Materials 28 - Putting approximating functions and value function iteration together

Laura Gáti

May 5, 2020

The setup of the problem makes the separate issues I'm facing clear:

$$V(x) = \max_u p(u, x) + \beta \, \mathbb{E} \, V(x') \tag{1}$$

1. Interpolate instead of discretize ← this is where I'm at

2. Interpolation may have to be shape-preserving

3. Compute expectation on RHS → quadrature

4. x is a vector → multivariate approximation

ALGORITHM: PARAMETRIC VALUE FUNCTION ITERATION

(Judd, *Numerical Methods*, Algorithm 12.5)

- Objective: Solve Bellman equation → find coefficients b^* such that the approximation $\hat{V}(x, b)$ is close enough.

- Initialization: Choose a functional form for $\hat{V}(x, b^0)$ and choose a grid of n interpolation nodes $X = \{x_1, \ldots, n_n\}$. Choose initial vector of coefficients b^0 and stopping criterion $\varepsilon > 0$.

Step 1 Maximization step

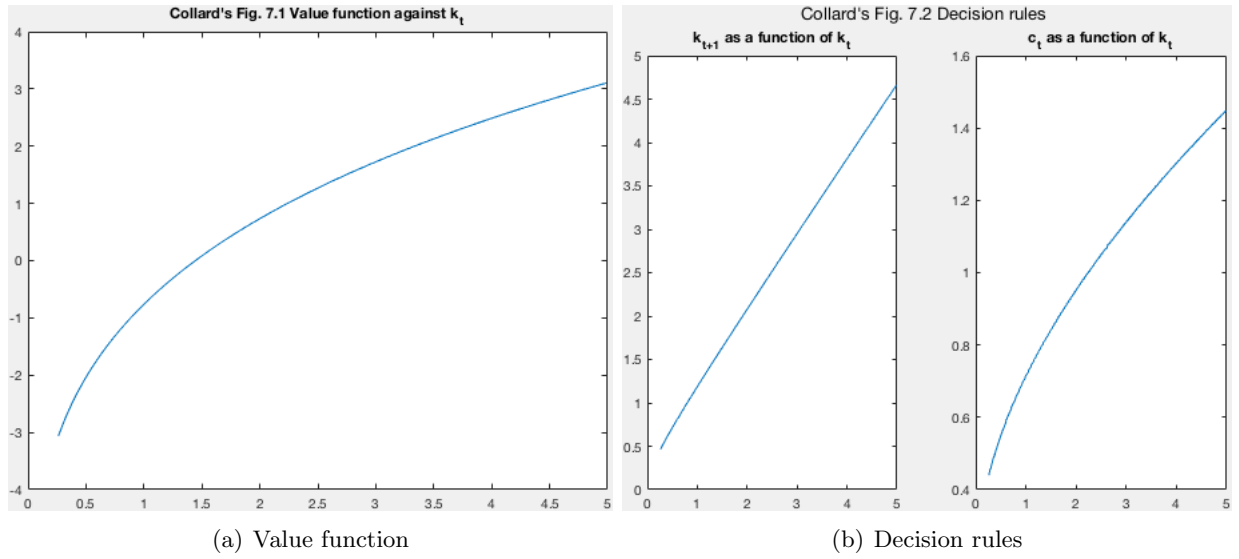Compute $v_j = T\hat{V}(\cdot, b^i)$ for $x_j \in X$.

