

You know, I don't know about that.

16 March 2020

- I wanna pause on for a sec b/c I can't seem to get it to work and I'm confused about how to get it time-varying anyway. So let's turn to estimation.

### Estimation of the anchoring function

The issue is that we wanna estimate the anchoring function together w/ the model. On the top of my head I can think of 3 ways of doing that:

- 1) IR-matching
- 2) likelihood-based (either MLE or Bayesian)
- 3) VAR-representation  $\Rightarrow$  exist? est. that!

$\hookrightarrow$  it would be a time-varying one.

- $\hookrightarrow$  I'm leaning toward #2 b/c 1) it's sexier  
2) it's more general than conditional on shocks  
3) a TV-VAR sounds challenging

The thing is:

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need to derive the log-likelihood of my model

Take the midsample model w/ no TR

In materials 21, this is eq. (5)-(10) + TR

$$\pi_t = kx_t - (1-\alpha)\beta f_{\alpha}(t) + [-\kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}]s_t$$

$$x_t = -b\bar{\pi}_t - b f_b(t) + [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}]s_t$$

$$f_{\alpha}(t) = \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} - b_1(f_3 - \alpha\beta h_x)^{-1}s_t$$

$$f_b(t) = \frac{1}{1-\beta} \bar{\pi}_{t-1} - b_1(f_3 - \beta h_x)^{-1}s_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d(\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + c$$

$$i_t = \gamma_{\pi}\pi_t + \gamma_x x_t + \bar{i}_t$$

this is a state-space model (believe it or not)

and I'm gonna eliminate some variables

$$\begin{aligned} \pi_t &= kx_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1} - (1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1}s_t \\ &\quad + [-\kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}]s_t \end{aligned}$$

$$\begin{aligned} x_t &= -b\pi_t - b f_b(t) - b\bar{i}_t - \frac{b}{1-\beta} \bar{\pi}_{t-1} - b b_1(f_3 - \beta h_x)^{-1}s_t \\ &\quad + [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}]s_t \end{aligned}$$

$$\pi_4 = \kappa x_+ - \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ [-(1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}] s_t$$

$$x_+ = -b\gamma_\pi \pi_4 - b\gamma_x x_+ - bi_r - \frac{b}{1-\beta} \bar{\pi}_{t-1} - b b_1(I_3 - \beta h_x)^{-1} s_t$$

$$+ [-(1-\beta)b_2(f_3 - \beta h_x)^{-1} - (1-\beta)b_3(f_3 - \beta h_x)^{-1} + 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$(1+b\gamma_x) x_+ = -b\gamma_\pi \pi_4 - \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$[-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$\Rightarrow x_+ = -\frac{b\gamma_\pi}{1+b\gamma_x} \pi_4 - \frac{1}{1+b\gamma_x} \frac{b}{1-\beta} \bar{\pi}_{t-1}$$

$$+ \frac{1}{1+b\gamma_x} [-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

Can even sub  $x_+$  out!

$$\pi_4 = -\frac{\kappa b\gamma_\pi}{1+b\gamma_x} \pi_4 - \frac{\kappa}{1+b\gamma_x} \frac{b}{(1-\beta)} \bar{\pi}_{t-1}$$

$$+ \frac{\kappa}{1+b\gamma_x} [-2e_2 - 2b_1(I_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 2\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1}] s_t$$

$$- \frac{(1-\alpha)\beta}{1-\alpha\beta} \bar{\pi}_{t-1}$$

$$+ [-(1-\alpha)\beta b_1(I_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1}] s_t$$

$$\Rightarrow \left(1 + \frac{\kappa b\gamma_\pi}{1+b\gamma_x}\right) \pi_4 = -\left(\frac{\kappa b}{(1+b\gamma_x)(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)}\right) \bar{\pi}_{t-1} + \text{stuff} \cdot s_t$$

$$\left(1 + \frac{k_2 Y_\pi}{1+bY_x}\right) \bar{\pi}_+ = - \left( \frac{k_2}{1+bY_x(1-\beta)} + \frac{(1-\alpha)\beta}{(1-\alpha\beta)} \right) \bar{\pi}_{t-1} +$$

$$+ \left\{ \frac{k}{1+bY_x} \left[ -2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \left[ -(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\} s_t$$

$$\frac{1+bY_x+k_2 Y_\pi}{1+bY_x} \bar{\pi}_+ = \text{same}$$

$$\Rightarrow \bar{\pi}_+ = - \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left( \frac{\frac{k_2(1-\alpha\beta)}{(1+bY_x)(1-\beta)} + \frac{(1-\alpha)\beta(1-\beta)(1+bY_x)}{(1+bY_x)(1-\beta)(1-\alpha\beta)} }{1+bY_x+k_2 Y_\pi} \right) \bar{\pi}_{t-1}$$

$$+ \left\{ \frac{k}{1+bY_x+k_2 Y_\pi} \left[ -2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left[ -(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\}$$

$$\Rightarrow \bar{\pi}_+ = - \frac{\frac{k_2(1-\alpha\beta)}{(1+bY_x+k_2 Y_\pi)} + \frac{\beta(1-\alpha)(1-\beta)(1+bY_x)}{(1+bY_x+k_2 Y_\pi)(1-\beta)(1-\alpha\beta)}}{(1+bY_x+k_2 Y_\pi)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left\{ \frac{k}{1+bY_x+k_2 Y_\pi} \left[ -2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_x}{1+bY_x+k_2 Y_\pi} \left[ -(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - K\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right\} s_t$$

Damn damn! So we have  $\xrightarrow{EA}$

$$\bar{\pi}_t = - \frac{k_2(\alpha-\alpha\beta) + \beta(1-\alpha)(1-\beta)(1+bY_X)}{(1+bY_X+k_2Y_H)(1-\beta)(1-\alpha\beta)} \bar{\pi}_{t-1} +$$

$$\left[ \frac{k}{1+bY_X+k_2Y_H} \left[ -2e_2 - 3b_1(f_3 - \beta h_x)^{-1} - (1-\beta)b_2(f_3 - \beta h_x)^{-1} + 3\beta b_3(f_3 - \beta h_x)^{-1} - 2e_1(f_3 - \beta h_x)^{-1} \right] \right.$$

$$\left. + \frac{1+bY_X}{1+bY_X+k_2Y_H} \left[ -(1-\alpha)\beta b_1(f_3 - \alpha\beta h_x)^{-1} s_1 - \kappa\alpha\beta(f_3 - \alpha\beta h_x)^{-1} - e_3(f_3 - \alpha\beta h_x)^{-1} \right] \right] s_t \xrightarrow{EB}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})) + c$$

$\hookrightarrow$  1 jump ( $\bar{\pi}_t$ ), 3 exogenous states ( $s_t = \begin{bmatrix} r_t \\ i_t \\ h_t \end{bmatrix}$ ) and 2

endogenous states  $\xi_t = \begin{bmatrix} \bar{\pi}_t \\ k_t^{-1} \end{bmatrix}$  ( $\propto \begin{bmatrix} \bar{\pi}_{t-1} \\ k_t^{-1} \end{bmatrix}$ ) so  $X_t = \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$  (states)

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = [A \ B] \begin{bmatrix} \xi_t \\ s_t \end{bmatrix}$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} (\bar{\pi}_t - (\bar{\pi}_{t-1} + b_1 s_{t-1}))$$

$$k_t^{-1} = k_{t-1}^{-1} + d (\bar{\pi}_t - (\bar{\pi}_{t-1} - b_1 s_{t-1})) + c$$

call this the state (?)  $f_{t-1}$

$$\pi_t = A \bar{\pi}_{t-1} + B s_t = [A \ B] \begin{bmatrix} \bar{s}_t \\ s_t \end{bmatrix}$$

$$Y_t = g x \cdot X_t$$

$$\bar{\pi}_t = \bar{\pi}_{t-1} + k_t^{-1} f c_{t-1}$$

$$X_{t+1} = h X_t + \eta \epsilon_t$$

$$k_t^{-1} = k_{t-1}^{-1} + d \cdot f c_{t-1} + c$$

$$f c_{t-1} = \pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1}$$

Several issues y'all:

- 1) fc state or jump? depends on  $\pi_t$  (a jump)  
 → there's gotta be some trick (like  $\pi_t$  is LEMP)  
 to make it a pure state

- 2.)  $\bar{\pi}$  nonlinear Lom!



ok - that's troubling. But let's pause it for a sec & let's read what Litterman has to say about MLE & log-likelihoods.

Supp. our VAR(p) looks like

$$y_t - \mu = A_1(y_{t-1} - \mu) + \dots + A_p(y_{t-p} - \mu) + u_t \quad (3.3.1)$$

then if the VAR(p) is Gaussian, that is

$$u \equiv \text{vec}(U) = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \sim N(0, I_T \otimes \frac{1}{T} I_n) , \text{ which}$$

p.75

equivalently means that the prob. density of  $u$  is

$$f_u(u) = \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp \left[ -\frac{1}{2} u' (I_T \otimes \Sigma_u^{-1}) u \right]$$

then we can use the fact that  $u = y - \mu^* - (x' \otimes I_K) \alpha$

where  $\alpha := \text{vec}(A)$ ,  $A := (A_1, \dots, A_p)$   $k \times kp$

$$k^2 p \times 1$$

$$Y^0 := (y_1 - \mu, \dots, y_T - \mu) \quad k \times T$$

$$Y_t^0 := \begin{bmatrix} y_t - \mu \\ y_{t-p+1} - \mu \end{bmatrix} \quad (k_p \times 1)$$

$$X := (Y_0^0, \dots, Y_{T-1}^0),$$

to write

$$\begin{aligned} f_y(y) &= \left| \frac{\partial y}{\partial u} \right| f_u(u) \\ &= \frac{1}{(2\pi)^{kT/2}} \left| I_T \otimes \Sigma_u \right|^{-1/2} \exp \left[ -\frac{1}{2} (y - \mu^* - (x' \otimes I_K) \alpha)' (I_T \otimes \Sigma_u^{-1}) \right. \\ &\quad \left. \cdot (y - \mu^* - (x' \otimes I_K) \alpha) \right] \quad (3.4.4) \end{aligned}$$

and

$$\ln L(\mu, \alpha, \Sigma_u) = -\frac{kT}{2} \ln(2\pi) - \frac{T}{2} \ln |\Sigma_u| - \frac{1}{2} \text{tr} (Y^0 - Ax)' \Sigma_u^{-1} (Y^0 - Ax)$$

is the log-likelihood

(3.4.5)

Given that the updating w/ endog. gain 18 March 2020 introduces non-linearities, I'm afraid that even a "simple" & quick estimation has to involve some form of particle filter. But let's see whether Litterpoli has anything interesting to say about 1) state-space models 2) non-linearities

### Litterpoli, State-space models (Ch. 13, p. 415 ff.)

$$\begin{aligned} z_{t+1} &= B_t z_t + F_t x_t + w_t \xrightarrow{\text{noise}} t=0,1,2, \dots \quad (13.2.1_1) \\ y_t &= H_t z_t + G_t x_t + v_t \quad t=1,2, \dots \quad (13.2.2) \end{aligned}$$

transition  
matrix      input  
measurement  
matrix      inputs/  
measurements/  
policy vars      measurement error

$$\text{and } \begin{bmatrix} w_t \\ v_t \end{bmatrix} \sim WN\left[0, V_L\right] \quad V_L = \begin{bmatrix} \frac{1}{2} w_t^2 & \frac{1}{2} w_t v_t \\ \frac{1}{2} v_t w_t & \frac{1}{2} v_t^2 \end{bmatrix}$$

### Nonlinear state-space models

$$\begin{aligned} z_{t+1} &= b_t(z_t, x_t, w_t, \delta_1) \\ y_t &= h_t(z_t, x_t, v_t, \delta_2) \end{aligned}$$

vectors of params

Example of nonlinear state-space is the "bilinear" model:

$$y_t = \alpha y_{t-1} + u_t + \beta y_{t-1} u_{t-1} \quad p. 427 w/ Refs.$$

$$\hookrightarrow z_{t+1} = B z_t + w_t + C \text{vec}(z_t z_t') \quad (13.2.33)$$

$$y_t = [I_k \ 0 \dots 0] z_t \quad (13.2.34)$$

$\Rightarrow$  Bounds of refs on bilinear systems, univariate & multivariate on p. 427 bottom.

### MLE of state-space models p. 434

Gather the time-invariant params from  $B, F, H_1, G_1, \Sigma_w, \Sigma_0$  and  $\Sigma_0 \& \mu_0$  in  $\delta$  as

$$\delta = \begin{bmatrix} \text{vec}[v, A_1, \dots, A_p] \\ \text{vech}(\Sigma_0) \end{bmatrix}$$

where  $\text{vech} = \text{"half-vectorization"}$ ,  $\text{vech}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{bmatrix} a \\ b \\ d \end{bmatrix}$

The log-likelihood for the Gaussian state-space model is:

$$\ln L(\delta | y_1, \dots, y_T) = -\frac{KT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_y(t|t-1)| - \frac{1}{2} \sum_{t=1}^T (y_t - y_{t|t-1})' \Sigma_y(t|t-1)^{-1} (y_t - y_{t|t-1}) \quad (13.4.1)$$

Denoting the first error  $e_t(\delta) := y_t - \hat{y}_{t|t-1}$

and  $\hat{\epsilon}_t(\delta) := \hat{E}_{y_t}(t|t-1)$ , we can rewrite this as

$$\ln l(\delta) = -\frac{kT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T [\ln |\hat{\epsilon}_t(\delta)| + e_t'(\delta) \hat{\epsilon}_t(\delta)^T e_t(\delta)] \quad (13.4.3)$$

which makes explicit

- 1) the dependence of  $\ln l$  on  $\delta$ ,
- 2) that all quantities in this  $\ln l$  are functions of  $\delta$  and can (most) be computed using the Kalman filter.

locally identified: when in a subspace of the param space,  $\delta$  is uniquely determined.

v. globally identified: when  $\delta$  is uniquely determined in the entire param space.

↪ identification: we need a min of  $-\ln l$ , so we need some sort of Hessian = pos. def.  $\Rightarrow$  the information matrix,  $= E[\text{Hessian}] = E \left[ \frac{\partial^2 (-\ln l)}{\partial \delta \partial \delta} \right]_{\delta_0}$

A quick note on DSEMs (dynamic simultaneous equations models | a.k.a. "linear systems") p. 323

essentially, these are linear VARMAX( $p, s, q$ ) models

$$A_0 y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 x_t + B_1 x_{t-1} + \dots + B_s x_{t-s} + w_t \quad (10.1.1)$$

- VARMAX( $p, s, q$ ) if  $w_t \sim WN$
- VARX( $p, s$ ) if  $w_t \sim WN$

VAR(p) models w/ time-varying coefficients p. 891ff.

periodic VARs  $\rightarrow$  e.g. w/ seasonal dummies

intervention models  $\rightarrow$  DGP<sub>1</sub> is replaced by DGP<sub>2</sub> at time T.

$$y_t = v_t + A_{1,t} y_{t-1} + \dots + A_{p,t} y_{t-p} + u_t \quad (12.2.1)$$

$\hookrightarrow WN(0, \Sigma_t)$

also time-varying  
(not identically distib.)

Rewrite the VAR( $p$ ) as a VAR(1)

$$Y_t = V_t + A_t Y_{t-1} + U_t \quad (12.2.2)$$

$$Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{kp \times 1}, \quad v_t := \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{kp \times 1}, \quad A_t := \begin{bmatrix} A_{1,1} & \dots & A_{p-1,1} & A_{p,1} \\ I_k & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & I_k & 0 \end{bmatrix}_{kp \times kp}, \quad u_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{kp \times 1}$$

And by recursive subst we get

$$Y_t = \left[ \prod_{j=0}^{h-1} A_{t-j} \right] Y_{t-h} + \sum_{i=0}^{h-1} \left[ \prod_{j=0}^{i-1} A_{t-j} \right] v_{t-i} + \sum_{i=0}^{h-1} \left[ \prod_{j=0}^{i-1} A_{t-j} \right] u_{t-i} \quad (12.2.3)$$

Defining  $J := [I_k \ 0]$  such that  $y_t = J Y_t$ , we can premultiply (12.2.3) by  $J$ , define

$$\bar{\Phi}_{it} := J \left[ \prod_{j=0}^{i-1} A_{t-j} \right] J' \quad \text{to get}$$

$$y_t = \mu_t + \sum_{i=0}^{\infty} \bar{\Phi}_{it} u_{t-i} \quad (12.2.4)$$

where  $\mu_t = E[y_t]$

$\Rightarrow$  Then the MSE (or FEV of the FE  $y_{t+h} - \hat{y}_t(h)$ ) is

$$\bar{\Phi}_t(h) := \sum_{i=0}^{h-1} \bar{\Phi}_{i,t+h} \bar{\Phi}_{t+h-i} \bar{\Phi}_{i,t+h}' \quad (12.2.10)$$

## MLE of TV-VAR

p. 394

$$\text{Write (12.2.1) as } y_t = B_t z_{t-1} + u_t \quad (12.2.11)$$

$$\text{where } B_t := [r_t, A_{1t}, \dots, A_{pt}], \quad z_{t-1} := (1, y_{t-1}')'$$

$B_t$  depend on the vector  $\gamma$  of time-invariant params.

