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The Perils of Taylor Rules[1]

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Quick SUM:

- Point: common wisdom is that active mon. pol. leads to stability (vs. passive to indeterminacy)
- but globally, this is not true.
- Globally, active policy is a source → you either converge to passive pol → or to a limit cycle around active pol
- and this is tricky b/c both are observationally equivalent w/ a stable eq (active).

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Since John Taylor's (1993, *Carnegie–Rochester Conf. Ser. Publ Policy* **39**, 195–214), seminal paper, a large literature has argued that active interest rate feedback rules, that is, rules that respond to increases in inflation with a more than one-for-one increase in the nominal interest rate, are stabilizing. In this paper, we argue that once the zero bound on nominal interest rates is taken into account, active interest rate feedback rules can easily lead to unexpected consequences. Specifically, we show that even if the steady state at which monetary policy is active is locally the unique equilibrium, typically there exist an infinite number of equilibrium trajectories originating arbitrarily close to that steady state that converge to a liquidity trap, that is, a steady state in which the nominal interest rate is near zero and inflation is possibly negative. *Journal of Economic Literature* Classification Numbers: E52, E31, E63.

1. INTRODUCTION

Since John Taylor's [21] seminal paper describing Federal Reserve policy, there has been a resurgence of interest in monetary policy rules that

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target the nominal rate. Much of the literature has explored the efficiency and dynamic effects of such policies, with particular attention to their stabilization properties. A central policy recommendation that has emerged from this body of research is that "active monetary policy," that is, a policy that strongly responds to the rate of inflation in setting the nominal interest rate, is stabilizing.[3] In an earlier paper (Benhabib *et al.* [2]), we argued that this result depends very much on the specification of the model, and that indeed often active monetary feedback policies lead to multiple equilibria under standard specifications, assumptions, and calibrations, including models with sticky prices, Taylor rules that allow for leads or lags, and Ricardian and non-Ricardian monetary–fiscal regimes. In this paper, we take an even stronger position and argue that active monetary policy generally leads to indeterminacy and multiple equilibria and that pursuing such a policy can easily lead to unexpected consequences even in the simplest and most innocuous monetary models, using the simplest and most standard assumptions.

Our method of analysis departs from the conventional local approach to study multiple equilibria that proceeds by linearizing around a steady state. The reason for this departure stems from the observation that the nominal rate must be constrained to be non-negative, since negative nominal rates are impossible. It immediately follows from this observation, as we illustrate below, that if there is a steady state with an active monetary policy, there must necessarily exist another steady state with a passive policy. As a result, local analysis is inadequate because paths of the economy diverging from one steady state can converge to the other steady state or to another attracting set, thus qualifying as equilibrium trajectories. We show these results in the context of flexible and sticky-price models both theoretically and through simulations of calibrated economies.

To intuitively illustrate the source of multiplicity, consider a simplified Taylor rule whereby the monetary authority sets the nominal rate as a non-decreasing function of inflation; $R = R(\pi)$, where R denotes the nominal interest rate and π denotes the rate of inflation. Combining this rule with the Fisher equation, $R = r + \pi$, where r is the real interest rate, yields

$$R(\pi) = r + \pi.$$

This steady-state relation is common to a wide range of monetary models with representative agents and with an infinite horizon, and it holds

[3] For papers arriving at this conclusion in the context of non-optimizing models, see Levin *et al.* [12] and Taylor [22, 23]; for optimizing models with flexible prices, see Leeper [11]; and for optimizing models with nominal frictions see Rotemberg and Woodford [16, 17], Christiano and Gust [7], and Clarida *et al.* [5].

irrespective of whether prices are flexible or sticky or of whether money enters the model through the utility function, the production function, or a cash-in-advance constraint. Suppose that there exists a steady state with active monetary policy, that is, a value of π that solves the above equation and satisfies $R'(\pi) > 1$. Suppose in addition that the feedback rule is continuous and respects the zero lower bound on nominal rates ($R(\pi) \geqslant 0$). Then there must exist another steady state in which monetary policy is passive, that is, a steady state in which $R'(\pi) < 1$ (Fig. 1). Note that for the existence of two solutions to the steady-state Fisher equation it is not crucial that the Taylor rule be continuous. It is sufficient that the Taylor rule is non-negative and non-decreasing and that one solution occurs at a value of π for which monetary policy is active. The bottom right panel of Fig. 1 displays a case in which there is a unique solution to the Fisher equation even though at that solution the feedback rule is active. The absence of a second solution results not because the Taylor rule is discontinuous but because it is non-monotonic. We will not explore the macroeconomic consequences of Taylor rules of this type because we believe that they are irrelevant, for it is implausible that the central bank will implement a discrete increase in the nominal interest rate in the context of declining inflation.

"If the Taylor-rule is
• non-decreasing ($\pi\uparrow \Rightarrow R\uparrow$)
• ZLB holds $R \geq 0$
⇒ then if an eqb ex. w/ $R'(\pi_1^*) > 1$
then an eqb w/ $R'(\pi_2^*) < 1$ also exists."

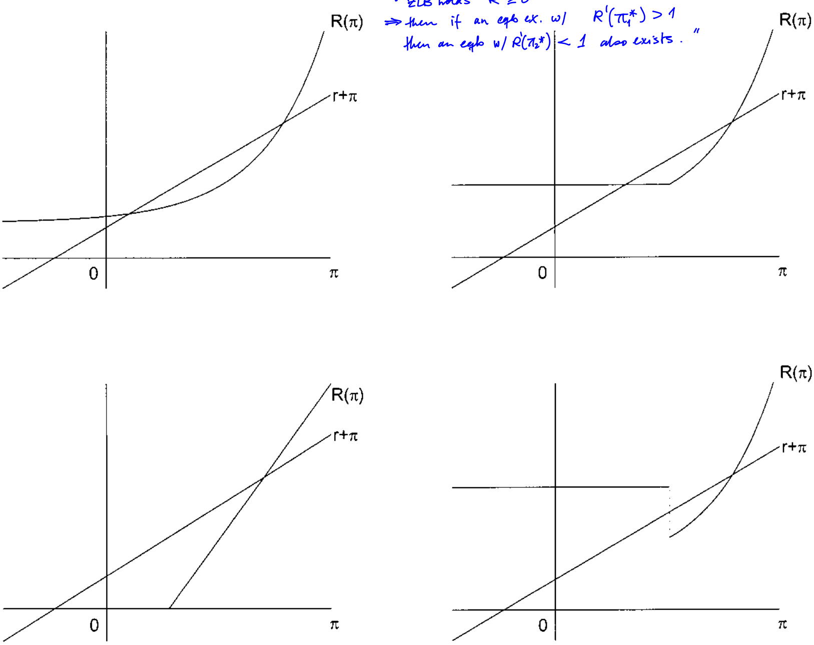

FIG. 1. Taylor Rules, zero bound on nomial rates, and multiple steady states.

The existence of multiple solutions to the steady-state Fisher equation immediately establishes the possibility of the existence of at least two steady-state equilibria. However, this need not be the case, for in general the equilibrium conditions will involve additional equations. In this paper, we show that the presence of a steady-state equilibrium at which monetary policy is active typically gives rise to at least one other steady-state equilibrium at which monetary policy is passive. But it would be naive to conclude that active interest rate rules are destabilizing solely because they give rise to multiple steady-state equilibria. First, although empirical studies show that in the past decades monetary policy in major industrialized countries can be described quite accurately by active interest rate feedback rules (e.g., Clarida *et al.* [6]), observed inflation dynamics are in general quite smooth, giving little credence to a model in which movements in inflation at business-cycle frequency are due to jumps from one steady state to another. Second, it is equally unconvincing that policy makers change the stance of monetary policy from active to passive at high frequencies.

The main result of this paper is that Taylor rules are destabilizing because the multiplicity of steady-state equilibria that they induce opens the door to a much larger class of equilibria. Specifically, we show that in general there exist an infinite number of equilibrium trajectories originating in the vicinity of the active steady state that converge either to the steady state at which monetary policy is passive (a saddle connection) or to a stable limit cycle around the active steady-state. Interestingly, along both the saddle connection and the limit cycle, the inflation rate fluctuates for long periods of time around the steady-state at which monetary policy is active. Thus, an econometrician using data generated from a saddle connection equilibrium to estimate the slope of the interest rate feedback rule may very well conclude that the economy is displaying stationary fluctuations around the active steady-state, even though the economy is in fact spiraling down into a liquidity trap.

Simulations of calibrated versions of a sticky-price model indicate that saddle connections from the active steady-state to the passive steady-state exist for empirically plausible parameterizations and are indeed the most typical pattern as they are robust to wide parameter perturbations. This type of equilibrium is of particular interest because it sheds light on the precise way in which economies may fall into liquidity traps. The results suggest that central banks that maintain an active monetary policy stance near a given inflation target are more likely to lead the economy into a deflationary spiral—like the one currently observed in Japan and, as some may argue, in the United States—than central banks that maintain a globally passive monetary stance such as an interest- or exchange-rate peg.

2. TAYLOR RULES, THE ZERO BOUND ON NOMINAL RATES, AND LIQUIDITY TRAPS: A SIMPLE EXAMPLE

In the introduction we point out that Taylor rules in combination with the zero bound on nominal rates may give rise to the existence of two steady states, in one of which the inflation rate and the nominal interest rate are below their intended targets and monetary policy is passive. In this section, we present a simple flexible-price model to show that these two steady states are connected by an equilibrium trajectory. Specifically, we demonstrate that the economy can slide from the intended steady state to the unintended one. We interpret this result as meaning that in the presence of a Taylor rule, a liquidity trap may emerge as an equilibrium outcome.

