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Chebyshev lattices, a unifying framework for cubature with Chebyshev weight function

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Abstract We present a multivariate extension to Clenshaw-Curtis quadrature based on Sloan's hyperinterpolation theory. At the centre of it, a cubature rule for integrals with Chebyshev weight function is needed. We introduce so called Chebyshev lattices as a generalising framework for the multitude of point sets that have been discussed in this context. This framework provides a uniform notation that extends easily to higher dimensions. In this paper we describe many known point sets as Chebyshev lattices.

In the introduction we briefly explain how convergence results from hyperinterpolation can be used in this context. After introducing Chebyshev lattices and the associated cubature rules, we show how most of the two- and three-dimensional point sets in this context can be described with this notation. The not so commonly known blending formulae from Godzina, which explicitly describe point sets in any number of dimensions, also fit in perfectly.

Keywords Multivariate Clenshaw-Curtis · Hyperinterpolation · Cubature · Chebyshev lattices · Morrow-Patterson points · Padua points · Godzina's blending formulae

Mathematics Subject Classification 65D30 · 65D32

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1 Introduction

Chebyshev polynomials are widely used for approximations in one dimension because the Chebyshev series of a function often rapidly converge. Once such an approximation is obtained, it can be used in the contexts of interpolation and numerical integration. The latter leads to so called interpolating quadrature rules (see e.g. [15]).

In the multivariate case, it is more difficult to find an approximation which interpolates the function at a specified number of points. The interpolation operation can therefore be replaced by a projection of the given function onto the polynomial function space. If the number of points exceeds the degrees of freedom in the function space, this approximation will generally not satisfy the interpolation criteria. It is therefore not an interpolation, but a *hyperinterpolation*, a concept introduced by Sloan in [12].

We are interested in approximating multivariate functions f on the s-dimensional cube $C_s := [-1, 1]^s$ by a polynomial of degree n. By $\mathcal{P}_n^s$ we denote the vector space of all polynomials of degree at most n in s dimension. We use product Chebyshev polynomials $\hat{T}_{\mathbf{h}}$ that have a degree $|\mathbf{h}| := \sum_{r=1}^{s} |h_r|$ and are defined as $\hat{T}_{\mathbf{h}}(\mathbf{x}) := \prod_{r=1}^{s} \hat{T}_{h_r}(x_r)$ where $\hat{T}_0(x) := 1$ and $\hat{T}_h(x) := \sqrt{2}\, T_h(x) = \sqrt{2}\cos(h \arccos(x))$. As a basis for $\mathcal{P}_n^s$ we will use these normalised product Chebyshev polynomials $\hat{\mathcal{T}}_n := \{\hat{T}_{\mathbf{h}}(\mathbf{x}), |\mathbf{h}| \leq n\}$. They are known to be orthogonal on the domain C_s with respect to the following continuous scalar product with Chebyshev weight function $\omega(\mathbf{x}) := \pi^{-s} \prod_{r=1}^{s} (1 - x_r^2)^{-\frac{1}{2}}$:

$$\langle f_1, f_2 \rangle_\omega := \int_{C_s} f_1(\mathbf{x})\, f_2(\mathbf{x})\, \omega(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Indeed, due to the product structure this breaks down to product of univariate integrals:

$$\langle \hat{T}_{\mathbf{h}_1}, \hat{T}_{\mathbf{h}_2} \rangle_\omega = \prod_{r=1}^{s} \int_{-1}^{1} \hat{T}_{h_{1,r}}(x_r)\, \hat{T}_{h_{2,r}}(x_r)\, \omega(x_r)\, \mathrm{d}x_r = \begin{cases} 0 & \text{if } \mathbf{h}_1 \neq \mathbf{h}_2, \\ 1 & \text{if } \mathbf{h}_1 = \mathbf{h}_2. \end{cases}$$

An approximation of f will be obtained by projecting the function f onto $\hat{\mathcal{T}}_n$:

$$P_n^s[f] := \sum_{\mathbf{h}, |\mathbf{h}| \leq n} \alpha_{\mathbf{h}}\, \hat{T}_{\mathbf{h}} \quad \text{where } \alpha_{\mathbf{h}} := \langle f, \hat{T}_{\mathbf{h}} \rangle_\omega.$$

Here $\alpha_{\mathbf{h}}$ can be seen as a Fourier coefficient of f in the Chebyshev basis of $\mathcal{P}_n^s$. In practice, instead of evaluating the integrals from the continuous scalar product, one might use a cubature rule with nodes $\mathbf{x}_\ell$ and corresponding weights w_ℓ to approximate

$$\alpha_{\mathbf{h}} \approx \sum_{\ell} w_\ell \underbrace{f(\mathbf{x}_\ell)\, \hat{T}_{\mathbf{h}}(\mathbf{x}_\ell)}_{g(\mathbf{x}_\ell)}. \tag{1}$$

Actually, it is this cubature rule for the integral with Chebyshev weight function that will be the main topic of this paper. This is similar to what was studied in [14].

