Restricted Perceptions Equilibria and Learning in Macroeconomics *

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Preliminary

1 Introduction

Since the 1970's Rational Expectations (RE) has become the dominant paradigm in macroeconomics. One reason for its popularity is the consistency it imposes between beliefs and outcomes. Under RE agents' subjective probability distribution coincides with the true distribution for the economy. Not surprisingly, a large literature objects to RE on the grounds that it requires agents possess unreasonable information and cognitive abilities. Instead many researchers (e.g. (Evans and Honkapohja 2001)) replace RE with an econometric forecasting model and ask under what conditions the forecasts converge to RE. The validity of RE is not just theoretical curiosum, (Branch 2004) and (Carroll 2003) demonstrate persistent heterogeneity in survey data on inflation expectations. Such phenomena can not be explained by RE models. Papers such as (Brock and Hommes 1997) generate heterogeneity by assuming agents make conscious choices between costly predictor functions, thereby, hypothesizing that deviations from rationality come from a weighing of benefits and costs.

These drawbacks, though, do not imply that there are no valid insights from the RE approach. In Muth's original formulation of the rational expectations hypothesis he advocated for subjective expectations which coincide with the true distribution. His argument rested on the joint determination of beliefs and the economy. This is

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the self-referential feature of most dynamic models: the economy depends on expectations which *should* depend on the structure of the economy. If agents' subjective beliefs did not take account of the economic structure then their forecasts would consistently perform badly. The great insight of the RE approach is that it hypothesizes an equilibrium to this self-referential process. Economic outcomes should not completely contradict the beliefs which generated these outcomes. If agents form their expectations by conditioning on the distribution (itself a function of those beliefs) then beliefs and outcomes are mutually consistent. The Rational Expectations Equilibrium is at the very heart of competitive or Walrasian equilibria.[1]

However, conditional expectations, or RE, are not the only form of beliefs which may lead to a fixed point of this self-referential property. A burgeoning literature argues that beliefs which satisfy a least squares orthogonality condition are also consistent with Muth's original hypothesis.[2] The least squares orthogonality condition in these models imposes that beliefs generate forecast errors which are orthogonal to an agent's forecasting model; that is, there is no discernible correlation between these forecast errors and an agent's model. Perhaps one should imagine agents' actions taking place in an economy but their beliefs (which generate the actions) exist in a perceived economy. Under this interpretation, the orthogonality condition guarantees that agents perceive their beliefs as consistent with the real world. Thus, agents can have misspecified (i.e. not RE) beliefs but within the context of their forecasting model they are unable to detect their misspecification.[3] An equilibrium between optimally misspecified beliefs and the stochastic process for the economy is called a *Restricted Perceptions Equilibrium* (RPE).[4]

This survey advocates for Restricted Perceptions Equilibria. We argue that RPE is a natural alternative to REE because it is consistent with Muth's original hypothesis and it allows for bounded rationality. One way this article advocates for RPE is by demonstrating the generality of RPE as it encompasses many forms of misspecified beliefs. We develop most of our arguments in the context of a linear stochastic self-referential economy driven by a VAR process in their forecasting models. Misspecification in this setting implies agents underparameterize by omitting variables and/or lags of the VAR process in their forecasting model. To demonstrate generality we highlight the connection between the Self-Confirming Equilibrium of (Sargent 1999) and RPE. We further demonstrate that the Consistent Expectations Equilibrium in (Hommes and Sorger 1998), where agents have linear beliefs in a non-linear model, is also an RPE.

[1] See (Ljungvist and Sargent 2000) and (Colander 1996).

[2] See, for example, (Anderson and Sonnenschein 1985), (Evans, Honkapohja, and Sargent 1993), (Evans and Honkapohja 2001), (Hommes and Sorger 1998), (Branch and McGough 2003), (Branch and Evans 2003a), (Sargent 1999).

[3] Of course, if agents step out of their forecasting model they might notice some structure to their forecast models.

[4] The name Restricted Perceptions Equilibrium was given by (Evans and Honkapohja 2001).

Misspecification is an important part of a complex and data rich environment such as a macroeconomy. VAR forecasters purposely limit the number of variables and/or lags because of degree of freedom and computational limitations. Economists and econometricians routinely take linear approximations to non-linear relations. In fact, (White 1994) argues that all econometric models are necessarily misspecified. In applied work there is disagreement about the true structure of the economy. So we should expect agents and policymakers to take actions based on misspecified models of the economy. An RPE simply requires that outcomes do not consistently contradict agents' beliefs.

We also advocate for RPE by demonstrating its implications. We show the implications in the context of the *Misspecification Equilibrium* (ME) of (Branch and Evans 2003a, 2003b). In a Misspecification Equilibrium agents underparameterize their forecasting model but only choose the best-performing statistical models. An ME is an RPE with agents endogenously choosing their form of misspecification. We argue for this approach since the nature of the economy dictates the appropriate form of misspecification. The advantage of ME is it makes the point that misspecification and economic outcomes are mutually dependent. We survey the main finding of (Branch and Evans 2003a) that in an ME agents can have heterogeneous expectations. The survey then shows how multiple equilibria can arise as in (Branch and Evans 2003b).

Finally, we present an example which show that in a simple multivariate *ad hoc* macro model with optimal monetary policy, the basins of attraction between multiple equilibria switch. The basins of attraction are determined by the interaction of expectation formation effects and the underlying exogenous stochastic process. Optimal monetary policy takes agents' beliefs as a given and then counters the exogenous process. Policy alters the direct effect of these exogenous factors thereby causing a switch to another equilibrium.

2 Towards an Equilibrium Concept Consistent with Muth

This section surveys the concept of a Restricted Perceptions Equilibrium (RPE). We argue that the RPE is consistent with Muth's Rational Expectations Hypothesis, and, because it captures reasonable cognitive and computational limitations of agents, is a natural alternative to the Rational Expectations Equilibrium.

It will prove useful to have a common economic framework for discussion. We consider economies that have a reduced-form with the following recursive expectational structure:

$$x_t = FE^*_{t-1}x_t + \gamma' z_t \tag{1}$$

where z_t has the vector autoregressive structure (VAR),

$$z_t = Az_{t-1} + \varepsilon_t$$

The operator E^*_{t-1} is an expectations operator; the superscript $*$ highlights that expectations may not be rational. Under rational expectations $E^* = E$, where E_{t-1} denotes the mathematical expectation conditioned on all available information through time $t-1$. We assume that z_t is a stationary VAR process and we will restrict attention to stationary solutions to (1).[5] In this paper we consider where x_t is both uni- and bivariate. There are many models in dynamic macroeconomics that have a reduced-form like (1). Examples include the cobweb model, the Lucas model and the Sargent-Wallace *ad hoc* model. We abstract from specific formulations at first to illustrate the generality of RPE. Below, we highlight implications in specific economic applications.

