

You know, I don't know about that.

16 March 2020

- I wanna pause on for a sec b/c I can't seem to get it to work and I'm confused about how to get it time-varying anyway. So let's turn to estimation.

Estimation of the anchoring function

The issue is that we wanna estimate the anchoring function together w/ the model. On the top of my head I can think of 3 ways of doing that:

- 1) IR-matching
- 2) likelihood-based (either MLE or Bayesian)
- 3) VAR-representation \Rightarrow exist? est. that!

\hookrightarrow it would be a time-varying one.

- \hookrightarrow I'm leaning toward #2 b/c 1) it's sexier
2) it's more general than conditional on shocks
3) a TV-VAR sounds challenging

The thing is:

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need to derive the log-likelihood of my model

Take the midsample model w/ no TR

In materials 21, this is eq. (5)-(10) + TR

$$\pi_t = kx_t - (1-\alpha)\beta f_{\alpha}(t) + [-\kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}]s_t$$

$$x_t = -b\bar{\pi}_t - b f_b(t) + [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}]s_t$$

$$f_{\alpha}(t) = \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} - b_1(f_3 - \alpha\beta h_x)^{-1}s_t$$

$$f_b(t) = \frac{1}{1-\beta} \bar{\pi}_{t-1} - b_1(f_3 - \beta h_x)^{-1}s_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d(\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + c$$

$$i_t = \gamma_{\pi}\pi_t + \gamma_x x_t + \bar{i}_t$$

this is a state-space model (believe it or not)

and I'm gonna eliminate some variables

$$\begin{aligned} \pi_t &= kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} - (1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1}s_t \\ &\quad + [-\kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}]s_t \end{aligned}$$

$$\begin{aligned} x_t &= -b\pi_t - b f_b(t) - b\bar{i}_t - \frac{b}{1-\beta} \bar{\pi}_{t-1} - b b_1(f_3 - \beta h_x)^{-1}s_t \\ &\quad + [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}]s_t \end{aligned}$$

$$\pi_4 = \kappa x_+ - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ [-(1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}] s_t$$

$$x_+ = -b\gamma_\pi \pi_4 - b\gamma_x x_+ - bi_r - \frac{b}{1-\beta} \bar{\pi}_{t-1} - b b_1(I_3 - \beta h_x)^{-1} s_t$$

$$+ [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} - (1-\beta)b_3(f_3 - \beta h_x)^{-1} + 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$(1+b\gamma_x) x_+ = -b\gamma_\pi \pi_4 - \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$[-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$\Rightarrow x_+ = -\frac{b\gamma_\pi}{1+b\gamma_x} \pi_4 - \frac{1}{1+b\gamma_x} \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$+ \frac{1}{1+b\gamma_x} [-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

Can even sub x_+ out!

$$\pi_4 = -\frac{\kappa b\gamma_\pi}{1+b\gamma_x} \pi_4 - \frac{\kappa}{1+b\gamma_x} \frac{b}{(1-\beta)} \bar{\pi}_{t-1}$$

$$+ \frac{\kappa}{1+b\gamma_x} [-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ [-(1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}] s_t$$

$$\Rightarrow \left(1 + \frac{\kappa b\gamma_\pi}{1+b\gamma_x}\right) \pi_4 = -\left(\frac{\kappa b}{(1+b\gamma_x)(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right) \bar{\pi}_{t-1} + \text{stuff} \cdot s_t$$

$$\left(1 + \frac{k_2 Y_\pi}{1+bY_x}\right) \bar{\pi}_+ = - \left(\frac{k_2}{1+bY_x(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)} \right) \bar{\pi}_{t-1} +$$

$$+ \left\{ \frac{k}{1+bY_x} \left[-2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \left[-(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\} s_t$$

$$\frac{1+bY_x+k_2 Y_\pi}{1+bY_x} \bar{\pi}_+ = \text{same}$$

$$\Rightarrow \bar{\pi}_+ = - \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left(\frac{\frac{k_2(1-\alpha\beta)}{(1+bY_x)(1-\beta)} + (1-\alpha)\beta(1-\beta)(1+bY_x)}{(1-\alpha\beta)} \right) \bar{\pi}_{t-1}$$

$$+ \left\{ \frac{k}{1+bY_x+k_2 Y_\pi} \left[-2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left[-(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\}$$

$$\Rightarrow \bar{\pi}_+ = - \frac{\frac{k_2(1-\alpha\beta)}{(1+bY_x+k_2 Y_\pi)} + \beta(1-\alpha)(1-\beta)(1+bY_x)}{(1+bY_x+k_2 Y_\pi)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left\{ \frac{k}{1+bY_x+k_2 Y_\pi} \left[-2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left[-(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\} s_t$$

Damn damn! So we have \xrightarrow{EA}

$$\bar{\pi}_t = - \frac{k_2(\alpha-\alpha\beta) + \beta(1-\alpha)(1-\beta)(1+bY_X)}{(1+bY_X+k_2Y_H)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left[\frac{k}{1+bY_X+k_2Y_H} \left[-2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_X}{1+bY_X+k_2Y_H} \left[-(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right] s_t \xrightarrow{EB}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})) + c$$

\hookrightarrow 1 jump ($\bar{\pi}_t$), 3 exogenous states ($s_t = \begin{bmatrix} r_t \\ i_t \\ h_t \end{bmatrix}$) and 2

endogenous states $\xi_t = \begin{bmatrix} \bar{\pi}_t \\ k_t^{-1} \end{bmatrix}$ ($\propto \begin{bmatrix} \bar{\pi}_{t-1} \\ k_t^{-1} \end{bmatrix}$) so $X_t = \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$ (states)

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = [A \ B] \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})) + c$$

call this the state (?) f_{t-1}

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = [A \ B] \begin{bmatrix} \bar{s}_t \\ s_t \end{bmatrix}$$

$$Y_t = g x \cdot X_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} f c_{t-1}$$

$$X_{t+1} = h X_t + \eta \epsilon_t$$

$$k_t^{-1} = k_{t-1}^{-1} + d \cdot f c_{t-1} + c$$

$$f c_{t-1} = \pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}$$

Several issues y'all:

- 1) fc state or jump? depends on π_t (a jump)
 → there's gotta be some trick (like π_t is LEMP)
 to make it a pure state

- 2.) $\bar{\pi}$ nonlinear Lom!



ok - that's troubling. But let's pause it for a sec & let's read what Litterford has to say about MLE & log-likelihoods.

Supp. our VAR(p) looks like

$$y_t - \mu = A_1(y_{t-1} - \mu) + \dots + A_p(y_{t-p} - \mu) + u_t \quad (3.3.1)$$

then if the VAR(p) is Gaussian, that is

$$u \equiv \text{vec}(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim N(0, I_T \otimes \frac{1}{T} I_n) , \text{ which}$$

p.75

equivalently means that the prob. density of u is

$$f_u(u) = \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp \left[-\frac{1}{2} u' (I_T \otimes \Sigma_u^{-1}) u \right]$$

then we can use the fact that $u = y - \mu^* - (x' \otimes I_K) \alpha$

where $\alpha := \text{vec}(A)$, $A := (A_1, \dots, A_p)$ $k \times kp$

$$k^2 p \times 1$$

$$Y^0 := (y_1 - \mu, \dots, y_T - \mu) \quad k \times T$$

$$Y_t^0 := \begin{bmatrix} y_t - \mu \\ y_{t-p+1} - \mu \end{bmatrix} \quad (k_p \times 1)$$

$$X := (Y_0^0, \dots, Y_{T-1}^0),$$

to write

$$\begin{aligned} f_y(y) &= \left| \frac{\partial y}{\partial u} \right| f_u(u) \\ &= \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp \left[-\frac{1}{2} (y - \mu^* - (x' \otimes I_K) \alpha)' (I_T \otimes \Sigma_u^{-1}) \right. \\ &\quad \left. \cdot (y - \mu^* - (x' \otimes I_K) \alpha) \right] \quad (3.4.4) \end{aligned}$$

and

$$\ln L(\mu, \alpha, \Sigma_u) = -\frac{kT}{2} \ln(2\pi) - \frac{T}{2} \ln |\Sigma_u| - \frac{1}{2} \text{tr} (Y^0 - Ax)' \Sigma_u^{-1} (Y^0 - Ax)$$

is the log-likelihood

(3.4.5)

Given that the updating w/ endog. gain 18 March 2020 introduces non-linearities, I'm afraid that even a "simple" & quick estimation has to involve some form of particle filter. But let's see whether Litterpol has anything interesting to say about 1) state-space models 2) non-linearities

Litterpol, State-space models (Ch. 13, p. 415 ff.)

$$\begin{aligned} z_{t+1} &= B_t z_t + F_t x_t + w_t \xrightarrow{\text{noise}} t=0,1,2, \dots \quad (13.2.1_1) \\ y_t &= H_t z_t + G_t x_t + v_t \quad t=1,2, \dots \quad (13.2.2) \end{aligned}$$

transition
matrix input
measurement
matrix inputs/
measurements/
policy vars measurement error

$$\text{and } \begin{bmatrix} w_t \\ v_t \end{bmatrix} \sim WN\left[0, V_L\right] \quad V_L = \begin{bmatrix} \frac{1}{2} w_t^2 & \frac{1}{2} w_t v_t \\ \frac{1}{2} v_t w_t & \frac{1}{2} v_t^2 \end{bmatrix}$$

Nonlinear state-space models

$$\begin{aligned} z_{t+1} &= b_t(z_t, x_t, w_t, \delta_1) \\ y_t &= h_t(z_t, x_t, v_t, \delta_2) \end{aligned}$$

vectors of params

Example of nonlinear state-space is the "bilinear" model:

$$y_t = \alpha y_{t-1} + u_t + \beta y_{t-1} u_{t-1} \quad p. 427 w/ Refs.$$

$$\hookrightarrow z_{t+1} = B z_t + w_t + C \text{vec}(z_t z_t') \quad (13.2.33)$$

$$y_t = [I_k \ 0 \dots 0] z_t \quad (13.2.34)$$

\Rightarrow Bounds of refs on bilinear systems, univariate & multivariate on p. 427 bottom.