The algebraic degree of cubature rule (1), i.e., the degree for which all polynomials up to that degree are approximated exactly with this weighted sum, is chosen to be $2n$. The above rule must thus be exact for all functions $g(\mathbf{x})$ with degrees up to $2n$, which can be ensured by verifying the rule for all polynomials from the basis, formally

$$\forall \hat{T}_{\mathbf{h}} \in \mathcal{T}_{2n}^{s} : \quad \sum_{\boldsymbol{\ell}} w_{\boldsymbol{\ell}} \ \hat{T}_{\mathbf{h}}(\mathbf{x}_{\boldsymbol{\ell}}) = \int_{C_s} \hat{T}_{\mathbf{h}}(\mathbf{x})\, \omega(\mathbf{x})\, \mathsf{d}\mathbf{x}. \tag{2}$$

The degree $2n$ of this rule might seem excessive. However, this is necessary to ensure exactness for all $f \in \mathcal{P}_n^s$ in (1), thus leading to an equality instead of an approximation. Based on such a cubature rule, a discrete scalar product can be introduced as follows

$$\langle f_1, f_2 \rangle_L := \sum_{\boldsymbol{\ell}} w_{\boldsymbol{\ell}} \ f_1(\mathbf{x}_{\boldsymbol{\ell}})\, f_2(\mathbf{x}_{\boldsymbol{\ell}}). \tag{3}$$

The final step in the derivation of a hyperinterpolation formula is the cubature rule based projection that uses the discrete scalar product instead of the continuous one

$$L_n^s[f] := \sum_{\mathbf{h}, |\mathbf{h}| \leq n} a_{\mathbf{h}} \, \hat{T}_{\mathbf{h}} \quad \text{where} \ \ a_{\mathbf{h}} := \langle f, \hat{T}_{\mathbf{h}} \rangle_L, \tag{4}$$

for which the following theorem can be used.

Theorem 1 *Given a function f that is continuous in C_s, let $L_n^s[f] \in \mathcal{P}_n^s$ be defined by (4), where the discrete inner product (3) has cubature points $\mathbf{x}_{\boldsymbol{\ell}} \in C_s$ and weights $w_{\boldsymbol{\ell}} > 0$ and satisfies (2). If $E_n[f] = \inf_{\psi \in \mathcal{P}_n^s} \|f - \psi\|_\infty$, the best uniform approximation of the function f by an element from the polynomial space $\mathcal{P}_n^s$, and $V = \int_{C_s} \omega(\mathbf{x})\, \mathsf{d}\mathbf{x}$ is the measure of the domain then $\|L_n^s[f]\|_2 \leq \sqrt{V}\, \|f\|_\infty$ and $\|L_n^s[f] - f\|_2 \leq 2\sqrt{V}\, E_n[f]$. The latter of these approximation errors reduces to zero as the degree n goes to infinity*:

$$\|L_n^s[f] - f\|_2 \to 0 \quad \textit{as} \ \ n \to \infty.$$

Proof See [12]. □

Theorem 1 guaranties that the approximation $L_n^s[f]$ converges to f as the degree n increases. Now, because the integrals of $\hat{T}_{\mathbf{h}}$ are known analytically, also the integral approximations based on $L_n^s[f]$ will converge to the exact integral

$$Q_n[f] := \int_{C_s} L_n^s[f]\, \mathsf{d}\mathbf{x} = \sum_{\mathbf{h}, |\mathbf{h}| \leq n} a_{\mathbf{h}} \int_{C_s} \hat{T}_{\mathbf{h}}(\mathbf{x})\, \mathsf{d}\mathbf{x} \xrightarrow{n \to \infty} \int_{C_s} f(\mathbf{x})\, \mathsf{d}\mathbf{x} =: I[f].$$

It is thus guarantied that approximating the Fourier coefficients with a cubature rule of sufficiently high degree leads to a converging function approximation and thus a converging integral approximation. It is known that this cubature rule is stable [14] in the sense that for $n \to \infty$, the sum of the absolute value of the weights is finite. Even more, this limit equals the volume of the domain C_s.

Note that, due to the weight function $\omega(\mathbf{x})$ in the continuous scalar product, this *inner* cubature rule uses a Chebyshev weighted setting. It should not be confused with the *outer* cubature rule that approximates the integral with constant weight function.

Details on the cubature rule from (1) were deferred on purpose. This rule approximates an integral with a Chebyshev weight function and is in fact the main topic of this paper. Section 2 will introduce a general framework for describing points and corresponding cubature rules for this Chebyshev weighted setting based on lattices. Sections 3 through 5 then show how most of the known point sets in two, three and s dimensions, can be described using this formalism. Section 6 finally concludes this paper and gives some remarks on future research directions in this context.

2 Chebyshev lattice rules

2.1 Motivation for the rule structure selection

The construction of cubature rules for approximating integrals over C_s with the Chebyshev weight function,

$$I[f] := \int_{C_s} f(\mathbf{x})\, \omega(\mathbf{x})\, \mathrm{d}\mathbf{x} \approx \sum_{\boldsymbol{\ell}} f(\mathbf{x}_{\boldsymbol{\ell}})\, w_{\boldsymbol{\ell}} =: Q[f], \tag{5}$$

received a lot of attention, especially in two dimensions, see, e.g., [3, 9, 11, 16]. It is the only known example of a centrally symmetric integral for which cubature rules of arbitrary algebraic degree are known that attain Möller's lower bound [2, 8, 9]. Furthermore, for a given degree, these rules are not unique. For degree $n = 11$ it was illustrated that there is an infinite number of rules [16] (see also Sect. 3.4).