Consider an endowment economy populated by a large number of identical infinitely lived households with preferences defined over consumption and over real balances and described by the utility function

$$\int_0^\infty e^{-rt} u(c, m)\, dt,$$

where c denotes consumption and m denotes real balances. The household's instant budget constraint is given by

$$c + \tau + \dot{a} = (R - \pi)\, a - Rm + y,$$

where τ denotes lump-sum taxes; a denotes real financial wealth, consisting of interest-bearing bonds and money balances; R is the nominal rate of return on bonds, π is the inflation rate; and y is a constant endowment. The right-hand side of the budget constraint represents the sources of income: real interest on the household's assets net of the opportunity cost of holding money and the endowment. The left-hand side shows the uses of income: consumption, tax payments, and increases in the stock of real wealth. Households are also subject to a borrowing limit of the form $\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\, ds} a(t) \geqslant 0$, which prevents them from engaging in Ponzi games. The household chooses paths for consumption, real balances, and wealth that satisfy the instant budget constraint, the no-Ponzi-game borrowing limit with equality, and the optimality conditions

$$u_c(c, m) = \lambda \tag{1}$$

$$u_m(c, m) = \lambda R \tag{2}$$

$$\dot{\lambda} = \lambda[r + \pi - R(\pi)], \tag{3}$$

where λ is a Lagrange multiplier associated with the instant budget constraint. Equilibrium in the goods market requires that consumption be equal to the endowment,

$$c = y. \tag{4}$$

Assuming that consumption and real balances are Edgeworth complements ($u_{cm} > 0$) and that the instant utility function is concave in real balances ($u_{mm} < 0$), Eqs. (1), (2), and (4) define a decreasing function linking λ and R:

shadow value of wealth is ↓ in R

$$\lambda = L(R); \qquad L' < 0. \tag{5}$$

Suppose that the monetary authority follows an interest rate feedback rule of the form

$$R = R(\pi),$$

where the function $R(\cdot)$ is positive, increasing, strictly convex, and differentiable. Suppose further that there exists an inflation rate π^* at which the steady-state Fisher equation is satisfied and at which the feedback rule is active; that is, $R(\pi^*) = r + \pi^*$ and $R'(\pi^*) > 1$. Then, as the top left panel of Fig. 1 illustrates, there exists an inflation rate $\pi^L < \pi^*$ such that the steady-state Fisher equation is satisfied and the interest rate rule is passive; that is, $R(\pi^L) = r + \pi^L$, $R'(\pi^L) < 1$. Combining this feedback rule with (3) and (5), we obtain the following first-order differential equation describing the equilibrium dynamics of inflation[4]

$$\dot{\pi} = \frac{-L(R(\pi))}{L'(R(\pi))\, R'(\pi))} [R(\pi) - \pi - r]. \tag{6}$$

Because $-L/(L'R')$ is always positive, the sign of $\dot{\pi}$ is the same as the sign of $R(\pi) - \pi - r$. Figure 2 illustrates the inflation dynamics implied by Eq. (6). The high-inflation, active steady state π^* is unstable, in the sense that trajectories initiating near π^* diverge from π^*. Thus, if one limits the analysis to equilibria in which π remains forever in a small neighborhood around π^*, then the only perfect-foresight equilibrium is the active steady

[4] In equilibrium the no-Ponzi-game condition must hold with equality. This will be the case if the fiscal authority follows a "Ricardian" policy whereby the present discounted value of future expected total government liabilities converges to zero regardless of the particular paths taken by inflation and nominal interest rates. We present an example of such fiscal policy in the next section.

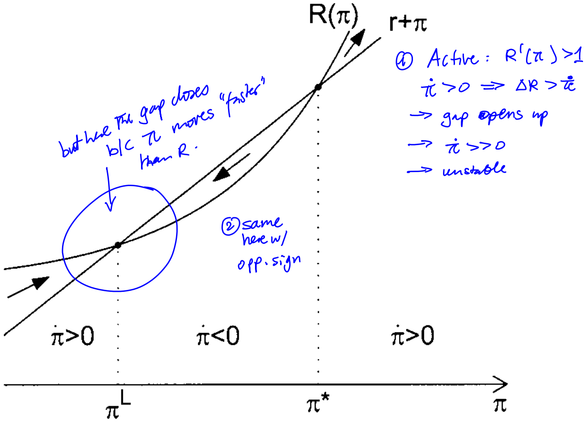

FIG. 2. The liquidity trap in a flexible price model.

state itself. However, if one allows equilibria in which π can take values in a larger neighborhood around π^* that includes the passive steady state π^L, then a large number of equilibrium trajectories become possible. In particular, any inflation path starting between π^L and π^* and satisfying (6) represents a perfect-foresight equilibrium. All such trajectories converge to the low-inflation, low-interest rate, passive steady state π^L. Note that equilibrium trajectories of this type can originate arbitrarily close to the high-inflation, high-interest rate, active steady state π^*. In this environment, all that is needed for the economy to fall into the liquidity trap is that people expect the economy to slide into a deflationary phase. Taylor rules in combination with the zero bound on nominal rates also give rise to a saddle connection in the context of discrete-time models. Schmitt-Grohé and Uribe [19] show this result in the context of a cash-in-advance, flexible-price model.

The simple flexible-price economy analyzed thus far conveys the main message of the paper in a direct and transparent way. However, most of the literature devoted to evaluating the stabilizing properties of Taylor rules includes as a central theoretical element the presence of nominal rigidities. Consequently, in the remainder of the paper we consider a model with price stickiness. In the model discussed in this section, the existence of equilibria originating close to the active steady state and converging to the passive steady state depends on the assumption that consumption and real balances are Edgeworth complements. However, as we will show below in the presence of sluggish price adjustment, a saddle path connection between the two steady states also emerges for preferences displaying Edgeworth substitutability as well as additive separability in consumption and real balances.

3. A STICKY-PRICE MODEL

The economy is assumed to be populated by a continuum of household–firm units indexed by j each of which produces a differentiated good Y^j. Firms have market power and set prices to maximize profits. The demand faced by firm j is given by $Y^d d(P^j/P)$, where Y^d denotes the level of aggregate demand, P^j the price firm j charges for the good it produces, and P the aggregate price level. Such a demand function can be derived by assuming that households have preferences over a composite good that is produced from differentiated intermediate goods via a Dixit–Stiglitz production function. The function $d(\cdot)$ is assumed to be twice continuously differentiable, to be decreasing, and to satisfy $d(1)=1$ and $d'(1)<-1$. The restriction imposed on $d'(1)$ is necessary for the firm's problem to be well defined in a symmetric equilibrium. The production of good j uses labor, h^j, supplied by household j, as the only input:

$$Y^j = y(h^j),$$

where $y(\cdot)$ is twice continuously differentiable, positive, strictly increasing, and strictly concave and satisfies the Inada conditions.

We introduce nominal price rigidity, following Rotemberg [15], by assuming that households face convex costs of adjusting prices. Specifically, the household's lifetime utility function is assumed to be of the form

$$U^j = \int_0^\infty e^{-rt}\left[u(c^j, m^j) - z(h^j) - \frac{\gamma}{2}\left(\frac{\dot{P}^j}{P^j} - \pi^*\right)^2\right] dt, \tag{7}$$

where c^j denotes consumption of the composite good by household j, $m^j \equiv M^j/P$ denotes real money balances held by household j, and M^j denotes nominal money balances. The utility function $u(\cdot,\cdot)$ is assumed to be twice continuously differentiable and to satisfy $u_c, u_m > 0$, $u_{cc}, u_{mm} < 0$, $u_{cc}u_{mm} - u_{cm}^2 > 0$, and $\lim_{c\to 0} u_c(c,m) = \lim_{m\to 0} u_m(c,m) = \infty$. To ensure normality of consumption and of real balances, we further assume that $u_{cc} - u_{cm}u_c/u_m < 0$ and $u_{mm} - u_{cm}u_m/u_c < 0$. The function $z(\cdot)$ measures the disutility of labor and is assumed to be twice continuously differentiable, increasing, and convex. The parameter γ measures the degree to which household–firm units dislike to deviate in their price-setting behavior from the constant rate of inflation $\pi^* > -r$.

