Work after 9 Nov 2019

Form the FEs correctly:

$$f_t^m = \hat{E}_t(\pi_{t+1} \mid \mathcal{I}_t^m) = \hat{E}_t(\pi_{t+1} \mid s_t, \bar{\pi}_{t-1})$$

$$f_t^e = \hat{E}_t(\pi_{t+1} \mid \mathcal{I}_t^e) = \hat{E}_t(\pi_{t+1} \mid s_t, \bar{\pi}_t)$$

$$\left.\begin{array}{l} FE_t^m = \pi_{t+1} - f_t^m \\ FE_t^e = \pi_{t+1} - f_t^e \end{array}\right\}$$ So in either case I just need to subtract F_t from $ysim_{t+1}$.

and note that these FE_t are realized at $t+1$.

The problem is that the FEs I construct this way aren't equal to the ones I get from the sim_learn.m code. This is puzzling b/c in principle they come from the same simulated π — same fcst of π.

The problem is that the FE coming out of sim_learn

1) always changes, despite IRF-ing & averaging
2) There are diffs btwn $FE^{shocked}$ & $FE^{unshocked}$ even before I impose the innov δ (!)

FEs are solved. 14 Nov 2019

I think that the criss-crossing of fcsts is well-understood: when cgain, you update your fcst too much and so your FE switches sign and voilà.
At a certain point, your FE is small enough so that no overupdating of expectations happens any more.

→ my bet is you can kill this overupdating / crisscrossing w/ a sufficiently low gain

Yes, w/ $\bar{g} = 0.1$ (instead of 0.145) you already have dgain & cgain similar at $t=5$

w/ $\bar{g} = 0.0145$ they're identical at $t=25$ too

- But you always get some overshooting, whether it's in the 2nd period (cgain) or later on (dgain)
- Moreover, it's puzzling that $i \uparrow$ as $\pi < 0$ in 2nd period

One way to get perfectly normal, RE-like responses is to set $\alpha = 1$ b/c then $fa \approx fb$. But even $\alpha = 0.99$ gets a quite sig diff btwn fa & fb & overshooting too!

$$f_a = \frac{a}{1-\alpha\beta} + b\left(I_{nx} - \alpha\beta\, hx\right)^{-1} s$$

What is $\frac{1}{1-\alpha\beta} = 50.2573$ and $\frac{1}{1-\beta} = 100$

for $\alpha = \beta = 0.99$?

$$(I_4 - \alpha\beta\, hx)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2.4275 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.0049 & 2.3069 & 1.4618 & 1 \end{bmatrix}$$

and $$(I_4 - \beta\, hx)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2.4631 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.0050 & 2.3643 & 1.4776 & 1 \end{bmatrix}$$

ha! The diff in f_a & f_b is most pronounced in the part that comes from the intercept!

I think this would change a bit as α moves away from 1 (PTO)

$\alpha = 0.5$, $\beta = 0.99$

$$\frac{1}{1-\alpha\beta} = 1.9802 \qquad \text{and} \qquad \frac{1}{1-\beta} = 100$$

$$(I_4 - \alpha\beta h_x)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.4225 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.0025 & 0.6827 & 0.7388 & 1 \end{bmatrix}$$

$$\text{and } (I_4 - \beta h_x)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2.4639 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.0050 & 2.3643 & 1.9776 & 1 \end{bmatrix}$$

Yes, now the part relating the slope is diff, but the part relating the intercept is even more diff!

$\Rightarrow$ <u>The lower α (the higher k, the less price stickiness)</u>
<u>the more f_b loads on the intercept both in absolute terms</u>
<u>(f_b reacts more) and relative terms (vs. the slope)</u>

$\Rightarrow$ This may be driving (some of) the overshooting b/c for the std param value of $\alpha = 0.5$, f_b is almost 50 times more driven by the intercept than the slope + shocks. ⇒ so overreaction in updating the intercept drives f_b, which is what drives x_t up for α gain.

Let's interpret

$$
\alpha\beta \cdot hx \quad \text{vs} \quad \beta \cdot hx
$$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.297 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0025 & 0.48 & 0.7388 & 0 \end{bmatrix}
\quad
\begin{array}{c} r^n \\ \bar{i} \\ u \\ i_{t-1} \end{array}
\quad
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.594 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.005 & 0.96 & 1.4776 & 0 \end{bmatrix}
$$

2 differences

$\left.\begin{array}{l} \text{1) effect of shocks on } i_t \\ \text{2) effect of } \bar{i} \text{ on } s \end{array}\right\}$ $\alpha < 1$ mutes these somewhat

→ you're just discounting shocks in the future more!

Also, when $\alpha = 1$, then $f_a(1)$ doesn't matter for (x, π)

$\text{“} E\pi$

The puzzling $i\uparrow$ at $t=2$

$A_a(3,1) = 0.5928 \rightarrow i\uparrow$ if $f_a(1) \uparrow$ $\text{“} E\pi$

$A_b(3,1) = -0.0978 \rightarrow i\downarrow$ if $f_b(1) \uparrow$

$\Rightarrow$ so when f_b moves a lot more than f_a, (which is in general not true for avg dgain), then $i\uparrow$ even when $E[\pi]$ is $\downarrow$ (!)

But why?

if $\psi_x \uparrow$ (now it's 0) then $A_a(3,1)\downarrow$ and $A_b(3,1)\downarrow$ too!

But $A_a(3,1)$ never <0, not even for $\psi_x = 5$.

when $\psi_x = 0$, it's b/c $\pi\downarrow$ when $f_b\uparrow$

$\rightarrow$ it seems like $i\uparrow$ in $t=2$ b/c π is $\uparrow$ from $t=1$ to $t=2$

:S

$$A_a(3,1) = \psi_\pi A_a(1,1) + \underset{=0}{\psi_x A_a(2,1)}$$

$$\underbrace{A_b(3,1) = \psi_\pi A_b(1,1)}_{\text{these are true!}}$$

$\Rightarrow$ ah I see: $i \uparrow$ at $t=2$ b/c it was going up much more at $t=1$ due to the innovation, but since π fell so much, this depressed i a lot. At $t=2$, since $\pi \uparrow$ (but is still <0), i is depressed below $0.6 \cdot 1$ " δ (it's only $\simeq 0.1$) but it's not depressed as much

Puzzling i-response

$$i_t = \underset{\downarrow}{\pi} + \underset{\uparrow}{\text{innovation } (\delta)}$$

Initially $|\delta| > |\pi|$

At $t=2$ $|\pi|$ shrinks so $i \uparrow$

What remains to be understood is why the overshooting happens regardless, just later:

- maybe whats going on is that $\downarrow E[\pi]$ are pushing stuff up but $i\uparrow$ is pushing them down, and i recedes faster

Check: if i-shock is iid, overshooting should happen at $t=2$

$\Rightarrow$ exactly, and it does!

The only thing that isn't a 100% clear is why the $\ominus$ reaction to expectations, when RE doesn't have this?

In RE:
$$x_t = E_t x_{t+1} - \sigma E_t(i_t - \pi_{t+1})$$
$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}$$
$$i_t = \psi_\pi \pi_t$$

$$x_t = -\sigma i_t + E_t x_{t+1} + \sigma E_t \pi_{t+1}$$

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}$$

$$x_t = -\sigma \psi_\pi \pi_t + E_t x_{t+1} + \sigma E_t \pi_{t+1}$$

$$x_t = -\sigma \psi_\pi \left[\kappa x_t + \beta E_t \pi_{t+1}\right] + E_t x_{t+1} + \sigma E_t \pi_{t+1}$$

$$x_t = -\sigma\psi_\pi\left[\kappa x_t + \beta E_t\pi_{t+1}\right] + E_t x_{t+1} + \sigma E_t\pi_{t+1}$$

$$(1+\sigma\psi_\pi\kappa)x_t = -\sigma\psi_\pi\beta E_t\pi_{t+1} + E_t x_{t+1} + \sigma E_t\pi_{t+1}$$

$$x_t = \underbrace{\frac{1}{w}\sigma(1-\psi_\pi\beta)}_{<0\,(!)}E_t\pi_{t+1} + \frac{1}{w}E_t x_{t+1}$$

$\Rightarrow$ RE has it too, only $E[\cdot]$ don't move as much

$$\Rightarrow \pi_t = \frac{\kappa}{w}\sigma(1-\psi_\pi\beta)E_t\pi_{t+1} + \frac{\kappa}{w}E_t x_{t+1} + \beta E_t\pi_{t+1}$$

$$\pi_t = \underbrace{\left[\frac{\kappa}{w}\sigma(1-\beta\psi_\pi)+\beta\right]}_{\text{likely }>0}E_t\pi_{t+1} + \frac{\kappa}{w}E_t x_{t+1}$$

(for current params $= 0.9298$)

RE:

$$x_t = \ominus E_t\pi_{t+1} + \oplus E_t x_{t+1}$$

$$\pi_t = \oplus E_t\pi_{t+1} + \oplus E_t x_{t+1}$$

Learning

$$x_t = \ominus E_t\pi^{fa}_{t+1} + \ominus E_t\pi^{fb}_{t+1} + \ominus E_t x^{fa}_{t+1} + \oplus E_t x^{fb}_{t+1}$$

$$\pi_t = \oplus E_t\pi^{fa}_{t+1} + \ominus E_t\pi^{fb}_{t+1} + \oplus E_t x^{fa}_{t+1} + \oplus E_t x^{fb}_{t+1}$$

$$x_t = \ominus E\pi + \ominus EX \qquad (1)$$

this is not a mistake

$$\pi_t = \oplus E\pi + \oplus EX \qquad (2)$$

① $\frac{-\vartheta\psi_\pi}{w} k\alpha\beta + \frac{1}{w}(1-\beta) = \frac{1-\beta-\vartheta\psi_\pi k\alpha\beta}{w} < 0$

② $\left(1-\frac{k\vartheta\psi_\pi}{w}\right)(1-\alpha)\beta + \frac{k}{w}\vartheta(1-\beta\psi_\pi) > 0$

⇒ why do we have this diff btwn RE & Learn? 15 Nov 2019

$\vartheta\psi_\pi k\alpha\beta \overset{!}{<} 1-\beta$

$\vartheta\psi_\pi k\alpha\beta + \beta < 1$

$(\vartheta\psi_\pi k\alpha + 1)\beta < 1$ but it's 1.1150.

In the RE world, x depends on E(x) only directly

My conjecture is that $E(\pi)$ in RE will incorporate $E(x)$ in some way. So π must depend stronger on $E(\pi)$ in RE than in learning.

RE: π on $E(\pi)$: $\frac{k\vartheta+\beta}{\vartheta\psi_\pi} = 0.7722$ under current params

Learn: $\left(1-\frac{k\vartheta\psi_\pi}{w}\right)(1-\alpha)\beta + \frac{k}{w}\vartheta(1-\beta\psi_\pi) = 0.33$ –"–

→ so yes, this is true

In fact, you can reason that in RE,

$\pi = \underset{+}{E(\pi)}$ only b/c only via

while in learning

$\pi = \underset{+}{E(\pi)}$ but part of this is $+ \rightarrow$ from f_a

$- \rightarrow$ from f_b

<u>Ryan meeting</u> <u>15 Nov 2019</u>

fix point where find the gain that min Var(FE)

data is generated by a gain = 0.145 gain,

given this, let an agent set a best gain

→ it must be lower than 0.145 !

Analog to RE for the gain param.

↳ Pooya Molavi's JMP does this, using a

Kullback - Leibler distance.

Stage 0: Establish that again learning causes excess volatility.

0.1: Do learning rule where I don't do RE-part

$PLM = \bar{\pi}_{t-1}$ and that's it.

0.2: Try learning the slope.

=> in those contexts, do I continue to get the flipflops?

2. Didn't quite get to the bottom of RE vs. learning loading on E(·)

→ connect to those equations

3. These features can become worse if I $\uparrow \psi_\pi$.

Do it in a week. Schedule to talk to Basu after.

Tell him: in learning models, there's this endemic instability. This can become worse if $\psi_\pi \uparrow$.

Here's how it works.

Work after

15 Nov: did "only-mean" PLM and "slope & constant"

✓ • check that the latter is correct

✓ • print figs w/o cutting them off

• polish explanation of $E(\cdot) \to z$

• do the fixed point thing, use Molavi.

Reading Molavi JMP

16 Nov 2019

I'm not superimpressed b/c the "constrained RE eqb" (CREE) is really just saying that give agents a set of models Θ and let them choose (their expectation formation) the subset Θ^* that min $H(\cdot)$ where $H(\theta, T)$ is the Kullback-Leibler distance between model θ and the ALM T. [Sometimes Θ^* is a singleton, Molavi calls this a pure CREE.)

He shows (thm 2 & 3) that Bayesian & adaptive learning coincide w/ a CREE in the LR

→ of course b/c the CREE must be = REE if the REE $\in \Theta$! This is why Molavi says that the "LR behavior of the econ is independent of ... the learning process" b/c they all converge to REE! (Unless they are not E-stable, which is the analog of Molavi's concept of REE $\notin \Theta$, i.e. when agents don't include the REE in the set of models they consider.)

→ ok so trying to solve for $\bar{g}^*$

$\bar{g}^* = \arg\min$ FEV

I have 2 ways to construct FEV.

1) analytically (don't know if possible)

2) numerically in Matlab

↳ here I'm confused whether the FEV is across time or cross-section ← I suppose that.

$$Var(X) = E(X^2) - E(X)^2$$

If $X = FE$, then $E(X) = 0$, so $FEV = E(FE^2)$

$$\mathrm{var}(X) = E\Big[\big(\underbrace{X - E[X]}_{FE}\big)^2\Big]$$

this is why

$FEV(x) = \mathrm{var}(X)$ when you initialize!

Otherwise $FEV = E\Big[\big(X_{t+k} - \underset{\substack{\uparrow \\ \text{fcst}}}{X_{t+k,t}}\big)\big(X_{t+k} - X_{t+k,t}\big)'\Big]$

Ok let's clarify one thing:

$$\underbrace{\mathrm{Var}(X) = E\big[(X - E[X])^2\big]}_{\text{it seems right now that } \mathrm{Var}(X) = FEV(X)} = E(X^2) - E(X)^2$$

are the same thing?!

Leaving that aside for a moment

- My fcsts are given by the $PLM(\bar{g})$
- $FE_{t-1} = \pi_t - PLM^e_{t-1}(\bar{g})$
- $FEV_{t-1} = E\Big[\big(\pi_t - PLM^e_{t-1}(\bar{g})\big)^2\Big]$

Let's take a general case: $PLM^e_{t-1}(\bar{g}) = \phi_{t-1}\begin{bmatrix} 1 \\ s_{t-1} \end{bmatrix}$

$\Rightarrow FEV_{t-1} = E\left[\left(\pi_t - \phi_{t-1}(\bar{g})\begin{bmatrix} 1 \\ s_{t-1} \end{bmatrix}\right)^2\right]$

$\phi_{t-1}(\bar{g}) = \left(\phi'_{t-2} + \bar{g}\left(\pi_t - \phi_{t-2}\begin{bmatrix} 1 \\ s_{t-2} \end{bmatrix}\right)\right)'$

I don't think I can solve the problem analytically b/c it's so recursive: to find $\bar{g}^*_t$, I need to have $\bar{g}^*_{t-1}$ etc.

Also I'm not sure if I should

- restrict agents to use the same $\bar{g}$ in every period
- make them optimize over $\bar{g}$ in every period.

Ok - what I have now is FEV across time → I make 'em min FEV for each history n

→ this gives me N $\bar{g}^*$'s, which I then average.

So far I got 0.00021076 $(2.1076 \cdot 10^{-4}) \approx 0.0002$

⇒ think more on this tomorrow!

RE vs learning: responses to $E(\cdot)$ 17 Nov 2019

RE:

$$x_t = -\sigma\psi_\pi\left(\kappa x_t + \beta E_t\pi_{t+1}\right) + E_t x_{t+1} + \sigma E_t\pi_{t+1}$$

$$= -\sigma\psi_\pi\kappa x_t - \sigma\beta\psi_\pi E_t\pi_{t+1} + E_t x_{t+1} + \sigma E_t\pi_{t+1}$$

$$(1+\sigma\psi_\pi\kappa)x_t = \sigma(1-\beta\psi_\pi)E_t\pi_{t+1} + E_t x_{t+1}$$

$$x_t = \frac{\sigma(1-\beta\psi_\pi)}{1+\sigma\psi_\pi\kappa}E_t\pi_{t+1} + \frac{1}{1+\sigma\psi_\pi\kappa}E_t x_{t+1}$$

$$\pi_t = \kappa\frac{\sigma(1-\beta\psi_\pi)}{1+\sigma\psi_\pi\kappa}E_t\pi_{t+1} \quad \frac{\kappa}{1+\sigma\psi_\pi\kappa}E_t x_{t+1} + \beta E_t\pi_{t+1}$$

$$\pi_t = \left(\frac{\kappa\sigma(1-\beta\psi_\pi)}{1+\sigma\psi_\pi\kappa} + \beta\right)E_t\pi_{t+1} + \frac{\kappa}{1+\sigma\psi_\pi\kappa}E_t x_{t+1}$$

Learning

$$x_t = \left(-\frac{\sigma\psi_\pi}{w}(1-\alpha)\beta + \frac{\sigma(1-\beta\psi_\pi)}{w}\right)E_t^{\alpha,\beta}\pi_\infty$$

$$\left(-\frac{\sigma\psi_\pi}{w}(\kappa\alpha\beta) + \frac{1-\beta}{w}\right)E_t^{\alpha,\beta}x_\infty$$

$$\pi_t = \left(\left(1-\frac{\kappa\sigma\psi_\pi}{w}\right)(1-\alpha)\beta + \frac{\kappa\sigma(1-\beta\psi_\pi)}{w}\right)E_t^{\alpha,\beta}\pi_\infty$$

$$+\left(\left(1-\frac{\kappa\sigma\psi_\pi}{w}\right)(\kappa\alpha\beta) + \frac{\kappa(1-\beta)}{w}\right)E_t^{\alpha,\beta}x_\infty$$

materials10 → parameter values:

- $$\frac{\kappa\sigma(1-\beta\psi_\pi)}{1+\sigma\psi_\pi\kappa} + \beta = \frac{\kappa\sigma - \cancel{\kappa\sigma\beta\psi} + \beta + \cancel{\beta\sigma\psi_\pi\kappa}}{1+\sigma\psi_\pi\kappa} > 0$$

- $$-\frac{\sigma\psi_\pi\kappa\alpha\beta}{1+\sigma\psi_\pi\kappa} + \frac{1-\beta}{1+\sigma\psi_\pi\kappa} \propto 1-\beta-\sigma\psi_\pi\kappa\alpha\beta$$

For this to be positive, we need $\beta + \sigma\psi_\pi\kappa\alpha\beta < 1$

For current params, this is $1.155 > 1$.

- $$\left(1-\frac{\kappa\sigma\psi_\pi}{1+\kappa\sigma\psi_\pi}\right)(1-\alpha)\beta + \frac{\kappa\sigma(1-\beta\psi_\pi)}{1+\sigma\psi_\pi\kappa}$$

$$\propto (1+\cancel{\kappa\sigma\psi_\pi}-\cancel{\kappa\sigma\psi_\pi})(1-\alpha)\beta + \kappa\sigma - \kappa\sigma\beta\psi_\pi$$

$$= (1+\alpha)\beta + \kappa\sigma(1-\beta\psi_\pi) = 1.4034 > 0$$

$$\underbrace{\beta+\alpha\beta}_{\approx 1.5} + \underbrace{\kappa\sigma}_{\approx 0} - \kappa\sigma\underbrace{\beta\psi_\pi}_{\approx\psi_\pi} > 0$$

even if $\psi_\pi = 5$

Ok, so now explain why, if I recursively substi into the RE system, why do I not get the learning system? Even though you can pull out the next term from the learning system to reduce to RE.