For several of the known cubature rules, the coordinates of most points are extrema of Chebyshev polynomials, hence a selection of points of a non-uniform grid. An advantage is that these points are known explicitly, but the grid-like structure reduces the number of distinct component values. This is typically something that one wants to avoid. Not only because for certain functions identical component values might not contain much new information about the function, but mainly due to the explosive growth of the number of points as the number of dimensions goes up. Our aim is to construct rules that do not have this disadvantage, are not limited to two/three dimensions and can be represented compactly, similar to traditional lattice rules [13].

2.2 Chebyshev lattices

Following [13], an integration lattice $\Lambda \subset \mathbb{R}^s$ is a discrete subset which is closed under addition and subtraction and contains $\mathbb{Z}^s$. Points from this subset can be described as linear combination of $k = \mathrm{rank}(\Lambda)$ linear independent generating vectors $\mathbf{g}_j \in \mathbb{R}^s$ for $j = 1, \ldots, k$. If we assume that Λ is an integration lattice, $\mathbf{g}_j$'s components are rational numbers. Writing them as an integer fraction $\mathbf{g}_j = \frac{\mathbf{z}_j}{d_j}$ where $\mathbf{z}_j \in \mathbb{Z}^s$ and $d_j \in \mathbb{Z}$, results in the so called canonical form. Using curly braces $\{\cdot\}$ only to denote the *set* of points,[1] leads to a very compact notation of these integration lattice

$$\Lambda = \left\{ \frac{\ell_1 \mathbf{z}_1}{d_1} + \cdots + \frac{\ell_k \mathbf{z}_k}{d_k} + \frac{\mathbf{z}_\Delta}{d_\Delta}, \text{with } \ell_1, \ldots, \ell_k \in \mathbb{Z} \right\}.$$

We have included an additional offset vector $\mathbf{z}_\Delta$ in this notation. This allows us to represent point sets that do not contain the origin. Unless stated otherwise, $\mathbf{z}_\Delta$ is assumed to be a zero vector $\mathbf{0}$ and Λ will then contain the origin.

A component-wise mapping $\mathbf{x} = \cos(\pi\, \mathbf{y})$, defined as $x_r = \cos(\pi\, y_r)$ for $r = 1, \ldots, s$, reduces the entire space $\mathbb{R}^s$ into a closed[2] hypercube C_s. Using a *closed* hypercube, the usual requirement for periodic functions is no longer necessary. Transforming the points from Λ with this leads to a family of point sets we have named *Chebyshev lattices*.

Definition 1 A s-dimensional rank-k *Chebyshev lattice* (*CL*) is described by the s-dimensional non-zero integer generating vectors $\mathbf{z}_1, \ldots, \mathbf{z}_k$ and $\mathbf{z}_\Delta$, positive integer denominators $d_1, \ldots, d_k$ and d_Δ. It can be written as a set satisfying

$$\chi = \left\{ \cos\left(\pi \left(\frac{\ell_1 \mathbf{z}_1}{d_1} + \cdots + \frac{\ell_k \mathbf{z}_k}{d_k} + \frac{\mathbf{z}_\Delta}{d_\Delta} \right)\right), \text{with } \ell_1, \ldots, \ell_k \in \mathbb{Z} \right\} \subset C_s.$$

In one dimension and for odd degrees n, a denominator $d_1 = \frac{n+1}{2}$ and generating vector $\mathbf{z}_1 = [a]$ (where a is relative prime to $(n+1)$) describes a Chebyshev lattice which corresponds to the extrema of Chebyshev polynomials. In two and three dimensions, specific generators lead to point sets that are already presented in literature and were discussed recently. In Sects. 3 through 5, we will show that most of them fit into our Chebyshev lattice-framework.

2.3 Chebyshev lattice rule

For a cubature rule based on a Chebyshev lattice, the points $\mathbf{x}_{\boldsymbol{\ell}}$ must be assigned a certain weight $w_{\boldsymbol{\ell}}$. Using ϕ (‘condition’) that evaluates to 1 if the condition is satisfied, 0 if not, a cubature rule based on this point set can be introduced as follows.

Definition 2 A *Chebyshev lattice rule* is a cubature rule, $Q[f]$ from (5), with nodes $\mathbf{x}_{\boldsymbol{\ell}}$ from a Chebyshev lattice and weights $w_{\boldsymbol{\ell}}$ defined as

$$w_{\boldsymbol{\ell}} = \frac{\tilde{w}_{\boldsymbol{\ell}}}{\tilde{W}}, \quad \text{where } \tilde{w}_{\boldsymbol{\ell}} = \prod_{r=1}^{s} \left(\frac{1}{2}\right)^{\phi(|x_{\boldsymbol{\ell},r}|=1)} \quad \text{and} \quad \tilde{W} = \sum_{\boldsymbol{\ell}} \tilde{w}_{\boldsymbol{\ell}}. \tag{6}$$

This weighting scheme uses equal weights for inner points and halves the weight for each component that is on the boundary: the weight is divided by the number of

[1] In a lattice context, $\{\cdot\}$ usually denotes taking components values modulo 1, but not here.