Let a^j denote the real value of household j's financial wealth, which consists of the sum of real money holdings and government bonds. Then a^j evolves according to the law of motion

$$\dot{a}^j = (R-\pi)\, a^j - Rm^j + \frac{P^j}{P}\, y(h^j) - \tau - c^j, \tag{8}$$

where R denotes the nominal interest rate on government bonds, π denotes the rate of change in the aggregate price level, and τ denotes real lump-sum taxes. The instant budget constraint (8) says that the change in household j's real wealth, $\dot{a}^j$, is equal to real interest earnings on wealth, $(R-\pi)\,a^j$, net of the opportunity cost of holding money, Rm^j, plus disposable income, $(P^j/P)\,y(h^j)-\tau$, minus consumption expenditure, c^j. Households are also subject to the following borrowing constraint that prevents them from engaging in Ponzi-type schemes:

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\,ds} a^j(t) \geqslant 0. \tag{9}$$

Given the price firm j charges for the good it produces, its sales are demand determined and equal to

$$y(h^j) = Y^d d\left(\frac{P^j}{P}\right). \tag{10}$$

Household j chooses nonnegative measurable functions of time for the control variables c^j, m^j, and h^j and absolutely continuous functions of time for the state variables P^j and a^j to maximize (7) subject to (8)–(10) taking as given $a^j(0)$, $P^j(0)$, and the time paths of τ, R, Y^d, and P. If the household's problem has an interior solution, then there exist an absolutely continuous function of time λ^j and a measurable function of time μ^j such that the following conditions are satisfied,[5]

IS

$$u_c(c^j, m^j) = \lambda^j \tag{11}$$

LM

$$u_m(c^j, m^j) = \lambda^j R \tag{12}$$

labor

$$z'(h^j) = \lambda^j \frac{P^j}{P}\, y'(h^j) - \mu^j y'(h^j) \tag{13}$$

wealth

$$\dot{\lambda}^j = \lambda^j\,(r+\pi-R) \tag{14}$$

[5] Note that one can express the household's problem in the standard form of an infinite horizon optimal control problem. Use Eq. (10) to eliminate h^j from (7) and (8) and introduce the variable $q^j \equiv \dot{P}^j$ to eliminate $\dot{P}^j$ from (7). The state vector function of the resulting problem is (a^j, P^j), and the control vector function is (c^j, m^j, q^j), with the additional evolution equation $\dot{P}^j = q^j$. One can then apply a standard version of Pontryagin's Maximum Principle for infinite horizon optimal control problems to show that if the household's problem has a solution, then there exists an absolutely continuous function of time λ^j satisfying (11), (12), (14), and (15), with h^j and μ^j eliminated using (10) and (13). (See, for instance, Seierstad and Sydsæter [20, Chapt. 3, Theorem 12].) It is clear from (13) that μ^j must be measurable. We confirmed numerically that the Hamiltonian satisfies the conditions of the Arrow sufficiency theorem for infinite horizons (Seierstad and Sydsæter [20, Chapt. 3, Theorem 14]) for a^j, P^j in a neighborhood of the equilibrium functions a, P.

$$\lambda^j \frac{P^j}{P}\, y(h^j) + \mu^j \frac{P^j}{P}\, Y^d d'\left(\frac{P^j}{P}\right) = \gamma r(\pi^j - \pi^*) - \gamma \dot{\pi}^j \tag{15}$$

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\, ds} a^j(t) = 0 \tag{16}$$

where $\pi^j \equiv \dot{P}^j / P^j$.

3.1. *Monetary and Fiscal Policy.*

The monetary authority is assumed to set the nominal interest rate as an increasing function of the inflation rate. Specifically, it conducts open market operations to ensure that

$$R = R(\pi) \equiv R^* e^{(A/R^*)(\pi - \pi^*)} \tag{17}$$

where R^*, A, and π^* are positive constants.[6] This specification of the feedback rule implies that the nominal interest rate is strictly positive and strictly increasing in the inflation rate. We will refer to monetary policy as active (passive) if the monetary authority raises the nominal interest rate by more (less) than one-for-one in response to an increase in the inflation rate, that is, if $R'(\pi) > (<) 1$.

The instant budget constraint of the government is given by

$$\dot{a} = (R - \pi)\, a - Rm - \tau, \tag{18}$$

where a denotes the real value of aggregate per capita government liabilities, which consist of real balances and bonds. This budget constraint says that the change in total government liabilities, $\dot{a}$, is equal to interest paid on outstanding real liabilities, $(R - \pi)\, a$, minus interest savings from the issuance of money, Rm, minus tax revenues, τ. The monetary–fiscal regime is assumed to be Ricardian in the sense of Benhabib *et al.* [2]. That is, the monetary–fiscal regime ensures that total government liabilities converge to zero in present discounted value for all (equilibrium or off-equilibrium) paths of the price level or other endogenous variables[7]:

like a gov. No-Ponzi

$$\lim_{t\to\infty} e^{-\int_0^t [R(s)-\pi(s)]\, ds} a(t) = 0. \tag{19}$$

[6] Note that we assume that the constant π^* appearing in the interest rate feedback rule is the same constant that plays a role in the household's cost of adjusting prices. We make this assumption for analytical convenience.

[7] As discussed in Benhabib *et al.* [2], an example of a Ricardian monetary–fiscal regime is an interest rate feedback rule like (17) in combination with the fiscal rule $\tau + Rm = \alpha a$; $\alpha > 0$. In the case in which $\alpha = R$, this fiscal rule corresponds to a balanced-budget requirement.

3.2. *Equilibrium*

In a symmetric equilibrium all household–firm units choose identical functions for consumption, asset holdings, and prices. Thus, we can drop the superscript j. In addition, the goods market must clear; that is,

IS
$$c = y(h). \tag{20}$$

Combining Eqs. (11) and (12) yields a liquidity preference function of the form

LM
$$m = m(c, R). \tag{21}$$

Given our maintained assumption about the normality of consumption and real balances, the demand for money is increasing in consumption and decreasing in the nominal interest rate. Using (17), (20), and (21) to eliminate R, c, and m from (11) yields the expression

$$h = h(\lambda, \pi), \tag{22}$$

where $h_\lambda < 0$, $h_\pi u_{cm} < 0$ if $u_{cm} \neq 0$, and $h_\pi = 0$ if $u_{cm} = 0$.[8]

Let $\eta \equiv d'(1) < -1$ denote the equilibrium price elasticity of the demand function faced by the individual firm. Using (13), (17), and (22) to eliminate μ, R, and h from Eqs. (14) and (15) yields

$$\dot{\lambda} = \lambda\,[r + \pi - R(\pi)] \tag{23}$$

$$\dot{\pi} = r(\pi - \pi^*) - \frac{y(h(\lambda, \pi))\,\lambda}{\gamma}\left[1 + \eta - \frac{\eta z'(h(\lambda, \pi))}{\lambda\, y'(h(\lambda, \pi))}\right]. \tag{24}$$

A perfect-foresight equilibrium is a pair of functions $\{\lambda, \pi\}$ satisfying (23) and (24). Given the equilibrium functions $\{\lambda, \pi\}$, the corresponding equilibrium functions $\{h, c, R, m\}$ are uniquely determined by (22), (20), (17), and (21), respectively. The assumed Ricardian nature of the monetary–fiscal regime requires that the fiscal authority sets taxes in such a way that, given paths for R, π, and m and an initial condition $a(0)$, the path for a implied by Eq. (18) satisfies the transversality condition (19).

[8] To see this, note that $h_\lambda = [u_{mm} - (u_m/u_c)\,u_{cm}]/[y'(u_{cc}u_{mm} - u_{cm}^2)]$. The assumed concavity of the instant utility function and normality of consumption imply, respectively, that the denominator of this expression is positive and that the numerator is negative. Also, $h_\pi = -h_\lambda u_{cm} m_R R'(\pi)$, which is of the opposite sign of u_{cm}.

4. STEADY-STATE EQUILIBRIA

A steady-state equilibrium is defined as a pair of constant functions $\{\lambda, \pi\}$ satisfying Eqs. (23) and (24); that is,

TR:

$$0 = r + \pi - R^* e^{(A/R^*)(\pi - \pi^*)} \tag{25}$$

$$0 = r(\pi - \pi^*) - \frac{\lambda y(h(\lambda, \pi))}{\gamma}\left(1 + \eta - \eta \frac{z'(h(\lambda, \pi))}{\lambda y'(h(\lambda, \pi))}\right). \tag{26}$$

Recalling that $R^* = r + \pi^*$, it is clear from (25) that in general there exist two steady-state levels of inflation, π^* and $\bar{\pi}$, with $\bar{\pi} < (>)\, \pi^*$ if $A > (<) 1$. If $A = 1$, then π^* is the unique steady-state level of inflation. Note that if $A > 1$, then monetary policy is active at π^* and passive at $\bar{\pi}$. Conversely, if $A < 1$, monetary policy is passive at π^* and active at $\bar{\pi}$. (always passive at lower π)

The steady-state level of λ associated with π^*, λ^*, is given by the solution to

$$\frac{1+\eta}{\eta}\lambda = \frac{z'(h(\lambda, \pi^*))}{y'(h(\lambda, \pi^*))}.$$

$\pi = \pi^*$, so (26) reduces to this.

Because the right-hand side of this expression is positive and decreasing in λ, λ^* exists and is unique. The steady-state value of λ associated with $\bar{\pi}$ is the solution to

$\pi = \bar{\pi}$, not π^*.

$$\frac{1+\eta}{\eta}\lambda = \frac{z'(h(\lambda, \bar{\pi}))}{y'(h(\lambda, \bar{\pi}))} - \frac{r\gamma}{\eta}\frac{(\pi^* - \bar{\pi})}{y(h(\lambda, \bar{\pi}))}.$$

If $A < 1$, then $\pi^* - \bar{\pi} < 0$ and hence the right-hand side of this expression is decreasing in λ. Therefore, if a steady-state value of λ exists, it is unique. On the other hand, if $A > 1$, then $\pi^* - \bar{\pi} > 0$ and the right-hand side of the above expression may not be monotone in λ. Thus, multiple steady-state values of λ may exist.

5. LOCAL EQUILIBRIA

We now consider perfect-foresight equilibria in which λ and π remain bounded in a small neighborhood around the steady state (λ^*, π^*) and converge asymptotically to it.

