8
Adaptive Expectations (1990's)

Least squares adaptation

This chapter modifies self-confirming equilibria to attain adaptive models in the style of Sims (1988) and Chung (1990) and studies whether they converge to self-confirming equilibria. I withhold knowledge of population regressions from agents, and require them to learn by updating least squares regressions as time passes. The government sets the inflation rate at the first-period recommendation of the Phelps problem for its current estimate of the Phillips curve.[1]

Starting from the self-confirming equilibrium models of Chapter 7, we in effect alter a free parameter that determines a gain sequence governing the rate at which past observations are discounted. If the gain is set to implement least squares learning, we eventually get nothing new from these models because under suitable assumptions on parameter values, they converge to self-confirming equilibria and leave us stuck at the Nash equilibrium outcome. However, if we set a constant gain, making agents discount past observations, new outcomes can emerge. Agents' discounting of past observations arrests convergence to a self-confirming equilibrium and can sustain paths that look like Volcker terminating inflation.

A purpose of this chapter is to replicate, analyze, and reinterpret simulations like those of Chung (1990) and Sims (1988). I add two things to their work. First, I relate their systems to the self-confirming equilibria of Chapter 7, and apply theorems

[1] Appendix B relates this type of adaptive model to Kreps's idea of anticipated utility.

from the literature on least squares learning to determine the limiting behavior of the models under alternative specifications of the government's adaptation algorithm.[2] Second, I interpret the behavior of the simulations in terms of recurrent dynamics that escape a self-confirming equilibrium and sustain close to Ramsey outcomes during long episodes. These escapes reflect how the approximating models use a unit root to approximate a constant and are reminiscent of Chapter 7's equilibrium with misspecified forecasting.

Before presenting the details of our adaptive models, the next section provides an overview of the issues and the analytical methods.

Primer on recursive algorithms

Within a compact notation, this section introduces the main concepts of this chapter and describes how our adaptive models connect to self-confirming equilibria.

A self-confirming equilibrium is determined by a government's beliefs about some population moments and associated regression coefficients. For the classical identification scheme, these beliefs are measured by $(\gamma, EX_CX_C', EUX_C)$. Time t values of these are among the economy's state variables in the adaptive models of this chapter. They disappear as state variables in a self-confirming equilibrium because they are constants.

A self-confirming equilibrium under the classical identification satisfies a set of moment conditions

$$ER_{XC}^{-1}(\gamma)\left[U_tX_{Ct}' - (X_{Ct}X_{Ct})'\gamma\right] = 0 \tag{67a}$$
$$EX_{Ct}X_{Ct}' - R_{XC}(\gamma) = 0, \tag{67b}$$

where the mathematical expectation is taken with respect to a distribution of (U_t, X_{Ct}) that depends on γ through the solution

[2] See Evans and Honkapohja's (1998b) handbook chapter and their forthcoming monograph (1998c) for comprehensive treatments of learning in macroeconomics.

$h(\gamma)$ of the Phelps problem. Self-reference surfaces in the dependence of this distribution on γ. For convenience, assemble the unknowns into the vector

$$\phi = \begin{bmatrix} \gamma \\ \mathrm{col}(R_{XC}) \end{bmatrix},$$

where $\mathrm{col}(R_{XC})$ is a vector formed by stacking the columns of R_{XC}. The moment equations can be written as

$$E\left[F(\phi,\zeta)\right] = 0, \tag{68}$$

where ζ is a random vector. For arbitrary ϕ, define

$$b(\phi) = E\left[F(\phi,\zeta)\right]. \tag{69}$$

A self-confirming equilibrium has a set of beliefs ϕ_f satisfying

$$b(\phi_f) = 0. \tag{70}$$

The next subsections describe alternative recursive algorithms for solving (70). A change of perspective converts these computational algorithms into models of real-time adaptation.

Iteration

Compute a sequence $\{\phi_k\}$ of estimates of ϕ from

$$\phi_{k+1} = \phi_k + ab(\phi_k), \tag{71}$$

where the distribution used for evaluating the expectation defining $b(\phi_k)$ in (68) is evaluated at the estimate ϕ_k, and $a > 0$ is a step size. This represents the iterative algorithm described in chapter 7. Each step requires evaluating the mathematical expectation $b(\phi) = E\left[F(\phi,\zeta)\right]$, the reason we reported the moment formulas in chapter 7.

Stochastic approximations

A random version of (71) can be obtained by substituting $F(\phi_n, \zeta_n)$ for its mean $b(\phi_n)$ and manipulating the step size to perform the averaging. In particular, compute the stochastic process

$$\phi_{n+1} = \phi_n + a_n F(\phi_n, \zeta), \tag{72}$$

where $\{a_n\}$ is a positive sequence of scalars satisfying

$$a_n > 0, \quad \sum_{n=0}^{\infty} a_n = +\infty. \tag{73}$$

Define artificial time

$$t_n = \sum_{k=0}^{n} a_k \tag{74}$$

and form the sampled processes $\phi(t_n) = \phi_n$. Interpolate $\phi(t_n)$ to get a continuous time process $\phi^o(t)$ (typically a piecewise linear interpolation). Then obtain a continuous time process that approximates $\phi^o(t)$ as $n \to +\infty$ and use it to study the limiting behavior of the original ϕ_n process (72).

We get different approximating processes by adjusting the rate of decrease of the gain sequence $\{a_n\}$ in (73). Different gain sequences affect details of the approximation through the mapping (74) from real discrete time n to artificial time t_n.

Mean dynamics

Classic stochastic approximation algorithms (Kushner and Clark (1978) and Ljung (1977)) set $a_n \sim \frac{1}{n}$ (at least $\forall\, t \geq N$ for some $N > 0$). That permits strong statements about the almost sure convergence of (72) to a zero of $b(\phi)$. For $a_n \sim \frac{1}{n}$, as $n \to \infty$, the stochastic process $\phi^o(t)$ approaches the solution of an ordinary differential equation

$$\frac{d\phi^o(t)}{dt} = b(\phi^o). \tag{75}$$

Equation (75) generates the mean dynamics. Using an argument sketched in the appendix to Chapter 3, when $a_n \sim \frac{1}{n}$, a law of large numbers makes the random term in the continuous time approximation converge to zero fast enough that the mean dynamics (75) describe the tail behavior of the stochastic process ϕ_n in (72). If the algorithm (almost surely) converges, it is to a zero of the mean dynamics, $0 = b(\phi)$, a self-confirming equilibrium. The ODE contains information about the local and global stability of the algorithm (72). We present and analyze this ODE for some examples later in this chapter.[3]

Constant gain

We are also interested in versions of the algorithm (72) with $a_n = \epsilon > 0\ \forall n$. Limit theorems about such constant gain algorithms use a weaker notion of convergence (convergence in distribution) than those for the classic stochastic approximation, where $a_n \sim \frac{1}{n}$ facilitates almost sure convergence. With a constant gain $a_n = \epsilon$, limit theorems are about small noise limits as $\epsilon \to 0$ and as $n\epsilon \to +\infty$.

Again define artificial time using (74) and form a family of processes

$$\phi_{n+1}^{\epsilon} = \phi_n^{\epsilon} + \epsilon F(\phi_n^{\epsilon}, \zeta_n). \tag{76}$$

Form $\phi^{\epsilon}(t)$ by interpolating $\phi^{\epsilon}(t_n)$, and study small ϵ limits of the family. Kushner and Dupuis (1987), Kushner and Yin (1997), and Gulinsky and Veretennikov (1993) verified conditions under which as $\epsilon \to 0$ and $\epsilon n \to \infty$, the ϕ_n^{ϵ} process converges in distribution to the zeros of the mean dynamics (75). The restrictions on the mean dynamics (75) needed for convergence match those from the classic stochastic approximation ($a_n \sim \frac{1}{n}$) theory.