[2] This also differs from the usual setting where the entire space is reduced to an open hypercube $[0, 1)^s$ using a ‘modulo 1’ operation and functions are assumed to be periodic.

neighbouring regions it is sharing its weight with. In three dimension for example, points on the faces have half, points on edges a quarter and points on the vertices one eight of the weight of the inner points. The normalisation constant is needed to ensure the exactness of the cubature rule for a constant function.

3 Relation with known two-dimensional cubature rules

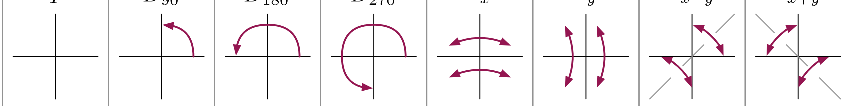
3.1 Morrow-Patterson points as full-rank Chebyshev lattice

It is easy to show that the Morrow-Patterson point set [9] for degrees $n = 4\nu - 3$ fit into the Chebyshev lattice rule framework from Sect. 2: it corresponds to a rank-2 Chebyshev lattice rule. For $n = 4\nu - 1$ however, a nonzero offset $\mathbf{z}_\Delta$ will be needed because $[1, 1]$, the cosine mapped origin, is not part of the point set (see Fig. 2(a). This clarifies the need for the offset vector $\mathbf{z}_\Delta$ in Definition 1. As mentioned before, several of the projections onto the coordinate axes coincide, as shown in Fig. 2(a).

The Morrow-Patterson cubature rule and its corresponding Chebyshev lattice notations are summarised in Table 1. This is the first in a series of tables that lists the generating vectors $\mathbf{z}_j$, offset vector $\mathbf{z}_\Delta$, denominators d_j and d_Δ and the number of points N as function of the degree n. For the two-dimensional point sets, they also give the number of distinct projections onto the coordinate axes and information on the symmetry of the set, using the same notation as in, e.g., [3] and summarised in Fig. 1.

3.2 The Padua points, a rank-1 Chebyshev lattice

The Padua points can be formalised by a generating curve [1], as illustrated in Fig. 2(b). Considering the fourth family (the other families would result in a rotated

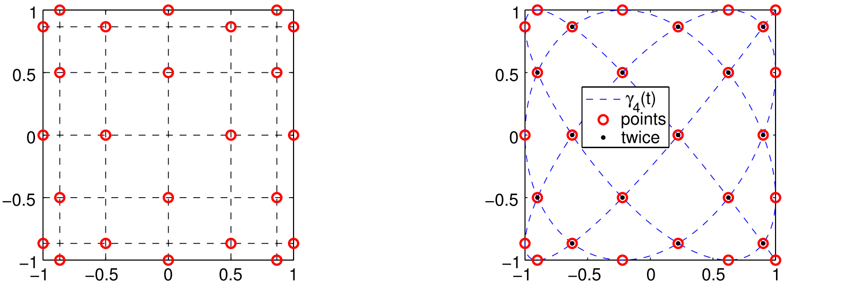


	I	D_{90}	D_{180}	D_{270}	S_x	S_y	S_{x-y}	S_{x+y}
S_2	✓		✓				✓	✓
D_2	✓		✓		✓	✓		
R_4	✓	✓	✓	✓				
D_4	✓	✓	✓	✓	✓	✓	✓	✓

Fig. 1 Overview of the symmetries that can be observed in two dimensional point sets. At the bottom are some named groups where the checkmarks indicate which symmetries are included

Table 1 Morrow-Patterson cubature rule in the Chebyshev lattice rule formalism

n	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_\Delta$	d_1, d_2, d_Δ	N	$\#x$,$\#y$-proj	Sym.
$4\nu - 3$	$[1, 1]$	$[0, 2]$	$[0, 0]$	$2\nu - 1$	$2\nu^2$	2ν	S_2
$4\nu - 1$	$[1, 1]$	$[0, 2]$	$[0, 1]$	2ν	$2\nu(\nu + 1)$	$2\nu + 1$	D_4



(a) Morrow-Patterson points ($N = 24$). Both x- and y-axis have 7 distinct projections. Note that $[1, 1]$ is not in the set, so $\mathbf{z}_\Delta \neq \mathbf{0}$.

(b) Padua points ($N = 28$) and generating curve $\gamma_4(t)$. Projections onto the coordinate axes here result in 8 and 7 distinct values.

Fig. 2 Illustration of two-dimensional point sets of degree $n = 11$ ($4\nu - 1$ type with $\nu = 3$)

Table 2 Padua cubature rule described as Chebyshev lattice rule for odd degrees n

n	$\mathbf{z}_1$	d_1	N	#x-proj	#y-proj	Sym.
$4\nu - 3$	$[2\nu - 1, 2\nu]$	$2\nu\,(2\nu - 1)$	$\nu\,(2\nu + 1)$	$2\nu + 1$	2ν	S_x
$4\nu - 1$	$[2\nu, 2\nu + 1]$	$2\nu\,(2\nu + 1)$	$(\nu + 1)\,(2\nu + 1)$	$2\nu + 2$	$2\nu + 1$	S_y

and/or reflected lattice) and an integer μ, this curve can be written parametrically as

$$\gamma_4(t) = [\cos((\mu + 1)\,t), \cos((\mu + 2)\,t)] \quad \text{where} \quad t \in [0, \pi]. \tag{7}$$