→ it seems that LIE holds for the idiosyncratic expectation $\hat{E}^i_t \hat{E}^i_{t+1} = \hat{E}^i_t$ (in fact, this IS anticipated utility!) but not for the average expectation: $\hat{E}_t \hat{E}_{t+1} \neq \hat{E}_t$ ⇒ it's a little bit like the distinction btw PLM & ALM b/c indis act based on $\hat{E}^i_t \hat{E}^i_{t+1} = \hat{E}^i_t$, i.e. thinking that LIE holds, but in the actual law of motion $\hat{E}_t \hat{E}_{t+1}$ turns out not to equal $\hat{E}_t$ since updating happens!

19 Nov 2019

For Susanto, use

- from materials 10: "A more concise rephrasing"
- I think I wanna show IRFs from Dgain & cgain against RE for std params for the 3 shocks (take iid shocks ∀ except monpol.

Ryan meeting

20 Nov 2015

(500 years)

↳ do for FEV-min T = 5·400

→ question: maybe distrib isn't ergodic ⇒ dispersion in $\bar{g}^*$ is too large

Fabio Milani & Preston } have estimated gains

so if I generate a data sequence from RE, and I allow agents to choose gain, optimal is 0.

Ryan conjectures:

"If you do T=5·400 → will you squeeze the dispersion and shrink the mean? Yes."

↳ But if this exercise gives $\bar{g}^* = 0$ then it's not the right model-based notion of what the $\bar{g}$ should be.)

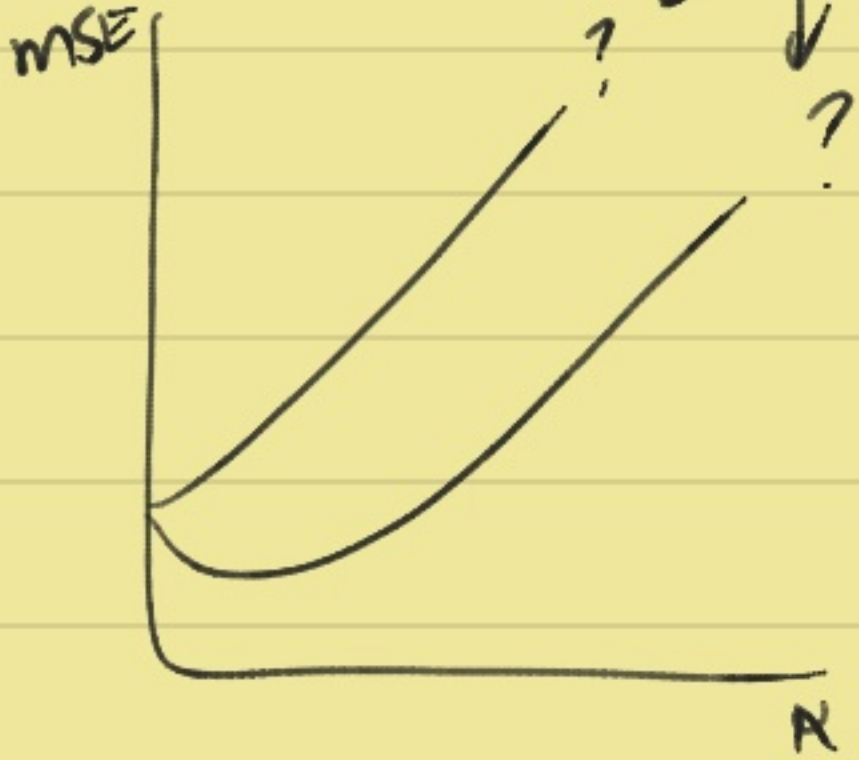
⇒ Ryan says something like: our exercise here is to get a model-based notion of what the gain should be. But we do not yet know what the right notion is.

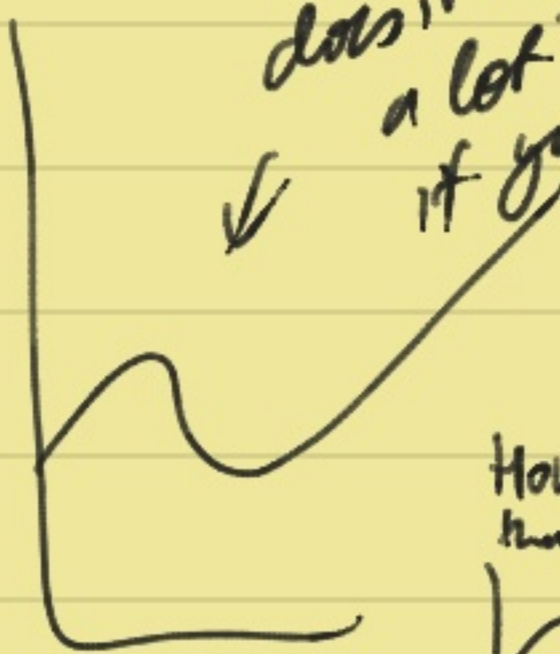
→ maybe b/c that's saying that the learning model is not optimal.

(?)

Plot this

We don't know yet what this fct looks like

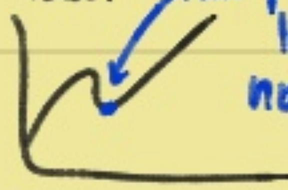




does it wobble a lot?

if yes, taking average is necessary but Ryan thinks it won't.

However it could be that



this point is a local, but not a global min!

→ allow slightly neg. κ to see if it comes back up. That is, if the MSE function looks like this:

If not minimize for reasonable κ, can $\pi - E(\cdot)$ look anchored?

↳ What he means by this is the following: We have found that the "puzzling IRFs" ("overshooting") is an endemic feature of learning. It decreases when gains are smaller. So we are trying to find a way to pick a reasonable gain k - either from a model & optimality perspective or from the data. And the question is: supp. we have a reasonable k, k^R.

1.) Do we get overshooting for that?

The point is that the IRFs of the econ for k^R are going to be the model's prediction for what unanchored expectations look like.

If overshooting is endemic for k^R, then for anchoring to be a good model, you'd need the overshooting IRFs to fit data.

$\Rightarrow$ So, as Ryan said, this can put me in a dilemma, or, I could call it a crossroads: maybe the anchored E model isn't a good model of $E(\cdot)$? Maybe they have to learn about x and/or i too, or maybe something entirely different...

Work after 20 Nov 2019

Let's gather some reasonable numbers for the gain.

Estimates	Calibration	
	0.002	Eusepi & Preston (2011)
$g < 2(1-\beta)$ for beliefs to be stable. For $\beta = 0.99$, $g < 0.02$.		Eusepi, Giannoni, Preston (2019) Limits
$\hat{g} = 0.05$		
0.145		CEMP
0.0183		Milani (2007)
	0.1, 0.05, 0.03	Williams (2003)
0.062		Branch & Evans (2006)
	0.075, 0.05, 0.025	Orphanides & Williams (2004)
	0.02	Orphanides & Williams (2005)

Avg = 0.05

Avg w/o CEMP & avg estimate w/o CEMP = 0.04

Let's interpret the gain number:

- Eusepi & Preston (2011)

  Data T quarters old receives the weight

  $(1-k)^T$

- Milani (2007)

  $\frac{1}{\text{gain}} \approx$ "how many past observations agents use to form expectations"

Eusepi & Preston (2011) → 500 quarters (⇒ 125 years!)
avg w/o CEMP → 24 quarters (⇒ 6 years)
CEMP → 7 quarters (< 2 years!)

I think it's reasonable that humans shouldn't use more than approx 50 years of data on avg (200 quarters)

↳ $\min(\bar{g}) = 0.005$

They also shouldn't use less than 5 years (20 quarters)

↳ $\max(\bar{g}) = 0.05$

$\bar{g} \in [0.005, 0.05]$, and in particular $0.02 \to 12$ years seems reasonable.

When burnin = 0, T = 400, N = 100 (48 sec)

$\bar{g}^* = 0.0005$ $Var(\bar{g}^*_n) = 1.2602 \cdot 10^{-6}$

→ but it varies per run!

burnin = 50, T = 400, N = 100

0.00018217 1.8037e-07

and still varies

burnin = 1600, T = 400, N = 100 (48 sec)

$\bar{g}^* = 0.00033$ $Var(\bar{g}^*_n) = 6.0104e-07$

varies still.

burnin = 0, T = 2000, N = 100 (150 sec)

$\bar{g}^* = 5.27773e-05$ $Var(\bar{g}^*_n) = 1.5801e-08$

$\bar{g}^* = 6.4184e-05$ $Var(\bar{g}^*_n) = 1.5136e-08$

2.6731e-05 N = 1 (2 sec)

4.6657e-05

1.511e-05

1.0546e-05

<u>3 obs:</u> 1) burnin doesn't matter (why? b/c doesn't impact FEV)

2) longer T decreases $\bar{g}^*$

3) decreases dispersion across n

} Ryan's conjecture was right.

Moving on to MSE:

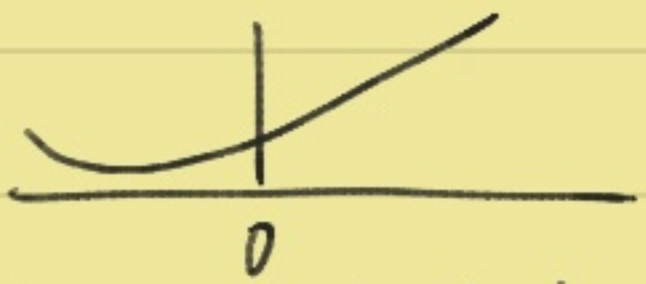
Note $MSE(y_j) = \frac{1}{T}\sum_{t=t_0}^{T}(y_{j,t} - \hat{y}_{j,t})^2$ (Brandt & Evans 2006)

MSE = FEV b/c $FE = x_t - x_{t|t-1}$

$Var(FE) = E[FE^2]$ = mean squared error

plotting this thing has a very clear shape:

it has a 2nd deriv > 0 (convex) and takes a min



either slightly to the left of 0

or slightly to the right

⇒ but maybe this means that if I increase T, the min will be pushed to 0 "from both sides"?

Any interesting finding: the more volatile the environment (the higher Σ = VC(shocks)) the lower $\bar{g}^*$ (more negative)

Ok so what Ryan had had in mind was something like: sim Y using $\bar{g} = 0.145$.
Now, for different values of $\bar{g}$, compute forecasts of Y_{t+1} at each t, allowing only $\bar{\pi}$ to update and thus fcsts, but not Y. What's the $\bar{g}$ that mins the FEV?

→ For this I obtain $\bar{g}^* = 1.003e-05$ $Var(\bar{g}^*) = 3.357e-16$
whereas for my exercise I get $7.2424e-05$ $1.7218e-8$
(burnin = 0, T = 2000, N = 100)

The figs show that ideally you'd get $\approx -1.5 \cdot 10^{-3}$
→ and this is consistent no matter how long T is
(actually that's not too surprising given that here T doesn't make the FEV larger b/c the data is created given $\bar{g} = 0.145$)

Why am I getting negative values in both exercises?
Why is a $\bar{g}^* = 0$ not good? What other notion to use?

Given the relationship between the gain and the Kalman gain, is there no KF-related notion of optimality we could use?

→ In fact, for Ryan's minFEV, I get $\bar{g}^* < 0$ even when the DGP is RE!

Scheinkman & Xiong, 2003 "Overconfidence & Speculative Bubbles"
setup on optimism & pessimism on financial markets
→ this stuff relates to my confirmation bias idea

<u>Understand why minFEV</u> <u>23 Nov 2015</u>

<u>Ryan's method</u>: y = generated by gain $= 0.145$

Now set $k = \bar{g}^* = \text{argmin } FEV(y(\bar{g} = 0.145))$.

<u>My method</u>: $k = \bar{g} = \text{argmin } FEV(y(\bar{g}))$

1.) I don't understand why Ryan said that I didn't go all the way to the fixed point. To me it seems like my method does go to the fixed point, his doesn't!

2.) Why did Ryan call his method "analog to RE"?

Interpretation of Ryan's method:
"Data is what it is. Let me update my beliefs so that they have as little errors given the structure of the data as possible."
→ I think that's why: b/c I wanna choose the optimal gain such that my expectations are close to model-consistent.

Interpretation of my method:
"Give me the gain that makes my beliefs model-consistent, internalizing that my beliefs affect the DGP."
→ to me my way still seems closer to a fixed point.

What would I expect the two methods to yield?
- Ryan expected his method to yield a lower $\bar{g}^*$ than $\bar{g}=0.145$, the one using which data was generated. But why should it? This means Ryan expected that FEs are smaller when $\bar{g} < \bar{g}^{DGP}$. I don't think this is

a general statement, i.e. I think he only expected that compared to $\bar{g}^{CEMP} = 0.145$. Why do I think that?
B/c fcsts are $\hat{E}_t \pi_{t+1} = \bar{\pi}_{t-1} + b_1 s_t$
where $\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$

- So the statement is that even as data g is generated w/ $k_t = \bar{g}^{CEMP}$, that leads to FEs b/c the PLM doesn't coincide w/ the ALM. So if $k_t = \bar{g}^{CEMP}$ and FEs switch sign, that means that by lowering $\bar{g}$, we could lower FE b/c agents are overupdating.
- On the other hand, even $k_t = 0$ will not be optimal, b/c then $\bar{\pi}_t = \bar{\pi}_{t-1} = \bar{\pi}$, so $\hat{E}_t \pi_{t+1} = \bar{\pi} + b_1 s_t$ which yields permanent FEs when the data is generated by a gain of $k = \bar{g}^{CEMP}$. If $k_t = 0$, and initialized at RE, $\hat{E}_t \pi_{t+1} = 0 + b_1 s_t$ = RE fcst. So if the DGP is RE, then $\bar{g}^*$ better be 0, and it's a problem that that's not what I find!
- Also, this shows that for the learning DGP, $k^* = 0$

cannot be optimal.

So for Ryan's method, we'd expect an interior k^*.

What does it mean if this is < 0?

fcsts are $\hat{E}_t \pi_{t+1} = \bar{\pi}_{t-1} + b_1 s_t$

where $\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1}(\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$

If $k < 0$, then if I underestimated π_t (fcst $< \pi_t$)

then I lower $\bar{\pi}$, that is, I lower my fcst even further. So that makes no sense!

$\Rightarrow$ it must be that these negative values are a result of a coding error, which potentially also explains why RE DGP doesn't give me $k^* = 0$.

- So what do I expect my method to give?

Reinterpretation of my method: "give me the gain that gives me the sequence of Y for which the FEV given that gain is minimized" $\rightarrow$ i.e. "find the gain $\bar{g}^*$ that generates the sequence Y w/ minimal FEV associated w/ it"

→ but this should give you $k^* = 0$ b/c then the $\bar{\pi}_0 = 0$ means that PLM = ALM = RE, which again shows you why this isn't a good notion of an optimal gain!

⇒ Ryan's notion is better than mine

→ in fact I think Branch & Evans are doing something similar when estimating the gain, so I should obtain figures that look like theirs:

But: neither my nor Ryan's should give $k^* < 0$.
So there's sthg wrong in both!

So now I'm focusing on Ryan's method alone:

1) I'm getting ≈ 70% of $\bar{g}^* < 0$ (!)

2) Increasing T makes 1) $\bar{g}^* \to 0$ from below 2) shrinks $\text{Var}(\bar{g}^*_n)$

3) Increasing N doesn't do anything

→ in any case, $\bar{g}^* < 0$ about 60-70% of the time!

More observations:

1) If I make the shocks be meaned around ± some number, k^*s become positive!

2) Changing Σ doesn't really do anything.

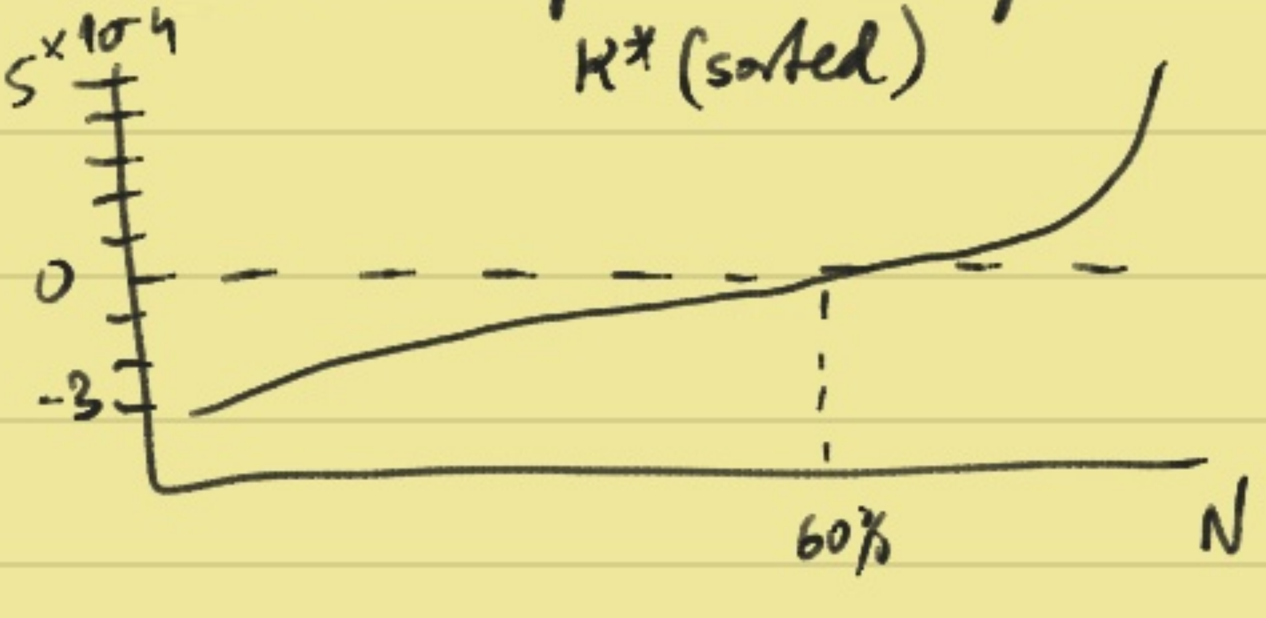
Why is this happening? And why doesn't the sign(mean(innovs)) matter? (→ i.e. why doesn't it matter whether shocks are centered around 1 or −1?)

→ Try shutting off all shocks except 1 → see how that affects k^*.

If in the objective function, I shut the effect of shocks on FE off, I obtain all $k^* > 0$

If you $T\uparrow$, $k^* \to 0$ but always from below

and the distrib of k^* always is skewed like this:



Reading Gerko 2019 2 Dec 2019

I just understood something: $T(\phi)$ is the mapping from the PLM to the ALM. That is, given a belief ϕ, what ALM does the PLM imply, i.e. how would the model expect observables to evolve (absent shocks) given these beliefs?

$T(\phi) = E(ALM)$ (it just disregards shocks, answering the question: "w/ these beliefs, on average, what will observables do?")

The E-stability condition is the differential equation

$$\frac{d}{d\tau}(\phi) = T(\phi) - \phi \qquad (2.8, \text{EH p. } 31)$$

$E(ALM(\phi)) - PLM$ (either in both cases w/, or in both cases w/o observables)

CEMP's criterion $\theta_t = PLM - E(ALM)$, so it's really the E-stability diff. equation $\theta_t = \frac{d}{d\tau}(\phi)$ (!)

Susanto meeting 2 Dec 2019

Work after 3 Dec 2019

Ball 1994 AER

Deflation causes recessions as we thought.

But disinflation (changing the grt of money ↓)

causes a boom.