$z_{t-1}$  depend on  $\beta$  of fixed params.

$y_t \sim N(0, \Sigma_t)$ , the log-likelihood is

$$\ln l(\gamma, \beta) = -\frac{kT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^T \ln |\Sigma_t| - \frac{1}{2} \sum_{t=1}^T u_t' \Sigma_t^{-1} u_t \quad (12.2.12)$$

where initial conditions have been ignored.

You can also derive an info-matrix.

So it seems like you just min -ln  $l(\gamma, \beta)$  !

It also seems like for certain special cases you can even derive the estimators in closed-form!

Ryan meeting

18 March 2010

from

not yet convinced that the procedure is sensible  
→ the only procedure that can work is

$f_{2,t}$  and more RHS-variables:  $k^{-1}, \bar{\pi}$

$$z_t = \bar{z}_t + f_{2,t} z_{t-1} + f_{4,t} h_t$$

values in  $f_{2,t}$  will be 0 or simply  
described by the LOMs

It may be too restrictive

A computationally intensive exercise is:

- long path for the int. rate (e.g.)
  - solve the model for next int. rate
  - check the target criterion, compute the resid
  - fmincon to min that resid
- ⇒ find simulated optimal plan.

- Simulate the Ramsey model
- Simulate the model w/ Taylor rule

→ see how close you can get

It might then be that most results won't  
be pencil & paper.

Comments:

- Concerning w/ linearity of (10)
  - "negative surprises cause me to be unanchored" shouldn't be "big mistakes" → so take the square
  - ↳ like smooth one better than jumpy
- Estim: filter the data → App paper w/ Robert  
 match the params to moments of data  
 → would give you results

They est an NK model: HP-filter both data and  
model, compute moments and try to match  
those.

## Work after

It's the primal economy paper w/ Robert and Ryan meant.  $\rightarrow$  ChauhanWorld.pdf in "literature."

$\Rightarrow$  they<sup>1)</sup> take data on real per-capita output, inflation, nominal interest rates & per-capita hours, <sup>(BK)</sup> 2) high-pass filter them, 3) use GMM to estimate the process  $\tilde{\tau}_t$  that best-matches the autocorrelation structure of the data.

App. D. p. 46 (Mac).

Wedges are an MA(14):

$$\tau_t = \Phi_\varepsilon(L) \varepsilon_t + \Phi_u(L) u_t$$

$$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2) \quad u_t \sim WN(0, \sigma_u^2)$$

$\Rightarrow$  params to be estimated are

$$\gamma_{ma} = (\Phi_\varepsilon, \Phi_u, \sigma_\varepsilon^2)$$

I think that the following is the target:  $t=8$

$$\tilde{\Sigma}_{\tau, T} = \text{rech} \left\{ \text{Var} [\tilde{q}_t^{\text{data}}, \dots, \tilde{q}_{t+k}^{\text{data}}] \right\} \quad q = y_t, \pi_t, b_t, w_t$$

Filting: Baxter-King filter w/ truncation horizon 32, lag-length 12

$\rightarrow \tilde{q}_t = BK_{32}(q_t)$ , so  $\tilde{q}_t$  is filtered data.

then, they do a trick in converting  $\gamma_{\text{ma}}$  into  $\gamma_{\text{ar}}$  to finally estimate  $\hat{\gamma}$  as:

$$\hat{\gamma}_{\text{ar}} = \underset{\gamma_{\text{ar}}}{\operatorname{argmin}} (\tilde{\Sigma}_T - \tilde{\Sigma}(\gamma_{\text{ar}}))' W^{-1} (\tilde{\Sigma}_T - \tilde{\Sigma}(\gamma_{\text{ar}}))$$

- $W$  is a diagonal matrix w/ the bootstrapped variances of  $\tilde{\Sigma}_T$  along the main diagonal.
- The model analogue  $\tilde{\Sigma}(\gamma_{\text{ar}})$  is computed after the model data has been similarly filtered as the data.

↳ Hold it there, now I know what I need to do.

↳ That and

- the numerical implementation of the target criterion  $\Rightarrow$  both are things to do once I have the big screen. So now give a last try to do the time-varying one.

## The ori - a first shot

Ryan wrote:

$$z_t = \bar{z}_t + f_{z,t} z_{t-1} + f_{u,t} u_t$$

I think he meant  $z_t = \bar{z}_t + h_{z,t} z_{t-1} + f_{z,t} u_t$

let me ignore  $r_t^n$  b/c it just blows things up and I just wanna see if it works. (ignore  $\bar{z}$  too.)

$$\begin{aligned} h_{\pi,t} \bar{a}_{t-1} + f_{\pi,t} u_t &= \kappa(h_{x,t} x_{t-1} + f_{x,t} u_t) \\ &\quad - (1-\alpha)\beta(h_{fa,t} f_a(t-1) + f_{fa,t} u_t) + exog_1 \cdot u_t = D \quad (9) \\ h_{x,t} x_{t-1} + f_{x,t} u_t &+ \beta(h_{i,t} i_{t-1} + f_{i,t} u_t) \\ &- \beta(h_{fb,t} f_b(t-1) + f_{fb,t} u_t) + exog_2 \cdot u_t = D \quad (10) \end{aligned}$$

↪

$$\begin{aligned} h_{\pi,t} \bar{a}_{t-1} - \kappa h_{x,t} x_{t-1} - (1-\alpha)\beta h_{fa,t} f_a(t-1) \\ + \underbrace{(f_{\pi,t} - \kappa f_{x,t} - (1-\alpha)\beta f_{fa,t} + exog_1)}_{u_t} = 0 \quad (10) \end{aligned}$$

This is exactly what I have in materials 21

$$\begin{aligned} h_{x,t} x_{t-1} + \beta h_{i,t} i_{t-1} - \beta h_{fb,t} f_b(t-1) \\ + (f_{x,t} + \beta f_{i,t} - \beta f_{fb,t} + exog_2) u_t = 0 \end{aligned}$$

$$z_t = \bar{z}_t + h_{\bar{z},t} z_{t-1} + f_{\bar{z},t} u_t$$

$$h_{fat,t} f_{at}(t-1) + f_{fat,t} u_t - \frac{1}{1-\alpha\beta} (h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + f_{\bar{\pi},t-1} u_{t-1}) + exog_3 u_t$$

$$h_{fat,t} f_{at}(t-1) - \frac{1}{1-\alpha\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + \underbrace{(f_{fat,t} + exog_3) u_t - \frac{1}{1-\alpha\beta} f_{\bar{\pi},t-1} u_{t-1}}_{\text{same as in materials 21}} = 0 \quad (9)$$

$$h_{fb,t} f_b(t-1) - \frac{1}{1-\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2} + \underbrace{(f_{fb,t} + exog_3) u_t - \frac{1}{1-\beta} f_{\bar{\pi},t-1} u_{t-1}}_{(10)} = 0$$

Honestly, I don't even think I will complete this b/c

we can see that if, in say (10),  $h_{fb,t} = 0$  and  $h_{\bar{\pi},t-1} = 0$

then we're back to exactly the system I had in mat21.

So suppose  $h_{fb,t} \neq 0$ . But then in each iteration we have an earlier iteration of itself. Now I might argue

$h_{\bar{\pi}} = h_x = h_i = 0$ , but if I ass that for  $h_{fat,t} = h_{fb,t} = 0$  then again the link between (9)-(10) and (5)-(6) is broken. Then (10) would give an additional constraint

$$h_{fb,t} f_b(t-1) = \frac{1}{1-\beta} h_{\bar{\pi},t-1} \bar{\pi}_{t-2}$$

In two unknowns. Mm... You'd get a proliferation of unknowns...! I'll stop here, this wasn't for endog states, merde!

① GMM of midsimple

19 March 2020

② Numerical implementation of target criterion

① GMM of midsimple

sim\_learnM.m w/ PLM = constant-only

→ Need to:

1) add smooth function as a third criterion

2) To have "midsimple", we need that  $x$  (and  $i$ )

aren't learned, so input  $a = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ ,  $b = b^{\text{RE}}$

( $\Rightarrow$   $a$ , I've learned one thing

(comparing w/ IRFs of materials 9)

⇒ when only  $\pi$  is learned, IRFs are less

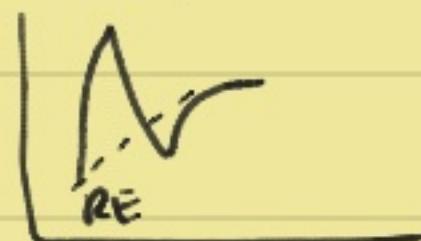
oscillatory in the sense that they overshoot

only once:

when  $(\pi, x, i)^T$  learned



vs.



→ did I ever conclude this? I thought so, but now it doesn't seem like it. In materials? I just state that vector vs. scalar learning is pretty much the same

... And it almost is, just not quite.  
⇒ so mental note!

Learning only  $\text{Lom}(\alpha)$  dampens the oscillations b/c  $E(\cdot)$  in NKIS & NKPC are moving less, and therefore  $E(i+k)$  is moving less. I think more importantly since the bulk of action is in  $x$ , the fact that  $\text{FE}(x)$  is now cut out gets rid of the FE that was most oscillatory.

Ok I just wasted an hour trying to customize buys in Matlab ...

Back to 1) the anchoring function:

$$k_t^{-1} - k_{t-1}^{-1} = c + d(FE)$$

has the interpretation that the gain decreases when  $FE < 0$ . Ryan is right: the easiest thing that makes  $\Delta k_t^{-1}$  big when  $FE$  is big in absolute value is  $FE^2$

$$\rightarrow k_t^{-1} - k_{t-1}^{-1} = c + d FE_t^2$$

↑ does this even make sense?

> Not really: it's saying that the gain always changes, even if the  $FE^2 = 0$ .

↳ So then gather data and implement

- HP
- BK
- Hamilton filters

But there is a problem: for this current form

$$k_t^{-1} - k_{t-1}^{-1} = (c + d \cdot FE^2),$$

it always explodes, even more so if  $c=0$ .

And it's b/c  $k \rightarrow 0$ , so I guess  $k^{-1}$  explodes.

↳ Need to be smart about this!

Normally, a gain would be

20 March 2020

$$k_{t+1} = k_t + 1$$

so I could just do  $k_t = k_{t-1} + \frac{1}{FE^2}$

so that if in the limit  $FE^2 \rightarrow \infty$ ,  $k_t = k_{t-1}$  (gain)

then one could have

$d < 1$

$$k_t = k_{t-1} + d \frac{1}{FE^2} \quad \text{in this case I guess } d \text{ small}$$

$$\text{or } k_t = k_{t-1} + \frac{1}{d FE^2} \quad \text{in this case I guess } d \text{ big}$$

↳ both of them work and exactly the way I hoped

$$\text{or even } k_t = k_{t-1} + \left(\frac{1}{d FE}\right)^2 = k_{t-1} + (d FE)^{-2}$$

works.

## HP-filter

A time series  $y_t$  (in logs) is

$$y_t = g_t + c_t$$

$\uparrow$                $\uparrow$   
 growth component      cyclical component

Obtain  $g_t$  as

$$\min_{\{g_t\}_{t=1}^T} \left\{ \sum_{t=1}^T c_t^2 + \lambda \sum_{t=1}^T [(g_t - g_{t-1}) - (g_{t-1} - g_{t-2})]^2 \right\}$$

where  $\lambda = 1600$ .

Hamilton provides a closed-form solution to this problem as  $g^* = (H'H + \lambda Q'Q)^{-1} H'y$  (2)

where  $\tilde{T} := T+2$

$$y = (y_T, y_{T-1}, \dots, y_1)' \quad q = (g_T, g_{T-1}, \dots, g_0, g_{-1})'$$

$T \times 1$                                      $\tilde{T} \times 1$

$$\text{and } H = \begin{bmatrix} I_T & 0 \\ 0 & I_{\tilde{T}} \end{bmatrix} \quad \text{and } Q = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{bmatrix}$$

## Hamilton filter

21 March 2020

$$\text{Idea: } c = y_{t+h} - E[y_{t+h} | y_t]$$

choose  $h=8$  for quarterly data.

How to calc  $E[y_{t+h} | y_t]$ ?

just regress  $y_{t+h}$  on  $y_t, y_{t-1}, y_{t-2}, y_{t-3}$   
and take the fitted value from that reg.

In fact, Hamilton's eq (21) gives us  $c$  directly:

$$c = \hat{v}_{t+h} = y_{t+h} - \hat{\beta}_0 - \hat{\beta}_1 y_t - \hat{\beta}_2 y_{t-1} - \hat{\beta}_3 y_{t-2} - \hat{\beta}_4 y_{t-3}$$

$$\text{So: } Y = y_{t+h}, \quad X = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}^{\text{4x1}} \Rightarrow \beta = (X'X)^{-1} X' Y$$

$$\text{and } \hat{v}_{t+h} = y_{t+h} - \beta X$$

## Baxter & King filter (BK filter, bandpass filter)

Def: business cycle = cyclical components of a ts that last between 6 and 32 quarters in duration

Notation:  $\text{BP}_k(p, q) =$  filter that processes cycles between  $q=32$  and  $p=6$  cycle length, and is truncated at leads/lags  $k$ . ( $\rightarrow$  recommend  $k=12$ )

$\hookrightarrow$  Such a filter is a MA( $k$ ).

So we're interested in constructing the filtered series

$$y_t^* = \sum_{k=-N}^K a_k y_{t-k} \quad (1)$$

and the question is how to obtain  $a_k$ ?

(where the MA is symmetric, i.e.  $a_k = a_{-k}$ )

Eq. (10) gives the answer:

$$a_h^{BP} = (\bar{b}_h - b_h) + (\bar{\theta} - \underline{\theta})$$

where the bars denote things pertaining to low-pass filters that only allow stuff below the frequency  $\bar{\omega}$  &  $\underline{\omega}$ ,

i.e periodicities  $\bar{p}$  &  $\underline{p}$ ,

where  $p = \frac{2\pi}{\omega}$

$\theta$  is a normalizing constant (coming from truncation)

$$\theta = \left(1 - \sum_{h=-K}^K b_h\right) \frac{1}{2K+1}$$

and  $b_0 = \frac{\omega}{\pi}$  and  $b_h = \frac{\sin(h\omega)}{h\pi}$  for  $h=1, 2, \dots$  (7)

Ok, so we have  $\underline{p} = 32$ ,  $\bar{p} = 6$

$$\Rightarrow \underline{\omega} = \frac{2\pi}{\underline{p}} = \frac{2\pi}{32} = \frac{\pi}{16}, \quad \bar{\omega} = \frac{2\pi}{\bar{p}} = \frac{\pi}{3}$$

## Autocorrelations of VARs (Lütkepohl, p. 21)

For a stationary VAR(1)

$$y_t = v + A_1 y_{t-1} + u_t \quad \hookrightarrow N(0, \Sigma_u)$$

define  $\Gamma_y(h) \equiv E[(y_t - \mu)(y_{t+h} - \mu)']$

where  $\mu := E[y_t]$ . Then if you know  $\Gamma_y(0)$ ,  
you can compute

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) \text{ recursively! } (2.1.31)$$

If you know  $A_1$  and  $\Sigma_u$ , you can compute

$$\text{vec} \Gamma_y(0) = (I_k^2 - A_1 \otimes A_1)^{-1} \text{vec} \Sigma_u \quad (2.1.32)$$

But I don't want to est a VAR... I just want  
the empirical auto-covariance matrix of the data  
and of the simulated data.

So, the 6MM of the anchoring fit works

23 March 2020

so far. Issues.

1) I'm not sure if I should model the raw data w/ a time series. So e.g. if they were a VAR(1) then I could estimate a RF-VAR, I could bootstrap using the residuals and I could estimate the anterior structure as in (2.1.32) & (2.1.31) in *filterpost*.

This issue shows up for  $W = \begin{bmatrix} \hat{\sigma}_{act_1}^2 & & \\ & \ddots & \\ & & \hat{\sigma}_{act_{15}}^2 \end{bmatrix}$   
b/c  $\hat{\sigma}_{act_i}^2$  are super small, thus  $W^{-1}$  is huge,  
and therefore the estimation does not move.

2) Filters may not be working?

→ I think now they are!

↳ check bootstrap! ✓ It's fine.

Write up materials 22 - peter ✓

Read "Stochastic" On it!

I'm not sure what I'm learning from 24 March 2020  
the fMM so far, so until I talk to Ryan, I'll  
postpone it and turn to questions of implementation  
of the target condition.

### Peter meeting

24 March 2020

- Ask Clough

- Intro: CBs always talk about anchoring,  
e.g. (2) formulates that!

- Where to go #1:

- Suggestions

- (1): A common theme: does it make sense  
to have  $b_t = k_t^{-1}$  and just track the  
evolution of that.

(2) : Eq. (2)

If you work w/ nonlinear diff eqs, they

are harder but not impossible to solve

→ mit-shock business is all about that

w/ perfect foresight

- take FOCs

- keep 'em in nonlinear form

- ass. the econ is hit by 1-time shock

- trace out transition to new st. st.