The corresponding Padua points are equidistant sampled points of this curve $\gamma_4(t)$

$$\left\{\gamma_4\left(\frac{\ell_1 \pi}{d_1}\right)\right\} \quad \text{with} \quad \begin{cases} \ell_1 = 0, \ldots, d_1, \\ d_1 = (\mu + 1)(\mu + 2). \end{cases} \tag{8}$$

Due to the construction, the sampled points coincide with the self intersections of the curve $\gamma_4(t)$ (plus the remaining boundary points). Therefore, the number of Padua points is less than d_1 from (8), namely $N = \frac{1}{2}(\mu + 2)(\mu + 3)$. Note that projections onto the coordinate axes are not distinct: several points have equal component values. The Padua cubature rule, with weights from Definition 2, has a degree $n = 2\mu + 1$.

This Padua point set is a rank-1 Chebyshev lattice with $\mathbf{z}_1 = [\mu + 1, \mu + 2]$ and $d_1 = (\mu + 1)(\mu + 2)$ as summarised in Table 2, where the two general forms, $n = 4\nu - 3$ and $n = 4\nu - 1$, are separated to show the difference in symmetry properties.

3.3 Minimal D_4-invariant cubature formulae

In [3] cubature rules of degrees $n = 4\nu - 1$ and different symmetry properties are discussed. Using $\alpha = -\frac{1}{2}$, these correspond to the Chebyshev weighted integrals that are considered here. It is easy to see in Fig. 3(a) that, except for the diagonals, these rules can be described by a Chebyshev lattice rule. It yields the same generators as

Table 3 Minimal D_4 invariant cubature formulae [3] and best fitting Chebyshev lattice

(a) D_4 invariant formulae

n	N	$\#x$,$\#y$-proj	Sym.
3	4	2	D_4
7	12	7	D_4
$4\nu - 1$	$2\nu(\nu+1)$	$4\nu + 1$	D_4

(b) Chebyshev lattice resembling the point set from (a) as much as possible

n	$\mathbf{z}_1$	$\mathbf{z}_2$	d_1, d_2	N	$\#x$,$\#y$-proj	Sym.
$4\nu - 1$	$[1, 1]$	$[0, 2]$	2ν	$2\nu(\nu+1)+1$	$2\nu + 1$	D_4

the Morrow-Patterson-set (see Table 1), but without the shift. The remaining points are to be found as zeros of a known polynomial of degree 2ν.

Note that the *nearest* Chebyshev lattice, as shown in Table 3, has only one extra point, thus leading to a near-minimal rule. The disadvantage of the extra point in the Chebyshev lattice is however often compensated by the explicit expression for the weights in Definition 2. They are known in advance and require far less work than the linear system that needs to be solved to determine the weights for the point sets in [3].

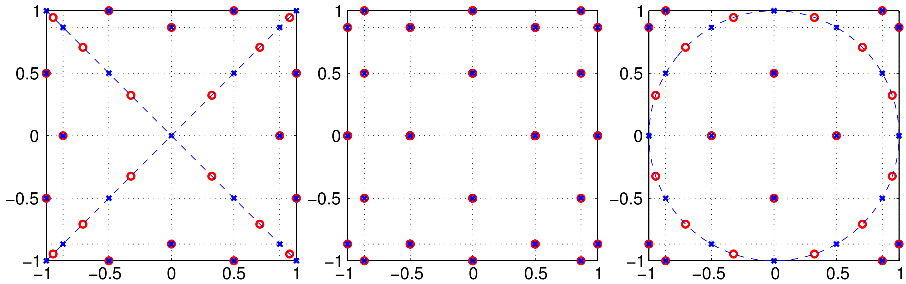
3.4 Three degree 11, fully symmetric cubature rules on a square

Using bifurcation analysis of the non-linear equations that characterise a degree $n = 11$ rotation invariant cubature rule on the square C_2 for the Chebyshev weight function, three fully symmetrical cubature rules were found in [16]. These point sets are shown in Fig. 3, together with the *nearest* Chebyshev lattice that could be found. It is clear from this figure that they have several points in common. Solution 'b' is an example of the D_4-invariant rules from Sect. 3.3 while solution 'c' is the Morrow-Patterson point set already discussed in Sect. 3.1. These degree $n = 11$ formulae and accompanying Chebyshev lattices are shown in Fig. 3 and summarised in Tables 1, 3 and 4.

The unexpected structure of the points that are not shared, the cross and circle structure in Fig. 3, can be explained. It turns out that for solution 'b' and its Chebyshev lattice are instances of a parametrized point set with 28 points. For two special values of the parameter, some of these points coincide, leading to these two sets with less than 28 points. For solution 'd' and the best fitting Chebyshev lattice, a similar parameterised point set can be found. For the same parameter values as for solution 'b', the two point sets from Fig. 3(c) are found.

3.5 Discrete Fourier analysis

Using discrete Fourier analysis on a rhombus in [6], a point set for integrals with Chebyshev weight function is derived. Degrees $n = 4\nu - 3$ lead to the Morrow-



(a) Solution 'b'. (b) Solution 'c'. (c) Solution 'd'.