I think the idea is this:

$$x_t = m_t - E_t m_{t+1} \quad \text{(roughly)}$$

↙ all future M^s

↓ deflation causes x_t ↓

but disinflation is $\dot{m}$ ↓ so m_t doesn't change,

only m_{t+1} does (here's where credibility comes in: $E_t m_{t+1} = m_{t+1}$)

↳ this is exactly the contractionary $E_t \pi_{t+1}$ I get!

$$x_t = -\sigma \psi_\pi \pi_t + \sum_{T=t}^{\infty} \beta^{T-t} \left((1-\beta) x_{T+1} + \underbrace{\sigma(1-\beta\psi_\pi)}_{<0} \pi_{T+1} \right)$$

I think that the intuition is that a

credible disinflation means a movement in expectations

only, but no current values.
I think Whelan's intuition is that when $E_t \pi_{t+1} \downarrow$,
$P_t \downarrow$ a bit and so consumers are richer $\left(\frac{M_t}{P_t} \uparrow\right)$
$\rightarrow x_t \uparrow$

And this is also what Ball's intuitive proof circles around: firms choose lower prices b/c they anticipate future decreases in the increase of m.

↳ So, like for me, it's all about expectations moving and credibility: ppl have to believe mon. pol. (in my case, know & believe the Taylor-rule).

One could also summarize Ball's argument as
A disinflation has two effects: 1) contractionary via the interest rate 2) expansionary via expectations
For the expectations channel to dominate, Ball figured you need to do policy quickly, so "1) doesn't get to move".

In my case, you just need expectations to move strongly enough.

Whelan adds 2 things:

- Since Ball himself said that disinflationary booms don't fit the data well, it must be that CB's have credibility problems.
  - → But, though likely during Great Inflation, is it still likely now?
- Something else must be wrong w/ the NK model.

One thing I'm a little worried about is that $E(\cdot)$ moving strengthens the Ball effect, and $E(\cdot)$ move a lot when unanchored (again) which is exactly a measure of the CB <u>not</u> being credible. — ah but that may be fine actually, b/c I think Ball needs credibility for $E(\cdot)$ to move. I can get them to move otherhow.

Instrument instability seems to happen here in the sense that expectations are the instrument. In particular, the FE oscillates.

4 Dec 2019

Let's try to write out the fcst error. It's

$$FE_{t-1} = \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$$

where $\bar{\pi}_{t-1} = \bar{\pi}_{t-2} + k^{-1}(\underbrace{\pi_{t-1} - (\bar{\pi}_{t-2} + b_1 s_{t-2})}_{FE_{t-2}})$

$$FE_{t-1} = \pi_t - \left[\bar{\pi}_{t-2} + k^{-1} FE_{t-2} + b_1 s_{t-1}\right]$$

$$FE_{t-1} = \pi_t - \left[\bar{\pi}_{t-3} + k^{-1} FE_{t-3} + k^{-1} FE_{t-2} + b_1 s_{t-1}\right]$$

$$FE_{t-1} = \pi_t - b_1 s_{t-1} - \underset{0}{\overset{\shortparallel}{\bar{\pi}_0}} - k^{-1} \sum_{s=0}^{t-2} FE_s \quad \Big| \pm FE_j$$

$j = t-2, \dots, "1"$

$$\Delta FE_{t-1} = \underbrace{\pi_t - b_1 s_{t-1}} - k^{-1} \sum_{s=0}^{t-2} \Delta FE_s$$

ignore these and switch $t-1$ to t for simplicity

$$\Delta FE_t = -k^{-1}\left[\Delta FE_{t-1} + \Delta FE_{t-2} + \dots\right]$$

so the weights are k^{-1} for all and 1 for ΔFE_t

or $k < -1$ for ΔFE_t and 1 for all the rest.

$k \Delta FE_t + \Delta FE_{t-1} + \Delta FE_{t-2} + \dots$ leads to this characteristic equation

$$k x^n + x^{n-1} + \dots + x + 1 = 0$$

If we only had one lag, this would be

$$kx + 1 = 0$$

$$x = -k^{-1} > -1$$ That would be stable but oscillating

Two lags

$$k x^2 + x + 1 = 0 \qquad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4k}}{2k}$$

For the roots to be real $1 - 4k \overset{!}{>} 0 \Rightarrow 1 > 4k$

$k \overset{!}{<} \frac{1}{4} = 0.25$ It usually is, even CEMP's.

If $k = 0.25$, $x_1 = x_2 = -\frac{1}{2k} = -2$

→ that would be unstable.

For $k \in (0, 0.25)$, both roots are < -1 always!

So the system is unstable.

My guess is that the more lags you include, the closer you will get to stability, but you will have (potentially all) roots < 0, which is why we get the oscillation.

But what this doesn't account for is the role of ψ_π in getting the oscillation. What seems to be clear is that the oscillations in FE are driving it. So in a sense I'm not even sure if it's instrument instability or simply instability.

The connection to ψ_π must come via the role of i in π via x.

$$\left.\begin{aligned} x &= -\bar{\sigma} i_t + E(\text{stuff}) \\ \pi_t &= \kappa x_t + E(\text{stuff}) \\ i_t &= \psi_\pi \pi_t \end{aligned}\right\}$$

In period 1, a shock hits, E moves a little, (x, π) move, i moves.

Evening of period 1: E(·) adjusts. 2 things influence the adjustment: 1) $\kappa = \bar{g}$ the size 2) ψ_π the size of Ball's disinflationary boom effect (direction of E-adjustment)

$$FE = \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$$

$\uparrow$ k governs the change here

ψ_π governs the change in <u>both</u>, and the bigger it is, the more $E(\cdot)$ move compared to π, opening up FEs.

→ it's as if $E(\cdot)$ were the policy instrument?
Changing i works partly via its effect on x_t and π_t today, but its main effect is via $E(\cdot)$.
So in that sense it's like instrument instability b/c the instrument becomes unstable, <u>but</u>: it's also not like instr. instab. b/c a too high ψ_π makes FEs unstable thereby rendering the objective variables unstable too.

Ryan meeting 4 Dec 2019

- Gerko: send to Ryan

- Data-IRFs oscillated for Ryan early on → so it's not quite the case that empirical IRFs never oscillate ...

- L'Huillier estimated $\hat{\psi}_\pi \approx 1.001$ or sthg

  Note: it's Blanchard, L'Huillier Lorenzoni News & Noise, AER 2013 and they get $\hat{\psi}_\pi = 1.0137$, $\hat{\psi}_x = 0.005$

- Try $\psi_\pi < 1$ → it seems like if for a fwd-looking system $\psi_\pi > 1$ gives stability, for a bw-looking one, $\psi_\pi < 1$ will.

- Consider flex price model

$$\left.\begin{aligned} r_t &= i_t - E_t[\pi_{t+1}] \\ i_t &= \phi \pi_t \end{aligned}\right\} \text{ do expectations pan out similarly here?}$$

- The big picture question is:

  Do we take the model's implications seriously and explore what they imply for policy?

  OR: do we change sthg about the model? 2 options

  1) change E-formation 2) change policy (e.g. have $E(\pi)$ in TR instead of π)

If you change policy, then you can make a statement like: "central bankers say they have $E(\pi)$ in TR, and look, indeed it works better than π"

If you change $E(\cdot)$-formation, you can say: "std learning implies this, but here's a learning that works".

Think of ways we can change policy

· $E(\pi)$ instead of π in TR

Think of ways we can change E-formation.

<u>Work after</u> 5 Dec 2019

One thing I've tried is to add interest rate smoothing, $\rho > 0$.

It doesn't change much b/c on the net it acts like ρ_i, the persistence of the mon pol shock, except that it has a contractionary effect (stuck w/ initial shock for longer) and an expansionary effect (stuck w/ expansionary policy reaction for longer). In gen, these two seem to balance so we still have the overshooting.

Peter meeting — 5 Dec 2019

A big-picture comment:
What you might worry about in learning:
Systematic pattern of FEs
→ serially corr errors in the same direction

Peter Ireland similar paper to Ball:
Stopping Inflations, Big & Small 1997 JMCB

Ric. Eq. seems to hold today? Chris Sims
30-yr T-bill i-rate is low despite high debt

→ ask if
alternative expectation-formation schemes that
deliver more appealing dynamics for
E(·) and thus for observables.

Work after 5 Dec 2019

Reading Townsend (1983): "Forecasting the forecasts of others" which seems to be the first paper to introduce dispersed info!

One claim he makes is that w/ a normal signal extraction problem:

- forecasts $E[v_t | \Omega_t]$ (info set) is correlated w/ $E[v_{t-1} | \Omega_{t-1}]$ (i.e. fcsts serially corr) b/c both contain v_{t-1}
- FEs $[E(\theta_t | \Omega_t) - \theta_t]$ will be serially corr too b/c it'll be an MA of past random vars v_t and ε_t.

Another is that in a simple model of fcsting the fcsts of others, FEs of those forecasting the fcsts of others (market 2) exhibit <u>damped oscillations</u> (!)

→ This may be part of the reason for me too b/c in this learning model agents don't know other people's fcsts but are implicitly fcsting them when fcsting aggregates.

To do's:

① Add $E(\pi)$ instead of π in TR

② Try $\psi_\pi < 1$

③ The Townsend analysis suggests that IRFs to EE-learning won't give oscillations b/c there agents aren't testing each others' tests

④ Do IRFs to vector learning from the spinoff LH paper

⑤ Figure out alternative learning that exhibits no oscillations

- can Bayesian learning do it?

  ↳ refs in Ryan's class reading list

Susanto commented on   6 Dec 2019

oscillating IRFs: 1) They can and do arise more frequently than you'd think b/c published IRFs look "nice" b/c a lot of work goes into making them nice

2) check local projections which estimates the MA process directly. Since a VAR estimates an AR and imposes

restrictions on the VC matrix (and the AR-matrix), it's likely to give you more smooth IRFs, while those of local projections are likely to be more choppy. Check Valerie Ramey's Handbook chapter on IRFs! It also has a bunch of replication files.

Adding $E(\pi)$ to TR — 7 Dec 2019

$$x_t = -\sigma\left(\psi_\pi \hat{E}_t \pi_{t+1} + \psi_x x_t + \underbrace{\rho i_{t-1} + \bar{i}_t}_{}\right) + \Sigma(\cdot)$$

clearly the shocks will be the same, ignore 'em

$$(1+\sigma\psi_x)x_t = -\sigma\psi_\pi \hat{E}_t \pi_{t+1} + \Sigma$$

$$(1+\sigma\psi_x)x_t = -\sigma\psi_\pi \hat{E}_t \pi_{t+1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\Big[(1-\beta)x_{T+1} + \sigma\pi_{T+1}$$
$$-\sigma\beta\left(\psi_\pi \hat{E}_{T+1}\pi_{T+2} + \psi_x x_{T+1} + \text{shocks}\right)\Big]$$

$$(1+\sigma\psi_x)x_t = -\sigma\psi_\pi \underline{\hat{E}_t \pi_{t+1}} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta-\sigma\beta\psi_x)x_{T+1} + \sigma\pi_{T+1} - \sigma\beta\psi_\pi \underline{\hat{E}_{T+1}\pi_{T+2}}\right]$$

now these two pose difficulties

b/c I don't know if I can use LIE on $\hat{E}_t$ or not.

Let's see if we can shove σi_t into the expectation instead.

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} \right]$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[-\sigma \beta i_{T+1} \right]$$

$$= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[-\sigma i_T \right]$$

$$\Rightarrow x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma i_t \right]$$

$$= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma (\psi_\pi \hat{E}_T \pi_{T+1} + \psi_x x_T) \right.$$ + shocks unchanged

$$(1+\sigma\psi_x) x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta-\sigma\psi_x\beta) x_{T+1} + \underbrace{\sigma \pi_{T+1} - \sigma \psi_\pi \hat{E}_T \pi_{T+1}}_{\text{LIE is still a question}} + \text{shocks unchanged} \right]$$

but at least this expression is neater.

So $\hat{E}^i_t \hat{E}^i_{t+1} = \hat{E}^i_t$, i.e. the idio $\hat{E}^i$ fulfills LIE b/c that is anticipated utility. There are 2 questions: if LIE holds for $\hat{E}^i$, why don't HHs use the recursive representation?

→ Preston would argue that b/c then they would ignore their wealth b/c they don't know they're identical. ok fine so they don't use LIE and so we get the LH expression when we aggregate.

But it seems to me - again! - from Preston 2005 p 16 that

that $\hat{E}$ doesn't satisfy LIE – not on the aggregate.
So also CEMP define anticipated utility as $\hat{E}^{+}_{t-1} \bar{\pi}_{T} = \bar{\pi}_{t}$
i.e. as an idiosyncratic statement.
However it seems to me like maybe actually even the agg $\hat{E}(\cdot)$ satisfied it somehow! so when we compute expectations I think we make use of it?
In particular, the modeler knows, as CEMP state, that $\hat{E}^{+}_{t} = \hat{E}_{t}$, i.e. that agents hold identical beliefs.
So then $\hat{E}_t$ will fulfill LIE too. So maybe it's really just the case that agg beliefs just "inherit" the discounted sums.

Anyway – the usual confusion about LH is there: it seems like everything fulfills LIE and yet still since LH forecasts are optimal for agents who don't know that they're identical, agg $\hat{E}$ "inherit" them.

yes this is the stance of my thinking later on too (8 Jan 2020)

Will do so after that detour:

$$(1+b\psi_x)x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\Big[(1-\beta)x_{T+1} + \underbrace{b\pi_{T+1} - b\psi_\pi \hat{E}_T \pi_{T+1}}_{=(b-b\psi_\pi)\pi_{T+1} \text{ here}} + \text{shocks unchanged}\Big]$$

$$(1+b\psi_x)x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\Big[(1-\beta-\beta b\psi_x)x_{T+1} + b\underbrace{(1-\psi_\pi)}_{\text{well this will always be } <0}\pi_{T+1}\Big]$$

So then: $$x_t = \underbrace{\frac{1}{1+b\psi_x}\big[b(1-\psi_\pi), (1-\beta-\beta b\psi_x), 0\big]}_{=g_{xb}} f_\beta \qquad g_{xa} = [0\ 0\ 0]$$

$$\pi_t = \kappa x_t + \big[(1-\alpha)\beta, \kappa\alpha\beta, 0\big] f_\alpha$$

$$\pi_t = \underbrace{\frac{\kappa}{1+b\psi_x}\big[b(1-\psi_\pi), 1-\beta, 0\big]}_{\kappa g_{xb}} f_\beta + \underbrace{\big[(1-\alpha)\beta, \kappa\alpha\beta, 0\big]}_{=g_{\pi a}} f_\alpha$$

$$x_t = g_{xb} f_\beta + \overset{[0\ 0\ 0]}{g_{xa}} f_a$$

$$\pi_t = \kappa \cdot g_{xb} \cdot f_\beta + g_{\pi a} \cdot f_a$$

$$i_t = (\psi_\pi g_{\pi a} + \psi_x g_{xa}) f_a + (\psi_\pi g_{\pi b} + \psi_x g_{xb}) f_\beta$$

+ shocks that are unchanged

actually that's not quite true,

see Mathematica materials12.m

or Notes 10 Dec 2019.

Stuff got lost here: Kozlowski et al: agents
are learning the distribution of shocks, $g(\cdot)$.
For Koz et al, this is crucial b/c when a tail event occurs, it leads to agents to estim a shock distrib $\hat{g}(\cdot)$ w/ fat tails (potentially asymmetric?)

They estimate $\hat{g}(\cdot)$ using a normal kernel density estimator, I guess the most std. non-parametric estimator:

$$\hat{g}_t(x) = \frac{1}{n_t} \sum_{s=0}^{n_t - 1} \Omega(x - x_{t-s}, \Sigma_t)$$

x_t? not clear but can check kernel estimators or mean, $\bar{x}$? (see Baum lecture 3)

obs at time t

multivariate normal density

VC matrix (aka "bandwidth matrix")

x = $d \times 1$ shock vector

Σ_t = diagonal matrix, I think like this:

$$\underset{d \times d}{\Sigma} = \begin{pmatrix} \hat{\sigma}_1 & & & \\ & \hat{\sigma}_2 & & \\ & & \ddots & \\ & & & \hat{\sigma}_d \end{pmatrix} \cdot \left(\frac{4}{(2+d)n_t} \right)^{\left(\frac{1}{4+d}\right)}$$

where $\hat{\sigma}_j$ = sample std dev of shock j

So 2 main differences to adaptive learning:

1) here agents learn about shocks, not observables

2) learning is nonparametric b/c they learn a distrib

↳ (1) is not so interesting for me b/c exog. shocks have no connection to central bank credibility, so that one can no longer talk about anchoring

↳ I expect (2) to be more interesting, yet maybe not too different to my setting (qualitatively) b/c I can imagine a generalization of my adaptive learning framework in which agents don't know the distrib of the LOM (they don't know that gx is linear and normal) ⇒ that would be similar to learning slope and constant I guess except it would be less restrictive on their "prior" if you will.

→ But one could try it out!

What is not clear to me is what would be the analogue to the gain? The bandwidth matrix?

Read Collins-Dufresne et al (2016)

Honestly, Bayesian learning seems very analogous to adaptive (frequentist) learning except that the updating rule is Bayes' rule, the prior matters AND one difference that might matter is that while in adaptive learning, the anticipated utility ass means that in every period, agents treat the PLM as if it was the ALM, Bayesian learning can account for the uncertainty around the correctness of the PLM (this is referred to as parameter uncertainty)

↳ and it often refers to the case where agents fit using an average of different models

↳ can this mitigate overshooting?

Or is this simply the Bayesian counterpart to the gain/anchoring?

The problem is that this is an asset pricing paper and doesn't describe how Bayesian learning works, nor does it give references.

Evans, Honkapohja & Sargent (2013)

(In the book Macro at the Service of Public Policy – which I haven't downloaded but I did save the ref.)

Muth market model (6.1)

$$p_t = \mu + \alpha E^*_{t-1} p_t + \delta z_{t-1} + \eta_t \, ; \quad \eta_t \sim WN(0, \bar{\sigma}^2_\eta)$$

$$z_t = \rho z_{t-1} + w_t \qquad w_t \sim iid(0, \sigma^2_w)$$

$$\mu = 0$$

REE model for (6.1) is

$$p_t = \bar{\beta} z_{t-1} + \eta_t \qquad w/ \ \bar{\beta} = \frac{\delta}{1-\alpha}$$

Now EHS consider a constant parameter learning rule that's estimated using Bayesian techniques:

the PLM is

$$\sigma^2_\eta \neq \bar{\sigma}^2_\eta$$

$$p_t = b_{t-1} z_{t-1} + \eta_t \qquad w/ \ \eta_t \sim N(0, \sigma^2_\eta)$$

Agents' prior distrib for β is $\beta \sim N(b_0, V_0)$ and updates are

$$b_t = b_{t-1} + \frac{V_{t-1} z_{t-1}}{\sigma^2_\eta + V_{t-1} z^2_{t-1}} (p_t - b_{t-1} z_{t-1})$$

$$V_t = V_{t-1} - \frac{z^2_{t-1} V^2_{t-1}}{\sigma^2_\eta + V_{t-1} z^2_{t-1}}$$ using the Kalman filter.

Prop. 6.1 Bayesian analog to E-stability

Convergence to REE w/ prob 1 if $\alpha < 1$

and $V_t = \dfrac{\sigma_\eta^2}{(t+1)S_t - z_t^2} \to 0$ irrespective of whether σ_η^2 is correct!

WTF is S_t...?

I'm a little suspicious of this b/c it says that if $V_t \to 0$ then the Kalman gain goes to zero too and you don't get that w/ the KF (instead $\Sigma_{t|t} \to \Sigma$, the st. st. and so the gain $K_{t|t}$ goes to K, the st. st gain) $\Rightarrow$ this is normally how the KF is analogous w/ constant gain learning and it always keeps a tiny variance around RE.