MLE of state-space models p. 434

Gather the time-invariant params from $B, F, H_1, G_1, \Sigma_w, \Sigma_0$ and $\Sigma_0 \& \mu_0$ in δ as

$$\delta = \begin{bmatrix} \text{vec}[v, A_1, \dots, A_p] \\ \text{vech}(\Sigma_0) \end{bmatrix}$$

where $\text{vech} = \text{"half-vectorization"}$, $\text{vech}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{bmatrix} a \\ b \\ d \end{bmatrix}$

The log-likelihood for the Gaussian state-space model is:

$$\ln L(\delta | y_1, \dots, y_T) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_y(t|t-1)| - \frac{1}{2} \sum_{t=1}^T (y_t - y_{t|t-1})' \Sigma_y(t|t-1)^{-1} (y_t - y_{t|t-1}) \quad (13.4.1)$$

Denoting the first error $e_t(\delta) := y_t - \hat{y}_{t|t-1}$

and $\hat{\epsilon}_t(\delta) := \hat{E}_{y_t}(t|t-1)$, we can rewrite this as

$$\ln l(\delta) = -\frac{kT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T [\ln |\hat{\epsilon}_t(\delta)| + e_t'(\delta) \hat{\epsilon}_t(\delta)^T e_t(\delta)] \quad (13.4.3)$$

which makes explicit

- 1) the dependence of $\ln l$ on δ ,
- 2) that all quantities in this $\ln l$ are functions of δ and can (most) be computed using the Kalman filter.

locally identified: when in a subspace of the param space, δ is uniquely determined.

v. globally identified: when δ is uniquely determined in the entire param space.

↪ identification: we need a min of $-\ln l$, so we need some sort of Hessian = pos. def. \Rightarrow the information matrix, $= E[\text{Hessian}] = E \left[\frac{\partial^2 (-\ln l)}{\partial \delta \partial \delta} \right]_{\delta_0}$

A quick note on DSEMs (dynamic simultaneous equations models | a.k.a. "linear systems") p. 323

essentially, these are linear VARMAX(p, s, q) models

$$A_0 y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + B_1 x_{t-1} + \dots + B_s x_{t-s} + w_t \quad (10.1.1)$$

- VARMAX(p, s, q) if $w_t \sim WN$
- VARX(p, s) if $w_t \sim WN$.

VAR(p) models w/ time-varying coefficients p. 891ff.

periodic VARs \rightarrow e.g. w/ seasonal dummies

intervention models \rightarrow DGP₁ is replaced by DGP₂ at time T.

$$y_t = v_t + A_{1,t} y_{t-1} + \dots + A_{p,t} y_{t-p} + u_t \quad (12.2.1)$$

$\hookrightarrow WN(0, \Sigma_t)$

also time-varying
(not identically distib.)

Rewrite the VAR(p) as a VAR(1)

$$Y_t = V_t + A_t Y_{t-1} + U_t \quad (12.2.2)$$

$$Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{kp \times 1}, \quad v_t := \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{kp \times 1}, \quad A_t := \begin{bmatrix} A_{1,1} & \dots & A_{p-1,1} & A_{p,1} \\ I_k & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_k & 0 \end{bmatrix}_{kp \times kp}, \quad u_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{kp \times 1}$$

And by recursive subst we get

$$Y_t = \left[\prod_{j=0}^{h-1} A_{t-j} \right] Y_{t-h} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j} \right] v_{t-i} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j} \right] u_{t-i} \quad (12.2.3)$$

Defining $J := [I_k \ 0]$ such that $y_t = J Y_t$, we can premultiply (12.2.3) by J , define

$$\bar{\Phi}_{it} := J \left[\prod_{j=0}^{i-1} A_{t-j} \right] J' \quad \text{to get}$$

$$y_t = \mu_t + \sum_{i=0}^{\infty} \bar{\Phi}_{it} u_{t-i} \quad (12.2.4)$$

where $\mu_t = E[y_t]$

\Rightarrow Then the MSE (or FEV of the FE $y_{t+h} - \hat{y}_t(h)$) is

$$\bar{\Phi}_t(h) := \sum_{i=0}^{h-1} \bar{\Phi}_{i,t+h} \bar{\Phi}_{t+h-i} \bar{\Phi}_{i,t+h}' \quad (12.2.10)$$

MLE of TV-VAR

p. 394

$$\text{Write (12.2.1) as } y_t = B_t z_{t-1} + u_t \quad (12.2.11)$$

$$\text{where } B_t := [Y_t, A_{1t}, \dots, A_{pt}], \quad z_{t-1} := (1, Y_{t-1})'$$

B_t depend on the vector γ of time-invariant params.

z_{t-1} depend on B of fixed params.

$y_t | u_t \sim N(0, \Sigma_t)$, the log-likelihood is

$$\ln l(\gamma, \beta) = -\frac{kT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_t| - \frac{1}{2} \sum_{t=1}^T u_t' \Sigma_t^{-1} u_t \quad (12.2.12)$$

where initial conditions have been ignored.

You can also derive an info-matrix.

So it seems like you just min -ln $l(\gamma, \beta)$!

It also seems like for certain special cases you can even derive the estimators in closed-form!

Ryan meeting

18 March 2010

from

not yet convinced that the procedure is sensible
→ the only procedure that can work is

$f_{2,t}$ and more RHS-variables: $k^{-1}, \bar{\pi}$

$$z_t = \bar{z}_t + f_{2,t} z_{t-1} + f_{4,t} h_t$$

values in $f_{2,t}$ will be 0 or simply
described by the LOMs

It may be too restrictive

A computationally intensive exercise is:

- long path for the int. rate (e.g.)
 - solve the model for next int. rate
 - check the target criterion, compute the resid
 - fmincon to min that resid
- ⇒ find simulated optimal plan.

- Simulate the Ramsey model
- Simulate the model w/ Taylor rule

→ see how close you can get

It might then be that most results won't
be pencil & paper.

Comments:

- Concerning w/ linearity of (10)
 - "negative surprises cause me to be unanchored" shouldn't be "big mistakes" → so take the square
 - ↳ like smooth one better than jumpy
- Estim: filter the data → App paper w/ Robert
 match the params to moments of data
 → would give you results

They est an NK model: HP-filter both data and
model, compute moments and try to match
those.

Work after

It's the primal economy paper w/ Robert and Ryan meant. \rightarrow ChauhanWorld.pdf in "literature."

\Rightarrow they¹⁾ take data on real per-capita output, inflation, nominal interest rates & per-capita hours, ^(BK) 2) high-pass filter them, 3) use GMM to estimate the process $\tilde{\tau}_t$ that best-matches the autocorrelation structure of the data.

App. D. p. 46 (Mac).

Wedges are an MA(14):

$$\tau_t = \Phi_\varepsilon(L) \varepsilon_t + \Phi_u(L) u_t$$

$$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2) \quad u_t \sim WN(0, \sigma_u^2)$$

\Rightarrow params to be estimated are

$$\gamma_{ma} = (\Phi_\varepsilon, \Phi_u, \sigma_\varepsilon^2)$$

I think that the following is the target: $t=8$

$$\tilde{\Sigma}_{\tau, T} = \text{rech} \left\{ \text{Var} [\tilde{q}_t^{\text{data}}, \dots, \tilde{q}_{t+k}^{\text{data}}] \right\} \quad q = y_t, \pi_t, b_t, w_t$$

Filting: Baxter-King filter w/ truncation horizon 32, lag-length 12

$\rightarrow \tilde{q}_t = BK_{32}(q_t)$, so \tilde{q}_t is filtered data.

then, they do a trick in converting γ_{ma} into γ_{ar} to finally estimate $\hat{\gamma}$ as:

$$\hat{\gamma}_{\text{ar}} = \underset{\gamma_{\text{ar}}}{\operatorname{argmin}} (\tilde{\Sigma}_T - \tilde{\Sigma}(\gamma_{\text{ar}}))' W^{-1} (\tilde{\Sigma}_T - \tilde{\Sigma}(\gamma_{\text{ar}}))$$

- W is a diagonal matrix w/ the bootstrapped variances of $\tilde{\Sigma}_T$ along the main diagonal.
- The model analogue $\tilde{\Sigma}(\gamma_{\text{ar}})$ is computed after the model data has been similarly filtered as the data.

↳ Hold it there, now I know what I need to do.

↳ That and

- the numerical implementation of the target criterion \Rightarrow both are things to do once I have the big screen. So now give a last try to do the time-varying one.