[3] See Brock and LeBaron (1996) Brock and Hommes (1997) for models driven by stable mean dynamics far from rational expectations equilibria and by locally unstable adaptation near them.

Escape routes

Another use of the constant gain apparatus is more pertinent for this chapter, namely, the application of the theory of large deviations to characterize excursions of (76) away from ϕ_f. The main purpose of this chapter is to study outcomes that emerge when the government is endowed with a constant-gain learning algorithm that impedes convergence to a self-confirming equilibrium. We are as interested in movements away from a self-confirming equilibrium as in those toward one.

The theory of large deviations characterizes excursions away from ϕ_f by using the following objects: a log moment generating function of an averaged version of the innovation process $F(\phi_n, \zeta_n)$; the Legendre transform of that log moment generating function; and an action functional defined in terms of the Legendre transformation. Where θ is a vector conformable to F, the log moment generating function $H(\theta, \phi)$ is designed to approximate[4]

$$H(\theta, \phi) = \log E \exp\left(\theta' F(\phi, \zeta)\right). \tag{77}$$

Here the mathematical expectation E is taken over the distribution of ζ. The Legendre transform of H is

$$L(\beta, \phi) = \sup_{\theta}[\theta' \beta - H(\theta, \phi)]. \tag{78}$$

[4] See Dupuis and Kushner (1987, p. 225) and Kushner and Yin (1997, p. 275) for the technical details. They assume that for each $\delta > 0$, the following limit exists uniformly in ϕ_i, α_i in any compact set:

$$\sum_{i=0}^{T/\delta-1} \delta H(\alpha_i, \phi_i) = \lim_N \frac{\delta}{N} \log E \exp \sum_{i=0}^{T/\delta-1} \alpha_i' \sum_{j=iN}^{iN+N-1} F(\phi_i, \zeta_j).$$

The action functional $S(T, \phi)$ is

$$S(T,\phi) = \begin{cases} \int_0^T L\left(\frac{d}{ds}\phi(s), \phi(s)\right) ds & \text{if } \phi(s) \text{ is abs. cts. and } \phi(0) = \phi_f; \\ \infty & \text{otherwise.} \end{cases} \tag{79}$$

Dupuis and Kushner describe a deterministic programming problem for finding the escape route along which paths of the algorithm move away from a self-confirming equilibrium ϕ_f. Let D be a compact set containing ϕ_f. Let ∂D be the boundary of this set. Let $C[0,T]$ be the space of continuous functions $\phi(t)$ on the interval $[0,T]$. The escape route $\phi(t)$ solves:

$$\inf_{T>0} \inf_{\phi \in A} S(T, \phi) \tag{80}$$

where

$$A = \{\phi(\cdot) \in C[0,T], \phi(T) \in \partial D\}.$$

Assume that the minimizer $\tilde{\phi}(\cdot)$ is unique, and let t_D^ϵ be the time that $\phi^\epsilon(t)$ first leaves D. Dupuis and Kushner (1987, p. 242) show that for all $\delta > 0$,

$$\lim_{\epsilon \to 0} \text{Prob}\left(|\phi^\epsilon(t_D^\epsilon) - \tilde{\phi}(T)| > \delta\right) = 0. \tag{81}$$

While the mean dynamics don't depend on the noise around the mean dynamics, the escape routes do. Not only do the noises add random fluctuations around (75); they contribute another set of paths, the escape routes.[5] The most interesting features of the simulations below come from movements along escape routes.

Simplification of action functional

While the escape route calculations promise cheap information about central tendencies of our stochastic algorithms, it can be difficult to calculate the action functional (79). There

[5] See Freidlin and Wentzell (1984), especially chapter 4, and Dupuis and Kushner (1985, 1989).

are specializations and modifications of the algorithm (72) that simplify the action functional. A most important one makes $F(\phi, \zeta) = b(\phi) + \sigma(\phi)\zeta$, where ζ_n is stationary and Gaussian, but not necessarily serially uncorrelated. Define $R = \sum_j E\zeta_t\zeta_{t-j}$ Then Dupuis and Kushner[6] report the following formula for the action functional:

$$S(T, \phi) = .5 \int_0^T \left(\frac{d}{ds}\phi - b(\phi)\right)' \left[\sigma(\phi)R\sigma(\phi)'\right]^+ \left(\frac{d}{ds}\phi - b(\phi)\right) h(s) ds \quad (82)$$

where $(\cdot)^+$ is the generalized inverse (used to get around possible stochastic singularity).[7]

From computation to adaptation

While the preceding description treats the recursive formulas for ϕ as algorithms to approximate a self-confirming equilibrium, the same mathematics tell what we can get from modifying our models of self-confirming equilibrium to incorporate adaptation. Key facts for us are: (1) gain sequences that implement versions of least squares make the mean dynamics pull the economy toward self-confirming equilibria; (2) gain sequences that fall off more slowly than least squares, and thereby discount the past faster, increase the frequency with which escape dynamics influence outcomes.[8]

[6] See Dupuis and Kushner, 1985, especially the remark on the top of page 678. See also Kushner and Clark (1978) and Kushner and Yin (1992) for descriptions of K-W algorithms.

[7] The function $h(s)$ depends on the form of the gain function; for example, $h(s) = \exp s$ if $\gamma = 1$ in $a_n = a_0/(n^\gamma)$, and $h(s) = 1$ if $\gamma < 1$. See Dupuis and Kushner (1989, p. 1113.) Also see Kushner and Yin (1997, chapter 10).

[8] Here is a brief history of the ideas in this section. Lucas and Prescott (1971) dismissed iterating on (70) as a computational strategy, but Townsend (1983) used it. Woodford (1990) and Marcet and Sargent (1989a, 1989b) used the mean dynamics (75) associated with a stochastic approximation algorithm to establish conditions for the convergence of least squares learning to rational expectations in models with self-reference. Both Woodford and Marcet and Sargent required continuity of $b(\phi)$. In-Koo Cho (1997a, 1997b) studied problems with discontinuous $b(\phi)$ inherited from discontinuous decision rules (e.g., trigger strategies in

Adaptation with the classical identification

In a self-confirming equilibrium, the government solves the Phelps problem only at the equilibrium values of its perceived Phillips curve and implements the recommendations of a unique Phelps rule as time passes. In contrast, I now assume that the government solves a new Phelps problem and uses a different decision rule each period as it adapts to new information about the Phillips curve.[9]

The government's beliefs and behavior

The government arrives at time t with a model (32) and an estimate γ_{t-1} of the coefficients γ. It sets the systematic part of inflation $\hat{y}_t$ by solving the Phelps problem with $\gamma = \gamma_{t-1}$. This produces the outcome[10]

$$y_t = h(\gamma_{t-1})X_{t-1} + v_{2t}. \tag{83}$$

In forming $\hat{y}_t = h(\gamma_{t-1})X_{t-1}$, the government acts at t as if its model for the Phillips curve for all $j \geq 0$ is

$$U_{t+j} = \gamma_{t-1}X_{C,t+j} + \varepsilon_{C,t+j}. \tag{84}$$

models of credibility and search problems). To make least squares learning approach rational expectations, he used algorithms with $\frac{1}{\log n} < a_n < \frac{1}{\sqrt{n}}$. This led to a diffusion approximation to (72). For Cho's settings, the assumptions about gains that lead to the diffusion approximation are important in promoting sufficient experimentation to discover a rational expectations equilibrium. Cho obtained convergence in distribution to rational expectations by driving the weight on the noise term to zero. Kandori, Mailath, and Rob (1993) use related mathematics. Roger Myerson (1998) used an escape route calculation in a voting problem.

[9] The government behaves adaptively in the sense that it updates γ_t recursively via least squares, but forms its decisions using the same mapping $h(\cdot)$ of γ_t that was appropriate (i.e., optimal given its beliefs) in a self-confirming equilibrium in which γ was time invariant. This is what control scientists mean when they speak of (suboptimal) adaptive control, and is an example of Kreps's anticipated utility.