Fig. 3 The three fully symmetric point sets originating from bifurcation analysis [16], shown as red circles (∘), and the best fitting Chebyshev lattices drawn with blue crosses (×)

Table 4 Degree $n = 11$ cubature rule [16] (solution 'd') and best fitting Chebyshev lattice

(a) solution 'd'

n	N	#x,#y-proj	Sym.
11	24	13	D_4

(b) Chebyshev lattice resembling the above points sets as much as possible

n	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_\Delta$	d_1, d_2, d_Δ	N	#x,#y-proj	Sym.
11	[1, 1]	[0, 2]	[0, 1]	7	24	7	D_4

Patterson point set, while for degrees $n = 4\nu - 1$, this results in the D_4 invariant rule obtained in [3] and mentioned in Sect. 3.3.

4 Relation with known three-dimensional cubature rules

4.1 Noskov

The first note of a explicit three-dimensional cubature rule for integrals with Chebyshev weight function is presented in [10, Example 4]. This paper briefly describes a cubature rule with $n = 4\nu - 1$ that can be reformulated as the Chebyshev lattice from Table 5.

4.2 3-D Morrow-Patterson extension: rank-3 Chebyshev lattice

In [7], a three-dimensional point set similar to the Morrow-Patterson points is presented. It extends the idea of the Cartesian product of Chebyshev-Lobatto points $C_m = \{\cos \frac{j\pi}{m},\ j = 0, \ldots, m\}$ with odd (O) and even (E) indices j to three variables.

Table 5 The three-dimensional rule as described in [10]

ν	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	$\mathbf{z}_\Delta$	d_1, d_2, d_3, d_Δ	N
$4\nu - 1$	$[1, 1, 1]$	$[2, 0, 0]$	$[0, 0, 2]$	$[1, 0, 0]$	2ν	$2\nu^3 + 3\nu^2 + \nu$

Table 6 Description of the 3-D extension to Morrow-Patterson [7] with Chebyshev lattice rules as function of the degree n. Only the (E, E, E) and (E, E, O) configuration are given, the others are identical, up to a coordinate change. If a point (x, y, z) is in the set, so are the ones described, where p_i^{xyz} is the i-th component of a permutation of the variables in superscript. The last column lists the number of symmetries, 48 can be considered to be fully symmetric

(a) (E, E, E)-configuration

n	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	d_1, d_2, d_3	N	Symmetries	#
$4\nu - 3$	$[1, 1, 1]$	$[0, 2, 0]$	$[0, 0, 2]$	$2\nu - 1$	$2\nu^3$	$\pm(p_1^{xyz}, p_2^{xyz}, p_3^{xyz})$	12
$4\nu - 1$	$[1, 1, 1]$	$[0, 2, 0]$	$[0, 0, 2]$	2ν	$\nu^3 + (\nu + 1)^3$	$(\pm p_1^{xyz}, \pm p_2^{xyz}, \pm p_3^{xyz})$	48

(b) (E, E, O)-configuration

n	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$	$\mathbf{z}_\Delta$	d_1, d_2, d_3, d_Δ	N	Symmetries	#
$4\nu - 3$	$[1, 1, 1]$	$[0, 2, 0]$	$[0, 0, 2]$	$[0, 0, 1]$	$2\nu - 1$	$2\nu^3$	$\pm(p_1^{xy}, p_2^{xy}, z)$ $\pm(-p_1^{xz}, p_2^{xz}, y)$ $\pm(p_1^{yz}, -p_2^{yz}, x)$	12
$4\nu - 1$	$[1, 1, 1]$	$[0, 2, 0]$	$[0, 0, 2]$	$[0, 0, 1]$	2ν	$\nu(\nu + 1)^2 + (\nu + 1)\nu^2$	$(\pm p_1^{xy}, \pm p_2^{xy}, \pm z)$	16

These point sets, with the weights from (2), can also be described by the Chebyshev lattice rule formalism as in Table 6. For some of the configurations, denoted by (X, X, X) with $X \in \{O, E\}$, an offset vector is required.

Although these point sets were labeled ‘new’ in [7], except for the (E, E, E) configuration with $n = 4\nu - 1$, these are examples of Noskov’s point set and, disregarding coordinate changes and reflections, also blending formulae in three dimensions (see Sect. 5). For degree $n = 4\nu - 1$, the (E, E, E) configuration contains more points than the ones found with the blending approach.

4.3 Discrete Fourier analysis

Similar to the two-dimensional point set, in [6] a three-dimensional point set is derived. This is identical to the (E, E, E) configuration described in [7].

5 Higher-dimensional cubature rules: blending formula

The principle of blending, first used in [9], was explored further in [4] (part of this work is published in [5]). The rules are derived from a blend of the nodes X^A and

X^B coming from two (univariate) quadrature rules Q^A and Q^B

$$Q^A[f] = \sum_l w_l^A\, f(x_l^A), \qquad Q^B[f] = \sum_l w_l^B\, f(x_l^B). \tag{9}$$

