European Economic Review 66 (2014) 39–62

Contents lists available at ScienceDirect

European Economic Review

journal homepage: www.elsevier.com/locate/eer

Optimal monetary policy when agents are learning

Krisztina Molnár [a,*], Sergio Santoro [b]

[a] Norwegian School of Economics (NHH), Bergen, Norway
[b] Economic Outlook and Monetary Policy Department, Bank of Italy, Rome

ARTICLE INFO

Article history:
Received 30 October 2012
Accepted 27 August 2013
Available online 13 November 2013

JEL classification:
C62
D83
D84
E52

Keywords:
Optimal monetary policy
Learning
Rational expectations

ABSTRACT

We derive optimal monetary policy in a sticky price model when private agents follow adaptive learning. We show that this slight departure from rationality has important implications for policy design. The central bank faces a new intertemporal trade-off, not present under rational expectations: it is optimal to forego stabilizing the economy in the present in order to facilitate private sector learning and thus ease the future intratemporal inflation-output gap trade-offs. The policy recommendation is robust: the welfare loss entailed by optimal policy under learning if the private sector actually has rational expectations is much smaller than if the central bank mistakenly assumes rational expectations when in fact agents are learning.

1. Introduction

> *Monetary policy makers can affect private-sector expectations through their actions and statements, but the need to think about such things significantly complicates the policymakers' task.* (Bernanke, 2004)

Optimal monetary policy design is extensively studied under the assumption of rational expectations (REs). Despite the fact that the role of deviations from RE is emphasized in several theoretical and empirical papers,[1] the influence of less-than-rational expectations on the optimal policy conduct is not yet well understood. Instead, earlier literature examined the robustness of Taylor rules derived under RE and have shown that slight deviations from rationality are important for policy design. Taylor rules that are optimal or guarantee determinacy under RE, can lead to instability if private expectations follow adaptive learning (see Bullard and Mitra, 2002; Evans and Honkapohja, 2003a, 2003b, 2006).

In this paper, we investigate the interaction between departures from RE and monetary policy from a different angle: instead of examining the asymptotic behavior of Taylor rules, we address the issue of how a rational central bank (CB) should optimally conduct monetary policy if the private sector forms expectations with adaptive learning. We assume that the CB is rational within the model, knows how private agents form their expectations and takes their expectation formation scheme into account when solving its control problem. We conduct our analysis in a standard dynamic stochastic general equilibrium (DSGE) model with nominal rigidities in order to facilitate comparison with the earlier literature.

* Corresponding author. Tel.: +47 559 59368.
E-mail addresses: krisztina.molnar@nhh.no (K. Molnár), sergio.santoro@bancaditalia.it (S. Santoro).

[1] See for example Marcet and Nicolini (2003), Milani (2007), and Slobodyan and Wouters (2012).

0014-2921/$ - see front matter
http://dx.doi.org/10.1016/j.euroecorev.2013.08.012

The main contribution of this paper is to derive the optimal solution analytically. The advantage of closed-form solutions is to provide a better understanding of policy trade-offs. There is a well known *intratemporal* inflation-output gap trade-off. We show that a slight departure from RE introduces a new *intertemporal* trade-off. In period t the CB foregoes stabilizing the economy in the way that would be optimal under RE and discretion, in order to anchor future inflation expectations better, hence easing the future intratemporal inflation-output gap trade-off. Hence a slight departure from rationality is not only relevant for stability of the equilibrium, but inherently changes policy design. Our quantitative analysis shows that incorporating the intertemporal tradeoff into policymaking increases welfare substantially even if the departure from RE equilibrium is small.

Our policy recommendation is that stabilizing private inflation expectations is more important when these deviate from rationality than under RE. Earlier literature analyzing the welfare effect of different Taylor rules have also shown that the CB should act against inflation beliefs more aggressively than what is suggested by an RE model (see for example Ferrero, 2007; Orphanides and Williams, 2005b, 2005c). Our analytical solutions rationalize these earlier numerical results.

In our setup the central bank can manipulate agents' expectations, but over time expectations become consistent with the central bank's policy. Therefore in the limit the Lucas critique does not apply: the central bank cannot manipulate expectations of the private sector indefinitely, in the limit agents do not make systematic mistakes.

Our results provide a rationale for the general practice by CB to closely monitor private sector expectations. Under RE this is not justified, since expectations are pinned down by the model and the monetary policy rule. Once we depart from rationality though, expectations become a state variable, therefore optimal policy should depend on private expectations.

Assuming that the CB knows and makes active use of the exact form of private expectations is undoubtedly a very strong hypothesis. In reality, there is still a lively debate about how to model private sector expectations; we account for this by performing two kinds of robustness checks, one under Knightian and the other under probabilistic uncertainty about private expectations. We compare the optimal learning rule derived in our paper to the time consistent optimal rule derived under RE. Our result is that when the CB is uncertain about the nature of expectations formation in the sense of Knight (1921),[2] the optimal learning rules derived in our paper are more robust. When the CB has instead a probability distribution defined over the set of possible forms of private expectations, the expected welfare losses are smaller under the optimal learning rules even if the CB assigns only a very small probability to the possibility that agents use learning instead of RE.

When expectations are rational, a credible CB can manipulate them by committing to a future course of action; under adaptive learning there is no such role for promises, since beliefs are affected only by past occurrences. Nevertheless, Sargent (1999), chapter 5, obtains the remarkable result that the optimal policy in the Phelps problem[3] is such that a CB which is patient enough can replicate the commitment solution under RE asymptotically. In our setup optimal policy does not replicate the commitment solution, but there is a qualitative similarity: the impulse response to a cost-push shock is similar to the commitment case in the sense that the contemporaneous impact of a cost-push shock on inflation is small (compared to the case of discretionary policy under RE) and inflation reverts to the equilibrium in a sluggish manner. This similarity is stronger when the CB is more patient. Both under RE and learning, this pattern results from CB's ability to directly manipulate private expectations, even if the channels used are quite different. Under commitment, the policymaker uses *credible promises about the future*, while under learning, the pattern results from the impact that *past actions* have on beliefs. Thus, the ability to manipulate future private sector expectations through the learning algorithm plays a role similar to a commitment device under RE, hence eases the future short-run trade-off between inflation and the output gap.

So far there is a fairly small literature that examines optimal policy when private agents deviate from rationality in a different way then our paper. In these papers the optimization can only be done numerically, because agents learn in a nonlinear fashion. Their results support our main policy recommendation: if the private sector is not fully rational there is an increased concern for stabilizing inflation expectations. In a closely related paper, Gaspar et al. (2006) focus on the case of private agents learning about the persistence of inflation when firms index to lagged inflation. Their numerical simulations show that an optimally behaving CB aims to anchor inflation expectations better. Similar numerical findings are reported in Mele et al. (2012), where agents form their beliefs assuming that macro variables depend on lagged output gap, as in the RE equilibrium under commitment (see Clarida et al., 1999).

The rest of the paper is organized as follows. In Section 2, after briefly recalling the discretionary optimal policy when expectations are rational, we show the existence of the new intertemporal trade-off under learning. Section 3 characterizes the optimal allocations (and the interest rate rule that supports them) when agents use constant gain learning, explaining how the presence of the intertemporal trade-off increases the CB aggressiveness against inflation beliefs. Section 4 relaxes the assumption that expectations follow constant gain learning and shows that our main results remain valid under decreasing gain learning. Section 5 argues that the optimal policy rule derived in the previous sections is robust to uncertainty about the agents' expectations formation mechanism and Section 6 concludes.

[2] Knightian uncertainty refers to the impossibility of forming a probability assessment of the possible states of the world.

[3] Phelps (1967) formulates a control problem for a natural rate model with a rational CB and private agents endowed with a mechanical forecasting rule, known to the CB.

2. The model

We consider the baseline version of the New Keynesian model; in this framework, the economy is characterized by two structural equations.[4] The first one is an IS equation:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1}), \tag{1}$$

where x_t, r_t and π_t denote the time t output gap (i.e., the difference between actual and natural output), the short-term nominal interest rate and inflation, respectively; σ is a parameter of the household's utility function, representing risk aversion.

The operator E_t^* represents the private agents' expectation conditional on the time t information set, which is not necessarily rational. The above equation is derived by loglinearizing the household's Euler equation and imposing the equilibrium condition that consumption equals output. To simplify notation we abstract from having a shock in the IS curve.[5] This simplification is without loss of generality because the CB can always move the interest rate in such a way to offset a shock to the IS curve, therefore this shock does not affect optimal allocations.[6] In the working paper version of this paper[7] we adopt a model with a shock in the IS curve and the results we get are the same as in the simpler model presented here.

The second equation is the so-called New Keynesian Phillips Curve (NKPC):

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t + u_t, \tag{2}$$

where β denotes the subjective discount rate, κ is a function of structural parameters, and $u_t \sim N(0, \sigma_u^2)$ is a white noise cost-push shock; this relation is obtained from optimal pricing decisions of monopolistically competitive firms whose prices are staggered à la Calvo (1983).[8] We assume an *iid* cost-push shock, which is empirically supported by Milani (2006), who shows that when E^* is modeled with learning, it endogenously generates persistence in inflation data, so that a strongly autocorrelated cost-push shock becomes redundant. Assuming an AR(1) process for u_t instead would make the problem analytically not tractable. We can consider the *iid* cost-push shock as a *lower bound* on the effects of learning on monetary policy design; a more persistent shock would make inflation and inflation expectations more persistent and would increase the importance of stabilizing inflation expectations.

The loss function of the CB is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2), \tag{3}$$

where α is the relative weight put by the CB on the objective of output gap stabilization.[9] Differently from the private sector, the CB has RE. It seems reasonable to assume that it is better informed than the other agents in the economy, provided that central banks in real world dedicate significant effort and human capital investment to understand the economic environment; studying the polar case in which the CB has complete knowledge of the economy, and especially that knows and makes active use of the exact form of private expectations, allows us to derive useful insights, despite being a strong hypothesis. In Section 5 we show that our main conclusions carry through in a more realistic setup, where the CB is uncertain about the nature of agents' beliefs.

2.1. Benchmark: discretionary solution under rational expectations and learning

In this subsection we briefly recall the optimal monetary policy when the CB takes private sector's beliefs as given; in Kreps (1998) terminology this is equivalent to assuming that the monetary authority is an anticipated utility maximizer.[10] We will use this anticipated utility policy (AU, hereafter) as a benchmark to illustrate the effects of taking into account the evolution of private sector's beliefs in the design of optimal policy.

[4] For details of the derivation of the structural equations of the New Keynesian model under RE see, among others, Yun (1996), Clarida et al. (1999) and Woodford (2003). As pointed out in Preston (2005), when departing from RE the microfounded structural equations of the New Keynesian model should include also forecasts of macroeconomic conditions many periods into the future (infinite horizon learning), and not only one step ahead (Euler equation learning). We follow much of the learning literature in adopting Euler equation learning for analytical tractability.

[5] This shock is typically a demand disturbance or the natural real rate of interest (i.e., the real interest rate that would hold in the absence of any nominal rigidity).

[6] See for example Clarida et al. (1999). When learning of private agents is introduced to this model, as for example in Evans and Honkapohja (2003b), a well-specified regression has the same functional form of the equilibrium law of motion under RE and does not include this shock among the regressors.

[7] See CESifo Working Paper No. 3072.

[8] In other words, the probability that a firm in period t can reset the price is constant over time and across firms.

[9] As is shown in Rotemberg and Woodford (1997), Eq. (3) can be obtained as a quadratic approximation to the expected household's utility function; in this case, α is a function of structural parameters.

[10] We slightly abuse the definition of anticipated utility, since Kreps (1998) deals with a decision maker who does not take into account how she will revise in the future *her own beliefs*; instead, we consider a decision maker who does not take into account how *other agents* in the economy will revise in the future *their beliefs*.

The policy problem is to minimize the social welfare loss (3), subject to the structural equations (1) and (2), and given the private sector's expectations:

$$\begin{aligned} &\min_{\{\pi_t, x_t, r_t\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2) \\ &\text{s.t.} \quad \begin{array}{l} (1),\ (2) \\ E_t^* \pi_{t+1}, E_t^* x_{t+1} \text{ given for } \forall t \end{array} \end{aligned} \tag{4}$$

As shown in Clarida et al. (1999), the optimality condition to this problem (at time t) is

$$\frac{\kappa}{\alpha}\pi_t + x_t = 0. \tag{5}$$

Using (5), Evans and Honkapohja (2003b) derive the following law of motion for inflation and the output gap, and the interest rate rule that implements these allocations:

$$\pi_t^{AU} = \frac{\alpha\beta}{\alpha+\kappa^2} E_t^* \pi_{t+1} + \frac{\alpha}{\alpha+\kappa^2} u_t \tag{6a}$$

$$x_t^{AU} = -\frac{\kappa\beta}{\alpha+\kappa^2} E_t^* \pi_{t+1} - \frac{\kappa}{\alpha+\kappa^2} u_t. \tag{6b}$$

$$r_t = \delta_\pi^{AU} E_t^* \pi_{t+1} + \delta_x^{AU} E_t^* x_{t+1} + \delta_u^{AU} u_t, \tag{6c}$$

where

$$\begin{aligned} \delta_\pi^{AU} &= 1 + \sigma \frac{\kappa\beta}{\alpha+\kappa^2} \\ \delta_x^{AU} &= \sigma \\ \delta_u^{AU} &= \sigma \frac{\kappa}{\alpha+\kappa^2}. \end{aligned}$$

In the terminology introduced in Evans and Honkapohja (2003b), this is an *expectations-based reaction function*; they show that this rule guarantees not only determinacy under RE, but also convergence to the RE equilibrium when expectations E_t^* evolve according to least squares learning.