The ori - a first shot

Ryan wrote:

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{uz,t} u_t$$

I think he meant $z_t = \bar{z}_t + h_{z,t} z_{t-1} + f_{z,t} u_t$

let me ignore r_t^n b/c it just blows things up and I just wanna see if it works. (ignore \bar{z} too.)

$$\begin{aligned} h_{\pi,t} \bar{a}_{t-1} + f_{\pi,t} u_t &= \kappa(h_{x,t} x_{t-1} + f_{x,t} u_t) \\ &\quad - (1-\alpha)\beta(h_{fa,t} f_a(t-1) + f_{fa,t} u_t) + exog_1 \cdot u_t = D \quad (9) \\ h_{x,t} x_{t-1} + f_{x,t} u_t &+ \beta(h_{i,t} i_{t-1} + f_{i,t} u_t) \\ &- \beta(h_{fb,t} f_b(t-1) + f_{fb,t} u_t) + exog_2 \cdot u_t = D \quad (10) \end{aligned}$$

↪

$$\begin{aligned} h_{\pi,t} \bar{a}_{t-1} - \kappa h_{x,t} x_{t-1} - (1-\alpha)\beta h_{fa,t} f_a(t-1) \\ + \underbrace{(f_{\pi,t} - \kappa f_{x,t} - (1-\alpha)\beta f_{fa,t} + exog_1)}_{u_t} = 0 \quad (10) \end{aligned}$$

This is exactly what I have in materials 21

$$\begin{aligned} h_{x,t} x_{t-1} + \beta h_{i,t} i_{t-1} - \beta h_{fb,t} f_b(t-1) \\ + (f_{x,t} + \beta f_{i,t} - \beta f_{fb,t} + exog_2) u_t = 0 \end{aligned}$$

$$z_t = \bar{z}_t + h_{\bar{z},t} z_{t-1} + f_{\bar{z},t} u_t$$

$$h_{fat,t} f_{at}(t-1) + f_{fat,t} u_t - \frac{1}{1-\alpha\beta} \left(h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + f_{\bar{\pi},t-1} u_{t-1} \right) + exog_3 u_t$$

$$h_{fat,t} f_{at}(t-1) - \frac{1}{1-\alpha\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + \underbrace{(f_{fat,t} + exog_3) u_t - \frac{1}{1-\alpha\beta} f_{\bar{\pi},t-1} u_{t-1}}_{\text{same as in materials 21}} = 0 \quad (9)$$

$$h_{fb,t} f_b(t-1) - \frac{1}{1-\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + \underbrace{(f_{fb,t} + exog_3) u_t - \frac{1}{1-\beta} f_{\bar{\pi},t-1} u_{t-1}}_{(10)} = 0$$

Honestly, I don't even think I will complete this b/c

we can see that if, in say (10), $h_{fb,t} = 0$ and $h_{\bar{\pi},t-1} = 0$

then we're back to exactly the system I had in mat21.

So suppose $h_{fb,t} \neq 0$. But then in each iteration we have an earlier iteration of itself. Now I might argue

$h_{\bar{\pi}} = h_x = h_i = 0$, but if I ass that for $h_{fat,t} = h_{fb,t} = 0$ then again the link between (9)-(10) and (5)-(6) is broken. Then (10) would give an additional constraint

$$h_{fb,t} f_b(t-1) = \frac{1}{1-\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2}$$

In two unknowns. Mm... You'd get a proliferation of unknowns...! I'll stop here, this wasn't for endog states, merde!

① GMM of midsimple

19 March 2020

② Numerical implementation of target criterion

① GMM of midsimple

sim_learnM.m w/ PLM = constant-only

→ Need to:

1) add smooth function as a third criterion

2) To have "midsimple", we need that x (and i)

aren't learned, so input $a = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$, $b = b^{\text{RE}}$

(\Rightarrow a_i , I've learned one thing

(comparing w/ IRFs of materials 9)

⇒ when only π is learned, IRFs are less

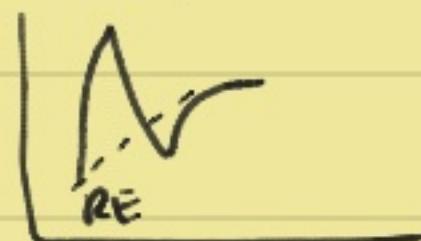
oscillatory in the sense that they overshoot

only once:

when $(\pi, x, i)^T$ learned



vs.



→ did I ever conclude this? I thought so, but now it doesn't seem like it. In materials? I just state that vector vs. scalar learning is pretty much the same

... And it almost is, just not quite.
⇒ so mental note!

Learning only $\text{Lom}(\alpha)$ dampens the oscillations b/c $E(\cdot)$ in NKIS & NKPC are moving less, and therefore $E(i+k)$ is moving less. I think more importantly since the bulk of action is in x , the fact that $\text{FE}(x)$ is now cut out gets rid of the FE that was most oscillatory.

Ok I just wasted an hour trying to customize buys in Matlab ...

Back to 1) the anchoring function:

$$k_t^{-1} - k_{t-1}^{-1} = c + d(FE)$$

has the interpretation that the gain decreases when $FE < 0$. Ryan is right: the easiest thing that makes Δk_t^{-1} big when FE is big in absolute value is FE^2

$$\rightarrow k_t^{-1} - k_{t-1}^{-1} = c + d FE_t^2$$

↑ does this even make sense?

> Not really: it's saying that the gain always changes, even if the $FE^2 = 0$.

↳ So then gather data and implement

- HP
- BK
- Hamilton filters

But there is a problem: for this current form

$$k_t^{-1} - k_{t-1}^{-1} = (c + d \cdot FE^2),$$

it always explodes, even more so if $c=0$.

And it's b/c $k \rightarrow 0$, so I guess k^{-1} explodes.

↳ Need to be smart about this!

Normally, a gain would be

20 March 2020

$$k_{t+1} = k_t + 1$$

$$\text{so I could just do } k_t = k_{t-1} + \frac{1}{FE^2}$$

so that if in the limit $FE^2 \rightarrow \infty$, $k_t = k_{t-1}$ (gain)

then one could have

$d < 1$

$$k_t = k_{t-1} + d \frac{1}{FE^2} \quad \text{in this case I guess } d \text{ small}$$

$$\text{or } k_t = k_{t-1} + \frac{1}{d FE^2} \quad \text{in this case I guess } d \text{ big}$$

$d > 1$

↳ both of them work and exactly the way I hoped

$$\text{or even } k_t = k_{t-1} + \left(\frac{1}{d FE}\right)^2 = k_{t-1} + (d FE)^{-2}$$

works.

HP-filter

A time series y_t (in logs) is

$$y_t = g_t + c_t$$

\uparrow \uparrow
 growth component cyclical component

Obtain g_t as

$$\min_{\{g_t\}_{t=1}^T} \left\{ \sum_{t=1}^T c_t^2 + \lambda \sum_{t=1}^T [(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2 \right\}$$

where $\lambda = 1600$.

Hamilton provides a closed-form solution to this problem as $g^* = (H'H + \lambda Q'Q)^{-1} H'y$ (2)

where $\tilde{T} := T+2$

$$y = (y_T, y_{T-1}, \dots, y_1)' \quad q = (g_T, g_{T-1}, \dots, g_0, g_{-1})'$$

$T \times 1$ $\tilde{T} \times 1$

$$\text{and } H = \begin{bmatrix} I_T & 0 \\ 0 & I_{\tilde{T}} \end{bmatrix} \quad \text{and } Q = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{bmatrix}$$

Hamilton filter

21 March 2020

$$\text{Idea: } c = y_{t+h} - E[y_{t+h} | y_t]$$

choose $h=8$ for quarterly data.

How to calc $E[y_{t+h} | y_t]$?

just regress y_{t+h} on $y_t, y_{t-1}, y_{t-2}, y_{t-3}$
and take the fitted value from that reg.

In fact, Hamilton's eq (21) gives us c directly:

$$c = \hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

$$\text{So: } Y = y_{t+h}, \quad X = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}^{\text{4x1}} \Rightarrow \beta = (X'X)^{-1} X' Y$$

$$\text{and } \hat{v}_{t+h} = y_{t+h} - \beta X$$

Baxter & King filter (BK filter, bandpass filter)

Def: business cycle = cyclical components of a ts that last between 6 and 32 quarters in duration

Notation: $\text{BP}_k(p, q) =$ filter that processes cycles between $q=32$ and $p=6$ cycle length, and is truncated at leads/lags k . (\rightarrow recommend $k=12$)

\hookrightarrow Such a filter is a MA(k).

So we're interested in constructing the filtered series

$$y_t^* = \sum_{k=-N}^K a_k y_{t-k} \quad (1)$$

and the question is how to obtain a_k ?

(where the MA is symmetric, i.e. $a_k = a_{-k}$)

Eq. (10) gives the answer:

$$a_h^{BP} = (\bar{b}_h - b_h) + (\bar{\theta} - \underline{\theta})$$

where the bars denote things pertaining to low-pass filters that only allow stuff below the frequency $\bar{\omega}$ & $\underline{\omega}$,

i.e periodicities \bar{p} & \underline{p} ,

where $p = \frac{2\pi}{\omega}$

θ is a normalizing constant (coming from truncation)

$$\theta = \left(1 - \sum_{h=-K}^K b_h\right) \frac{1}{2K+1}$$

and $b_0 = \frac{\omega}{\pi}$ and $b_h = \frac{\sin(h\omega)}{h\pi}$ for $h=1, 2, \dots$ (7)

Ok, so we have $\underline{p} = 32$, $\bar{p} = 6$

$$\Rightarrow \underline{\omega} = \frac{2\pi}{\underline{p}} = \frac{2\pi}{32} = \frac{\pi}{16}, \quad \bar{\omega} = \frac{2\pi}{\bar{p}} = \frac{\pi}{3}$$

Autocorrelations of VARs (Lütkepohl, p. 21)

For a stationary VAR(1)

$$y_t = v + A_1 y_{t-1} + u_t \quad \hookrightarrow N(0, \Sigma_u)$$

define $\Gamma_y(h) \equiv E[(y_t - \mu)(y_{t+h} - \mu)']$

where $\mu := E[y_t]$. Then if you know $\Gamma_y(0)$,
you can compute

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) \text{ recursively! } (2.1.31)$$

If you know A_1 and Σ_u , you can compute

$$\text{vec} \Gamma_y(0) = (I_k^2 - A_1 \otimes A_1)^{-1} \text{vec} \Sigma_u \quad (2.1.32)$$

But I don't want to est a VAR... I just want
the empirical auto-covariance matrix of the data
and of the simulated data.