[10] For alternative ways to set up problems like this using active control, where the decision maker balances control and experimentation, see Edward C. Prescott (1967), Volker Wreiland (1995), and Kenneth Kasa (1996).

The content of (84) is that in forecasting the future at t, the government pretends that the coefficients γ_{t-1} will forever govern the dynamics.

The government's procedure for reestimation falsifies this pretense as it updates γ_t via the recursive least squares algorithm (RLS)

$$\gamma_t = \gamma_{t-1} + g_t R_{XC,t}^{-1} X_{Ct} \left(U_t - \gamma_{t-1}' X_{Ct} \right) \tag{85a}$$

$$R_{XC,t} = R_{XC,t-1} + g_t \left(X_{Ct} X_{Ct}' - R_{XC,t-1} \right). \tag{85b}$$

In (85), $\{g_t\}$ is a gain sequence of positive scalars. Formula (85) is a stochastic approximation algorithm that is a good method for estimating a γ that satisfies the moment conditions $EX_{Ct}(U_t - \gamma' X_{Ct}) = 0$ from (50) if (X_{Ct}, U_t) form a stationary stochastic process. The right side of equation (85a) is a weighting matrix $(R_{XC,t}^{-1})$ times the one-period value of the object $X_{Ct}(U_t - \gamma X_{Ct})$, whose expected value the orthogonality conditions set to zero. Equation (85b) is a recursive algorithm for estimating the second moment matrix of the regressors X_{Ct}. We obtain different models of learning with alternative gain sequences. For least squares, $g_t = 1/t$. So-called constant gain algorithms set $g_t = g_o > 0$ and thereby discount past observations. Discounting past observations is a good idea when the government believes that the Phillips curve wanders over time.

RLS and the Kalman filter

Alternative gain sequences $\{g_t\}$ affect the motion over time of the coefficients γ and can be interpreted in terms of connections between RLS and the Kalman filter. These connections are spelled out in an Appendix A. The Kalman filter uses Bayes' law to update beliefs about a hidden Gaussian state variable – in this case γ – as observations accrue. Recursive least squares corresponds to a special setting of parameter values when the Kalman filter is applied to the random-walk-in-coefficients model

$$\tilde{\gamma}_t = \tilde{\gamma}_{t-1} + w_t \tag{86}$$

where $\tilde{\gamma}$ is the value of the parameter in the time varying version of the model

$$U_t = \tilde{\gamma}_t' X_{Ct} + \varepsilon_{Ct} \tag{87}$$

where $Ew_t w_t' = R_{1t}, E\varepsilon_{Ct}^2 = R_{2t}$; $E\epsilon_{Ct}\epsilon_{Cs} = 0$ for $t \neq s$; $E\epsilon_{Ct}v_s = 0$ for all t and s; and $Ew_t w_s' = 0$ for all $t \neq s$. The Kalman filter assumes that beliefs are summarized by (86), (87); by given sequences of variances $\{R_{1t}, R_{2t}\}_{t \geq 1}$; and by initial conditions for $\gamma_0 = E\tilde{\gamma}_0$, $P_0 = E(\gamma_0 - \tilde{\gamma}_0)(\gamma_0 - \tilde{\gamma}_0)'$. The filter also assumes that $(\{w_t\}, \{\varepsilon_t\}, x_0)$ are jointly Gaussian. The Kalman filter is a recursive formula for $\gamma_t = E\tilde{\gamma}_t | J_t$, where J_t is the sigma algebra generated by observations through date t; and a recursive formula for $P_t = E(\gamma_t - \tilde{\gamma}_t)(\gamma_t - \tilde{\gamma}_t)'$. For our application, the information set J_t is the history $[U_s, y_s, s \leq t]$.

The RLS algorithm with a decreasing gain $g_t = \frac{1}{t}$ corresponds to the Kalman filter under the specification $R_{1t} \equiv 0$ and R_{2t} being set to a constant. The RLS with a constant gain $g_t = g_o$ corresponds to a Kalman filter under the specification that $R_{1t} = \left(\frac{g_o}{1-g_o}\right) P_{t-1}$, R_{2t} a constant. These comparisons describe the government's model uncertainty and are useful when we choose initial conditions for some simulations below.

Private sector beliefs

There are two convenient ways to model the public's beliefs in an adaptive model under the classical identification scheme for the Phillips curve. One is to follow Sims (1988) and Chung (1990) and let the public know the government's rule $h(\gamma_{t-1})$ at each t. Another would be to let the public also use a recursive least squares algorithm to estimate α_t each period in the public's forecasting model $y_t = \alpha_{t-1} X_{t-1} + e_{yt}$. The RLS algorithm would be

$$\alpha_t = \alpha_{t-1} + \ell_t R_{X,t}^{-1} X_t (y_t - \alpha_{t-1}' X_t) \tag{88a}$$

$$R_{X,t} = R_{X,t-1} + \ell_t (X_t X_t' - R_{X,t-1}), \tag{88b}$$

where $\{\ell_t\}$ is another gain sequence.

An advantage of assuming that the public knows $h(\gamma_{t-1})$ is that it eliminates α_{t-1} from the state of the system. We shall work with this reduced system. We can interpret it as the outcome of a system with Fed watchers who give the public an accurate estimate of the government's decision rule at each t.

System evolution

The dynamic system is completed by the law of motion that determines (U_t, y_t). The adaptive version of the model under the classical identification scheme assumes that unemployment continues to be determined by (48), but now inflation is determined by $y_t = h(\gamma_{t-1}) + v_{2t}$, instead of $y_t = h(\gamma) + v_{2t}$. This means that the entire system can be written in the form (53) with the amendment that γ_{t-1} appears on the right side at t, replacing γ. Under the assumption that the public knows the government's rule, the system formed by (85) and this adaptive counterpart to (48) is a stochastic difference equation in the variables $[\gamma_{t-1}, R_{XC,t-1}, U_t, y_t]$. The components $[\gamma_{t-1}, R_{XC,t-1}]$ of the system summarize the government's beliefs.

We want to study two aspects of the adaptive system: (1) the potential limit points of these beliefs under a learning rule with a gain that eventually behaves like $\frac{1}{t}$; and (2) how the system behaves under a constant gain learning rule. The mean dynamics determine what we know about $\frac{1}{t}$ gains. If they exist, the limit points are self-confirming equilibria. The escape route dynamics provide the interesting new behavior under constant gain algorithms. We mainly rely on computer simulations to study the constant gain dynamics.

Mean dynamics

Stochastic approximation methods deliver an ordinary differential equation (ODE) whose behavior about a fixed point contains much information about the limiting behavior of the stochastic difference equations governing our adaptive system. The ODE can be derived mechanically by imitating the steps we used in applying Margaret Bray's idea to our model in Chapter 3.[11]

Application of those steps to the system formed by (85) and the adaptive version of (48) leads to the ordinary differential equation system

$$\frac{d}{dt}\gamma = R_{XC}^{-1} M_{XC}(\gamma)\left[T(h(\gamma)) - \gamma\right] \tag{89a}$$

$$\frac{d}{dt}R_{XC} = M_{XC}(\gamma) - R_{XC}. \tag{89b}$$

In (89a) and (89b), $M_X(\gamma) = EX_{Ct}X_{Ct}'$ is computed from the stationary distribution of (48) at a fixed value of γ. A fixed point of the ODE (89) evidently satisfies

$$\begin{aligned} R_{XC} &= M_{XC}(\gamma) \\ \gamma &= T(h(\gamma)). \end{aligned} \tag{90}$$

Here R_{XC} is the unconditional covariance matrix for X_C computed at the associated value of γ satisfying the last two equations of (90). Because $\gamma = T(h(\gamma))$, a fixed point of the ODE is a self-confirming equilibrium.