Linearizing Eqs. (23) and (24) around (λ^*, π^*), we obtain the system

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\pi} \end{pmatrix} = J \begin{pmatrix} \lambda - \lambda^* \\ \pi - \pi^* \end{pmatrix}, \tag{27}$$

where

$$J = \begin{bmatrix} 0 & u_c(1-A) \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{21} = \frac{y\eta}{y'\gamma}\left[\left(z'' - \frac{z'y''}{y'}\right)h_\lambda - \frac{z'}{\lambda}\right] > 0$$

$$J_{22} = r + \frac{y\eta}{y'\gamma}\left(z'' - \frac{z'y''}{y'}\right)h_\pi.$$

The sign of the coefficient J_{22} depends on the sign of h_π, which in turn depends on whether consumption and real balances are Edgeworth substitutes or complements. Specifically, J_{22} is positive if $u_{cm} \geqslant 0$ and may be negative if $u_{cm} < 0$.[9]

If monetary policy is active at π^* $(A > 1)$, then the determinant of J is positive, so that the real part of the roots of J have the same sign. Since both λ and π are jump variables, the equilibrium is locally determinate if and only if the trace of J is positive. It follows that if $u_{cm} \geqslant 0$, the equilibrium is locally determinate. If $u_{cm} < 0$, the equilibrium may be locally determinate or indeterminate.[10] If monetary policy is passive at π^*, $(A < 1)$, then the determinant of J is negative, so the real parts of the roots of J are of opposite sign. In this case, the equilibrium is locally indeterminate.

One may be tempted to conclude from the above analysis that if $u_{cm} \geqslant 0$, there is no indeterminacy problem under active monetary policy in the sense that there exists no equilibrium allocation other than the active steady state, starting in a small neighborhood around that steady state, with the property that λ and π remain forever bounded. Such a conclusion, however, would be misplaced because globally the picture may be quite different.

6. GLOBAL EQUILIBRIA

In order to characterize global equilibrium dynamics, in this section, we assume particular functional forms for preferences and technology. We

[9] As shown in Benhabib *et al.* [2], an aggregate supply schedule like the one given by the second row of (27) also arises in the context of a staggered price setting model with optimizing firms like Yun's [25] extension of Calvo [4]. In Calvo's original model, firms change prices according to a rule of thumb that results in an aggregate supply function in which $\dot{\pi}$ is a function only of aggregate demand ($J_{22} = 0$).

[10] Benhabib *et al.* [2] show that the economy with $u_{cm} < 0$ is similar to one without money in the utility function but money entering the production function.

assume that the instant utility function displays constant relative risk aversion in a composite good, which in turn is produced with consumption goods and real balances via a CES aggregator. Formally,

$$u(c, m) - z(h) = \frac{[(xc^q + (1-x)\, m^q)^{1/q}]^w}{w} - \frac{h^{1+v}}{1+v}; \qquad q, w \leqslant 1, \quad v > 0. \tag{28}$$

q & w

The restrictions imposed on q and w ensure that $u(\cdot, \cdot)$ is concave, c and m are normal goods, and the interest elasticity of money demand is strictly negative.[11] The production function takes the form

$$y(h) = h^{\alpha}; \qquad 0 < \alpha < 1.$$

In the recent related literature on determinacy of equilibrium under alternative specifications of Taylor rules, it is typically assumed that preferences are separable in consumption and real balances (e.g., Woodford [24], Bernanke and Woodford [3], and Clarida *et al.* [5]). We therefore characterize the equilibrium under this preference specification before turning to the more general case.

6.1. *Separable Preferences*

hm.

The case of separable preferences arises when the intra- and intertemporal elasticities of substitution take the same value, that is, when $q = w$. In this case the equilibrium conditions (23) and (24) become

$$\dot{\lambda} = \lambda[r + \pi - R^* e^{(A/R^*)(\pi - \pi^*)}] \tag{29}$$

$$\dot{\pi} = r(\pi - \pi^*) - \frac{1+\eta}{\gamma} \lambda^{\omega} x^{\alpha\theta} + \frac{\eta}{\alpha\gamma} \lambda^{\beta} x^{(1+v)\theta}, \tag{30}$$

where $\beta \equiv (1+v)/(\alpha(w-1)) < 0$, $\omega \equiv w/(w-1)$, and $\theta \equiv 1/(\alpha(1-w))$. Throughout this section we assume that

$$R^* = r + \pi^*.$$

[11] Note that the sign of u_{cm} equals the sign of $w - q$.

This expression implies that π^* solves (29) when $\dot{\lambda}=0$. From evaluating (30) at $\dot{\pi}=0$ and $\pi=\pi^*$, it follows that λ^* must satisfy

$$\frac{1+\eta}{\gamma}\lambda^{*\omega}x^{\alpha\theta}=\frac{\eta}{\alpha\gamma}\lambda^{*\beta}x^{(1+v)\theta}\equiv M<0.$$

Evaluating (30) at $\pi=\bar{\pi}$ and setting $\dot{\pi}=0$ yield the following expression defining the steady-state value of λ, $\bar{\lambda}$, associated with $\bar{\pi}$:

$$r(\bar{\pi}-\pi^*)=M(\bar{\lambda}^{\omega}-\bar{\lambda}^{\beta}).$$

Because $\bar{\pi}<\pi^*$ for $A>1$ and $\omega\geqslant 0$ for $w\leqslant 0$, it follows from this expression that if $A>1$ and $w\leqslant 0$, then $\bar{\lambda}$ exists and is unique.[12] Thus, in this section we assume that $w\leqslant 0$; that is, the intertemporal elasticity of substitution does not exceed unity.

The main result of this section is that in the economy described above there exist an infinite number of equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to the steady state at which monetary policy is passive or to a limit cycle. In Section 5, we showed that if one restricts the analysis to equilibria in which π and λ remain forever bounded in an arbitrarily small neighborhood of the active steady state, then the unique perfect-foresight equilibrium is the steady state itself. Thus, the picture that arises from a local analysis might wrongly lead one to conclude that active monetary policy is stabilizing when in fact it is not. The following proposition formalizes this result.

Proposition 1 (Global Indeterminacy under Active Monetary Policy and Separable Preferences). *Suppose preferences are separable in consumption and real balances ($q=w$). Then, for r and $A-1$ positive and sufficiently small, the equilibrium exhibits indeterminacy as follows: trajectories originating in the neighborhood of the steady state $(\lambda,\pi)=(\lambda^*,\pi^*)$, at which monetary policy is active, converge either to the other steady state, $(\bar{\lambda},\bar{\pi})$, at which monetary policy is passive or to a limit cycle around (λ^*,π^*). In the first case, there exists a saddle connection, and the dimension of indeterminacy is one, whereas in the latter case the dimension of indeterminacy is* 2.

↳ 2

Proof. See the Appendix. ∎

This result is likely also to arise in models with alternative specifications of the source of nominal rigidities. For example, in a model with staggered

[12] If $w=0$, $\bar{\lambda}$ may not exist for all parameterizations of the model. For r sufficiently close to zero or A sufficiently close to one (or both), $\bar{\lambda}$ always exists.

price setting like Yun's [25] extension of Calvo [4], the aggregate supply schedule takes a form that is qualitatively similar to (30). Thus, we conjecture that the Calvo–Yun model exhibits global indeterminacy of the kind described in Proposition 1 as well.[13]

Figure 3 illustrates the existence of a saddle connection from the steady state at which monetary policy is active to the steady state at which monetary policy is passive. For the computation of the equilibrium dynamics of π and λ, the assumed time unit is a quarter. The parameters R^*, π^*, and r were set at 0.06/4, 0.042/4, and 0.018/4, respectively. The parameter A was set at 1.5, so that at the active steady state the Taylor rule has the slope suggested by Taylor [21]. These parameter values imply that at the active steady state the nominal interest rate is 6% per year, which equals the average three-month Treasury Bill rate in the period 1960:1–1998:9, the inflation rate is 4.2% per year, which is consistent with the average U.S. inflation rate over the period 1960:Q1–1998:Q3 as measured by the GDP deflator, and the real discount rate is 1.8% per year. In addition, we set $w = q = -1$ so that the instant utility function is separable in consumption and real balances and the intertemporal elasticity of substitution equals 1/2. The parameter x was set at a value consistent with an annual consumption velocity of money of 3. The labor share, α, was set at 0.7, and the labor supply elasticity at 1. The value of η was chosen so that the implied markup of prices over marginal cost at the active steady states is 5%, which is consistent with the evidence presented by Basu and Fernald [1]. Finally, following Sbordone [18], we set γ, the parameter governing the disutility of deviating from the inflation target, at $-17.5(1+\eta)$. Table 1 summarizes the calibration. The inflation rate at the passive steady state is 0.7% per year, and the sensitivity of the Taylor rule with respect to inflation is 0.63. The active steady state is a source and the passive steady state is a saddle. Thus, the active steady state is locally the unique rational expectations equilibrium whereas around the passive steady state the equilibrium is indeterminate. The solid line in Fig. 3 displays the saddle path converging to the passive steady state. The dashed line corresponds to the unstable manifold diverging from the passive steady state.

Three features of Fig. 3 are noteworthy. First, the indeterminacy result established in Proposition 1 seems to hold not only for pairs (r, A) close to

[13] In Calvo's [4] original sticky-price model, the aggregate supply function takes the form $\dot{\pi} = f(\lambda)$. In this case, the equilibrium conditions (32) and (33) form a conservative Hamiltonian system whose Jacobian has a zero trace and a positive determinant under active monetary policy. Such a system gives rise to a continuum of cycles surrounding the active steady state. These cycles are enclosed by a homoclinic orbit formed by the connection of the stable and unstable manifolds of the passive steady state. The period of the cycles approaches infinity as the cycles get closer to the homoclinic orbit. ?