The expectations formation effect of an economy is parameterized by F. The parameter F governs the self-referential feature because it determines the effect expectations have on the state. We often differentiate between positive and negative feedback models. A positive feedback model is one where the state moves in the same direction as expectations. Conversely, a negative feedback model has the state move in an opposite direction. The cobweb model is a univariate reduced-form system with negative feedback, i.e. $F < 0$. The negative feedback arises because there is a lag from production to the market so that firms will forecast future prices when making supply decisions. If firms expect prices to be low they produce little and equilibrium prices will actually be high: hence, the negative feedback from expectations. This particular self-referential feature generates expectations driven oscillations. In business cycle models, though, the feedback is typically positive. [6]

2.1 Muth's Rational Expectations Hypothesis

(Muth 1961) is credited with the development of the Rational Expectations Hypothesis. Later, in works by Lucas and Sargent, among others, Rational Expectations (RE) became the dominant paradigm in dynamic macroeconomics. One aspect of RE that appealed to researchers was that in a Rational Expectations Equilibrium (REE) agents' subjective beliefs coincide with the true probability distribution of the economy. The REE is a deep equilibrium concept that rests on the self-referential feature of the model. In an REE there is a sense of self-fulfilling prophesy. Agents form beliefs by conditioning on the true distribution which, itself, depends on these beliefs. The obvious objections to the RE then are the strong informational and

[5] Though, one implication of Misspecified Equilibria is that learning dynamics that surround these stationary equilibria may not be stationary.

[6] When x is bivariate, we say that there is positive (negative) feedback when F is positive (negative) semi-definite.

cognitive assumptions required. RE assumes agents know the true distribution of the economy and are able to form conditional expectations with this information. The learning literature (e.g. (Evans and Honkapohja 2001)) examine under what conditions agents can learn this distribution if it was unknown. Moreover, it strains credulity to imagine a large economy populated by agents all coordinating on the precisely correct distribution. It is to these objections, while preserving the essence of Muth's argument, which we propose overcoming with an RPE.

It is worth emphasizing the relationship between subjective beliefs and actual outcomes in an REE for it is this insight that RPE exploits while allowing for bounded rationality. This connection is made most clearly by the approach of (Evans and Honkapohja 2001). Their approach conjectures subjective beliefs and finds the conditions under which these beliefs are supported by the actual stochastic process. For simplicity assume x_t is univariate and z_t is an $(n \times 1)$ VAR(1). Suppose agents believe the economy is a linear function of the exogenous processes:

$$x_t = a + b' z_{t-1} + c\varepsilon_t \tag{2}$$

(Evans and Honkapohja 2001) refer to such an equation as a Perceived Law of Motion (PLM) because it summarizes agents' subjective distribution. Agents take conditional expectations of the PLM:

$$E_{t-1} x_t = a + b' z_{t-1} \tag{3}$$

Given these beliefs, the actual law of motion (ALM) is computed by plugging (3) into (1),

$$x_t = Fa + Fb' z_{t-1} + \gamma' A z_{t-1} + \gamma' \varepsilon_t$$

The parameters $(a, b, c)'$ summarize agents' subjective beliefs. In an REE, these beliefs are reinforced so that a should coincide with Fa, b' coincide with $(Fb' + \gamma' A)$, etc. There exists a map from the space of beliefs to outcomes:

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} Fa \\ (Fb' + \gamma' A)' \\ \gamma' \end{pmatrix}$$

The connection between beliefs and outcomes is seen clearly in the ALM, suitably rewritten,

$$x_t = T((a, b, c)')' \, (1, z_{t-1}, \varepsilon_t) \tag{4}$$

Rational expectations are beliefs $(a, b, c)'$ which are consistent with the model's actual parameters of the model. Thus, an REE occurs under the fixed point $(a, b, c)' = T((a, b, c)')$. In this instance, the unique REE satisfies

$$x_t = (1 - F)^{-1} \gamma' A z_{t-1} + \gamma' \varepsilon_t$$

It is worth emphasizing that in an REE agents' expectations are unbiased. This implies an orthogonality between their forecasting model and the actual stochastic process. Throughout we exploit mappings from beliefs to outcomes in self-referential models to find alternative equilibrium concepts which preserve the consistency between the model and expectations.

The learning literature as typified by (Evans and Honkapohja 2001) studies whether agents could start with arbitrary values of (a, b, c) and learn the REE of $(0, (1 - F)^{-1}\gamma' A, \gamma')$. Evans and Honkapohja note that, in many models, the E-stability Principle determines if an REE is learnable. E-stability places a condition on how beliefs are adjusted and filtered through the T-map.

Now that we have highlighted the connection between beliefs and outcomes in an REE, we ask whether other equilibrium concepts can be consistent with Muth's original objectives while easing the strong information assumptions of RE. (Muth 1961) states his objective to be "...a theory of expectations and to show that the implications are – as a first approximation– consistent with the relevant data." (pg. 316) He goes on to phrase his hypothesis a bit more precisely as

> "...expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (or the "objective" probability distributions of outcomes)." (pg. 316)

This definition of Rational Expectations is the one that the literature has focused on: subjective expectations conditional on the true distribution of outcomes. To Muth, the only reason expectations might depart from outcomes is because of unforecastable events.

However, we argue that this theory of expectation formation is not the only definition consistent with Muth's stated hypothesis. After phrasing the REH, Muth then makes three assertions:

> "... (1) Information is scarce, and the economic system generally does not waste it. (2) The way expectations are formed depends specifically on the structure of the relevant system describing the economy. (3) A 'public prediction,' ... will have no substantial effect on the operation of the economic system (unless it is based on inside information)." (pg. 316)

Muth's hypothesis that agents' expectations are consistent with the underlying model is in the sense that: (1) agents do not (freely) dispose of useful information; (2) beliefs reflect the nature of the economy; and (3) beliefs and outcomes are directly related so that outcomes do not contradict beliefs. A 'public prediction' to Muth is a published forecast of a future event that depends on individual beliefs. The lack of an effect of a public expectation is as we state in point (3) –expectations should be based on all available information. Clearly, RE satisfies these assertions: under RE agents use all information available, the subjective distribution of beliefs coincides with the true economic distribution, and in an REE there is a correspondence between beliefs and outcomes.

We argue, though, that other expectation formation schemes are consistent with Muth's hypothesis. It is the contention of this paper that Muth's hypothesis is satisfied whenever each agent's forecast errors are uncorrelated with their forecasting model. An alternative way to think about a self-referential economy is that agents only understand the economy so far as their own subjective model; it is through this subjective model that they pass actual observations. Agents and their actions exist in economic models but their beliefs reside in forecasting models. Under Muth's assertion (2) forecasting models must reflect, in some dimension, the true time-series structure of the economy. If agents' beliefs within their forecasting model are not contradicted by actual outcomes then agents can not be freely disposing of useful information. There also has to be a correspondence between beliefs and outcomes since agents' beliefs are supported by the structure of the economy. It follows that *any* forecasting model that produces forecast errors uncorrelated with that same forecasting model is consistent with Muth's hypothesis.[7] In the equilibria we present below, we exploit this orthogonality between forecast errors and forecast models to derive an alternative equilibrium. This alternative to REE is in the spirit of Muth but, as we will see below, incorporates bounded rationality.

2.2 Misspecification Consistent with the Model: Restricted Perceptions Equilibrium

The previous subsection argued that agents exist conceptually inside forecasting models. The Muth hypothesis requires that these forecasting models be consistent with the underlying economic model only in the sense that agents are unable to tell that their model is distinct from the actual stochastic process. A *Restricted Perceptions Equilibrium*, first defined by (Evans and Honkapohja 2001), formalizes the notion that agents' beliefs come from misspecified forecasting models but agents can not detect their misspecification.[8]

To illustrate, we again consider the univariate self-referential model in the previous subsection. In an RPE agents form beliefs via a restricted version of the PLM in (2). Forecasting models are restricted from (2) by restricting the dimension of the vector of exogenous variables: by restricting the size of the set of explanatory variables we impose that agents underparameterize their forecasting models. The set

[7]We are really arguing for consistent expectations. Simon suggested labeling RE 'model consistent' expectations.