Stochastic approximation

Under the assumption that the gain sequence $\{g_t\}$ eventually behaves like t^{-1}, the following points are true:

(a) If the beliefs $\{\gamma_t, R_t\}$ converge, they converge to a rest point of (89).

(b) If a fixed point of the ODE is locally unstable, then the beliefs $\{\gamma_t, R_t\}$ cannot converge to that fixed point.

(c) If the ODE is globally stable about a fixed point, then modified versions of the laws of motion for beliefs exist that make them converge almost surely to the fixed point.[12]

[11] See Marcet and Sargent (1989a, 1989b) and Ljung (1977) for the steps.

[12] To obtain global convergence, the algorithms must include a projection facility that interrupts the basic algorithm whenever the (γ, R_{XC}) threaten to move

(d) This item requires some prior bookkeeping. The state of the system (89) is the list (γ, R_{XC}) measuring beliefs. Let $\mathrm{col}(R_{XC})$ denote the vector formed by stacking successive columns of the matrix R_{XC}. Then by stacking columns of matrices on both sides of (89) we obtain the representation

$$\frac{d}{dt}\begin{bmatrix} \gamma \\ \mathrm{col}(R_{XC}) \end{bmatrix} = g(\gamma, R_{XC}).$$

Define the Jacobian

$$\mathcal{H}(\gamma, R_{XC}) = \frac{d}{d(\gamma, \mathrm{col}R_{XC})'} g(\gamma, R_{XC}).$$

Local stability of the ODE about a fixed point is governed by the eigenvalues of $\mathcal{H}(\gamma_f, R_f)$, where (γ_f, R_f) is a fixed point. If all eigenvalues of $\mathcal{H}(\gamma_f, R_f)$ have negative real parts, then (γ, R_f) is a locally stable rest point of (89). Under some technical conditions, including that the gains not be absolutely summable, the eigenvalues of $\mathcal{H}(\gamma_f, R_f)$ govern the rate of convergence of the algorithm to a self-confirming equilibrium. A necessary condition for convergence at the rate $T^{.5}$ is that the eigenvalues of $\mathcal{H}(\gamma_f, R_f)$ be bounded from above in modulus by $-.5$.[13] When the eigenvalue of maximum modulus of $\mathcal{H}(\gamma_f, R_f)$ is between $-.5$ and 0, convergence to a self-confirming equilibrium occurs, but at a slower rate than $T^{.5}$.

outside a domain of attraction of the ODE. See Marcet and Sargent (1989a) and Ljung (1977). Some of the simulations below activate a projection facility. Evans and Honkapohja (1998a) discuss the role of the projection facility and results that can be attained without it. When they dispense with the projection facility they sometimes attain convergence of least squares learning algorithms to rational expectations equilibria with probabilities that are positive but less than unity. They display cases with a globally stable ODE that have convergence with probability one in the absence of a projection facility.

[13] See Marcet and Sargent (1995).

When the gain g_t converges to a constant $\bar{g} > 0$, (γ, R_{XC}) converges to a stationary stochastic process, and not to a fixed set of beliefs.[14]

The constant gain algorithm arrests the force for convergence to a self-confirming equilibrium. This opens the possibility that the inflation outcome from an adaptive version of the model will produce outcomes different from the Nash outcome. I shall explore this possibility using computer simulations. Before looking at the simulations, I describe an adaptive model under the Keynesian identification.

Adaptation with Keynesian identification

Government beliefs and behavior

In the adaptive model where the government fits a Phillips curve in the Keynesian direction, each period the government updates its estimate of β in the conjectured Phillips curve (35). The government uses RLS to update β_t, leading to

$$\begin{aligned} \beta_t &= \beta_{t-1} + g_t R_{XK,t}^{-1} X_{Kt} \left[y_t - \beta_{t-1}' X_{Kt} \right] \\ R_{XK,t} &= R_{XK,t-1} + g_t [X_{Kt} X_{Kt}' - R_{XK,t-1}]. \end{aligned} \tag{91}$$

Here $\{g_t\}$ again is the gain sequence, set at $\{t^{-1}\}$ by least squares.

After applying the invert-the-Phillips-curve operator $\gamma(\beta)$ to β_t to get γ_t, the model works in the way it does with the classical identification scheme. Based on its beliefs, the government computes a feedback rule $h(\gamma_{t-1})$, where $\gamma_{t-1} = \gamma(\beta_{t-1})$. The government sets y_t according to $y_t = h(\gamma_{t-1})X_{t-1} + v_{2t}$.

[14] See Cho (1997) for a study of learning of reputational equilibria in games where beliefs converge to a stochastic differential equation. Cho restricts the gain and other features of the learning dynamics to make the innovation variance of the stochastic differential equation shrink fast enough to attain convergence in distribution to a particular equilibrium.

Technical details

The first equation of (91) can be written

$$\beta_t = \beta_{t-1} + g_t R_{XK,t}^{-1} X_{Kt} X'_{Kt} \left\{ \begin{bmatrix} 0 \\ h \end{bmatrix} - \beta_{t-1} \right\}$$
$$+ g_t R_{XK,t}^{-1} X_{Kt} v_{2t}.$$

Think of forming the regression $v_{2t} = \phi' X_{Kt} + \varepsilon_{\phi,t}$, where $\varepsilon_{\phi,t}$ is orthogonal to X_{Kt}. Evidently, for a stationary X_{Kt} process, ϕ satisfies $\phi = \left(E X_{Kt} X'_{Kt}\right)^{-1} E X_{Kt} v_{2t}$.

Stochastic approximation shows how the limiting dynamics of the system are described by the associated differential equation system

$$\frac{d}{dt}\beta = R_{XK}^{-1} M_{XK}(\beta) \left(\begin{bmatrix} 0 \\ h \end{bmatrix} + \phi - \beta \right) \qquad (92a)$$

$$\frac{d}{dt} R_{XK} = M_{XK}(\beta) - R_{XK} \qquad (92b)$$

where M_{XK} is the moment matrix calculated from (48) at fixed values of $h(\gamma(\beta))$. A rest point of the ODE is evidently

$$\beta = S(\beta)$$
$$R_{XK} = M_{XK}(\beta),$$

where $S(\beta)$ is as defined in (57). Comparing these equations to those for our self-confirming equilibrium with a Keynesian identification, we see that a rest point is a self-confirming equilibrium.

Simulations

I have simulated both the Keynesian and the classical adaptive systems, using the Kalman filter to implement recursive least squares by setting $R_{1t} = \left(\frac{g_o}{1-g_o}\right) P_{t-1}$ and R_{2t} to a constant. Appendix A shows how the constant gain satisfies $g_o = 1 - \lambda$

where $\lambda \in (0,1)$ is a discount factor applied to past observations. Appendix A describes the initial value of P_0 that starts the government with the prior of someone whose only source of information is T periods of data from within a self-confirming equilibrium; T parameterizes the tightness of the government's prior distribution. I set R_2 at the value of $E\varepsilon_{Ct}^2$ associated with the self-confirming equilibrium.

All of the simulations set parameters of the true data generating process at the values used to construct the example at the end of Chapter 7: $\rho_1 = \rho_2 = 0, U^* = 5, \delta = .98, \sigma_1 = \sigma_2 = .3$. I initiate the coefficients at self-confirming equilibrium values and start the covariance matrix P_0 at a value for the asymptotic covariance matrix of someone with data from T periods in a self-confirming equilibrium. As our computed-by-hand example showed, because the ρ_j's are zero, at self-confirming equilibria of both Keynesian and classical types, $[\,U_t \quad y_t\,]$ is a serially uncorrelated process. Except for details, this example is the same as Sims's (1988) and is a good laboratory for studying how adaptation can generate serial correlation though there is none in the fundamentals.[15] The value of the discount factor δ is irrelevant within a self-confirming equilibrium (because the ρ_j's are zero), although it becomes relevant for the behavior of the adaptive systems.