The s-dimensional cubature rule then uses the union of two blends of these nodes

$$X^{\mathrm{I}} = \left\{(x_{l_1}^A, x_{l_2}^B, x_{l_3}^A, x_{l_4}^B, \ldots)\right\} \quad \text{and} \quad X^{\mathrm{II}} = \left\{(x_{l_1}^B, x_{l_2}^A, x_{l_3}^B, x_{l_4}^A, \ldots)\right\},$$

with suitable indexes l_r from (9). The weights are combined in a similar fashion

$$W^{\mathrm{I}} = \left\{\frac{1}{2} w_{l_1}^A w_{l_2}^B w_{l_3}^A w_{l_4}^B \cdots\right\} \quad \text{and} \quad W^{\mathrm{II}} = \left\{\frac{1}{2} w_{l_1}^B w_{l_2}^A w_{l_3}^B w_{l_4}^A \cdots\right\}.$$

For degrees $n = 4\nu - 1$, the two base quadrature rules use the zeros of the Chebyshev polynomial $T_\nu(x)$ and $T_{\nu+1}(x) - \rho_\nu T_{\nu-1}(x)$ where $\rho_1 = \frac{1}{2}$ and $\rho_\nu = \frac{1}{4}$ for $\nu = 2, 3, \ldots$. These zeros can be determined analytically, thus leading to the node sets

$$X^A = \left\{\cos\left(\pi \frac{2l-1}{2\nu}\right), l = 1, \ldots, \nu\right\}, \qquad X^B = \left\{\cos\left(\pi \frac{2l}{2\nu}\right), l = 0, \ldots, \nu\right\}.$$

Because these zeros are interleaved, the blended point set is a grid like point set where only one of two neighbouring grid points is in the set. In two dimensions, this resembles all white squares of a chess board. Note that because no corners are in the blends, the Chebyshev lattice needs a nonzero $\mathbf{z}_\Delta$.

Theorem 2 (Blending, degree $n = 4\nu - 1$) *The s-dimensional Chebyshev lattice rule with weights from* (6) *and generators, denominators and offset as follows*

$$\begin{aligned}
\mathbf{z}_1 &= [1,\ 1,\ 1,\ 1 \cdots\ 1,\ 1,\ 1],\\
\mathbf{z}_2 &= [0,\ 2,\ 0,\ 0 \cdots\ 0,\ 0,\ 0],\\
\mathbf{z}_3 &= [0,\ 0,\ 2,\ 0 \cdots\ 0,\ 0,\ 0],\\
&\ \ \vdots \qquad\qquad\qquad \ddots\\
\mathbf{z}_{s-1} &= [0,\ 0,\ 0,\ 0 \cdots\ 0,\ 2,\ 0],\\
\mathbf{z}_s &= [0,\ 0,\ 0,\ 0 \cdots\ 0,\ 0,\ 2],
\end{aligned} \tag{10}$$

$$d_1, \ldots, d_s = d_\Delta = 2\nu, \qquad \mathbf{z}_\Delta = [0, 1, 0, 1, \ldots],$$

describes a blending cubature rule of degree $n = 4\nu - 1$. The number of points is

$$N_n^s = \nu^{\lceil \frac{s}{2} \rceil} (\nu+1)^{\lfloor \frac{s}{2} \rfloor} + (\nu+1)^{\lceil \frac{s}{2} \rceil} \nu^{\lfloor \frac{s}{2} \rfloor} = \begin{cases} 2\,[\,\nu(\nu+1)\,]^{\frac{s}{2}}, & s \text{ even},\\ (2\nu+1)\,[\,\nu(\nu+1)\,]^{\frac{s-1}{2}}, & s \text{ odd}. \end{cases}$$

For degrees $n = 4\nu - 3$, the nodes are chosen from the zeros of $T_{\nu+1} \pm \sqrt{\rho_\nu}\, T_{\nu-1}$

$$X^E = \left\{\cos\left(\pi \frac{2l-1}{2\nu-1}\right), l = 1, \ldots, \nu\right\}, \qquad X^F = \left\{\cos\left(\pi \frac{2l-2}{2\nu-1}\right), l = 1, \ldots, \nu\right\}.$$

Each of these sets has one end point of $[-1,1]$, therefore some corners of C_s will be included in the blended sets. The offset vector will thus only be nonzero to ensure that the right corners are selected. Note that without offset, a rule with the same degree is achieved, but some components are mirrored compared to the original blending description.

Theorem 3 (Blending, degree $n = 4\nu - 3$) *The s-dimensional Chebyshev lattice rule with weights* (6), *generators* (10), *and denominators and offset as follows*

$$d_1, \ldots, d_s = d_\Delta = 2\nu - 1, \qquad \mathbf{z}_\Delta = [0, d_\Delta, 0, d_\Delta, \ldots],$$

describes a blending cubature rule of degree $n = 4\nu - 3$. The number of points is

$$N_n^s = 2\,\nu^s.$$

The degrees of the blending formulae in Theorems 2 and 3 was proven using ideal theory [4], however, the Chebyshev lattice notation clearly shows how this breaks down to the one-dimensional quadrature rules that were blended.