But maybe they are looking at dgain learning analogues here – that might be what they mean w/ "constant param learning". In the following section, they consider time-varying param β. It is useful to understand how time-varying param (TVP)

models are related to misspecification of fcst rules.
When agents think that βs may be time-varying they entertain the idea that they may be using a misspecified model. This is intimately connected to e-quilibrium learning b/c agents prefer to track the params because they are never sure whether a structural change may render their converged fcst-ing rule misspecified.

EHS show w/ a simple example (where agents aren't sure whether a constant param PLM or a TVP PLM is the right one) that if feedback effects from beliefs are sufficiently strong then agents may select the non-RE model b/c E(·) induce actual drifts in the ALM that induce the MSE of the non-RE PLM to be lower than the RE one. (This is like a self-confirming learning result.)

It's like an endog. e-quilibrium choice !!!

A note: EHS claim on p. 105 that

they guess that for this TVP learning case the stable param space will be $-1 < \alpha < 0.5$ b/c under LS learning:

1) $0.5 < \alpha < 1$ may yield slow convergence to RE

2) "$\alpha < -1$ a possible problem of overshooting can emerge when agents overparametrize the PLM"

$\Rightarrow$ oh really??

So when the feedback from expectations is negative and sufficiently strong, least-squares learning may yield overshooting???

You don't say!

So when RLS learning, so agents learn gx, then in terms of the NKPC, NKIS & TR, what do they know? 9 Dec 2019

they know hx

$$x_t = hx\, x_{t-1} + \eta\, \varepsilon_t$$

$$y_t = gx\, x_t$$

I think they always know the TR — what they don't seem to know is how shocks $\begin{bmatrix} r_t^n \\ \bar{i}_t \\ u_t \\ \rho i_{t-1} \end{bmatrix}$ affect observables $z_t = \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}$.

I think they must understand contemporaneous relationships between observables → so even when they learn the slope also, they know $i_t = \psi_\pi \pi_t$.

What they get wrong are forecasts $\hat{E}_t z_{t+1} = \hat{gx}\, hx\, s_t$.

If $\hat{gx} = a + gx$, then they shift expectations up/down.

$\uparrow$ constant

If $\hat{gx} = \hat{gx}$ ← constant & slope → so when a persistent shock hits, now agents might not know how that will impact observables in the future (in particular $\bar{i}_t \uparrow \longrightarrow \pi_t \downarrow \to i_{t+h} \downarrow$ will be less clear, so expectations move less.)

⇒ I'm not sure that a different updating scheme per se (KF or PF) would change anything about the speed of updating, or its size.

→ in this sense if they learned hx as well might help b/c then in the case of the persistent mon pol shocks they wouldn't actually be sure that it will persist.

→ It seems like my hunch is that the more they are uncertain about, the less expectations will move.

→ Is learning a distrib. helpful here? I'm not sure. I kinda feel that learning a functional form might... or might not?

⇒ I kinda want them not to know the TR!

↳ what if I put π_{t-1} in the Taylor-rule instead of π_t? (Or add a π_{t-1} term?)

So if $i_t = \psi_\pi \pi_{t-1} + \psi_x x_t + \bar{i}_t$ 10 Dec 2019

don't ignore shocks again b/c they might not be unchanged!

then $x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_t^N \right]$

$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_{t-1} + \hat{E}_t \sum \beta \left[(1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma\pi_{T+1} - \sigma\beta\pi_T + \sigma r_t^N \right]$

Or, if instead starting from $\star$,

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)x_{T+1} + \sigma\pi_{T+1} - \sigma i_t + \sigma r_t^N\right]$$

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)x_{T+1} + \sigma\pi_{T+1} - \sigma\left(\psi_\pi \pi_{T-1} + \psi_x x_T + \bar{i}_T\right) + \sigma r_t^N\right]$$

$$(1-\sigma\psi_x)x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta-\sigma\beta\psi_x)x_{T+1} + \sigma\pi_{T+1} \underline{- \sigma\psi_\pi \pi_{T-1}}\right]$$

$$- \sigma \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[\bar{i}_t - r_t^N\right]$$

$$-\sigma\psi_\pi \; \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\pi_{T-1} = -\sigma\psi_\pi\left[\pi_{t-1} + \beta\pi_t + \beta^2\pi_{t+1} + \dots\right]$$

So we get $-\sigma\psi_\pi(\pi_{t-1} + \beta\pi_t) + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\beta^2\pi_{T+1}$

So

$$(1+\sigma\psi_x)x_t = \underline{-\sigma\psi_\pi\pi_{t-1}} - \sigma\psi_\pi\beta\pi_t + \hat{E}_t \sum_{T=t}^{\infty}\beta^{T-t}\left[(1-\beta-\sigma\beta\psi_x)x_t + \sigma(1-\beta^2\psi_\pi)\pi_{T+1}\right]$$

lag term novel here

β's novel here and will dampen feedback

$$- \sigma \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[\bar{i}_T - r_T^N\right]$$

$$(1+\sigma\psi_x)x_T = \underline{-\sigma\psi_\pi\pi_{t-1}} - \sigma\psi_\pi\beta\left[kx_t + [(1-\alpha)\beta, k\alpha\beta, 0]f_\alpha + \hat{E}_t\sum\alpha\beta^{T-t}u_T\right]$$

$$+ \left[\sigma(1-\beta^2\psi_\pi), 1-\beta-\sigma\beta\psi_x, 0\right]f_\beta - \sigma\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\bar{i}_T - r_T^N\right]$$

$$(1+\sigma\psi_x + k\sigma\psi_\pi\beta)x_t = -\sigma\psi_\pi\pi_{t-1} - \sigma\psi_\pi\beta\left[(1-\alpha)\beta, k\alpha\beta, 0\right]f_\alpha$$

$$+ \left[\sigma(1-\beta^2\psi_\pi), 1-\beta-\sigma\beta\psi_x, 0\right]f_\beta - \sigma\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[\bar{i}_T - r_T^N\right] - \sigma\psi_\pi\beta\hat{E}_t\sum\alpha\beta^{T-t}u_T$$

$$\Rightarrow (1+\sigma\psi_x + k\sigma\psi_\pi\beta)x_t = -\sigma\psi_\pi\pi_{t-1} - \sigma\psi_\pi\beta\left[(1-\alpha)\beta, k\alpha\beta, 0\right]f_\alpha$$

$$+ \left[\sigma(1-\beta^2\psi_\pi), 1-\beta-\sigma\beta\psi_x, 0\right]f_\beta - \sigma[-1\;1\;0\;0\;0](I_4-\beta h_x)^{-1}s_t$$

$$-\sigma\psi_\pi[0\;0\;0\;0\;1]s_t$$

$$-\sigma\psi_\pi\beta[0\;0\;1\;0\;0](I_4-\alpha\beta h_x)^{-1}s_t$$

So $x_t = \frac{1}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)} \cdot \Big\{ -\sigma\psi_\pi\pi_{t-1} - \sigma\psi_\pi\beta\,[(1-\alpha)\beta, k\alpha\beta, 0]\, f\alpha$

$\hookrightarrow -\sigma\psi_\pi[0\ 0\ 0\ 0\ 1]\, s_t$

$+ [\sigma(1-\beta^2\psi_\pi), 1-\beta-\sigma\beta\psi_x, 0]\, f_\beta - \sigma[-1\ 1\ 0\ 0\ 0](I_4-\beta hx)^{-1}s_t - \sigma\psi_\pi\beta[0\ 0\ 1\ 0\ 0](I_4-\alpha\beta hx)^{-1}s_t \Big\}$

β^2

$\Rightarrow$ coeffa = unchanged $[(1-\alpha)\beta, k\alpha\beta, 0]$ coefspil $= [0\ 0\ 0\ 0\ 1]$

coeffb $= [\sigma(1-\beta^2\psi_\pi), 1-\beta-\beta\sigma\psi_x, 0]$

coefsa = unchanged $[0\ 0\ 1\ 0\ 0][I_4-\alpha\beta hx]^{-1}$

coefsb = unchanged $[-1\ 1\ 0\ 0\ 0][I_4-\beta hx]^{-1}$

coefss = unchanged $[0\ 1\ 0\ \rho\ 0]$ (shocks in TR)

$x_t = \frac{1}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)} \cdot \Big\{ -\sigma\psi_\pi\pi_{t-1} - \sigma\psi_\pi\beta$ coeffa $\cdot\, fa$

$\hookrightarrow -\sigma\psi_\pi[0\ 0\ 0\ 0\ 1]\, s_t$

$+$ coeffb $\cdot\, f_b + [-\sigma$ coefsb $- \sigma\psi_\pi\beta$ coefsa$] \cdot s_t \Big\}$

$x_t = \cancel{\frac{-\sigma\psi_\pi}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)}\pi_{t-1}} - \frac{\sigma\psi_\pi\beta}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)}$ coeffa $\cdot\, f\alpha$

$+ \frac{1}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)}$ coeffb $\cdot\, f_\beta - \frac{\sigma\psi_\pi}{1+\sigma\psi_x+k\sigma\psi_\pi\beta}$ coefspil $[0\ 0\ 0\ 0\ 1]\, s_t$

$+ \frac{1}{(1+\sigma\psi_x+k\sigma\psi_\pi\beta)}[-\sigma$ coefsb $- \sigma\psi_\pi\beta$ coefsa$] \cdot s_t$

$$\Rightarrow \pi_t = \cancel{\frac{-\zeta\psi_\pi \kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\pi_{t-1}} - \frac{\zeta\psi_\pi\beta\kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\,coeffa\cdot f\alpha$$

$$+ \frac{\kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\,coeffb\cdot f_\beta - \frac{\kappa\cdot\zeta\psi_\pi}{1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta}\begin{bmatrix}0 & 0 & 0 & 0 & 1\end{bmatrix} s_t$$

$$+ \frac{\kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\left[-\zeta\,coefsb - \zeta\psi_\pi\beta\,coefsa\right]\cdot s_t$$

$$+ \underbrace{\left[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0\right]\cdot fa}_{coeffa\cdot fa} + \underbrace{\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\left[I_4-\alpha\beta h x\right]^{-1}s_t}_{coefsa\cdot s_t}$$

$$\boxed{\begin{aligned}\Rightarrow \pi_t = &\frac{-\zeta\psi_\pi \kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\pi_{t-1} + \left(1-\frac{\zeta\psi_\pi\beta\kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\right)coeffa\cdot f\alpha\\ &+\frac{\kappa}{(1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta)}\,coeffb\cdot f_\beta + \Bigg\{-\underbrace{\frac{\kappa\zeta\psi_\pi}{1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta}\begin{bmatrix}0 & 0 & 0 & 0 & 1\end{bmatrix}}_{}\ \ coefspil\\ &+\frac{-\kappa\zeta}{1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta}\,coefsb + \left(1-\frac{\kappa\zeta\psi_\pi\beta}{1+\zeta\psi_x+\kappa\zeta\psi_\pi\beta}\right)coefsa\Bigg\}s_t\end{aligned}}$$

what is kinda funny is that I forgot that π_{t-1} is a new state, so it's a 5th entry in s_t.

→ the way I'm gonna deal with it is as a zero in s_t

and thus in coefsa & coefsb, & coefss

b/c π_{t-1} doesn't enter in the expectations

$$z_t = new \cdot \pi_{t-1} + A_a \cdot f_a + A_b \cdot f_b + A_s \cdot s_t$$

↑ bigger size b/c hx is 5×5

Or, one can avoid having to redesign the learning code and instead incorporate the effects of π_{t-1} into A_s! This is what I'll do!

↳ change in previous pages using red!

Ok done, but there's even more overshooting in IRFs. Is something wrong?

- Also RE-i oscillates a bit b/c it doesn't move on impact (THIS is instrument instability: it reacts to inflation w/ a lag)
- But it doesn't make sense that the shock doesn't have bigger impact effects both for RE and learning

Erm – I think that w/ π_{t-1} in TR instead of π_t
should have larger IRFs for RE on impact

So I was looking at Basu lectures and what do I find in Lect 22 (in a part we haven't covered)? $\Rightarrow$ disinflationary booms!

11 Dec 2019

slide 36: $\pi_t - \beta E_t \pi_{t+1} = \kappa (\hat{Y}_t - \hat{Y}_t^f)$

1) $\pi_t \uparrow \Rightarrow x_t \uparrow$ standard

2) $E_t \pi_{t+1} \downarrow$ (anticipated disinflation) $\Rightarrow x_t \uparrow$

slide 38: having a π_{t-1} term in NKPC would make disinflation costly

- this is equivalent to imperfect credibility or learning (!)

"We don't believe π_t will fall unless we see π_{t-1} fall"

The intuition of why it's costly is that in the disinflationary boom case, you could get $E_t \pi_{t+1}$ to move w/o π_t moving

$\Rightarrow$ w/ a lagged π term, you can't do that, and this will depress the output gap.

The weird thing is that I get exactly the opposite:

I add π_{t-1} then learning and I still get disinflationary booms.

→ I think I know why: b/c of what I outline in materials10Susanto: when ψ_π sufficiently large the expectations effect still outweighs the "I demand to see π fall"-effect.

There's an issue around the concept of credibility here: Ball: credibility is when $E_t\pi_{t+1}$ moves a lot. For me, that's exactly no credibility b/c agents are reassessing the target.

But I've gotten a little off track: I was gonna try to understand RE responses when π_{t-1} in TR.

The only reason the lag-π-TR can yield higher π is if due to lower movement in expectations have smaller effects on x → and yes: although $x\downarrow$ more on impact in π_t-TR, the cumulative fall in x is smaller in lag-π-TR.

What's going on? In the lag-π-TR RE model...

Although the CB won't respond on impact, it will in the 2nd period and more strongly so b/c of the "missed response" on impact. Agents w/ RE internalize this which is why $E(\pi)$ don't drop as much; and the cumulative drop in X is thus lower, leading π_t to drop less.

$\Rightarrow$ Ok, so surprising but makes sense!

Ok so to do: understand the learning responses

To do so, work hard to develop a general solution method so that adapting the model is quicker.

Also check then the pil-matrices!

Ok so: the goal is to get (quickly & painlessly) to

$$z_t = A_\alpha f_\alpha + A_b f_\beta + A_s \cdot s_t$$

Suppose we start from the std NKIS & NKPC

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left((1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n \right)$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right)$$

and the Taylor-rule will be something like

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \rho i_{t-1} + \bar{i}_t$$

As a first step, let's isolate the shocks / states

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left((1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} \right) + \sigma \sum_{T=t}^{\infty} \beta^{T-t} r_T^n$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} \right) + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} u_T$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + [0\ 1\ 0\ \rho]\, s_t$$

As a 2nd step, introduce LH expectations

$$x_t = -\sigma i_t + [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{nx} - \beta h x)^{-1} s_t$$

$$\pi_t = \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1\ 0] (I_{nx} - \alpha\beta h x)^{-1} s_t$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + [0\ 1\ 0\ \rho]\, s_t$$

Already this should be enough to solve things!

$$x_t = -\sigma i_t + [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0](I_{nx} - \beta h x)^{-1} s_t$$

$$\pi_t = \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1\ 0](I_{nx} - \alpha\beta h x)^{-1} s_t$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + [0\ 1\ 0\ \rho] s_t$$

$\Rightarrow$

$$x_t = -\sigma i_t + c_{xb} \cdot f_\beta + c_{xs} \cdot s_t$$

$$\pi_t = \kappa x_t + c_{\pi a} \cdot f_\alpha + c_{\pi s} \cdot s_t$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + c_{is} \cdot s_t$$

where c_{xy} stands for "coefficient of x on y" and are:

$c_{xb} = [\sigma, 1-\beta, -\sigma\beta]$ $(1 \times ny)$

$c_{xs} = \sigma[1\ 0\ 0\ 0](I_{nx} - \beta h x)^{-1}$ $(1 \times nx)$

$c_{\pi a} = [(1-\alpha)\beta, \kappa\alpha\beta, 0]$ $(1 \times ny)$

$c_{\pi s} = [0\ 0\ 1\ 0](I_{nx} - \alpha\beta h x)^{-1}$ $(1 \times nx)$

$c_{is} = [0\ 1\ 0\ \rho]$ $(1 \times nx)$

Then Mathematica can solve this (or you can) and it should be easier to manipulate the model by tweaking the c-matrices.

$$x_t = -\sigma i_t + c_{xb} \cdot f_\beta + c_{xs} \cdot s_t$$
$$\pi_t = \kappa x_t + c_{\pi a} \cdot f_\alpha + c_{\pi s} \cdot s_t$$
$$i_t = \psi_\pi \pi_t + \psi_x x_t + c_{is} \cdot s_t$$

$$x_t = -\sigma(\psi_\pi \pi_t + \psi_x x_t + c_{is} \cdot s_t) + c_{xb} \cdot f_\beta + c_{xs} \cdot s_t$$
$$\pi_t = \kappa x_t + c_{\pi a} \cdot f_\alpha + c_{\pi s} \cdot s_t$$

$$x_t = \frac{1}{1+\sigma\psi_x}\left[-\sigma\psi_\pi\pi_t - \sigma c_{is} \cdot s_t + c_{xb} \cdot f_\beta + c_{xs} \cdot s_t\right]$$

$$x_t = \frac{1}{1+\sigma\psi_x}\Big[-\sigma\psi_\pi(\kappa x_t + c_{\pi a} \cdot f_\alpha + c_{\pi s} \cdot s_t) - \sigma c_{is} \cdot s_t$$
$$+ c_{xb} \cdot f_\beta + c_{xs} \cdot s_t\Big]$$

$$= \frac{1}{1+\sigma\psi_x}\Big[-\sigma\psi_\pi\kappa x_t - \sigma\psi_\pi c_{\pi a} f_\alpha - \sigma\psi_\pi c_{\pi s} \cdot s_t - \sigma c_{is} \cdot s_t$$
$$+ c_{xb} \cdot f_\beta + c_{xs} \cdot s_t\Big]$$

$$= \frac{-\sigma\psi_\pi\kappa}{1+\sigma\psi_x} x_t - \frac{\sigma\psi_\pi}{1+\sigma\psi_x} c_{\pi a} f_\alpha + \frac{1}{1+\sigma\psi_x}\left[-\sigma\psi_\pi c_{\pi s} - \sigma c_{is} + c_{xs}\right] s_t$$
$$+ \frac{1}{1+\sigma\psi_x} \cdot c_{xb} \cdot f_\beta$$

$$\underbrace{\left(1 + \frac{\sigma\psi_\pi\kappa}{1+\sigma\psi_x}\right)}_{\frac{1+\sigma\psi_x+\sigma\psi_\pi\kappa}{1+\sigma\psi_x}} x_t = -\frac{\sigma\psi_\pi}{1+\sigma\psi_x} c_{\pi a} f_\alpha + \frac{1}{1+\sigma\psi_x} c_{xb} f_\beta$$
$$+ \frac{1}{1+\sigma\psi_x}\left[-\sigma\psi_\pi c_{\pi s} - \sigma c_{is} + c_{xs}\right] s_t$$

$$\left(\frac{1+\beta\psi_x+\beta\psi_\pi k}{\cancel{1+\beta\psi_x}}\right)x_t = -\frac{\beta\psi_\pi}{\cancel{1+\beta\psi_x}}\, C_{\pi a} f_\alpha + \frac{1}{\cancel{1+\beta\psi_x}}\, C_{xb}\cdot f_\beta$$

$$+ \frac{1}{\cancel{1+\beta\psi_x}}\left[-\beta\psi_\pi C_{\pi s} - \beta C_{is} + C_{xs}\right] s_t$$

$$x_t = \frac{-\beta\psi_\pi}{1+\beta\psi_x+\beta\psi_\pi k}\, C_{\pi a}\cdot f_\alpha + \frac{1}{1+\beta\psi_x+\beta\psi_\pi k}\, C_{xb}\cdot f_\beta$$

$$+ \frac{1}{1+\beta\psi_x+\beta\psi_\pi k}\left[-\beta\psi_\pi C_{\pi s} - \beta C_{is} + C_{xs}\right] s_t$$

Now check this against bottom of Mathematica, materials 12.nb. It's correct (solGEN)

Of course it doesn't equal my old solution. I wonder if it's b/c I'm somehow not adequately dealing w/ the mon. pol shock i → it should maybe enter the $E(\cdot)$ in the NKPC? But then again - one could argue the same about cost push shocks? It certainly seems like only the MP shock has different loadings (2nd column of A_s) and in terms of LH-E(.), x & i have different ones (1st & 3rd column of A_b).