Classical adaptive simulations

In our simulation of the classical adaptive system, the government's Phillips curve is of the form $U_t = \gamma_{t-1} X_{Ct} + \epsilon_{Ct}$, where $X_{Ct} = [y_t, U_{t-1}, U_{t-2}, y_{t-1}, y_{t-2}, 1]$. Soon our attention will turn to the covariation of the estimated constant and the sum of weights on current and lagged inflation y.

Recall that the classical self-confirming equilibrium has serially uncorrelated fluctuations of (U, y) around means of $(5,5)$,

[15] The example is not identical to Sims's (1988) because he made the government fit a static Phillips curve. Therefore, my discussion of the activation of the induction hypothesis does not apply to Sims's work. Chung (1990) has the government fit a distributed lag Phillips curve.

and that the Keynesian self-confirming equilibrium has fluctuations around means of $(5, 10)$. Figures 8.1 and 8.2 report simulations of the classical system under two specifications, least squares and constant gain. Thus, in Figure 8.1, we set $\lambda = 1, T = 800$; in Figure 8.2, we set $\lambda = .975, T = 300$.

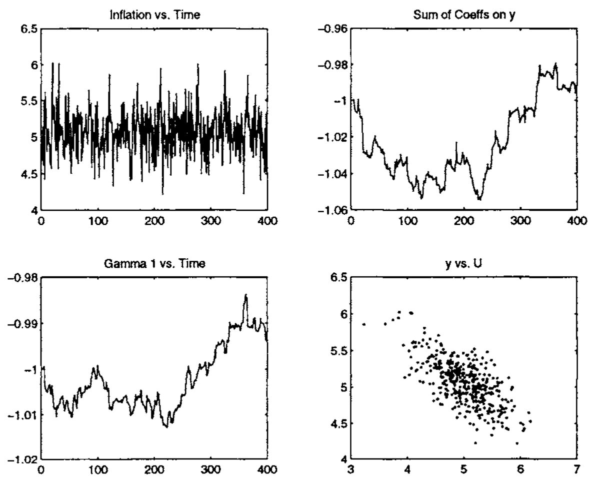

Figure 8.1. Simulation of classical adaptive model under decreasing gain (least squares).

The $\{t^{-1}\}$-gain Figure 8.1 resembles a simulation of a self confirming equilibrium. But the long (1,000 periods) Figure 8.2 simulation under constant gain displays aspects of behavior reported by Sims and Chung. The inflation panel shows that inflation starts near the self-confirming equilibrium value, but then drops almost to zero and stays there for a long time. Over time, inflation slowly heads back toward the self-confirming value of 5, only to be propelled back toward zero again. The mean dynamics that pull the system toward the self-confirming equilibrium are opposed by a recurrent force that sends the inflation rate close to the Ramsey outcome (zero inflation). Inspection of the simulated v_t series, which I do not report, revealed that no large shocks prompt the large initial stabilizations or the subsequent smaller ones.

Evidence that the mean dynamics operate slowly is supplied by the eigenvalues of the Jacobian $\mathcal{H}$ described above. For the model under the classical identification with these parameter values, there is a repeated eigenvalue of $-.5$ of multiplicity 6, as well as a repeated eigenvalue of -1 associated with the dynamics of the components of R_{XC}. Note that one repeated eigenvalue occurs at the boundary of the region within which convergence occurs at a rate corresponding to the square root of the sample size.

Other panels of Figure 8.2a and 8.2b assemble clues about features of the government's beliefs that prompt it to reduce inflation along the simulated sample path. The second and third panels of Figure 8.2a show time series of the sum of coefficients and the constant, respectively, in the estimated Phillips curve. How these two move together is the key to the behavior of inflation. During the first dramatic stabilization episode, the sum of coefficients on current and lagged y, shown in panel 2 of Figure 8.2a, jumps from its self-confirming value of -1 to nearly zero. Simultaneously, the constant in the estimated Phillips curve drops. These co-movements affect the solution of the Phelps problem, and cause the constant in the decision rule for inflation to behave as in the fourth panel. The constant in the decision rule h, shown in the fourth panel of Figure 8.2a, essentially determines the behavior of inflation. As we saw in Chapter 5, a value near zero of the sum of coefficients on inflation activates the induction hypothesis and makes the Phelps problem advise the government to reduce inflation. Values of the sum of coefficients near zero seem to occupy a dominating position along an escape route for most likely deviations away from a self-confirming equilibrium.

Figures 8.2b, 8.2c, and 8.2d add more information about the dynamics of the escape route from the self-confirming equilibrium.[16]

Figure 8.2b plots the sum of coefficients again, and below it the standard error of that sum from the time t covariance matrix of coefficients (denoted P_{t-1} in the Kalman filter formulation of Appendix A). It also plots the slope of the contemporaneous Phillips curve and its standard error. The behavior of the standard errors shows us why the system takes this particular route away from the self-confirming equilibrium. Within the self-confirming equilibrium, the standard error of the sum of weights is about .13. As the initial 80 observations accumulate, the system stays close to the self-confirming equilibrium, but the standard error of the sum of weights approximately doubles. Simultaneously, the standard error on the constant increases (shown in the third panel of Figure 8.2a). Now look at Figures 8.2c and 8.2d, which show 95% and 99% confidence ellipsoids around the sum of the coefficients on current and lagged inflation y (the ordinate) and the constant (the coordinate) in the estimated Phillips curve at various dates t. The vertical line at ordinate zero indicates where the induction hypothesis is satisfied. Figure 8.2d plots the confidence ellipsoids for dates surrounding the first stabilization. When near the self-confirming equilibrium early in the simulation, the confidence ellipsoid reveals a trade-off reflected in a negative correlation between the constant and the sum of the weights. Note how the ellipsoid is tilted along an axis that connects the self-confirming equilibrium value $(0, 10)$ with the value of the sum of weights of zero that would activate the induction hypothesis. Figure 8.2c

[16] As mentioned above, a deterministic intertemporal cost minimization problem determines a path around which the stochastic process for beliefs puts most probability. The state of beliefs is a large dimensional object, including the coefficient vector and the covariance matrix of that coefficient vector. The confidence ellipses in the text summarize the escape route from the simulations to highlight the action of the induction hypothesis. The moving confidence ellipsoids in Figure 8.2 and 8.3 are projections of the confidence ellipsoids of (γ, R_{XC}) and (β, R_{XK}), respectively, onto smaller spaces.

shows how the confidence ellipsoid grows as observations up to about period 80 accrue, while remaining centered at the self-confirming values. The growing confidence ellipsoids makes the government's econometricians more open to interpreting the data as consistent with Solow and Tobin's distributed lag version of the natural-rate hypothesis.[17] By chance, this happens in the eighties, as Figure 8.2d shows when the Phelps problem directs the government to stabilize.

Figures 8.2c and 8.2d show how after the stabilization occurs, the government's behavior generates a string of serially correlated observations that affirm the induction hypothesis. Notice how the confidence ellipsoids center around the induction hypothesis and how they shrink. Evidently, the stabilization generates observations that temporarily add credibility to the induction hypothesis that prompted it. Our earlier analysis shows that this situation is not self-confirming in the technical sense but is nevertheless reinforcing.

[17] The growing confidence ellipsoids make it as likely that the data drive the estimated Phillips curve away from the induction hypothesis as towards it. The behavior induced by the Phelps problem prevents persistent movements in a direction pointing away from the induction hypothesis.