So, the 6MM of the anchoring fit works

23 March 2020

so far. Issues.

1) I'm not sure if I should model the raw data w/ a time series. So e.g. if they were a VAR(1) then I could estimate a RF-VAR, I could bootstrap using the residuals and I could estimate the anterior structure as in (2.1.32) & (2.1.31) in *filterpost*.

This issue shows up for $W = \begin{bmatrix} \hat{\sigma}_{act_1}^2 & & \\ & \ddots & \\ & & \hat{\sigma}_{act_{15}}^2 \end{bmatrix}$
b/c $\hat{\sigma}_{act_i}^2$ are super small, thus W^{-1} is huge,
and therefore the estimation does not move.

2) Filters may not be working?

→ I think now they are!

↳ check bootstrap! ✓ It's fine.

Write up materials 22 - peter ✓

Read "Stochastic" On it!

I'm not sure what I'm learning from 24 March 2020
the fMM so far, so until I talk to Ryan, I'll
postpone it and turn to questions of implementation
of the target condition.

Peter meeting

24 March 2020

- Ask Clough

- Intro: CBs always talk about anchoring,
e.g. (2) formulates that!

- Where to go #1:

- Suggestions

- (1): A common theme: does it make sense
to have $b_t = k_t^{-1}$ and just track the
evolution of that.

(2) : Eq. (2)

If you work w/ nonlinear diff eqs, they

are harder but not impossible to solve

→ mit-shock business is all about that

w/ perfect foresight

- take FOCs

- keep 'em in nonlinear form

- ass. the econ is hit by 1-time shock

- trace out transition to new st. st.

A precise way of putting it:

any dynamic econ: all vars $\rightarrow X_t$

$$F(X_t, X_{t-1}, \varepsilon_t) = 0$$

If F has a lin form, $A X_t + B X_{t-1} + C \varepsilon_t = 0$

then we can solve it immediately quite
easily.

If F is nonlinear, but nonstochastic:

$$F(x_t, x_{t+1}) = 0$$

is harder, but doable.

↳ Sect 2. point (1)

"Can we analytic methods to describe eqs
in opt policy & key variables"

& simulate vars under those policies.

⇒ Now you do answer

- 1) How close does a TR come to opt. policy.
- 2) Opt TR will try to do the same thing
as opt policy and you'll see what they
both are trying to do.

Suggestion overall:

- simulate numerically the model under
some kind of optimal pol that's more flex
than a TR.

2nd suggestion w/ nonlinearity:

Analogy: stochastic Ramsey model
(neoclassical growth model)

no closed-form sol once depreciation isn't
 $= 1$ and not $\log U$. So to solve this:

- loglin

- discretize the state-space

- TFP follows not an AR(1) but
one of 3 values

- Then decision rules are also 3-form

tophi (continues working w/ full-blown non-linear
model but replace the stock, light sources

$$u = \begin{cases} u_{\text{high}} \\ u_{\text{low}} \end{cases} \quad r^n = \begin{cases} r^{\text{high}} \\ r^{\text{low}} \end{cases}$$

\Rightarrow 4 states of the world

\hookrightarrow calculating $E(\cdot)$ of a future term in (1)
may be easier

2 option: take (1)

$$f_t = f_{t-1} + h_t (\pi_t - f_{t-1})$$

and login

Another way to solve a DSGE model

login can's that decision rules are lin
functions of the states

→ well you can say that they are
quadratic function of the states

↳ so instead of Taylor-approx, maybe the
right thing is a cubic or spline

In eq (3) I try to approx something of
unknown form w/ a class

Simulate the model under some optimal targeting rule & compare w/ behavior under an optimized TR economy.

Draft for end of semester
→ she'll read it then
and afterwards too.

↳ talk to Ryan now:
what is the most promising route?

Tomorrow noon → send Peter email for major
summer

Work after

Takeaways:

- 1) In my introduction, I talk about how CB-ers often talk about anchoring. The target criterion (eq. (2) in materials 22 peters) formalizes this!
- 2) Small detail: replace $h_t = k_t^{-1}$
- 3.) The main point: need to identify the set of results (and avenues) that can be achieved and then written up. For him it seems to be that this set of results is the target criterion and its numerical implementation so we can make statements about the optimal plan and how it compares to the optimal Taylor rule.

↳ he seemed adamant that I settle this w/ Ryan

4) Taking the analogy of the neoclassical growth model, no closed-form sol exists for the nonlinear model unless $\log u$ and $\beta = 1$. So if we wanted a tighter analytical result, we could 1) either loglin, or
2) discretize the stochastic disturbance

e.g. $u_t = \begin{cases} u^{\text{high}} \\ u^{\text{low}} \end{cases}$

↪ this way we might more fully describe the target criterion.

5) As for the loglin, which is a Taylor-Approx, maybe a higher order approx (ubic or spline) would be sensible.

Here's the deal: I don't think I wanna pursue further analytical work. Not w/ the discretization, not w/ a higher-order approx, although both might work. I think that building on the existing

work, I want to go on to simulating the optimal Ramsey policy and comparing it to the opt. TR.
⇒ I think that that is 1) feasible 2) makes sense in terms of storytelling & also is likely to provide additional inputs to that storytelling.

I will continue reading Stochastic Opt. in Continuous Time, not b/c I really hope for insights for more analytical work, more for my intellectual completion - as diff. egs contributed too.

So: numerical implementation

25 March 2020

of the target criterion

- I wonder if I couldn't do it via value function iteration → can I not solve the nonlinear system numerically and obtain the optimal paths like that?
→ which would allow me to back out i^{Ramsey} from NKIS?

$$g_{\pi} : \quad k_t = k_{t-1} + \underbrace{(d \cdot f c_{t-1})^{-2}}_{=g}$$

$$k_t = k_{t-1} + d^{-2} (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-2}$$

$$\text{Then } g_{\pi}(t) = d^{-2}(-2) (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-3}$$

$$= -2 (d f c_{t-1})^{-2} f c_{t-1}^{-1}$$

Now what I need to solve is the fact that when $\{i_t\}$ is exog, the model sol

$y = \text{function}(\text{matrices}, f_a, f_b)$ changes.

In particular, if we go back to the model summary in materials 1. in (1)-(3)

we know that (3) doesn't hold (R), and

$\{i_t\}$ influences the econ via the NKIS curve (1).

$$x_t = -3i_t + [\beta, (1-\beta), -3\beta] f_\beta + 3[1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_t$$

$$\pi_t = k k_t + [(1-\alpha)\beta, \kappa \alpha \beta, 0] f_a + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_t$$

$$\begin{bmatrix} 0 & 1 \\ -k & 1 \\ 1 & -k \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ k_t \end{bmatrix} = \begin{bmatrix} -3i_t + [\beta, (1-\beta), -3\beta] f_\beta + 3[1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_t \\ [(1-\alpha)\beta, \kappa \alpha \beta, 0] f_a + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_t \end{bmatrix}$$

So

27 March

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix}^{-1} \left[-\beta_{11} \begin{bmatrix} \beta, (1-\beta), -\beta\beta \end{bmatrix} f_\beta + \beta \begin{bmatrix} 1, 0, 0 \end{bmatrix} (f_{nx} - \beta h_x)^T s_1 \right. \\ \left. - \begin{bmatrix} (1-\alpha)\beta, \alpha\beta, 0 \end{bmatrix} f_\alpha + \begin{bmatrix} 0, 0, 1 \end{bmatrix} (f_{nx} - \beta h_x)^T s_2 \right]$$

$$\begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

$$= \begin{bmatrix} -k & -1 \\ -1 & 0 \end{bmatrix} \frac{1}{-1} = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$$

where f_β, f_α unchanged.

Ryan meeting

25 March 2020

$$k_t = k_{t-1} + \frac{1}{(df_e)^2} - c$$



so that when fe large,

the gain grows

Calibration: $E(fe)^{RE}$ would be the size of c .

• check if drawing w/ replacement!

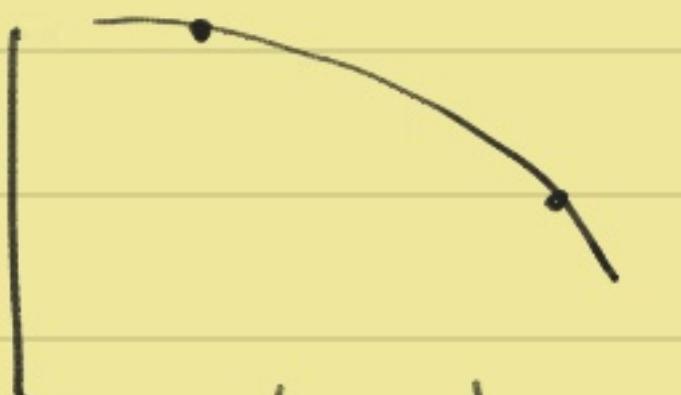
- take new data: run a VAR
 - bootstrap from residuals
- } What Ryan did in the paper w/ Robert

- put sim data in filter
- g should
 - tend to go down
 - but when it goes up, it goes up a lot
(i.e. for very large F_E , it \uparrow a lot.)
- check if you're close to the target moments

$$k_t = k_{t-1} + \hat{g}(f_e^t)$$

\hat{g} is a spline w/ x nodes

piecewise linear w/ x nodes



small | med | large F_E



S | M | L

Each node comes w/ 2 params

\rightarrow so can est a 4-param family

\rightarrow here you just est 3 slopes

→ eventually you could even est. what a small, large, FE is.

shape-preserving splines: are either always convex or always concave

Judd, the textbook, Numerical Methods for Econ

piece-wise linear interpolation

spline / quadratic spline

basis functions

Econ Librarian: Sonia Ensins

Makes sense:

Hamilton \hat{d} lower b/c Ver \uparrow , so FEs lower.