Need to understand why the two solutions for the A-matrices differ. 12 Dec 2019

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n \right] \quad (1)$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta\, x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right] \quad (2)$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \rho i_{t-1} + \bar{i}_t \quad (3)$$

The old sol was:

$$x_t = -\sigma\left[\psi_\pi \pi_t + \psi_x x_t + \rho i_{t-1} + \bar{i}_t\right]$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta \left[\psi_\pi \pi_{T+1} + \psi_x x_{T+1} + \rho i_T + \bar{i}_{T+1}\right] + \sigma r_T^n \right]$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t - \sigma\left[\rho i_{t-1} + \bar{i}_t\right]$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta\psi_\pi) \pi_{T+1} - \sigma\beta\left[\rho i_T + \bar{i}_{T+1}\right] + \sigma r_T^n \right]$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta\psi_\pi) \pi_{T+1} - \sigma\left[\rho i_{T-1} + \bar{i}_T\right] + \sigma r_T^n \right]$$

So we can write (1) & (2) as

$$
\left.\begin{aligned}
(1+b\psi_x)x_t &= -b\psi_\pi \pi_t \\
&+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta-b\beta\psi_x)x_{T+1} + b(1-\beta\psi_\pi)\pi_{T+1} - b[\rho i_{T-1} + \bar{i}_T] + b r_T^n\right] \\
\pi_t &= \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\left[\kappa\alpha\beta\, x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\right]
\end{aligned}\right\}
$$

$\Leftrightarrow$

$$
\left.\begin{aligned}
x_t &= \frac{-b\psi_\pi}{1+b\psi_x}\pi_t + \frac{1}{1+b\psi_x}\left[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0\right] f_\beta \\
&+ \frac{1}{1+b\psi_x}(-b)\left[-1 \;\; 1 \;\; 0 \;\; \rho\right](I_{hx} - \beta hx)^{-1} s_t \qquad (*) \\
\pi_t &= \kappa x_t + \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] f_\alpha + \left[0 \; 0 \; 1 \; 0\right](I_{hx} - \alpha\beta hx)^{-1} s_t
\end{aligned}\right\}
$$

Hm. Anyway, so then

$$
\begin{aligned}
(1+b\psi_x)x_t &= \\
&-b\psi_\pi \left\{\kappa x_t + \left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] f_\alpha + \left[0 \; 0 \; 1 \; 0\right](I_{hx} - \alpha\beta hx)^{-1} s_t\right\} \\
&+ \left[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0\right] f_\beta + (-b)\left[-1 \;\; 1 \;\; 0 \;\; \rho\right](I_{hx} - \beta hx)^{-1} s_t
\end{aligned}
$$

$$
\begin{aligned}
(1+b\psi_x + b\kappa\psi_\pi)x_t &= \\
&-b\psi_\pi\left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] f_\alpha - b\psi_\pi\left[0 \; 0 \; 1 \; 0\right](I_{hx} - \alpha\beta hx)^{-1} s_t \\
&+ \left[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0\right] f_\beta + \frac{(-b)}{w}\left[-1 \;\; 1 \;\; 0 \;\; \rho\right](I_{hx} - \beta hx)^{-1} s_t
\end{aligned}
$$

$$
\begin{aligned}
x_t &= \frac{-b\psi_\pi}{w}\left[(1-\alpha)\beta, \kappa\alpha\beta, 0\right] f_\alpha + \frac{1}{w}\left[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0\right] f_\beta \\
&+ \left[\frac{-b\psi_\pi}{w}\left[0 \; 0 \; 1 \; 0\right](I_{hx} - \alpha\beta hx)^{-1} + \frac{(-b)}{w}\left[-1 \;\; 1 \;\; 0 \;\; \rho\right](I_{hx} - \beta hx)^{-1}\right] s_t
\end{aligned}
$$

x^*, old method

Now let's solve using the new method (for x only)

$$\left.\begin{aligned} x_t &= -\sigma i_t + [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{hx} - \beta h x)^{-1} s_t \\ \pi_t &= \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1\ 0] (I_{hx} - \alpha\beta h x)^{-1} s_t \\ i_t &= \psi_\pi \pi_t + \psi_x x_t + [0\ 1\ 0\ \rho] s_t \end{aligned}\right\}$$

So if I solve for x, I get

$$x_t = -\sigma [\psi_\pi \pi_t + \psi_x x_t + [0\ 1\ 0\ \rho] s_t]$$
$$+ [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{hx} - \beta h x)^{-1} s_t$$

$$(1+\sigma\psi_x) x_t = -\sigma \psi_\pi \pi_t - \sigma [0\ 1\ 0\ \rho] s_t$$
$$+ [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{hx} - \beta h x)^{-1} s_t$$

$$(1+\sigma\psi_x) x_t = -\sigma [0\ 1\ 0\ \rho] s_t$$
$$-\sigma\psi_\pi \left\{ \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1\ 0] (I_{hx} - \alpha\beta h x)^{-1} s_t \right\}$$
$$+ [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{hx} - \beta h x)^{-1} s_t$$

$$(1+\sigma\psi_x) x_t = -\sigma [0\ 1\ 0\ \rho] s_t - \sigma\psi_\pi \kappa x_t$$
$$-\sigma\psi_\pi [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha - \sigma\psi_\pi [0\ 0\ 1\ 0] (I_{hx} - \alpha\beta h x)^{-1} s_t$$
$$+ [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1\ 0\ 0\ 0] (I_{hx} - \beta h x)^{-1} s_t$$

$$(1+\sigma\psi_x + \sigma\psi_\pi \kappa) x_t = RHS$$

$$x_t = -\frac{b}{w}[0\ 1\ 0\ \rho]\, s_t$$
$$-\frac{b\psi_\pi}{w}[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha - \frac{b\psi_\pi}{w}[0\ 0\ 1\ 0](I_{hx}-\alpha\beta h_x)^{-1} s_t$$
$$+\frac{1}{w}[b,\ 1-\beta,\ -b\beta]\, f_\beta + \frac{b}{w}[1\ 0\ 0\ 0](I_{hx}-\beta h_x)^{-1} s_t$$

$$x_t = -\frac{b\psi_\pi}{w}[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + \frac{1}{w}[b,\ 1-\beta,\ -b\beta]\, f_\beta$$
$$+\Big\{-\frac{b\psi_\pi}{w}[0\ 0\ 1\ 0](I_{hx}-\alpha\beta h_x)^{-1} + \frac{b}{w}[1\ 0\ 0\ 0](I_{hx}-\beta h_x)^{-1}$$
$$-\frac{b}{w}[0\ 1\ 0\ \rho]\Big\}\, s_t$$

x^* using new method

Let's compare old and new:

$$x_t = -\frac{b\psi_\pi}{w}[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + \frac{1}{w}\underline{[b(1-\beta\psi_\pi),\ (1-\beta-b\beta\psi_x),\ 0]}\, f_\beta$$
$$+\Big[-\frac{b\psi_\pi}{w}[0\ 0\ 1\ 0](I_{hx}-\alpha\beta h_x)^{-1} + \underline{\left(\frac{-b}{w}\right)[-1\ 1\ 0\ \rho]}\,(I_{hx}-\beta h_x)^{-1}\Big] s_t$$

clearly one diff is the loadings on f_β

the second thing is this matrix on shocks

$$x_t = -\frac{b\psi_\pi}{w}[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + \frac{1}{w}\underline{[b,\ 1-\beta,\ -b\beta]}\, f_\beta$$
$$+\Big\{-\frac{b\psi_\pi}{w}[0\ 0\ 1\ 0](I_{hx}-\alpha\beta h_x)^{-1} + \underline{\frac{b}{w}[1\ 0\ 0\ 0]}(I_{hx}-\beta h_x)^{-1}$$
$$\underline{-\frac{b}{w}[0\ 1\ 0\ \rho]}\Big\}\, s_t$$

I'm thinking right now that the two systems should be different but give rise to the same dynamics.

Another way to summarize the two methods is:

<u>Old method</u>: sub in i_t to reduce the system to

$$M \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = N \qquad (M \ 2\times2 \ , N \ 2\times1)$$

<u>New method</u> don't sub in i_t, just write

$$P \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = Q \qquad (P \ 3\times3, \ Q \ 3\times1)$$

I think maybe this way of writing makes clear that the two approaches might treat shocks and expectations differently, for which reason the old method may still be preferable.

If this is so, then I should still be able to write a general solution procedure using M and N.

General method based on the old method
(M-N method)

Plugging in the TR-rule, from system (*) we have:

$$(1+\sigma\psi_x)x_t = -\sigma\psi_\pi\pi_t + [\sigma(1-\beta\psi_\pi), (1-\beta-\sigma\beta\psi_x), 0]f_\beta$$
$$-\sigma\,[-1\ \ 1\ \ 0\ \ \rho]\,(I_{hx}-\beta hx)^{-1}s_t$$

$$\pi_t = \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0]f_a + [0\ 0\ 1\ 0](I_{hx}-\alpha\beta hx)^{-1}s_t$$

$\Leftrightarrow$

$$\begin{bmatrix}\sigma\psi_\pi ; 1+\sigma\psi_x \\ 1 ; -\kappa\end{bmatrix}\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix}\overbrace{[\sigma(1-\beta\psi_\pi), (1-\beta-\sigma\beta\psi_x), 0]}^{cxb}f_\beta \ \overbrace{-\sigma[-1\ 1\ 0\ \rho](I_{hx}-\beta hx)^{-1}}^{cxs}s_t \\ \underbrace{[(1-\alpha)\beta, \kappa\alpha\beta, 0]}_{c\pi a}f_a + \underbrace{[0\ 0\ 1\ 0](I_{hx}-\alpha\beta hx)^{-1}}_{c\pi s}s_t\end{bmatrix}$$

Or, more compactly,

$$\underbrace{\begin{bmatrix}\sigma\psi_\pi ; 1+\sigma\psi_x \\ 1 ; -\kappa\end{bmatrix}}_{M}\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \underbrace{\begin{bmatrix}c_{xb} f_b + c_{xs}\cdot s_t \\ c_{\pi a} f_a + c_{\pi s}\cdot s_t\end{bmatrix}}_{N}$$

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = M^{-1}N$$

$\rightarrow i_t = \psi_\pi\cdot\pi^* + \psi_x\cdot x^* + \overbrace{[0\ 1\ 0\ \rho]}^{cis}s_t$

$\rightarrow$ get g_{xy} as coeff of x on y for $x,y = \pi, x, i$

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Alright, then if we have π_{t-1} in the TR, then using method M-N,

$$i_t = \psi_\pi \pi_{t-1} + \psi_x x_t + \rho i_{t-1} + \bar{i}_t$$

$$i_t = \psi_x x_t + \underbrace{[0\ 1\ 0\ \rho\ \psi_\pi]}_{\text{new } C_{is}} \underset{5\times 1}{s_t}$$

$$x_t = -\sigma(\psi_x x_t + C_{is}^{new} s_t)$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n\right]$$

while $\pi_t = \kappa x_t + \underbrace{[(1-\alpha)\beta, \kappa\alpha\beta, 0] f_a}_{C_{pa}(\text{unchanged})} + \underbrace{[0\,0\,1\,0\,0](I_{nx} - \alpha\beta h_x)^{-1}}_{\text{new } C_{ps}} s_t$

So $(1+\sigma\psi_x) x_t = \rho i_{t-1} + \bar{i}_t + \psi_\pi \pi_{t-1}$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta\left[\psi_x x_{T+1} + \bar{i}_{T+1} + \rho i_T + \psi_\pi \pi_T\right] + \sigma r_T^n\right]$$

$$(1+\sigma\psi_x) x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma\pi_{T+1} - \sigma\left[\bar{i}_T + \rho i_{T-1} + \psi_\pi \pi_{T-1} - r_T^n\right]\right]$$
$$= \underbrace{[\sigma, 1-\beta-\sigma\beta\psi_x, 0] f_\beta}_{\text{new } C_{xb}} \underbrace{-\sigma[-1, 1, 0, \rho, \psi_\pi](I_{nx} - \beta h_x)^{-1}}_{\text{new } C_{xs}} s_t$$

$$(1+\sigma\psi_x) x_t = C_{xb} f_\beta + C_{xs} \cdot s_t$$

$$\pi_t = \kappa x_t + C_{pa} \cdot f_\alpha + C_{ps} \cdot s_t$$

$$(1+\sigma\psi_x)x_t = C_{xb} f_\beta + C_{xs} \cdot s_t$$

$$\pi_t = \kappa x_t + C_{pa} \cdot f_\alpha + C_{ps} \cdot s_t$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 1+\sigma\psi_x \\ 1 & -\kappa \end{bmatrix}}_{\substack{M \\ \text{(new)}}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} C_{xb} f_\beta + C_{xs} \cdot s_t \\ C_{pa} \cdot f_\alpha + \underbrace{C_{ps} \cdot s_t}_{\substack{N \\ \text{(new)}}} \end{bmatrix}$$

And also new is $i_t = \psi_x \cdot x_t^* + C_{is} \cdot s_t$

Erm – IRFs are explosive… but at the same time, when I look at the old sol, that sure don't look good!

↳ but it has a grain of truth… b/c at a certain point the lagged π becomes sthg we need to expect (here) – Let's reconsider…

$$i_t = \psi_\pi \pi_{t-1} + \psi_x x_t + \rho i_{t-1} + \bar{i}_t$$

$$\pi_t = \kappa x_t + \underbrace{[(1-\alpha)\beta, \kappa\alpha\beta, 0]}_{C_{pa}\text{(unchanged)}} f_a + \underbrace{[0\,0\,1\,0\,0](I_{nx} - \alpha\beta h_x)^{-1}}_{\text{new } C_{ps}} s_t$$

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} + \sigma\pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n \right]$$

$$x_t = -b\left(\psi_\pi \pi_{t-1} + \psi_x x_t + \rho i_{t-1} + \bar{i}_t\right)$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)x_{T+1} + b\pi_{T+1} - b\beta i_{T+1} + b r_T^n\right]$$

$$(1+b\psi_x)x_t = -b\left(\psi_\pi \pi_{t-1} + \rho i_{t-1} + \bar{i}_t\right)$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)x_{T+1} + b\pi_{T+1} - b\beta\left[\psi_\pi \pi_T + \psi_x x_{T+1} + \rho i_T + \bar{i}_{T+1}\right] + b r_T^n\right]$$

$\Leftrightarrow$

$$(1+b\psi_x)x_t = -b\left(\psi_\pi \pi_{t-1} + \rho i_{t-1} + \bar{i}_t\right)$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta-b\beta\psi_x)x_{T+1} + b\pi_{T+1} - b\beta\left[\psi_\pi \pi_T + \rho i_T + \bar{i}_{T+1}\right] + b r_T^n\right]$$

$\Leftrightarrow$

$$(1+b\psi_x)x_t = \hat{E}\sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta-b\beta\psi_x)x_{T+1} + b\pi_{T+1} - b\left[\underbrace{\psi_\pi \pi_{T-1} + \rho i_{T-1}} + \bar{i}_T + r_T^n\right]\right]$$

here the questions emerge, and they emerge for i_{T-1} as well!

$$\hat{E}\sum_{T=t}^{\infty} \beta^{T-t}\left[\psi_\pi \pi_{T-1} + \rho i_{T-1}\right]$$
$$= \underbrace{(\psi_\pi \pi_{t-1} + \rho i_{t-1}) + \beta(\psi_\pi \pi_t + \rho i_t)}_{\text{known}} + \underbrace{\beta^2(\psi_\pi \pi_{t+1} + \rho i_{t+1}) + \ldots}_{\text{unknown}}$$

$$= \text{known stuff} + \beta^2\left[\hat{E}\sum_{T=t}^{\infty} \beta^{T-t}\left(\psi_\pi \pi_{T+1} + \rho i_{T+1}\right)\right]$$

This is the problem!

In fact it hints at the fact that even the model w/ lagged-i is wrong b/c it doesn't take into account that **i is an endogenous state** → agents only know the LOM of exog states. I think! – right? agents should only know exog states? Or? But only knowing exog states amounts to ... not knowing a subset of hx ... omg ... but I think that's correct b/c looking at the lag-i model, the bottom part of hx (the row pertaining to i_{t-1}) is just a combo of elements or gx ... it wouldn't be consistent w/ learning gx if agents actually knew it in hx ...

Look: materials6, p. 2 (eq 16): $hx(4,:) = gx(4,:)$.

⇒ so we need indeed to account for $hx(4,:)$ not known!

It shouldn't really matter for results so far b/c I've in general set $\rho = 0$, shutting off i_{t-1}. But need to correct!

So taking another step back ... model with lagged i ...

$$\left.\begin{aligned} x_t &= -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^N \right\} \\ \pi_t &= \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right\} \\ i_t &= \psi_\pi \pi_t + \psi_x x_t + \rho i_{t-1} + \bar{i}_t \end{aligned}\right\}$$

$$(1+\sigma\psi_x) x_t = -\sigma\left[\psi \pi_t + \rho i_{t-1} + \bar{i}_t\right]$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta \left[\psi_\pi \pi_{T+1} + \psi_x x_{T+1} + \rho i_T + \bar{i}_{T+1}\right] + r_T^N \right\}$$

$\Leftrightarrow$

$$(1+\sigma\psi_x) x_t = -\sigma\left[\psi_\pi \pi_t + \rho i_{t-1} + \bar{i}_t\right]$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta\psi_\pi) \pi_{T+1} - \sigma\beta\left[\rho i_T + \bar{i}_{T+1}\right] + r_T^N \right\}$$

$\Leftrightarrow$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t - \sigma\rho i_{t-1}$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta\psi_\pi) \pi_{T+1} - \sigma\beta\rho i_T - \sigma\left[-r_T^N + \bar{i}_T\right] \right\}$$

$\uparrow$ now I'm worried that this gives us an endless recursion ...

But: as the simulation code makes clear [e.g. sim_learnLH.m], when $\hat{E}_t z_{t+s}$ is formed, z_t is not realized yet – does that help us?

Not really, b/c x_t, π_t aren't either. No – but no proof, just pull it out!

$$(1+\sigma\psi_x)x_t = -\sigma\psi_\pi\pi_t - \sigma\rho i_{t-1}$$
$$+\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left\{(1-\beta-\sigma\beta\psi_x)x_{T+1} + \sigma(1-\beta\psi_\pi)\pi_{T+1} - \sigma\beta\rho i_T - \sigma\left[-r_T^N + \bar{i}_T\right]\right\}$$
$$\Rightarrow (1+\sigma\psi_x)x_t = -\sigma\psi_\pi\pi_t - \sigma\rho i_{t-1} - \sigma\beta\rho i_t$$
$$+\hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\left\{(1-\beta-\sigma\beta\psi_x)x_{T+1} + \sigma(1-\beta\psi_\pi)\pi_{T+1} - \sigma\beta^2\rho i_{T+1} - \sigma\left[-r_T^N + \bar{i}_T\right]\right\}$$

b/c $\quad E\sum\beta^{T-t}\beta i_T = \beta i_t + \beta^2 i_{t+1} + \beta^3 + i_{t+2} + \dots$

$$= \beta i_t + \beta^2\sum_{T=t}^{\infty}\beta^{T-t} i_{T+1}$$

Now the problem is that by the same token that not subbing i out of f_β before was wrong, it still should be wrong...