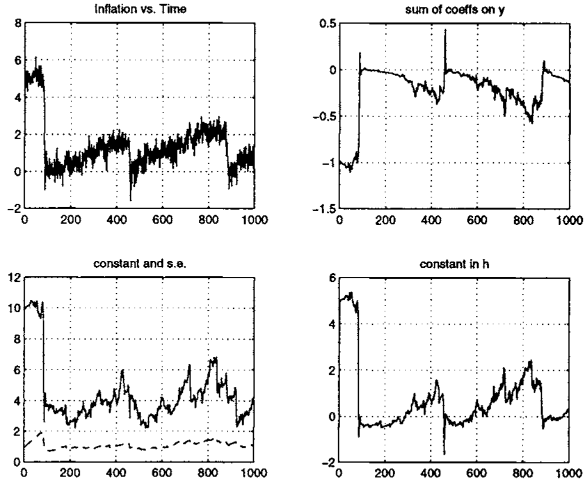

Figure 8.2a. Simulation of classical adaptive model under constant gain, $\delta = .98, \lambda = .975$.

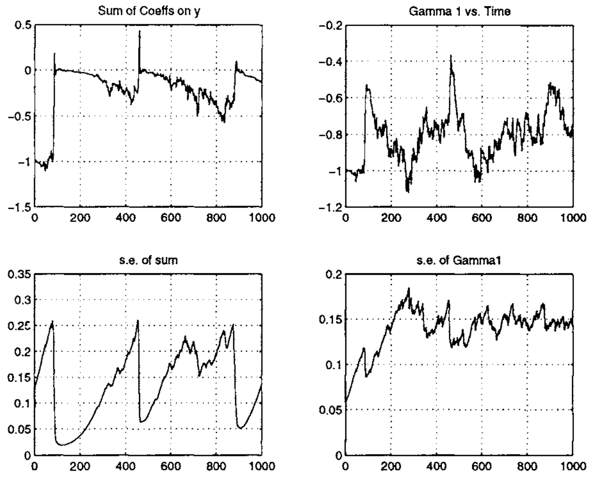

Figure 8.2b. Simulation of classical adaptive model under constant gain, $\delta = .98, \lambda = .975$.

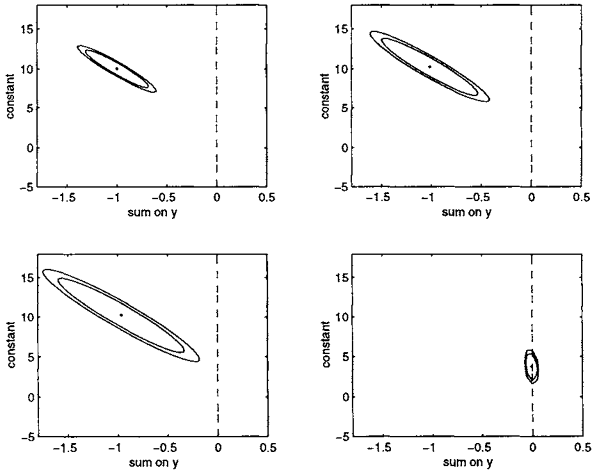

Figure 8.2c. Confidence ellipses around sum of weights on y and constant in Phillips curve, evaluated at observation numbers 5, 40, 80, and 100, for classical adaptive model.

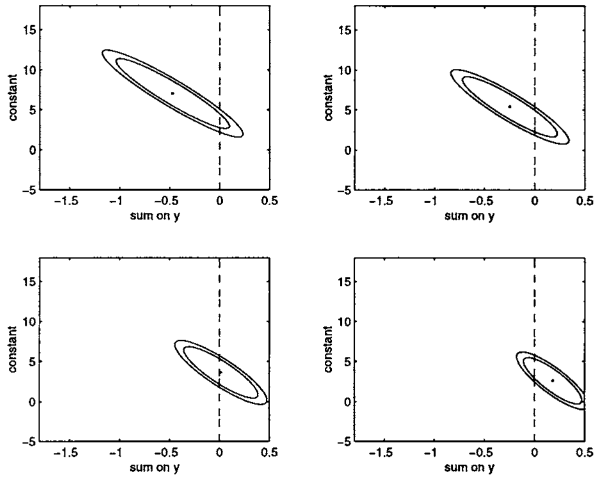

Figure 8.2d. Confidence ellipses around sum of weights on y and constant in Phillips curve, evaluated at observation numbers 84, 85, 86, 87, for classical adaptive model.

Relation to equilibria under forecast misspecification

The simulation of the adaptive system exhibits a feature reminiscent of the Chapter 6 and 7 equilibria with optimal misspecified forecasts. There, an incorrect forecasting model without a constant but with a unit root in the inflation forecasting equation, could closely approximate a true model that includes a constant. The sense of approximation there was subtle, because of how both the true and approximating models influenced one another.[18] An approximation mechanism with a similar flavor operates during periods in our simulations that have near-Ramsey outcomes. The approximation is not to a fixed model for the inflation process, but to one that changes as the government's beliefs influence the actual inflation process through the Phelps problem.

[18] Recall how the adaptive expectations parameter C disappeared from the list of free parameters.

Simulation with Keynesian adaptation

In simulations of the adaptive Keynesian system, the government's estimated Phillips curve is $y_t = \beta_{t-1} X_{Kt} + \epsilon_{Kt}$, where $X_{Kt} = [U_t, U_{t-1}, U_{t-2}, y_{t-1}, y_{t-2}, 1]$. As in our study of the system under the classical identification scheme, I focus on the covariation of the estimate of the constant and the sum of weights on lagged inflation.

Figures 8.3a–8.3d display a simulation of the adaptive system under the Keynesian identification. This simulation sets $\lambda = .99, T = 300$. The inflation rate starts with a long spell near the self-confirming value of 10, then after a period of turbulence drops to near the Ramsey level and stays there for a long time. There is eventually a burst of inflation back toward a neighborhood of the self-confirming value, but this episode is again followed by a long spell near the Ramsey outcome. The remaining panels of Figures 8.3a and 8.3b again allow us to interpret the stabilization in terms of the eventual activation of the induction hypothesis, together with movements in the estimated slope γ_1 of the short-run Phillips curve. The system evidently spends much time away from the self-confirming equilibrium, and, like the classical system, recurrently escapes to the Ramsey outcome.

For the system under the Keynesian identification scheme, the mean dynamics in the vicinity of the self-confirming equilibrium issue a warning: the Keynesian counterpart to the Jacobian $\mathcal{H}$ contains a repeated eigenvalue of $-.2$.[19] The $-.2$ eigenvalue leads us to expect a very slow rate of convergence to a self-confirming equilibrium under least squares learning. This sustains long episodes near the Ramsey outcome, because it evidently takes much time for the mean dynamics to drive the system away from the Ramsey outcome.

The various panels of Figure 8.3 again shed light on the escape route away from the self-confirming equilibrium. They tell

[19] Of multiplicity six. There is a repeated eigenvalue of -1 associated with R_{XK}.

a story much like that for the classical identification scheme. The early part of the sample has inflation near the self-confirming equilibrium value. But the data from this period foster growing doubt about the location of the Phillips curve and put higher weight in the direction of the induction hypothesis which, with the Keynesian identification, manifests itself when the sum of the weights on lagged inflation equals one. The confidence ellipsoids gradually spread, while remaining centered on the self-confirming equilibrium values. Eventually, by chance some observations arrive that push the government's estimated Phillips curve toward the induction hypothesis. After the Phelps problem induces a stabilization, the confidence ellipsoids quickly move toward and collapse around the induction hypothesis.

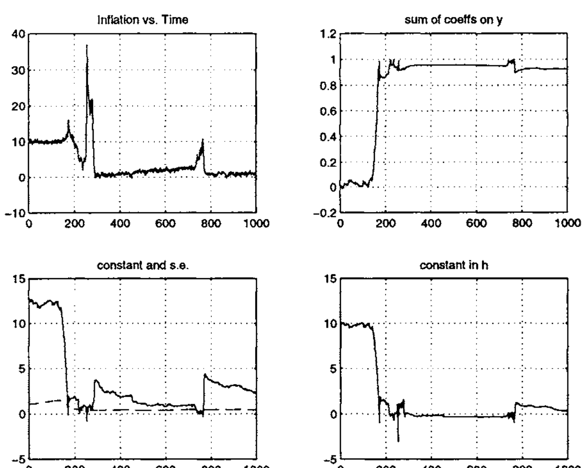

Figure 8.3a. Simulation of Keynesian adaptive model under constant gain, $\lambda = .99$.