4 states, so VFI could be done.

Work after

26 March 2020

EBSCO host: lauragati
empirical - 77 methods

GMM

1) close to target?

No, not at all! loss is huge (5.4×10^8)!

2) w/ replacement

randperm really wasn't w/ replacement

randi (maxvalue, a×b vector) is w/ replaced

but it's still blowing up ñ

Maybe I do need to estimate a VAR?

I'm surprised b/c Ryan said that it sounds like my procedure is valid.

3) An anchoring function w/ the feature that

a) it generally decreases b) grows a lot when PE large

$$k_t = k_{t-1} + d - c f e_{t-1}^2$$

If $c f e^2 < d$, again $\rightarrow a)$ $\rightarrow g_\pi = -2 c f e_{t-1}$

For $c f e^2 > d$, (gain $\rightarrow b)$ Dam. Real bad.

try

$$k_+ = k_{+\infty} + \frac{1}{(d+k)} - c$$

Not good either.

27 March 2020

$$x_+ = -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+$$

$$\pi_+ = kx_+ + \text{stuff}_3 f_a + \text{stuff}_4 s_+$$

$$1x_+ + 0\pi_+ = -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+$$

$$-kx_+ + 1\pi_+ = \text{stuff}_3 f_a + \text{stuff}_4 s_+$$

$$\begin{bmatrix} 0\pi_+ + 1x_+ \\ 1\pi_+ - kx_+ \end{bmatrix} = \begin{bmatrix} -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+ \\ \text{stuff}_3 f_a + \text{stuff}_4 s_+ \end{bmatrix}$$

2×1

2×1

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} \pi_+ \\ x_+ \end{bmatrix}}_{2 \times 1}$$

I actually think they are the same -

28 March 2020

it's just that since π is exploding, small numerical

errors become considerable.

Strange that x doesn't explode.

→ Like I thought yesterday, pi-x-given_i.m should be correct. It's the target criterion that may not be.

Or actually: $f_a(3)$ and $f_b(3)$ may not be correct.

29 March 2020

I think I know what the problem is. $f_{a/b} \dots m$ just computes $\hat{E}_+ \sum_T (\alpha) \beta^{T+2}_{T+1}$ and now in pi-x-given_i.m there may be a discrepancy in that $f_b(3)$ is used, and yet the A-matrices that specified that $f_b(3)$ won't used before are no longer valid.

→ Need to make an assumption on how $f_b(3)$ is formed. The most natural is to say that the C-matrices announces the future path of i , i.e. it's like an

exogenous sequence. On the 29 March, I had:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - \kappa \end{bmatrix}^{-1} \left[-\beta i_{t+1} [\beta, (1-\beta), -\beta\beta] f_\beta + \beta [1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_n \right. \\ \left. - [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_n \right]$$

$$-\beta\beta f_\beta = -\beta\beta E_t \sum_{T=0}^{\infty} \beta^{T-t} i_{T+1} \quad \text{together with 4,}$$

this gives

$$\begin{aligned} & -\beta \left[i_t + \beta E_t \sum_{T=0}^{\infty} \beta^{T-t} i_{T+1} \right] \\ &= -\beta \left[i_t + E_t \sum_{T=0}^{\infty} \beta^{T-t+1} i_{T+1} \right] \\ &= -\beta \left[E_t \sum_{T=t}^{\infty} \beta^{T-t} i_T \right] \end{aligned}$$

Since the i -sequence may not be an AR(1), it may be completely arbitrary, I should simply write $\text{discsum_i} = \sum_{T=t}^{T+H} \beta^{T-t} i_T =: \text{dsi}$

So $\pi_{t+1} \text{ given } i_m$ modifies to

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - \kappa \end{bmatrix}^{-1} \left[-\beta \text{dsi} + [\beta, (1-\beta), 0] f_\beta + \beta [1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_n \right. \\ \left. - [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_n \right]$$

and stuff_1 modifies to $[\beta, 1-\beta, 0]$

still exploding...

I'm wondering if (A.1) - (A.3) yield an i-sequence, and I might that i-sequence into pi-x-giving, should I get the same $\{\pi_+, x_+\}_+^T$?

Initially I thought yes, but now I think no b/c A_a, A_b were derived w/ the assumption that $f_0(3) = \gamma_1 f_0(1) + \gamma_2 f_0(2) + \text{expected i-shock}$
→ I turned off i-shock, but that actually makes things explode faster!

Well, it depends. B/c if you set all $i_i = 0$, $\forall i$, then to make \hat{E}_{i+k} constant even under the TR-regime, you need to set $\rho_i = \sigma_i = 0$.

But at least it makes sense that x explodes not from π , b/c $\dot{\pi} = \kappa x + (\text{stiff } \perp i)$
 $\hookrightarrow \text{small}$

Nun, if $T + t$ small, then things don't have time to explode and therefore formation works. Otherwise

somewhat loss=0 already initially. Why?

initial loss decreases in H

so when $H \geq 36$, loss = 0 (when $T=10$)

But this seems to be extremely robust across T .

initial loss is \wedge -shaped in T

so when $T=1$, loss = 0 (when $H=35$)

$T=10$, loss = 49.479

$T=100$, loss = 1.4761

\rightarrow after that, it's 1.48 for $\forall T$.

But this is not robust to H .

E.g. if $H=5$, loss ↑ in T monotonically.

\rightarrow maybe I should take the vector of results as moments, not the biggest one.

That doesn't help w/ loss-explosions either, but I wonder if makes the loss function well-behaved?

So I'm back to wondering that $\{\bar{a}_i, \bar{x}_i | i\}$ shouldn't explode - I can accept that they aren't the same as $\{\bar{a}_i, \bar{x}_i | i^{\text{TR}}\}$ b/c $E(i)$ isn't the same, but I guess they shouldn't always explode.

→ they're exploding b/c f_a & f_b are exploding. Of course it's only $f_a(1) \& f_b(1)$, the others are numerically 0 (e^{-16}), which makes sense b/c $a_2 = a_3 = 0$ & $b = g \times h \bar{x} = \text{zeros}$ now that I set all elements of $h \bar{x} = 0$.

Somehow the learning isn't E-stable (indeed $a = \bar{a}$ explodes) when you just feed in a i -sequence b/c it's as if the TR was just specified as $i_+ = \bar{i}_+$.

→ If there was some way of letting the private sector know that i isn't chosen randomly, that it follows a rule...

Need to find a way to have the
private sector internalize the target criterion.

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→ the TC needs to be a model equation

But if that is so, then the expectation of the RHS
needs to be the public's expectation!

But before I do this, this is the time to verify that
the TC of (B.1) in materials23 is correct!

I solve for TC in materials18. (Notes 6, p. 95 ff.)

$$d = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ (\pi_t^2 + \lambda x_t^2) \right.$$

$$+ \psi_{1,t} (\pi_t - \kappa x_t + (1-\alpha)\beta f_a(t) + exog)$$

$$+ \rho_{2,t} (x_t + \beta \bar{x} + \beta f_B(t) + exog)$$

$$+ \psi_{3,t} (f_a(t) - \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + exog)$$

$$+ \psi_{4,t} (f_B(t) - \frac{1}{1-\beta} \bar{x}_{t-1} + exog)$$

$$+ \psi_{5,t} (\hat{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1} [\pi_t - (\bar{\pi}_{t-1} + b_s s_{t-1})])$$

$$+ \psi_{6,t} (k_t^{-1} - k_{t-1}^{-1} + g(\bar{\pi}_t - \bar{x}_{t-1} - b_s s_{t-1}))$$

↑ This is a change compared to materials18.

$$\begin{aligned}
d &= E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ (\pi_t^2 + 2x_t^2) \right. \\
&\quad + \psi_{1,t} (\pi_t - kx_t - (1-\alpha)\beta f_a(t) + exog) \\
&\quad + \psi_{2,t} (x_t + 2i_t - 2f_B(t) + exog) \\
&\quad + \psi_{3,t} (f_a(t) - \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + exog) \\
&\quad + \psi_{4,t} \left(f_b(t) - \frac{1}{1-\beta} \bar{\pi}_{t-1} + exog \right) \\
&\quad + \psi_{5,t} \left(\hat{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_t s_{t-1})) \right) \\
&\quad \left. + \psi_{6,t} (k_t^{-1} - \underline{k_{t-1}^{-1}} - g(\bar{\pi}_t - \bar{\pi}_{t-1} - b_t s_{t-1})) \right)
\end{aligned}$$