[8]The first development of an RPE was by (Anderson and Sonnenschein 1985) who called their equilibrium a Rational Expectations Equilibrium with Econometric Models. Other early RPE models include (Marcet and Sargent 1989), (Sargent 1991), (Evans, Honkapohja and Sargent 1993).

of underparameterized forecasting models is:

$$\mathcal{F} = \{x_t = a + b'\hat{z}_{t-1} + \varepsilon_t : \dim(\hat{z}) < \dim(z)\} \tag{5}$$

An object in the set $\mathcal{F}$ is a PLM that must omit at least one variable or lag.[9] For the purposes of defining an RPE, we assume that all agents select one of these underparameterized models. Below, we describe a mechanism for allowing agents to choose only the best performing statistical models from this set.

Underparameterization is a reasonable approach to expectation formation. In VAR forecasting many professional forecasters limit the number of variables and/or lags in their statistical models. Because of degrees of freedom restrictions, computational costs, etc. it may be necessary to forecast based on a parsimonious specification. The Muth hypothesis requires that agents construct these forecasts so that they are unable to detect their underparameterization within the context of the forecasting model. In other words, the parameters (a, b') are formed as the optimal linear projection of the forecasting model on the state x.[10] That is, (a, b') must satisfy the least-squares orthogonality condition:

$$E\hat{z}_{t-1}\left(x_t - (a + b'\hat{z}_{t-1})\right) = 0 \tag{6}$$

The condition (6) requires agents' forecast errors, $x_t - (a + b'\hat{z}_{t-1})$, be uncorrelated with the forecasting model $\hat{z}_{t-1}$. Within the agent's perceived world they are unable to improve on their forecast. Clearly, this is consistent with Muth's hypothesis.

A solution to (6) is non-trivial because the model is self-referential. When beliefs are underparameterized so that $E_{t-1}x_t = a + b'\hat{z}_{t-1}$, then the actual law of motion for the economy is

$$x_t = Fa + Fb'\hat{z}_{t-1} + \gamma' A z_{t-1} + \gamma'\varepsilon_t \tag{7}$$

Agents' beliefs (a, b) affect the actual law of motion which feed back onto agents' beliefs via the orthogonality condition (6). An RPE is a fixed point to this mapping. The RPE is distinct from the REE because the PLM does not include RE as a special case; it is not possible for agents' beliefs to coincide with the actual distribution of the economy.

The RPE is a reasonable equilibrium outcome because it accounts for realistic departures from RE. In dynamic models it is likely that agents will restrict their information sets. An RPE restricts information sets in a way that is consistent with the true economic structure. The remainder of this survey describes economic implications of this alternative to REE.

[9] z can represent the stacked form of a VAR(1) so that omitting an element of z could be either a variable or a lag.

[10] Because the ε_t are iid zero-mean omitting the parameter c has no bearing on our definition.

2.3 Related Equilibrium Concepts

We turn to a brief discussion of the connection between RPE and other closely related equilibrium concepts.

good idea

2.3.1 Self-Confirming Equilibrium

(Sargent 1999) shows existence of a *Self-Confirming Equilibrium* (SCE) in a simple dynamic macroeconomic model.[11] In Sargent's model, the economy is governed by the natural rate hypothesis –an expectations-augmented phillips curve implies there is no long-run tradeoff between inflation and unemployment. The government, though, (mistakenly) believes the economy follows a statistical phillips curve with a long-run tradeoff between inflation and unemployment. The government sets inflation by solving an optimal control problem with their statistical phillips curve as one constraint. The actual time-series of the economy is determined by the government's misperceptions and the expectations-augmented phillips curve. In Sargent's model the SCE has the government exploit their misspecified phillips curve thereby producing excessive inflation.

Like the model above, decisions in Sargent's model are based on a misperceived forecasting model. Muth's hypothesis requires that the government not be able to detect their misspecification. Sargent defines a SCE exactly as we define an RPE: the government's beliefs must be uncorrelated with their forecast errors. The primary difference, though, between the RPE and the SCE is that the SCE can correspond with the REE while the RPE can not. In the RPE, rational expectations can not be nested within the set of restricted PLM's. In Sargent's SCE it is possible that the data exactly confirm the governments beliefs and the economy is in an REE.

One reason the SCE in Sargent's model coincides with the REE is because agents in his model have rational expectations. In other models, such as (Williams 2003) it is assumed that the private agents have restricted perceptions, of the type in (5), and in that case the SCE and the RPE exactly coincide.[12]

[11]SCE were first developed by (Fudenberg and Levine 1998) and applied in game theoretic settings. In an SCE players correctly guess opponents play however their assumptions on play out of equilibrium may be wrong.

[12]In (Adam 2004) an RPE exists (with an additional stability condition) where agents choose between two models, one of which is consistent with RE. Adam finds that additional persistence from economic shocks arises and such a model can explain key covariance relationships of the U.S. economy.

2.4 Consistent Expectations Equilibria

Most dynamic macroeconomic models are actually non-linear. The infinite-horizon representative agent model, overlapping generations model, dynamic stochastic general equilibrium models all produce dynamic equations which are non-linear. Frequently, these non-linear laws of motion come from Euler equations which govern how the consumer will (optimally) choose its consumption sequence. Analytic solutions to these non-linear models are difficult to obtain. Instead the predominant solution method takes a linear approximation to the non-linear model.[13] These approximations bound the stochastic process to a neighborhood of a steady-state. So long as the noise has sufficiently small support then the system will stay bounded in this neighborhood and the linearization is reasonable.

Unfortunately, a number of papers casts doubt on whether economies stay bounded in a neighborhood of a unique steady-state. For example, (Grandmont 1985), (Grandmont, Pintus, and deVilder 1998), (Cazavillan, Braga, and Pintus 1998), (Benhabib and Rustichini 1994) and (Benhabib, Schmitt-Grohe, and Uribe 2001) show that the dynamics in many standard models may be quite complicated and not restricted to neighborhoods of the steady-states. (Hommes and Sorger 1998) are motivated along these lines to assume agents have linear beliefs in a non-linear world. They ask whether these beliefs can be consistent with the non-linear model just as the RPE shows underparameterized models can be consistent with the economy. This is a compelling question since most economists construct linear solutions to these models and most econometric forecasting is based on linear models. If agents are boundedly rational then they should also form linear forecasts.

A *Consistent Expectations Equilibrium* (CEE) is a stochastic process and a set of beliefs such that agents use a linear forecasting rule and are unable to detect that the true model is non-linear. Take, as an example, the model in (Branch and McGough 2003) who suppose that the true model is

$$x_t = G(x^e_{t+1}, \eta_t)$$

but beliefs are formed according to the linear forecasting rule

$$x_t = bx_{t-1} + \varepsilon_t$$

The function G can arise from an Euler equation. A CEE occurs when the autocorrelation coefficients of the perceived model align with the actual autocorrelation coefficients from the data $\{x_t\}$:

$$b^j = corr(x_t, x_{t-j})$$

[13]There are , on the other hand, a number of approaches to computing higher-order taylor expansions. See (Judd 1998).