Step 2 Fitting step

Using your choice of approximation method, compute the updated vector of coefficients b^{i+1} such that $\hat{V}(x, b^{i+1})$ approximates the (v_i, x_i) data.
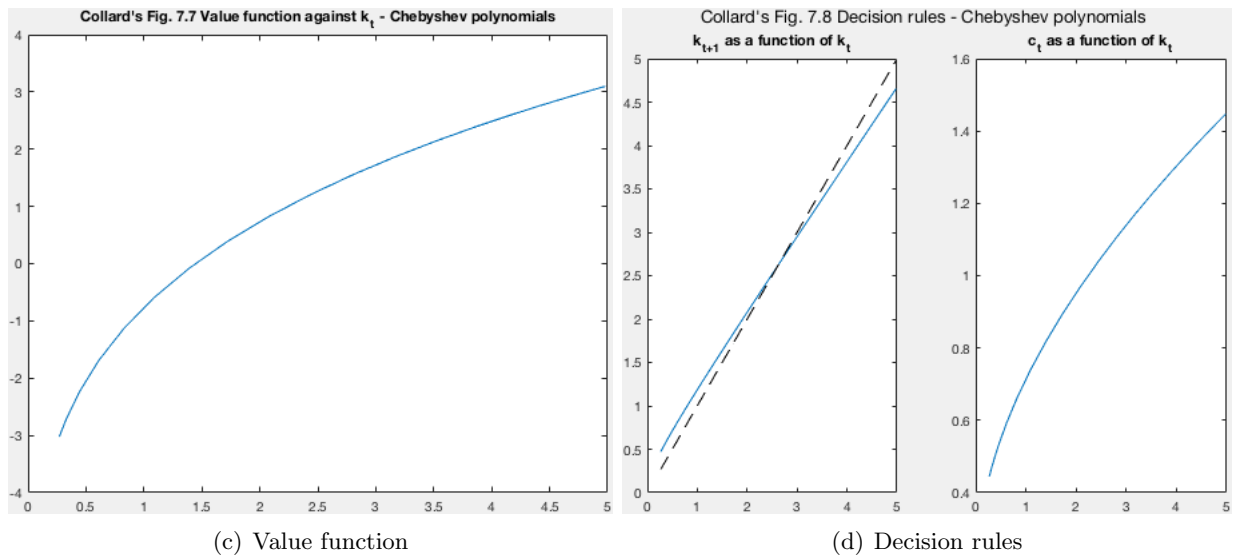
Step 3 If $||\hat{V}(x, b^i) - \hat{V}(x, b^{i+1}) < \varepsilon||$, stop; else go to Step 1.
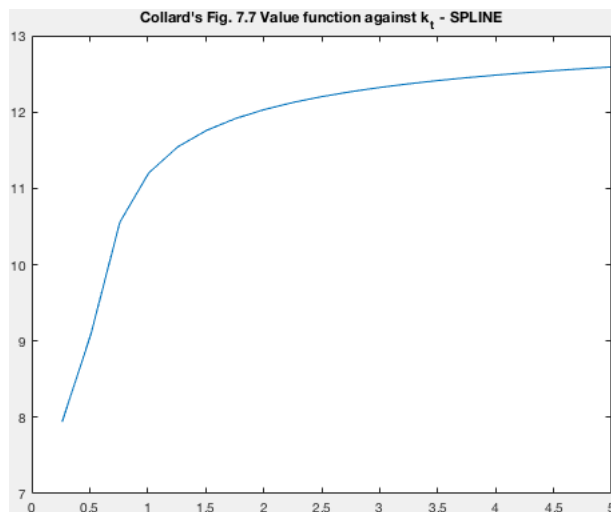
1 Optimal growth model