Proof To achieve this algebraic degree n, all polynomials from $\mathcal{P}_n^s$ must be integrated exactly. Due to the construction of the Chebyshev lattice, it is advisable to use Chebyshev polynomials to show this. Formally, it boils down to proving that

$$\forall T_{\mathbf{h}} \in \mathcal{T}_n^s: \quad Q[T_{\mathbf{h}}] := \sum_{\boldsymbol{\ell}} w_{\boldsymbol{\ell}}\, T_{\mathbf{h}}(\mathbf{x}_{\boldsymbol{\ell}}) = \int_{[-1,1]^s} T_{\mathbf{h}}(\mathbf{x})\,\omega(\mathbf{x})\,\mathrm{d}\mathbf{x} =: I!\,[T_{\mathbf{h}}].$$

Due to the structure in the generators (10), points from the blending set depend only on one or two components of the vector $\boldsymbol{\ell}$, which becomes clear by writing a point as

$$\mathbf{x}_{\boldsymbol{\ell}} = \left[\cos\left(\pi\frac{\ell_1 + z_{\Delta,1}}{d_1}\right), \cos\left(\pi\frac{\ell_1 + 2\,\ell_2 + z_{\Delta,2}}{d_2}\right), \ldots, \cos\left(\pi\frac{\ell_1 + 2\,\ell_s + z_{\Delta,s}}{d_s}\right)\right].$$

Consider now the approximation $Q[T_{\mathbf{h}}]$, using the suitable '$\boldsymbol{\ell}$'-indices, this becomes

$$Q_{\mathbf{h}} = \sum_{\ell_1} \cdots \sum_{\ell_s} w_{\boldsymbol{\ell}} \prod_{r=1}^{s} \cos(h_r \,\arccos(x_{\boldsymbol{\ell},r})).$$

Now, the r-th component of $\mathbf{x}_{\boldsymbol{\ell}}$, denoted by $x_{\boldsymbol{\ell},r}$ depends only on ℓ_1 and ℓ_r so, using $d_1 = d_2 = \cdots = d_s$, the sums and the product can be reordered to

$$Q_{\mathbf{h}} = \sum_{\ell_1} w_{\ell_1} \cos\left(\frac{h_1\pi}{d_1}(\ell_1 + z_{\Delta,1})\right) \prod_{r=2}^{s} \sum_{\ell_r} w_{\ell_r} \cos\left(\frac{h_r\pi}{d_1}(\ell_1 + 2\,\ell_r + z_{\Delta,r})\right).$$

The indexes ℓ_r were specified so that they do not produce duplicate points. The first, ℓ_1 has a range $[0, d_1]$ and because $z_{\Delta,1} = 0$ for all degrees, the offset component can be removed there. For the other components of $\boldsymbol{\ell}$, the integer parts inside the cosine can be re-indexed with m because $(\ell_1 + z_{\Delta,r})$ and $(\ell_1 + 2\,\ell_r + z_{\Delta,r})$ have the same parity. This allows us to specify explicit bounds for the integral approximation

$$Q_{\mathbf{h}} = \sum_{\ell_1=0}^{d_1} w_{\ell_1} \cos\left(\frac{h_1\,\pi}{d_1}\ell_1\right) \prod_{r=2}^{s} q_r \quad \text{with}$$

$$q_r = \begin{cases} \sum_{m=0}^{\lfloor \frac{d_1}{2} \rfloor} \; w_{2m} \cos(\frac{h_r\,\pi}{d_1}(2m)) & \text{for } (\ell_1 + z_{\Delta,r}) \text{ even}, \\ \sum_{m=0}^{\lfloor \frac{d_1-1}{2} \rfloor} \; w_{2m+1} \cos(\frac{h_r\,\pi}{d_1}(2m+1)) & \text{for } (\ell_1 + z_{\Delta,r}) \text{ odd}. \end{cases}$$

So far we haven't mentioned anything about the weights, but their relative values will be 1 or $\frac{1}{2}$ depending on the position of one-dimensional points described with the new index m. Depending on the parity of d_1 (and thus the degree n), this leads to four one-dimensional quadrature rules that, using a line under and/or above the sum to indicate that the first and/or last term of the sum is halved, can be summarised as

	$n = 4\nu - 1, d_1 = 2\nu$	$n = 4\nu - 3, d_1 = 2\nu - 1$
$(\ell_1 + z_{\Delta,r})$ even	$\overline{\underline{\sum}}_{m=0}^{\nu} \cos\big(\frac{h_r\,\pi}{d_1}(2m)\big)$	$\underline{\sum}_{m=0}^{\nu-1} \cos\big(\frac{h_r\,\pi}{d_1}(2m)\big)$
$(\ell_1 + z_{\Delta,r})$ odd	$\sum_{m=0}^{\nu-1} \cos\big(\frac{h_r\,\pi}{d_1}(2m+1)\big)$	$\underline{\sum}_{m=0}^{\nu-1} \cos\big(\frac{h_r\,\pi}{d_1}(2m+1)\big)$

Now the cubature rule is decomposed into independent quadrature rules over the different dimensions and it is not surprising that they are identical to the rules with nodes X^A, X^B, X^E and X^F that were construct by Godzina [4]. As all those quadrature rules have a degree n, the blended rule is exact for Chebyshev polynomials where $h_r \leq n$ for $r = 1, \ldots, s$. This includes the polynomials with total degree up to n, which concludes this proof. □

6 Conclusion and future work

In this paper we introduced Chebyshev lattices as a compact way to describe some point sets that are known for (hyper-) interpolation and cubature. It clearly shows relations between known point sets and hints at higher dimensional generalisations. This new framework also allows for systematic searches for "better" point sets through the lattice description. These searches and the efficient FFT-enabled implementations of multivariate Clenshaw-Curtis integration and interpolation are the subject of ongoing research.

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