It can be shown that for the case where $j = 1$ this also satisfies an orthogonality condition:

$$Ex_{t-1}(x_t - bx_{t-1}) = 0$$

Again, because of the self-referential feature of the model–x_t depends on bx_{t-1}– existence of such equilibria are non-trivial. (Hommes, Sorger, and Wagener 2003) provide an interesting example where a CEE exists in the OLG model of (Grandmont 1985).

The intuitive properties of a CEE can be illustrated graphically. Figure 1, excerpted from (Branch and McGough 2003), shows a non-linear function symmetric about an (unstable) steady-state. The line running through G is the linear belief function. Given these linear beliefs, and realizations of the random variable η_t, the function G produces outcomes denoted by the circles. As the figure illustrates these realizations are scattered about the linear beliefs of the agents. Without knowing that G is non-linear agents will think they have represented the stochastic process well with their trend line. The orthogonality condition shows that this is an example of an RPE where agents have their functional form incorrect; within the context of their forecasting model agents are unable to detect their misspecification. A CEE, though, also requires that higher order autocorrelations coincide with agents' beliefs. It is this restriction which differentiates a CEE and RPE. A CEE imposes stronger restrictions than an RPE.

INSERT FIGURE 1 HERE

3 Extending the RPE: Misspecification Equilibrium

One criticism of the RPE as a suitable alternative to REE is that the form of misspecification is *ad hoc.* (Branch and Evans 2003a,b) develop an extension of the RPE called a *Misspecification Equilibrium* (ME) which endogenizes the underparameterization. The set $\mathcal{F}$ in (5) consists of all underparameterized forecasting models. A more restricted set of models would consist of all underparameterized forecasting models in $\mathcal{F}$ where the parameters are computed via the least squares orthogonality condition. An equilibrium most consistent with Muth's hypothesis will have agents only choose the best performing statistical models from the set of all underparameterized models. Of course, since these models are self-referential, 'best performing' is dependent on how many agents forecast with a given model. By allowing agents to endogenously choose their forecasting model we can search for a Nash equilibrium in forecasting models.

Denote u^j as a selector matrix that picks out those elements of z_t used by the predictor indexed by j.[14] If there are K predictors then there exists a $K \times 1$ vector of predictor proportions $n' = (n_j)_{j=1}^K$; where n_j is the proportion of agents who use predictor j. Expectations in a reduced-form model such as (1) are a weighted average of underparameterized predictors,

$$E_{t-1}^* x_t = \sum_{j=1}^{K} n_j b^j u^j z_{t-1}$$

Attached to each predictor j is a fitness measure EU^j.

To close the model we need a method for determining the proportions n_j. Here we follow (Brock and Hommes 1997) in assuming that predictor choice is given by a discrete choice among the (optimal) forecasting functions.[15] In (Brock and Hommes 1997) agents choose between rational and naive expectations with the probability they will choose a given predictor given by a multinomial logit (MNL) law of motion. In (Branch and Evans 2003a,b) we follow their approach and specify,

$$n_j = \frac{\exp(\alpha EU^j)}{\sum_{k=1}^K \exp(\alpha EU^k)} \tag{8}$$

Below we consider several applications of this set-up where fitness is defined as unconditional mean profits or mean square forecast error. In either case the fitness measures $EU^j, j = 1, ..., K$ depend on the stochastic process x_t. Because the model is self-referential the data x_t also depend on the choices n_j. From this logic it follows that predictor fitness measures depend on n and so we can define the function $G_j(n) = EU^j$.[16] The predictor proportions in (8) depend on $G_j(n)$ and, hence, on n itself. Rewriting (8),

$$n_j = \frac{\exp(\alpha G_j(n))}{\sum_{k=1}^K \exp(\alpha G_k(n))} \tag{9}$$

A *Misspecification Equilibrium* (ME) is a vector n which is a fixed point of the mapping defined by (9). An ME is an RPE where agents only choose the best of the best misspecified models. An ME is a powerful equilibrium concept because it is most consistent with Muth's hypothesis but also allows for bounded rationality. As we will see in the remainder of this survey the ME makes several important implications.

Below we survey three applications. The first shows how heterogeneity may arise as an equilibrium outcome, the second explores the existence of multiple equilibria, and the third examines the joint determination of optimal monetary policy and misspecification.

[14] u^j consists of zeros and ones so that if one wanted to pick out the first element of z then $u = (1, 0, ..., 0)$.

[15] (Brock and Durlauf 2001) examine discrete choices dependent on expectations of the actions of peers. They find that multiple equilibria can arise.

[16] This function maps from the unit simplex into the set of real numbers.

3.1 Cobweb Model

Heterogeneity in expectations is an element of most surveys such as the University of Michigan's Survey of Consumers (see (Branch 2004)). However, models with fully optimizing agents have been unable to derive heterogeneous expectations endogenously. In (Branch and Evans 2003a) heterogeneity arises in an ME with agents split between underparameterized models. (Branch and Evans 2003a) call heterogeneity that arises in a Misspecification Equilibrium as $\alpha \to \infty$ *Intrinsic Heterogeneity*. The parameter α is often called the 'intensity of choice'. It plays a key role in the stability and bifurcation analysis of (Brock and Hommes 1997). Because the discrete choice is based on a random utility model, the parameter α is inversely related to the variance of the noise in the random utility term. The neoclassical case has $\beta = \infty$. This is 'neoclassical' because no agent will ever choose a predictor which performs poorly relative to the opportunities. We focus on the properties of ME as $\alpha \to \infty$ because this is where all agents only choose best performing statistical models. We highlight the results in (Branch and Evans 2003a) by focusing on the case where z_t is bivariate.

The cobweb model is a simple framework of supply and demand where supply has a one-period production lag. (Branch and Evans 2003a) adapt the Cobweb model of (Muth 1961):

$$\begin{aligned} D(p_t) &= H - Bp_t + \delta' z_t && (10) \\ S(E^*_{t-1}p_t) &= CE^*_{t-1}p_t && (11) \end{aligned}$$

where z_t is bivariate and follows

$$z_t = Az_{t-1} + \varepsilon_t$$

The equilibrium price process is

$$p_t = -\frac{C}{B}E^*_{t-1}p_t + \frac{\delta'}{B}z_t \tag{12}$$

where H has been normalized to zero. Equilibrium price (12) has the same reduced-form as (1) with $F = -C/B$, $\gamma' = \delta'/B$. In the cobweb model, $F < 0$ implying 'negative feedback'. Expectations produce a movement in price in the opposite direction. It is this self-referential feature of the cobweb model which makes heterogeneity possible as an equilibrium outcome.