A precise way of putting it:

any dynamic econ: all vars  $\rightarrow X_t$

$$F(X_t, X_{t-1}, \varepsilon_t) = 0$$

If  $F$  has a lin form,  $A X_t + B X_{t-1} + C \varepsilon_t = 0$

then we can solve it immediately quite  
easily.

If  $F$  is nonlinear, but nonstochastic:

$$F(x_t, x_{t+1}) = 0$$

is harder, but doable.

↳ Sect 2. point (1)

"Can we analytic methods to describe eqs  
in opt policy & key variables"

& simulate vars under those policies.

⇒ Now you do answer

- 1) How close does a TR come to opt. policy.
- 2) Opt TR will try to do the same thing  
as opt policy and you'll see what they  
both are trying to do.

Suggestion overall:

- simulate numerically the model under  
some kind of optimal pol that's more flex  
than a TR.

2<sup>nd</sup> suggestion w/ nonlinearity:

Analogy: stochastic Ramsey model  
(neoclassical growth model)

no closed-form sol once depreciation isn't  
 $= 1$  and not  $\log U$ . So to solve this:

- loglin

- discretize the state-space

- TFP follows not an AR(1) but  
one of 3 values

- Then decision rules are also 3-form

tophi (continues working w/ full-blown non-linear  
model but replace the stock, light sources

$$u = \begin{cases} u_{\text{high}} \\ u_{\text{low}} \end{cases} \quad r^n = \begin{cases} r^{\text{high}} \\ r^{\text{low}} \end{cases}$$

$\Rightarrow$  4 states of the world

$\hookrightarrow$  calculating  $E(\cdot)$  of a future term in (1)  
may be easier

2 option: take (1)

$$f_t = f_{t-1} + h_t (\pi_t - f_{t-1})$$

and login

Another way to solve a DSGE model

login can's that decision rules are lin  
functions of the states

→ well you can say that they are  
quadratic function of the states

↳ so instead of Taylor-approx, maybe the  
right thing is a cubic or spline

In eq (3) I try to approx something of  
unknown form w/ a class

Simulate the model under some optimal targeting rule & compare w/ behavior under an optimized TR economy.

Draft for end of semester  
→ she'll read it then  
and afterwards too.

↳ talk to Ryan now:  
what is the most promising route?

Tomorrow noon → send Peter email for major  
summer

## Work after

Takeaways:

- 1) In my introduction, I talk about how CB-ers often talk about anchoring. The target criterion (eq. (2) in materials 22 peters) formalizes this!
- 2) Small detail: replace  $h_t = k_t^{-1}$
- 3.) The main point: need to identify the set of results (and avenues) that can be achieved and then written up. For him it seems to be that this set of results is the target criterion and its numerical implementation so we can make statements about the optimal plan and how it compares to the optimal Taylor rule.

↳ he seemed adamant that I settle this w/ Ryan

4) Taking the analogy of the neoclassical growth model, no closed-form sol exists for the nonlinear model unless  $\log u$  and  $\beta = 1$ . So if we wanted a tighter analytical result, we could 1) either loglin, or  
2) discretize the stochastic disturbance

e.g.  $u_t = \begin{cases} u^{\text{high}} \\ u^{\text{low}} \end{cases}$

↪ this way we might more fully describe the target criterion.

5) As for the loglin, which is a Taylor-Approx, maybe a higher order approx (ubic or spline) would be sensible.

Here's the deal: I don't think I wanna pursue further analytical work. Not w/ the discretization, not w/ a higher-order approx, although both might work. I think that building on the existing

work, I want to go on to simulating the optimal Ramsey policy and comparing it to the opt. TR.  
⇒ I think that that is 1) feasible 2) makes sense in terms of storytelling & also is likely to provide additional inputs to that storytelling.

I will continue reading Stochastic Opt. in Continuous Time, not b/c I really hope for insights for more analytical work, more for my intellectual completion - as diff. egs contributed too.

So: numerical implementation

25 March 2020

of the target criterion

- I wonder if I couldn't do it via value function iteration → can I not solve the nonlinear system numerically and obtain the optimal paths like that?  
→ which would allow me to back out  $i^{\text{Ramsey}}$  from NKIS?

$$g_{\pi} : \quad k_t = k_{t-1} + \underbrace{(d \cdot f c_{t-1})^{-2}}_{=g}$$

$$k_t = k_{t-1} + d^{-2} (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-2}$$

$$\text{Then } g_{\pi}(t) = d^{-2}(-2) (\pi_t - \bar{\pi}_{t-1} - b_1 s_{t-1})^{-3}$$

$$= -2 (d f c_{t-1})^{-2} f c_{t-1}^{-1}$$

Now what I need to solve is the fact that when  $\{i_t\}$  is exog, the model sol

$y = \text{function}(\text{matrices}, f_a, f_b)$  changes.

In particular, if we go back to the model summary in materials 17.m (1) - (3)

we know that (3) doesn't hold ( $R$ ), and

$\{i_t\}$  influences the econ via the NKIS curve (1).

$$x_t = -3i_t + [\beta, (1-\beta), -3\beta] f_\beta + 3[1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_t$$

$$\pi_t = k k_t + [(1-\alpha)\beta, \kappa \alpha \beta, 0] f_a + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_t$$

$$\begin{bmatrix} 0 & 1 \\ -k & 1 \\ 1 & -k \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ k_t \end{bmatrix} = \begin{bmatrix} -3i_t + [\beta, (1-\beta), -3\beta] f_\beta + 3[1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_t \\ [(1-\alpha)\beta, \kappa \alpha \beta, 0] f_a + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_t \end{bmatrix}$$

So

27 March

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix}^{-1} \left[ -\beta_{11} \begin{bmatrix} \beta, (1-\beta), -\beta\beta \end{bmatrix} f_\beta + \beta \begin{bmatrix} 1, 0, 0 \end{bmatrix} (f_{nx} - \beta h_x)^T s_1 \right. \\ \left. + \begin{bmatrix} (1-\alpha)\beta, \alpha\beta, 0 \end{bmatrix} f_\alpha + \begin{bmatrix} 0, 0, 1 \end{bmatrix} (f_{nx} - \beta h_x)^T s_2 \right]$$

$$\begin{bmatrix} 0 & 1 \\ -k & 1 \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

$$= \begin{bmatrix} -k & -1 \\ -1 & 0 \end{bmatrix} \frac{1}{-1} = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$$

where  $f_\beta, f_\alpha$  unchanged.

Ryan meeting

25 March 2020

$$k_t = k_{t-1} + \frac{1}{(df_e)^2} - c$$



so that when  $fe$  large,

the gain grows

Calibration:  $E(fe)^{RE}$  would be the size of  $c$ .

• check if drawing w/ replacement!

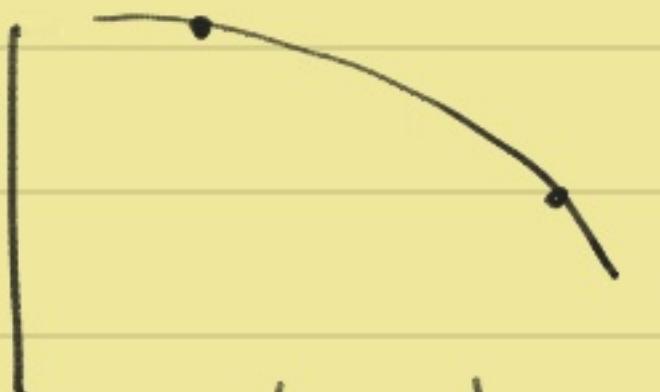
- take new data: run a VAR
  - bootstrap from residuals
- } What Ryan did in the paper w/ Robert

- put sim data in filter
- $g$  should
  - tend to go down
  - but when it goes up, it goes up a lot  
(i.e. for very large  $F_E$ , it  $\uparrow$  a lot.)
- check if you're close to the target moments

$$k_t = k_{t-1} + \hat{g}(f_e^t)$$

$\hat{g}$  is a spline w/  $x$  nodes

piecewise linear w/  $x$  nodes



small | med | large  $F_E$

Each node comes w/ 2 params  
 → so can est a 4-param family



→ here you just  
est 3 slopes

→ eventually you could even est. what a small, large, FE is.

shape-preserving splines: are either always convex or always concave

Judd, the textbook, Numerical Methods for Econ

piece-wise linear interpolation

spline / quadratic spline

basis functions

Econ Librarian: Sonia Ensins

Makes sense:

Hamilton  $\hat{d}$  lower b/c Ver  $\uparrow$ , so FEs lower.

4 states, so VFI could be done.

Work after

26 March 2020

EBSCO host: lauragati  
empirical - 77 methods

GMM

1) close to target?

No, not at all! loss is huge ( $5.4 \times 10^8$ )!

2) w/ replacement

randperm really wasn't w/ replacement

randi (maxvalue, a×b vector) is w/ replaced

but it's still blowing up ñ

maybe I do need to estimate a VAR?

I'm surprised b/c Ryan said that it sounds like my procedure is valid.

3) An anchoring function w/ the feature that

a) it generally decreases b) grows a lot when PE large

$$k_t = k_{t-1} + d - c f e_{t-1}^2$$

$c f e^2 < d$ , again  $\rightarrow a)$   $\rightarrow g_\pi = -2 c f e_{t-1}$

For  $c f e^2 > d$ , (again  $\rightarrow b)$  Dam. Real bad.

try

$$k_+ = k_{+\infty} + \frac{1}{(d+k)} - c$$

Not good either.

27 March 2020

$$x_+ = -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+$$

$$\pi_+ = kx_+ + \text{stuff}_3 f_a + \text{stuff}_4 s_+$$

$$1x_+ + 0\pi_+ = -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+$$

$$-kx_+ + 1\pi_+ = \text{stuff}_3 f_a + \text{stuff}_4 s_+$$

$$\begin{bmatrix} 0\pi_+ + 1x_+ \\ 1\pi_+ - kx_+ \end{bmatrix} = \begin{bmatrix} -2i_+ + \text{stuff}_1 f_b + \text{stuff}_2 s_+ \\ \text{stuff}_3 f_a + \text{stuff}_4 s_+ \end{bmatrix}$$

$2 \times 1$

$2 \times 1$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} \pi_+ \\ x_+ \end{bmatrix}}_{2 \times 1}$$

I actually think they are the same -

28 March 2020

it's just that since  $\pi$  is exploding, small numerical

errors become considerable.

Strange that  $x$  doesn't explode.

→ Like I thought yesterday,  $\text{pi-x-given\_i.m}$  should be correct. It's the target criterion that may not be.

Or actually:  $f_a(3)$  and  $f_b(3)$  may not be correct.

29 March 2020

I think I know what the problem is.  $f_{a/b} \dots m$  just computes  $\hat{E}_+ \sum_T (\alpha) \beta^{T+2}_{T+1}$  and now in  $\text{pi-x-given\_i.m}$  there may be a discrepancy in that  $f_b(3)$  is used, and yet the A-matrices that specified that  $f_b(3)$  won't used before are no longer valid.

→ Need to make an assumption on how  $f_b(3)$  is formed. The most natural is to say that the C-matrices announces the future path of  $i$ , i.e. it's like an

exogenous sequence. On the 29 March, I had:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - \kappa \end{bmatrix}^{-1} \left[ -\beta i_{t+1} [\beta, (1-\beta), -\beta\beta] f_\beta + \beta [1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_n \right. \\ \left. [(\alpha-\kappa)\beta, \kappa\beta, 0] f_\alpha + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_n \right]$$

$$-\beta\beta f_\beta = -\beta\beta E_t \sum_{T=0}^{\infty} \beta^{T-t} i_{T+1} \quad \text{together with 4,}$$

this gives

$$\begin{aligned} & -\beta \left[ i_t + \beta E_t \sum_{T=0}^{\infty} \beta^{T-t} i_{T+1} \right] \\ &= -\beta \left[ i_t + E_t \sum_{T=0}^{\infty} \beta^{T-t+1} i_{T+1} \right] \\ &= -\beta \left[ E_t \sum_{T=t}^{\infty} \beta^{T-t} i_T \right] \end{aligned}$$

Since the  $i$ -sequence may not be an AR(1), it may be completely arbitrary, I should simply write  $\text{discsum}_i = \sum_{T=t}^{T+H} \beta^{T-t} i_T =: \text{dsi}$

So  $\pi_{t+1} \text{ given } i_m$  modifies to

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - \kappa \end{bmatrix}^{-1} \left[ -\beta \text{dsi} + [\beta, (1-\beta), 0] f_\beta + \beta [1, 0, 0] (f_{nx} - \beta h_x)^{-1} s_n \right. \\ \left. [(\alpha-\kappa)\beta, \kappa\beta, 0] f_\alpha + [0, 0, 1] (f_{nx} - \beta h_x)^{-1} s_n \right]$$

and  $\text{stuff}_1$  modifies to  $[\beta, 1-\beta, 0]$

still exploding...

I'm wondering if (A.1) - (A.3) yield an i-sequence, and I might that i-sequence into pi-x-giving, should I get the same  $\{\pi_+, x_+\}_+^T$ ?

Initially I thought yes, but now I think no b/c  $A_a, A_b$  were derived w/ the assumption that  $f_0(3) = \gamma_1 f_0(1) + \gamma_2 f_0(2) + \text{expected i-shock}$   
→ I turned off i-shock, but that actually makes things explode faster!

Well, it depends. B/c if you set all  $i_i = 0$ ,  $\forall i$ , then to make  $\hat{E}_{+i+k}$  constant even under the TR-regime, you need to set  $\rho_i = \sigma_i = 0$ .

But at least it makes sense that  $x$  explodes not from  $\pi$ , b/c  $\dot{\pi} = \kappa x + (\text{stiff } \perp i)$   
 $\hookrightarrow \text{small}$

Nun, if  $T + t$  small, then things don't have time to explode and therefore formation works. Otherwise

somewhat loss=0 already initially. Why?

initial loss decreases in  $H$

so when  $H \geq 36$ , loss = 0 (when  $T=10$ )

But this seems to be extremely robust across  $T$ .

initial loss is  $\wedge$ -shaped in  $T$

so when  $T=1$ , loss = 0 (when  $H=35$ )

$T=10$ , loss = 49.479

$T=100$ , loss = 1.4761

$\rightarrow$  after that, it's 1.48 for  $\forall T$ .

But this is not robust to  $H$ .

E.g. if  $H=5$ , loss ↑ in  $T$  monotonically.

$\rightarrow$  maybe I should take the vector of results as moments, not the biggest one.

That doesn't help w/ loss-explosions either, but I wonder if makes the loss function well-behaved?

So I'm back to wondering that  $\{\bar{a}_i, \bar{x}_i | i\}$  shouldn't explode - I can accept that they aren't the same as  $\{\bar{a}_i, \bar{x}_i | i^{\text{TR}}\}$  b/c  $E(i)$  isn't the same, but I guess they shouldn't always explode.

→ they're exploding b/c  $f_a$  &  $f_b$  are exploding. Of course it's only  $f_a(1) \& f_b(1)$ , the others are numerically 0 ( $e^{-16}$ ), which makes sense b/c  $a_2 = a_3 = 0$  &  $b = g \times h \bar{x} = \text{zeros}$  now that I set all elements of  $h \bar{x} = 0$ .

Somehow the learning isn't E-stable (indeed  $a = \bar{a}$  explodes) when you just feed in a  $i$ -sequence b/c it's as if the TR was just specified as  $i_+ = \bar{i}_+$ .

→ If there was some way of letting the private sector know that  $i$  isn't chosen randomly, that it follows a rule...

Need to find a way to have the  
private sector internalize the target criterion.

30 March 2020

→ the TC needs to be a model equation

But if that is so, then the expectation of the RHS  
needs to be the public's expectation!

But before I do this, this is the time to verify that  
the TC of (B.1) in materials23 is correct!