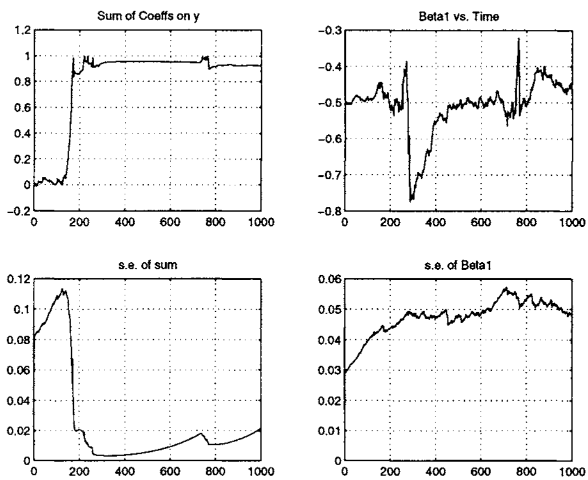

Figure 8.3b. Simulation of Keynesian adaptive model under constant gain, $\lambda = .99$.

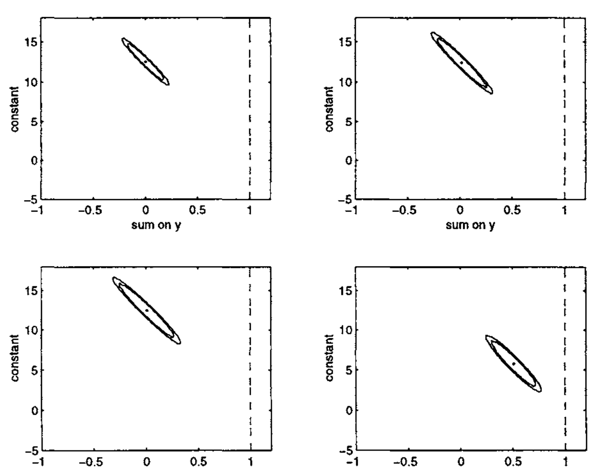

Figure 8.3c. Confidence ellipses around sum of weights on y and constant in Phillips curve, evaluated at observation numbers 2, 80, 120, and 160, for Keynesian adaptive model.

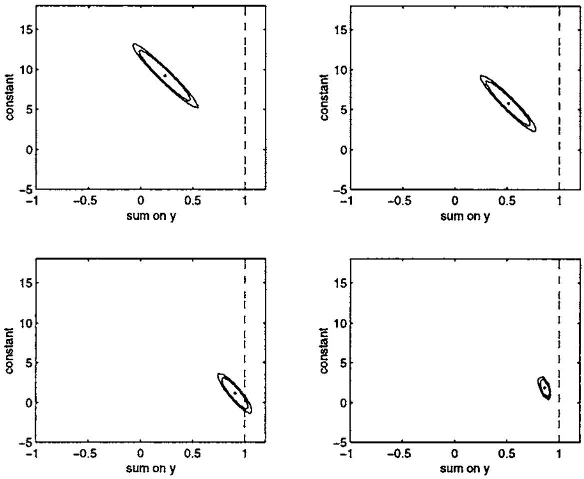

Figure 8.3d. Confidence ellipses around sum of weights on y and constant in Phillips curve, evaluated at observation numbers 150, 160, 170, and 200, for Keynesian adaptive model.

Role of discount factor

The recurrent stabilizations of inflation toward the Ramsey value of 0 depend on the value of the discount factor δ being near 1. Other simulations that I have performed, but do not report, show how decreasing δ causes the value of inflation observed during the recurrent periods of low inflation to rise. This pattern is consistent with the workings of the Phelps problem in conjunction with the induction hypothesis.

Conclusions

My simulations have features that Sims (1988) and Chung (1990) reported for related systems. For long periods, adaptive governments learn to generate better than Nash or self-confirming outcomes. These results come from the recurrent dynamics induced by adaptation. The mean dynamics that under least squares drive the system toward a self-confirming equilibrium continue to operate under adaptation, but noise lets the adaptive system recurrently escape from a self-confirming equilibrium.[20] Starting from a self-confirming equilibrium, an adaptive algorithm gradually makes the government put enough weight on the induction hypothesis that chance observations eventually promote better than Nash outcomes.

Adaptation makes the government's beliefs a hidden state imparting serial correlation into $[\,U_t \quad y_t\,]$. An outside forecaster would do well to use a random coefficients model, or to make the constant adjustments noted by Lucas (1976).

Our adaptive models thus contain underpinnings for vindicating econometric policy evaluation. It is time to leave the laboratory and turn to history. In the next chapter I take the historical data as inputs and use them to generate parameter estimates and residuals. How the model matches the data, and how it misses, will vindicate or indict econometric policy evaluation.

Appendix A: RLS and the Kalman filter

The Kalman filter

Recall the basic Kalman filter from Ljung and Söderström (1983) or Ljung (1992).[21] The Kalman filter is a recursive algorithm for computing least squares estimators of a sequence of hidden state vectors described by a linear stochastic difference equation. The statistical model governing the hidden state ξ_t

[20] The mechanism of escape is similar to the switches between neighborhoods of rational expectations equilibria that occur in Evans and Honkapohja (1993), except here the escape from an equilibrium is not to another equilibrium.
[21] See Ljung, Pflug, and Walk (1992).

and the observations z_t, ϕ_t follows. Let ξ be a vector of coefficients and φ a vector of regressors. The model is

$$\xi_t = \xi_{t-1} + w_t \ , \quad E w_t w_t' = R_{1t} \tag{93a}$$

$$z_t = \xi_t' \varphi_t + e_t \ , \quad E e_t e_t' = R_{2t}. \tag{93b}$$

Here ξ_t, φ_t are each $(n \times 1)$. The aim is to compute $\hat{\xi}_t \equiv E\xi_t | J_t$, where J_t is the time t information set consisting of $z_s, \phi_s, s = 0, \ldots, t$, and $E(\cdot)$ is the least squares projection operator. In our applications, in the classical case, z_t is U_t, φ_t is X_{Ct}, and ξ_t is γ_t; in the Keynesian case, z_t is y_t, φ_t is X_{Kt}, and ξ_t is β_t.[22]

The Kalman filter takes the form of the following recursions for $\hat{\xi}_t, P_t$:

$$\hat{\xi}_t = \hat{\xi}_{t-1} + L_t \left[z_t - \varphi_t' \hat{\xi}_{t-1} \right] \tag{94a}$$

$$L_t = \frac{P_{t-1} \, \varphi_t}{R_{2t} + \varphi_t' \, P_{t-1} \, \varphi_t} \tag{94b}$$

$$P_t = P_{t-1} - \frac{P_{t-1} \, \varphi_t \, \varphi_t' \, P_{t-1}}{R_{2t} + \varphi_t' \, P_{t-1} \, \varphi_t} + R_{1t}, \tag{94c}$$

where $P_t = E(\xi_t - \hat{\xi}_t)(\xi_t - \hat{\xi}_t)'$. The recursions are initialized from $P_0, \hat{\xi}_0$.

Consider the case of homoskedastic measurement noise with $R_{2t} = R_2$. It is useful to note that the structure of (94) implies that all that matters in terms of the initial condition is the ratio $\frac{P_0}{R_2}$, so that with an appropriate choice of P_0, R_2 normalizes.