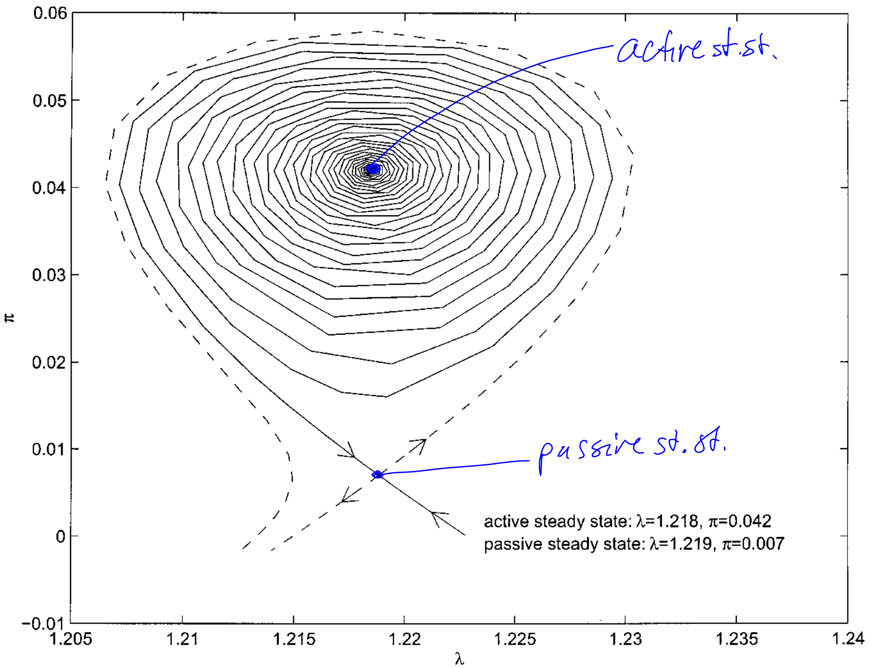

FIG. 3. Separable preferences: Saddle connection from the active to the passive steady state.

(0, 1) but also for empirically relevant values. Second, the saddle connection is not inconsistent with the observation that the inflation rate fluctuates for long periods of time in a region in which monetary policy is active, as has been argued in the case of the U.S. economy since the Volcker era (see Clarida *et al.* [5] and Rotemberg and Woodford [16]). In our calibrated economy monetary policy is active for all inflation rates exceeding 2.6% per year. Third, one argument for restricting attention to local dynamics is that observed inflation fluctuations at business-cycle frequencies are relatively small. The global dynamics illustrated in Fig. 3 suggest that the short-term fluctuations in the inflation rate along the saddle

TABLE 1

Calibration

R^*	π^*	r	A	w	q	c/m	α	v	$\frac{\eta}{1+\eta}$	γ
0.06/4	0.042/4	0.018/4	1.5	−1	−1	3/4	0.7	1	1.05	350

Note. The time unit is one quarter. $x/(1-x)=(c/m)^{1-q}/R^*$.

connection are empirically plausible, with a maximum annual inflation rate of 5.7% and a minimum of 0.7%.

The dynamics are robust to wide variations in parameter values. Fig. 4 illustrates that the saddle path connecting the steady state at which monetary policy is active with the steady state at which policy is passive does not disappear if: (a) A, the slope of the Taylor rule at $\pi = \pi^*$, is increased from the baseline value of 1.5 to a value of 2, which, as some authors may argue, reflects more closely the stance of U.S. monetary policy in the post-Volcker era (see, again, Clarida *et al.* [5] and Rotemberg and Woodford [16]). (b) π^*, the inflation rate associated with the active steady state, is set at 3% per year. This case illustrates that the global indeterminacy result does not hinge in any important way on the inflation rate being high at the active steady state. (Note that the inflation rate at the corresponding passive steady state is negative.) (c) γ, the parameter governing the cost of deviating from the inflation target, is reduced from its baseline value of 350 to 35. Although not noticeable in the figure, for such a low value of γ, the economy converges from the vicinity of the active steady state to the passive steady state at a much higher speed than under the baseline calibration. (d) The annual consumption velocity of money is

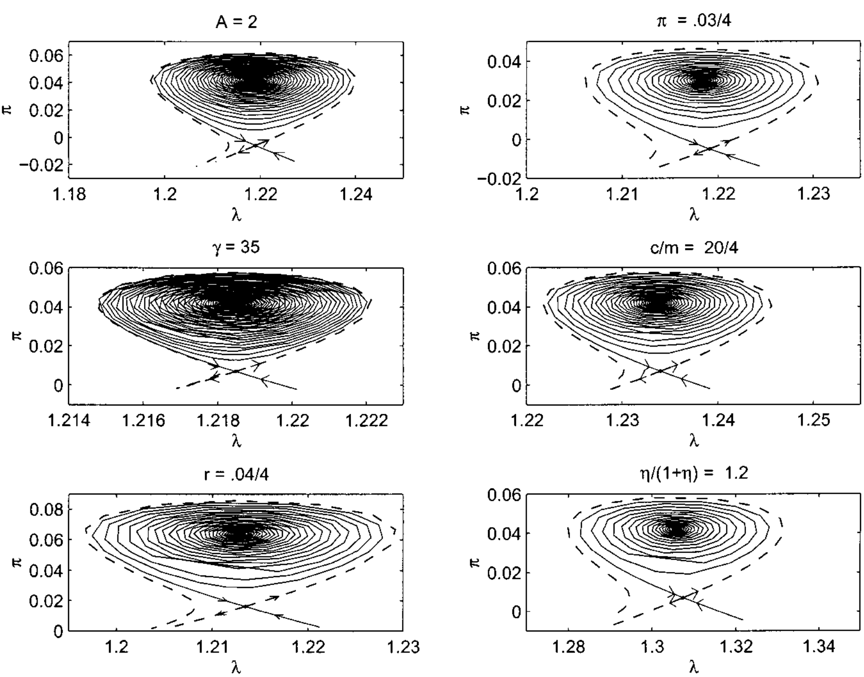

FIG. 4. Separable preferences: Sensitivity analysis.

increased from 3 to 20. This result is of particular interest in light of the view that as a result of financial innovation agents are increasingly able to perform transactions without money. (e) The discount rate, r, takes a value of 4% per year, a value commonly used in the real-business-cycle literature (Prescott [14]). (f) A markup of prices over marginal cost of 20% is assumed. This number reflects the upper range of available empirical estimates (see Basu and Fernald [1]).

6.2. *Non-separable Preferences*

In this section, we consider preference specifications for which the intra- and intertemporal elasticities of substitution are different ($q \neq w$). In this case, the equilibrium conditions (23) and (24) can be written as

$$\dot{\lambda} = \lambda[r + \pi - R^* e^{(A/R^*)(\pi - \pi^*)}] \tag{31}$$

$$\begin{aligned} \dot{\pi} = r(\pi - \pi^*) &- \frac{1+\eta}{\gamma} x^{\alpha\theta} \lambda^{\omega} \left[(R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} e^{(A/R^*)\chi(\pi - \pi^*)} + 1 \right]^{\alpha\xi} \\ &+ \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} \lambda^{\beta} \left[(R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} e^{(A/R^*)\chi(\pi - \pi^*)} + 1 \right]^{(1+v)\xi}, \end{aligned} \tag{32}$$

where β and ω are defined as in the previous section and $\chi \equiv q/(q-1)$, $\xi \equiv (w-q)/[\alpha q(1-w)] \neq 0$, and $\theta \equiv w/[\alpha q(1-w)]$. Let λ^* be the steady-state value of λ associated with $\pi = \pi^*$, $A = 1$, $r = r^c$, and $R^* = \pi^* + r^c$ (with r^c to be determined below). Then, by Eq. (32), λ^* is implicitly defined by

$$\begin{aligned} &\frac{1+\eta}{\gamma} x^{\alpha\theta} (\lambda^*)^{\omega} \left((R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} + 1 \right)^{\alpha\xi} \\ &= \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} (\lambda^*)^{\beta} \left((R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} + 1 \right)^{\xi(1+v)} \equiv M. \end{aligned}$$

The parameter r^c is defined as the value of r that makes the trace of the Jacobian of the system (31)–(32) equal to zero for $A = 1$. That is, r^c is implicitly given by

$$r^c = -B\xi \left(\frac{1}{r^c + \pi^*} \right) \chi M (1 + v - \alpha), \tag{33}$$

where

$$B \equiv (R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} \Big/ \left[1 + (R^*)^{\chi} \left(\frac{1-x}{x}\right)^{1-\chi} \right].$$

Inspection of (33) reveals that the existence of a positive r^c depends on parameter values. For example, one can show that a positive r^c always exists if $q \in (0, 1)$ and $q - w, \pi^* > 0$. Throughout this section, we assume that R^* is fixed and equal to $r^c + \pi^*$. When $(r, A) = (r^c, 1)$, the point $(\lambda, \pi) = (\lambda^*, \pi^*)$ is the unique steady state of the system (31) and (32). At that point, monetary policy is neither active nor passive ($R'(\pi^*) = 1$). For parameter configurations in which $(r, A) \neq (r^c, 1)$, the economy displays in general either none or two steady-state values of π. When two steady-state values of π exist, the larger of them corresponds to an active monetary policy stance and the smaller one to a passive stance. In addition, each steady-state value of π is associated with one or two steady-state values of λ. The following lemma shows that under the assumption that the intertemporal elasticity of substitution is less than one ($w < 0$), each steady-state value of π is associated with a unique steady-state value of λ. For this reason and because it is clearly the case of greatest empirical relevance, in what follows we assume that $w < 0$. The lemma also shows that the steady state at which monetary policy is active is either a sink or a source, while the steady state at which monetary policy is passive is always a saddle.