1.1 Optimal growth - value function iteration with discretization

(a) Value function

(b) Decision rules

1.2 Optimal growth - value function iteration with Chebyshev polynomial interpolation

(c) Value function

(d) Decision rules

1.3   Optimal growth - value function iteration with cubic spline interpolation

(e) Value function
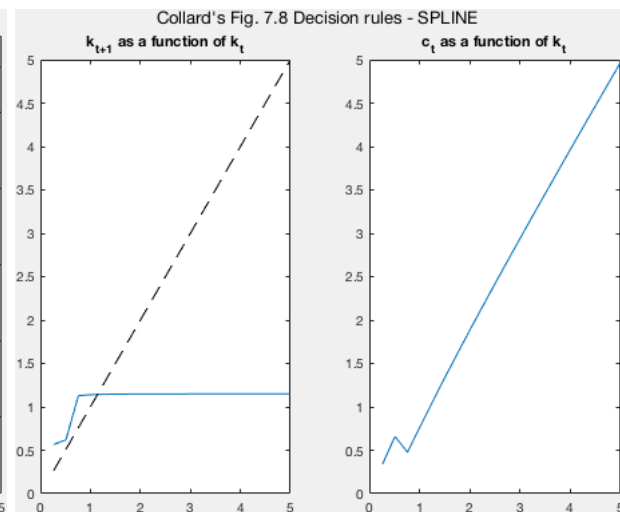
(f) Decision rules

A   Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{A.1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{A.2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \qquad \text{(if imposed)} \tag{A.3}$$

$$\text{PLM:} \qquad \hat{\mathbb{E}}_t z_{t+h} = a_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \tag{A.4}$$

$$\text{Updating:} \qquad a_t = a_{t-1} + k_t^{-1}\big(z_t - (a_{t-1} + b s_{t-1})\big) \tag{A.5}$$

$$\text{Anchoring function:} \qquad k_t = k_{t-1} + \mathbf{g}(fe_{t-1}^2) \tag{A.6}$$

$$\text{Forecast error:} \qquad fe_{t-1} = z_t - (a_{t-1} + b s_{t-1}) \tag{A.7}$$

$$\text{LH expectations:} \qquad f_a(t) = \frac{1}{1-\alpha\beta} a_{t-1} + b(\mathbb{I}_{nx} - \alpha\beta h)^{-1} s_t \qquad f_b(t) = \frac{1}{1-\beta} a_{t-1} + b(\mathbb{I}_{nx} - \beta h)^{-1} s_t \tag{A.8}$$

This notation captures vector learning (z learned) for intercept only. For scalar learning, $a_t = \begin{pmatrix} \bar{\pi}_t & 0 & 0 \end{pmatrix}'$ and b_1 designates the first row of b. The observables (π, x) are determined as:

$$x_t = -\sigma i_t + \begin{bmatrix} \sigma & 1-\beta & -\sigma\beta \end{bmatrix} f_b + \sigma \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (\mathbb{I}_{nx} - \beta h_x)^{-1} s_t \tag{A.9}$$

$$\pi_t = \kappa x_t + \begin{bmatrix} (1-\alpha)\beta & \kappa\alpha\beta & 0 \end{bmatrix} f_a + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (\mathbb{I}_{nx} - \alpha\beta h_x)^{-1} s_t \tag{A.10}$$

B   Target criterion

The target criterion in the simplified model (scalar learning of inflation intercept only, $k_t^{-1} = \mathbf{g}(fe_{t-1})$):

$$\pi_t = -\frac{\lambda_x}{\kappa}\left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + ((\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}))\mathbf{g}_\pi(t) \right) \right.$$

$$\left(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} (1 - k_{t+1+j}^{-1} - (\pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j})\mathbf{g}_{\bar{\pi}}(t+j)) \right) \Bigg\} \tag{B.1}$$

where I'm using the notation that $\prod_{j=0}^0 \equiv 1$. For interpretation purposes, let me rewrite this as follows:

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t + \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t) \right) \mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i}$$

$$- \frac{\lambda_x}{\kappa}\frac{(1-\alpha)\beta}{1-\alpha\beta}\left(k_t^{-1} + fe_{t|t-1}^{eve}\mathbf{g}_\pi(t) \right)\left(\mathbb{E}_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} (k_{t+1+j}^{-1} + fe_{t+1+j|t+j}^{eve})\mathbf{g}_{\bar{\pi}}(t+j) \right) \tag{B.2}$$

Interpretation: tradeoffs from discretion in RE + effect of current level and change of the gain on future tradeoffs + effect of future expected levels and changes of the gain on future tradeoffs