What to do now?

I think that I need to return to the simplest model and the question of M-N method vs PQ-method (old vs. new) and how these relate to ass's about what is known and what is learned. 29 Dec 2019

A generic representation of the model is the state-space form:
$$X_t = hx \cdot X_{t-1} + \eta\,\varepsilon_t$$
$$Y_t = gx \cdot X_t \qquad \text{(caps denote vectors)}$$

As long as all X exog, agents think

$$X_t = h_x \cdot X_{t-1} + \eta \varepsilon_t \qquad \text{(correct)}$$

$$Y_t = \hat{a} + \hat{g} \cdot X_t \iff Y_t = \hat{\phi} X_t$$

This representation is the solution to

$$E_t\left(f_x X_t + f_y Y_t + f_{xp} X_{t+1} + f_{yp} Y_{t+1} \right) = 0$$

(possibly w/ wrong signs somewhere)

So if the agents know f_x, f_y, f_{xp}, f_{yp}, which they should given that these are loglin FOCs, then why can't they just do LU-decomp to solve the model & find g_x? Although right now I'm not sure they know the fs b/c even though they obtain the same FOCs they don't know that others did so too.

But that's what I don't get: do they then know the NKPC, the NKIS, the TR? More generally, do they know the relationships between the elements of Y? Right now I think no - when I work w/ PC, IS, TR, I'm working w/ agg rules that govern dynamics, but that doesn't mean that agents know those rules.

But that should imply that the PQ - method (the new) should be the same ... unless it embodies an expectational assumption somewhere that I don't see yet.

Ignore shocks for a moment:

The most basic system is:

$$\left.\begin{aligned} x_t &= -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} \right\} \\ \pi_t &= \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} \right\} \\ i_t &= \psi_\pi \pi_t + \psi_x x_t \end{aligned}\right\}$$

The only way there can be a difference between the two methods is if the computer doesn't understand that $\hat{E}_t i_{T+1} = \hat{E}_t(\psi_\pi \pi_T + \psi_x x_{T+1})$

$$x_t + \sigma i_t = [\sigma \quad 1-\beta \quad -\sigma\beta] f_\beta$$

<u>30 Dec 2019</u>

$$\pi_t - \kappa x_t = [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha$$

$$-\psi_\pi \pi_t - \psi_x x_t + i_t = 0$$

$$\underset{3\times 3}{\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 0 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} [\sigma, 1-\beta, -\sigma\beta] f_\beta \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha \\ 0 \end{bmatrix}}$$

vs old method

$$x_t = -\sigma(\psi_\pi \pi_t + \psi_x x_t) + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta(\psi_\pi \pi_{T+1} + \psi_x x_{T+1}) \right\}$$

$$\pi_t - \kappa x_t = [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] f_a$$

$$\sigma\psi_\pi \pi_t + (1+\sigma\psi_x) x_t = \hat{E} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + (\sigma - \sigma\beta\psi_\pi)\pi_{T+1} \right\}$$

So

$$\sigma\psi_\pi \pi_t + (1+\sigma\psi_x) x_t = [\sigma(1-\beta\psi_\pi),\ (1-\beta-\sigma\beta\psi_x),\ 0] f_b$$

$$\pi_t - \kappa x_t = [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] f_a$$

$$\begin{bmatrix} \sigma\psi_\pi, & 1+\sigma\psi_x \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} [\sigma(1-\beta\psi_\pi),\ (1-\beta-\sigma\beta\psi_x),\ 0] f_b \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] f_a \end{bmatrix}$$

So

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \sigma\psi_\pi & 1+\sigma\psi_x \\ 1 & -\kappa \end{bmatrix}^{-1} \begin{bmatrix} [\sigma(1-\beta\psi_\pi),\ (1-\beta-\sigma\beta\psi_x),\ 0] f_b \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] f_a \end{bmatrix} \quad (M\text{-}N)$$

vs.

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 0 \end{bmatrix}^{-1} \begin{bmatrix} [\sigma,\ 1-\beta,\ -\sigma\beta] f_b \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] f_a \\ 0 \end{bmatrix} \quad (P\text{-}Q)$$

In Mathematica (materials12.nb) $\psi := \gamma_\pi$ and $\delta := \gamma_x$

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma\psi & 1+\sigma\delta \\ 1 & -k \end{bmatrix}^{-1}}_{=:M} \underbrace{\begin{bmatrix} [\sigma(1-\beta\psi),\ (1-\beta-\sigma\beta\delta),\ 0]\, f_\beta \\ [(1-\alpha)\beta,\ k\alpha\beta,\ 0]\, f_a \end{bmatrix}}_{=:N} \quad (M\text{-}N)$$

vs.

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -k & 0 \\ -\psi & -\delta & 0 \end{bmatrix}^{-1}}_{=:P} \underbrace{\begin{bmatrix} [\sigma,\ 1-\beta,\ -\sigma\beta]\, f_\beta \\ [(1-\alpha)\beta,\ k\alpha\beta,\ 0]\, f_a \\ 0 \end{bmatrix}}_{=:Q} \quad (P\text{-}Q)$$

$\Leftrightarrow$

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{k}{1+\sigma\delta+k\sigma\psi} & \frac{1+\sigma\delta}{1+\sigma\delta+k\sigma\psi} \\ \frac{1}{1+\sigma\delta+k\sigma\psi} & \frac{-\sigma\psi}{1+\sigma\delta+k\sigma\psi} \end{bmatrix} \underbrace{\begin{bmatrix} [\sigma(1-\beta\psi),\ (1-\beta-\sigma\beta\delta),\ 0]\, f_\beta \\ [(1-\alpha)\beta,\ k\alpha\beta,\ 0]\, f_a \end{bmatrix}}_{=:N} \quad (M\text{-}N)$$

vs.

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & \frac{-\sigma\delta}{-\delta\sigma-k\sigma\psi} & \frac{k\sigma}{-\delta\sigma-k\sigma\psi} \\ 0 & \frac{\sigma\psi}{-\delta\sigma-k\sigma\psi} & \frac{\sigma}{-\delta\sigma-k\sigma\psi} \\ \frac{-\sigma-k\psi}{-\delta\sigma-k\sigma\psi} & \frac{-\psi}{-\delta\sigma-k\sigma\psi} & \frac{-1}{-\delta\sigma-k\sigma\psi} \end{bmatrix} \underbrace{\begin{bmatrix} [\sigma,\ 1-\beta,\ -\sigma\beta]\, f_\beta \\ [(1-\alpha)\beta,\ k\alpha\beta,\ 0]\, f_a \\ 0 \end{bmatrix}}_{=:Q} \quad (P\text{-}Q)$$

Let's solve the following, very stupid system:

$x_t = -\sigma i_t$

$\pi_t = \kappa x_t$

$i_t = \psi \pi_t + \delta x_t$

M-N method: $x_t = -\sigma(\psi \pi_t + \delta x_t) = -\sigma\psi\pi_t - \sigma\delta x_t$

$\pi_t = \kappa x_t$

Hmm. Maybe the problem w/ the P-Q method is that the TR has a 0 RHS. Look:

$$\sigma\psi \pi_t + (1+\sigma\delta) x_t = 0$$

$$\pi_t - \kappa x_t = 0$$

$$\begin{bmatrix} \sigma\psi, & 1+\sigma\delta \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} -\kappa & -(1+\sigma\delta) \\ -1 & \sigma\psi \end{bmatrix} \frac{1}{-\sigma\psi\kappa - (1+\sigma\delta)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is a <u>homogenous</u> system! (RHS vector is the null)

A homo system has 1 sol or ∞, the latter if # vars > # of eqs. So the fa & fb act like a nonhomogen part!

Consider the following system: X

$x_t = -\beta i_t + a$

$\pi_t = k x_t + b$

$i_t = \psi \pi_t + \delta x_t + c$

Now I bet M-N & P-Q yield the same sol:

$x_t = -\beta \psi \pi_t - \beta \delta x_t - \beta c + a$

$\pi_t = k x_t + b$

$\Rightarrow \quad \beta\psi \pi_t + (1+\beta\delta) x_t = -\beta c + a$

$\pi_t - k x_t = b$

$$\begin{bmatrix} \beta\psi , & 1+\beta\delta \\ 1 , & -k \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} -\beta c + a \\ b \end{bmatrix}$$

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \beta\psi , & 1+\beta\delta \\ 1 , & -k \end{bmatrix}^{-1} \begin{bmatrix} -\beta c + a \\ b \end{bmatrix} \qquad (M-N)$$

$$x_t = -\sigma i_t + a$$
$$\pi_t = \kappa x_t + b$$
$$i_t = \psi \pi_t + \delta x_t + c$$
$$\Rightarrow \begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi & -\delta & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi & -\delta & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (P-Q)$$

You can see in Mathematica that the two sols are the same! (I've checked for π & x, for i it's not so obvious.)

If you set $c = 0$, you still get $MN = PQ$.

So this homo vs non-homo thing isn't the thing.

Plus w/ shocks the TR is also non-homo anyway.

So it has to do w/ expectations. 31 Dec 2019

The most basic system again: supp this doesn't ex

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \cancel{\sigma\beta i_{T+1}} \right\}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} \right\}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t$$

Now I bet that the two methods give the same sol

$$x_t = -\sigma(\psi_\pi \pi_t + \psi_x x_t) + [\sigma, 1-\beta, 0] f_\beta$$

$$\pi_t = \kappa x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha$$

$$\begin{bmatrix} \sigma\psi_\pi & (1+\sigma\psi_x) \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} [\sigma, 1-\beta, 0] f_\beta \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha \end{bmatrix} \quad (M\text{-}N)$$

$$\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} [\sigma, 1-\beta, 0] f_\beta \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha \\ 0 \end{bmatrix} \quad (P\text{-}Q)$$

Note that this corresponds to the RHS of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ w/ $c=0$

and we've already seen that in that case the sols

(M-N) & (P-Q) correspond.

I think this causes a problem b/c it's as if you changed the independent RHS in $AB = C$.

What this suggests to me is that

$$\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + b\pi_{T+1} - b\beta i_{T+1} \right\} \neq$$

$$\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + b\pi_{T+1} - b\beta \left(\psi_\pi \pi_{T+1} + \psi_x x_{T+1} \right) \right\}$$

$\Leftrightarrow$

$$[b,\ 1-\beta,\ -b\beta] f_\beta \neq [b - b\beta\psi_\pi,\ 1-\beta-b\beta\psi_x,\ 0] f_\beta$$

$$\Leftrightarrow [b,\ 1-\beta,\ 0] f_\beta + [0,\ 0,\ -b\beta] f_\beta \neq$$

$$[b,\ 1-\beta,\ 0] f_\beta + [-b\beta\psi_\pi,\ -b\beta\psi_x,\ 0] f_\beta$$

$$\Leftrightarrow [0,\ 0,\ -b\beta] f_\beta \neq [-b\beta\psi_\pi,\ -b\beta\psi_x,\ 0] f_\beta \qquad | : (-b\beta)$$

$$\Leftrightarrow [0,\ 0,\ 1] f_\beta \neq [\psi_\pi,\ \psi_x,\ 0] f_\beta$$

which is weird!

No wait – it makes perfect sense that the computer doesn't realize this and this is why I get different things!

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1)$$

this is what the computer doesn't internalize.

To do: 1) show on Mathematica that w/ this piece of info, The M-N & P-Q sols coincide → Yes! materials12.nb

2) Correct general method

3) Check why on Matlab the two still don't give the same dynamics. ⇒ b/c shocks weren't treated correctly I think (condition (*) wasn't correct yet)

2 Jan 2019

2) Now take the errors back in:

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n \right\}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right\}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t$$

(no i-rate smoothing)

M-N method: Sub out i_t:

$$x_t = -\sigma(\psi_\pi \pi_t + \psi_x x_t + \bar{i}_t) + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta(\psi_\pi \pi_{T+1} + \psi_x x_{T+1} + \bar{i}_{T+1}) + \sigma r_T^n \right\}$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \pi_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ \sigma(1-\beta\psi_\pi)\pi_{T+1} + (1-\beta-\sigma\beta\psi_x) x_{T+1} - \sigma \bar{i}_T + \sigma r_T^n \right\}$$

$$\sigma\psi_\pi \pi_t + (1+\sigma\psi_x) x_t = [\sigma(1-\beta\psi_\pi), (1-\beta-\sigma\beta\psi_x), 0] f_\beta - \sigma[-1, 1, 0](I_{nx} - \beta h_x)^{-1} s_t$$

$$\pi_t - \kappa x_t = [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1](I_{nx} - \alpha\beta h_x)^{-1} s_t$$

$$\underbrace{\begin{bmatrix} \sigma\psi_\pi & 1+\sigma\psi_x \\ 1 & -\kappa \end{bmatrix}}_{M} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \underbrace{\begin{bmatrix} [\sigma(1-\beta\psi_\pi),\ (1-\beta-\sigma\beta\psi_x),\ 0]\, f_\beta - \overbrace{\sigma[-1,1,0]\overbrace{(I_{hx}-\beta hx)^{-1}}^{=: f_{\gamma\beta} = f_{cb}}}^{=d1} s_t \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + \underbrace{[0\ 0\ 1]\underbrace{(I_{hx}-\alpha\beta hx)^{-1}}_{=: f_{\gamma a} = f_{ca}}}_{=d2} s_t \end{bmatrix}}_{N}$$

and then $i_t = \psi_\pi \pi_t^* + \psi_x x_t^* + \underbrace{[0\ 1\ 0]\, s_t}_{=d3}$

P-Q method

$$x_t + \sigma i_t = [\sigma,\ 1-\beta,\ -\sigma\beta]\, f_\beta + \sigma[1\ 0\ 0](I_{nx}-\beta hx)^{-1} s_t$$

$$\pi_t - \kappa x_t = [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + [0\ 0\ 1](I_{nx}-\alpha\beta hx)^{-1} s_t$$

$$-\psi_\pi \pi_t - \psi_x x_t + i_t = [0\ 1\ 0]\, s_t$$

$$\underbrace{\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 1 \end{bmatrix}}_{P} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \underbrace{\begin{bmatrix} [\sigma,\ 1-\beta,\ -\sigma\beta]\, f_\beta + \overbrace{\sigma[1\ 0\ 0]\overbrace{(I_{nx}-\beta hx)^{-1}}^{=f_{cb}}}^{=d4} s_t \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0]\, f_\alpha + \underbrace{[0\ 0\ 1]\underbrace{(I_{nx}-\alpha\beta hx)^{-1}}_{=f_{ca}}}_{=d5} s_t \\ [0\ 1\ 0]\, s_t \end{bmatrix}}_{Q}$$

$= d6$

w/ the extra condition that:

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1) + \underbrace{\overbrace{[0\ 1\ 0]\overbrace{(I_{nx}-\beta hx)^{-1}}^{=f_{cb}}}^{=d7} s_t}_{(*)}$$

(*) the new term that's supp to account for the error term

Mathematica has some trouble here so there are 2 ways I can check whether the two sols coincide:

1) $i^*_{NM} - i^*_{PQ} = -d5 + (d2 - d4)\Psi$
$$+ (d1 - d3 + d7\,\mathfrak{z}\beta)(\delta + \kappa\Psi)$$
$$+ d6(1 + \delta\mathfrak{z} + \kappa\mathfrak{z}\Psi)$$

2) Let Mathematica give you M^{-1} and P^{-1} and calculate $i^*_{NM} - i^*_{PQ}$ analytically from there.

1) $-[0\ 0\ 1](I_{nx} - \alpha\beta h_x)^{-1} s_t$
$$+ \left\{\left([0\ 0\ 1](I_{nx} - \alpha\beta h_x)^{-1} s_t - \mathfrak{z}[1\ 0\ 0](I_{nx} - \beta h_x)^{-1} s_t\right)\Psi\right\}$$
$$+ (\delta + \kappa\Psi)\Big\{-\mathfrak{z}[-1, 1, 0](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0]\,s_t$$
$$+ [0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t\Big\}$$
$$+ (1 + \delta\mathfrak{z} + \kappa\mathfrak{z}\Psi)[0\ 1\ 0]\,s_t$$

$$= \left\{-[0\ 0\ 1](I_{nx} - \alpha\beta h_x)^{-1} + (1 + \delta\mathfrak{z} + \kappa\mathfrak{z}\Psi)[0\ 1\ 0]\right\} s_t$$
$$+ \Psi\left\{[0\ 0\ 1](I_{nx} - \alpha\beta h_x)^{-1} - \mathfrak{z}[1\ 0\ 0](I_{nx} - \beta h_x)^{-1}\right\} s_t$$
$$+ (\delta + \kappa\Psi)\left\{[\mathfrak{z}, 1 + \mathfrak{z}, 0](I_{nx} - \beta h_x)^{-1} - [0\ 1\ 0]\right\} s_t$$

$$= \{ -[0\ 0\ 1](I_{nx} - \alpha\beta h x)^{-1} + (1+\delta b + kb\psi)[0\ 1\ 0] \} S_t$$

$$+ \psi \{ [0\ 0\ 1](I_{nx} - \alpha\beta h x)^{-1} - b[1\ 0\ 0](I_{nx} - \beta h x)^{-1} \} S_t$$

$$+ (\delta + k\psi) \{ [b, 1+b, 0](I_{nx} - \beta h x)^{-1} - [0\ 1\ 0] \} S_t$$

$$= \{ \underline{[0\ 0\ {-1}](I_{nx} - \alpha\beta h x)^{-1}} + [0, 1+\delta b + kb\psi, 0]$$

$$+ \underline{[0\ 0\ \psi](I_{nx} - \alpha\beta h x)^{-1}} + [-b\ 0\ 0](I_{nx} - \beta h x)^{-1}$$

$$+ [b(\delta + k\psi), (\delta + k\psi)(1+b), 0](I_{nx} - \beta h x)^{-1} + [0, -\delta - k\psi, 0] \} S_t$$

$$= \{ [0\ 0\ \psi - 1](I_{nx} - \alpha\beta h x)^{-1}$$

$$+ [-b + b(\delta + k\psi), (\delta + k\psi)(1+b), 0](I_{nx} - \beta h x)^{-1}$$

$$+ [0, 1 + \delta b + kb\psi - \delta - k\psi, 0] \} S_t$$

Honestly that doesn't look like it was $= 0$.