Lemma 1. *Suppose $w < 0$. Then, the steady states of the system* (31) *and* (32) *satisfy*: (i) *for each steady-state value of π there exists a unique steady-state value of λ; and* (ii) *the steady state at which monetary policy is active is either a sink or a source and the steady state at which monetary policy is passive is always a saddle.*

Proof. See the Appendix. ∎

The next proposition contains the main result of this subsection. Namely, if the steady state at which monetary policy is active is locally the unique equilibrium (i.e., the steady state is a source), then the equilibrium is globally indeterminate. Specifically, there exist equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converge either to a limit cycle or to the other steady state, at which monetary policy is passive.

Proposition 2 (Global Indeterminacy under Active Monetary Policy and Non-separable Preferences). *For parameter specifications (r, A) sufficiently close to $(r^c, 1)$, the economy with non-separable preferences exhibits indeterminacy as follows: There always exist an infinite number of equilibrium trajectories originating arbitrarily close to the steady state at which monetary policy is active that converges to:* (i) *that steady state,* (ii) *a limit cycle, or* (iii) *the other steady state at which monetary policy is passive. In cases* (i) *and* (ii) *the dimension of indeterminacy is* 2, *whereas in case* (iii) *it is* 1.

Proof. See the Appendix. ∎

The following corollary establishes parameter restrictions under which attracting limit cycles exist around the steady state at which monetary policy is active.

COROLLARY 1 (Periodic Equilibria). *If* $-(1-B)/(B(1+v+\alpha))<\xi<0$, *then there exists a region in the neighborhood of* $(r, A)=(r^c, 1)$ *for which the active steady state is a source surrounded by a stable limit cycle. On the other hand, if* $\xi>0$ *or* $\xi<-(1-B)/(B(1+v+\alpha))$, *then stable limit cycles do not exist.*

Proof. See the Appendix. ∎

It is important to recall that the equilibrium remains globally indeterminate even if limit cycles do not exist. This is because in that case there always exists an equilibrium trajectory connecting the active steady state with the passive one. In fact, as shown in Figs. 5 and 6, a saddle connection is the typical pattern that arises under plausible parameterizations of the model with non-separable preferences. In both figures, the calibration is the same as that used in the economy with separable preferences, summarized

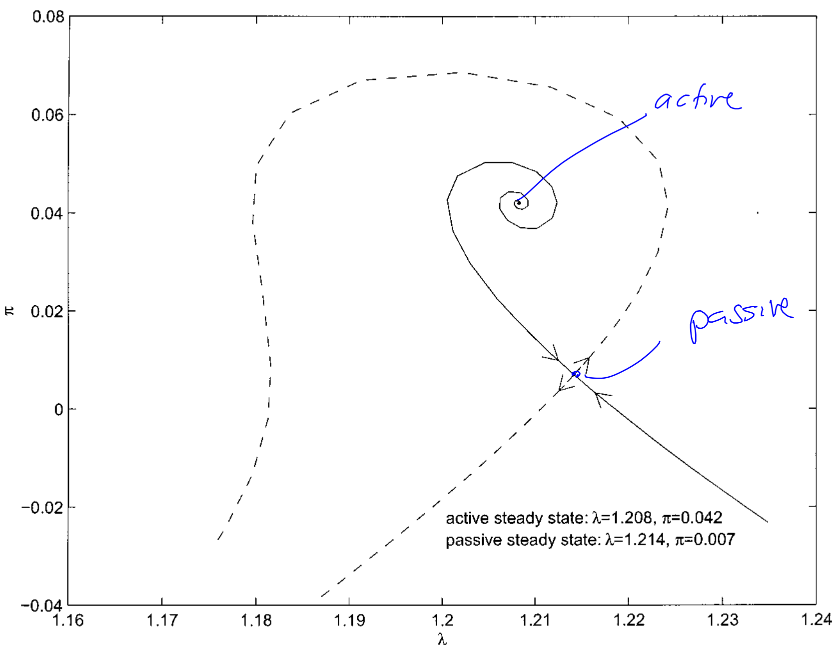

FIG. 5. Non-separable preferences, $w>q$: Saddle connection from the active to the passive steady state.

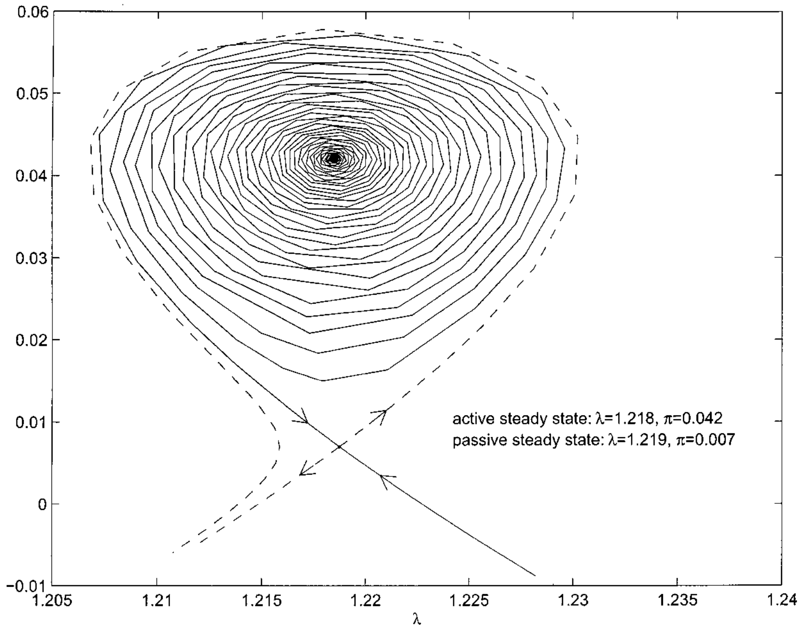

FIG. 6. Non-separable preferences, $w<q$: Saddle connection from the active to the passive steady state.

in Table 1, except, of course, that now the intratemporal elasticity of substitution between consumption and real balances, $1/(1-q)$, is assumed to be different from the intertemporal elasticity of substitution, $1/(1-w)$. In both figures, the intertemporal elasticity of substitution takes its baseline value of 0.5. In Fig. 5, q is set at -9, a value consistent with a log–log interest elasticity of money demand of -0.1.[14] In this case $w>q$, which implies that consumption and real balances are Edgeworth complements ($u_{cm}>0$).[15] In Fig. 6, q is set at -0.975, which corresponds to a log–log interest elasticity of money demand of $-1/2$. In this case $w<q$, so consumption and money are Edgeworth substitutes ($u_{cm}<0$). Under both parameterizations, the active steady state is locally the unique perfect-foresight equilibrium (i.e., the active steady state is a source). However, as the figures suggest, from a global perspective it is clear that an infinite number of trajectories originating arbitrarily close to the active steady state and on the saddle connection can be supported as equilibrium outcomes because they converge to the passive steady state.

[14] Given the particular functional form assumed for preferences in this subsection, the liquidity preference function (22) takes the form $m(c,R)=[x/(1-x)]^{1/(q-1)}R^{1/(q-1)}c$.

[15] When $u_{cm}>0$, the economy is equivalent to a cash-in-advance economy with cash and credit goods like the one developed by Lucas and Stokey [13].

The pattern illustrated in Figs. 5 and 6 is unchanged for values of q between -0.975 and -9, the two values assumed in the figures. As q is increased above -0.975, the active steady state becomes a sink and thus the equilibrium is locally indeterminate. If $w > q$ ($u_{cm} > 0$), the simulation results are, as in the case of separable preferences, robust to wide variations in other parameter values. In particular, a saddle path connecting the active steady state with the passive one continues to exist for more aggressive Taylor rules ($A \geqslant 2$) and lower costs of price adjustment (for example, $\gamma = 35$). In the case $q > w$ ($u_{cm} < 0$), parameter variations may or may not eliminate the saddle connection. However, when the saddle connection disappears, it is typically replaced by a situation in which the active steady state is a sink, which is locally a more severe case of indeterminacy.

7. FINAL REMARKS

This paper shows that when a global analysis is undertaken in a sticky-price model, the existence of a steady state with active monetary policy generically leads to global indeterminacy. Although the propositions above are proven for specific functional forms to facilitate checking for non-degeneracies, it is clear from the general structure of the equilibrium conditions that generically alternative specifications for smooth preferences and the interest rate feedback rule will give rise to similar results, as long as the feedback rule ensures the existence of a steady state with an active monetary policy. The main results of the paper also obtain in flexible-price versions of the model (Schmitt-Grohé and Uribe [19]). In this case, equilibrium dynamics are described by a scalar system where again a continuous feedback rule generating a steady state with active monetary policy implies the existence of a passive steady state, with all the implications for global indeterminacy, quite independent of the structure of preferences and production.