I solve for TC in materials18. (Notes 6, p. 95 ff.)

$$d = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ (\pi_t^2 + \lambda x_t^2) \right.$$

$$+ \psi_{1,t} (\pi_t - \kappa x_t + (1-\alpha)\beta f_a(t) + exog)$$

$$+ \rho_{2,t} (x_t + \beta \bar{x}_t + \beta f_B(t) + exog)$$

$$+ \psi_{3,t} (f_a(t) - \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + exog)$$

$$+ \psi_{4,t} (f_B(t) - \frac{1}{1-\beta} \bar{x}_{t-1} + exog)$$

$$+ \psi_{5,t} (\hat{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1} [\pi_t - (\bar{\pi}_{t-1} + b_s s_{t-1})])$$

$$+ \psi_{6,t} (k_t^{-1} - k_{t-1}^{-1} + g(\bar{\pi}_t - \bar{x}_{t-1} - b_s s_{t-1}))$$

↑ This is a change compared to materials18.

$$\begin{aligned}
d &= E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ (\pi_t^2 + 2x_t^2) \right. \\
&\quad + \psi_{1,t} (\pi_t - kx_t - (1-\alpha)\beta f_a(t) + exog) \\
&\quad + \psi_{2,t} (x_t + 2i_t - 2f_B(t) + exog) \\
&\quad + \psi_{3,t} (f_a(t) - \frac{1}{1-\alpha\beta} \bar{\pi}_{t-1} + exog) \\
&\quad + \psi_{4,t} \left( f_b(t) - \frac{1}{1-\beta} \bar{\pi}_{t-1} + exog \right) \\
&\quad + \psi_{5,t} \left( \hat{\pi}_t - \bar{\pi}_{t-1} - k_t^{-1} (\pi_t - (\bar{\pi}_{t-1} + b_t s_{t-1})) \right) \\
&\quad \left. + \psi_{6,t} (k_t^{-1} - \underline{k_{t-1}^{-1}} - g(\bar{\pi}_t - \bar{\pi}_{t-1} - b_t s_{t-1})) \right)
\end{aligned}$$

FD(S):

$$\pi: 2\pi_t + \psi_{1,t} - k_t^{-1} \psi_{2,t} - \psi_{6,t} g_\pi(t) = 0 \quad (1)$$

$$x: 2\lambda x_t - k \psi_{1,t} + \psi_{2,t} = 0 \rightarrow \underline{\psi_{1,t} = \frac{2\lambda}{k} x_t} \quad (2)$$

$$i: \underline{3\psi_{2,t} = 0} \quad (3)$$

$$f_a: -\psi_{1,t} (1-\alpha)\beta + \psi_{3,t} = 0 \rightarrow \underline{\psi_{3,t} = \frac{(1-\alpha)\beta}{2\frac{\lambda}{k}} x_t} \quad (4)$$

$$f_b: \psi_{4,t} + 3\psi_{2,t} = 0 \rightarrow \underline{\psi_{4,t} = 0} \quad (5)$$

$$\begin{aligned}
\bar{\pi}: & \psi_{3,t-1} \left[ -\frac{1}{1-\alpha\beta} \right] + \psi_{3,t} + \psi_{5,t-1} \left[ -1 + k_{t-1}^{-1} \right] \\
& + \psi_{6,t-1} g_{\bar{\pi}}(t) = 0 \quad (6)
\end{aligned}$$

$$k^{-1}: -\psi_{5,t} (\pi_t - (\bar{\pi}_{t-1} + b_t s_{t-1})) + \psi_{6,t} - \psi_{6,t-1} = 0 \quad (7)$$

$$2\pi_t + \frac{2\beta}{K} x_t - k_T^{-1} \varphi_{5,t} - \varphi_{6,t} g_{\bar{\pi}}(t) = 0 \quad (1)$$

$$-\frac{2(1-\alpha)\beta\frac{\beta}{K}}{1-\alpha\beta} x_{t+1} + \varphi_{3,t} - \varphi_{5,t+1} (1 - k_{t+1}^{-1}) \\ + \varphi_{6,t+1} g_{\bar{\pi}}(t) = 0 \quad (2)$$

$$-\varphi_{5,t} (\pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})) + \varphi_{6,t} - \varphi_{6,t+1} = 0 \quad (3)$$

now

Note: Eq. (1) is exactly Eq. (I), p. 95 of Notes 6.

Eq (2) is exactly Eq (II).

Eq (3) is (III) + an additional  $\varphi_{6,t+1}$  term

that comes from redefining the anchoring function

$$\text{as } h_t = \underline{h_{t-1}} + g(\text{fe})$$

$$\text{NBW} \rightarrow \underline{\varphi_{6,t+1}}$$

The issue that now comes up is the one I anticipated in materials 18, Notes 6 before 95: if the anchoring function has  $h_{t-1}^{-1}$  in it, then  $\varphi_{6,t+1}$  shows up and causes trouble. Let me simplify the system:

Define  $c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{\kappa}$  and  $fe_t = \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$ ,  
 Then: Where I'm annoyingly using a different notation than usual ...

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1} \varphi_{5,t} - g_\pi(t) \varphi_{6,t} = 0 \quad (1)$$

$$c x_{t+1} + \varphi_{5,t} - (1 - k_{t+1}^{-1}) \varphi_{5,t+1} + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0 \quad (2)$$

$$-fe_t \cdot \varphi_{5,t} + \varphi_{6,t} - \varphi_{6,t+1} = 0 \quad (3)$$

From (3),  $\varphi_{5,t} = \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t}$ .

Plug into (1)

$$2\pi_t + \frac{2\lambda}{\kappa} x_t - k_t^{-1} \left( \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t} \right) - g_\pi(t) \varphi_{6,t} = 0$$

$$\Leftrightarrow 2\pi_t + \frac{2\lambda}{\kappa} x_t - \left( \frac{k_t^{-1}}{fe_t} + g_\pi(t) \right) \varphi_{6,t} + \frac{k_t^{-1}}{fe_t} \varphi_{6,t+1} = 0 \quad (1')$$

and plug into (2)

$$cx_{t+1} + \frac{\varphi_{6,t} - \varphi_{6,t+1}}{fe_t} - (1 - k_{t+1}^{-1}) \left( \frac{\varphi_{6,t+1} - \varphi_{6,t+2}}{fe_{t+1}} \right) + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0$$

This I should be able to solve for  $\varphi_{6,t}$ .

$$cx_{t+1} + \frac{y_{6,t} - y_{6,t+1}}{fe_t} - (1 - k_{t+1}^{-1}) \left( \frac{p_{6,t+1} - p_{6,t+2}}{fe_{t+1}} \right) + g_{\bar{n}}(t) p_{6,t+2} = 0$$

$$(x_{t+1} + \frac{1}{fe_t} y_{6,t} + \left[ -\frac{1}{fe_t} - \frac{(1 - k_{t+1}^{-1})}{fe_{t+1}} + g_{\bar{n}}(t) \right] y_{6,t+1} + \frac{(1 - k_{t+1}^{-1})}{fe_{t+1}} y_{6,t+2}$$

(6')

$$cx_{t+1} + \alpha_t y_{6,t} + \beta_t y_{6,t+1} + \gamma_t y_{6,t+2} = 0$$

Now the question is: solve forward or backward?

$1 - k^{-1}$  is always  $< 1$ , and in the limit it  $\rightarrow 1$ .

$fe$  doesn't have a limit, it's Normal and in the limit its variance is  $\Sigma$  ( $= \text{var}(\text{shocks})$ ), mean zero.

For the most part, then, and close to the limit, if  $\Sigma$ 's elements are  $\beta_i^2 \leq 1$ , then  $fe$  (in the limit) will be mainly (or always?) below 1. So  $\frac{1}{fe} > 1$

$$cfe_t x_{t+1} + y_{6,t} + \left[ -1 - \frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) + fe_t g_{\bar{n}}(t) \right] p_{6,t+1} - \underbrace{\frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) p_{6,t+2}}$$

Here,  $\frac{fe_t}{fe_{t+1}} \approx 1$  in expectation, so  $\frac{fe_t}{fe_{t+1}} (1 - k_{t+1}^{-1}) < 1^{<1}$

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \left[ 1 + \frac{f_{t+1}(1-k_{t+1}^{-1}) - f_{t+1} g_{\bar{n}}(t)}{f_{t+1}} \right] \varphi_{6,t+1} - \underbrace{\frac{f_{t+1}(1-k_{t+1}^{-1})}{f_{t+1}}}_{<1} \varphi_{6,t+2}$$

$=: \alpha_+$

In the limit,  
this needs to  
be  $> 1$  in abs. value. But it's not.

$$\lim \left[ 1 + \frac{f_{t+1}(1-k_{t+1}^{-1}) - f_{t+1} g_{\bar{n}}(t)}{f_{t+1}} \right] \rightarrow 1 + 1 - \Sigma \cdot (\text{some small number, might even be zero})$$

$> 1$

Can I factor this using lag polynomials then? I need to solve both backward & forward.

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \alpha_+ \varphi_{6,t+1} + \beta_+ \varphi_{6,t+2} \quad |: \alpha_+$$

$|\alpha_+| > 1 \quad |\beta_+| < 1$

$$\frac{1}{\alpha_+} \varphi_{6,t} = -c f_{t+1} x_{t+1} + \varphi_{6,t+1} + \frac{\beta_+}{\alpha_+} \varphi_{6,t+2}$$

$$\varphi_{6,t+1} = \underbrace{\frac{c f_{t+1} x_{t+1}}{\alpha_+}}_{<1} - \underbrace{\frac{\beta_+}{\alpha_+} \varphi_{6,t+2}}_{<1}$$

6"

Before we go there, let me just check the target criterion as it now is, i.e. if  $\varphi_{6,t+1} = 0$ .

If  $\varphi_{6,t+1} = 0$ , the system is

$$2\pi_t + \frac{2\lambda}{k} x_t - k_t^{-1} \varphi_{5,t} - g_{\bar{\pi}}(t) \varphi_{6,t} = 0 \quad (1)$$

$$c x_{t+1} + \varphi_{5,t} - (1 - k_{t+1}^{-1}) \varphi_{5,t+1} + g_{\bar{\pi}}(t) \varphi_{6,t+1} = 0 \quad (2)$$

$$-f_{\ell,t} \cdot \varphi_{5,t} + \varphi_{6,t} - \cancel{\varphi_{6,t+1}} = 0 \quad (3)$$

and eq (3) corresponds to Notes b, p. 95, eq (1).

$$\rightarrow \varphi_{6,t} = f_{\ell,t} \varphi_{5,t}$$

$$\Rightarrow (1): 2\pi_t + \frac{2\lambda}{k} x_t - (k_t^{-1} + g_{\bar{\pi}}(t) f_{\ell,t}) \varphi_{5,t} = 0 \quad (1')$$

$$\Rightarrow (2) c x_{t+1} + \varphi_{5,t} + [-(1 - k_{t+1}^{-1}) + g_{\bar{\pi}}(t) f_{\ell,t+1}] \varphi_{5,t+1} = 0 \quad (2')$$

$$\varphi_{5,t} = -c x_{t+1} + [1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{\ell,t+1}] \varphi_{5,t+1}$$

To be compatible w/ Notes b, p. 96, let me redefine

$$C := -c \quad \text{and} \quad \alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{\ell,t+1}$$

and assume that  $\alpha_t < 1$  for most  $t$ . Which is

likely actually b/c  $\alpha_{t+1} \rightarrow 1$  as  $t \rightarrow \infty$  if  $k_t^{-1} \rightarrow 0$

and  $g_{\bar{\pi}}(t) \rightarrow 0$ . For any  $k_t^{-1}, g_{\bar{\pi}}(t) > 0$ ,  $\alpha_{t+1} < 1$ .

It can be  $< -1$  though! Suppose  $k_{t+1}^{-1} \approx 1$ . Then  $1 - k_{t+1}^{-1}$

is close to 0. If  $g_{\bar{\pi}}(t) f_{\ell,t+1} > 1$  by enough, then this could happen. But it's not likely b/c 1)  $f_{\ell} < 1$  if  $\sum \leq 1$ ; 2)

$\hat{g}_N(t)$  is likely to be not just  $< 1$ , but quite small since the overall level of the gain is  $< 1$ , so its change is likely to be an order of magnitude smaller.

$\Rightarrow$  So I think  $\alpha_t < 1 \quad \forall t$  is safe to assume.

$$\begin{aligned}
 \psi_{S,t} &= Cx_{t+1} + \alpha_{t+1} \psi_{S,t+1} && | \text{Iter. fwd.} \\
 &= Cx_{t+1} + \alpha_{t+1} [Cx_{t+2} + \alpha_{t+2} \psi_{S,t+2}] \\
 &= Cx_{t+1} + \alpha_{t+1} [Cx_{t+2} + \alpha_{t+2} (Cx_{t+3} + \alpha_{t+3} \psi_{S,t+3})] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \\
 &\quad + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \psi_{S,t+3}] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} \\
 &\quad + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} [Cx_{t+4} + \alpha_{t+4} \psi_{S,t+4}]] \\
 &= C [x_{t+1} + \alpha_{t+1} x_{t+2} + \alpha_{t+1} \alpha_{t+2} x_{t+3} + \alpha_{t+1} \alpha_{t+2} \alpha_{t+3} x_{t+4} \\
 &\quad + \underbrace{\alpha_{t+1} \alpha_{t+2} \alpha_{t+3} \alpha_{t+4} \psi_{S,t+4}}_{\rightarrow 0 \text{ as } t \rightarrow \infty}
 \end{aligned}$$

$$\psi_{S,t} = Cx_{t+1} + C \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j}$$

So that (1') becomes:

$$2\pi_t + \frac{2\lambda}{k} x_t - (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left( c x_{t+1} + c \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right) = 0$$

$$\Rightarrow 2\pi_t = -\frac{2\lambda}{k} x_t + c (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left( x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right)$$

Since  $c = \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k}$ , we have

$$\pi_t = -\frac{\lambda}{k} x_t + \frac{(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{k} (k_t^{-1} + g_{\bar{\pi}}(t) f_{e_t}) \left( x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=1}^{i-1} \alpha_{t+j} \right)$$

Recalling that  $\alpha_{t+1} := 1 - k_{t+1}^{-1} - g_{\bar{\pi}}(t) f_{e_{t+1}}$ , this is:

$$\begin{aligned} \pi_t &= -\frac{\lambda}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{e_t} g_{\bar{\pi}}(t) \right) \right. \\ &\quad \left. \left( x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - \underbrace{k_{t+1+j}}_{=} - g_{\bar{\pi}}(t+j) f_{e_{t+1+j}} \right) \right) \right\} \end{aligned}$$

☞ Note this is not 1, as in p.97, Notes 6.

And lastly I defined  $f_{e_t} \equiv \pi_t - (\bar{\pi}_{t-1} + b_1 s_{t-1})$ , so

$$\begin{aligned} \pi_t &= -\frac{\lambda}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right. \\ &\quad \left. \left( x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - \underbrace{k_{t+1+j}}_{=} - g_{\bar{\pi}}(t+j) (\bar{\pi}_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j}) \right) \right) \right\} \\ &\quad \text{was } + \text{ } j \quad \text{=} \text{ minus, not plus!} \end{aligned}$$

so I write

$$\pi_t = -\frac{\gamma}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right.$$

$\underset{\text{= minus, not plus!}}{=}$

$$\left. \left( x_{t+1} + \sum_{i=2}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - \underset{\text{= minus, not plus!}}{\cancel{k_{t+j}}} \right) \underset{\text{= minus, not plus!}}{\cancel{g_{\bar{\pi}}(t+j)}} \left( \pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j} \right) \right) \right\}$$

changing the  $i$ -index and using the notation  $\sum_{j=0}^{\infty} := 1$ ,  
as

$$\pi_t = -\frac{\gamma}{k} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\bar{\pi}}(t) \right) \right.$$

$\underset{\text{= minus, not plus!}}{=}$

$$\left. \left( \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - \underset{\text{= minus, not plus!}}{\cancel{k_{t+j}}} \right) \underset{\text{= minus, not plus!}}{\cancel{g_{\bar{\pi}}(t+j)}} \left( \pi_{t+1+j} - \bar{\pi}_{t+j} - b_1 s_{t+j} \right) \right) \right\}$$

where I'm highlight the changes vis-à-vis p. 97 of  
Notes 6, and I'll adopt these changes to  
materials 23, (B. 1)

and note also that  $-g_{\bar{\pi}}(t+j)$  got lost in materials 18  
and thus the paper too! Need to correct!

✓ = corrected in materials 23

Let me write it in a way good to interpret:

$$\pi_t = -\frac{\gamma}{K} \left\{ x_t - \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + (\bar{\pi}_t - \bar{\pi}_{t-1} - b_1 s_{t-1}) g_{\pi}(t) \right) \right. \\ \left. \left( E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - k_{t+j+1}^{-1} - (\bar{\pi}_{t+j} - \bar{\pi}_{t+j-1} - b_1 s_{t+j}) g_{\bar{\pi}}(t+j) \right) \right) \right\}$$

Which allows the even neater formulation

$$\pi_t = -\frac{\gamma}{K} x_t + \underbrace{\frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right)}_{E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - k_{t+j+1}^{-1} - f_{t+j+1:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)}$$

$$E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - k_{t+j+1}^{-1} - f_{t+j+1:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)$$

"discretion tradeoff + tradeoff coming from how the gain today responded to latest FE + tradeoff coming from how the gain is expected to change respond to future FE"

Rewrite once more

*effect of current level and change of gain on future tradeoffs*

$$\pi_t = -\frac{\gamma}{K} x_t + \underbrace{\frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right) E_t \sum_{i=1}^{\infty} x_{t+i}}_{- \frac{\gamma}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{t+1:t-1}^{eve} g_{\pi}(t) \right) E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( k_{t+j+1}^{-1} + f_{t+j+1:t+j}^{eve} g_{\bar{\pi}}(t+j) \right)}$$

*effect of future expected levels and changes of the gain on future tradeoffs, given current level & change*

Taking the 2<sup>nd</sup> expression again to consider how one could simplify it,

$$\pi_t = -\frac{\lambda}{K} x_t + \frac{\lambda}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{t+1:t+1}^{eve} g_{\bar{\pi}}(t) \right) .$$

$$E_t \sum_{i=1}^{\infty} x_{t+i} \prod_{j=0}^{i-1} \left( 1 - k_{t+n+j}^{-1} - f_{t+j+1:t+j+1}^{eve} g_{\bar{\pi}}(t+j) \right)$$

reveals that only the orange part is not available at time  $t$  to the CB. Evaluating even  $E_t x_{t+k}$   $k > 0$  is difficult b/c it requires evaluating  $E_t k_{t+k}^{-1}$

But: if I were a rule that the CB relies on agents' expectations (otherwise it couldn't implement the TC).

But then anticipated utility would bind:

$$\hat{E}_t k_{t+k}^{-1} = k_t^{-1} \quad (\text{is that true?})$$

$$\hat{E}_t f_{t+1:k+1:k}^{eve} = 0$$

$$\hat{E}_t g_{\bar{\pi}}(t+k) = 0 \quad (\text{is that true?})$$

So the "expectations-based TC" would be

$$\pi_t = -\frac{\lambda}{K} x_t + \frac{\lambda}{K} \frac{(1-\alpha)\beta}{1-\alpha\beta} \left( k_t^{-1} + f_{t+1:t+1}^{eve} g_{\bar{\pi}}(t) \right) \cdot \underbrace{\sum_{i=1}^{\infty} b_e h^x i^{-1} s_t (1 - k_t^{-1})^{i-1}}_{\text{check later.}}$$

Let's go back to the question of solving (6'):

$$\varphi_{6,t+1} = \frac{c f_{t+1} x_{t+1}}{\alpha_t} - \underbrace{\frac{1}{\alpha_t} \varphi_{6,t}}_{<1} - \underbrace{\frac{\beta_t}{\alpha_t} \varphi_{6,t+2}}_{<1} \quad \boxed{|\alpha_t| > 1 \quad |\beta_t| < 1}$$

Time-out: write this (following Hamilton) as a std 2nd order diff eq:

$$\beta_t \varphi_{6,t+2} + \alpha_t \varphi_{6,t+1} - \varphi_{6,t} = c f_{t+1} x_{t+1}$$

$$(\beta_t + \alpha_t L - L^2) \varphi_{6,t+2} = c f_{t+1} x_{t+1} \quad | : \beta_t$$

$$(1 + \underbrace{\frac{\alpha_t}{\beta_t} L}_{>1} - \underbrace{\frac{1}{\beta_t} L^2}_{>1}) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

$$(1 - (-\frac{\alpha_t}{\beta_t}) L - \frac{1}{\beta_t} L^2) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L) \varphi_{6,t+2} = \frac{c f_{t+1} x_{t+1}}{\beta_t}$$

let me drop the 6 and roll backward

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$$\beta_{t-2} \varphi_t + \alpha_{t-2} \varphi_{t-1} - \varphi_{t-2} = c f_{t-2} x_{t-1}$$

$$\varphi_t = - \underbrace{\frac{\alpha_{t-2}}{\beta_{t-2}}}_{>1} \varphi_{t-1} + \varphi_{t-2} + c f_{t-2} x_{t-1}$$

can't solve bw.

But can't solve for either!

Let me step back: can we simplify this somehow?

$$\varphi_{6,t} = -c f_{t+} x_{t+1} + \left[ 1 + \frac{f_{t+}(1-k_{t+1}^{-1}) - f_{t+} g_{\bar{\pi}}(t)}{f_{t+1}} \right] \varphi_{6,t+1} - \underbrace{\frac{f_{t+}(1-k_{t+1}^{-1})}{f_{t+1}} p_{6,t+2}}_{< 1}$$

Obs:

1. If  $f_{t+} = 0$ ,  $\varphi_{6,t} = \varphi_{6,t+1}$ . I guess  $k_{t+}$  is  $\downarrow$  then.
2. If  $f_{t+1} = 0$ ,  $\varphi_{6,t} = \infty$  ( $\varphi_{6,t+1} - \varphi_{6,t+2} = \infty \cdot 0$ ) ?
3. If  $f_{t+} = \infty$ ,  $\varphi_{6,t} = \infty$
4. If  $f_{t+1} = \infty$ ,  $\varphi_{6,t} = -c f_{t+} x_{t+1} + (1 - f_{t+} g_{\bar{\pi}}(t)) \varphi_{6,t+1}$

An unorthodox thing to do: what if we solved (1')

$$2\pi_t + \frac{2}{k} x_t - \left( \frac{k_{t+1}^{-1}}{f_{t+}} + g_{\bar{\pi}}(t) \right) \varphi_{6,t} + \frac{k_{t+1}^{-1}}{f_{t+}} p_{6,t+1} = 0 \quad (1')$$

for  $\varphi_{6,t}$ , and plugged that in (6')?

$$\varphi_{6,t} = \frac{\frac{k_{t+1}^{-1}}{f_{t+}}}{\underbrace{\frac{k_{t+1}^{-1}}{f_{t+}} + g_{\bar{\pi}}(t)}_{< 1}} p_{6,t+1} - \gamma_t \quad \gamma_t := 2(\pi_t + \frac{2}{k} x_t)$$

$$=: \delta_t \quad \varphi_{6,t} = \delta_t p_{6,t+1} - \gamma_t$$

$$\gamma_{6,t} = \delta_+ \gamma_{6,t+1} - \gamma_+$$

$$\begin{aligned}
\gamma_{6,t} &= \delta_+ \left[ \delta_{t+1} \gamma_{6,t+2} - \gamma_{t+1} \right] - \gamma_+ \\
&= \delta_+ \left[ \delta_{t+1} \left[ \delta_{t+2} \gamma_{6,t+3} - \gamma_{t+2} \right] - \gamma_{t+1} \right] - \gamma_+ \\
&= \delta_+ \delta_{t+1} \delta_{t+2} \gamma_{6,t+3} - \left[ \delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= \delta_+ \delta_{t+1} \delta_{t+2} \delta_{t+3} \gamma_{6,t+4} \\
&\quad - \delta_+ \delta_{t+1} \delta_{t+2} \gamma_{t+3} \\
&\quad - \left[ \delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= - \left[ \delta_+ \delta_{t+1} \delta_{t+2} \gamma_{t+3} + \delta_+ \delta_{t+1} \gamma_{t+2} + \delta_+ \gamma_{t+1} + \gamma_+ \right] \\
&= - \gamma_+ - \sum_{i=1}^{\infty} \delta_{t+i} \prod_{j=0}^{i-1} \delta_{t+j}
\end{aligned}$$

$$\gamma_{6,t} = - \sum_{i=0}^{\infty} \delta_{t+i} \prod_{j=0}^{i-1} \delta_{t+j} \quad (\text{again using the } \prod_{j=0}^0 = 1 \text{ notation})$$

$$\gamma_{6,t} = -2E \sum_{i=0}^{\infty} \left( \pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{k_{t+j}}{f_{t+j}} \frac{k_{t+j}^{-1}}{f_{t+j} + g_{\pi}(t+j)}$$

Sol of (1') for  $\gamma_{6,t}$ .

This implies (plugging into (6')):

$$E_t \sum_{i=0}^{\infty} \left( \pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}^{-1}}{f_{t+j}}}{\frac{k_{t+j}^{-1}}{f_{t+j}} + g_{\pi}(t+j)}$$

$$= E_t c f_{t+1} x_{t+1}$$

$$- E_t \left[ 1 + \frac{f_{t+1} (1 - k_{t+1}^{-1}) - f_{t+1} g_{\pi}(t)}{f_{t+1}} \right] \left( \sum_{i=0}^{\infty} \left( \pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j+1}^{-1}}{f_{t+j+1}}}{\frac{k_{t+j+1}^{-1}}{f_{t+j+1}} + g_{\pi}(t+j+1)} \right)$$

$$- E_t \frac{f_{t+1} (1 - k_{t+1}^{-1})}{f_{t+1}} \left( \sum_{i=0}^{\infty} \left( \pi_{t+2+i} + \frac{\lambda}{\kappa} x_{t+2+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+2+j}^{-1}}{f_{t+2+j}}}{\frac{k_{t+2+j}^{-1}}{f_{t+2+j}} + g_{\pi}(t+2+j)} \right)$$

The target criterion if the anchoring function is specified in terms of changes in the gain.

(I divided them by -2.)

(Bignly)

Back to (6')

$$\varphi_{6,t+1} = \underbrace{c_{t+1} x_{t+1}}_{<1} + \underbrace{\frac{1}{\alpha_t} \varphi_{6,t}}_{<1} - \underbrace{\frac{\beta_t}{\alpha_t} \varphi_{6,t+2}}_{<1}$$

Let's do a really crazy thing: sub frods & backwards

$$\varphi_t = c_{t-1} x_t + \underbrace{a_{t-1} \varphi_{t-1}}_{<1} + \underbrace{b_{t-1} \varphi_{t+1}}_{<1}$$

let me call  $b_{t-1} = -\frac{\beta_t}{\alpha_t}$

$$\varphi_t = c_{t-1} x_t$$

Iter 1

(Mathematica)

$$+ a_{t-1} (c_{t-2} x_{t-1} + a_{t-2} \varphi_{t-2} + b_{t-2} \varphi_t)$$

$$+ b_{t-1} (c_t x_{t+1} + a_t \varphi_t + b_t \varphi_{t+2})$$

$$= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1}$$

✓  $+ a_{t-1} a_{t-2} \varphi_{t-2}$

✓  $- a_{t-1} b_{t-2} \varphi_t + b_{t-1} a_t \varphi_t$

✓  $+ b_{t-1} b_t \varphi_{t+2}$

→ I shouldn't have plugged it here. I should have saved myself the bold-circled terms.

Iter 2  $\Rightarrow$  LHS  
 $= c_{t-1} x_t + a_{t-1} c_{t-2} x_{t-1} + b_{t-1} c_t x_{t+1} \quad 1 - a_{t-1} b_{t-2} - b_{t-1} a_t$

$+ a_{t-1} a_{t-2} (c_{t-3} x_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1})$

$- (a_{t-1} b_{t-2} + b_{t-1} a_t) (c_{t-1} x_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1})$

$+ b_{t-1} b_t (c_{t+1} x_{t+2} + a_{t+1} \varphi_{t+1} + b_{t+1} \varphi_{t+3})$

$$= C_{t-1} X_t + a_{t-1} C_{t-2} X_{t-1} + b_{t-1} c_t X_{t+1}$$

$$+ a_{t-1} a_{t-2} (C_{t-3} X_{t-2} + a_{t-3} \varphi_{t-3} + b_{t-3} \varphi_{t-1})$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) (C_{t-1} X_t + a_{t-1} \varphi_{t-1} + b_{t-1} \varphi_{t+1})$$

$$+ b_{t-1} b_t (C_{t+1} X_{t+2} + a_{t+1} \varphi_{t+1} + b_{t+1} \varphi_{t+3})$$

$$= C_{t-1} X_t + a_{t-1} C_{t-2} X_{t-1} + b_{t-1} c_t X_{t+1}$$

$$\checkmark a_{t-1} a_{t-2} C_{t-3} X_{t-2}$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) C_{t-1} X_t$$

$$+ b_{t-1} b_t C_{t+1} X_{t+2} \checkmark$$

$$+ a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3} \checkmark$$

$$+ a_{t-1} a_{t-2} b_{t-3} \varphi_{t-1} \checkmark$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1} \varphi_{t-1}$$

$$+ (a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} \varphi_{t+1}$$

$$+ b_{t-1} b_t a_{t+1} \varphi_{t+1} \checkmark$$

$$+ b_{t-1} b_t b_{t+1} \varphi_{t+3} \checkmark$$

these are all 0

in the limit

$$\text{the } x \text{ parts: } a_{t-1} C_{t-2} X_{t-1} + a_{t-1} a_{t-2} C_{t-3} X_{t-2}$$

$$+ b_{t-1} C_t X_{t+1} + b_{t-1} b_t C_{t+1} X_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) C_{t-1} X_t$$

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} \varphi_{t-3}$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) \varphi_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) \varphi_{t+1}$$

$$+ b_{t-1} b_t b_{t+1} \varphi_{t+3}$$

|| Now plug only the  $x$ -term. kr 3

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} (c_{t-4} x_{t-3} + \dots)$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) (c_{t-2} x_{t-1} + \dots)$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) (c_t x_{t+1} + \dots)$$

$$+ b_{t-1} b_t b_{t+1} (c_{t+2} x_{t+3} + \dots)$$

So only the  $x$  parts:

$$= a_{t-1} c_{t-2} x_{t-1} + a_{t-1} a_{t-2} c_{t-3} x_{t-2}$$

$$+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2}$$

$$(1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ a_{t-1} a_{t-2} a_{t-3} (c_{t-4} x_{t-3} + \dots)$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) (c_{t-2} x_{t-1} + \dots)$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) (c_t x_{t+1} + \dots)$$

$$+ b_{t-1} b_t b_{t+1} (c_{t+2} x_{t+3} + \dots)$$

$$= \underline{a_{t-1} c_{t-2} x_{t-1}} + \underline{a_{t-1} a_{t-2} c_{t-3} x_{t-2}} + \underline{a_{t-1} a_{t-2} a_{t-3} c_{t-4} x_{t-3}}$$

$$\underline{+ b_{t-1} c_t x_{t+1} + b_{t-1} b_t c_{t+1} x_{t+2} + b_{t-1} b_t b_{t+1} c_{t+2} x_{t+3}}$$

$$+ (1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

→ there will be a backward sum & a forward sum

$$BS := \sum_{i=1}^{\infty} x_{t-i} c_{t-1-i} \prod_{j=1}^i a_{t-j} \quad \blacksquare$$

$$FS := \sum_{i=1}^{\infty} x_{t+i} c_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j} \quad \blacksquare$$

and one or more cross-sums. Let's look at those.

$$+ (1 + a_{t-1} b_{t-2} + b_{t-1} a_t) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1} b_{t-2} + b_{t-1} a_t) a_{t-1}) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} + b_{t-1} a_t) b_{t-1} + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

Let's multiply out partly

$$(1 + a_{t-1} b_{t-2} + a_t b_{t-1}) c_{t-1} x_t$$

$$+ (a_{t-1} a_{t-2} b_{t-3} + (a_{t-1}^2 b_{t-2} + a_{t-1} a_t b_{t-1})) c_{t-2} x_{t-1}$$

$$+ ((a_{t-1} b_{t-2} b_{t-1} + a_t b_{t-1}^2) + b_{t-1} b_t a_{t+1}) c_t x_{t+1}$$

$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ a_{t-1} (a_{t-2} b_{t-3} + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-2} x_{t-1}$$

$$+ b_{t-1} (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}} + b_t a_{t+1}) c_t x_{t+1}$$

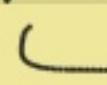
$$= (1 + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) c_{t-1} x_t$$

$$+ (a_{t-2} b_{t-3} + \underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}}) a_{t+1} c_{t-2} x_{t-1}$$

$$+ (\underline{a_{t-1} b_{t-2}} + \underline{a_t b_{t-1}} + b_t a_{t+1}) b_{t-1} c_t x_{t+1}$$

I think (and hope) that more and more powers and multiplications can be pulled out, so that the bw- & fw-cross terms  go to 0. In most case we'd be left with

$$x_t c_{t-1} (1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \dots \text{expanding sum})$$

 in fact,

I might even be able to neglect some or all of this sum. Or it may not be expanding. Try to verify w/ Mathematica. If it doesn't expand, I'll get:

$$\varphi_{t+} = BS + FS + CS$$

$$= \sum_{i=1}^{\infty} x_{t-i} c_{t-1-i} \prod_{j=1}^{i-1} a_{t-j}$$

$$+ \sum_{i=1}^{\infty} x_{t+i} c_{t-1+i} \prod_{j=-1}^{i-2} b_{t+j}$$

$$+ x_t c_{t-1} (1 + a_{t-1} b_{t-2} + a_t b_{t-1} + \dots \text{expanding sum})$$

$$\text{where } c_{t-1} = \frac{c f_{t-1}}{\alpha_t}$$

$$c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}$$

$$a_{t-1} = \frac{1}{\alpha_t}$$

$$a_t = \left[ 1 + \frac{f_{t-1}(1-k_{t-1}^{-1}) - f_{t-1}g_{t-1}(t)}{f_{t-1}} \right]$$

$$b_{t-1} = -\frac{\beta}{\alpha_t}$$

$$\beta_t = \frac{f_{t-1}(1-k_{t-1}^{-1})}{f_{t-1}}$$

But now the cross-terms simplify if we hadn't subbed in  $\varphi_{t+}$  on RHS, but on LHS we'd have

$$(1 - a_{t-1} b_{t-2} - b_{t-1} a_t) \varphi_{t+} = BS + FS + CS$$

and CS would be previous minus the bold stuff; i.e.

$$CS = a_{t-2} a_{t-1} b_{t-3} c_{t-2} x_{t-1} + a_{t+1} b_t b_{t-1} c_t x_{t+1}$$

Now the hope is that the CS-term will vanish.

Mathematica: Iter 3: new t-term shows up:

$$(a_{t-2}a_{t-1}b_{t-3}b_{t-2} + a_t a_{t+1}b_{t-1}b_t) \varphi_{6,t}$$

→ have to subtract from LHS!

And in Iter 5 it will appear again.

So instead of subtracting them always from the LHS,

I'll assume that already 4 multiplications of any

a or b combos is close enough to 0 so I don't

need to worry (i.e. from Iter 3 onward.)

Ok, so I'm letting all  $\varphi_{6,+ik} \ k \in \mathbb{R} \rightarrow 0$

Let me write down what I get after 5 iters

in Mathematica, where I took  $\varphi_{6,+}$  on the LHS

only in Iter 3: Coefficients:

$$x_{t-5}: a_{t-5}a_{t-4}a_{t-3}a_{t-2}a_{t-1}c_{t-6}$$

$$x_{t-4}: a_{t-4}a_{t-3}a_{t-2}a_{t-1}c_{t-5}$$

Good up to  $x_{t-3}$ :

$x_{t-3} :$ 

$$a_{t-3} a_{t-2} a_{t-1} c_{t-4}$$

$$+ a_{t-4} a_{t-3} a_{t-2} a_{t-1} b_{t-5} c_{t-4}$$

$$+ a_{t-3}^2 a_{t-2} a_{t-1} b_{t-4} c_{t-4}$$

$$+ a_{t-3} a_{t-2}^2 a_{t-1} b_{t-3} c_{t-4}$$

 $x_{t-2} :$ 

$$a_{t-2} a_{t-1} c_{t-3}$$

$$+ a_{t-3} a_{t-2} a_{t-1} b_{t-4} c_{t-3}$$

$$a_{t-2}^2 a_{t-1} b_{t-3} c_{t-3}$$

 $x_{t-1} :$ 

$$a_{t-1} c_{t-2}$$

$$+ a_{t-2} a_{t-1} b_{t-3} c_{t-2}$$

$$+ a_{t-3} a_{t-2} a_{t-1} b_{t-4} b_{t-3} c_{t-2}$$

$$+ a_{t-2}^2 a_{t-1} b_{t-3} c_{t-2}$$

It's impossible to see any pattern!

My only chance is to make restrictions on the coefficients. In particular I'm interested in doing so for  $a$  &  $b$ , not so much for  $c$ .

$$c_{t+1} = \frac{c f_{t+}}{\alpha_t} \quad c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta}$$

$$a_{t+1} = \frac{1}{\alpha_t} \quad a_t = \left[ 1 + \frac{f_{t+}(1-k_{t+1}^{-1}) - f_{t+}g_{\bar{n}}(t)}{f_{t+1}} \right]$$

$$b_{t+1} = -\frac{\beta_+}{\alpha_t} \quad \beta_+ = \frac{f_{t+}(1-k_{t+1}^{-1})}{f_{t+1}}$$

But  $c_{t+1} = c f_{t+} a_{t+1}$ , so if I have any restriction that says that  $a \cdot b = 0$ , then  $b \cdot c = 0$  too.  
 (which will kill the forward sum FS).

$$\text{So } a_{t+1} b_{t+1} = -\frac{1}{\alpha_t} \frac{\beta_+}{\alpha_t}$$

Can't make this case that  $\alpha_t^2 \rightarrow \infty$  b/c then  
 $a_{t+1} c_{t+1} = 0$  too, kill everything.

What one can do is to make a time-dependent argument.

$$\lim_{t \rightarrow \infty} \beta_+ \approx 1 \quad \lim_{t \rightarrow 0} \beta_+ = 0 \quad \rightarrow \lim_{t \rightarrow \infty} b_+ = -\frac{1}{2}, \quad \lim_{t \rightarrow 0} b_+ = \frac{0}{0}$$

$$\lim_{t \rightarrow \infty} \alpha_t \approx 2 \quad \lim_{t \rightarrow 0} \alpha_t \approx 0 \quad \rightarrow \lim_{t \rightarrow \infty} a_+ = \frac{1}{2}, \quad \lim_{t \rightarrow 0} a_+ = \infty$$

$$\lim_{t \rightarrow \infty} c_+ = \frac{c \infty}{2} \quad \lim_{t \rightarrow 0} c_+ = \infty$$

Ok, so one can't ever solve that. All I can do is interpret it: the target criterion under anchoring function specified in gain changes, not levels, takes the form of two equations:

$$y_{6,1} = -c f_{\pi} x_{2,-1} + \left[ 1 + \frac{f_{\pi} (1 - k_{+,-1}^{-1}) - f_{\pi} g_{\bar{\pi}}(t)}{f_{\pi,-1}} \right] y_{6,1+1} - \frac{f_{\pi} (1 - k_{+,-1}^{-1})}{f_{\pi,-1}} p_{6,1+2} \quad (6')$$

$$2\pi_+ + \frac{2\lambda}{K} x_+ - \left( \frac{k_+^{-1}}{f_{\pi}} + g_{\bar{\pi}}(t) \right) y_{6,1} + \frac{k_+^{-1}}{f_{\pi}} p_{6,1+1} = 0 \quad (1')$$

One way to represent this is (big ugly)

$$c = \frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K}$$

on p. 65. The other is in the 3-eq-form: (+ or -?)  
(from p. 53)

Define  $c = -\frac{2(1-\alpha)\beta}{1-\alpha\beta} \frac{\lambda}{K}$  and  $f_{\pi} = \pi_+ - (\bar{\pi}_{+-1} + b_1 s_{1-1})$ ,

then:

When I'm annoyingly using a different notation than usual...

$$2\pi_+ + \frac{2\lambda}{K} x_+ - k_+^{-1} p_{5,1} - g_{\bar{\pi}}(t) y_{6,1} = 0 \quad (1)$$

$$c x_{2,-1} + y_{5,1} - (1 - k_{+,-1}^{-1}) p_{5,1+1} + g_{\bar{\pi}}(t) y_{6,1+1} = 0 \quad (6)$$

$$f_{\pi} \cdot y_{5,1} = y_{6,1} - y_{6,1+1} \quad (7)$$

(1) says "discretion + learning ( $y_{5,1}$ ) × effect of stance of anchoring  
+ "effect of how anchoring changes" where (7) says that if learning

is a strong constraint now, then at least anchoring constraint will be relaxed strongly tomorrow.

Interpreting (bigraph) is an alternative way: the CB can follow a target criterion as computing the anchoring multipliers  $\gamma_{6,t}$  for  $t$ ,  $t+1$  and  $t+2$  and then evaluating (6'). Compute the multiplier as the solution to (1') as:

$$\gamma_{6,t} = -2E \sum_{i=0}^{\infty} \left( \pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} \right) \prod_{j=0}^{i-1} \frac{\frac{k_{t+j}}{f_{t+j}}}{\frac{k_{T+j}}{f_{T+j}} + g_T(t+j)} \quad (\text{sol } 1')$$

Interpretation:

$$\gamma_{6,t} = 0 \quad \text{if} \quad \pi_{t+i} + \frac{\lambda}{\kappa} x_{t+i} = 0 \quad \forall i \quad \text{or}$$

$$k_{t+j}^{-1} = 0 \quad \forall j$$

i.e. if you hit the target or expectations are anchored. But exp. can't be anchored if you don't hit the target (nice result can happen - you hit the target but there was a shock that got expectations unanchored).

1 minute (sol 1') suggests that

$$\varphi_{6,t} = -2E_t(\pi_t + \frac{\beta}{\mu}x_t) + \varphi_{6,t+1}$$

So if  $\varphi_{6,t} = 0$  b/c  $(\pi_t + \frac{\beta}{\mu}x_t) = 0$  bt, then  $\varphi_{6,t+1} = 0$  for

But  $\varphi_{6,t} = 0 \not\Rightarrow \varphi_{6,t+1} = 0$ .

$$\text{then } \varphi_{6,t} = -2(\pi_t + \frac{\beta}{\mu}x_t) - 2E_t(\pi_{t+1} + \frac{\beta}{\mu}x_{t+1}) + \varphi_{6,t+2}$$

$$\text{so } \varphi_{6,t+2} = \varphi_{6,t} + 2(\pi_t + \frac{\beta}{\mu}x_t) + 2E_t(\pi_{t+1} + \frac{\beta}{\mu}x_{t+1})$$

$$\varphi_{6,t+1} = \varphi_{6,t} + 2(\pi_t + \frac{\beta}{\mu}x_t)$$

Plugging this in (6')

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \left[ 1 + \frac{f_{t+1}(1 - k_{t+1}^{-1}) - f_t g_{\pi}^{-1}(t)}{f_{t+1}} \right] \varphi_{6,t+1} - \frac{f_{t+1}(1 - k_{t+1}^{-1})}{f_{t+1}} \varphi_{6,t+2}$$

$$\varphi_{6,t} = -c f_{t+1} x_{t+1} + \alpha_t (\varphi_{6,t} + 2(\pi_t + \frac{\beta}{\mu}x_t))$$

$$- \beta_t (\varphi_{6,t} + 2(\pi_t + \frac{\beta}{\mu}x_t) + 2E_t(\pi_{t+1} + \frac{\beta}{\mu}x_{t+1}))$$

$$(1 - \alpha_t + \beta_t) \varphi_{6,t} = -c f_{t+1} x_{t+1} + 2(\alpha_t - \beta_t)(\pi_t + \frac{\beta}{\mu}x_t)$$

$$- 2\beta_t E_t(\pi_{t+1} + \frac{\beta}{\mu}x_{t+1})$$

$$\varphi_{t+1} = \frac{-c f_{t+1} x_{t+1} + 2(\alpha_t - \beta_t)(\pi_t + \frac{\gamma}{\lambda} x_t) - 2\beta_t E_t (\pi_{t+1} + \frac{\gamma}{\lambda} x_{t+1})}{1 - \alpha_t - \beta_t}$$

$$\alpha_t - \beta_t = \left[ 1 - \frac{f_{t+1}(1 - k_{t+1}^{-1}) - f_{t+1}g_{\bar{\pi}}(t)}{f_{t+1}} \right] - \frac{f_{t+1}(1 - k_{t+1}^{-1})}{f_{t+1}}$$

$$= 1 - f_{t+1}g_{\bar{\pi}}(t)$$

$$\Rightarrow \varphi_{t+1} = \frac{-c f_{t+1} x_{t+1} + 2(1 - f_{t+1}g_{\bar{\pi}}(t))(\pi_t + \frac{\gamma}{\lambda} x_t) - 2 \frac{f_{t+1}(1 - k_{t+1}^{-1})(E_t \pi_{t+1} + \frac{\gamma}{\lambda} x_{t+1})}{f_{t+1}}}{f_{t+1} g_{\bar{\pi}}(t)}$$

(finite)

Which is neat I guess b/c you don't need to evaluate all the future periods.

Can we plug this in (1')? I don't really think so b/c we used knowledge of the sol. to (1') to derive this. Hmmm. Let it rest.

 stuff potentially to be used for interpretation in the paper. Otherwise for the implementation of the target criterion, I'll stick w/ anchoring functions in levels.

Ryan meeting

1 April 2020

Conjecture a sequence of  $\{x_i\}$  s.t. A.1. & A.3

function to find the sequence of  $\{x_i\}$  for which  
A.1. & A.3 hold

Think of the  $\hat{E}$  you'd compute over period 50.

think of A.1. & A.3 as residual eq.

and (A.4) holding exactly

You can also do the opposite: solve for  
expectations that satisfy all the processes.

Ryan thinks

A.1. & A.3. need to be residual

A.2 we don't know

A.5, 6, 7 can be used exactly

Start w/ sim you know works  $\rightarrow$  check results of 1-7  
 $\rightarrow$  should be zero. Then perturb your sequence & see

## Work after

### My comments/ interpretation :

It seems like there is a language for these kinds of problems:

residual equations vs. those that are fulfilled exactly.  
And it also seems (Ryan was suggesting) that there is a right/wrong decision about which equation to treat as residual or exactly fulfilled equation.

It seems to me that

- 1) initially I treated all equations as exact (or something)
- 2) Need to figure out which to treat how.

Ignore the target crit. for a sec, and

2 April 2020

just consider the problem of simulating the model given an  $i$ -sequence. So far I treated that as a simulation exercise given 7 exact equations. But maybe

the sequence of  $i$  should already be an optimization  
- one that minimizes the residual coming from one  
of the model equations.

I feel that if I treated (A.1) as a residual eq,  
and minimized a resid, then maybe things wouldn't  
blow up in my face.

→ It feels to me that -as Lynn suggested- if I treat  
 $i$  (A.3) as a residual eq (b/c I'm adjusting  $\{i_+\}$ )  
and I treat (A.1) as a residual eq. (b/c given  $\{i_+\}$   
and Exp I don't let (A.3) determine  $\{x_+\}$ , instead I  
treat  $\{x_i\}$ ) then I need to optimize over  
 $\{i_+, x_+\} = \text{argmin} \text{ resid}(A.1)$

My fear is that I have too many d.o.f. here  
b/c  $\{i_+\}$  and  $\{x_+\}$  can adjust to fulfill a single resid.  
This may be fixed in w/ the fact that (A.3) is exact  
by default. Implicitly, it says,  $\{i_+\} = \text{WN}(0, 2^2)$ , and  
this is always fulfilled no matter what. If I'm

cumbersome to find an interest-rate-reaction-function,  
i.e. its functional form, then I could experiment  
w/ something similar to estimating  $g(\cdot)$

$$i_t = r(\pi_t, x_t, \hat{E}_t(\pi_{t+1}), \pi_{t-1}, \dots, e_t)$$

that would also allow me to treat (A.3) as  
a resid, since the residual

$$i_t - r(\cdot)$$

would be well defined.

But one step at a time.

Step (-1) Treat (A.1) as a resid and min  $\{i_t, x_t\}$

Step (1) Postulate TR as well (A.3 is a model  $\mathcal{G}$ )

and now see if you can obtain  $\{i_t^{\text{TR}}\}$   
by making \* Step \* a resid.

Need to rework command-implement-target-criterion.

But, even before that, I need a file that does  
steps (-2) and (1): command-sim-given-Seg.

I'm also creating sim-learn-LH-given-seq.m  
which will do the sim given the input sequences.

objective-seq.m tells me that A.1 & A.2 are not fulfilled, so I have to treat them as residual eqs  
In fact, (A.3) isn't either!

So what I have now is:

feed in exog  $\{i_t\}$ -series  $\Rightarrow$  min reads of (A1)(A3)  
 $\Rightarrow$  obtain  $\{\hat{x}_t\}$  which is close to  $\{\hat{x}_t^{TR}\}$ , but it's  
not equal, not even  $\{i_t\}$ .

Can it be that this is right?

So the interpretation of this  $\{i_t\}$  would be that it,  
more or less, implements the same (on the TR  
would have implemented). Can I get closer to  $\{\hat{x}_t^{TR}\}$

by adding d.o.f., i.e.  $\{x_t, \pi_t\}$  as optimized over?  
But if I raise T to 100,  $\hat{x}_t$  explodes again!

Maybe by using  $\pi_i(x)$ -given-in.m, I'm still de facto treating (A.1) & (A.2) as holding exactly.

↳ I need a flexible code that implements (A.9)-(A.10) ( $\equiv$ (A.1)&(A.2)), allowing me to specify what is a residual equation and what not.

A9A10.m

Observations:

3 April 2010

To take (A.1) under Taylor-rule and compare it to (A.1) under exeq  $\{i_t\}$  is NOT the right concept of a residual eq b/c I'm still de facto postulating an (A.1) which generates  $x_t$ , and it holds.

No, the residuals should be generated in sim-learn.H, given-seq.m, or even deeper, and be output to objective-seq.m.

## Model equations

$$x_t = z_i^t + [z \ 1 - \beta - 2\beta] f_{\beta} \left[ + z [1 \ 0 \ 0] (I_{nx} - \beta h_x)^{-1} s_t \right] \quad (A.9)$$

$$a_t = k x_t + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0, 0, 1] (I_{nx} - \alpha\beta h_x)^{-1} s_t \quad (A.10)$$

$$f_{\ell_{t+1}} = z_t - (a_{t+1} + b s_{t+1}) \quad \text{exact} \quad (A.7)$$

$$k_t = k_{t+1} + g(f_{\ell_{t+1}}^2) \quad \text{exact} \quad (A.6)$$

$$a_t = a_{t+1} + k_t^{-1} (f_{\ell_{t+1}}) \quad \text{exact} \quad (A.5)$$

$$f_{al}(t) = \frac{1}{1-\alpha\beta} a_{t+1} + b(I_{nx} - \alpha\beta h_x)^{-1} s_t, \quad f_{bl}(t) = \frac{1}{1-\alpha\beta} a_{t+1} + b(I_{nx} - \beta h_x)^{-1} s_t$$

exact (A.8)

→ i.e. I take the expectations formation as given.

If  $i_t = \{i_t\}$  (A.3), then (A.9) determines  $\{x_t\}$  exactly.

→ (A.9) can only be a residual eq if  $\{a_t\}$  is also input

Suppose  $\{x_t, i_t\}$  input. (A.9) has a res<sub>AG</sub>. Then (A.10) determines  $\{a_t\}$  exactly. → (A.10) can only be a residual eq if  $\{f_{al}\}$  is also input.

Let's try to rephrase this:

We have 2 equations in the 3 unknowns  $x_+, \bar{x}_+, i_+$

$$x_+ = \beta i_+ + [2 \ 1 - \beta - 2\beta] f_\beta + \beta [1 \ 0 \ 0] (I_{nx} - \beta h_x)^{-1} s_+ \quad (\text{A.9})$$

$$\bar{x}_+ = K x_+ + [(1-\alpha)\beta, \kappa\alpha\beta, 0] f_\alpha + [0, 0, 1] (I_{nx} - \alpha\beta h_x)^{-1} s_+^U \quad (\text{A.10})$$

If we set

(A.3)  $i_+ = TR$ , we have a  $3 \times 3$  system.

If we set

(A.3)  $i_+ = \{i_+\}$ , we have a  $3 \times 3$  system,  
but it might (and does) explode.

If we set

(A.3)  $i_+ = \{i_+\}$  AND a new equation

$x_+ = \{x_+\}$ , then we have  $4 \times 3$   
and we need to add  $\text{res}_{A9} \rightarrow$  (A.9) a residual eq.

If we set

(A.3)  $i_+ = \{i_+\}$  AND  $x_+ = \{x_+\}$  and  $\bar{x}_+ = \{\bar{x}_+\}$ ,

then we have a  $5 \times 3$ , therefore we  
need to add  $\text{res}_{A9}$  and  $\text{res}_{A10} \rightarrow$  (A.9), (A.10)  
residual eqs.

the thing is, that I don't think I can treat (A.9) as a residual eq unless I interpret as  $\{x_t\}$  exogenous b/c then the system is undetermined:

$$(A.9) \quad x_t + \text{res}_{A.9}(t) = -\gamma u_t + \overset{x_{\text{given}}}{\text{given}}$$

$\curvearrowleft ? \curvearrowright$

What confuses me though is that according to this analysis, when I specify  $i_t = \{i_t^R\}$  or  $i_t = \{i_t\}$ , then all equations should be exact b/c otherwise I'm undetermined! So things aren't blowing up b/c I'm overdetermined, unlike what I suspected before. Nonetheless I feel that adding residuals gives additional "wiggle-room".

Believe it or not but it seems to work. I think I am finally doing it correctly this time (AGATO.m allows residuals to form depending on selection) and simulations aren't

blowing up in my face; I can almost perfectly replicate the Taylor-rule simulated series too.  
May!

Ok - but then I need to add the TC as a new model equation. That means that I now have

$$x_t = -\gamma f_t + \text{stuff} \quad (\text{A.9})$$

$$\pi_t = K x_t + \text{stuff} \quad (\text{A.10})$$

$$\pi_t = x_t, x_{t+1}, \dots + \text{stuff} \quad (\text{TC})$$

Let's simplify first and use the RE distortion  $\bar{\pi}_t$ ,

$$\bar{\pi}_t = -\frac{\lambda}{K} x_t \quad (\text{RE-TC})$$

Maybe it would suffice to input  $\{f_t\}$ , let (A.9) & (A.10) be exact (determine  $\{x_t\}, \{\bar{\pi}_t\}$ ) and have (RE-TC) be the residual eq, with  $\text{res}_{\text{TC}} = -\bar{\pi}_t - \frac{\lambda}{K} x_t$ .

Yo - just noticed that function just stops prematurely!

15 000 (instead of default, 3000)

→ It needs a bunch of function evaluations,  
ESPECIALLY if you start at the TR - sequences,  
I guess b/c it's harder to move the loss fun.

What I don't get is that if I add a new model  
equation (the RE-TC) with a residual attached  
to it, I get pretty much the same thing.  
→ similar sequences and similar loss  
except having TC as a residual eq makes  
inputting  $\{i_4\}$  not as explosive. But I think it's  
pretty much exploding, it just takes longer.

↳ It makes sense though that it should explode

$$x_+ = -\alpha i_+ \quad \leftarrow \text{stuff}$$

$$i_4 = -k(x_+)$$

$$\text{res}_{TC} = -i_4 - \frac{\partial x}{\partial k}(x_+) \quad \Rightarrow \text{there isn't enough wiggle-room.}$$

One reason there isn't really a difference I guess is b/c the TC just adds another restriction on the sequences, but to get done enough, they need to move in the same direction along the odd dimensions.

4 April 2020

Let me repeat my imposition of the anchoring TC in words:

- 1) During the simulation, for each  $t$ :  
calculate  $\pi_t, x_t, i_t$  given model eqs (w/o TC)  
and given exog. sequences. Where specified,  
split out residuals.  $\Rightarrow \{\text{res}_t\}_{t=1}^T$  given sim
- 2) After the simulation, compute resids to anch TC  
 $\Rightarrow \{\text{res anch TC}_t\}_{t=1}^{T-H}$ , add these to  $\{\text{res}_t\}_{t=1}^T$
- 3) Choose sequences to min max  $\|\{\text{res}_t\}_{t=1}^{T-H}\|$   
In my favorite spec., I chose  $\{\pi_t, x_t, i_t\}_{t=1}^T$ , and res consists of res AG, M10 and TC.

As I note in materials 24, questions/notes, point 1:  
I'm evaluating the CB's expectation may by simply  
imputing future values in the simulation. This is  
saying that conditional on  $\{s_{t+j}\}_{j=1}^T$ , this is  
the CB's expected  $x_{t+j}$  etc. But the CB doesn't  
have perfect foresight concerning  $s_{t+j}$ .

Instead, it thinks  $E_t s_{t+j} = h x^{j-1} s_t$ . So replace  
in target and all  $s_{t+j}$  w/  $h x^{j-1} s_t$ .

- Need to rerun & save figures for anchoring TC.
- Wanna do some interpretation. (except  $\pi_1, x_1, \dots$ )
- Then turn to VFI & spline.

Note: we still have the issue that the solver stops  
prematurely: loss is on the order of  $e+00$ .

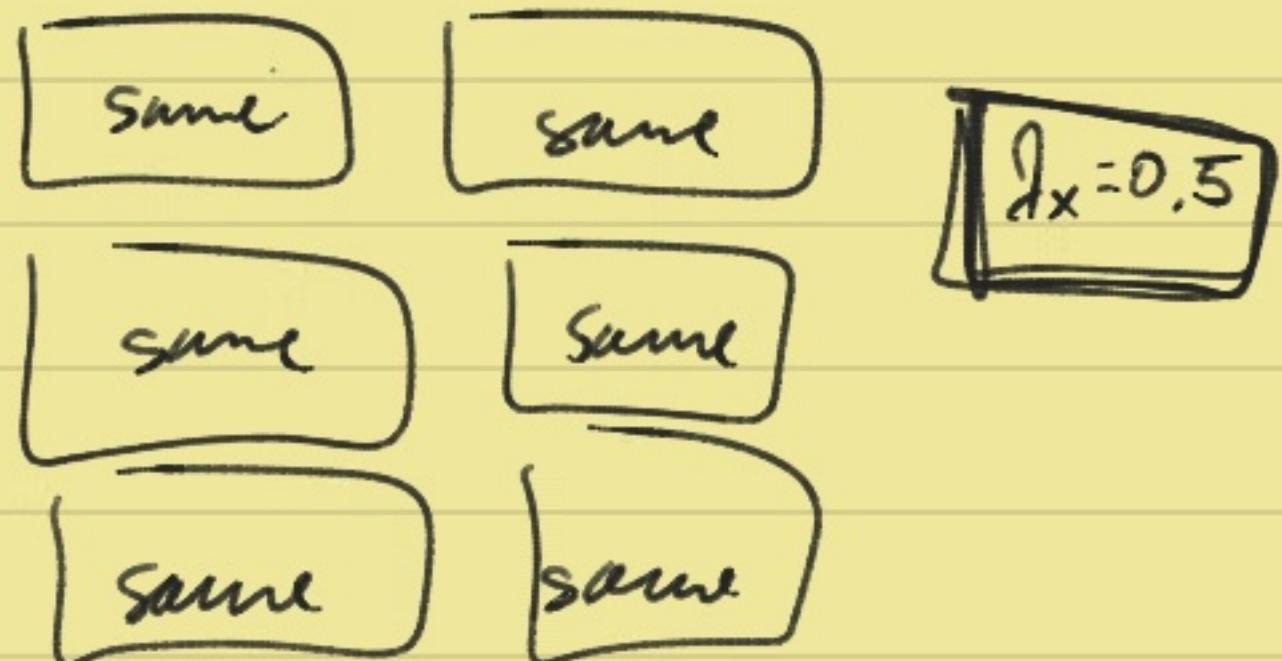
## Interpretation:

### ① General things:

- 1.1) Initializing at  $\{i_t\}^{\text{TR}}$  gives you sequences that resemble the TR, but often  $i$  is more volatile (makes sense b/c in a TR econ, knowledge of the TR helps to stabilize)
- 1.2) Initializing away from  $\{i_t\}^{\text{TR}}$  often gives you less volatile series overall
- 1.3) In general, it's better to optimize over more sequences than fewer b/c you have more margins of adjustment (d.o.f.). This is why under (RE-)TC, when inputting only  $\{i_t\}$ , you no longer explode but you face more than if you'd input  $\{x_t\}$  or  $\{\pi_t, x_t\}$  too.

Contrast no-TC w/ RE-TC & anch-TC. Only makes sense if  $\lambda_x \neq 0$ .

No TC



RE-TC

$\approx 63$

local min  
 $e+01$

local min  
 $e+02$

$\approx 240$

$\approx 15$

Premat  
 $e+01$

local min  
 $e+01$

$\approx 25$

1.4

Premat  
 $e+00$

local min  
 $e+00$

2.8

Anti TC

Premat  
 $e+08$

Local min  
 $e+03$

$\approx 60$

Premat  
 $e+01$

Premat  
 $e+01$

$\approx 13$

$\approx 60$

Premat  
 $e+01$

Premat  
 $e+01$

$\approx 80$

TR

Rand seq.

## RE-TC vs. no-TC

↳ if it can, it tries to stabilize  $x_+$  more  
b/c when  $\lambda_x > 0$ , for any  $x$ -deviation,  $\pi_t$   
needs to deviate  $\frac{\lambda_x}{K} > 1$ .

→ if you can move  $\pi_t$  on its own you make  
use of it to clamp down on  $x_+$

## anch-TC vs. RE-TC

↳ the same  $\{i_t, x_t\}$  leads to much more volatility  
 $x_+$  b/c the TC bites you more.

↳  $\frac{\lambda_x}{K}$  gets applied "more times"

⇒ anch-TC behaves like a much stronger  
version of RE-TC

⇒ If they can, TCs make  $\{i_t\}$  more less: an analogy  
of a less-aggressive TR.

⇒ the TR outcome often converges, even when not initial.  
→ the TR seems to do not too badly!

## VFI

5 April 2020

Note: Ryan did keep saying 'fsolve it' (and not 'fixme') and now I find in PS6 something that looks like VFI and he's fsolve-ing. Hm.

⇒ this may not be VFI / PVI. In fact, it's closer to what I'm doing: take some approx  $c_t$ s, compute residuals and zero them out using fsolve.

I remembered there was a comment in Peter's notes on VFI. I'm not sure if the following is it, but maybe: (p. 70, own-notes.pdf)

$$V(k_t, t) = \max_{c_t} u(c_t) + \beta V(k_{t+1}, t+1)$$

Easiest way to solve is guess & verify.

If you can't fully guess, you can take an initial guess for  $V_{t+1}$ . Solve for  $r_t = \text{stuff} + f(V_{t+1})$ , and plug  $V_t$ , rolled forward, as a 2<sup>nd</sup> guess for  $V_{t+1}$ . After  $\times$  iterations, it will converge.

Yeah baby! p. 73-76 convinces that it's VFI!

Interesting things Peter goes on to say:

If you don't know that the guess is true, use the iterative algorithm:

1) Start w/ a simple guess  $\rightarrow$  plug in RHS

2) get LHS  $\rightarrow$  plug into RHS as a 2<sup>nd</sup> guess

... iterate, converges!

Suppose for the problem

$$V(k_t; t) = \max_{c_t} \ln(c_t) + \beta V(k_{t+1}, t+1)$$

we start w/ the extremely simple guess

1)  $V_0(k_t) = 0$  | Plug in RHS

2)  $V_1(k_t) = \max_{c_t} \ln(c_t)$  s.t.  $k_t^\alpha \geq c_t + k_{t+1}$   
 $= \alpha \ln k_t$  (the updated guess)

You can see it converging to the functional form  
as ln appears.

3)  $V_2(k_t) = \max_{c_t} \ln(c_t) + \alpha \beta \ln k_{t+1}$

✓ since  $U_{t+1}(k_{t+1})$

The sol to this gives

$$V_2(k_t) = \ln\left(\frac{1}{1-\alpha\beta}\right) + \alpha \ln k_t + \alpha\beta \ln\left(\frac{\alpha\beta}{1-\alpha\beta}\right) + \alpha^2\beta \ln k_t \\ = E_2 + (\alpha + \alpha^2\beta) \ln k_t$$

4) Update once more

$$V_3(k_t) = \max_{c_t} \ln(c_t) + \beta E_2 + (\alpha\beta + (\alpha\beta)^2) \ln(k_{t+1})$$

$$\Rightarrow c_t = \frac{1}{1-\alpha\beta+(\alpha\beta)^2} k_t^\alpha$$

→ See the pattern emerging?

$$\text{Iter 1: } c_t = \frac{1}{1} k_t^\alpha$$

$$\text{Iter 2: } c_t = \frac{1}{1-\alpha\beta} k_t^\alpha$$

$$\text{Iter 3: } c_t = \frac{1}{1-\alpha\beta+(\alpha\beta)^2} k_t^\alpha$$

$$\dots c_t = \frac{1}{1-\alpha\beta} k_t^\alpha$$

Repeating enough times makes the value function converge. VFI isn't as useful here in this ex.

b/c the value function has a closed-form sol  
It's more useful when a closed-form doesn't  
exist and yet you can still recover a list of  
numbers which converges to a value.

It looks like Stokey, Lucas, Prescott book shows  
that  $\exists$  a unique fixed point to which this  
converges, i.e. convergence is guaranteed.

Judd's book p. 411 (Sect. III, Chap. 12.3)  
also has a section on VFI.

The VFI algorithm is presented on p. 413.

## Value function iteration algorithm, p. 413.

Objective: solve the Bellman eq. (12.3.4)

$$V_i = \max_u \left[ \pi(x_i, u) + \beta \sum_{j=1}^n q_{ij}(u) V_j \right] \quad i = 1, \dots, n \quad (12.3.4)$$

$V^*$  := the solution. the optimal policy  $u^*$  satisfies

$$u_i^* = \arg \max_u \left[ \pi(x_i, u) + \beta \sum_j q_{ij}(u) V_j \right] \quad i = 1, \dots, n \\ =: Q_l V^+ \quad (12.3.8) \quad (12.3.6)$$

Initialization. Initial guess  $V^0$ , choose stopping criterion  $\text{crit} > 0$ .

Step 1. For  $i = 1, \dots, n$ , compute ("for all gridpoints...")

$$V_i^{l+1} = \max_{u \in D} \pi(x_i, u) + \beta \sum_{j=1}^n q_{ij}(u) V_j^l \quad \text{expected transition} \\ \text{see (12.3.11)}$$

Step 2. If  $\|V^{l+1} - V^l\| < \text{crit} (1-\beta)$ , go to step 3.

Else, go back to Step 4.

Step 3. Compute the final solution, setting

$$u^* = Q_l \cdot V^{l+1} \quad (\text{RHS of BE} = \text{policy function})$$

$$P_i^* = \pi(x_i, u_i^*) \quad i = 1, \dots, n$$

$$V^* = (I - \beta Q^{u^*}) P^* \quad \text{STOP.}$$

where:

$\mathcal{Q}^l$  := the mapping from tomorrow's value function  
to optimal policy.  $u^{l+1} = \mathcal{Q}^l V^l$  (12.3.8)

$T$  := the mapping from tomorrow's value function  
to today's:  $V^{l+1} = T V^l$  (12.3.9)

$$l = 0, 1, 2, \dots$$

(I think there's some notational inconsistency here:

$$\text{B } u^l = \mathcal{Q}^l \cdot V^{l+1} \text{ or } u^{l+1} = \mathcal{Q}^l \cdot V^l ?$$

$$\text{Same w/ } V^l = T V^{l+1} \text{ or } V^{l+1} = T V^l$$

→ seems like  $\mathcal{Q}^l$  and  $T$  sometimes denote their own  
inverses.)

p. 409

$X_i$  := the set  $x_i, i=1, \dots, n$  of  $n$  states (= gridpoints?)

$u$  := control

$q_{ij}^t(u) := \text{prob}[x_j | x_i, u]$  → transition prob to  $x_j$

given state  $x_i$  and control  $u$

$Q^t(u) := \text{prob distib of state at } t+1, \text{ if the time } t \text{ state is } x_i$   
and the control is  $u$ .

Define  $V_i^t := V(x_i, t)$

and I use (12.3.4)

$$V_i = \max_u \left[ \pi(x_i, u) + \beta \sum_{j=1}^n q_{ij}(u) V_j \right] \quad i = 1, \dots, n$$

evaluated at policy  $\pi$  forever:

$$V^\pi = \rho^\pi + \beta Q^\pi V^\pi$$

↑ seems to be something like the payoff  
function given state & policy ( $\rightarrow$  utility fn)

$$\Rightarrow V^\pi = (1 - \beta Q^\pi)^{-1} \rho^\pi \quad (12.3.5)$$

See also Eric Sims' great VFT notes, called  
valfun-iter-eric-sims.pdf !

And I should fix the stuff I did b/c it solves  
a problem specified by  $F(x) = 0$ , for  $x$ .

## Value function iteration algorithm

6 April 2020

based on Eric Sims and in words:

% Create the grid, gridmat, somehow.