In the cobweb model the appropriate fitness measure for forecasting is expected profits $E\pi^j$. The forecasting models for the bivariate case are,

$$\begin{aligned} E^1_{t-1}x_t &= b^1 z_{1,t-1} && (13) \\ E^2_{t-1}x_t &= b^2 z_{2,t-1} && (14) \end{aligned}$$

Aggregate beliefs are,

$$E^*_{t-1}x_t = nb^1 z_{1,t-1} + (1-n)b^2 z_{2,t-1}$$

where n is the proportion incorporating z into their beliefs. Given these beliefs, plugging in expectations into the equilibrium price equation leads to the following reduced-form for the economy,

$$x_t = \xi_1 z_{1,t-1} + \xi_2 z_{2,t-1} + v_t \tag{15}$$

for appropriately defined ξ_1, ξ_2 and zero-mean iid v_t.[17] Note that ξ_1, ξ_2 depend on n. The form of (15) highlights the dual effects of z_t. z_t has a direct effect given by $\gamma' A$, but it also has an indirect expectation formation effect. It is this tension that produces interesting implications of an ME. The parameters b^1, b^2 are computed according to the orthogonality condition above and using the reduced-form (15) are given by

$$\begin{aligned} b^1 &= \xi_1 + \xi_2 \rho && (16) \\ b^2 &= \xi_2 + \xi_1 \tilde{\rho} && (17) \end{aligned}$$

where $\rho = E z_1 z_2 / E z_1^2$ and $\tilde{\rho} = E z_1 z_2 / E z_2^2$ are correlation coefficients.[18]

A couple of brief notes are useful. These expectations take into account the correlation between the forecasting model and the omitted variable. Optimal projections will 'tease' out of the price process as much information about z_t as is statistically possible. Note also how an RPE arises in (16)-(17) given predictor proportions n: ξ_1, ξ_2 depend on b^1, b^2 which depend on ξ_1, ξ_2. An ME yields, in addition to the RPE, an equilibrium in n.

Given the reduced-form and the RPE belief parameters $b^j, j = 1, 2$, it is straightforward to compute mean profits.[19] We follow (Brock and Hommes 1997) in making a convenient reduction in the size of the state when there are only two predictors. Define $\tilde{G}(n) = E\pi^1 - E\pi^2$ as the relative profit differences as a function of n. For heterogeneity to arise this function must be positive at $n = 0$ –so that agents have an incentive to deviate from the consensus forecast– and negative at $n = 1$. (Brock and Hommes 1997) illustrated that it is possible to define the mapping for n in terms of the profit difference $\tilde{G}$:

$$n = \frac{1}{2} \tanh \left[\frac{\alpha}{2} \tilde{G}(n) \right] + \frac{1}{2}$$

This function maps the unit interval into itself.[20] Since the tanh is continuous, Brouwer's theorem guarantees that there exists at least one misspecification equilibrium. Moreover, if $\tilde{G}(n)$ is a monotonically decreasing function then n will also be monotonically decreasing and there will exist a unique ME. Figure 2 illustrates one possibility.

[17] $\xi_1 = \gamma_1 a_{11} + \gamma_2 a_{21} + n b^1$, $\xi_2 = \gamma_1 a_{12} + \gamma_2 a_{22} + (1 - n) b^2$.

[18] See (Branch and Evans 2003a) for details.

[19] The Muth cobweb model assumes a quadratic cost function.

[20] The tanh is continuous, increasing, symmetric about the origin and asymptotes at -1 and 1.

INSERT FIGURE 2 HERE

Figure 2 plots the profit difference function (bottom panel) and the predictor proportion mapping (top panel). This is for a particular parameterization presented in (Branch and Evans 2003a). Notice that because the profit difference is positive at $n = 0$ an agent would have an incentive to use model 1 since it returns higher mean profits. The opposite is true at $n = 1$ where agents will want to mass from predictor 1 to predictor 2. These forces imply that the only equilibrium is where both predictors fare equally well in terms of average profit. Both predictors return the same mean profit at the proportion where $\tilde{G}$ crosses the horizontal axis. This intuition is demonstrated in the top panel. It plots the aggregate best-response mapping for agents. This panel plots the predictor proportion mapping for various values of the 'intensity of choice' α. As $\alpha \to \infty$ the equilibrium proportion of agents tends to the point where both predictors fare equally well.

That agents only choose the best performing models was the motivation for an ME. Here we illustrate that it is possible for agents to be split across multiple models. In a model, such as the cobweb, where there is negative feedback from expectations the equilibrium dynamics of the model push each predictor to have the same average return; agents will be split heterogeneously across these predictors. The existence of heterogeneity is a significant result in (Branch and Evans 2003a) because previous models were unable to generate heterogeneity across predictors without assuming a finite 'intensity of choice'. That heterogeneity arises as an equilibrium outcome illustrates how small deviations from full rationality can lead to observationally important phenomena.

3.2 Lucas Model

In this subsection we present an overview of (Branch and Evans 2003b). The Lucas-type monetary model shares the same reduced-form as the cobweb model. However, in this model $0 < F < 1$ so that there is 'positive feedback' from expectations to the state. Positive feedback suggests coordination and the possibility of multiple equilibria.

The Lucas model is based on a general equilibrium framework where firms make signal extractions of local versus global price shocks. The model consists of an aggregate demand (AD) and aggregate supply (AS) relationship:

$$
\begin{aligned}
AS: \; q_t &= \phi(p_t - E^*_{t-1}p_t) + \beta_1 z_t && (18)\\
AD: \; q_t &= m_t - p_t + \beta_2 z_t + w_t && (19)\\
m_t &= p_{t-1} + \delta' z_t + u_t && (20)\\
z_t &= A z_{t-1} + \varepsilon_t && (21)
\end{aligned}
$$

where $z_t \in \mathbb{R}^2$ is a vector of exogenous disturbances that hit the economy, w_t is white noise, and m_t is the money supply. The reduced-form of this model is

$$\pi_t = \frac{\phi}{1+\phi} E^*_{t-1}\pi_t + \frac{(\delta + \beta_2 - \beta_1)'}{1+\phi} z_t + \frac{1}{1+\phi}(w_t + u_t)$$

where $\pi_t = p_t - p_{t-1}$. This again takes the same reduced-form as (1).

The details of the model are identical to the cobweb model with the exception of the sign of F.[21] Additionally, we define predictor fitness in terms of mean square error:

$$EU^j = -E(x_t - b^j z_{j,t-1})^2, j = 1, 2$$

The positive feedback implies that agents' expectations are reinforced by the stochastic process. The coordinating forces suggests that multiple equilibria may be present.

A necessary condition for multiple equilibria is that the function $\tilde{G}(n_1) = EU^1 - EU^2$ takes the values $\tilde{G}(0) < 0$ and $\tilde{G}(1) > 0$. Under these conditions agents have an incentive to stick with the consensus forecasts. Figure 3 illustrates this case for a particular parameterization of (Branch and Evans 2003b). The bottom panel shows that $\tilde{G}$ is monotonically increasing and crosses the horizontal axis. The top panel illustrates how this translates into the MNL mapping. Here there are three equilibria: there are the two homogenous expectations equilibria and a third interior equilibrium. The outside equilibria, at $n = 0, 1$, are stable in the sense that the mapping is (locally) contracting at those points. This implies that under a learning rule the system will tend to be confined to neighborhoods of these two points.

INSERT FIGURE 3 HERE

This result is interesting because it suggests that there can be a coordinating incentive in misspecified models. Even though agents are free to choose their misspecification, the equilibrium forces of the model drive them to the same model. Of course, one hallmark of Post-Walrasian macroeconomics is the assertion that the economy may not have institutions capable of coordinating on such an equilibrium. (Branch and Evans 2003b) explore this issue by replacing optimal linear projections with recursive least squares estimates. That paper shows that under certain learning rules, the economy may generate endogenous dynamics which switch between neighborhoods of these two stable equilibria. This occurs under a type of weighted least squares called constant gain learning. Constant gain learning has agents place greater weight on recent than distant observations. So a particularly large shock to the economy (by an omitted variable) may lead agents to believe the other forecasting model is superior. Because the asymptotic stochastic properties of each equilibria are distinct it is shown that endogenous volatility in inflation can arise; this is an important finding because one empirical regularity is that inflation volatility has declined

[21]Beliefs are again of the form (16)-(17).

considerably in the U.S. since the mid-1980's. The next section explores how optimal monetary policy might interact with these coordinating Misspecification Equilibria.

4 Policy Implications

The intuition for the ME results in the cobweb model and lucas model surround the dual effects that exogenous disturbances have on recursive expectational models. An exogenous stochastic process has a direct effect (γ') on the state (x_t) but it also has an indirect expectation formation effect (F and b^j). The papers in (Branch and Evans 2003a,b) highlight the tension between these dual effects. If there is a negative feedback in the model then the indirect effect may work to counteract the influence of the direct effect. In the Lucas model, where there is positive feedback, the indirect and direct effects work in tandem.

Many business cycle models also depend on a government control, such as monetary or fiscal policy. If this policy is conducted optimally (e.g. (Woodford 2003)) then one would expect the government to counteract the direct effect of exogenous disturbances. This opens the possibility for a third effect of exogenous disturbances: policy feedback. Given the parameters of the model, optimal policy prescribes a reaction for the government's control (say nominal interest rates) to exogenous disturbances. This reaction must take into account the expectations of agents and how they are distributed across misspecified forecasting models: the indirect effect has bearing on the policy effect.[22] But, the policy effect alters how the exogenous disturbances are correlated with the state x_t, so policy also influences the predictor proportions. This intuition suggests that a fully specified model will have a joint determination of optimal policy and a Misspecification Equilibrium. We explore this issue by presenting a highly stylized model as an example. A formal model is being developed in work in progress.

4.1 A simple model

To illustrate how optimal policy may affect the Misspecification Equilibrium we consider a slight alteration of (1):

$$\begin{aligned} x_t &= FE^*_{t-1}x_t + \gamma' z_t + Pi_t && (22) \\ z_t &= Az_{t-1} + \varepsilon_t && (23) \end{aligned}$$

We assume x_t is bivariate and consists of output and inflation. Notice that (22) now depends on the nominal interest rate which is under the control of the central bank. We continue to assume z_t is a stationary bivariate VAR(1). Then F, γ' are (2×2).

[22] We assume that the policy authority is able to commit to such a policy.

This model is inspired by the Sargent-Wallace model where expectations affect both an IS and AS curve. Expectations are multivariate and so misspecification can occur in multiple dimensions. To keep the analysis close to the previous section, we assume that agents underparameterize by omitting a component of z but their beliefs have the same form for both components of x_t[23]:

$$\begin{aligned} E^1_{t-1}x_t &= b^1 z_{1,t-1} && (24) \\ E^2_{t-1}x_t &= b^2 z_{2,t-1} && (25) \end{aligned}$$

now b^j is (2×1). In an RPE these expectations are set so that the forecast errors are orthogonal to the forecasting model. Before such an equilibrium can be computed we require a specification for optimal policy.

As before, the law of motion can be re-written in the form,

$$x_t = \xi_1(n)z_{1,t-1} + \xi_2(n)z_{2,t-1} + Pi_t + \gamma'\varepsilon_t \tag{26}$$

where the ξ's have been written to emphasize their dependence on the predictor proportion n. The government's problem is to choose a sequence $\{i_t\}_{t=0}^{\infty}$ in order to maximize their objective function subject to the law of motion for the economy taking n as given.[24] We assume that policymaker's care about minimizing inflation and output variance:

$$\max_{\{i_t\}} E_0 \textstyle\sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \omega y_t^2\right)$$

$$s.t. \quad x_t = \xi_1(n)z_{1,t-1} + \xi_2(n)z_{2,t-1} + Pi_t + \gamma'\varepsilon_t$$

Optimal policy in this case takes the form:

$$i_t = -G'z_{t-1} \tag{27}$$

where G is (2×1).

Then the law of motion is,

$$x_t = \xi_1(n)z_{1,t-1} + \xi_2(n)z_{2,t-1} + PG'z_{t-1} + \gamma'\varepsilon_t \tag{28}$$

In an RPE beliefs must satisfy the orthogonality condition:

$$Ez_{j,t-1}\left(x_t - b^j z_{j,t-1}\right) = 0 \tag{29}$$

In this model, the effect of the exogenous disturbances depends on the direct effect $(\gamma'A)$ but also on the policy effect (PG').

[23]One could assume alternatively that z_1 is used to forecast inflation and z_2 to forecast output.

[24]In equilibrium, policy and beliefs, n, are jointly determined.

As in the lucas model we assume that predictor fitness is measured by mean square error. Mean square error is now with respect to two state variables. In this survey article we are providing an illustration and so we choose the simplest case: agents care equally about inflation and output variance. The 'intensity of choice' is now a vector $\alpha = (\alpha_1, \alpha_2)'$. Fix $\alpha_1 = \alpha_2$ and denote $\tilde{G}(n) = EU^1 - EU^2$ as the (2×1) vector of mean forecast errors. Predictor choice is governed by,

$$n = \frac{1}{2} \tanh \left[\frac{\alpha'}{2} \tilde{G}(n) \right] + \frac{1}{2}$$

4.2 Misspecification Equilibrium with no policy

We illustrate our results by picking parameters and computing the fitness difference $\tilde{G}$ and the best-response mapping for n. We set the parameters to be:

$$F = \begin{bmatrix} .7 & -.3 \\ .2 & .8 \end{bmatrix} \quad A = \begin{bmatrix} .2 & .2 \\ .2 & .7 \end{bmatrix}$$

$$\gamma = \begin{bmatrix} .1 & .01 \\ .01 & .8 \end{bmatrix} \quad P = \begin{bmatrix} .1 \\ .05 \end{bmatrix}$$

We choose z_t so that the asymptotic stochastic properties are $Ez_1^2 = .6161, Ez_2^2 = 7.0689, Ez_1 z_2 = 1.3589$. These were chosen not for an insightful economic interpretation but to illustrate policy implications of Misspecification Equilibria.

Figure 4 plots the predictor proportion mapping when policy is set to zero, that is interest rates have no effect on the model. The figure demonstrates that because of the positive feedback of expectations there are again two stable coordinating equilibria at $n = 0$ and $n = 1$. Because the z_2 component has much higher variance the basin of attraction is much higher for the $n = 0$ equilibrium than the $n = 1$ equilibrium. If we were to replace optimal linear projections with an adaptive learning rule the economy would spend, asymptotically, more time near the high variance equilibrium. It is possible, though, that agents will coordinate at times near the low variance equilibrium.

INSERT FIGURE 4 HERE

4.3 Misspecification Equilibrium with optimal policy

Our interest with optimal policy is what effect there is on the equilibrium properties when the government chooses its control by taking into account expectations and model misspecification. To explore this issue we present the same plot above, except now we include the predictor proportion mapping when policy is non-zero and takes account of n.

Figure 5 again includes the predictor proportion mapping for the no-policy case but also includes the mapping for the policy case. As the figure illustrates the value for the interior equilibrium switches with the addition of optimal policy. The two stable coordinating equilibria still exist and will always exist regardless of policy in this model. The coordinating equilibria are a result of the positive feedback.

