Materials 30 - Parameterized expectations

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Let me note for everyone who is a little slow of mind: this is a projection method to solve systems of nonlinear difference/differential equations. To clean up the confusion in my mind: projection belongs to the class of global methods, which should almost more aptly be called "functional equation methods" because their take on the problem is to zero out (collocation) or minimize weighted residuals (least squares, method of moments, Galerkin method) of a functional equation.[1] The argument of this minimization is the regression coefficient on a basis of an approximated ("parameterized") function - this is the "projection" concept. The basis can vary: spectral approximation uses polynomials (e.g. Chebyshev) over the whole domain, finite element approximation uses polynomials over N-dimensional subdomains (simplices/rectangles - now I understand the triangulation business you made me look into in my first semester!). If I understand correctly, in the paper with Ozge, the basis is $s(x_t)$, the set of elementwise x_t, x_t^2, x_t^3 including a subset of the cross-products.

Procedure to solve model via parameterizing $E_t \equiv \mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(k^{-1}_{t+1+j} + fe^{eve}_{t+1+j|t+j})\mathbf{g}_{\bar{\pi},\mathbf{t+j}}$ in (B.1). Let $\hat{E}_t$ denote the approximated E_t. Then:

1. Conjecture an initial expectation $\hat{E}_t = \beta^0 s(X_t)$

   β^0 are the coefficients that I plan to initialize as the unit vector.

   $s(X_t)$ is the basis, $X_t = (k_t, \bar{\pi}_{t-1}, r^n_t, u_t)$ is the state vector. Initially, I'll just take the first, second and third powers of X_t, ignoring all cross-terms. Can deal with those later.

2. Solve model equations given conjectured $\hat{E}_t$ for a given sequence of shocks

   I think this means for me to evaluate model equations (A.1)-(B.1) (without (A.3)) given input sequences, shocks and $\hat{E}_t$. I think this also implies a zeroing-out-the-residuals step so that model equations are solved. (Anticipate a hiccup here - what if the solver gets stuck on this step?) Solving the equations delivers $\{v_t\}$, the simulated history of endogenous variables ("synthetic time series").

[1] Value function iteration is also a global method. The difference to projection is only that it solves the functional equation by iteration, not by projection.

3. Compute realized analogues to E_t given $\{v_t\}$

   I think this means using the values in v_t to evaluate E_t. But how exactly? Do I just, for each t, input all the $t + j$ values $j = t + 1, \ldots, T$ from the synthetic series $\{v_t\}$?

4. Update β regressing the synthetic E_t on $s(X_t)$.

   Lol, the only step that seems straightforward to me: $\beta^{i+1} = (s(X_t)'s(X_t))^{-1}s(X_t)'E_t$.

   Then I guess I iterate until convergence by evaluating at every step $||\beta^i - \beta^{i+1}||$.

A Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \qquad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \qquad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + b s_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \qquad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \qquad fe_{t-1} = z_t - (a_{t-1} + b s_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \qquad f_a(t) = \frac{1}{1-\alpha\beta} a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta} a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1} s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \begin{pmatrix} \bar{\pi}_t & 0 & 0 \end{pmatrix}'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix} \sigma & 1-\beta & -\sigma\beta \end{bmatrix} f_b + \sigma \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1} s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix} (1-\alpha)\beta & \kappa\alpha\beta & 0 \end{bmatrix} f_a + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1} s_t \tag{A.10}$$

B Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\Bigg\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t)\Big) \\ \Bigg(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j))\Bigg)\Bigg\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^{0} \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \\ -\frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\Big(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t)\Big)\Bigg(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1}(k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j)\Bigg) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs