

You know, I don't know about that.

16 March 2020

- I wanna pause on for a sec b/c I can't seem to get it to work and I'm confused about how to get it time-varying anyway. So let's turn to estimation.

Estimation of the anchoring function

The issue is that we wanna estimate the anchoring for together w/ the model. On the top of my head I can think of 3 ways of doing that:

- 1) IR-matching
- 2) Likelihood-based (either MLE or Bayesian)
- 3) VAR-representation \Rightarrow exist? est. that!

\hookrightarrow it would be a time-varying one.

- \hookrightarrow I'm leaning toward #2 b/c
- 1) it's sexier
 - 2) it's more general than conditional on shocks
 - 3) a TV-VAR sounds challenging

The thing is:

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need to derive the log-likelihood of my model

Take the midsample model w/ the TR

In materials 21, this is eq. (5) - (10) + TR

$$\pi_t = kx_t - (1-\alpha)\beta f_a(t) + [-k\alpha\beta(I_3 - \alpha\beta h_x)^{-1} - e_3(I_3 - \alpha\beta h_x)^{-1}]S_t$$

$$x_t = -b_1 i_t - b_2 f_b(t) + [-(1-\beta)b_2(I_3 - \beta h_x)^{-1} + \beta b_3(I_3 - \beta h_x)^{-1} - b_4(I_3 - \beta h_x)^{-1}]S_t$$

$$f_a(t) = \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} - b_1(I_3 - \alpha\beta h_x)^{-1} S_t$$

$$f_b(t) = \frac{1}{1-\beta} \bar{\pi}_{t-1} - b_1(I_3 - \beta h_x)^{-1} S_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_1 S_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 S_{t-1})) + c$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t$$

this is a state-space model (believe it or not)

and I'm gonna eliminate some variables

$$\pi_t = kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} - (1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} S_t + [-k\alpha\beta(I_3 - \alpha\beta h_x)^{-1} - e_3(I_3 - \alpha\beta h_x)^{-1}]S_t$$

$$x_t = -b_1 \psi_\pi \bar{\pi}_t - b_1 \psi_x x_t - b_1 \bar{i}_t - \frac{b_1}{1-\beta} \bar{\pi}_{t-1} - b_1 b_1(I_3 - \beta h_x)^{-1} S_t + [-(1-\beta)b_2(I_3 - \beta h_x)^{-1} + \beta b_3(I_3 - \beta h_x)^{-1} - b_4(I_3 - \beta h_x)^{-1}]S_t$$

$$\pi_t = k x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h x)^{-1} s_t - k\alpha\beta (I_3 - \alpha\beta h x)^{-1} - e_3 (I_3 - \alpha\beta h x)^{-1} \right] s_t$$

$$x_t = -b\psi_\pi \pi_t - b\psi_x x_t - b\bar{i}_t - \frac{b}{1-\beta} \bar{\pi}_{t-1} - b b_1 (I_3 - \beta h x)^{-1} s_t$$

$$+ \left[-(1-\beta)b_2 (I_3 - \beta h x)^{-1} + b\beta b_3 (I_3 - \beta h x)^{-1} - b e_1 (I_3 - \beta h x)^{-1} \right] s_t$$

$$(1+b\psi_x)x_t = -b\psi_\pi \pi_t - \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$\left[-b e_2 - b b_1 (I_3 - \beta h x)^{-1} - (1-\beta)b_2 (I_3 - \beta h x)^{-1} + b\beta b_3 (I_3 - \beta h x)^{-1} - b e_1 (I_3 - \beta h x)^{-1} \right] s_t$$

$$\Rightarrow x_t = -\frac{b\psi_\pi}{1+b\psi_x} \pi_t - \frac{1}{1+b\psi_x} \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$+ \frac{1}{1+b\psi_x} \left[-b e_2 - b b_1 (I_3 - \beta h x)^{-1} - (1-\beta)b_2 (I_3 - \beta h x)^{-1} + b\beta b_3 (I_3 - \beta h x)^{-1} - b e_1 (I_3 - \beta h x)^{-1} \right] s_t$$

Can even sub x_t out!

$$\pi_t = -\frac{k b \psi_\pi}{1+b\psi_x} \pi_t - \frac{k}{1+b\psi_x} \frac{b}{(1-\beta)} \bar{\pi}_{t-1}$$

$$+ \frac{k}{1+b\psi_x} \left[-b e_2 - b b_1 (I_3 - \beta h x)^{-1} - (1-\beta)b_2 (I_3 - \beta h x)^{-1} + b\beta b_3 (I_3 - \beta h x)^{-1} - b e_1 (I_3 - \beta h x)^{-1} \right] s_t$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ \left[-(1-\alpha)\beta b_1 (I_3 - \alpha\beta h x)^{-1} s_t - k\alpha\beta (I_3 - \alpha\beta h x)^{-1} - e_3 (I_3 - \alpha\beta h x)^{-1} \right] s_t$$

$$\Rightarrow \left(1 + \frac{k b \psi_\pi}{1+b\psi_x} \right) \pi_t = - \left(\frac{k b}{1+b\psi_x} \frac{1}{(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)} \right) \bar{\pi}_{t-1} + \text{stuff} \cdot s_t$$

$$\left(1 + \frac{k\beta\gamma_\pi}{1+\beta\gamma_x}\right) \pi_t = - \left(\frac{k\beta}{1+\beta\gamma_x(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)} \right) \bar{\pi}_{t-1} +$$

$$\left\{ \frac{k}{1+\beta\gamma_x} \left[-\beta e_2 - \beta b_1(I_3 - \beta h x)^{-1} - (1-\beta)b_2(I_3 - \beta h x)^{-1} + \beta b_3(I_3 - \beta h x)^{-1} - \beta e_1(I_3 - \beta h x)^{-1} \right] \right. \\ \left. + \left[-(1-\alpha)\beta b_1(I_3 - \alpha\beta h x)^{-1} s_1 - \kappa\alpha\beta(I_3 - \alpha\beta h x)^{-1} - e_3(I_3 - \alpha\beta h x)^{-1} \right] \right\} s_t$$

$$\frac{1+\beta\gamma_x + k\beta\gamma_\pi}{1+\beta\gamma_x} \pi_t = \text{same}$$

$$\Rightarrow \pi_t = - \frac{1+\beta\gamma_x}{1+\beta\gamma_x + k\beta\gamma_\pi} \left(\frac{k\beta(1-\alpha\beta) + (1-\alpha)\beta(1-\beta)(1+\beta\gamma_x)}{(1+\beta\gamma_x)(1-\beta)(1-\alpha\beta)} \right) \bar{\pi}_{t-1}$$

$$+ \left\{ \frac{k}{1+\beta\gamma_x + k\beta\gamma_\pi} \left[-\beta e_2 - \beta b_1(I_3 - \beta h x)^{-1} - (1-\beta)b_2(I_3 - \beta h x)^{-1} + \beta b_3(I_3 - \beta h x)^{-1} - \beta e_1(I_3 - \beta h x)^{-1} \right] \right. \\ \left. + \frac{1+\beta\gamma_x}{1+\beta\gamma_x + k\beta\gamma_\pi} \left[-(1-\alpha)\beta b_1(I_3 - \alpha\beta h x)^{-1} s_1 - \kappa\alpha\beta(I_3 - \alpha\beta h x)^{-1} - e_3(I_3 - \alpha\beta h x)^{-1} \right] \right\}$$

$$\Rightarrow \pi_t = - \frac{k\beta(1-\alpha\beta) + \beta(1-\alpha)(1-\beta)(1+\beta\gamma_x)}{(1+\beta\gamma_x + k\beta\gamma_\pi)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left\{ \frac{k}{1+\beta\gamma_x + k\beta\gamma_\pi} \left[-\beta e_2 - \beta b_1(I_3 - \beta h x)^{-1} - (1-\beta)b_2(I_3 - \beta h x)^{-1} + \beta b_3(I_3 - \beta h x)^{-1} - \beta e_1(I_3 - \beta h x)^{-1} \right] \right. \\ \left. + \frac{1+\beta\gamma_x}{1+\beta\gamma_x + k\beta\gamma_\pi} \left[-(1-\alpha)\beta b_1(I_3 - \alpha\beta h x)^{-1} s_1 - \kappa\alpha\beta(I_3 - \alpha\beta h x)^{-1} - e_3(I_3 - \alpha\beta h x)^{-1} \right] \right\} s_t$$

Damn damn! So we have ΞA

$$\pi_t = - \frac{k_2(1-\alpha\beta) + \beta(1-\alpha)(1-\beta)(1+b_2\gamma_x)}{(1+b_2\gamma_x + k_2\gamma_x)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left[\frac{k}{1+b_2\gamma_x + k_2\gamma_x} \left[-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(I_3 - \beta h_x)^{-1} + 2\beta b_3(I_3 - \beta h_x)^{-1} - 2e_1(I_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+b_2\gamma_x}{1+b_2\gamma_x + k_2\gamma_x} \left[-(1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} s_t - k\alpha\beta(I_3 - \alpha\beta h_x)^{-1} - e_3(I_3 - \alpha\beta h_x)^{-1} \right] \right\} s_t$$

ΞB

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + c$$

\hookrightarrow 1 jump (π_t), 3 exog states ($s_t = \begin{bmatrix} r_t^h \\ i_t \\ u_t \end{bmatrix}$) and 2

endog states $\xi_t = \begin{bmatrix} \bar{\pi}_t \\ k_t^{-1} \end{bmatrix}$ (or $\begin{bmatrix} \bar{\pi}_{t-1} \\ k_t^{-1} \end{bmatrix}$) so $X_t = \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$
(States)

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + c$$

call this the state (?) ξ_{t-1}

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \bar{\pi}_t \\ s_t \end{bmatrix} \quad Y_t = g'x_t \cdot X_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} f_{t-1} \quad X_{t+1} = h'x_t X_t + \eta_t \epsilon_t$$

$$k_t^{-1} = k_{t-1}^{-1} + d \cdot f_{t-1} + c$$

$$f_{t-1} = \pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}$$

Several issues y'all:

1) f_t state or jump? depends on π_t (a jump)
 \rightarrow there's gotta be some trick (like π_t is LEMP)
 to make it a pure state

2.) $\bar{\pi}_t$ nonlinear Lom!

✓
 ok - that's troubling. But let's pause it for a sec & let's read what Litterpohl has to say about MLE & log-likelihoods.

Supp. our VAR(p) looks like

$$y_{t-m} = A_1 (y_{t-1} - \mu) + \dots + A_p (y_{t-p} - \mu) + u_t \quad (3.3.1)$$

then if the VAR(p) is Gaussian, that is

p. 75

$$u \equiv \text{vec}(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim N(0, I_T \otimes \Sigma_u), \quad \text{which}$$

equivalently means that the prob. density of u is

$$f_u(u) = \frac{1}{(2\pi)^{kT/2}} |I_T \otimes \Phi_u|^{-1/2} \exp \left[-\frac{1}{2} u' (I_T \otimes \Phi_u^{-1}) u \right]$$

then we can use the fact that $u = y - \mu^* - (X' \otimes I_k) \alpha$

$$\text{where } \alpha := \text{vec}(A), \quad A := (A_1, \dots, A_p) \quad k \times k_p$$

$$k^2_p \times 1$$

$$Y^0 := (y_1 - \mu, \dots, y_T - \mu) \quad k \times T$$

$$Y_k^0 := \begin{bmatrix} y_1 - \mu \\ \vdots \\ y_{t-p+1} - \mu \end{bmatrix} \quad (k_p \times 1)$$

$$X := (Y_0^0, \dots, Y_{T-1}^0),$$

to write

$$f_y(y) = \left| \frac{\partial y}{\partial u} \right| f_u(u)$$

$$= \frac{1}{(2\pi)^{kT/2}} |I_T \otimes \Phi_u|^{-1/2} \exp \left[-\frac{1}{2} (y - \mu^* - (X' \otimes I_k) \alpha)' (I_T \otimes \Phi_u^{-1}) \cdot (y - \mu^* - (X' \otimes I_k) \alpha) \right] \quad (3.44)$$

and

$$\ln L(\mu, \alpha, \Phi_u) = -\frac{kT}{2} \ln(2\pi) - \frac{T}{2} \ln |\Phi_u| - \frac{1}{2} \text{tr} (Y^0 - AX)' \Phi_u^{-1} (Y^0 - AX)$$

is the log-likelihood.

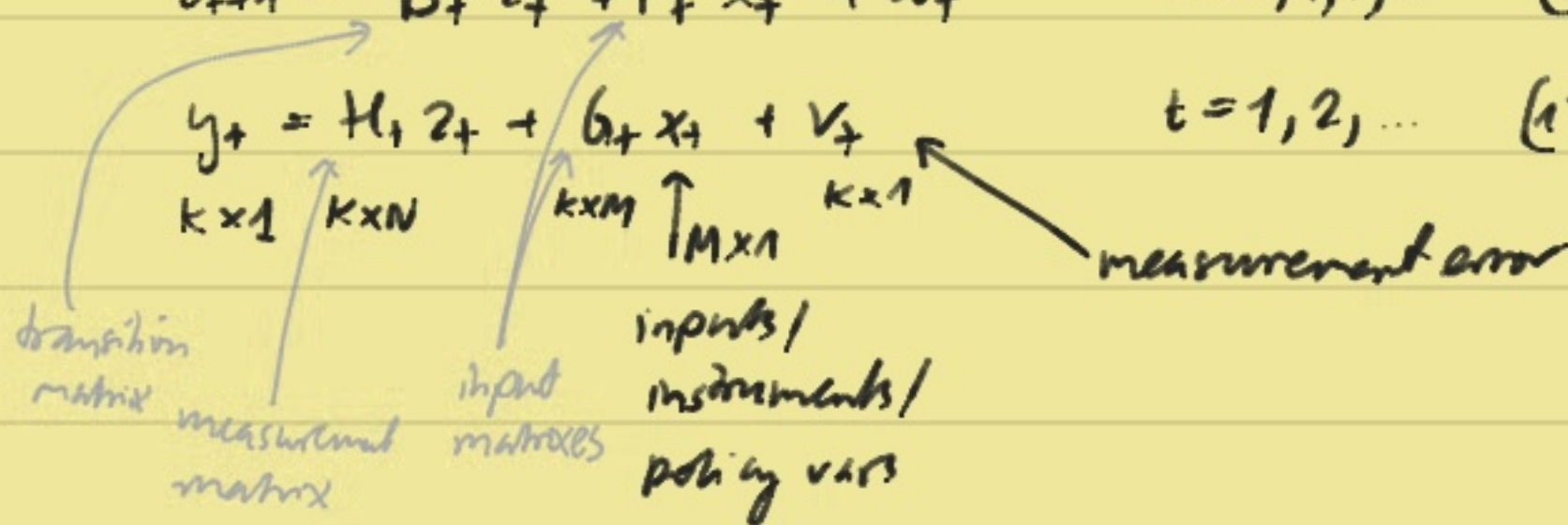
$$(3.4.5)$$

Given that the updating eq. w/ endogenous gain 18 March 2020 introduces non-linearities, I'm afraid that even a "simple" & quick estimation has to involve some form of particle filter. But let's see whether Lütkepohl has anything interesting to say about 1) state-space models 2) non-linearities

Lütkepohl, State-space models (Ch. 13, p. 415 ff.)

$$\begin{matrix} N \times 1 & N \times N & N \times M & N \times 1 & \text{noise} \\ z_{t+1} = B_t z_t + F_t x_t + w_t \end{matrix} \quad t = 0, 1, 2, \dots \quad (13.2.1a)$$

$$y_t = H_t z_t + G_t x_t + v_t \quad t = 1, 2, \dots \quad (13.7.2)$$



and $\begin{bmatrix} w_t \\ v_t \end{bmatrix} \sim WN[0, VC_t]$ $VC_t = \begin{bmatrix} \Sigma_{w_t} & \Sigma_{w_t v_t} \\ \Sigma_{v_t w_t} & \Sigma_{v_t} \end{bmatrix}$

Nonlinear state-space models

$$z_{t+1} = b_t(z_t, x_t, w_t, \delta_1)$$

$$y_t = h_t(z_t, x_t, v_t, \delta_2)$$

vectors of params

Example of nonlin state-space is the "bilinear" model:

$$y_t = \alpha y_{t-1} + u_t + \beta y_{t-1} u_{t-1} \quad \text{p. 427 w/ Refs.}$$

$$\hookrightarrow z_{t+1} = B z_t + w_t + C \text{vec}(z_t z_t') \quad (13.2.33)$$

$$y_t = [I_k \ 0 \dots 0] z_t \quad (13.2.34)$$

\Rightarrow Bundle of refs on bilinear systems, univariate & multivariate on p. 427 bottom.

MLE of state-space models p. 434

Gather the time-invariant params from $B, F, H, G, \Sigma_w, \Sigma_0$ and Σ_0 & μ_0 in δ as

$$\delta = \begin{bmatrix} \text{vec}[v, A_1, \dots, A_p] \\ \text{vech}(\Sigma_u) \end{bmatrix}$$

where vech = "half-vectorization", $\text{vech}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{bmatrix} a \\ b \\ d \end{bmatrix}$.

The log-likelihood for the Gaussian state-space model is:

$$\ln l(\delta | y_1, \dots, y_T) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_y(t|t-1)| \\ - \frac{1}{2} \sum_{t=1}^T (y_t - y_{t|t-1})' \Sigma_y(t|t-1)^{-1} (y_t - y_{t|t-1}) \quad (13.4.1)$$

Denoting the first error $e_t(\delta) := y_t - y_{t|t-1}$
 and $\Phi_t(\delta) := \Phi_y(t|t-1)$, we can rewrite this as

$$\ln l(\delta) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \left[\ln |\Phi_t(\delta)| + e_t'(\delta) \Phi_t(\delta)^{-1} e_t(\delta) \right] \quad (13.4.3)$$

which makes explicit

- 1) the dependence of $\ln l$ on δ ,
- 2) that all quantities in the $\ln l$ are functions of δ and can (must) be computed using the Kalman filter.

locally identified when in a subspace of the param space, δ is uniquely determined.

vs. **globally identified** when δ is uniquely determined in the entire param space.

↳ identification: we need a min of $-\ln l$, so we need some sort of Hessian = pos. def. \Rightarrow the

information matrix, $= E[\text{Hessian}] = E \left[\frac{\partial^2 (-\ln l)}{\partial \delta \partial \delta} \Big|_{\delta_0} \right]$

A quick note on DSEMs (dynamic simultaneous equations models a.k.a. "linear systems") p. 323

essentially, these are linear VARMAX(p, s, q) models

$$A_0 y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + B_1 x_{t-1} + \dots + B_s x_{t-s} + w_t \quad (10.1.1)$$

• VARMAX(p, s, q) is w_t is MA(q)

• VARX(p, s) if $w_t \sim WN$.

VAR(p) models w/ time-varying coefficients p. 891ff.

periodic VARs \rightarrow eg. w/ seasonal dummies

intervention models \rightarrow DBP_1 is replaced by DBP_2 at time τ .

$$y_t = v_t + A_{1t} y_{t-1} + \dots + A_{pt} y_{t-p} + u_t \quad (12.2.1)$$

$\hookrightarrow WN(0, \Sigma_t)$

also time-varying
(not identically distrib!)

Rewrite the VAR(p) as a VAR(1)

$$Y_t = v_t + A_t Y_{t-1} + u_t \quad (12.2.2)$$

$$Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{k_p \times 1}, \quad v_t := \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{k_p \times 1}, \quad A_t := \begin{bmatrix} A_{1,t} & \dots & A_{p-1,t} & A_{p,t} \\ I_k & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_k & 0 \end{bmatrix}_{k_p \times k_p}, \quad u_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{k_p \times 1}$$

And by recursive sub we get

$$Y_t = \left[\prod_{j=0}^{h-1} A_{t-j} \right] Y_{t-h} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j} \right] v_{t-i} + \sum_{i=0}^{h-1} \left[\prod_{j=0}^{i-1} A_{t-j} \right] u_{t-i} \quad (12.2.3)$$

Defining $J := [I_k \ 0]$ such that $y_t = J Y_t$, we can premultiply (12.2.3) by J , define

$$\bar{\Phi}_{it} := J \left[\prod_{j=0}^{i-1} A_{t-j} \right] J'$$

$$y_t = \mu_t + \sum_{i=0}^{\infty} \bar{\Phi}_{it} u_{t-i} \quad (12.2.4)$$

where $\mu_t = E[y_t]$.

\Rightarrow Then the MSE (or FEV of the FE $y_{t+h} - y_t(h)$) is

$$\Sigma_t(h) := \sum_{i=0}^{h-1} \bar{\Phi}_{i,t+h} \bar{\Phi}_{t+h-i} \bar{\Phi}_{i,t+h}' \quad (12.2.10)$$

MLE of TV-VAR p. 394

Write (12.2.1) as $y_t = B_t z_{t-1} + u_t$ (12.2.11)

where $B_t := [v_t, A_{1t}, \dots, A_{pt}]$, $z_{t-1} := (1, y_{t-1}')$

B_t depend on the vector γ of time-invariant params.

Σ_t depend on β of fixed params.

If $u_t \sim N(0, \Sigma_t)$, the log-likelihood is

$$\ln l(\gamma, \beta) = -\frac{KT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_t| - \frac{1}{2} \sum_{t=1}^T u_t' \Sigma_t^{-1} u_t \quad (12.2.12)$$

where initial conditions have been ignored.

You can also derive an info-matrix.

So it seems like you just $\min -\ln l(\gamma, \beta)$!

It also seems like for certain special cases you can even derive the estimators in closed-form!

Ryan meeting

18 March 2020

Not yet convinced that the procedure is sensible
→ the only procedure that can work is

$f_{z,t}$ and more RHS-variables: $k^{-1}, \bar{\pi}$

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

things in $f_{z,t}$ will be 0 or simply
described by the LOMs

It may be too restrictive.

A computationally intensive exercise is:

- conj. path for the int. rate (exog.)
 - solve the model for that int. rate
 - check the target criterion, compute the resid
 - fmincon to min that resid
- ⇒ find simulated optimal plan.

- Simulate the Ramsey model
- Simulate the model w/ Taylor-rule
- see how close you can get

It might then be that most results won't be pencil & paper.

Comments:

- Concerning w/ linearity of (10)
 - "negative surprises cause me to be unanchored" shouldn't be "big mistakes" → so take the square
 - ↳ like smooth one better than jumpy
- Estimation: filter the data ^{→ App paper w/ Robert}
 - match the parameters to ^{moments} covs of data
 - ⇒ would give you results.

They est an NK model: HP-filter both data and model, compute moments and try to match those.