FD(S):

$$\pi: 2\pi_t + \psi_{1,t} - k_t^{-1} \psi_{2,t} - \psi_{6,t} g_\pi(t) = 0 \quad (1)$$

$$x: 2\lambda x_t - k \psi_{1,t} + \psi_{2,t} = 0 \rightarrow \underline{\psi_{1,t} = \frac{2\lambda}{k} x_t} \quad (2)$$

$$i: \underline{3\psi_{2,t} = 0} \quad (3)$$

$$f_a: -\psi_{1,t} (1-\alpha)\beta + \psi_{3,t} = 0 \rightarrow \underline{\psi_{3,t} = \frac{(1-\alpha)\beta}{2\frac{\lambda}{k}} x_t} \quad (4)$$

$$f_b: \psi_{4,t} + 3\psi_{2,t} = 0 \rightarrow \underline{\psi_{4,t} = 0} \quad (5)$$

$$\begin{aligned}
\bar{\pi}: & \psi_{3,t-1} \left[-\frac{1}{1-\alpha\beta} \right] + \psi_{3,t} + \psi_{5,t-1} \left[-1 + k_{t-1}^{-1} \right] \\
& + \psi_{6,t-1} g_{\bar{\pi}}(t) = 0 \quad (6)
\end{aligned}$$

$$k^{-1}: -\psi_{5,t} (\pi_t - (\bar{\pi}_{t-1} + b_t s_{t-1})) + \psi_{6,t} - \psi_{6,t-1} = 0 \quad (7)$$

$$2\pi_t + \frac{2\beta}{K} x_t - k_t^{-1} \varphi_{5,t} - \varphi_{6,t} g_{\bar{\pi}}(t) = 0 \quad (1)$$

$$-\frac{2(1-\alpha)\beta\frac{\beta}{K}x_{t+1}}{1-\alpha\beta} + \varphi_{3,t} - \varphi_{5,t+1}(1-k_{t+1}^{-1}) \\ + \varphi_{6,t+1} g_{\bar{\pi}}(t) = 0 \quad (2)$$

$$-\varphi_{5,t}(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + \varphi_{6,t} - \varphi_{6,t+1} = 0 \quad (3)$$

now

Note: Eq. (1) is exactly Eq. (I), p. 95 of Notes 6.

Eq (2) is exactly Eq (II).

Eq (3) is (III) + an additional $\varphi_{6,t+1}$ term

that comes from redefining the anchoring function

$$\text{as } h_t = \underline{h_{t-1}} + g(f_t)$$

$$\text{NBW} \rightarrow \underline{\varphi_{6,t+1}}$$

The issue that now comes up is the one I anticipated in materials 18, Notes 6 before 95: if the anchoring function has h_{t-1}^{-1} in it, then $\varphi_{6,t+1}$ shows up and causes trouble. Let me simplify the system:

Define $c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{\kappa}$ and $fe_t = \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$,
 Then: Where I'm annoyingly using a different notation than usual ...

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1} \varphi_{5,t} - g_\pi(t) \varphi_{6,t} = 0 \quad (1)$$

$$c x_{t+1} + \varphi_{5,t} - (1 - k_{t+1}^{-1}) \varphi_{5,t+1} + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0 \quad (2)$$

$$-fe_t \cdot \varphi_{5,t} + \varphi_{6,t} - \varphi_{6,t+1} = 0 \quad (3)$$

From (3), $\varphi_{5,t} = \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t}$.

Plug into (1)

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1} \left(\frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t} \right) - g_\pi(t) \varphi_{6,t} = 0$$

$$\Leftrightarrow 2\pi_t + \frac{2\lambda}{\kappa} x_t - \left(\frac{k_t^{-1}}{fe_t} + g_\pi(t) \right) \varphi_{6,t} + \frac{k_t^{-1}}{fe_t} \varphi_{6,t+1} = 0 \quad (1')$$

and plug into (2)

$$cx_{t+1} + \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t} - (1 - k_{t+1}^{-1}) \left(\frac{\varphi_{6,t+1} - \varphi_{6,t+2}}{fe_{t+1}} \right) + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0$$

This I should be able to solve for $\varphi_{6,t}$.

$$cx_{t+1} + \frac{y_{6,t} - y_{6,t+1}}{fe_t} - (1 - k_{t+1}^{-1}) \left(\frac{p_{6,t+1} - p_{6,t+2}}{fe_{t+1}} \right) + g_{\bar{n}}(t) p_{6,t+2} = 0$$

$$(x_{t+1} + \frac{1}{fe_t} y_{6,t} + \left[-\frac{1}{fe_t} - \frac{(1 - k_{t+1}^{-1})}{fe_{t+1}} + g_{\bar{n}}(t) \right] y_{6,t+1} + \frac{(1 - k_{t+1}^{-1})}{fe_{t+1}} y_{6,t+2}$$

(6')

$$cx_{t+1} + \alpha_t y_{6,t} + \beta_t y_{6,t+1} + \delta_{t+1} y_{6,t+2} = 0$$

Now the question is: solve forward or backward?

$1 - k^{-1}$ is always < 1 , and in the limit it $\rightarrow 1$.

fe doesn't have a limit, it's Normal and in the limit its variance is Σ ($= \text{var}(\text{shocks})$), mean zero.

For the most part, then, and close to the limit, if Σ 's elements are $\beta_i^2 \leq 1$, then fe (in the limit) will be mainly (or always?) below 1. So $\frac{1}{fe} > 1$

$$cfe_t x_{t+1} + y_{6,t} + \left[-1 - \frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) + fe_t g_{\bar{n}}(t) \right] p_{6,t+1} - \underbrace{\frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) p_{6,t+2}}$$

Here, $\frac{fe_t}{fe_{t+1}} \approx 1$ in expectation, so $\frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) < 1$ < 1

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \left[1 + \frac{f_{t+1}(1-k_{t+1}^{-1}) - f_{t+1} g_{\bar{n}}(t)}{f_{t+1}} \right] \varphi_{6,t+1} - \underbrace{\frac{f_{t+1}(1-k_{t+1}^{-1})}{f_{t+1}}}_{<1} \varphi_{6,t+2}$$

$=: \alpha_+$

In the limit,
this needs to
be > 1 in abs. value. But it's not.

$$\lim \left[1 + \frac{f_{t+1}(1-k_{t+1}^{-1}) - f_{t+1} g_{\bar{n}}(t)}{f_{t+1}} \right] \rightarrow 1 + 1 - \Sigma \cdot (\text{some small number, might even be zero})$$

> 1

Can I factor this using lag polynomials then? I need to solve both backward & forward.

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \alpha_+ \varphi_{6,t+1} + \beta_+ \varphi_{6,t+2} \quad |: \alpha_+$$

$$|\alpha_+| > 1 \quad |\beta_+| < 1$$

$$\frac{1}{\alpha_+} \varphi_{6,t} = -c f_{t+1} x_{t+1} + \varphi_{6,t+1} + \frac{\beta_+}{\alpha_+} \varphi_{6,t+2}$$

$$\varphi_{6,t+1} = \underbrace{\frac{c f_{t+1} x_{t+1}}{\alpha_+}}_{<1} - \underbrace{\frac{\beta_+}{\alpha_+} \varphi_{6,t+2}}_{<1}$$

6"

Before we go there, let me just check the target criterion as it now is, i.e. if $\varphi_{6,t+1} = 0$.

If $\varphi_{6,t+1} = 0$, the system is

$$2\pi_t + \frac{2\lambda}{k} x_t - k_t^{-1} \varphi_{5,t} - g_{\bar{\pi}}(t) \varphi_{6,t} = 0 \quad (1)$$

$$c x_{t+1} + \varphi_{5,t} - (1 - k_{t+1}^{-1}) \varphi_{5,t+1} + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0 \quad (6)$$

$$-f_{\ell,t} \cdot \varphi_{5,t} + \varphi_{6,t} - \cancel{\varphi_{6,t+1}} = 0 \quad (7)$$

and eq (7) corresponds to Notes b, p.95, eq (1).

$$\rightarrow \varphi_{6,t} = f_{\ell,t} \varphi_{5,t}$$

$$\Rightarrow (1): 2\pi_t + \frac{2\lambda}{k} x_t - (k_t^{-1} + g_{\bar{\pi}}(t) f_{\ell,t}) \varphi_{5,t} = 0 \quad (1')$$

$$\Rightarrow (6) c x_{t+1} + \varphi_{5,t} + [-(1 - k_{t+1}^{-1}) + g_{\bar{\pi}}(t) f_{\ell,t+1}] \varphi_{5,t+1} = 0 \quad (6')$$

$$\varphi_{5,t} = -c x_{t+1} + [1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{\ell,t+1}] \varphi_{5,t+1}$$

To be compatible w/ Notes b, p. 96, let me redefine

$$C := -c \quad \text{and} \quad \alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{\ell,t+1}$$

and assume that $\alpha_t < 1$ for most t . Which is

likely actually b/c $\alpha_{t+1} \rightarrow 1$ as $t \rightarrow \infty$ if $k_t^{-1} \rightarrow 0$

and $g_{\bar{\pi}}(t) \rightarrow 0$. For any $k_t^{-1}, g_{\bar{\pi}}(t) > 0$, $\alpha_{t+1} < 1$.

It can be < -1 though! Suppose $k_{t+1}^{-1} \approx 1$. Then $1 - k_{t+1}^{-1}$

is close to 0. If $g_{\bar{\pi}}(t) f_{\ell,t+1} > 1$ by enough, then this could happen. But it's not likely b/c 1) $f_{\ell} < 1$ if $\sum \leq 1$; 2)

$\hat{g}_N(t)$ is likely to be not just < 1 , but quite small since the overall level of the gain is < 1 , so its change is likely to be an order of magnitude smaller.