Recursive least squares

In special cases, the Kalman filter leads to a stochastic approximation algorithm called recursive least squares (RLS). Letting the prediction error at the true parameter at time t be $\epsilon(t, \xi) = z_t - \phi_t' \xi$, RLS minimizes a loss function

$$V_T(\xi) = \sum_{t=1}^{T} \kappa_{T,t} \ell(\epsilon(t, \xi), t)$$

[22] Thus in the respective cases $\hat{\gamma}_t = E_t \, \gamma_t$ or $\hat{\beta}_t = E_t \, \beta_t$. Note that $\hat{\gamma}_t$ or $\hat{\beta}_t$ is conditioned on time t information.

where $\epsilon(t,\xi) \equiv z_t - \phi_t\xi$ and $\kappa_{T,t}$ discounts past observations geometrically according to

$$\kappa_{T,t} = \lambda^{T-t}, \quad 0 < \lambda \leq 1$$

and where the one-period loss function is $\ell_t = \alpha_t(z_t - \phi_t'\hat{\xi}_{t-1})^2$. The RLS algorithm is

$$\hat{\xi}_t = \hat{\xi}_{t-1} + \alpha_t g_t R_t^{-1}\phi_t\left(z_t - \phi_t'\hat{\xi}_{t-1}\right)$$

$$R_t = R_{t-1} + g_t\left(\alpha_t\phi_t\phi_t' - R_{t-1}\right),$$

where the gain g_t is defined as

$$g_t = \left[\sum_{s=1}^{t} \kappa_{t,s}\right]^{-1}.$$

A constant forgetting factor $\lambda < 1$, leads to $g_t = \left(\frac{1-\lambda^t}{1-\lambda}\right) \rightarrow (1-\lambda)$ as $t \rightarrow +\infty$; $\lambda = 1$ leads to $g_t = \frac{1}{t}$.

Matching RLS to the Kalman filter

Ljung (1992) pointed out that for a special choice of R_{1t}, R_{2t}, the Kalman filter becomes the RLS algorithm. In particular, let

$$R_{1t} = (1/\lambda - 1)P_{t-1} \tag{95a}$$

$$R_{2t} = \lambda/\alpha_t. \tag{95b}$$

For us, the following two cases are interesting. First, when $\lambda = 1$, $R_{1t} = 0$, and $R_{2t} = 1/\alpha_t$. Second, when $\lambda < 1$, R_{1t} is proportional to the last estimate P_{t-1}.

We will be considering homoskedastic cases in which $\alpha_t = \alpha$ is constant, which makes the RLS algorithm take the form of the stochastic approximation algorithm

$$\hat{\xi}_t = \hat{\xi}_{t-1} + g_t R_t^{-1}\phi_t(z_t - \phi_t'\hat{\xi}_{t-1})$$

$$R_t = R_{t-1} + g_t \left(\phi_t \phi_t' - R_{t-1} \right).$$

As Ljung (1992) remarked, by leading to a constant gain g_t, a forgetting factor operates much like a nonzero R_{1t} coupled with a gain that eventually decreases as $1/t$.

We get an adaptive model by assuming that at time t, $\hat{\gamma}_{t-1}$ is used to solve the Phelps problem, which implies

$$y_t = h(\hat{\gamma}_{t-1}) X_{t-1} + v_{2t}$$

where $\hat{\gamma}_{t-1}$ is based on information up through $t-1$, namely U_{t-1}, X_{Ct-1}, and recall that X_{t-1} includes only information dated $t-1$ and earlier.

Initial conditions for simulations

For our simulations, we shall set R_{1t}, R_{2t} according to (95) with an initial P_0 chosen as follows for the classical case. Associated with a self-confirming equilibrium is a moment matrix $Q = EX_{Ct}X_{Ct}'$, and an associated variance decomposition

$$EU_t^2 = \gamma' Q \gamma + E\varepsilon_{Ct}^2. \tag{96}$$

We set $R_{2t} = \sigma_{Ct}^2 \equiv E\varepsilon_{Ct}^2$, and choose $P_0 = \frac{\sigma_{Ct}^2}{T} Q^{-1}$, where T is a positive integer.[23] This sets P_0 equal to the value of the covariance matrix that would be estimated for γ by a government that had observed a data record of length T from a self-confirming equilibrium. This procedure gives us a simple one-parameter specification of the government's initial uncertainty about γ.

For the Keynesian model, we set $Q = EX_{Kt}X_{Kt}'$ from a self-confirming equilibrium, and chose R_{2t} according to the variance decomposition like (96) for the Keynesian direction of fit.

[23] Note how with this setting for P_0, the choice of σ_{Ct}^2 has no influence on the behavior of the Kalman filter (94), and amounts just to a normalization.

Appendix B: Anticipated utility

David Kreps (1998) calls our adaptive models of the 1990's anticipated utility models. These models induce transient dynamics by assuming that decision makers adapt a temporarily misspecified model to incorporate the most recent observations and reoptimize along the way. This is a small modification of a rational expectations.

Boiler plate recursive rational expectations model

Assume the following rational expectations model as a base. A representative agent solves the problem whose Bellman equation is

$$v(k,K) = \max_u\{R(k,K,u) + \beta E v(k',K')\} \tag{97}$$

subject to the transition laws

$$k' = g(k,u,\epsilon') \tag{98}$$

$$K' = G(K,\epsilon'), \tag{99}$$

where $R(k,K,u)$ is the one period return function and u is the agent's control vector, and $\beta \in (0,1)$. Variables without primes denote this period values and those with primes denote next period values; ϵ is a random vector drawn from the c.d.f. $\text{Prob}(\epsilon \leq \bar{\epsilon}) = F(\bar{\epsilon})$.

Here k is the state vector of the representative agent, and K is the average state vector over all agents. The solution of the problem is a timeless policy function

$$u = p(k,K;G). \tag{100}$$

Substituting (100) into the law of motion (99), and setting $k = K$ gives

$$K' = g(K, p(K,K;G), \epsilon') \equiv T(G)(K,\epsilon'). \tag{101}$$

This constructs a mapping $T(G)$ from the perceived law of motion G for K to the actual law of motion $T(G)$ for K. A rational expectations equilibrium is a fixed point of this mapping.

Notice that calendar time makes no explicit appearance in this description. The laws that link next period values (with primes) to present ones are timeless. An anticipated utility model makes calendar time appear in those laws by withdrawing knowledge of G.

Anticipated utility model

In Kreps's (1998) sense, an anticipated utility model for this setting could be obtained as follows. In place of (99), the representative agent believes

$$K' = G_t(K, \epsilon'), \tag{102}$$

where for the first time in this appendix, calendar time t appears. The perceived law of motion G_t is formed by an adaptive scheme feeding back on the history of outcomes:

$$G_t = f(G_{t-1}, K_{t-1}). \tag{103}$$

The actual law of motion for the system at t would be

$$K_{t+1} = T(G_t)(K_t, \epsilon_{t+1}), \tag{104}$$

where $T(G)$ is the same operator T defined implicitly via (101), namely, the one derived assuming that G was time-invariant.

We make several observations:

(1) Looked at one way, the recursive structure makes the original rational expectations equilibrium static. The same Bellman equation describes the agent's optimum problem at all dates. These dynamics are a special case of statics.

(2) Kreps's term anticipated utility applies because the setup injects dynamics only by making the representative agent's continuation value time-dated by becoming $E_t v_t(k', K')$, where E_t denotes the expectation, conditional on k, K, taken with

respect to the probability distribution induced by the time t model G_t.

(3) Because it retains the same functional $p(K, k; G)$ mapping the belief G into the agent's policy, the anticipated utility model is misspecified, at least temporarily (but see point (4)). The misspecification comes from the fact that the functional $p(K, k; G)$ was derived on the assumption of a time-invariant G.

(4) The model is arranged so that, depending on details of the recursive learning scheme (103), the misspecification vanishes as $t \to \infty$, because $G_t \to G = T(G)$.

(5) Compared to Bayesian or robust decision makers, these agents ignore their period-by-period model misspecification. A robust decision maker would exercise caution in the face of misspecifications by using a worst case analysis. A Bayesian decision maker would know more about the environment.[24]

[24] See Bray and Kreps (1986).