INSERT FIGURE 5 HERE

That the value of the unstable equilibrium will switch under optimal policy may not seem significant because the point is unstable. We focus on stable equilibria because under a suitable learning process we expect the system to spend most of its time near one of these equilibria. However, as discussed above, where the unstable equilibrium lies governs the 'size' of the basin of attraction. When the interior equilibrium switched from a relatively high value of n to a relatively low value of n the basin of attraction increased for the $n = 1$ equilibrium. This equilibrium is also the low volatility equilibrium. Under an adaptive learning rule the asymptotic variance of the economy under optimal policy which takes expectations into account will be lower than when not taking expectations into account.

This finding is intuitive when placed in the context of the tension between indirect and direct effects of exogenous disturbances. By having policy counteract exogenous disturbances they will cause the z_t to have the opposite effect if left to their own devices. Thus, if without policy agents find model 2 more appealing under optimal policy they will now find model 1 more appealing. Clearly, these results are stylized and informal. In work in progress we develop the details formally and in a less rigged model.

5 Conclusion

This survey summarizes and argues in favor of Restricted Perceptions Equilibria (RPE) as a boundedly rational alternative to Rational Expectations. Misspecification is a reasonable assumption in macroeconomics. We focus on cases where agents underparameterize their forecasting models by omitting a variable and/or a lag. In data rich environments it is likely that agents will underparameterize. In fact, most VAR forecasters purposely limit the number of variables and lags. In an RPE agents underparameterize optimally in the sense that they are unable to detect their misspecification within the context of their forecasting model. We argue that such an equilibrium is consistent with Muth's original hypothesis. In self-referential models consistency between beliefs and outcomes are a desirable feature. It turns out that

Rational Expectations are not the only type of beliefs which are consistent with the underlying model. This paper demonstrated that in an RPE beliefs are also consistent with the data generating process.

We explore the implications of RPE by reviewing the Misspecification Equilibrium (ME) of (Branch and Evans 2003a,b). In an ME agents endogenously choose their form of misspecification so that they only choose best performing statistical models. In a model with negative feedback from expectations, like the cobweb model, agents may be distributed heterogeneously across forecasting models. In business cycle models, such as a Lucas-type or Sargent-Wallace *ad hoc* model, there is positive feedback and multiple Misspecification Equilibria.

Besides highlighting results on RPE in the literature this paper also argues that there are important policy interaction effects. We provided an example where policy, taking the form of misspecification as given, alters the stochastic properties of the economy. By altering the stochastic properties of the economy agents endogenous choice of misspecified model will also change. A fully specified equilibrium prescribes a joint determination of policy and misspecification. Future work intends to explore this implication more formally. In particular, the literature on monetary policy during the 1970's highlights two alternative explanations of the stagflation: misspecification on the part of agents (e.g. (Orphanides and Williams 2003)) and misspecification on the part of the government about the structure of the economy (see (Sargent 1999) and (Bullard and Eusepi 2003)). The approach advocated in this paper suggests that an additional form of misspecification may be empirically important: agents optimally choose underparameterized forecasting models and policymaker's form incorrect assumptions about the form of misspecification. The story in mind is that policymaker's during the 1970's, by assuming agents were not coordinating on oil prices, accommodated supply shocks which altered the effects of oil prices to the extent that agents would then incorporate oil prices into their forecasting models. In the 1980's, though, policymaker's correctly realized that agents were predominantly reacting to supply shocks, they then optimally responded to those shocks, altering the effects of the shocks and leading agents to coordinate on other forecasting models. Such a story can explain why oil price shocks lost their potency during the 1990's.

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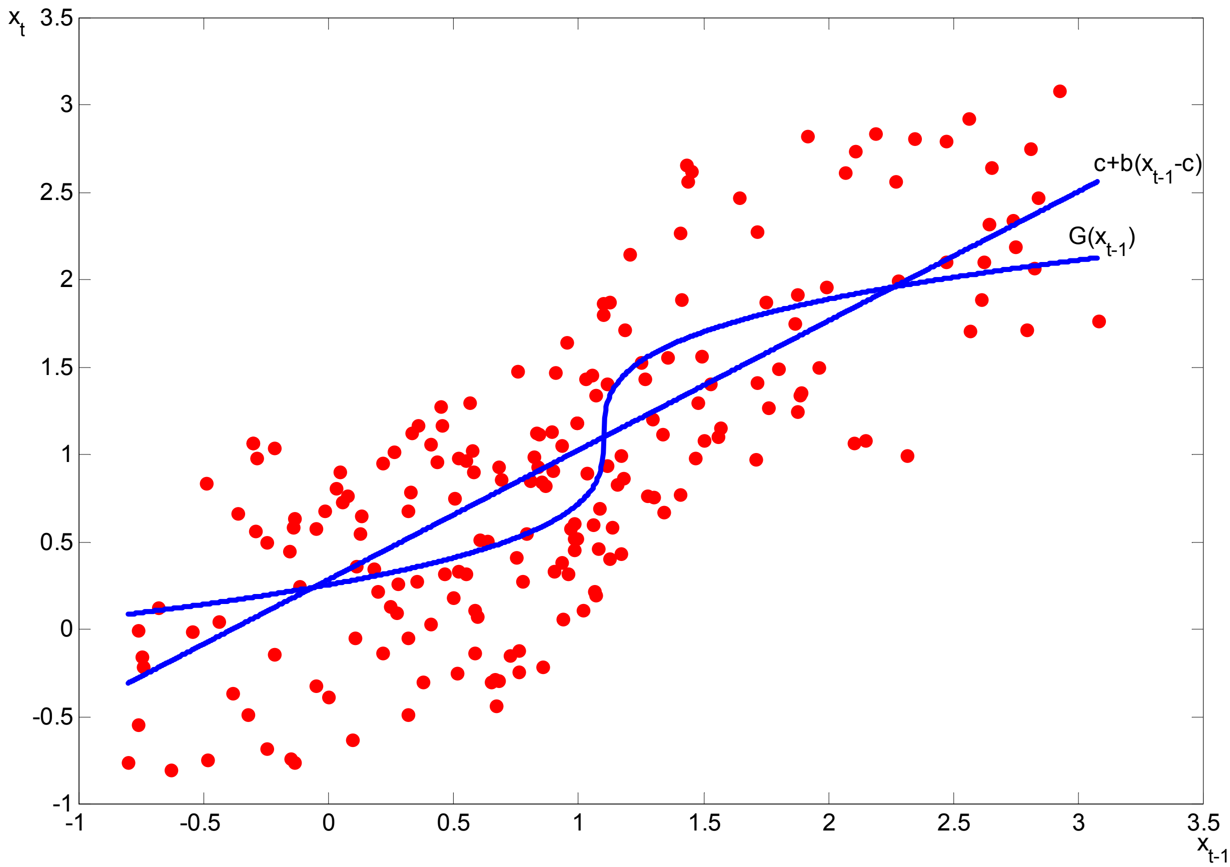

Figure 1. Consistent Expectations Equilibrium. (Excerpted from (Branch and McGough 2003).