2) NM: $\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \underset{2\times 2}{M^{-1}} \underset{2\times 1}{N} = \begin{bmatrix} m1 & m2 \\ m3 & m4 \end{bmatrix} \begin{bmatrix} n1 \\ n2 \end{bmatrix}$

$\pi_t = m1 \cdot n1 + m2 \cdot n2 \qquad x_t = m3 \cdot n1 + m4 \cdot n2$

$$i^*_{t,\,NM} = \psi_\pi (m1 \cdot n1 + m2 \cdot n2) + \psi_x (m3 \cdot n1 + m4 \cdot n2) + [0\ 1\ 0] S_t$$

$$M^{-1} = \begin{bmatrix} \frac{k}{w} & \frac{1+\delta b}{w} \\ \frac{1}{w} & \frac{-b\psi}{w} \end{bmatrix}$$

$$PQ = \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} p11 & p12 & p13 \\ p21 & p22 & p23 \\ p31 & p32 & p33 \end{bmatrix} \begin{bmatrix} Q1 \\ Q2 \\ Q3 \end{bmatrix}$$

$$i_{PQ}^{*} = p31 \cdot Q1 + p32 \cdot Q2 + p33 \cdot P3$$

$$= M^{-1}$$

$$P^{-1} = \begin{bmatrix} \frac{k}{w} & \frac{1+\delta b}{w} & -\frac{kb}{w} \\ \frac{1}{w} & \frac{-b\psi}{w} & -\frac{b}{w} \\ \frac{\delta + k\psi}{w} & \frac{\psi}{w} & \frac{1}{w} \end{bmatrix}$$

$\underline{\underline{MN}}$:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{k}{w} & \frac{1+\delta b}{w} \\ \frac{1}{w} & \frac{-b\psi}{w} \end{bmatrix} \begin{bmatrix} [b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0] f_\beta - b[-1,1,0](I_{hx}-\beta hx)^{-1} s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0] f_\alpha + [0\ 0\ 1](I_{hx}-\alpha\beta hx)^{-1} s_t \end{bmatrix}$$

$$\pi_t = \frac{k}{w}[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0] f_\beta - \frac{kb}{w}[-1,1,0](I_{hx}-\beta hx)^{-1} s_t$$
$$+ \frac{1+\delta b}{w}[(1-\alpha)\beta, k\alpha\beta, 0] f_\alpha + \frac{1+\delta b}{w}[0\ 0\ 1](I_{hx}-\alpha\beta hx)^{-1} s_t$$

PQ:

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} \frac{\kappa}{w} & \frac{1+\delta b}{w} & -\frac{\kappa b}{w} \\ \frac{1}{w} & -\frac{b\psi}{w} & -\frac{b}{w} \\ \frac{\delta+\kappa\psi}{w} & \frac{\psi}{w} & \frac{1}{w} \end{bmatrix} \begin{bmatrix} [b, 1-\beta, -b\beta] f_\beta + b[1\ 0\ 0](I_{nx}-\beta h x)^{-1} s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0\ 0\ 1](I_{nx}-\alpha\beta h x)^{-1} s_t \\ [0\ 1\ 0] s_t \end{bmatrix}$$

$$\pi_t^{PQ} = \frac{\kappa}{w}[b, 1-\beta, -b\beta] f_\beta + \frac{\kappa b}{w}[1\ 0\ 0](I_{nx}-\beta h x)^{-1} s_t$$
$$+ \frac{1+\delta b}{w}[(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + \frac{1+\delta b}{w}[0\ 0\ 1](I_{nx}-\alpha\beta h x)^{-1} s_t$$
$$- \frac{\kappa b}{w}[0\ 1\ 0] s_t$$

Compare w/ π_{MN}:

$$\pi_t^{MN} = \frac{\kappa}{w}[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0] f_\beta - \frac{\kappa b}{w}[-1, 1, 0](I_{nx}-\beta h x)^{-1} s_t$$
$$+ \frac{1+\delta b}{w}[(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + \frac{1+\delta b}{w}[0\ 0\ 1](I_{nx}-\alpha\beta h x)^{-1} s_t$$

$$\pi_t^{PQ} - \pi_t^{MN} = \frac{\kappa}{w}[b, 1-\beta, -b\beta] f_\beta + \frac{\kappa b}{w}[1\ 0\ 0](I_{nx}-\beta h x)^{-1} s_t$$
$$- \frac{\kappa b}{w}[0\ 1\ 0] s_t$$
$$- \left\{ \frac{\kappa}{w}[b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0] f_\beta - \frac{\kappa b}{w}[-1, 1, 0](I_{nx}-\beta h x)^{-1} s_t \right\}$$

$$= \frac{\kappa}{w}[b, 1-\beta, -b\beta] f_\beta + \frac{\kappa b}{w}[1\ 0\ 0](I_{nx}-\beta h x)^{-1} s_t - \frac{\kappa b}{w}[0\ 1\ 0] s_t$$
$$- \left\{ \frac{\kappa}{w}[b, 1-\beta, 0] f_\beta + \frac{\kappa}{w}[-b\beta\psi_\pi, -b\beta\psi_x, 0] f_\beta + \frac{\kappa b}{w}[1, 0, 0](I_{nx}-\beta h x)^{-1} s_t \right.$$
$$\left. + \frac{\kappa b}{w}[0, -1, 0](I_{nx}-\beta h x)^{-1} s_t \right\}$$

$$= \cancel{\frac{k}{w}[b, 1-\beta, 0] f_\beta} + \cancel{\frac{kb}{w}[1\ 0\ 0](I_{nx} - \beta h x)^{-1} S_t} - \frac{kb}{w}[0\ 1\ 0] S_t$$

$$+ \frac{k}{w}[0, 0, -b\beta] f_\beta$$

$$- \left\{ \cancel{\frac{k}{w}[b, 1-\beta, 0] f_\beta} + \frac{k}{w}[-b\beta\psi_\pi, -b\beta\psi_x, 0] f_\beta + \cancel{\frac{kb}{w}[1, 0, 0](I_{nx} - \beta h x)^{-1} S_t} \right.$$

$$\left. [0, -1, 0](I_{nx} - \beta h x)^{-1} S_t \right\}$$

$$= -\frac{kb}{w}[0\ 1\ 0] S_t + \cancel{\frac{k}{w}[0, 0, -b\beta] f_\beta} - \left\{ \cancel{\frac{k}{w}[-b\beta\psi_\pi, -b\beta\psi_x, 0] f_\beta} \right.$$

$$\left. + \frac{kb}{w}[0, -1, 0](I_{nx} - \beta h x)^{-1} S_t \right\}$$

$$= -\frac{kb}{w}[0\ 1\ 0] S_t - \left\{ + \frac{kb}{w}[0\ -1\ 0](I_{nx} - \beta h x)^{-1} S_t \right\}$$

$$+ \frac{k}{w}(-b\beta)[0\ 1\ 0](I_{nx} - \beta h x)^{-1} S_t$$

Recall that from (*)

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1) + [0\ 1\ 0](I_{nx} - \beta h x)^{-1} S_t$$

which is where the last term comes from.

Take the last line, divide by $\frac{kb}{w}$

$$-\bar{i}_t + \frac{1}{1-\beta\rho_i}\bar{i}_t - \frac{\beta}{1-\beta\rho_i}\bar{i}_t = \left[\frac{-(1-\beta\rho_i) + 1 - \beta}{1-\beta\rho_i}\right]\bar{i}_t$$

$$= \frac{\beta\rho_i - \beta}{1-\beta\rho_i}\bar{i}_t = \frac{\beta(\rho_i - 1)}{1-\beta\rho_i}\bar{i}_t \quad \text{that sho' ain't 0.}$$

Let's compare the 2 systems:

MN

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix}\frac{k}{w} & \frac{1+\delta\sigma}{w} \\ \frac{1}{w} & \frac{-\sigma\psi}{w}\end{bmatrix}\begin{bmatrix}[\sigma(1-\beta\psi_\pi), (1-\beta-\sigma\beta\psi_x), 0]f_\beta + \sigma[1,-1,0](I_{nx}-\beta hx)^{-1}s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0]f_\alpha + [0\ 0\ 1](I_{nx}-\alpha\beta hx)^{-1}s_t\end{bmatrix}$$

PQ

$$\begin{bmatrix}\pi_t \\ x_t \\ i_t\end{bmatrix} = \begin{bmatrix}\frac{k}{w} & \frac{1+\delta\sigma}{w} & -\frac{k\sigma}{w} \\ \frac{1}{w} & \frac{-\sigma\psi}{w} & -\frac{\sigma}{w} \\ \frac{\delta+k\psi}{w} & \frac{\psi}{w} & \frac{1}{w}\end{bmatrix}\begin{bmatrix}[\sigma, 1-\beta, -\sigma\beta]f_\beta + \sigma[1\ 0\ 0](I_{nx}-\beta hx)^{-1}s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0]f_\alpha + [0\ 0\ 1](I_{nx}-\alpha\beta hx)^{-1}s_t \\ [0\ 1\ 0]s_t\end{bmatrix}$$

Copy w/ changes: (show only stuff for π & that are different)

$$\begin{bmatrix}\pi_t \\ x_t\end{bmatrix} = \begin{bmatrix}\frac{k}{w} & \cdot \\ \cdot & \cdot\end{bmatrix}\begin{bmatrix}\cancel{[\sigma, 1-\beta, 0]f_\beta} + [-\sigma\beta\psi_\pi, -\sigma\beta\psi_x, 0]f_\beta + \cancel{\sigma[1,0,0](I_{nx}-\beta hx)^{-1}s_t} \\ + \sigma[0,-1,0](I_{nx}-\beta hx)^{-1}s_t \\ \cdot\end{bmatrix}$$

$$\begin{bmatrix}\pi_t \\ x_t \\ i_t\end{bmatrix} = \begin{bmatrix}\frac{k}{w} & \cdot & -\frac{k\sigma}{w} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}\cancel{[\sigma, 1-\beta, 0]f_\beta} + [0,0,-\sigma\beta]f_\beta + \cancel{\sigma[1,0,0](I_{nx}-\beta hx)^{-1}s_t} \\ \cdot \\ [0\ 1\ 0]s_t\end{bmatrix}$$

So what we really need is

$$-\frac{k\sigma\beta}{w}[\psi_\pi, \psi_x, 0]f_\beta + \frac{k\sigma}{w}[0,-1,0](I_{nx}-\beta hx)^{-1}s_t$$

$$\overset{!}{=} -\frac{k\sigma\beta}{w}[0,0,1]f_\beta + \frac{k\sigma}{w}[0,-1,0]s_t$$

which is exactly what I obtained before.

Can we rewrite this using tricks on the sum such that it works?

$$-\bar{i}_t + \frac{1}{1-\beta\rho_i}\bar{i}_t - \frac{\beta}{1-\beta\rho_i}\bar{i}_t = -\bar{i}_t + \sum_{T=t}^{\infty}(\beta\rho_i)^{T-t}\bar{i}_t - \sum_{T=t}^{\infty}(\beta\rho_i)^{T-t}\beta\bar{i}_t$$

$$= \sum_{T=t}^{\infty}\left\{(\beta\rho_i)^{T-t}\bar{i}_t\right\} - \bar{i}_t + \sum_{T=t}^{\infty}\left\{\beta^{T-t+1}\rho_i^{T-t}\bar{i}_t\right\}$$

$$= \bar{i}_t + \beta\rho_i\bar{i}_{t+1} + (\beta\rho_i)^2\bar{i}_{t+2} + \dots$$

$$-\bar{i}_t$$

$$+\beta\bar{i}_t + \beta^2\rho_i\bar{i}_{t+1} + \beta^3\rho_i^2\bar{i}_{t+2} + \dots$$

Ah wait ... can it be that the $f_\beta(3,1)$-term only is i_{t+1} onward while the last element of Q has the t-term?

Exactly! Look: Preston 2005, p. 40 (mac)

$$f_\beta := \hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}z_{T+1} \quad \text{which is} \quad \hat{E}_t\sum_{T=t}^{\infty}\beta^{T-t}\bar{i}_{T+1}$$

And like here, I see that that works 3 Jan 2019

except that somehow I have a β too many!

unless $(*)$ doesn't treat the errors quite right!

If $(*)$ had to divide errors by β, we'd be fine:

(*) was

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1) + [0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t$$

But here the errors are $\sum_{T=t}^{\infty} (\beta\ \rho_i)^{T-t}\ \bar{i}_t$

while they should be $\sum_{T=t}^{\infty} \beta^{T-t} \rho_i^{T-t+1}\ \bar{i}_{T+1}$ since they are future expectations

$$= \frac{1}{\beta} \sum_{T=t}^{\infty} (\beta^{T-t+1} \rho_i)\ \bar{i}_{T+1}$$

$$= \frac{1}{\beta} \left[\beta\rho_i\, i_{t+1} + (\beta\rho_i)^2\, i_{t+2} + \dots \right]$$

$$= \frac{1}{\beta} \Big[\underbrace{[0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0]\, s_t}_{\text{the full sum minus the first term}} \Big]$$

$\Rightarrow$ (*) is

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1)$$
$$+ \frac{1}{\beta} \left[[0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0]\, s_t \right]$$

then what we need becomes:

$$-\frac{\kappa_3\beta}{w} [\psi_\pi, \psi_x, 0] f_\beta + \frac{\kappa_3}{w} [0, -1, 0](I_{nx} - \beta h_x)^{-1} s_t$$

$$\stackrel{!}{=} -\frac{\kappa_3\beta}{w} [0, 0, 1] f_\beta + \frac{\kappa_3}{w} [0, -1, 0]\, s_t$$

which is

$$-\cancel{\frac{k\mathfrak{b}\beta}{w}[\psi_\pi, \psi_x, 0] f_p} + \frac{k\mathfrak{b}}{w}\cancel{[0,-1,0](I_{nx}-\beta h_x)^{-1}} s_+$$

$$\stackrel{!}{=} \qquad + \frac{k\mathfrak{b}}{w}\cancel{[0,-1,0]} s_+$$

$$\cancel{-\frac{k\mathfrak{b}\beta}{w}\left[\psi_\pi f_p(1,1) + \psi_x f_\beta(2,1)\right]}$$

$$-\frac{k\mathfrak{b}\beta}{w}\frac{1}{\beta}\left[\cancel{[0\ 1\ 0](I_{nx}-\beta h_x)^{-1}} s_+ - \cancel{[0\ 1\ 0]} s_+\right]$$

Yeah!!! So indeed the condition in PQ, (*) needs to be:

$$f_\beta(3,1) = \psi_\pi f_p(1,1) + \psi_x f_\beta(2,1)$$
$$+ \frac{1}{\beta}\left[[0\ 1\ 0](I_{nx}-\beta h_x)^{-1} s_+ - [0\ 1\ 0] s_+\right]$$

Ok, so I've found the equality between the MN - PQ methods even when shocks are present. The problem is that they are too intensive for Mathematica. So either I find a way to simplify them or I just do the MN-method. The thing is that the PQ condition (*) may change when changing the model and therefore I think that the MN-method is more robust.

So what Mathematica (materials12d.nb) now does is it calculates M^{-1} to give

MN

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{k}{w} & \frac{1+\delta b}{w} \\ \frac{1}{w} & \frac{-b\psi}{w} \end{bmatrix} \begin{bmatrix} [b(1-\beta\psi_\pi), (1-\beta-b\beta\psi_x), 0] f_\beta + b \overbrace{[1, -1, 0](I_{nx}-\beta h_x)^{-1} s_t}^{=d1} \\ [(1-\alpha)\beta, k\alpha\beta, 0] f_\alpha + \underbrace{[0\ 0\ 1](I_{nx}-\alpha\beta h_x)^{-1} s_t}_{=d2} \end{bmatrix}$$

which it solves for (π_t, x_t) and for i as

$$i_t = \psi \pi_t^* + \delta \cdot x_t^* + \underbrace{[0\ 1\ 0] s_t}_{=d6}$$

Then it collects the gs and assembles

$$A_a = \begin{pmatrix} gpa \\ gxa \\ gia \end{pmatrix} \qquad A_b = \begin{pmatrix} gpb \\ gxb \\ gib \end{pmatrix}$$

$$A_s = \begin{pmatrix} gps \\ gxs \\ gis \end{pmatrix}$$

Check that $gis = \psi gps + \delta gxs + d6$ ✓

And checked these A-matrices against the old ones in materials6.m, they are identical. 4 Jan 2015 matrices_A_intrate_smoothing2.m is WRONG

matrices_A_intrate_smoothing3.m is correct as long as $\rho = 0$

To do next: check what happens to the system once we have interest rate smoothing, and then eventually $E[\pi]$ in TR or π_{t-1} in TR.

So: <u>adding ρi_{t-1}</u>

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta i_{T+1} + \sigma r_T^n \right\}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right\}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t + \rho i_{t-1}$$

$\Leftrightarrow$

$$x_t = -\sigma(\psi_\pi \pi_t + \psi_x x_t + \bar{i}_t + \rho i_{t-1})$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta\psi_\pi) \pi_{T+1} - \sigma\beta \bar{i}_{T+1} - \underline{\sigma\beta\rho i_T} + \sigma r_T^n \right\}$$

NKPC unchanged

A problem b/c even if you sub the TR in again, i_{T-1} will show up and since there are ∞ terms, expectations of i_{T+1} <u>will</u> show up.

⟹ it looks to me as if this forced us to use the PQ method, either w/ an augmented (∗) or w/ an additional linking eq.

$$x_t = -\mathfrak{z} i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \mathfrak{z} \pi_{T+1} - \mathfrak{z}\beta i_{T+1} + \mathfrak{z} r_T^n \right\}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \kappa\alpha\beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} + u_T \right\}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t + \rho i_{t-1}$$

$$\begin{bmatrix} 0 & 1 & \mathfrak{z} \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} [\mathfrak{z},\ 1-\beta,\ -\mathfrak{z}\beta] f_\beta + \mathfrak{z}[1\ 0\ 0\ \underline{0}](I_{nx} - \beta h_x)^{-1} s_t \\ [(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] + [0\ 0\ 1\ \underline{0}](I_{nx} - \alpha\beta h_x)^{-1} s_t \\ [0\ 1\ 0\ \underline{\rho}] s_t \end{bmatrix}$$

So far this is identical to the PQ method, except s_t is 4×1.

(∗) used to be:

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1) + \frac{1}{\beta}\left[[0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0] s_t \right]$$

An easy (but likely not quite correct) modification would be:

$$f_\beta(3,1) = \psi_\pi f_\beta(1,1) + \psi_x f_\beta(2,1) + \frac{1}{\beta}\left[[0\ 1\ 0\ \underline{\rho}](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0\ \underline{\rho}] s_t \right]$$

1. This takes into account expectations of future i_{t-1} ✓

2. but it doesn't take into account that $i'_{t-1} = i_t$ ✗

→ I'm scared that this treats "il_t" as an exog. state, ignoring that $\hat{E}_t[i_{t+1}] = \hat{E}_t[il_{t+2}] = \hat{E}_t[i'_t]$

This worry is not true if hx accounts for this which - oh wonder - it does! Recall that

$$hx_{il} = gx_i \quad \text{(more or less...)}$$

So the "easy fix" of (*) may be correct.

The worry then is that the "il-row" of hx may reveal parts of gx?

B/c now I'm saying that putting il into the exog. state block is ok for expectations of il as long as hx encompasses the structural relationship between i_t and i_{t-1}, which I think it does.

→ Ryan 24 Oct 2019 suggests that the "il-row" of hx ≠ i-row of gx b/c of indirect & direct effects → check on Matlab!

Bad news: 6 Jan 2020

hx(4,:) = gx(4,:) in Matlab

and $gx_{3,4} = hx_{3,4} \neq \rho$ b/c since i_{t-1} is an endog. state, it doesn't depend on s_t as $[0\ 0\ 0\ \rho]$, instead it depends on s_t as it does, thru the effects of other shocks.

$\Rightarrow$ thus we do have the problem that hx reveals a part of gx. Which necessitates a learning where

a) either only the exog part of hx is known

b) none of hx is known ...

But in any case, the first step is to figure out the PQ method in Mathematica.

-> materials12e.nb ✓

Type it up in Latex materials12e.tex. ✓

And compared in materials6.m too - it works ✓

:)

So now w/ the PQ method working for the baseline model, let's try to tackle extensions where the Taylor-rule is modified:

1) intrate smoothing: ρi_{t-1} or $\rho \cdot il_t$ w/ $il_t = i_{t-1}$

2) pil: π_{t-1} in TR

3) Epi: $\hat{E}_t[\pi_{t+1}]$ in TR

In each case the NKPC is unchanged, so I just need to see how the NKIS and TR change and how (*) has to change to take them into account.

Epi may not be so diff b/c it doesn't introduce a new, endog. state, but pil will.