APPENDIX

Proof of Proposition 1

Preliminaries

To prove Proposition 1 we apply the following theorem due to Kopell and Howard [9]:

THEOREM (Kopell and Howard [9, Theorem 7.1.]). *Let $\dot{X} = F_{\mu,\nu}(X)$ be a two-parameter family of ordinary differential equations on R^2, F smooth in all of its four arguments, such that $F_{\mu,\nu}(0) = 0$. Also assume*:

1. *$dF_{0,0}(0)$ has a double zero eigenvalue and a single eigenvector e.*

2. *The mapping $(\mu, v) \to (\det dF_{\mu,v}(0), \operatorname{tr} dF_{\mu,v}(0))$ has a nonzero Jacobian at $(\mu, v) = (0, 0)$.*

3. *Let $Q(X, X)$ be the 2×1 vector containing the terms quadratic in the x_i and independent of (μ, v) in a Taylor series expansion of $F_{\mu,v}(X)$ around 0. Then $[dF_{(0,0)}(0), Q(e, e)]$ has rank 2.*

Then there is a curve $f(\mu, v) = 0$ such that if $f(\mu_0, v_0) = 0$, then $\dot{X} = F_{\mu_0, v_0}(X)$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in (X, μ, v) space) is on the boundary of a two-parameter family of periodic solutions. For all $|\mu|$, $|v|$ sufficiently small, if $\dot{X} = F_{\mu,v}(X)$ has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical points.

To apply this theorem, we must first perform several changes of variables and a Taylor series expansion of the equilibrium conditions around the steady state. Let $p \equiv \pi - \pi^*$ and $n \equiv \ln(\lambda/\lambda^*)$. Then the equilibrium conditions (29) and (30) can be expressed as

$$\dot{n} = R^* + p - R^* e^{(A/R^*)\,p} \tag{34}$$

$$\begin{aligned} \dot{p} = rp &- \gamma^{-1}(1+\eta)(\lambda^*)^{\omega}\, e^{\omega n} x^{\alpha\theta} \\ &+ \alpha^{-1}\gamma^{-1}\eta(\lambda^*)^{\beta}\, e^{\beta n} x^{\theta(1+v)}. \end{aligned} \tag{35}$$

Defining $y = p/[M(\beta - \omega)]$ and $s = R^* n$, we have

$$\dot{s} = R^{*2} + M(\beta - \omega)\, R^* y - R^{*2} e^{(A/R^*)\, M(\beta - \omega)\, y}$$

$$\dot{y} = ry + (e^{(\beta/R^*)\,s} - e^{(\omega/R^*)\,s})/(\beta - \omega).$$

We now take a Taylor series expansion of these two equations around $(s, y) = (0, 0)$, which yields

$$\dot{s} = R^*(1 - A)\, M(\beta - \omega)\, y - \frac{[AM(\beta - \omega)]^2}{2}\, y^2 - \frac{[AM(\beta - \omega)]^3}{3!\, R^*}\, y^3 - \cdot \tag{36}$$

$$\dot{y} = ry + \frac{s}{R^*} + \frac{1}{\beta - \omega}\left(\frac{(\beta^2 - \omega^2)}{2R^{*2}}\, s^2 + \frac{(\beta^3 - \omega^3)}{3!\, R^{*3}}\, s^3 + \cdots\right) \tag{37}$$

with Jacobian

$$J = \begin{bmatrix} 0 & R^*(1 - A)\, M(\beta - \omega) \\ 1/R^* & r \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 0 & 0 \\ 1/R^* & 0 \end{bmatrix}$$

when $r = (1 - A) = 0$. We are now ready to prove Proposition 1.

Proof of Proposition 1

We prove the proposition by showing that for $(r, 1 - A)$ small enough, the system of differential Eqs. (36) and (37) satisfies the hypotheses of the Kopell–Howard theorem stated above. Let $\mu \equiv r$, $v \equiv 1 - A$, and $X \equiv [s; y]$. Then, the system (36), (37) can be expressed as $\dot{X} = F_{\mu, v}(X)$. We have that $dF_{0,0}(0) = [0\ 0;\ 1/R^*\ 0]$. Clearly, $dF_{0,0}$ has a double zero eigenvalue and a single eigenvector $e = [0; 1]$. The Jacobian of the mapping $(\mu, v) \rightarrow (\det dF_{\mu, v}(0), \operatorname{tr} dF_{\mu, v}(0))$ at $(\mu, v) = (0, 0)$ is given by $[0\ -M(\beta - \omega);\ 1\ 0]$ and is different from zero. Note that neither λ^* nor M depends on μ or v. The vector $Q(e, e)$ is given by $[-M^2(\beta - \omega)^2/2;\ 0]$. It follows that $[dF_{0,0}(0)\ Q(e, e)]$ has rank 2. The proposition follows from the facts that the active steady state is a source, the passive steady state is a saddle, and both s and y are jump variables.

Proof of Lemma 1

(i) $w < 0$ implies that $\omega > 0$. Given a steady-state value $\bar{\pi}$ the uniqueness of the associated steady-state value of λ follows directly from evaluating (32) at $\dot{\pi} = 0$ and $\pi = \bar{\pi}$ and recalling that $\beta, 1 + \eta < 0$. (ii) By definition, monetary policy is active (passive) at a given steady state $(\bar{\lambda}, \bar{\pi})$ if and only if $Ae^{(A/R^*)\pi - \bar{\pi}^*} > (<)\, 1$. Let J denote the Jacobian of (31)–(32). Then $J_{11} = 0$. Therefore, the determinant of J is given by $-J_{21}J_{12}$. The element J_{12} equals $\bar{\lambda}[1 - Ae^{(A/R^*)(\bar{\pi} - \pi^*)}]$, which is negative (positive) if monetary policy is active (passive). The element J_{21} is given by

$$J_{21} = -\omega \frac{1+\eta}{\gamma} x^{\alpha\theta} \bar{\lambda}^{\omega - 1} \left[(R^*)^{\chi} \left(\frac{1-x}{x} \right)^{1-\chi} e^{(A/R^*)\chi(\bar{\pi} - \pi^*)} + 1 \right]^{\alpha\xi}$$

$$+ \beta \frac{\eta}{\alpha\gamma} x^{(1+v)\theta} \bar{\lambda}^{\beta - 1} \left[(R^*)^{\chi} \left(\frac{1-x}{x} \right)^{1-\chi} e^{(A/R^*)\chi(\bar{\pi} - \pi^*)} + 1 \right]^{(1+v)\xi},$$

which is clearly positive. Therefore, the determinant of J is positive (negative) if and only if monetary policy is active (passive).

Proof of Proposition 2

We prove the proposition by applying a theorem and a lemma from Kuznetsov [10] that together allow us to transform the system of equilibrium conditions into a simpler, topologically equivalent planar system of differential equations with known bifurcation diagram. Technically, we show that the system (31)–(32) exhibits a Bogdanov–Takens (double-zero) bifurcation at $(r, A) = (r^c, 1)$.

Preliminaries

Let $n \equiv \ln(\lambda/\lambda^*)$ and $y \equiv (\pi - \pi^*)/[M(\beta - \omega)]$. Then, equilibrium conditions (31) and (32) can be written as

$$\dot{y} = ry - \left(\frac{e^{\omega n}}{\beta - \omega}\right)(Be^{(A/R^*)\chi M(\beta-\omega)y} + 1 - B)^{\alpha\xi}$$
$$+ \left(\frac{e^{\beta n}}{\beta - \omega}\right)(Be^{(A/R^*)\chi M(\beta-\omega)y} + 1 - B)^{(1+v)\xi} \tag{38}$$

$$\dot{n} = r + \pi^* + M(\beta - \omega)\, y - R^* e^{(A/R^*)M(\beta-\omega)y}. \tag{39}$$

Taking a Taylor series expansion of the right-hand side of this system around $(y, n) = (0, 0)$ yields

$$\dot{y} = \left[r + B\xi \frac{A}{R^*}\chi M(1 + v - \alpha)\right] y + n$$
$$+ \frac{1}{2} B\xi(\beta - \omega)\left(\frac{A}{R^*}\chi M\right)^2 [(1 + v - \alpha)(1 - B) + B\xi((1+v)^2 - \alpha^2)]\, y^2$$
$$+ B\xi \frac{A}{R^*}\chi M[\beta(1+v) - \omega\alpha]\, yn + \frac{1}{2}(\beta + \omega)\, n^2 + \cdots \tag{40}$$

$$\dot{n} = (r + \pi^* - R^*) + M(\beta - \omega)(1 - A)\, y - \frac{1}{2} R^* \left(\frac{A}{R^*} M(\beta - \omega)\right)^2 y^2 - \cdots \tag{41}$$

The Jacobian of this system is

$$\begin{bmatrix} r + B\xi \dfrac{A}{R^*}\chi M(1 + v - \alpha) & 1 \\ M(\beta - \omega)(1 - A) & 0 \end{bmatrix}.$$

At $(r, A) = (r^c, 1)$ this Jacobian collapses to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has two zero eigenvalues (the Bogdanov–Takens condition). We now state the aforementioned theorem and lemma from Kuznetsov [10].