\Rightarrow So I think $\alpha_t < 1 \quad \forall t$ is safe to assume.

$$\begin{aligned}
 \psi_{S,t} &= Cx_{t+1} + \alpha_{t+1} \psi_{S,t+1} && | \text{Iter. fwd.} \\
 &= Cx_{t+1} + \alpha_{t+1} [Cx_{t+2} + \alpha_{t+2} \psi_{S,t+2}] \\
 &= Cx_{t+1} + \alpha_{t+1} [Cx_{t+2} + \alpha_{t+2} (Cx_{t+3} + \alpha_{t+3} \psi_{S,t+3})] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \\
 &\quad + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \psi_{S,t+3}] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \\
 &\quad + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} [Cx_{t+4} + \alpha_{t+4} \psi_{S,t+4}]] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} x_{t+4} \\
 &\quad + \underbrace{\alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \alpha_{t+4} \psi_{S,t+4}}_{\rightarrow 0 \text{ as } t \rightarrow \infty}
 \end{aligned}$$

$$\psi_{S,t} = Cx_{t+1} + C \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j}$$

So that (1') becomes:

$$2\pi_t + \frac{2\lambda}{k} x_t - (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left(c x_{t+1} + c \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right) = 0$$

$$\Rightarrow 2\pi_t = -\frac{2\lambda}{k} x_t + c (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right)$$

Since $c = \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k}$, we have

$$\pi_t = -\frac{\lambda}{k} x_t + \frac{(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right)$$

Recalling that $\alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{e_{t+1}}$, this is:

$$\begin{aligned} \pi_t &= -\frac{\lambda}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{e_t} g_{\bar{\pi}}(t) \right) \right. \\ &\quad \left. \left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - \underbrace{k_{t+1+j}}_{=} - g_{\bar{\pi}}(t+j) f_{e_{t+1+j}} \right) \right) \right\} \end{aligned}$$

☞ Note this is not 1, as in p.97, Notes 6.

And lastly I defined $f_{e_t} \equiv \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$, so

$$\begin{aligned} \pi_t &= -\frac{\lambda}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right. \\ &\quad \left. \left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - \underbrace{k_{t+1+j}}_{=} - g_{\bar{\pi}}(t+j) (\bar{\pi}_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}) \right) \right) \right\} \\ &\quad \text{was } + \text{ } j \quad \text{=} \text{ minus, not plus!} \end{aligned}$$

so I write

$$\pi_t = -\frac{\gamma}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right.$$

$\underset{\text{= minus, not plus!}}{=}$

$$\left. \left(x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - \underset{\text{= minus, not plus!}}{\cancel{k_{t+j}}} \right) \underset{\text{= minus, not plus!}}{\cancel{g_{\bar{\pi}}(t+j)}} \left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j} \right) \right) \right\}$$

changing the i -index and using the notation $\sum_{j=0}^{\infty} := 1$,
as

$$\pi_t = -\frac{\gamma}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right.$$

$\underset{\text{= minus, not plus!}}{=}$

$$\left. \left(\sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - \underset{\text{= minus, not plus!}}{\cancel{k_{t+j}}} \right) \underset{\text{= minus, not plus!}}{\cancel{g_{\bar{\pi}}(t+j)}} \left(\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j} \right) \right) \right\}$$

where I'm highlight the changes vis-à-vis p. 97 of
Notes 6, and I'll adopt these changes to
materials 23, (B. 1)

and note also that $-g_{\bar{\pi}}(t+j)$ got lost in materials 18
and thus the paper too! Need to correct!

✓ = corrected in materials 23

Let me write it in a way good to interpret:

$$\pi_t = -\frac{\gamma}{K} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\pi}(t) \right) \right. \\ \left. \left(E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+j+1}^{-1} - \bar{\pi}_{t+j} - b_1 s_{t+j} \right) g_{\bar{\pi}}(t+j) \right) \right\}$$

Which allows the even neater formulation

$$\pi_t = -\frac{\gamma}{K} x_t + \underbrace{\frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right)}_{E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+j+1}^{-1} - f_{t+1+j:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)}$$

$$E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+j+1}^{-1} - f_{t+1+j:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)$$

"discretion tradeoff + tradeoff coming from how the gain today responded to latest FE + tradeoff coming from how the gain is expected to change respond to future FE"

Rewrite once more

effect of current level and change of gain on future tradeoffs

$$\pi_t = -\frac{\gamma}{K} x_t + \underbrace{\frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right) E_t \sum_{i=1}^{\infty} x_{t+i}}_{- \frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right) E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(k_{t+j+1}^{-1} + f_{t+j+1:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)}$$

effect of future expected levels and changes of the gain on future tradeoffs, given current level & change

Taking the 2nd expression again to consider how one could simplify it,

$$\pi_t = -\frac{\lambda}{K} x_t + \frac{\lambda}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{t+1:t+1}^{eve} g_{\bar{\pi}}(t) \right) .$$

$$E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left(1 - k_{t+n+j}^{-1} - f_{t+j+1:t+j+1}^{eve} g_{\bar{\pi}}(t+j) \right)$$

reveals that only the orange part is not available at time t to the CB. Evaluating even $E_t x_{t+k}$ $k > 0$ is difficult b/c it requires evaluating $E_t k_{t+k}^{-1}$

But: if I were a rule that the CB relies on agents' expectations (otherwise it couldn't implement the TC).

But then anticipated utility would bind:

$$\hat{E}_t k_{t+k}^{-1} = k_t^{-1} \quad (\text{is that true?})$$

$$\hat{E}_t f_{t+1:t+k}^{eve} = 0$$

$$\hat{E}_t g_{\bar{\pi}}(t+k) = 0 \quad (\text{is that true?})$$

So the "expectations-based TC" would be

$$\pi_t = -\frac{\lambda}{K} x_t + \frac{\lambda}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left(k_t^{-1} + f_{t+1:t+1}^{eve} g_{\bar{\pi}}(t) \right) \cdot \underbrace{\sum_{i=1}^{\infty} b_e h^x i^{-1} s_t (1 - k_t^{-1})^{i-1}}_{\text{check later.}}$$

Let's go back to the question of solving (6'):

$$\varphi_{6,t+1} = \frac{c f_{t+1} x_{t+1}}{\alpha_t} - \underbrace{\frac{1}{\alpha_t} \varphi_{6,t}}_{<1} - \underbrace{\frac{\beta_t}{\alpha_t} \varphi_{6,t+2}}_{<1} \quad \boxed{|\alpha_t| > 1 \quad |\beta_t| < 1}$$

Time-out: write this (following Hamilton) as a std 2nd order diff eq:

$$\beta_t \varphi_{6,t+2} + \alpha_t \varphi_{6,t+1} - \varphi_{6,t} = c f_{t+1} x_{t+1}$$

$$(\beta_t + \alpha_t L - L^2) \varphi_{6,t+2} = c f_{t+1} x_{t+1} \quad | : \beta_t$$

$$(1 + \underbrace{\frac{\alpha_t}{\beta_t} L}_{>1} - \underbrace{\frac{1}{\beta_t} L^2}_{>1}) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

$$(1 - (-\frac{\alpha_t}{\beta_t}) L - \frac{1}{\beta_t} L^2) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

let me drop the 6 and roll backward

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$$\beta_{t-2} \varphi_t + \alpha_{t-2} \varphi_{t-1} - \varphi_{t-2} = c f_{t-2} x_{t-1}$$

$$\varphi_t = - \underbrace{\frac{\alpha_{t-2}}{\beta_{t-2}}}_{>1} \varphi_{t-1} + \varphi_{t-2} + c f_{t-2} x_{t-1}$$

can't solve bw.

But can't solve for either!

Let me step back: can we simplify this somehow?

$$\varphi_{6,t} = -c f_{t+} x_{t+1} + \left[1 + \frac{f_{t+}(1-k_{t+1}^{-1}) - f_{t+} g_{\bar{\pi}}(t)}{f_{t+1}} \right] \varphi_{6,t+1} - \underbrace{\frac{f_{t+}(1-k_{t+1}^{-1})}{f_{t+1}} p_{6,t+2}}_{< 1}$$

Obs:

1. If $f_{t+} = 0$, $\varphi_{6,t} = \varphi_{6,t+1}$. I guess k_{t+} is \downarrow then.
2. If $f_{t+1} = 0$, $\varphi_{6,t} = \infty$ ($\varphi_{6,t+1} - \varphi_{6,t+2} = \infty \cdot 0$) ?
3. If $f_{t+} = \infty$, $\varphi_{6,t} = \infty$
4. If $f_{t+1} = \infty$, $\varphi_{6,t} = -c f_{t+} x_{t+1} + (1 - f_{t+} g_{\bar{\pi}}(t)) \varphi_{6,t+1}$

An unorthodox thing to do: what if we solved (1')

$$2\pi_t + \frac{2}{k} x_t - \left(\frac{k_{t+1}^{-1}}{f_{t+}} + g_{\bar{\pi}}(t) \right) \varphi_{6,t} + \frac{k_{t+1}^{-1}}{f_{t+}} p_{6,t+1} = 0 \quad (1')$$

for $\varphi_{6,t}$, and plugged that in (6')?

$$\varphi_{6,t} = \frac{\frac{k_{t+1}^{-1}}{f_{t+}}}{\underbrace{\frac{k_{t+1}^{-1}}{f_{t+}} + g_{\bar{\pi}}(t)}_{< 1}} p_{6,t+1} - \gamma_t \quad \gamma_t := 2(\pi_t + \frac{2}{k} x_t)$$

$$=: \delta_t \quad \varphi_{6,t} = \delta_t p_{6,t+1} - \gamma_t$$

$$\gamma_{6,t} = \delta_+ \gamma_{6,t+1} - \gamma_+$$

$$\begin{aligned}
\gamma_{6,t} &= \delta_+ \left[\delta_{t+1} \gamma_{6,t+2} - \gamma_{t+1} \right] - \gamma_+ \\
&= \delta_+ \left[\delta_{t+1} \left[\delta_{t+2} \gamma_{6,t+3} - \gamma_{t+2} \right] - \gamma_{t+1} \right] - \gamma_+ \\
&= \delta_+ \delta_{t+1} \delta_{t+2} \gamma_{6,t+3} - \left[\delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= \delta_+ \delta_{t+1} \delta_{t+2} \delta_{t+3} \gamma_{6,t+4} \\
&\quad - \delta_+ \delta_{t+1} \delta_{t+2} \gamma_{t+3} \\
&\quad - \left[\delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= - \left[\delta_+ \delta_{t+1} \delta_{t+2} \gamma_{t+3} + \delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= - \gamma_+ - \sum_{i=1}^{\infty} \delta_{t+i} \prod_{j=0}^{i-1} \delta_{t+j}
\end{aligned}$$

$$\gamma_{6,t} = - \sum_{i=0}^{\infty} \delta_{t+i} \prod_{j=0}^{i-1} \delta_{t+j} \quad (\text{again using the } \prod_{j=0}^0 = 1 \text{ notation})$$

$$\gamma_{6,t} = -2E \sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{k_{t+j}}{f_{t+j}} \frac{k_{t+j}^{-1}}{f_{t+j} + g_{\pi}(t+j)}$$

Sol of (1') for $\gamma_{6,t}$.