$V = \text{zeros}(\text{ngrid}, 1)$  % initial guess

tol = 0.01; maxiter = 300;

while dif > tol & iter < maxiter

for i = 1 : ngrid

$x = \text{gridmat}(i, 1)$  ( $x = \text{states at } t$ )

$x_p(i, :) = \text{fminbnd}(@(\text{valfun\_t}, k_{\min}, k_{\max})$

$V_p(i, 1) = -\text{valfun\_t}(x_p)$

$\uparrow V_{t+1} = V_t(x_{t+1})$  (-1 b/c we did a min)

end

$\text{dif} = \text{norm}(V_p - V)$

$V = V_p$

$\text{iter} = \text{iter} + 1$

end

How to implement VFI in my context?

I feel I can't just take eqs. (A1)-(A7), and  $\pi$  (B1) b/c that doesn't fulfill a notion of optimality.

I feel like I need to take the CB's loss function and use that (E1) as the value function, sthg like this

$$V(x_+, t) = \max_{\{\bar{a}_+, l_+, \bar{\pi}_+, k_+^{-1}\}} (\bar{a}_+^2 + \lambda x_+^2) + \beta E_+ V(x_{++1}, t+1)$$

s.t. model equations

And what I think I can do is I think I can sort out  $x, f_a(t), f_b(t)$ , so have

$$\{\bar{a}_+, l_+, \bar{\pi}_+, k_+^{-1}\} = \operatorname{argmax} V(x_+, t)$$

so we only have 2 states,  $x_+ = \{\bar{\pi}_+, k_+^{-1}\}$ .

But for VFI unfortunately the exog states count too...

## Policy function iteration Algorithm

Judd, p. 416

Initialization: Choose stopping criterion,  $\text{cnt} > 0$ .

Either: initial guess  $V^0$  and go to step 1,  
or: initial guess  $U^0$  and go to step 2.

Step 1:  $U^{l+1} = Q_U V^l$

Step 2:  $P_i^{l+1} = \pi(x_i, U_i^{l+1}) \quad i = 1, \dots, n$

Step 3:  $V^{l+1} = (I - \beta Q U^{l+1})^{-1} P^{l+1}$

Step 4: If  $\|V^{l+1} - V^l\| < \text{cnt}$ , stop, else go to Step 1.

Obs 1: Start by guessing  $V$  or  $U$  depending on where your guess is better  $\rightarrow$  helps speeding up!

Obs 2: Note how in step 3, PFI assumes the most recent policy will be continued forever. This is allows it to compute an associated, new value function w/o solving the costly maximization step.

Obs 3: The costly step now is Step 3, bc we invert.  
 $\hookrightarrow$  This can be omitted in "modified policy iteration".

## Modified policy function iteration Judd, p. 417

Replace step 3 by the following steps:

$$3.1: w^0 = V^k$$

$$3.2: w^{j+1} = \rho^{l+1} + \beta Q^{k^{l+1}} w^j, \quad j=0, \dots, k$$

$$3.3: V^{l+1} = W^{k+1} \quad (12.4.2)$$

Policy iteration is the limiting case of modified policy iteration as  $k \rightarrow \infty$ .

What I wanna do today is work thru 7 April 2020

Peter's VPI Example here and on Matlab. (matlab524)

Social planner's problem:

Peter, own-notes, p. 68

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{s.t. to given and}$$

$$\{k_t\}_{t=1}^{\infty} \quad \text{RC: } k_t^\alpha - c_t \geq k_{t+1} \quad \forall t = 0, 1, 2, \dots \quad (\text{com})$$

Set up the Bellman eq:

$$V(k_t; t) = \max_{c_t} \ln(c_t) + \beta \left( \underbrace{k_t^\alpha - c_t}_{\text{time-invariant}}; t+1 \right) \quad (BE)$$

$$\text{Green: } V(k_t; t) = v(k_t) = E + F \ln(k_t) \quad (\text{Play rule (B)})$$

$$V(k_+, t) = \max_{c_+} u(c_+) + \beta \tilde{E}(k_+^\alpha - c_+, t+1) \quad (\text{BE})$$

Given:  $V(k_+, t) = v(k_+) + E + F \ln(k_+)$

$$\rightarrow E + F \ln(k_+) = \max_{c_+} u(c_+) + \beta E + \beta F \ln(k_{++1})$$

$$E + F \ln(k_+) = \max_{c_+} u(c_+) + \beta E + \beta F \ln(k_+^\alpha - c_+)$$

FOC:

$$(c_+) \quad 0 = \frac{1}{c_+} + \beta F \frac{1}{k_+^\alpha - c_+} (-1) \quad (\text{FOC})$$

ENV:

$$(k_+) \quad F \frac{1}{k_+} = \beta F \frac{1}{k_+^\alpha - c_+} \propto k_+^{\alpha-1} \quad (\text{ENV})$$

(BE), (FOC), (ENV), and (LOM)  $\rightarrow$  4 eqs in 4 unknowns.

$$\text{solve (FOC) for } c_+ \rightarrow \frac{1}{c_+} = \beta F \frac{1}{k_+^\alpha - c_+} \Leftrightarrow k_+^\alpha - c_+ = \beta F c_+$$

$$\rightarrow k_+^\alpha = (\beta F + 1) c_+ \rightarrow c_+ = \underbrace{\frac{1}{1+\beta F} k_+^\alpha}_{(1)} \quad (1)$$

$$\text{Sub (1) into ENV: } F = \beta F \frac{1}{k_+^\alpha - \frac{1}{1+\beta F} k_+^\alpha} \propto k_+^\alpha$$

$$\rightarrow F = \beta F \frac{1}{\left(1 - \frac{1}{1+\beta F}\right) k_+^\alpha} \propto k_+^\alpha$$

$$1 = \beta \frac{1}{\left(1 - \frac{1}{1+\beta F}\right)^\alpha} \Leftrightarrow \frac{1+\beta F-1}{1+\beta F} = \alpha \beta$$

$$\frac{\beta F}{1+\beta F} = \alpha F \rightarrow F = \alpha(1+\beta F) = \alpha + \alpha \beta F$$

$$\rightarrow F(1-\alpha\beta) = \alpha \Rightarrow F = \frac{\alpha}{1-\alpha\beta} \quad (13)$$

Sub  $F$  back into (11):  $c_t = \frac{1}{1+\beta F} k_t^\alpha$

$$\rightarrow c_t = \frac{1}{1 + \frac{\alpha\beta}{1-\alpha\beta}} k_t^\alpha = \frac{1}{\frac{1-\alpha\beta+\alpha\beta}{1-\alpha\beta}} k_t^\alpha$$

$$\underline{c_t = (1-\alpha\beta) k_t^\alpha} \quad (14)$$

Sub this into constraint (12m):

$$k_{t+1} = k_t^\alpha - c_t = k_t^\alpha - (1-\alpha\beta) k_t^\alpha$$

$$\underline{k_{t+1} = \alpha\beta k_t^\alpha} \quad (15)$$

Now need to answer: which of these is the policy function?  
And what's the other?

Still need to solve for  $E$ :

Sub (13), (4), (15) into (BE):

$\uparrow \text{sol } F \quad \uparrow \text{sol } C \quad \uparrow \text{sol } k_{t+1}$

$$V(k_t, t) = \max_{C_t} \ln(C_t) + \beta V(k_{t+1}, t+1)$$

$$E + F \ln(k_t) = \ln[(1-\alpha\beta)k_t^\alpha] + \beta E + \beta F \ln[\alpha\beta k_t^\alpha]$$

$$= E + F \ln(k_t) = \ln(1-\alpha\beta) + \alpha \ln(k_t) + \beta E + \beta F \ln(\alpha\beta) + \alpha\beta F L(k_t)$$

$$\Theta (1-\beta)E = \ln(1-\alpha\beta) + \beta F \ln(\alpha\beta) + (\underbrace{\alpha + \alpha\beta F - F}_{= \alpha + (\alpha\beta - 1)F}) \ln(k_t)$$

$$= \alpha - \frac{(1-\alpha\beta)\alpha}{1-\alpha\beta} = 0$$

$$\hookrightarrow E = \frac{1}{1-\beta} \left\{ \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) \right\}$$

Okay, so I've characterized the sol to the deterministic optimal growth model using dynamic programming (value fn, Bellman eq.) using guess & verify.  
Now let's turn to the VFI formulation!

Again we face the same problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{s.t. } k_0 \text{ given and}$$

$$RC: k_t^\alpha - c_t \geq k_{t+1} \quad \forall t = 0, 1, 2, \dots \quad (\text{com})$$

Set up the Bellmann eq:

$$V(k_t; t) = \max_{c_t} \ln(c_t) + \beta \underbrace{k_{t+1}}_{(BE)}$$

Let  $V^l$  denote the iteration index,  $l = 0, 1, 2, \dots$

Step 0. Initial guess:  $V_0(k_t) = 0$

Structure is

$$\underbrace{V^{l+1}(k_t)}_{\text{update}} = \max_{c_t} \ln(c_t) + \beta \underbrace{V^l(k_{t+1})}_{\text{current guess} = \text{RHS}}$$

At every step subject to the constraints!  
so here

$$V'(k_t) = \max_{c_t} \ln(c_t) \quad \text{s.t. } k_t^\alpha \geq c_t + k_{t+1}$$

PTO for sol.

$$V'(k_t) = \ln(k_t^\alpha)$$

$$U = \ln(c_t) + \beta \mathbb{E}_t \left( k_t^\alpha - c_{t+1} - k_{t+1} \right)$$

$$c_t: \frac{1}{c_t} = \lambda_t$$

$$k_{t+1}: -\lambda_t + \beta E_t \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} = 0$$

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} \alpha k_{t+1}^{\alpha-1}$$

I think the point is that we're somehow not looking at an intertemporal budget now; setting  $V(k_{t+1})=0$  kind of says that we don't value future  $k$ 's

and so  $\max_{c_t} \ln(c_t)$  s.t.  $k_t^\alpha \geq c_t + k_{t+1}$

has a maximum where  $k_{t+1}=0$ , so the constraint determines  $c_t = k_t^\alpha$

$$\text{Step 1. } V'(k_t) = \ln(k_t^\alpha) - \alpha \ln(k_t)$$

$$V(k_t, t) = \max_{c_t} \ln(c_t) + \beta V(k_{t+1}, t+1)$$

$$\Rightarrow V^2(k_t) = \max_{c_t} \ln(c_t) + \beta \alpha \ln(k_{t+1})$$

s.t.  $k_t^\alpha \geq c_t + k_{t+1}$

Now  $k_{t+1}$  now shows up, so can plug it.

$$V^2(k_t) = \max_{c_t} \ln(c_t) + \alpha \beta \ln(k_t^\alpha - c_t)$$

$$V^2(k_t) = \max_{c_t} \ln(c_t) + \alpha\beta \ln(k_t^\alpha - c_t)$$

$$\Rightarrow FOC : \frac{1}{c_t} + \alpha\beta \frac{1}{k_t^\alpha - c_t} (-1) = 0$$

$$\Rightarrow \alpha\beta c_t = k_t^\alpha - c_t$$

$$c_t = \frac{1}{1+\alpha\beta} k_t^\alpha \quad | \text{ think policy fn.}$$

$$\text{The constraint implies } k_{t+1} = k_t^\alpha - c_t$$

$$\Rightarrow k_{t+1} = \left(1 - \frac{1}{1+\alpha\beta}\right) k_t^\alpha$$

$$k_{t+1} = \frac{\alpha\beta}{1+\alpha\beta} k_t^\alpha \quad \begin{matrix} \text{What's this called?} \\ \text{transition? State pol'y?} \end{matrix}$$

$$V^2(k_t) = \max_{c_t} \ln(c_t) + \alpha\beta \ln(k_{t+1})$$

$$= \ln\left(\frac{1}{1+\alpha\beta} k_t^\alpha\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta} k_t^\alpha\right)$$

$$V^2(k_t) = \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha \ln(k_t) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha^2\beta \ln(k_t)$$

$$V^2(k_t) = \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + (\alpha + \alpha^2\beta) \ln(k_t)$$

Step 2.

$$V^3(k_t) = \max_{c_t} \ln(c_t) + \beta V^2(k_{t+1})$$

$$\text{s.t. } k_t^\alpha \geq c_t + k_{t+1}$$

$$V^2(k_t) = \underbrace{\ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1-\alpha\beta}\right)}_{= E} + (\alpha + \alpha^2\beta) \ln(k_t)$$

(hasn't converged yet)

$$\rightarrow V^3(k_t) = \max_{c_t} \ln(c_t) + \beta [E + (\alpha + \alpha^2\beta) \ln(k_{t+1})]$$

$$= \max_{c_t} \ln(c_t) + \beta E + (\alpha\beta + \alpha^2\beta^2) \ln(k_t^\alpha - c_t)$$

FOC:

$$\frac{1}{c_t} = \frac{\alpha\beta + \alpha^2\beta^2}{k_t^\alpha - c_t} \Rightarrow k_t^\alpha - c_t = (\alpha\beta + \alpha^2\beta^2) c_t$$

$$(1 + \alpha\beta + \alpha^2\beta^2) c_t = k_t^\alpha$$

$$c_t = \frac{1}{1 + \alpha\beta + \alpha^2\beta^2} k_t^\alpha$$

the policy fct is  
converging.

$$V^{l+1} = p^l + \beta V^l \quad \text{"value } l+1 = \text{payoff}(l) + \beta V^l \text{"}$$

$$u^l(x) = \operatorname{argmax} (p^l + \beta V^l) \quad x = \text{state}$$

policy fct = control as fct of state as sol to  $\max p + \beta V^l$

Ah! In a deterministic setting,  $Q = I$

so  $V^{l+1} = P^l + \beta V^l$

which when it converges indeed gives

$$V = P + \beta V$$

$$V = (1 - \beta)^{-1} P$$

My feeling is also that w/o stochasticity, we may not need a grid (the index  $i$  in Judd's notation becomes redundant) b/c there's no expectation to evaluate.

$P$  is just the payoff function as a function of the control, in Peter's case  $\ln(c_t)$ .

→ so the question that remains is how can I solve the maximization step numerically in each iter?

In each iter, I need to max  $p(c_t) + \beta V(k_{t+1})$

$p(c_t) = \ln(c_t)$ ,  $V(k_{t+1})$  I just plug, I guess.

Peter meeting

7 April 2020

materials 24 - peter

Instead of AR(1) shocks : suppose both iid,  
and each can take on only "high" or "low"  
→ each shock is one of 4 values  
complete history of shocks could be summarized  
by  $4^T$   
↳ discretizity is the easiest step  
the 2nd easiest step is a Markov chain  
w/ finite possible values.  
→ Tauchen does that.

Say that it's a challenge to find  $\{t_t\}$  s.t.  
all eqs hold  $\Rightarrow$  optimise over  $\{x_t, t_t\}$   
and use  $\{t_t^*\}$  as an initial point for the real  
optimization.

The problem is the same as in the std theory of optimization: trying to find an  $\alpha$ -dimensional object  $\rightarrow$  not possible to represent on computer  
 $\Rightarrow$  This is why ppl discretize

Can do the following:

- 1) Simulate the model  $N$  times using a Taylor-rule.  
 $\rightarrow (\gamma_{\pi}, \gamma_x) = \text{argmin residual}(\pi)$   
 these are Taylor-rule coeffs that get us close to implementing the TC as possible (con-sectionally, NOT conditional on a shock seq.  $\{s_t\}$ ). Treat the TR as a 1<sup>st</sup> order Taylor-approx\* of optimal, nonlinear rule.
- 2) Simulate the model  $N$  times using  $(\gamma_{\pi}^*, \gamma_x^*)$  and the 2<sup>nd</sup> order terms  $\bar{\pi}_t^2, x_t^2, x_t \bar{\pi}_t$ , with coefficients  $\gamma_{\pi^2}, \gamma_{x^2}, \gamma_{\pi x}$   
 $\rightarrow \gamma_i = \text{argmin residual}(\pi_i) \quad i = \pi, x, \pi^2, x^2, \pi x.$

→ choose the 5 params of that quadratic. The TC would still not hold, so repeat w/ 3<sup>rd</sup> order terms, etc.

⇒ this would characterize a Taylor-approx to the opt nonlinear int-rate rule.

→ But  $x_{t+1}/x_t$ , etc might be in there. But you could do the same thing adding those terms.

\* Another approx is Fourier of sine & cosine; the approx doesn't need to be a Taylor-approx. This would be more numerically stable than mine, which is important!

→ other approaches may be more numerically stable in the first place, so might prefer those.

Ryan might have meant: w/o a target and treat (A3) as residual, given a  $\{f_t\}$ , and miss residual in (TR) !

Or, keeping (TR) as exactly fulfilled, treat (A3) as a residual.

↳ this would be the check that it works.

The main point is that my issue is that it's an uncountably co-dim problem.  $\rightarrow$  Need to simplify somewhere

- $\rightarrow$  treating the rule as an approx (Taylor-, Fourier - or whatever) "continuous shocks (simulated cross-section)" is one way to do this
- $\rightarrow$  treating the rule as exact but the shocks as an approx (discretization) is the other way.

The last thing is numerical stability: it's ok if a procedure is slow if we know it will (slowly but surely) deliver a result. But if it only delivers sometimes, and unreliable stuff, then we waste running time. So need to keep that in mind!