THEOREM (Normal Form Representation, Kuznetsov [10, Theorem 8.4]). *Suppose that a planar system*

$$\dot{x} = f(x, \alpha), \qquad x \in R^2, \quad \alpha \in R^2,$$

with smooth f, has at $\alpha = 0$ the equilibrium $x = 0$ with a double zero eigenvalue. Via a Taylor series expansion around $x = 0$ and transformation of variables, this system can be expressed as

$$\begin{aligned} \dot{y}_1 &= y_2 + a_{00}(\alpha) + a_{10}(\alpha)\, y_1 + a_{01}(\alpha)\, y_2 \\ &\quad + \tfrac{1}{2} a_{20}(\alpha)\, y_1^2 + a_{11}(\alpha)\, y_1 y_2 + \tfrac{1}{2} a_{02}(\alpha)\, y_2^2 + P_1(y, \alpha) \\ \dot{y}_2 &= b_{00}(\alpha) + b_{10}(\alpha)\, y_1 + b_{01}(\alpha)\, y_2 \\ &\quad + \tfrac{1}{2} b_{20}(\alpha)\, y_1^2 + b_{11}(\alpha)\, y_1 y_2 + \tfrac{1}{2} b_{02}(\alpha)\, y_2^2 + P_2(y, \alpha), \end{aligned}$$

where $a_{lk}(\alpha)$, $b_{lk}(\alpha)$, and $P_{1,2}(y, \alpha) = O(\|y\|)^3$ are smooth functions of their arguments. Assume that

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0$$

and that the following nondegeneracy conditions are satisfied:

(BT.0) *the Jacobian matrix* $\frac{\partial f}{\partial x}(0, 0) \neq 0$;

(BT.1) $a_{20}(0) + b_{11}(0) \neq 0$;

(BT.2) $b_{20}(0) \neq 0$;

(BT.3) *the map*

$$(x, \alpha) \mapsto \left(f(x, \alpha), \operatorname{tr}\left(\frac{\partial f(x, \alpha)}{\partial x}\right), \det\left(\frac{\partial f(x, \alpha)}{\partial x}\right)\right)$$

is regular at the point $(x, \alpha) = (0, 0)$.

Then there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparameterization, *and smooth invertible parameter changes, which together reduce the system to*

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 + O(\|\eta\|^3), \end{aligned}$$

where $s = \operatorname{sign}[b_{20}(0)(a_{20}(0) + b_{11}(0))] = \pm 1$.

The explicit steps of the transformation of variables are given in Kuznetsov [10]. We note that β_1 and β_2 are functions of α satisfying $\beta_1(\alpha) = \beta_2(\alpha) = 0$ for $\alpha = 0$.

LEMMA (Effect of Higher-Order Terms, Kuznetsov [10, Lemma 8.8]). *The system*

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2 + O(\|\eta\|^3) \end{aligned}$$

is locally topologically equivalent near the origin to the system

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2. \end{aligned}$$

We are now ready to prove Proposition 2.

Proof of Proposition 2

We first show that the system (31), (32) of equilibrium conditions of the economy with non-separable preferences is in general locally (i.e., near $(r, A) = (r^c, 1)$) topologically equivalent near the steady state $(\lambda, \pi) = (\lambda^*, \pi^*)$ to the system

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2. \end{aligned} \tag{42}$$

The first step is to show that the conditions of Theorem 8.4 of Kuznetsov [10] are satisfied by the transformation of (31), (32) given by (38), (39). Let $x \equiv (y, n)$ and $\alpha \equiv (1 - A, r - r^c)$. Then the system (38), (39) can be expressed as $\dot{x} = f(x, \alpha)$. We have shown above that (38), (39) has at $\alpha = 0$ the equilibrium $x = 0$ with a non-zero Jacobian. Thus, BT.0 is satisfied. We have also shown that at $(x, \alpha) = (0, 0)$ the Jacobian has a double zero eigenvalue. It is clear from (40), (41) that

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0.$$

Also, $a_{20}(0) = B\xi(\beta - \omega)(\frac{1}{R^*}\chi M)^2\,[(1+v-\alpha)(1-B) + B\xi((1+v)^2 - \alpha^2)]$ and $b_{20}(0) = -R^*[1/R^* M(\beta - \omega)]^2$ are in general non-zero while $b_{11}(0) = 0$. Therefore, BT.1 and BT.2 are satisfied. The Jacobian of the mapping $(x, \alpha) \mapsto (f(x, \alpha), \operatorname{tr}(\frac{\partial f(x,\alpha)}{\partial x}), \det(\frac{\partial f(x,\alpha)}{\partial x}))$ at $(x, \alpha) = (0, 0)$ is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{20}(0) & a_{11}(0) & r^c & 1 \\ -b_{20}(0) & 0 & -1 & 0 \end{bmatrix},$$

where $a_{11}(0) = B\xi\chi M R^{*-1}[\beta(1+v) - \omega\alpha]$. The determinant of this Jacobian is equal to $-b_{20}(0)\,r^c$, which is in general different from zero, so that the map is regular at $(x, \alpha) = (0, 0)$ and condition BT.3 is satisfied. The claim that the equilibrium conditions have the normal form representation given by (42) follows from the theorem and the lemma stated above. Proposition 2 then follows directly from Lemma 1 and the properties of the the bifurcation diagram of (42) (see Kuznetsov [10, Sect. 8.4.2] for the case in which the coefficient on $\eta_1\eta_2$ is -1 and Guckenheimer and Holmes [8, Sect. 7.3] for the case in which the coefficient on $\eta_1\eta_2$ is $+1$).

Proof of Corollary 1

The existence of stable limit cycles depends on the sign of the coefficient of $\eta_1\eta_2$ in (42), which is equal to the sign of the parameter s defined in Theorem 8.4 of Kuznetsov [10] stated above. As shown in Kuznetsov [10], if s is negative there exists a region in the vicinity of $(r, A) = (r^c, 1)$ for which stable limit cycles emerge. If s is positive, then stable limit cycles do not exist. In the economy with non-separable preferences, $s = -\operatorname{sign}(a_{20}(0))$, where $a_{20}(0)$ is given in terms of the structural parameters of the economy in the proof of Proposition 2.

REFERENCES

1. S. Basu and J. G. Fernald, Returns to scale in U.S. production: Estimates and implications, *J. Polit. Econ.* **105** (1997), 249–283.
2. J. Benhabib, S. Schmitt-Grohé, and M. Uribe, “Monetary Policy and Multiple Equilibria,” C. V. Starr Center Working Paper No. 98-02, 1998.
3. B. Bernanke and M. Woodford, Inflation forecasts and monetary policy, *J. Money, Credit, Banking* **29** (1997), 653–684.
4. G. A. Calvo, Staggered prices in a utility-maximizing framework, *J. Monet. Econ.* **12** (1983), 383–98.
5. R. Clarida, J. Galí, and M. Gertler, Monetary policy rules and macroeconomic stability: Evidence and some theory, mimeo, New York University, May 1997.

6. R. Clarida, J. Galí, and M. Gertler, Monetary policy rules in practice: some international evidence, *Europ. Econ. Rev.* **42** (1998), 1033–1067.
7. L. Christiano and C. Gust, Interest rate targeting rules, mimeo, Northwestern University, March 1998.
8. J. Guckenheimer and P. Holmes, “Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,” Springer-Verlag, New York, 1983.
9. N. Kopell and L. N. Howard, Bifurcations and trajectories joining critical points, *Adv. Math.* **18** (1975), 306–358.
10. Y. A. Kuznetsov, “Elements of Applied Bifurcation Theory,” Springer-Verlag, New York, 1995.
11. E. Leeper, Equilibria under “active” and “passive” monetary and fiscal policies, *J. Monet. Econ.* **27** (1991), 129–147.
12. A. Levin, V. Wieland, and J. Williams, Robustness of simple monetary policy rules under model uncertainty, *in* “Monetary Policy Rules” (J. B. Taylor, Ed.), The Univ. of Chicago Press, Chicago, 1998, forthcoming.
13. R. E. Lucas, Jr. and N. Stokey, Money and interest in a cash-in-advance economy, *Econometrica* **55** (1987), 491–514.
14. E. C. Prescott, Theory ahead of business cycle measurement, *Fed. Res. Bank Minn. Quart. Rev.* **10** (1986), 9–22.
15. J. J. Rotemberg, Sticky prices in the United States, *J. Polit. Econ.* **90** (1982), 1187–1211.
16. J. J. Rotemberg and M. Woodford, An optimization-based econometric framework for the evaluation of monetary policy, *in* “NBER Macroeconomics Annual 1997” (B. S. Bernanke and J. J. Rotemberg, Eds.), pp. 297–346, MIT Press, Cambridge, MA, 1997.
17. J. J. Rotemberg and M. Woodford, Interest-rate rules in an estimated sticky price model, *in* “Monetary Policy Rules” (J. B. Taylor, Ed.), The Univ. of Chicago Press, Chicago, 1998, forthcoming.
18. A. Sbordone, Prices and unit labor costs: A new test of price stickiness, mimeo, Rutgers University, 1998.
19. S. Schmitt-Grohé and M. Uribe, Price-level determinacy and monetary policy under a balanced-budget requirement, *J. Monet. Econ.* **45** (2000), 211–246.
20. A. Seierstad and K. Sydsæter, “Optimal Control Theory with Economic Applications,” North-Holland, Amsterdam, 1987.
21. J. B. Taylor, Discretion versus rules in practice, *Carnegie-Rochester Conf. Ser. Public Policy* **39** (1993), 195–214.
22. J. B. Taylor, A historical analysis of monetary policy rules, *in* “Monetary Policy Rules” (J. B. Taylor, Ed.), The Univ. Chicago Press, Chicago, 1998a, forthcoming.
23. J. B. Taylor, The robustness and efficiency of monetary policy rules as guidelines for interest rate setting by the European Central Bank, mimeo, Stanford University, 1998b.
24. M. Woodford, “Control of Public Debt: A Requirement for Price Stability,” NBER Working Paper 5684, July 1996.
25. T. Yun, Nominal price rigidity, money supply endogeneity, and business cycles, *J. Monet. Econ.* **37** (1996), 345–370.