1) $$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta i_{T+1} + \sigma r_T^n \right\}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t + \rho i_{t-1}$$

2) $$i_t = \psi_x \pi_{t-1} + \psi_x x_t + \bar{i}_t$$

3) $$i_t = \psi_x \hat{E}_t \pi_{t+1} + \psi_x x_t + \bar{i}_t$$

$$
\begin{bmatrix} 0 & 1 & b \\ 1 & -\kappa & 0 \\ -\psi_\pi & -\psi_x & 1 \end{bmatrix}
\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}
=
\begin{bmatrix}
[b,\ 1-\beta,\ -b\beta]\, f_\beta + b\,[1\ 0\ 0\ \underline{0}]\,(I_{nx}-\beta h_x)^{-1} s_t \\
[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] + [0\ 0\ 1\ \underline{0}]\,(I_{nx}-\alpha\beta h_x)^{-1} s_t \\
[0\ 1\ 0\ \underline{\rho}]\, s_t
\end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 1 & b \\ 1 & -\kappa & 0 \\ 0 & -\psi_x & 1 \end{bmatrix}
\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}
=
\begin{bmatrix}
[b,\ 1-\beta,\ -b\beta]\, f_\beta + b\,[1\ 0\ 0\ \underline{0}]\,(I_{nx}-\beta h_x)^{-1} s_t \\
[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] + [0\ 0\ 1\ \underline{0}]\,(I_{nx}-\alpha\beta h_x)^{-1} s_t \\
[0\ 1\ 0\ \psi_\pi]\, s_t
\end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 1 & b \\ 1 & -\kappa & 0 \\ 0 & -\psi_x & 1 \end{bmatrix}
\begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}
=
\begin{bmatrix}
[b,\ 1-\beta,\ -b\beta]\, f_\beta + b\,[1\ 0\ 0\ ]\,(I_{nx}-\beta h_x)^{-1} s_t \\
[(1-\alpha)\beta,\ \kappa\alpha\beta,\ 0] + [0\ 0\ 1\ ]\,(I_{nx}-\alpha\beta h_x)^{-1} s_t \\
[0\ 1\ 0\ \ ]\, s_t + \psi_\pi \underbrace{\hat{E}_t[\pi_{t+1}]}
\end{bmatrix}
$$

Epi → need to be evaluated as a 1-period-ahead fcst in learning-rule and (* -Epi) is:

$$f_\beta(3,1) = \cancel{\Psi_\pi f_p(1,1)} + \Psi_x f_\beta(2,1)$$
$$+ \frac{1}{\beta}\left[[0\ 1\ 0](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0]\, s_t \right]$$
$$+ \hat{E}_t \pi_{t+1}$$ ← this ain't right I think (7 Jan 2020)

→ so the Epi-model can be done directly w/ these green modifications, and the learning code needs to spit out $\hat{E}_t \pi_{t+1}$, and the compact notation needs to be $z_t = A_a f_a + A_b f_b + A_s \cdot s_t + A_e \cdot \hat{E}_t \pi_{t+1}$

→ As for il and pil, the PQ-method can deal w/ the general solution, and I suggest that the modified (*)

$$f_\beta(3,1) = \Psi_\pi f_p(1,1) + \Psi_x f_\beta(2,1) \qquad (\text{* -il})$$
$$+ \frac{1}{\beta}\left[[0\ 1\ 0\ \rho](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0\ \rho]\, s_t \right]$$

and the implied for pil

$$f_\beta(3,1) = \cancel{\Psi_\pi f_p(1,1)} + \Psi_x f_\beta(2,1) \qquad (\text{* - pil})$$
$$+ \frac{1}{\beta}\left[[0\ 1\ 0\ \Psi_\pi](I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0\ \Psi_\pi]\, s_t \right]$$

might also work. The uncertainty though is hx!

For the pil-model, MN would look like:

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta i_{T+1} + \sigma r_T^n \right\}$$

$$i_t = \psi_x \pi_{t-1} + \psi_x x_t + \bar{i}_t$$

$$\Rightarrow x_t = -\sigma \left(\psi_x \pi_{t-1} + \psi_x x_t + \bar{i}_t \right)$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta \left(\psi_\pi \pi_T + \psi_x x_{T+1} + \bar{i}_{T+1} \right) + \sigma r_T^n \right\}$$

$$(1+\sigma\psi_x) x_t = -\sigma \psi_x \pi_{t-1}$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma \pi_{T+1} \underbrace{- \sigma \beta \psi_\pi \pi_T}_{\text{this is the prob.}} + \sigma [1, -1, 0, 0] s_T \right\}$$

$$\Rightarrow -\sigma\beta\psi_\pi \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \pi_T = -\sigma\beta\psi_\pi \left[\pi_t + \beta \pi_{t+1} + \beta^2 \pi_{t+2} + \dots \right]$$

$$= -\sigma\beta\psi_\pi \pi_t + (-\sigma\beta\psi_\pi) \beta \left[\pi_{t+1} + \beta \pi_{t+2} + \dots \right]$$

So I can write the NKIS as:

$$(1+\sigma\psi_x) x_t = -\sigma \psi_x \pi_{t-1} - \sigma\beta\psi_\pi \pi_t$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta^2\psi_\pi \pi_{T+1} + \sigma [1, -1, 0, 0] s_T \right\}$$

$$\sigma\beta\psi_\pi \pi_t + (1+\sigma\psi_x) x_t = -\sigma \psi_x \pi_{t-1}$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma(1-\beta^2\psi_\pi) \pi_{T+1} + \sigma [1, -1, 0, 0] s_T \right\}$$

$\Rightarrow$ So I think the MN-method could deal w/ this ...?

It's a new day and I no longer think that it's necessarily a problem that parts of hx equal parts of gx. It can be a problem but it doesn't have to be if we <u>assume that agents don't know that gx(4,:) = hx(4,:)</u>. Then hx doesn't reveal anything about gx. And I think that implicitly I've been making this assumption.

<u>7 Jan 2020</u>

But: if I continue making this assumption then I can solve all 3 extensions: il (PQ-method), pil and Epi (both MN and PQ methods).

Since il is actually the hardest (MN sol doesn't exist) I'll reverse the order:

1) Epi: materials12f1 .nb/.m/.tex

matrices_A_12f1.m

2) pil: materials12f2 .nb/.m/.tex

matrices_A_12f2.m

3) il: materials12f3 .nb/.m/.tex

matrices_A_12f3.m

① Epi

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta i_{T+1} + \sigma r_T^n \right\}$$

$$i_t = \psi_x \hat{E}_t \pi_{t+1} + \psi_x x_t + \bar{i}_t$$

$\Rightarrow$ PQ:
$$x_t + \sigma i_t = [\sigma, 1-\beta, -\sigma\beta] f_\beta + \sigma [1,0,0] (I_{nx} - \beta h_x)^{-1} s_t$$

$$\pi_t - \kappa x_t = [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0,0,1] (I_{nx} - \alpha\beta h_x)^{-1} s_t$$

$$-\psi_x x_t + i_t = \psi_x \hat{E}_t \pi_{t+1} + [0\ 1\ 0] s_t$$

$$\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ 0 & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} [\sigma, 1-\beta, -\sigma\beta] f_\beta + C_{xs} \cdot s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + C_{\pi s} \cdot s_t \\ C_{is} \cdot s_t + \underline{\psi_\pi \cdot \hat{E}_t \pi_{t+1}} \end{bmatrix}$$

and (*-Epi) is

$$f_\beta(3) = \psi_x f_\beta(2)$$
$$+ \frac{1}{\beta} \left[[0\ 1\ 0] (I_{nx} - \beta h_x)^{-1} s_t - [0\ 1\ 0] s_t \right]$$
$$+ \underline{\psi_\pi \hat{E}_t \sum_{T=t}^{\infty} \pi_{T+2}} \quad \leftarrow \text{now what?}$$

b/c

$$\hat{E}_t i_{t+1} = \psi_x \hat{E}_t x_{t+1} + C_{is} \hat{E}_t s_{t+1} + \psi_\pi \hat{E}_t \hat{E}_{t+1} \pi_{t+2}$$

$$\hat{E}_t i_{t+2} = \psi_x \beta \hat{E}_t x_{t+2} + C_{is} \hat{E}_t s_{t+2} + \psi_\pi \hat{E}_t \pi_{t+3}$$

$$\Rightarrow MN: \quad + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta \left(\psi_\pi \hat{E}_T \pi_{T+1} + \psi_x x_{T+1} + \bar{i}_{T+1} \right) + \sigma r_T^n \right\}$$

$$x_t = -\sigma \psi_x Epi - \sigma \psi_x x_t - \sigma \bar{i}_t$$

$$(1+\sigma\psi_x) x_t = -\sigma\psi_\pi \hat{E}_t \pi_{t+1} + \sigma [1 \; -1 \; 0] (I_{nx} - \beta h_x)^{-1} s_t$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta\psi_\pi \hat{E}_T \pi_{T+2} \right\}$$

Ass. that LIE holds for $\hat{E}$ (which I think is true by anticipated utility)

$$\Rightarrow (1+\sigma\psi_x) x_t = -\sigma\psi_\pi \hat{E}_t \pi_{t+1} + \sigma [1 \; -1 \; 0] (I_{nx} - \beta h_x)^{-1} s_t$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + \sigma \pi_{T+1} - \underbrace{\sigma\beta\psi_\pi \quad \pi_{T+2}} \right\}$$

$$\Rightarrow (1+\sigma\psi_x) x_t = \cancel{-\sigma\psi_\pi \hat{E}_t \pi_{t+1}} + \sigma [1 \; -1 \; 0] (I_{nx} - \beta h_x)^{-1} s_t$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\sigma\beta\psi_x) x_{T+1} + (\sigma - \sigma\psi_\pi) \pi_{T+1} \right\}$$

$$\begin{bmatrix} 0 & (1+\sigma\psi_x) \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} [\sigma(1-\psi_\pi), 1-\beta-\sigma\beta\psi_x, 0] f_b + d_{xs} \cdot s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_a + d_{ps} \cdot s_t \end{bmatrix}$$

and in both cases the compact notation is

$$z_t = A_a \cdot f_a + A_b \cdot f_b + \underline{A_c} \cdot Epi$$

$\hat{E}_t \pi_{t+1}$

Try (*-Epi) again:

$$f_\beta(3) = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} i_{T+1}$$

$$= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\psi_x x_{T+1} + c_{is} s_{T+1} + \psi_\pi \hat{E}_t \pi_{T+2} \right]$$

$$= \psi_x f_\beta(2) + c_{is}\left[\rho s_t + \beta \rho^2 s_t + \dots\right] + \psi_\pi \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \hat{E}_t \pi_{T+2}$$

$$\to \psi_\pi \left[\pi_{t+2} + \beta \pi_{t+3} + \dots \right]$$

$$= \frac{1}{\beta} \psi_\pi \left[\pi_{t+1} + \beta \pi_{t+2} + \beta^2 \pi_{t+3} + \dots - \pi_{t+1} \right]$$

$$= \frac{1}{\beta} \psi_\pi \left[f_\beta(1) - \hat{E}_t \pi_{t+1} \right]$$

So (*-Epi) is:

$$f_\beta(3) = \psi_x f_\beta(2) + \underline{\psi_\pi/\beta}\, f_\beta(1) - \underline{\psi_\pi/\beta\, \hat{E}_t} \pi_{t+1}$$
$$+ \frac{1}{\beta} c_{is} \left[(I_{nx} - \beta h_x)^{-1} s_t - s_t \right]$$

This is the correct (*-Epi) !

$\Rightarrow$ ok, the two are equal in Mathematica! :)

$\Rightarrow$ need to adapt the learning code

sim_learnLH_12f1.m

I haven't changed the EE-learning b/c we don't care so much about it.

② pil-Extension

MN

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta i_{T+1} + \sigma r_T^n \right\}$$

$$i_t = \psi_\pi \pi_{t-1} + \psi_x x_t + \bar{i}_t$$

$$\Rightarrow x_t = -\sigma \left(\psi_x \pi_{t-1} + \psi_x x_t + \bar{i}_t \right)$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta) x_{T+1} + \sigma \pi_{T+1} - \sigma \beta \left(\psi_\pi \pi_T + \psi_x x_{T+1} + \bar{i}_{T+1} \right) + \sigma r_T^n \right\}$$

$$(1+\sigma\psi_x) x_t = -\sigma \psi_\pi \pi_{t-1}$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\beta\sigma\psi_x) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta\psi_\pi \pi_T - \sigma \bar{i}_T + \sigma r_T^n \right\}$$

$$\Leftrightarrow (1+\sigma\psi_x) x_t = -\sigma \psi_\pi \pi_{t-1} - \sigma \psi_\pi \beta \pi_t$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\beta\sigma\psi_x) x_{T+1} + \sigma \pi_{T+1} - \sigma\beta^2\psi_\pi \pi_{T+1} - \sigma \bar{i}_T + \sigma r_T^n \right\}$$

$$\beta \sigma \psi_\pi \pi_t + (1+\sigma\psi_x) x_t = -\sigma \psi_\pi \pi_{t-1}$$

$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\beta\sigma\psi_x) x_{T+1} + \sigma (1-\beta^2\psi_\pi) \pi_{T+1} - \sigma \bar{i}_T + \sigma r_T^n \right\}$$

$$\sigma\psi_\pi \pi_t + (1+\sigma\psi_x)x_t = -\sigma\psi_\pi \pi_{t-1}$$
$$+ \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ (1-\beta-\beta\sigma\psi_x) x_{T+1} + \sigma(1-\beta^2\psi_\pi)\pi_{T+1} - \sigma \bar{i}_T + \sigma r_T^N \right\}$$

$$\rightarrow \begin{bmatrix} \sigma\psi_\pi \beta & 1+\sigma\psi_x \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} [\sigma(1-\beta^2\psi_\pi), (1-\beta-\beta\sigma\psi_x), 0] f_\beta + \overset{1\times 4}{d_{xs}} s_t + [0 \; 0 \; 0 \; -\sigma\psi_\pi] s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + \overset{1\times 4}{d_{\pi s}} s_t \end{bmatrix}$$

PQ

$\sigma\beta^2\psi_\pi$

$$\begin{bmatrix} 0 & 1 & \sigma \\ 1 & -\kappa & 0 \\ 0 & -\psi_x & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} = \begin{bmatrix} [\sigma, 1-\beta, -\sigma\beta] f_\beta + \overset{c_{xs} \text{ is } 1\times 4}{\sigma[1,0,0,0](I_{nx}-\beta h_x)^{-1}} s_t \\ [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + \overset{1\times 4}{c_{\pi s}} s_t \\ \underbrace{[0, 1, 0, \psi_\pi]}_{c_{is} \text{ is new}} s_t \end{bmatrix}$$

And (*-pil)

$$f_\beta(3) = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} i_{T+1} = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left\{ \psi_\pi \pi_T + \psi_x x_{T+1} + \bar{i}_{T+1} \right\}$$
$$= \psi_x f_\beta(2) + \frac{1}{\beta} \left\{ [0,1,0,0](I_{nx}-\beta h_x)^{-1} s_t - [0,1,0,0] s_t \right\}$$
$$+ \underbrace{\psi_\pi [\pi_t + \beta \hat{E}_t \pi_{t+1} + \beta^2 \hat{E}_t \pi_{t+2} + \dots]}_{\psi_\pi \pi_t + \beta[\hat{E}_t \pi_{t+1} + \beta \hat{E}_t \pi_{t+2} + \dots]}$$
$$= \psi_\pi \pi_t + \beta \psi_\pi f_\beta(1)$$

$$f_\beta(3) = \beta\psi_\pi f_\beta(1) + \psi_x f_\beta(2) + \psi_\pi \pi_t$$
$$+ \frac{1}{\beta} \left\{ [0,1,0,0](I_{nx}-\beta h_x)^{-1} s_t - [0,1,0,0] s_t \right\}$$

now what?

what if we took this from $-\sigma\beta f_\beta(3)$ over to the LHS as $\sigma\beta\psi_\pi \pi_t$ and cancelled it in (*)?

↳ works! ☺

Ok, so pil also implemented in Matlab!

1) Tomorrow: do "il"-extension.

2) Do something about command_IRF_many_learning.m so that one can switch between the TRs easily.

3) Now it's the end of the day and I'm thinking that agents knowing hx might still cause trouble b/c no matter what I ass, I need to be consistent. If they know hx but don't realize that $\pi_{t-1} = pil$, then they should be able to fcst all of pil, even pil_{t+k} $k \geq 2$! If they aren't able to fcst pil_{t+k}, then they realize the linking equation but then don't know a part of hx.

4) In materials12, I really need to spell out this ↑ problem, and also the problem of whether they know relationships between jumps. Do they know model equations??

③ il - extension 8 Jan 2020

MN doesn't work here, we know that.

PQ

$b + \frac{b\beta\rho}{1-\rho\beta}$

$$\begin{bmatrix}0 & 1 & b\\ 1 & -k & 0\\ -\psi_\pi & -\psi_x & 1\end{bmatrix}\begin{bmatrix}\pi_t\\ x_t\\ i_t\end{bmatrix} = \begin{bmatrix}[b, 1-\beta, -b\beta] f_\beta + b[1,0,0,\underline{0}](I_{nx}-\beta h_x)^{-1}s_t \\ [(1-\alpha)\beta, k\alpha\beta, 0] f_\alpha + [0,0,1,\underline{0}](I_{nx}-\alpha\beta h_x)^{-1}s_t \\ [0,1,0,\underline{\rho}]s_t\end{bmatrix}$$

$(x - il)$ is

$$f_\beta(3) = \hat{E}_t\sum_{T=t}^{\infty} \beta^{T-t} i_{T+1} = \hat{E}_t\sum_{T=t}^{\infty} \beta^{T-t}\left[\psi_\pi \pi_{T+1} + \psi_x x_{T+1} + \bar{i}_{T+1} + \rho i_T\right]$$

$$= \psi_\pi f_\beta(1) + \psi_x f_\beta(2) + \frac{1}{\beta}\left\{[0,1,0,0](I_{nx}-\beta h_x)^{-1}s_t - [0,1,0,0]s_t\right\}$$

$$+ \underbrace{\rho \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T}$$

$$\rho[i_t + \beta i_{t+1} + \beta^2 i_{t+2} + \dots]$$

$$= \rho i_t + \rho\beta[i_{t+1} + \beta i_{t+2} + \dots]$$

$$= \rho i_t + \rho\beta f_\beta(3)$$

$$(1-\rho\beta) f_\beta(3) = \psi_\pi f_\beta(1) + \psi_x f_\beta(2) + \frac{1}{\beta}\left\{[0,1,0,0](I_{nx}-\beta h_x)^{-1}s_t - [0,1,0,0]s_t\right\}$$

$$+ \rho i_t$$

$$f_\beta(3) = \frac{1}{\underline{1-\rho\beta}}\left[\psi_\pi f_\beta(1) + \psi_x f_\beta(2) + \frac{1}{\beta}\left\{[0,1,0,\underline{0}](I_{nx}-\beta h_x)^{-1}s_t - [0,1,0,\underline{0}]s_t\right\}\right]$$

$$+ \frac{\rho}{1-\rho\beta} i_t$$

again, like in pil, we subtract $-\frac{b\beta\rho}{1-\rho\beta} i_t$ in 1st row of P.

Interesting: I get the same A-matrices I got w/

1) original int rate smoothing

2) int rate smoothing 3 (which used the MN method, treating it as an exog state)

this breaks down if $\rho \neq 0$, in which case the two old codes still coincide. Why?

1) b/c ... intrate_smoothing.m = ... intrate_smoothing3.m
→ both use the MN method, except 3 is more explicit about it

2) ...12f3.m however uses the PO-method so actually it's great news that it equals the other two when $\rho = 0$ b/c it means that it's correct, and it should also not equal the other two when $\rho \neq 0$ b/c of the different info assumption!

⇒ Info ass's:

MN: since you sub in i_t and treat all future i's as exog shocks, the info ass you make is that agents don't realize that $i_t = il_{t+1}$, but they know

(all of) hx, so they fcst the shock, but not the jump.

PQ: here the info ass you're making is that agents

realize that $i_t = il_{t+1}$, and so they are fcsting

il as i_t, not realizing that they can use hx.

Neither is really kosher!

$\Rightarrow$ the difference is really what they are using to fcst

an endog. state which equals a lagged jump;

gx, which they don't know, or hx, which they know.