If agents have RE (i.e., if $E_t^* = E_t$), the system of Eqs. (6) collapses to

$$\pi_t^{RE} = \frac{\alpha}{\kappa^2+\alpha} u_t, \quad x_t^{RE} = -\frac{\kappa}{\kappa^2+\alpha} u_t,$$

which is the optimal policy under discretion derived in Clarida et al. (1999).

2.2. Optimal policy under learning

If private agents follow learning, a fully rational CB could do better than our benchmark (6c). In this section we show how optimal monetary policy is modified when the monetary authority optimizes taking into account its effect on private sector expectations.

We assume that the private sector's expectations are formed according to the adaptive learning literature.[11] Agents do not know the exact process followed by the endogenous variables, but recursively estimate a Perceived Law of Motion (PLM) consistent with the law of motion that the CB would implement under RE. As shown in Clarida et al. (1999), the optimal allocations of the discretion and the commitment solution under RE have different functional forms and are therefore associated with different PLMs. In this paper, we restrict our attention to the discretionary case. In particular, we assume that agents believe that inflation and the output gap are continuous invariant functions of the cost-push shock only, $\pi_t = \pi(u_t)$ and $x_t = x(u_t)$.[12] This hypothesis, together with the *iid* nature of the shock, implies that the conditional and unconditional expectations of inflation and output gap coincide, and are perceived by the agents as constants. Hence, it is natural to assume that agents estimate them using their sample means[13]:

$$E_t^* \pi_{t+1} \equiv a_t = a_{t-1} + \gamma_t (\pi_{t-1} - a_{t-1}) \tag{7}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + \gamma_t (x_{t-1} - b_{t-1}), \tag{8}$$

where γ_t is a deterministic sequence of gains in the interval (0, 1), which governs how responsive estimate revisions are to new data. In the next two sections, we will be more explicit on the precise form taken by γ_t.

[11] The modern literature on this topic was initiated by Marcet and Sargent (1989), who were the first to apply stochastic approximation techniques to study the convergence of learning algorithms. For an extensive monograph on this paradigm, see Evans and Honkapohja (2001).

[12] In the terminology of Evans and Honkapohja (2001) chapter 11, the PLM is a noisy steady state.

[13] To be precise, in the algorithms (7) and (8), the observations are weighted geometrically if $\gamma_t = \gamma$, while if $\gamma_t = 1/t$ all observations receive equal weight.

We choose Eqs. (7) and (8) to model the private sector's PLM since they are consistent with the optimal discretionary RE solution in our setup; hence, it is the correct PLM if the CB has no credibility, which is the case under adaptive learning.[14]

To analyze the optimal control problem faced by the CB, we suppose that the policymakers take the structure of the economy (Eqs. (1) and (2)) as given; moreover, we assume that the CB knows how private agents' expectations are formed, and takes into account its ability to influence the evolution of the beliefs. Hence, the CB problem can be stated as follows:

$$\min_{\{\pi_t, x_t, r_t, a_{t+1}, b_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t(\pi_t^2 + \alpha x_t^2)$$
$$\text{s.t.} \quad \begin{array}{l} (1),\ (2),\ (7),\ (8), \\ a_0,\ b_0 \text{ given.} \end{array} \tag{9}$$

Note that contrary to our benchmark problem (4), the CB now also takes first order conditions with respect to private expectations. When expectations depart from rationality and follow a law of motion, they become natural state variables.

Assuming that the CB knows the exact learning algorithm followed by private agents is a strong hypothesis. In real life, there is still no consensus about how we should model private expectations. Nevertheless, we think it is important to examine how the policy recommendation changes if private agents depart slightly from rationality, and the monetary authority takes this departure into account. In Section 5, we relax this assumption and examine the robustness of our results when the CB is uncertain about how the private sector forms its expectations.

The first-order conditions at every $t \geq 0$ are

$$\lambda_{1t} = 0 \tag{10}$$

$$2\pi_t - \lambda_{2t} + \gamma_{t+1}\lambda_{3t} = 0 \tag{11}$$

$$2\alpha x_t + \kappa\lambda_{2t} - \lambda_{1t} + \gamma_{t+1}\lambda_{4t} = 0, \tag{12}$$

$$E_t\left[\frac{\beta}{\sigma}\lambda_{1t+1} + \beta^2\lambda_{2t+1} + \beta(1-\gamma_{t+2})\lambda_{3t+1}\right] = \lambda_{3t}, \tag{13}$$

$$E_t[\beta\lambda_{1t+1} + \beta(1-\gamma_{t+2})\lambda_{4t+1}] = \lambda_{4t}, \tag{14}$$

where λ_{it}, $i = 1, \ldots, 4$ denote the Lagrange multipliers associated with (1), (2), (7) and (8), respectively. The necessary conditions for an optimum are the first-order conditions, the structural Eqs. (1) and (2) and the laws of motion of private agents' beliefs, (7) and (8). Note that the optimality conditions are not time invariant if the γ_t depends on time; however, because it is exogenous and deterministic, the policy function that solves the optimality conditions does not depend on the period when the CB optimizes, even if it is not time invariant. Thus, the optimal policy characterized above is time consistent, in the sense of Lucas and Stokey (1983) and Alvarez et al. (2004).[15] Combining Eqs. (10) and (14), we get

$$\lambda_{4t} = \beta(1-\gamma_{t+2})E_t[\lambda_{4t+1}],$$

which can be solved forward, implying that the only bounded solution is

$$\lambda_{4t} = 0. \tag{15}$$

Since Eqs. (10) and (15) prescribe that the Lagrange multipliers on the IS curve and on the law of motion of output gap beliefs must be zero along an equilibrium path, constraints (1) and (8) are irrelevant.[16]

If we put together Eqs. (10)–(12) and (15), we derive the following optimality condition:

$$2\pi_t + 2\frac{\alpha}{\kappa}x_t + \gamma_{t+1}\lambda_{3,t} = 0, \tag{16}$$

where $\lambda_{3,t}$ is the Lagrange multiplier on the evolution of inflation expectations.

From (16) we can isolate two trade-offs faced by the CB in designing the optimal policy. When $\gamma_{t+1} = 0$, namely when expectations are constant and, consequently, cannot be manipulated by the monetary authority, (16) simplifies to

$$\frac{\kappa}{\alpha}\pi_t + x_t = 0, \tag{17}$$

which is identical to the optimality condition derived in the RE optimal monetary policy literature when the CB sets the optimal plan taking the private sector's expectations as given (i.e., in the discretionary case). When a cost-push shock is

[14] If we had assumed a *hybrid* NKPC, motivated by indexation to past inflation among firms, a model consistent PLM of private agents should also have included lagged inflation (as in Gaspar et al., 2006). We think the Gaspar et al. (2006) analysis is important and more research is needed on how the exact nature of expectation formation modifies the optimal policy recommendation. Nevertheless, not assuming indexation not only enables us to derive closed-form solutions, but is also supported by empirical evidence. There is a recent strand of empirical literature that argues that the presence of indexation is not a robust feature of the data; see Benati (2008) and Cogley and Sbordone (2005), among others. Furthermore, Woodford (2007) questions the necessity (and the correctness) of price indexation to replicate inflation dynamics, especially when expectations are not rational.

[15] A problem solved at t is said to be time consistent for $t+1$ if the continuation from $t+1$ of the optimal allocations chosen at t solves it in $t+1$; moreover, in period zero it is time consistent if the problem in period t is time consistent for $t+1$ for all $t \geq 0$.

[16] See Eusepi et al. (2012) for a more general model where the IS curve is a relevant constraint on policy design under learning.

present, (17) represents a well-known *intratemporal trade-off*between stabilization of inflation at t and the output gap at t: because of the nonzero term u_t in the Phillips Curve (2), π_t and x_t cannot be set contemporaneously equal to zero in every period. Clarida et al. (1999) describe (17) as implying a “lean against the wind” policy: in other words, if the output gap (inflation) is above target, it is optimal to deflate the economy (contract demand below capacity).

Under learning (i.e., when $\gamma_{t+1} > 0$), the CB faces an additional *intertemporal trade-off* between optimal behavior in t and in later periods, generated by its ability to manipulate future values of inflation expectations. The CB has to take into account how its choice about inflation/output at time t influences inflation expectations, and thus future intratemporal trade-offs between inflation/output.

The term $\gamma_{t+1}\lambda_{3,t}$ shows an important difference compared with earlier results: the optimal decision should be conditional on the current stance of inflation expectations. The interpretation of this term is very simple: Eq. (7) implies that a change in pi_t will influence the next period's inflation expectations, a_{t+1}, by a factor γ_{t+1}, and a change in inflation expectations affects welfare losses by a factor $\lambda_{3,t}$. The sign of $\lambda_{3,t}$ depends on current inflation expectations: because target inflation is zero, an increase in inflation expectations drives them further away from the target when expectations are positive; this in turn increases welfare loss so the Lagrange multiplier on inflation expectations is positive. When inflation expectations are negative, the opposite occurs: increasing inflation expectations drives them closer to the steady state, thus $\lambda_{3,t}$ is negative.

When inflation expectations are positive (so $\lambda_{3,t} > 0$) and inflation is positive, the optimal contraction of x_t is harsher than under discretionary policy. It is well documented in the literature that disinflations have real costs.[17] Brayton and Tinsley (1996) and Erceg and Levin (2003) argue that disinflation can be costly because of slowly adjusting expectations. Our results show that, under learning, it is indeed optimal to incur high output losses (compared with discretionary policy) in order to contain inflation expectations. Moreover, the higher inflation expectations are, the higher $\lambda_{3,t}$ is and the bigger the output loss the CB should engineer in order to bring down inflation.

When inflation expectations are negative ($\lambda_{3,t} < 0$), (16) implies that the lean against the wind policy is not always optimal. If, for example, inflation is positive but inflation expectations are sufficiently negative, the optimal value of x_t can be zero or even positive.

Let us summarize our first result for later reference:

Result 1. *Learning introduces an intertemporal trade-off not present under rational expectations.*

3. Constant gain learning

In this section, we assume that agents' beliefs are updated according to a *constant gain* algorithm, namely that $\gamma_t = \gamma \in (0,1)$ for any t.[18] In Section 4 we will relax this assumption and examine how optimal policy changes when agents follow decreasing gain learning.

Under constant gain learning optimal allocations implemented by the CB are as follows.

Proposition 1. *There exists a unique solution of the control problem (9) with $\gamma_t = \gamma$, and the policy function for inflation associated to it has the form*:

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t. \tag{18}$$

The coefficient c_π^{cg} can be characterized as follows:

$$-\text{if } \gamma \in (0,1), \text{then } 0 < c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$$

$$-\text{if } \gamma = 0, \text{ i.e. if expectations are constant } c_\pi^{cg} = \frac{\alpha\beta}{\alpha+\kappa^2},$$

and

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2 + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma(1-\gamma)(\alpha\beta - (\kappa^2+\alpha)c_\pi^{cg}).}$$

In the terminology of the adaptive learning literature (18) is called the actual law of motion (ALM) of inflation.

Under the optimal policy, an increase in inflation expectations (a_t) increases current inflation, but less than proportionally, because $c_\pi^{cg} \le \alpha\beta/(\alpha+\kappa)^2 < 1$. c_π^{cg} is not necessarily monotone in γ, a higher value of the tracking parameter γ has two opposing effects on it. On the one hand, with an increasing γ, current inflation impacts future inflation expectations more strongly (see (7)), therefore the incentive of the CB to lower the feedback from inflation expectations to inflation (c_π^{cg}) increases. On the other hand, a higher γ reduces the impact of current expectations on future expectations

[17] For evidence on the costs of ending moderate inflations, see for example Ball (1994). Note that our model is valid only around the steady state, so it cannot be used to model hyperinflationary episodes.

[18] As discussed extensively in the learning literature, private agents are likely to use such a learning scheme if they believe structural changes are going to occur.

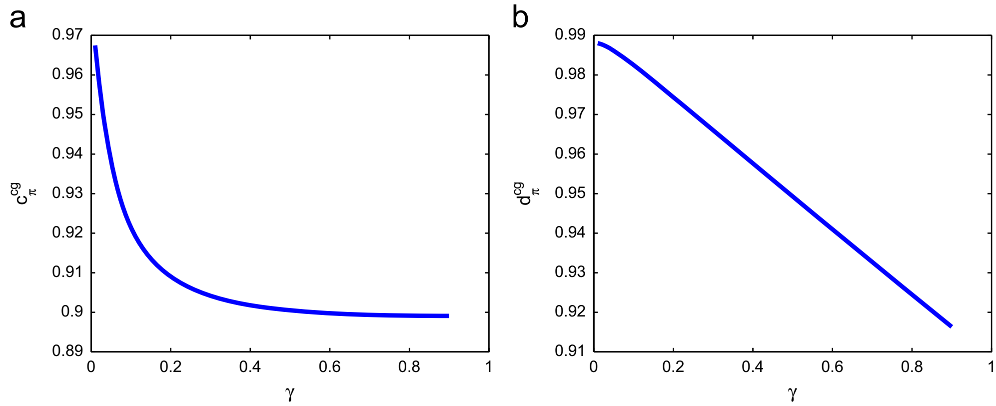

Fig. 1. Feedback parameters in the ALM for inflation $\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t$ as a function of γ. (a) Parameter of inflation expectations and (b) parameter of cost-push shock.

(see (7)), which reduces the benefits from a reduction of the expectations, so there is an incentive to set a higher c_π^{cg}. In Fig. 1 we show that for the calibration of Woodford (1999), with $\beta = 0.99$, $\sigma = 0.157$, $\kappa = 0.024$ and $\alpha = 0.04$ the first effect dominates, making c_π^{cg} a monotonically decreasing function of γ. We find that the second effect dominates only for some parameter combinations with very high tracking parameters.[19] For empirically plausible values of the tracking parameter we find that c_π^{cg} is always a decreasing function of the gain parameter.[20] In the following analysis we will therefore assume that c_π^{cg} is decreasing in γ: the higher is γ, the less the CB is allowing high inflation expectations to pass through to inflation.

Using the structural equation (2) we can derive the optimal allocation of the output gap:

$$x_t = c_x^{cg} a_t + d_x^{cg} u_t,$$
$$c_x^{cg} = -\frac{\beta - c_\pi^{cg}}{\kappa}, \quad d_x^{cg} = -\frac{1 - d_\pi^{cg}}{\kappa}. \tag{19}$$

When the private sector expects inflation to be positive the policymaker contracts economic activity because $c_\pi^{cg} < \alpha\beta/(\alpha+\kappa^2)$ (see Proposition 1) implies $c_x^{cg} < -\kappa/(\beta\alpha+\kappa^2)$. The higher the tracking parameter of private agents, the bigger the optimal output contraction. Through the inflation-output tradeoff (see (2)) this policy drives the actual inflation rate sufficiently lower than the expected one, thus lowering future inflation expectations.

Using (18) and (19) in (1) we can derive the nominal interest rate rule, which is an expectations-based reaction function:

$$r_t = \delta_\pi^{cg} a_t + \delta_x^{cg} b_t + \delta_u^{cg} u_t,$$
$$\delta_\pi^{cg} = 1 + \sigma\frac{\beta - c_\pi^{cg}}{\kappa}, \quad \delta_x^{cg} = \sigma, \quad \delta_u^{cg} = \sigma\frac{1 - d_\pi^{cg}}{\kappa}. \tag{20}$$

When inflation expectations are positive , the central bank raises the interest rate $\delta_\pi^{cg} > 0$. A higher γ calls for more aggressive behavior from the CB, i.e., a higher $\delta_\pi^{cg} > 0$. Because $c_\pi^{cg} < \beta$ (see Proposition 1) δ_π^{cg} is always greater than 1. In response to a rise in expected inflation, optimal policy should raise the nominal interest rate sufficiently to increase the real interest rate. In other words, the Taylor principle emphasized in Clarida et al. (1999) holds.

Optimal policy responds to a positive cost-push shock in a similar fashion as to positive inflation expectations. After a positive cost-push the CB raises the interest rate to induce an output contraction in order to decrease inflation. When γ is high the CB raises the nominal interest rate more (see 20) to contract the economic activity more strongly (see 19). Fig. 1 shows that when γ is high a positive cost-push shock increases inflation to a smaller extent.

Finally, the coefficient on output expectations b_t in the interest rate rule (20) is such that its effect on the output gap in the IS curve is fully neutralized. This is the reason why b_t is not a true state variable (its lagrange multiplier is zero, see (15)).[21]

Plugging (18) into (7), we get

$$a_{t+1} = a_t + \gamma(c_\pi^{cg} - 1)a_t + \gamma d_\pi^{cg} u_t$$
$$= (1 - \gamma(1 - c_\pi^{cg}))a_t + \gamma d_\pi^{cg} u_t,$$

[19] With different parameterizations, characterized by a higher κ and a lower α, the relationship would indeed be nonmonotonic, with c_π^{cg} being a decreasing function of γ for smaller values of the tracking parameter, and increasing when γ is big.

[20] For examples of estimates of γ for the US see Milani (2007), Orphanides and Williams (2005a), and Branch and Evans (2006). For a wider set of countries see Molnar and Reppa (2013).

[21] This result is analogous to the result, that by an appropriate interest rate rule the central bank can completely counteract shocks that are present solely in the New Keynesian IS curve, therefore these are not true state variables. Typical examples are the real interest rate shock and the demand shock.

which is a stationary AR(1).[22] Thus, as the literature on adaptive learning has already demonstrated, in the presence of random shocks the constant gain specification of the updating algorithm prevents the expectations from asymptotically converging to a precise value: instead,

$$a_t \sim N\left(0, \frac{\gamma^2 (d_\pi^{cg})^2}{1-(1-\gamma(1-c_\pi^{cg}))^2}\sigma_u^2\right).$$

3.1. Consequences of the intertemporal tradeoff

The main difference compared to the earlier literature, which treats the CB as an anticipated utility maximizer, is that our policy incorporates the effect of the intertemporal tradeoff arising from the dynamics of private agents' expectations. In order to understand the consequences of this intertemporal tradeoff, this section compares our constant gain optimal policy (CG) to the AU rule (6c) of Evans and Honkapohja (2003b). Both rules ensure E-stability of learning, but because (6c) does not take the evolution of private expectations into account, the policy functions and their welfare implications are very different.

In the optimal interest rate rule (20), the coefficient of the output gap expectations is the same as in the AU rule (6c), while the other two coefficients are typically different.

Proposition 1 implies $\delta_\pi^{cg} > \delta_\pi^{mp}$: the optimal interest rate response to out of equilibrium inflation expectations is more aggressive than the interest rate response of AU. As a result CG induces a smaller increase in inflation than AU ($c_\pi^{cg} < c_\pi^{mp}$). So when the CB takes its ability to influence agents' beliefs into account, it chooses to undercut future inflation expectations more than it would do otherwise.

From Proposition 1 it also follows that $\delta_u^{cg} > \delta_u^{mp}$: after a positive cost-push shock, the optimally behaving CB raises the interest rate higher than an anticipated utility maximizer CB would. A bigger increase of the interest rate in turn decreases output and consequently inflation to a bigger extent. Thus, the more aggressive interest rate response of CG to the cost-push shock dampens the influence of the cost-push shock on inflation. In fact, $c_\pi^{cg} < \alpha\beta/(\kappa^2+\alpha)$ from Proposition 1 implies that $d_\pi^{cg} < \alpha/(\kappa^2+\alpha)$, so in the ALM of inflation the coefficient on the cost-push shock is smaller under CG (see 18) than under AU (see 6). Since CG allows for a smaller feedback from the cost-push shock to inflation than AU, it speeds up agents' learning about the true equilibrium level of inflation.

Let us summarize the main policy differences of CG and AU below:

Result 2. *When the central bank takes into account not only the intratemporal trade-off but also the intertemporal trade-off, it accommodates less the effect of out of equilibrium inflation expectations and noisy cost-push shocks on inflation. In this way, optimal policy facilitates learning of the private sector.*

It is also worth noting that optimal policy decreases the autocorrelation of inflation compared with AU.[23] The optimal rule's strong feedback to inflation expectations dampens the interaction between inflation and expectations. This lowers the persistence of a shock's effect on expectations and on inflation. This result is analogous to the findings of Gaspar et al. (2006) in a different model. They show that when firms index their prices to past inflation it is optimal to decrease inflation persistence.

Because of the inflation output tradeoff, the flip-side of Result 2 is that compared to AU, CG allows for a higher feedback from out of equilibrium expectations and noisy cost-push shocks to the output gap. Under CG both coefficients in the ALM of x_t are higher in absolute value than under AU (compare (19) and (6)).

3.1.1. Welfare loss analysis

To obtain a quantitative measure of the welfare gains of internalizing the intertemporal tradeoff we present a numerical welfare loss analysis. We report consumption equivalents (following Adam and Billi (2007)), because welfare losses in utility terms are hard to interpret: for a given monetary policy rule we calculate the cumulative utility losses resulting from deviations from the steady state allocation and then express the equivalent percentage decrease of the steady state consumption that results inn the same cumulative utility loss. We use the calibration of Woodford (1999): $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$ and $\sigma = 0.157$.[24] We perform a Monte Carlo with simulation length 10,000 and a cross-sectional sample size of 1000. Cost-push shocks are drawn from a normal distribution with 0 mean and variance 0.1.

---

[22] In fact, because $0 < c_\pi^{cg} < 1$, it immediately follows that $0 < (1-\gamma(1-c_\pi^{cg})) < 1$.

[23] It can be easily derived that the autocorrelation of inflation under constant gain with AU is

$$E\pi_t^{AU}\pi_{t-1}^{AU} = \left(\frac{\alpha\beta}{\alpha+\kappa^2}\right)^2 \left(1-\gamma+\gamma\frac{\alpha\beta}{\alpha+\kappa^2}\right)\sigma_{a_{AU}}^2 + \frac{\alpha\beta}{\alpha+\kappa^2}\left(\frac{\alpha}{\alpha+\kappa^2}\right)^2 \gamma\sigma_u^2$$

while under the optimal rule $E\pi_t^{CG}\pi_{t-1}^{CG} = (c_\pi^{cg})^2(1-\gamma+\gamma c_\pi^{cg})\sigma_{a_{CG}}^2 + c_\pi^{cg}(d_\pi^{cg})^2\gamma\sigma_u^2$. We have already seen that $\sigma_{a_{CG}}^2 < \sigma_{a_{AU}}^2$, $c_\pi^{cg} < \alpha\beta/(\alpha+\kappa^2)$ and $d_\pi^{cg} < \alpha/(\alpha+\kappa^2)$, thus $E\pi_t^{CG}\pi_{t-1}^{CG} < E\pi_t^{AU}\pi_{t-1}^{AU}$.

[24] Similar consumption equivalents are obtained using other standard calibrations, like Clarida et al. (2000) and McCallum and Nelson (1999). Results not reported here.

Table 1
Consumption equivalents using CG and AU under constant gain learning.

γ	Loss under optimal policy (c^{CG})	Loss under AU policy (c^{AU})	$\frac{c^{AU}-c^{CG}}{c^{CG}}$	$\frac{c_\pi^{AU}-c_\pi^{CG}}{c_\pi^{CG}}$	$\frac{c_x^{AU}-c_x^{CG}}{c_x^{CG}}$
$a_0=0$					
0.03	0.0135	0.0136	0.5%	1.4%	−43.7%
0.1	0.0188	0.0211	12.6%	26.3%	−88.8%
$a_0=0.5$					
0.03	0.0391	0.0411	5.1%	14.2%	−86.3%
0.1	0.0342	0.0429	25.5%	56.4%	−92.8%

Woodford (1999) calibration.

Table 1 compares consumption equivalents under CG and AU for constant gain learning with two different tracking parameters and two different initial conditions. $\gamma=0.03$ is a tracking parameter characteristic of the US,[25] while $\gamma=0.1$ is more typical in Latin American and Transition economies.[26]

The results are in the range of the original estimates of Lucas (1987): consumption losses resulting from cyclical fluctuations are small. Naturally, consumption equivalents are higher if we start the economy away from the RE equilibrium (see rows 3–4 compared to rows 1–2). The higher the tracking parameter, the higher the consumption equivalents are, both under CG and AU, because of the higher variance of inflation expectations (see also Fig. 2). A higher inflation expectations implies higher variance of inflation and output, both under CG (see Eqs. (18) and (19) and under AU (see Eq. (18)), and higher consumption equivalents.[27]

The first two rows show that the gain from internalizing the intertemporal tradeoff can be nonnegligible even if initial inflation expectations are at the RE equilibrium, and expectations stay close to the RE equilibrium. Column three shows that for $\gamma=0.03$ the gain from using an optimal interest rate rule is 0.5% while for $\gamma=0.1$ the gain is 12.6%. The intuition behind is that for a higher γ CG proportionally decreases more the variance of inflation expectations compared to AU (see Fig. 2), therefore for a higher γ the percentage gain of using CG is also higher.

Column three in the last two rows show that the gain of using CG instead of AU is higher when initial expectations are further away from the RE equilibrium. This is because the main advantage of the optimal rule is that it helps private agents to learn the equilibrium inflation faster than the AU rule.

Fig. 3 complements Table 1 by plotting the transition path of cumulative consumption equivalents, CG divided by AU. We can se that the long-run gains of CG in containing inflation expectations come at a cost in the short run. In the first periods, the CG yields *ex-post* higher cumulative welfare losses expressed in consumption terms than the AU rule; later, however, our rule starts generating smaller welfare losses. These findings are consistent with Results 1 and 2: because of the intertemporal trade-off, it is optimal to react to out of equilibrium inflation expectations more aggressively than the AU rule in order to undercut more future expectations, even if it results in short-term output gap losses. As soon as inflation expectations become small enough, this initial loss is more than compensated.[28]

Another way to gauge what the intertemporal trade-off implies for welfare is to calculate separately the equivalent permanent consumption losses caused by inflation or output gap variation separately. The last two columns of Table 1 show, that optimal policy lowers inflation variation at the cost of higher output gap variation. The higher is the tracking parameter, the more pronounced is the difference between CG and AU.[29]

3.2. Comparison with the commitment solution

In this section, we show that the optimal policy response to a supply shock under learning is qualitatively similar to that of the commitment solution under RE. However, despite the similarities in short-run behavior, the two equilibria are different in the limit. The learning equilibrium intrinsically depends on how private agents learn.

Fig. 4 displays the impulse response function of inflation to a unit shock under CG and discretionary RE policy. In the optimal RE discretionary policy, inflation rises on impact and immediately reverts to the steady state once the *iid* shock dies out. Under learning the policymaker engineers a smaller initial response of inflation; in subsequent periods inflation

[25] See estimates of Milani (2007), Orphanides and Williams (2005a), and Branch and Evans (2006).

[26] See Molnar and Reppa (2013).

[27] Inflation and output gap variance can be expressed as a linear function of the variance of the cost-push shock, therefore the absolute value of consumption equivalents is bigger for a bigger σ_u^2, but the ratio of consumption equivalents under CG and AU is not sensitive to the choice of σ_u^2.

[28] Results not reported here show that the further away initial expectations are from the RE equilibrium, the larger the long-run gains are and the bigger are the short-run costs of using the optimal rule. For details, see the working paper version of this paper.

[29] In the framework of Gaspar et al. (2006), the CB engineers a lower welfare loss in inflation without a significant cost in output. This result is difficult to compare with ours because the presence of indexation changes their setup along three important dimensions: the CB wants to stabilize a *quasi-difference* of inflation instead of inflation itself, the NKPC is of the hybrid type, and in our model agents learn about the expected value of inflation while in Gaspar et al. (2006) agents learn about the persistence of inflation.

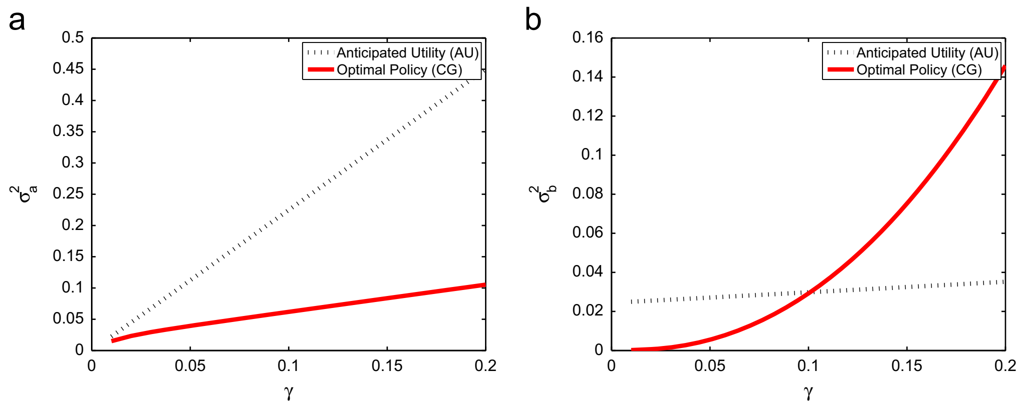

Fig. 2. Variance of inflation and output gap expectations under CG and AU policy as a function of γ. (a) Variance of inflation expectations and (b) variance of output gap expectations.

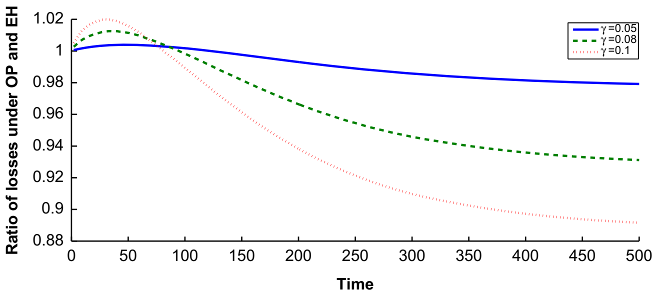

Fig. 3. The dynamics of welfare losses (ratio of cumulative consumption equivalents under CG and AU) under constant gain learning, with initial beliefs at the rational equilibrium.

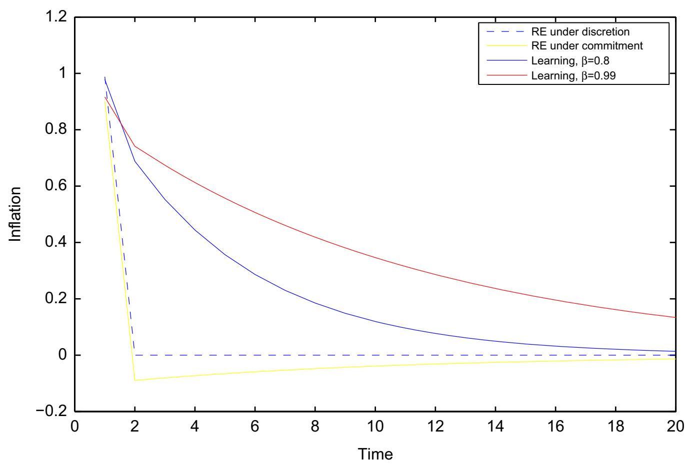

Fig. 4. Impulse response of inflation for an initial cost-push shock $u=1$. Solid line: Optimal policy under learning and private agents following learning with $\gamma=0.9$. Dashed line: Optimal discretionary policy under RE with private agents having rational expectations. Initial conditions: $a_0=0$, $\pi_0=0$, $x_0=0$.

gradually converges back to the steady state value. Gali (2003) shows a *similar disinflation path for the optimal policy under RE and commitment*: a smaller initial inflation compared with the discretionary case, in exchange for a more persistent deviation from the steady state later.[30]

These similarities arise because under both learning and RE commitment the CB can directly manipulate private expectations even if the channels used are quite different. Under commitment, the policymaker uses a *credible promise on the future* to obtain an immediate decline in inflation expectations and thus in inflation; moreover, the necessity to fulfill

[30] This behavior of optimal policy under commitment leads to welfare gains over discretion because of the convexity of the loss function; this preference for slower but milder adjustment to shocks is at the heart of the stabilization bias.

past commitments introduces additional inertia in inflation and output. Under learning, we observe a smaller initial response of inflation relative to the RE discretionary case because optimal policy dampens the inflation response to the cost-push shock to *ease private agents' learning* (Result 2), and the past-dependent nature of private sector beliefs imparts sluggishness on the system. In this sense, we can say that the ability to manipulate future private sector expectations through the learning algorithm plays a role similar to a commitment device under RE, hence easing the short-run trade-off between inflation and the output gap.

One difference compared with the impulse response of inflation under full commitment RE is that there is no overshooting of inflation under learning. Commitment policy under RE engineers a sequence of negative inflation after the first period, yet a positive sequence under learning. A second difference is that the full commitment is characterized by a smaller output decrease compared with RE discretionary policy (see Clarida et al., 1999), while under learning the initial decrease of output is bigger than under discretion and RE. The reason for this is that while under RE the commitment of the CB can improve the current terms of the inflation-output trade-off, under learning monetary policy can only influence future expectations and can improve only future inflation-output trade-offs.

Sargent (1999), chapter 5, shows a similarity between the optimal policy under adaptive learning and the RE commitment solution in the Phelps problem: optimal monetary policy drives the economy close to the Ramsey optimum, and when the discount factor β equals 1, optimal policy under learning replicates the Ramsey equilibrium. In the Phelps problem, the discretion and commitment outcome of inflation have the same functional form, therefore when agents learn in this functional form they can converge to both equilibria. A sufficiently patient CB is willing to incur higher short-term losses for the opportunity to drive private expectations to the welfare-improving Ramsey equilibrium.

In our model, discretionary and commitment solutions under RE have a different functional form; hence the equilibrium depends on how agents learn, and Sargent's result does not hold anymore. However, in our case an increase in the discount factor also makes the optimal disinflationary path under learning move closer to the commitment solution. This can be seen in Fig. 4: as β gets closer to 1, the initial response of inflation becomes milder and the path back to the steady state longer.

The findings in this subsection strengthen the point that when we abandon the RE paradigm, several issues arise in monetary policy design that are not present when agents are fully rational, and the implications for policymaking go beyond the asymptotic learnability criterion: as we showed, the equilibrium law of motion of optimal inflation can be significantly affected by the way agents learn, and careful consideration of private sector beliefs can play a role qualitatively similar to a commitment device even in the absence of CB credibility.

4. Decreasing gain learning

In this section, we relax the assumption of constant gain learning and show that our main results remain valid also with decreasing gain learning.[31]

We assume agents use the following decreasing gain learning rules (henceforth DG):

$$E_t^*\pi_{t+1} \equiv a_t = a_{t-1} + t^{-1}(\pi_{t-1} - a_{t-1}), \tag{21}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + t^{-1}(x_{t-1} - b_{t-1}), \tag{22}$$

that are Eqs. (7) and (8) with $\gamma_t = t^{-1}$. Under certain conditions on the values used to initialize this algorithm (see Evans and Honkapohja, 2001), it is equivalent to estimating the conditional expectations of inflation and output gap every period with OLS.[32]

In the Appendix, we derive the following optimal allocations.

Proposition 2. *The solution of the control problem (9) with $\gamma_t = 1/t$ yields the following policy function for inflation*:

$$\pi_t = c_{\pi,t}^{dg} a_t + d_{\pi,t}^{dg} u_t, \tag{23}$$

where $c_{\pi,t}^{dg}$ and $d_{\pi,t}^{dg}$ are deterministic functions of time characterized as follows:

$$-\lim_{t\to\infty} c_{\pi,t}^{dg} \text{ exists, and is given by } \lim_{t\to\infty} c_{\pi,t}^{dg} = \frac{\alpha\beta}{\alpha+\kappa^2};$$

$$-\text{if } t<\infty, \text{then } c_{\pi,t}^{dg} < \frac{\alpha\beta}{\alpha+\kappa^2}$$

[31] Decreasing gain algorithms place equal weight on all observations, which is optimal in stationary environments.

[32] Note that, because the conditional expectations of inflation and output gap are assumed by the learners to be constant, the OLS estimates are just the sample averages of the two.

and

$$d_{\pi,t}^{dg} = \frac{P_{1,t}}{c_{\pi,t+1}^{dg}\frac{1}{t+1} - A_{11,t}},$$

where the matrices $P_{1,t}$ and $A_{11,t}$ are defined in the Appendix.

With decreasing gain, during the transition Result 2 holds: there is a new intertemporal trade-off, therefore it is optimal to decrease the effect of out of equilibrium expectations on inflation compared with the AU rule (Eq. (6)) in order to drive future inflation expectations closer to the equilibrium. This relaxes the future intratemporal inflation-output gap trade-off embedded in the Phillips curve. The ALM for the output gap is

$$x_t = c_{x,t}^{dg} a_t + d_{x,t}^{dg} u_t,$$
$$c_{x,t}^{dg} = -\frac{\beta - c_{\pi,t}^{dg}}{\kappa}, \quad d_{x,t}^{dg} = -\frac{1 - d_{\pi,t}^{dg}}{\kappa}. \tag{24}$$

As in the constant gain case, if the private sector expects inflation to be positive, the optimal CB contracts economic activity more than the AU rule.[33] The CB is ready to pay a short-term cost represented by a wider current output gap in order to contain future inflation expectations.

The nominal interest rate rule is

$$r_t = \delta_{\pi,t}^{dg} a_t + \delta_x^{dg} b_t + \delta_{ut}^{dg} u_t,$$
$$\delta_{\pi,t}^{dg} = 1 + \sigma \frac{\beta - c_{\pi,t}^{dg}}{\kappa}, \quad \delta_x^{dg} = \sigma, \quad \delta_{ut}^{dg} = \sigma \frac{1 - d_{\pi,t}^{dg}}{\kappa}. \tag{25}$$

Because $c_{\pi,t}^{dg} < \beta$ (see Proposition 2) $\delta_{\pi,t}^{dg}$ is always bigger than 1; hence, the Taylor principle holds. In the Appendix, the following results are derived.

Proposition 3. *Assume that $t < \infty$; then*:

$$-\delta_{\pi,t}^{dg} > \delta_\pi^{AU}, \delta_{ut}^{dg} > \delta_u^{AU},$$
$$-\lim_{t\to\infty} \delta_{\pi,t}^{dg} = \delta_\pi^{AU}, \lim_{t\to\infty} \delta_{ut}^{dg} = \delta_u^{AU}.$$

During the transition, the optimal interest rate rule is similar to the constant gain rule: it reacts more aggressively to out of equilibrium expectations (and cost-push shocks) than the AU rule.

An interesting result is that the coefficient on inflation expectations in the interest rate rule (25) is *time-varying*, reflecting the fact that the CB's incentives to manipulate agents' beliefs evolve over time. This implies that during the transition, optimal policy should be time-varying even in a stationary environment. This coefficient can be characterized as follows:

Proposition 4. *Let $\delta_{\pi,t}^{dg}$ be given by $1 - \sigma(c_{\pi,t}^{dg} - \beta)/\kappa$; then, there exists a $T < \infty$ such that $\{\delta_{\pi,t}^{dg}\}_{t=T}^{\infty}$ is a monotonic decreasing sequence.*

After time T, the bank dampens its aggressiveness in reacting to out of equilibrium inflation expectations (and cost-push shocks).[34] For empirically relevant coefficient estimates, time T is maximum a few quarters. Numerical analysis on the grid $\beta = 0.99$ and $\alpha \in [0.01, 2]$, $\kappa \in [0.01, 0.5]$ shows that T is typically very small.[35] We find that after the fourth period (from the fourth to the fifth period and so on) $\delta_{\pi,t}^{dg}$ always decreases, while in the first four periods $\delta_{\pi,t}^{dg}$ might increase (hump-shaped) for a combination of low values of α and high values of κ (see Fig. 5). Fig. 6 shows that for the Woodford (1999) calibration, $\delta_{\pi,t}^{dg}$ and $\delta_{u,t}^{dg}$ always decrease over time (i.e., $T=0$).[36]

Intuitively, this result means that it is optimal for the CB to react more aggressively to out of equilibrium inflation beliefs in the first periods, when agents pay more attention to new information and the CB's possibilities of influencing private expectations are therefore greater. This policy is beneficial even at the cost of larger short-term losses in terms of output gap variability. As time passes, expectations will be influenced to a lesser extent by the most recent realizations of the inflation rate, hence the CB's reaction will more closely reflect the case where the policymakers cannot manipulate expectations. As an example of this situation, consider a new CB governor is appointed, and agents start learning how this affects the equilibrium.

[33] From $c_{\pi,t}^{dg} < \alpha\beta/(\alpha + \kappa^2)$ it follows that $c_{x,t}^{dg} < -\kappa\beta/(\alpha + \kappa^2)$. Compare with the ALM under AU (6).

[34] From (25) it is easy to see that the change in $\delta_{u,t}^{dg}$ through time has the same sign as $\delta_{\pi,t}^{dg}$.

[35] We have chosen the grid to include typical calibrated values for the US and the euro area.

[36] $\delta_{\pi,t}^{dg}$ is always decreasing also for other calibrations widely adopted in the New Keynesian literature, such as those taken from Clarida et al. (2000) and McCallum and Nelson (1999).

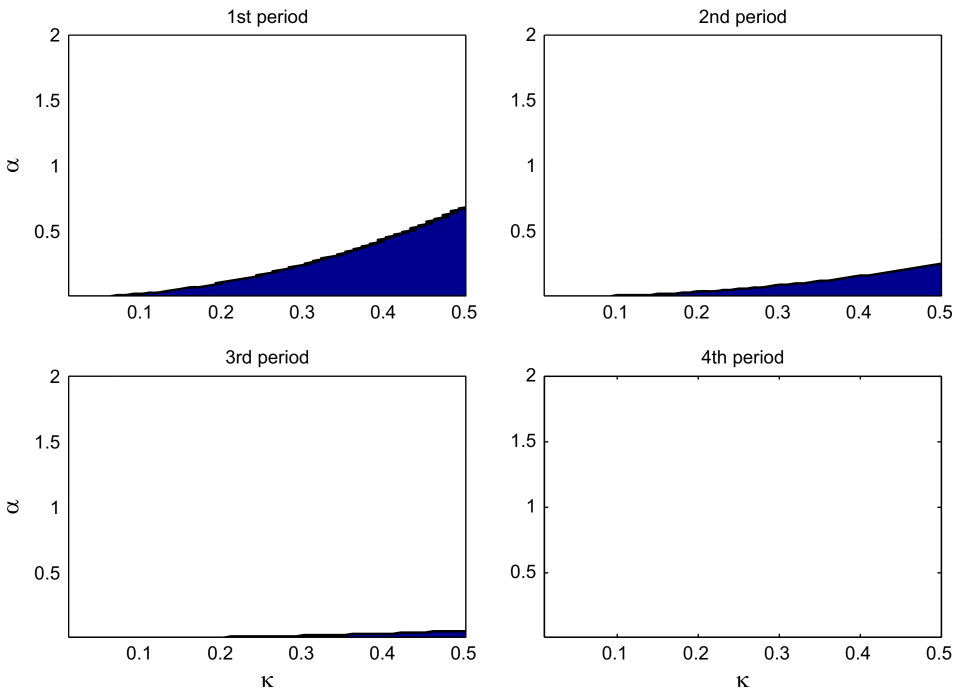

Fig. 5. Values of α and κ for which δ_π^{dg} is increasing in the first four periods. From the fourth period on δ_π^{dg} is always decreasing ($\beta = 0.99$).

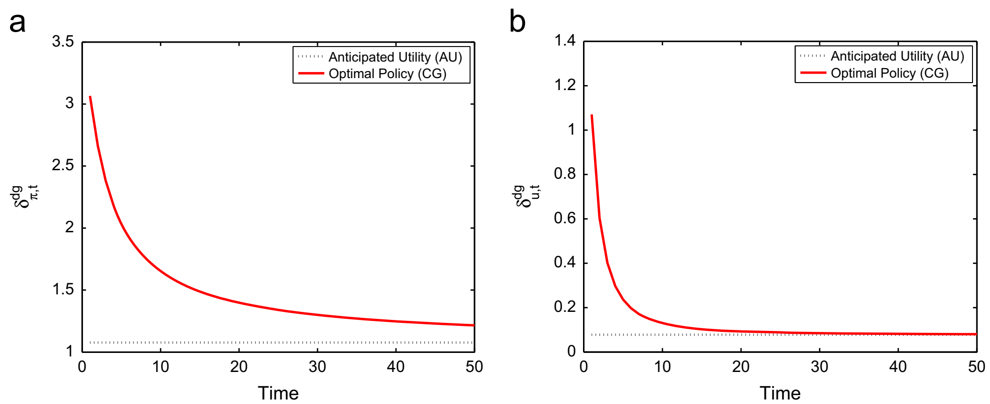

Fig. 6. Interest rate rule coefficients under decreasing gain learning $r_t = \delta_{\pi,t}^{dg} a_t + \delta_x^{dg} b_t + \delta_{ut}^{dg} u_t$. (a) Coefficient of inflation expectations and (b) coefficient of cost-push shock.

The CB can manipulate agents' expectations only in the short run, in the limit Lucas critique is satisfied, and agents do not make any systematic mistakes. To see this, note that the asymptotic properties of the inflation and output gap ALM, (23) and (24), depend on the limiting behavior of a_t, which is given by the stochastic recursive algorithm:

$$a_{t+1} = a_t + (t+1)^{-1}((c_{\pi t}^{dg} - 1)a_t + d_{\pi,t}^{dg} u_t). \tag{26}$$

We study its properties in the Appendix, where we use the stochastic approximation techniques[37] to prove the following Proposition:

Proposition 5. *Let a_t evolve according to (26); then, $a_t \to 0$ a.s.*

This result, together with the boundedness of $c_{\pi,t}^{dg}$, implies that $c_{\pi,t}^{dg} a_t$ goes to zero almost surely; moreover, it is easy to see that $d_{\pi,t}^{dg} \to \alpha/(\kappa^2 + \alpha)$, therefore we can conclude that $\pi_t \to \alpha/(\kappa^2 + \alpha v)$ almost surely, where v is a random variable with the

[37] For an extensive monograph on stochastic approximation, see Benveniste et al. (1990); the first paper to apply these techniques to learning models was Marcet and Sargent (1989).

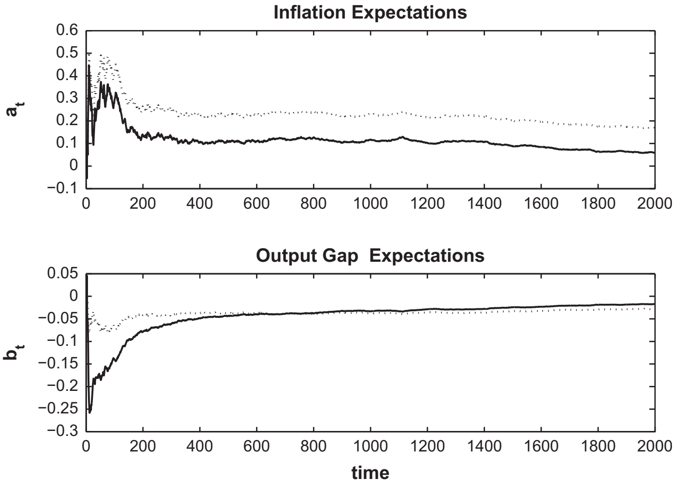

Fig. 7. Evolution inflation and output gap expectations under the optimal (solid line) and the AU rule (dashed line), when private agents follow decreasing gain learning.

same probability distribution as u_t. The equilibrium corresponds to the discretionary RE equilibrium, and private agents learn the unconditional expectation of inflation and output under discretionary RE.[38]

It follows from Proposition 3 that the optimal policy converges to the AU policy. Because in the limit expectations converge to a constant, it is intuitive that in the limit optimal policy behaves as if expectations were fixed. However, during the transition optimal policy results in substantially lower welfare losses. For the Woodford (1999) calibration, even if we start inflation expectations from the RE equilibrium, $a_0 = 0$, in the long run the consumption equivalent under the optimal rule is about 10% lower than that of AU. If initial expectations are slightly different from the long-run equilibrium then gains are even higher. For $a_0 = 1$ the welfare losses under the optimal policy are 42% lower than under AU. In the first period, the optimal interest rate rule (25) yields *ex-post* higher cumulative welfare losses expressed in terms of consumption than the AU rule; later, however, our rule starts generating smaller welfare losses. These findings are similar to the numerical results of the constant gain section.

An alternative way to examine the mechanisms at work is to look at the path of expectations. Both the optimal and the AU rule are E-stable under learning, so expectations converge to the discretionary REE; the difference is the speed of convergence. Fig. 7 shows a typical realization of the evolution of expectations under both rules. We can observe that inflation expectations converge faster and output gap expectations converge more slowly with our rule than with the AU one. This is a consequence of the intertemporal trade-off (Result 1): when the CB does take into account its influence on the learning algorithm, it has an incentive to undercut future inflation beliefs. However, because of the intratemporal trade-off between inflation and output, the cost of keeping inflation closer to its RE value is a wider output gap, and consequently a slower convergence of b to its RE value.

In this section, we have proved that our main results do not depend on what type of learning algorithm private agents follow. Our new results are that under decreasing gain learning, optimal policy should be time-varying: more aggressive on inflation initially and less in subsequent periods. In the limit, expectations converge to the discretionary RE equilibrium, and optimal policy is equivalent to the one derived under the assumption of constant expectations.

5. Robust policy advice

So far we have assumed that the CB has perfect knowledge about the learning algorithm followed by private agents. Since this is a rather strong assumption, in this section we relax it and ask what the policy recommendation is when the monetary authority is uncertain about the nature of private sector expectations.[39] In particular, we aim to define consistent policy advice on a set of private agents' expectations formation schemes empirically relevant for the US.

Let us conduct an experiment, in which we assume that the US Federal Reserve is uncertain about how the private sector forms its expectations but, relying on the empirical literature, it can define a relevant set of expectations, which includes both constant gain with a small gain and RE. The empirical literature on the US shows that a constant gain algorithm with a small tracking parameter is a good approximation of the data. For example, Milani (2007) estimates the New Keynesian model with adaptive learning using Bayesian methods and finds γ to be 0.0183. Orphanides and Williams (2005a) and

[38] Note that the PLM of private agents does not nest the commitment REE, only the discretionary REE.

[39] We have assumed that the CB perfectly observes all the relevant state variables of the system, namely the exogenous shocks and the agents' beliefs. It is possible to show that our main results extend to a more general framework, where the shocks or the expectations are not observable, and the CB has to solve a signal extraction problem to learn about them. For details, see the working paper version of this paper.

Table 2
Consumption equivalents under the optimal or a wrong rule, initial inflation expectations not RE.

Expectations	$\gamma=0.0183$	$\gamma=0.03$	$\gamma=0.04$	RE
Interest rate rule				
$\gamma=0.0183$	0.0409	0.0393	0.0384	0.0126
$\gamma=0.03$	0.0411	0.0391	0.0379	0.0126
$\gamma=0.04$	0.0415	0.0392	0.0378	0.0126
AU	0.0418	0.0411	0.0408	0.0126
The worst rule	AU	AU	AU	$\gamma=0.04$
% increase of cons. eq.[a]	2.17	4.88	7.41	0.02

Woodford (1999) calibration. Starting out of RE equilibrium: a_0=0.5. Consumption equivalents for a given underlying private sector expectation formation and a given interest rate rule.

[a] Worst rule compared with the optimal.

Branch and Evans (2006) calibrate γ to fit the Survey of Professional Forecasters and find that tracking parameters between 0.01 and 0.04 fit survey expectations well. We therefore consider tracking parameters in this range.

In addition, let us assume that the FED has no probability distribution over the possible forms of private expectations. Instead, it uses robust control and looks for the policy that minimizes the maximum loss.[40] We perform numerical Monte Carlo analysis to examine welfare losses when private expectations are taken from this set and the CB's interest rate rule is either an optimal rule for a given small gain parameter or the discretionary rule under RE (6c). Table 2 reports consumption equivalents, with initial inflation expectations being out of equilibrium ($a_0=0.5$).[41] In order to present the results in a compact way, the last line of Table 2 shows percentage increases in consumption equivalents of the worst case compared with the optimal rule for a given private expectation. For example if private agents use $\gamma=0.03$ and the central bank uses the correct CG interest rate rule with the same γ, the corresponding consumption equivalent is 0.039 percent, while under the AU interest rate rule it is 0.041 percent. The worst rule in this case is the AU rule, which gives a 4.88 percent increase compared to the optimal $\gamma=0.03$ rule.

The main result is that the worst-case scenario is using the AU interest rate rule when private agents are in fact learning. The min-max rule (following Hansen and Sargent (2007)), which minimizes the maximum loss, is the CG rule with $\gamma=0.0183$. With different initial values of inflation expectations the min-max rule changes, but it is always a rule that is optimal under learning. The reason why the min-max rule is always a learning rule, is that losses under RE caused by mistakenly using an optimal learning rule are smaller than losses associated with the use of AU when agents are in fact learning.

Under RE, all of these rules lead to a determinate equilibrium. The AU rule provides smaller losses than optimal learning rules (see last line of Table 2), and the reason for this is that learning rules allow for volatility in the output gap that is too high.[42]

When private agents are learning and the FED uses a misspecified learning rule, consumption equivalents increase but the loss is always smaller than losses associated with using the AU rule. The bigger the misperception of the monetary policy about γ is, the bigger the increase in consumption equivalents.

Let us now assume that the monetary authority is able to formulate a probability distribution over private expectations, using Table 2 we can also examine what is the minimum probability it must attribute to private agents following learning in order to choose a learning rule over the AU rule. In particular, let's assume that the FED's prior is that with probability p private agents follow constant gain learning with a given γ, and with probability $1-p$ agents have RE. Then, for any given tracking parameter we search a cutoff value of p for which the expected loss of using the CG rule is less than the welfare loss of the AU rule.

A surprising result is that the cutoff value of p is between 0.02% and 0.07%[43] This means that it is optimal to use the learning rule even if the CB attributes only a very small probability to agents following learning and a very high probability to RE.

6. Conclusions

We have shown that deviations of private sector beliefs from RE can have important policy implications that go beyond the possible failure to asymptotically converge to RE. Besides the well-known intratemporal inflation-output gap trade-off, when expectations are less than rational the CB faces an intertemporal trade-off between stabilizing the economy in the present and in the future. Optimal policy does not stabilize the economy in the present in order to influence future inflation expectations and ease future inflation-output gap trade-offs.

[40] For an extensive treatise on the use of robust control techniques in economics, see Hansen and Sargent (2007).

[41] For the case when initial expectations are at the RE equilibrium ($a_0=0$) see the working paper version of this paper. The differences between different policies are less pronounced, but the main result is the same.

[42] Since learning rules decrease the volatility of inflation and allow for higher volatility in the output gap, for small values of α learning rules do outperform the AU rule even under RE.

[43] Cutoff values for p are very low (1–1.5%) even when initial expectations are at the RE equilibrium.

We show that optimal policy derived under adaptive learning is more robust to uncertainty about expectation formation than optimal policy under RE. This suggests that central bankers should pay attention to the intertemporal trade-off because in practice they do face a substantial degree of uncertainty about private expectations.

Overall, our results suggest that the way private expectations are formed is a significant issue for policy design, and monetary policy should closely monitor private expectations. We hope this result will motivate more research interest in understanding how private expectations are formed in different environments. In most of the paper, we took the extreme view that the CB has perfect knowledge of private expectations; because this is a strong assumption we think it is an important avenue for future research to examine policy implications when the CB does not have full knowledge of the way private expectations are formed, such as for example in Woodford (2010).

Acknowledgements

We are especially grateful to our supervisors Albert Marcet and Ramon Marimon for very helpful comments and encouragement throughout this project. We are also indebted to Kosuke Aoki, Gernot Doppelhofer, Refet Gurkaynak, Seppo Honkapohja, Thijs van Rens, Alessandro Rebucci, José-Víctor Ríos-Rull, Stephan Sauer, and Ákos Valentinyi, the editor Eric M. Leeper and an anonymous referee for their comments. Any errors are our own. The views expressed herein are those of the authors, and do not necessarily reflect those of the Bank of Italy.

Appendix A. Constant gain learning

In this section, we provide an outline of the derivation of the inflation law of motion (18) and prove Proposition 1.

Combining the optimality conditions (10)–(13) and (15) we can write

$$\frac{\kappa}{\alpha}\pi_t + x_t = \beta E_t\left[\beta\gamma x_{t+1} + (1-\gamma)\left(\frac{\kappa}{\alpha}\pi_{t+1} + x_{t+1}\right)\right].$$

Using the Phillips curve (2) and the evolution of inflation expectations (7), we get

$$E_t[\pi_{t+1}] = A_{11}\pi_t + A_{12}a_t + P_1 u_t, \tag{27}$$

where

$$\begin{aligned} A_{11} &\equiv \frac{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))}{\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)}, \\ A_{12} &\equiv -\frac{\alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)}, \\ P_1 &\equiv -\frac{\alpha}{\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)}. \end{aligned}$$

Hence, at an optimum, the dynamics of the economy can be summarized by stacking Eqs. (7), (8) and (27) and obtaining the trivariate system:

$$E_t y_{t+1} = Ay_t + Pu_t, \tag{28}$$

where $y_t \equiv [\pi_t, a_t, b_t]'$, and

$$A \equiv \begin{pmatrix} A_{11} & A_{12} & 0 \\ \gamma & 1-\gamma & 0 \\ \frac{\gamma}{\kappa} & -\frac{\beta\gamma}{\kappa} & 1-\gamma \end{pmatrix}, \quad P = \begin{pmatrix} P_1 \\ 0 \\ -\frac{\gamma}{\kappa} \end{pmatrix}.$$

The three boundary conditions of the above system are

$$\begin{aligned} &a_0, b_0 \text{ given} \\ &\lim_{s\to\infty}|E_t\pi_{t+s}| < \infty. \end{aligned} \tag{29}$$

The last one is a result of the fact that, if there exists a solution to the problem (9) when the possible stochastic processes $\{\pi_t, x_t, r_t, a_{t+1}, b_{t+1}\}$ are restricted to be bounded, then this would also be the minimizer in the unrestricted case.[44]

Because A is block triangular, its eigenvalues are given by $1-\gamma$ and by the eigenvalues of

$$A_1 \equiv \begin{pmatrix} A_{11} & A_{12} \\ \gamma & 1-\gamma \end{pmatrix}. \tag{30}$$

In Lemma 1 we show that A_1 has one eigenvalue inside and one outside the unit circle, which implies (together with $(1-\gamma)\in(0,1)$) that we can invoke Proposition 1 of Blanchard and Kahn (1980) to conclude that the systems (28) and (29) have one and only one solution. In other words, there exists one and only one stochastic process for each of the three

[44] The proof is available from the author upon request.

variables of y such that (29) is satisfied. Moreover, note that $y_{1t} \equiv [\pi_t, a_t]'$ does not depend on b_t; therefore, the processes for inflation and a that solve (together with the process for b) the systems (28) and (29) are also a solution of the subsystem:

$$E_t y_{1t+1} = A_1 y_{1t} + (P_1, 0)' u_t,$$

together with the boundary conditions

$$a_0 \text{ given}, \quad \lim_{s\to\infty} |E_t \pi_{t+s}| < \infty.$$

By Lemma 1, we can invoke Proposition 1 of Blanchard and Kahn (1980) to conclude that the law of motion for inflation can be written in the form:

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t,$$

as stated in Proposition 1.

Lemma 1. *Let A_1 be given by Eq. (30) in the text; then it has an eigenvalue inside and one outside the unit circle.*

Proof. It is easy to see that both the trace and the determinant of the matrix A_1 are positive; hence, to conclude that it has two real eigenvalues, one outside and one inside the unit circle, it suffices to show[45]

$$\mu_1 + \mu_2 > 1 + \mu_1 \mu_2,$$

where μ_1, μ_2 are the eigenvalues of the matrix; in the case of A_1, the above condition can be written equivalently as

$$\frac{\kappa^2 + \alpha + \alpha\beta^2\gamma(1-\gamma(1-\beta))}{\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))} + 1 - \gamma \quad > 1 + \frac{\kappa^2 + \alpha + \alpha\beta^2\gamma(1-\gamma(1-\beta))}{\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))}(1-\gamma) + \frac{\alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))}\gamma,$$

where we have used the fact that the trace is equal to the sum of the eigenvalues, and that the determinant is equal to the product. After simplifying the above inequality, we get

$$-\gamma > -\gamma\left(\frac{\kappa^2 + \alpha + \alpha\beta^2\gamma(1-\gamma(1-\beta)) - \alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))}\right),$$

so that all we have to prove is that

$$\frac{\kappa^2 + \alpha + \alpha\beta^2\gamma(1-\gamma(1-\beta)) - \alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))} > 1.$$

Some tedious algebra shows that this is equivalent to the following expression:

$$\kappa^2(1-\beta(1-\gamma)) + \alpha(1-\beta)(1-\beta(1-\gamma(1-\beta))) > 0,$$

which is always true, because β and γ are assumed to be smaller than one. □

We now prove the rest of Proposition 1. First of all, we can guess that inflation follows the ALM (18) and use the optimality condition (27) and the method of undetermined coefficients to verify that c_π^{cg} must satisfy the following quadratic expression:

$$p_2 (c_\pi^{cg})^2 + p_1 c_\pi^{cg} + p_0 = 0,$$

where

$$\begin{aligned}
p_2 &\equiv \gamma[\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))],\\
p_1 &\equiv (1-\gamma)[\kappa^2\beta(1-\gamma) + \alpha\beta(1-\gamma(1-\beta))] - [\kappa^2 + \alpha + \alpha\beta^2\gamma(1-\gamma(1-\beta))],\\
p_0 &\equiv \alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta))),
\end{aligned}$$

and that:

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2 + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma(1-\gamma)(\alpha\beta - (\kappa^2+\alpha)c_\pi^{cg})}.$$

The polynomial in c_π^{cg} can be equivalently rewritten as follows:

$$c_\pi^{cg} = -\frac{p_0 + p_2 (c_\pi^{cg})^2}{p_1} \equiv f(c_\pi^{cg}).$$

We will prove that the function $f(\cdot)$, defined on the interval $[0, 1]$, is a contraction, so that it admits one and only one fixed point; moreover, because the two roots of the quadratic expression have the same sign (this is due to the fact that both p_2 and p_0 are positive), it follows that the other candidate value for c_π^{cg} is greater than one, which is not compatible with the boundary conditions.[46]

[45] LaSalle (1986).
[46] Because it would imply an exploding inflation.

First of all, we show that $f(\cdot)$, when defined on the interval $[0,1]$, takes values on the same interval.

Lemma 2. *$f(c_\pi^{cg})$ is strictly monotone increasing on the interval* $[0, 1]$.

Proof. Note that:

$$f'(c_\pi^{cg}) = \frac{2\gamma[\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}c_\pi^{cg},$$

which is positive if and only if the denominator is positive:

$$\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]\lesseqgtr 0.$$

After rearranging:

$$\kappa^2(1-\beta(1-\gamma)^2)+\alpha[1-\beta(1-\gamma)(1-\gamma(1-\beta))]+\alpha\beta^2\gamma(1-\gamma(1-\beta))\lesseqgtr 0,$$

which is always positive. Thus we have proved that $f(c_\pi^{cg})$ is strictly monotone increasing on the interval $[0, 1]$. □

Lemma 3. $f(c_\pi^{cg}) : [0,1]\to[0,1]$.

Proof. Because $f(c_\pi^{cg})$ is strictly monotone, increasing it suffices to show that $f(0)>0$ and $f(1)<1$:

$$f(0) = \frac{\alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]},$$

where the denominator is positive (see the preceding proof), and also the numerator is trivially positive. Thus $f(0)>0$:

$$f(1) = \frac{\gamma[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]+\alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}$$

After rearranging, we get

$$f(1)\lesseqgtr 1 \Longleftrightarrow 0\lesseqgtr\kappa^2(1-\beta(1-\gamma))+\alpha(1-\beta)(1-\beta(1-\gamma(1-\beta))),$$

but, as we argued above, the RHS of the last inequality is always positive; hence, $f(1)<1$. □

To show that $f(\cdot)$ is a contraction, it suffices to show that its derivative is bounded above by a number smaller than one: in fact, by the mean value theorem, we know that for any a,b, there exists a $c\in(a,b)$ such that:

$$|f(a)-f(b)| \le |f'(c)||a-b|,$$

and if $|f'(c)|\le M<1$ for any $c\in[0,1]$, we have the definition of a contraction.

Lemma 4. *For any* $x\in[0,1]$, $0<f'(x)\le f'(1)<1$.

Proof. First of all, note that:

$$f'(x) = \frac{2\gamma[\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}x,$$

is positive and increasing in x, so that $\max_{x\in[0,1]}f'(x)=f'(1)$; after some algebraic manipulation, we get

$$f'(1)\lesseqgtr 1 \Longleftrightarrow (1-\beta\gamma)\beta(1-\gamma(1-\beta))+\beta\gamma(1-\gamma(1-\beta))-1\lesseqgtr\frac{\kappa^2}{\alpha}\left(1-\beta\left(1-\gamma^2\right)\right)$$

Because $\beta,\gamma\in(0,1)$, we have

$$(1-\beta\gamma)\beta(1-\gamma(1-\beta))+\beta\gamma(1-\gamma(1-\beta))-1<1-\beta\gamma+\beta\gamma(1-\gamma(1-\beta))-1<0,$$

so that $f'(1)$ will be smaller than one ($\kappa^2/\alpha(1-\beta(1-\gamma^2))$ is always positive). □

Moreover, we prove the following result.

Lemma 5. *Let $f(\cdot)$ be defined as above; then,*

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right)\le\frac{\alpha\beta}{\kappa^2+\alpha}.$$

Proof. Note that:

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right) = \frac{\alpha\beta(1-\beta(1-\gamma)(1-\gamma(1-\beta)))}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}$$
$$+\frac{\gamma[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right)^2\gtrless\frac{\alpha\beta}{\kappa^2+\alpha},$$

if and only if:

$$\frac{(\kappa^2+\alpha)(1-\beta(1-\gamma)(1-\gamma(1-\beta)))+\gamma\left[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))\right]\frac{\alpha\beta}{\kappa^2+\alpha}}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}\gtreqless 1.$$

For $\gamma=0$ it is easy to verify that

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right)=\frac{\alpha\beta}{\kappa^2+\alpha}.$$

If $\gamma>0$, because the $\alpha\beta/(\alpha+\kappa^2)<\beta$, the LHS of the above inequality is smaller than:

$$\frac{(\kappa^2+\alpha)(1-\beta(1-\gamma)(1-\gamma(1-\beta)))+\beta\gamma[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]},$$

which is equal to one; in fact:

$$\frac{(\kappa^2+\alpha)(1-\beta(1-\gamma)(1-\gamma(1-\beta)))+\beta\gamma[\kappa^2\beta(1-\gamma)+[\alpha\beta(1-\gamma(1-\beta))]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}\gtreqless 1,$$

is equivalent to

$$-(\kappa^2+\alpha)\beta(1-\gamma)(1-\gamma(1-\beta))+(1-\gamma(1-\beta))[\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)]\gtreqless\alpha\beta^2\gamma(1-\gamma(1-\beta)).$$

But the LHS can simplified as

$$\alpha\beta(1-\gamma(1-\beta))(1-\gamma(1-\beta)-(1-\gamma)),$$

which is equal to

$$\alpha\beta^2\gamma(1-\gamma(1-\beta)).$$

Summing up, we showed that (if $\gamma>0$) the following holds:

$$\frac{(\kappa^2+\alpha)(1-\beta(1-\gamma)(1-\gamma(1-\beta)))+\beta\gamma[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}{\kappa^2+\alpha+\alpha\beta^2\gamma(1-\gamma(1-\beta))-(1-\gamma)[\kappa^2\beta(1-\gamma)+\alpha\beta(1-\gamma(1-\beta))]}=1,$$

which implies that:

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right)<\frac{\alpha\beta}{\kappa^2+\alpha}. \qquad \square$$

We are now ready to prove the Proposition.

Proof of Proposition 1. Combining the Lemmas 3 and 4 we obtain that $f(\cdot)$ is a contraction when defined on the interval [0, 1]; moreover, by Lemma 5 we get that f, when defined on $[0,\alpha\beta/(\kappa^2+\alpha)]$, takes values on the same interval. This result, together with Lemma 4 and with the inequality $\alpha\beta/(\kappa^2+\alpha)<1$, implies that $f(\cdot)$ is a contraction also when defined on the interval $[0,\alpha\beta/(\kappa^2+\alpha)]$ and, therefore, that the optimal c_π^{cg} must be between zero and $\alpha\beta/(\kappa^2+\alpha)$.

Finally, note that when $\gamma=0$, $f(c_\pi^{cg})$ collapses to $\alpha\beta/(\kappa^2+\alpha)$, which completes the proof. □

Appendix B. Decreasing gain learning

In this section, we prove Propositions 2, 4 and 5.

Proof of Proposition 2. To derive the optimal allocations, note that we can use first-order conditions (10)–(13) and (15) and $\gamma_t=1/t$ to rewrite:

$$\frac{\kappa}{\alpha}\pi_t+x_t=\beta E_t\left[\beta\frac{1}{t+1}x_{t+1}+\frac{\kappa}{\alpha}\pi_{t+1}+x_{t+1}\right].$$

Using (2) to substitute out x_t in the above equation, and then using the evolution of inflation expectations (21) we get

$$E_t[\pi_{t+1}]=A_{11,t}\pi_t+A_{12,t}a_t+P_{1,t}u_t, \tag{31}$$

where:

$$A_{11,t}\equiv\frac{\kappa^2+\alpha+\alpha\beta^2\frac{1}{t+1}\left(1+\beta\frac{1}{t+1}\right)}{\alpha\beta\left(1+\beta\frac{1}{t+1}\right)+\kappa^2\beta},$$

$$A_{12,t} \equiv -\frac{\alpha\beta\left[1-\beta\left(1-\frac{1}{t+1}\right)\left(1+\beta\frac{1}{t+1}\right)\right]}{\alpha\beta\left(1+\beta\frac{1}{t+1}\right)+\kappa^2\beta},$$

$$P_{1,t} \equiv -\frac{\alpha}{\alpha\beta\left(1+\beta\frac{1}{t+1}\right)+\kappa^2\beta}.$$

Hence, at an optimum, the dynamics of the economy can be summarized by stacking Eqs. (21), (22) and (31), and obtaining the trivariate system:

$$E_t y_{t+1} = A_t y_t + P_t u_t, \tag{32}$$

where $y_t \equiv [\pi_t, a_t, b_t]'$, and

$$A_t \equiv \begin{pmatrix} A_{11,t} & A_{12,t} & 0 \\ \frac{1}{t+1} & 1-\frac{1}{t+1} & 0 \\ \frac{\frac{1}{t+1}}{\kappa} & -\frac{\beta\frac{1}{t+1}}{\kappa} & 1-\frac{1}{t+1} \end{pmatrix}, \quad P_t = \begin{pmatrix} P_{1,t} \\ 0 \\ -\frac{\frac{1}{t+1}}{\kappa} \end{pmatrix}.$$

We can find the solution with the method of undetermined coefficients with the guess:[47]

$$\pi_t = c^{dg}_{\pi,t} a_t + d^{dg}_{\pi,t} u_t.$$

The sequence $\{c^{dg}_{\pi,t}\}$ must satisfy the nonlinear, nonautonomous first-order difference equation:

$$c^{dg}_{\pi,t} = \frac{c^{dg}_{\pi,t+1}\left(1-\frac{1}{t+1}\right) - A_{12,t}}{A_{11,t} - c^{dg}_{\pi,t+1}\frac{1}{t+1}}, \tag{33}$$

and the sequence $\{d^{dg}_{\pi,t}\}$ is defined as

$$d^{dg}_{\pi,t} = \frac{P_{1,t}}{c^{dg}_{\pi,t+1}\frac{1}{t+1} - A_{11,t}},$$

as stated in the Proposition. Clearly, once we solve for $c^{dg}_{\pi,t}$, finding the value of $d^{dg}_{\pi,t}$ is a trivial task. Of course, there exist infinite sequences that satisfy equation (33), one for each initial value $c^{dg}_{\pi,0}$. However, because the boundary conditions require π_t to stay bounded, we will concentrate on the solutions that do not explode. To characterize its properties, first note that if we solve forward the following difference equation:

$$c^{dg}_{\pi t} = \beta c^{dg}_{\pi t+1} + \frac{\alpha\beta}{\kappa^2+\alpha}(1-\beta),$$

we obtain one and only one bounded solution, i.e.,:

$$c^{dg}_{\pi t} = \frac{\alpha\beta}{\kappa^2+\alpha} \quad \forall t.$$

Moreover, we can rewrite the difference equation defining $c^{dg}_{\pi t}$ as

$$G_t \equiv A_{11,t} c^{dg}_{\pi,t} - c^{dg}_{\pi,t+1} = -\frac{1}{t+1} c^{dg}_{\pi,t+1} - A_{12,t} + \frac{1}{t+1} c^{dg}_{\pi,t} c^{dg}_{\pi,t+1} \equiv F_t.$$

If c^{dg}_{π} is bounded, it is easy to show that F has a limit:

$$\lim_{t\to\infty} F_t = -\lim_{t\to\infty} A_{12,t} = \frac{\alpha}{\kappa^2+\alpha}(1-\beta).$$

We can also show that the difference equation defined by G converges to

$$\beta^{-1} c^{dg}_{\pi,\tau} - c^{dg}_{\pi,\tau+1}.$$

Summing up, in the limit we have that c^{dg}_{π} evolves according to

$$c^{dg}_{\pi\tau} = \beta c^{dg}_{\pi\tau+1} + \frac{\alpha\beta}{\kappa^2+\alpha}(1-\beta),$$

which, as we state in the Proposition, has one and only one bounded solution:

$$c^{dg}_{\pi\tau} = \frac{\alpha\beta}{\kappa^2+\alpha}.$$

[47] This guess corresponds to the unique solution under constant gain learning.

We prove the last part of the statement by contradiction. Assume that there exists a $T<\infty$ such that $c_{\pi T}^{dg}\geq \alpha\beta/(\alpha+\kappa^2)$; we show that this implies that $c_{\pi t}^{dg}>\alpha\beta/(\alpha+\kappa^2)$ for any $t>T$. First of all, we can write

$$\frac{c_{\pi,T+1}^{dg}\left(1-\frac{1}{T+1}\right)-A_{12,T}}{A_{11,T}-c_{\pi,T+1}^{dg}\frac{1}{T+1}}=c_{\pi T}^{dg}\geq\frac{\alpha\beta}{\alpha+\kappa^2}.$$

Rearranging and simplifying, this turns out to be equivalent to

$$\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)c_{\pi T+1}^{dg}\geq\frac{\alpha\beta}{\alpha+\kappa^2}A_{11,T}+A_{12,T}. \tag{34}$$

Note that the RHS is equal to

$$\begin{aligned}\frac{\alpha\beta}{\alpha+\kappa^2}A_{11,T}+A_{12,T}&=\frac{\alpha\beta}{\alpha\beta\left(1+\beta\frac{1}{t+1}\right)+\kappa^2\beta}\left[\beta\left(1+\beta\frac{1}{t+1}\right)\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)\right]\\&=\frac{\alpha\beta}{\alpha+\kappa^2\left(1+\beta\frac{1}{t+1}\right)-1}\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)\\&>\frac{\alpha\beta}{\alpha+\kappa^2}\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right),\end{aligned}$$

where the last inequality is a result of the fact that $(1+\beta 1/(t+1))^{-1}<1$; putting together the last inequality and (34), we get

$$c_{\pi T+1}^{dg}>\frac{\alpha\beta}{\alpha+\kappa^2}.$$

Then, we can apply the above argument to $c_{\pi T+2}^{dg}$ as well and, proceeding by induction, conclude that $c_{\pi t}^{dg}>\alpha\beta/(\alpha+\kappa^2)$ for any $t>T$. However, it is possible to show that if we start from the limit value $\lim_{t\to\infty}c_{\pi t}^{dg}=\alpha\beta/(\alpha+\kappa^2)$, the value of $c_{\pi t}^{dg}$ implied by Eq. (33) for an arbitrary small $1/(t+1)$ is smaller than $\alpha\beta/(\alpha+\kappa^2)$,[48] which contradicts the result that $c_{\pi t}^{dg}>\alpha\beta/(\alpha+\kappa^2)$ for any $t>T$. Hence, we have showed that there is no $t<\infty$ such that $c_{\pi t}^{dg}\geq\alpha\beta/(\alpha+\kappa^2)$. □

To prove Proposition 4 we first state and prove the following technical lemma:

Lemma 6. *Let λ_1 be the smallest root of the second-order polynomial*:

$$\rho(p)\equiv\omega_2p^2+\omega_1p+\omega_0,$$

where:

$$\begin{aligned}&\omega_2\equiv-\gamma[(\kappa^2+\alpha)\beta+\alpha\beta^2\gamma],\\&\omega_1\equiv[(\kappa^2+\alpha)(1-\beta(1-\gamma))+\alpha\beta^2\gamma^2(1+\beta)],\\&\omega_0\equiv-\alpha\beta[1-\beta(1-\gamma+\beta\gamma-\beta\gamma^2)],\end{aligned}$$

and where the restrictions on the parameters α, β and κ are the same as those imposed in the rest of the paper. Then, there exists a $\overline{\gamma}\in(0,1)$ such that when $\gamma\in(0,\overline{\gamma}]$, the following holds:

$$\frac{\partial}{\partial\gamma}\lambda_1<0.$$

Proof. First of all, note that applying the implicit function theorem, we have

$$\frac{\partial}{\partial\gamma}\lambda_1=-\frac{\partial\rho/\partial\gamma}{\partial\rho/\partial p}\bigg|_{p=\lambda_1}\gtrless0\to\frac{\partial\rho}{\partial\gamma}\bigg|_{p=\lambda_1}\lessgtr0, \tag{35}$$

where we used the fact that, because $\omega_2<0$, $\partial\rho/\partial p|_{p=\lambda_1}>0$. Moreover, we have

$$\frac{\partial\rho}{\partial\gamma}\equiv\psi(p)=\varpi_2p^2+\varpi_1p+\varpi_0,$$

where

$$\begin{aligned}&\varpi_2\equiv-[(\kappa^2+\alpha)\beta+2\alpha\beta^2\gamma],\\&\varpi_1\equiv[(\kappa^2+\alpha)\beta+2\alpha\beta^2\gamma(1+\beta)],\\&\varpi_0\equiv\alpha\beta^2[\beta-2\beta\gamma-1].\end{aligned}$$

It is easy to show that, (i) there exists a $\overline{\gamma}_1$ such that, for $\gamma<\overline{\gamma}_1$, the largest root of $\rho(\cdot)$ is bigger than the largest root of $\psi(\cdot)$

[48] The proof is available from the authors upon request.

(actually, the former goes to infinity as γ goes to zero); (ii) there exists a $\overline{\gamma}_2$ such that, for $\gamma<\overline{\gamma}_2$, the quadratic polynomial:

$$\rho(p)-\psi(p),$$

has one positive and one negative root. Combining this result with the fact that both $\rho(\cdot)$ and $\psi(\cdot)$ are concave, we obtain that, for $\gamma<\overline{\gamma}\equiv\min\{\overline{\gamma}_1,\overline{\gamma}_2\}$, the smallest root of $\rho(\cdot)$ lies between the two roots of $\psi(\cdot)$; in other words:

$$\frac{\partial\rho}{\partial\gamma}\Big|_{p=\lambda_1}>0.$$

Using this result in (35) completes the proof. □

An immediate corollary of the above lemma is the following:

Corollary 1. *Let λ_{1t} be the smallest root of the second-order polynomial*:

$$\rho_t(p)\equiv\omega_{2t}p^2+\omega_{1t}p+\omega_{0t},$$

where

$$\omega_{2t}\equiv-\frac{1}{t+1}\left[(\kappa^2+\alpha)\beta+\alpha\beta^2\frac{1}{t+1}\right],$$

$$\omega_{1t}\equiv\left[(\kappa^2+\alpha)\left(1-\beta\left(1-\frac{1}{t+1}\right)\right)+\alpha\beta^2\left(\frac{1}{t+1}\right)^2(1+\beta)\right],$$

$$\omega_{0t}\equiv-\alpha\beta\left[1-\beta\left(1-\frac{1}{t+1}+\beta\frac{1}{t+1}-\beta\left(\frac{1}{t+1}\right)^2\right)\right].$$

Then, there exists a $T<\infty$ such that $\{\lambda_{1t}\}_{t=T}^{\infty}$ is a monotonic increasing sequence.

Proof. First of all, note that λ_{1t} and ω_{it}, $i=1,2,3$, are defined as the correspondent coefficient in the statement of Lemma 6, with γ replaced by $(t+1)^{-1}$; hence, $t+1\geq 2$ is equivalent to $\gamma\leq\overline{\gamma}$ implies $t+1\geq T+1$, where $T+1$ is the integer part of $1/\gamma$. Invoking the result of Lemma 6, we get that λ_{1t} increases as $(t+1)^{-1}$ decreases. □

We are now ready to prove Proposition 4.