This implies (plugging into (6')):

$$E_t \sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{f_{t+j}}}{\frac{k_{t+j}^{-1}}{f_{t+j}} + g_{\pi}(t+j)}$$

$$= E_t c f_t x_{t+1}$$

$$- E_t \left[1 + \frac{f_t c_t (1 - k_{t+1}^{-1}) - f_t g_{\pi}(t)}{f_{t+1}} \right] \left(\sum_{i=0}^{\infty} \left(\pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{f_{t+j}}}{\frac{k_{t+j}^{-1}}{f_{t+j}} + g_{\pi}(t+j)} \right)$$

$$- E_t \frac{f_t c_t (1 - k_{t+1}^{-1})}{f_{t+1}} \left(\sum_{i=0}^{\infty} \left(\pi_{t+2+i} + \frac{\lambda}{\kappa} x_{t+2+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+2+j}^{-1}}{f_{t+2+j}}}{\frac{k_{t+2+j}^{-1}}{f_{t+2+j}} + g_{\pi}(t+2+j)} \right)$$

The target criterion if the anchoring function is specified in terms of changes in the gain.
 (I divided them by -2.)

Back to (6')

$$\varphi_{6,t+1} = \underbrace{\frac{c}{\alpha_t} x_{t+1}}_{<1} + \underbrace{\frac{1}{\alpha_t} \varphi_{6,t}}_{<1} - \underbrace{\frac{\beta_t}{\alpha_t} \varphi_{6,t+2}}_{<1}$$

Let's do a really crazy thing: sub frods & backwards

$$\varphi_t = c_{t-1} x_t + \underbrace{a_{t-1} \varphi_{t-1}}_{<1} + \underbrace{b_{t-1} \varphi_{t+1}}_{<1} \quad \text{let me call } b_{t-1} = -\frac{\beta_t}{\alpha_t}$$

$$\varphi_t = c_{t-1} x_t$$

$$+ a_{t-1} (c_{t-2} x_{t-1} + a_{t-2} \varphi_{t-2} + b_{t-2} \varphi_t)$$

$$+ b_{t-1} (c_t x_{t+1} + a_t \varphi_t + b_t \varphi_{t+2})$$

$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$

$$+ a_{t-1} a_{t-2} \varphi_{t-2}$$

$$- a_{t-1} b_{t-2} \varphi_t$$

$$+ b_{t-1} a_t \varphi_t$$

$$+ b_{t-1} b_t \varphi_{t+2}$$

\Rightarrow I shouldn't have plugged it here. I would have saved myself the bold-circled terms.

\hookrightarrow LHS

$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1} \quad 1 - a_{t-1} b_{t-2} - b_{t-1} a_t$$

$$+ a_{t-1} a_{t-2} (c_{t-3} x_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1})$$

$$- (a_{t-1} b_{t-2} + b_{t-1} a_t) (c_{t-1} x_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1})$$

$$+ b_{t-1} b_t (c_{t+1} x_{t+2} + a_{t+1} \varphi_{t+1} + b_{t+1} \varphi_{t+3})$$

$$= C_{t-1} X_t + a_{t-1} C_{t-2} X_{t-1} + b_{t-1} c_t X_{t+1}$$

$$+ a_{t-1} a_{t-2} (C_{t-3} X_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1})$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) (C_{t-1} X_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1})$$

$$+ b_{t-1} b_t (C_{t+1} X_{t+2} + a_{t+1} \varphi_{t+1} + b_{t+1} \varphi_{t+3})$$

$$= C_{t-1} X_t + a_{t-1} C_{t-2} X_{t-1} + b_{t-1} c_t X_{t+1}$$

$$a_{t-1} a_{t-2} C_{t-3} X_{t-2}$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) C_{t-1} X_t$$

$$+ b_{t-1} b_t C_{t+1} X_{t+2}$$

$$+ a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3}$$

$$+ a_{t-1} a_{t-2} b_{t-3} \varphi_{t-1}$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1} \varphi_{t-1}$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} \varphi_{t+1}$$

$$+ b_{t-1} b_t a_{t+1} \varphi_{t+1}$$

$$+ b_{t-1} b_t b_{t+1} \varphi_{t+3}$$

these are all 0

in the limit

$$\text{the } x \text{ parts: } a_{t-1} C_{t-2} X_{t-1} + a_{t-1} a_{t-2} C_{t-3} X_{t-2}$$

$$+ b_{t-1} C_t X_{t+1} + b_{t-1} b_t C_{t+1} X_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) C_{t-1} X_t$$

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3}$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) \varphi_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) \varphi_{t+1}$$

$$+ b_{t-1} b_t b_{t+1} \varphi_{t+3} \quad || \text{Now plug only the } x\text{-term.}$$

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} (c_{t-4} x_{t-3} + \dots)$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) (c_{t-2} x_{t-1} + \dots)$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) (c_t x_{t+1} + \dots)$$

$$+ b_{t-1} b_t b_{t+1} (c_{t+2} x_{t+3} + \dots)$$

So only the x parts:

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} (c_{t-4} x_{t-3} + \dots)$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) (c_{t-2} x_{t-1} + \dots)$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) (c_t x_{t+1} + \dots)$$

$$+ b_{t-1} b_t b_{t+1} (c_{t+2} x_{t+3} + \dots)$$

$$= \underline{a_{t-1} c_{t-2} x_{t-1}} + \underline{a_{t-1} a_{t-2} c_{t-3} x_{t-2}} + \underline{a_{t-1} a_{t-2} a_{t-3} c_{t-4} x_{t-3}}$$

$$\underline{+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2} + b_{t-1} b_t b_{t+1} c_{t+2} x_{t+3}}$$

$$+ (1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

→ there will be a backward sum & a forward sum

$$BS := \sum_{i=1}^{\infty} x_{t-i} c_{t-1-i} \prod_{j=1}^i a_{t-j} \quad \blacksquare$$

$$FS := \sum_{i=1}^{\infty} x_{t+i} c_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j} \quad \blacksquare$$

and one or more cross-sums. Let's look at those.

$$+ (1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

Let's multiply out partly

$$(1 + a_{t-1} b_{t-2} + a_t b_{t-1}) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1}^2 b_{t-2} + a_{t-1} a_t b_{t-1})) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} b_{t-1} + a_t b_{t-1}^2) + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ a_{t-1} (a_{t-2} b_{t-3} + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-2} x_{t-1}$$

$$+ b_{t-1} (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}} + b_t a_{t+1}) c_t x_{t+1}$$

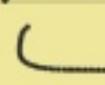
$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ (a_{t-2} b_{t-3} + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) a_{t+1} c_{t-2} x_{t-1}$$

$$+ (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}} + b_t a_{t+1}) b_{t-1} c_t x_{t+1}$$

I think (and hope) that more and more powers and multiplications can be pulled out, so that the bw- & fw-cross terms  go to 0. In most case we'd be left with

$$x_t c_{t-1} (1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \dots \text{expanding sum})$$

 in fact,

I might even be able to neglect some or all of this sum. Or it may not be expanding. Try to verify w/ Mathematica. If it doesn't expand, I'll get:

$$\varphi_{t+} = BS + FS + CS$$

$$= \sum_{i=1}^{\infty} x_{t-i} c_{t-1-i} \prod_{j=1}^{i-1} a_{t-j}$$

$$+ \sum_{i=1}^{\infty} x_{t+i} c_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j}$$

$$+ x_t c_{t-1} (1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \dots \text{expanding sum})$$

$$\text{where } c_{t-1} = \frac{c f_{t-1}}{\alpha_t}$$

$$c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}$$

$$a_{t-1} = \frac{1}{\alpha_t}$$

$$a_t = \left[1 + \frac{f_{t-1}(1-k_{t-1}^{-1}) - f_{t-1}g_{t-1}(t)}{f_{t-1}} \right]$$

$$b_{t-1} = -\frac{\beta}{\alpha_t}$$

$$\beta_t = \frac{f_{t-1}(1-k_{t-1}^{-1})}{f_{t-1}}$$

But now the cross-terms simplify if we hadn't subbed in φ_{t+} on RHS, but on LHS we'd have

$$(1 - a_{t-1} b_{t-2} - b_{t-1} a_t) \varphi_{t+} = BS + FS + CS$$

and CS would be previous minus the bold stuff; i.e.

$$CS = a_{t-2} a_{t-1} b_{t-3} c_{t-2} x_{t-1} + a_{t+1} b_t b_{t-1} c_t x_{t+1}$$

Now the hope is that the CS-term will vanish.