(t)\varphi_{6,t} = 0 \tag{C.2}$$

$$cx_{t+1} + \varphi_{5,t} - (1 - k_t^{-1})\varphi_{5,t+1} + \mathbf{g}_{\bar{\pi}}(t)\varphi_{6,t+1} = 0 \tag{C.3}$$

$$\varphi_{6,t} + \varphi_{6,t+1} = fe_t\varphi_{5,t} \tag{C.4}$$

where the red multiplier is the new element vis-a-vis the case where the anchoring function is specified in levels ($k_t^{-1} = \mathbf{g}(fe_{t-1})$, as in App. B), and I'm using the shorthand notation

$$c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}\frac{\lambda}{\kappa} \tag{C.5}$$

$$fe_t = \pi_t - \bar{\pi}_{t-1} - bs_{t-1} \tag{C.6}$$

(C.2) says that in anchoring, the discretion tradeoff is complemented with tradeoffs coming from learning ($\varphi_{5,t}$), which are more binding when expectations are unanchored (k_t^{-1} high). Moreover, the change in the anchoring of expectations imposes an additional constraint ($\varphi_{6,t}$), which is more strongly binding if the gain responds strongly to inflation ($\mathbf{g}_\pi(t)$). One can simplify this three-equation-system to:

$$\varphi_{6,t} = -cfe_tx_{t+1} + \left(1 + \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1}) - fe_t\mathbf{g}_{\bar{\pi}}(t)\right)\varphi_{6,t+1} - \frac{fe_t}{fe_{t+1}}(1 - k_{t+1}^{-1})\varphi_{6,t+2} \tag{C.7}$$

$$0 = 2\pi_t + 2\frac{\lambda}{\kappa}x_t - \left(\frac{k_t^{-1}}{fe_t} + \mathbf{g}_\pi(t)\right)\varphi_{6,t} + \frac{k_t^{-1}}{fe_t}\varphi_{6,t+1} \tag{C.8}$$

Unfortunately, I haven't been able to solve (C.7) for $\varphi_{6,t}$ and therefore I can't express the target criterion so nicely as before. The only thing I can say is to direct the targeting rule-following central bank to compute $\varphi_{6,t}$ as the solution to (C.8), and then evaluate (C.7) as a target criterion. The solution to (C.8) is given by:

$$\varphi_{6,t} = -2\,\mathbb{E}_t\sum_{i=0}^{\infty}(\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i})\prod_{j=0}^{i-1}\frac{\frac{k_{t+j}^{-1}}{fe_{t+j}}}{\frac{k_{t+j}^{-1}}{fe_{t+j}} + \mathbf{g}_\pi(t+j)} \tag{C.9}$$

Interpretation: the anchoring constraint is not binding ($\varphi_{6,t} = 0$) if the CB always hits the target ($\pi_{t+i} + \frac{\lambda_x}{\kappa}x_{t+i} = 0 \quad \forall i$); or expectations are always anchored ($k_{t+j}^{-1} = 0 \quad \forall j$).