Proof of Proposition 4. First of all, note that $\delta_{\pi t}^{dg}$ is decreasing if and only if $c_{\pi t}^{dg}$ is increasing; hence, we prove this latter statement. Recall that:

$$c_{\pi,t}^{dg}=\frac{c_{\pi,t+1}^{dg}\left(1-\frac{1}{t+1}\right)-A_{12,t}}{A_{11,t}-c_{\pi,t+1}^{dg}\frac{1}{t+1}},$$

which means that, for any finite t, we have:

$$c_{\pi t+1}^{dg}=\frac{A_{11,t}c_{\pi,t}^{dg}+A_{12,t}}{1-\frac{1}{t+1}\left(1-c_{\pi,t}^{dg}\right)}.$$

Because $1-1/(t+1)(1-c_{\pi,t}^{dg})$ is a positive expression, $c_{\pi t+1}^{dg}-c_{\pi t}^{dg}\gtreqless 0$ is equivalent to the second-order inequality:

$$\omega_{2t}(c_{\pi t}^{dg})^2+\omega_{1t}c_{\pi t}^{dg}+\omega_{0t}\gtreqless 0,$$

where

$$\omega_{2t}\equiv-\frac{1}{t+1}\left[(\kappa^2+\alpha)\beta+\alpha\beta^2\frac{1}{t+1}\right],$$

$$\omega_{1t}\equiv\left[(\kappa^2+\alpha)\left(1-\beta\left(1-\frac{1}{t+1}\right)\right)+\alpha\beta^2\left(\frac{1}{t+1}\right)^2(1+\beta)\right],$$

$$\omega_{0t}\equiv-\alpha\beta\left[1-\beta\left(1-\frac{1}{t+1}+\beta\frac{1}{t+1}-\beta\left(\frac{1}{t+1}\right)^2\right)\right].$$

Let λ_{1t}, λ_{2t} be the two roots of the above quadratic expression, such that $\lambda_{1t}<\lambda_{2t}$; because ω_{2t}, $\omega_{0t}<0$ for any t, and $-(\omega_{1t}/\omega_{2t})$ can be easily shown to be positive, we know that λ_{1t}, $\lambda_{2t}>0$ and that:

$$\lambda_{1t}<c_{\pi t}^{dg}<\lambda_{2t}\Longleftrightarrow c_{\pi t+1}^{dg}-c_{\pi t}^{dg}>0.$$

It is easy to see that $\lambda_{2t} > \alpha\beta/(\alpha+\kappa^2)$ for any t, which implies that:

$$\lambda_{1t} < c^{dg}_{\pi t} < \frac{\alpha\beta}{\alpha+\kappa^2} \Longleftrightarrow c^{dg}_{\pi t+1} - c^{dg}_{\pi t} > 0,$$

because we showed in Proposition 2 that $c^{dg}_{\pi t} < \alpha\beta/(\alpha+\kappa^2)$ for any finite t. Now assume, for the sake of contradiction, that $c^{dg}_{\pi t} \le \lambda_{1t}$ for some $\tau \ge T$, where T is the one defined in Corollary 1; then, $c^{dg}_{\pi\tau+1} \le c^{dg}_{\pi\tau}$ and, for Corollary 1, $\lambda_{1\tau+1} > \lambda_{1\tau}$.
Combining these two inequalities yields the conclusion that $c^{dg}_{\pi\tau+1} \le \lambda_{1\tau+1}$. Repeating the preceding line of reasoning infinitely many times implies that a subsequence of $\{c^{dg}_{\pi t}\}$ moves monotonically away from $\alpha\beta/\alpha+\kappa^2$, so that $\lim_{t\to\infty} c^{dg}_{\pi t}$, if it exists, is definitely smaller than $\alpha\beta/\alpha+\kappa^2$, contradicting Proposition 2. This completes the proof. □

Finally, we prove Proposition 5. First of all, we briefly describe some results of the stochastic approximation[49] that we will exploit in the proof.

Let us consider a stochastic recursive algorithm of the form:

$$\theta_t = \theta_{t-1} + \gamma_t Q(t, \theta_{t-1}, X_t), \tag{36}$$

where X_t is a state vector with an invariant limiting distribution, and γ_t is a sequence of gains; the stochastic approximation literature shows how, provided certain technical conditions are met, the asymptotic behavior of the stochastic difference equation (36) can be analyzed using the associated deterministic ODE:

$$\frac{d\theta}{d\tau} = h(\theta(\tau)), \tag{37}$$

where

$$h(\theta) \equiv \lim_{t\to\infty} EQ(t, \theta, X_t).$$

E represents the expectations taken over the invariant limiting distribution of X_t, for any fixed θ. In particular, it can be shown that the set of limiting points of (36) is given by the stable resting points of the ODE (37).

Proof of Proposition 5. Note that our equation (26) is a special case of (36), where the technical conditions are easily shown to be satisfied; moreover, it is also easy to see that:

$$h(a) = \lim_{t\to\infty} (c^{dg}_{\pi,t} - 1)a = \left(\frac{\alpha\beta}{\alpha+\kappa^2} - 1\right)a,$$

which has a unique possible resting point at $a^* = 0$. Because $\dfrac{\alpha\beta}{\alpha+\kappa^2} < 1$, we have that a^* is globally stable, which proves the statement. □

Appendix C. Comparison with AU rule

Proof of Proposition 3. First of all, note that:

$$\delta^{dg}_{\pi,t} \gtrless \delta^{AU}_{\pi} \Longleftrightarrow \sigma\frac{\beta - c^{dg}_{\pi,t}}{\kappa} \gtrless \sigma\frac{\kappa\beta}{\alpha+\kappa^2},$$

where the second inequality can be rewritten as

$$\frac{\beta}{\kappa} - \frac{\kappa\beta}{\alpha+\kappa^2} \gtrless \frac{c^{dg}_{\pi,t}}{\kappa}.$$

Rearranging the terms, we get

$$\delta^{dg}_{\pi,t} \gtrless \delta^{AU}_{\pi} \Longleftrightarrow \frac{\alpha\beta}{\alpha+\kappa^2} \gtrless c^{dg}_{\pi,t}.$$

Because we have shown in Proposition 2 that $t < \infty$ implies $c^{dg}_{\pi,t} < \alpha\beta/(\alpha+\kappa^2)$, we conclude that $\delta^{dg}_{\pi t} > \delta^{AU}_{\pi}$. Using a similar argument it is easy to show that:

$$\delta^{dg}_{ut} \gtrless \delta^{AU}_{u} \Longleftrightarrow \frac{\alpha}{\alpha+\kappa^2} \gtrless d^{dg}_{\pi,t},$$

which implies, because:

$$d^{cg}_{\pi} = \frac{\alpha}{\kappa^2 + \alpha + \alpha\beta^2\gamma^2(\beta - c^{cg}_{\pi}) + \beta\gamma(1-\gamma)(\alpha\beta - (\kappa^2+\alpha)c^{cg}_{\pi})} < \frac{\alpha}{\alpha+\kappa^2},$$

that $\delta^{dg}_{ut} > \delta^{AU}_{u}$ whenever $t < \infty$. Finally, note that Proposition 2 also shows that $\lim_{t\to\infty} c^{dg}_{\pi,t} = \frac{\alpha\beta}{\alpha+\kappa^2}$, which trivially yields $\lim_{t\to\infty} \delta^{dg}_{\pi,t} = \delta^{AU}_{\pi}$ and $\lim_{t\to\infty} \delta^{dg}_{ut} = \delta^{AU}_{u}$. □

[49] Ljung (1977) and Benveniste et al. (1990) provide a recent survey.

Appendix D. Supplementary data

Supplementary data associated with this article can be found in the online version at http://dx.doi.org/10.1016/j.euroecorev.2013.08.012.

References

Adam, K., Billi, R., 2007. Discretionary monetary policy and the zero lower bound on nominal interest rates. J. Monet. Econ. 54 (3), 728–752.
Alvarez, F., Kehoe, P.J., Neumeyer, P.A., 2004. The time consistency of optimal monetary and fiscal policies. Econometrica 72 (2), 541–567.
Ball, L., 1994. What determines the sacrifice ratio?. In: Mankiw, N.G. (Ed.), Monetary Policy, Chicago University Press, Chicago.
Benati, L., 2008. Inflation persistence across monetary regimes. Q. J. Econ. 123 (3), 1005–1060.
Benveniste, A., Métivier, M., Priouret, P., 1990. Adaptive Algorithms and Stochastic Approximations. Springer-Verlag, Berlin.
Bernanke, B.S., 2004. The logic of monetary policy, speech delivered by federal reserve governor ben bernanke to the national economists club in Washington D.C., The Federal Reserve Board.
Blanchard, O.J., Kahn, C.M., 1980. The solution of linear difference models under rational expectations. Econometrica 48 (5), 1305–1311.
Branch, W.A., Evans, G.W., 2006. A simple recursive forecasting model. Econ. Lett. 91 (2), 158–166.
Brayton, F., Tinsley, P., 1996. A guide to FRB/US: a macroeconomic model of the United States, Finance and Economics Discussion Series 96–42. Board of Governors of the Federal Reserve System.
Bullard, J., Mitra, K., 2002. Learning about monetary policy rules. J. Monet. Econ. 49 (6), 1105–1129.
Calvo, G., 1983. Staggered prices in a utility maximizing framework. J. Monet. Econ. 12 (3), 383–398.
Clarida, R., Gali, J., Gertler, M., 1999. The science of monetary policy: a new Keynesian perspective. J. Econ. Lit. 37 (2), 1661–1707.
Clarida, R., Gali, J., Gertler, M., 2000. Monetary policy rules and macroeconomic stability: evidence and some theory. Q. J. Econ. 115 (1), 147–180.
Cogley, T., Sbordone, A.M., 2005. A Search for a Structural Phillips Curve. Staff Report 203. Federal Reserve Bank of New York.
Erceg, C.J., Levin, A.T., 2003. Imperfect credibility and inflation persistence. J. Monet. Econ. 50 (4), 915–944.
Eusepi, S., Giannoni, M., Preston, B., 2012. Long-Term debt pricing and monetary policy transmission under imperfect knowledge, Discussion papers 8845, CEPR.
Evans, G.W., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press, Princeton.
Evans, G.W., Honkapohja, S., 2003a. Adaptive learning and monetary policy design. J. Money Credit Bank. 35 (6), 1045–1072.
Evans, G.W., Honkapohja, S., 2003b. Expectations and the stability problem for optimal monetary policies. Rev. Econ. Stud. 70 (4), 807–824.
Evans, G.W., Honkapohja, S., 2006. Monetary policy, expectations and commitment. Scand. J. Econ. 108 (1), 15–38.
Ferrero, G., 2007. Monetary policy, learning and the speed of convergence. J. Econ. Dyn. Control 31 (9), 3006–3041.
Gali, J., 2003. New perspectives on monetary policy, inflation, and the business cycle. In: Dewatripont, M., Hansen, L., Turnovsky, S. (Eds.), Advances in Economic Theory, Cambridge University Press, Cambridge.
Gaspar, V., Smets, F., Vestin, D., 2006. Adaptive learning, persistence, and optimal monetary policy. J. Eur. Econ. Assoc. 4 (2–3), 376–385.
Hansen, L.P., Sargent, T.J., 2007. Robustness. Princeton University Press, Princeton.
Knight, F.H., 1921. Risk Uncertainty and Profit. Houghton Mifflin Company, Boston, MA.
Kreps, D., 1998. Anticipated utility and dynamic choice. In: Jacobs, D.P., Kalai, E., Kamien, M. (Eds.), Frontiers of Research in Economic Theory, Cambridge University Press, Cambridge.
LaSalle, J.P., 1986. The Stability and Control of Discrete Processes. Springer-Verlag, New York.
Ljung, L., 1977. Analysis of recursive stochastic algorithms. IEEE Trans. Autom. Control AC-22, 551–575.
Lucas, R.E., 1987. Models of Business Cycles. Basil Blackwell, Oxford.
Lucas, R.E., Stokey, N.L., 1983. Optimal fiscal and monetary policy in an economy without capital. J. Monet. Econ. 12 (1), 55–93.
Marcet, A., Nicolini, J.P., 2003. Recurrent hyperinflations and learning. Am. Econ. Rev. 93 (5), 1476–1498.
Marcet, A., Sargent, T.J., 1989. Convergence of least squares learning mechanisms in self referential linear stochastic models. J. Econ. Theory 48 (2), 337–368.
McCallum, B., Nelson, E., 1999. Performance of operational optimal policy rules in an estimated semi-classical model. In: Taylor, J. (Ed.), Monetary Policy Rules, University of Chicago Press, Chicago.
Mele, A., Molnar, K., Santoro, S., 2012. The suboptimality of commitment equilibrium when agents are learning. Mimeo.
Milani, F., 2006. A Bayesian DSGE model with infinite-horizon learning: do "Mechanical" sources of persistence become superfluous?. Int. J. Cent. Bank. 2 (3), 87–106.
Milani, F., 2007. Expectations, learning and macroeconomic persistence. J. Monet. Econ. 54 (7), 2065–2082.
Molnar, K., Reppa, Z., 2013. Survey expectations of inflation and their dependence on the economic environment. Mimeo.
Orphanides, A., Williams, J.C., 2005a. The decline of activist stabilization policy: natural rate misperceptions, learning, and expectations. J. Econ. Dyn. Control 29 (11), 1927–1950.
Orphanides, A., Williams, J.C., 2005b. Imperfect knowledge, inflation expectations and monetary policy. In: Bernanke, B., Woodford, M. (Eds.), The Inflation-Targeting Debate, Chicago University Press, Chicago.
Orphanides, A., Williams, J.C., 2005c. Inflation scares and forecast-based monetary policy. Rev. Econ. Dyn. 8 (2), 498–527.
Phelps, E.S., 1967. Phillips curves, expectations of inflation and optimal unemployment over time. Economica 34 (135), 254–281.
Preston, B., 2005. Learning about monetary policy rules when long-horizon expectations matter. Int. J. Cent. Bank. 1 (2), 81–126.
Rotemberg, J.J., Woodford, M., 1997. An optimization-based econometric framework for the evaluation of monetary policy. In: Bernanke, B., Rotemberg, J.J. (Eds.), NBER Macroeconomics Annual 12, MIT Press, Cambridge (MA).
Sargent, T.J., 1999. The Conquest of American Inflation. Princeton University Press, Princeton.
Slobodyan, S., Wouters, R., 2012. Learning in an estimated medium-scale DSGE model. J. Econ. Dyn. Control 36 (1), 26–46.
Woodford, M., 1999. Optimal monetary policy inertia. Manch. Sch. Suppl. 67, 1–35.
Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press, Princeton.
Woodford, M., 2007. Interpreting inflation persistence: comments on the conference on "quantitative evidence on price determination". J. Money Credit Bank. 39 (Suppl. 1), 203–210.
Woodford, M., 2010. Robustly optimal monetary policy under near-rational expectations. Am. Econ. Rev. 100 (1), 274–303.
Yun, T., 1996. Nominal price rigidity, money supply endogeneity, and business cycles. J. Monet. Econ. 37 (2–3), 345–370.