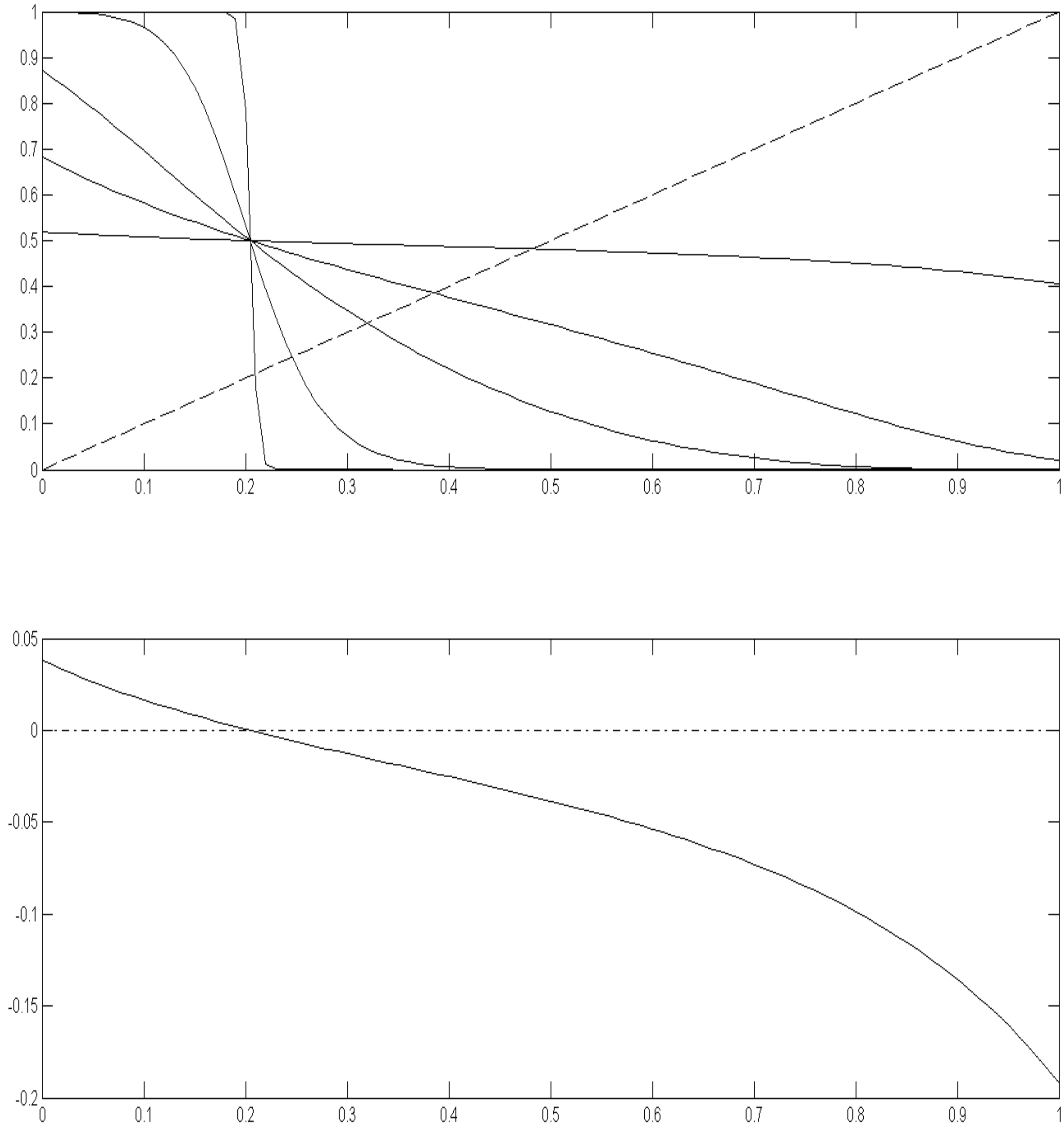

Figure 2. Unique Misspecification Equilibrium and Intrinsic Heterogeneity in the cobweb model. (Excerpted from (Branch and Evans 2003a)).

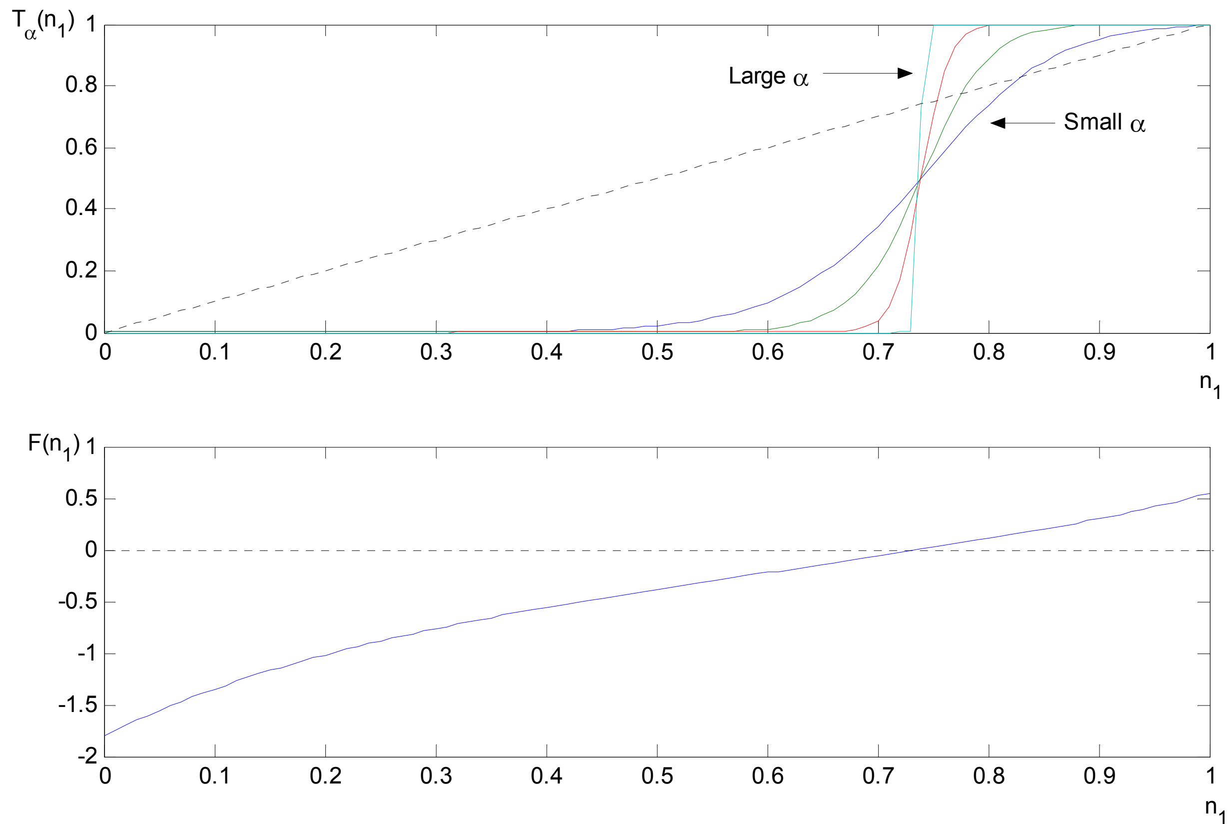

Figure 3. Multiple Misspecification Equilibria in the Lucas Model. (Excerpted from (Branch and Evans 2003b).

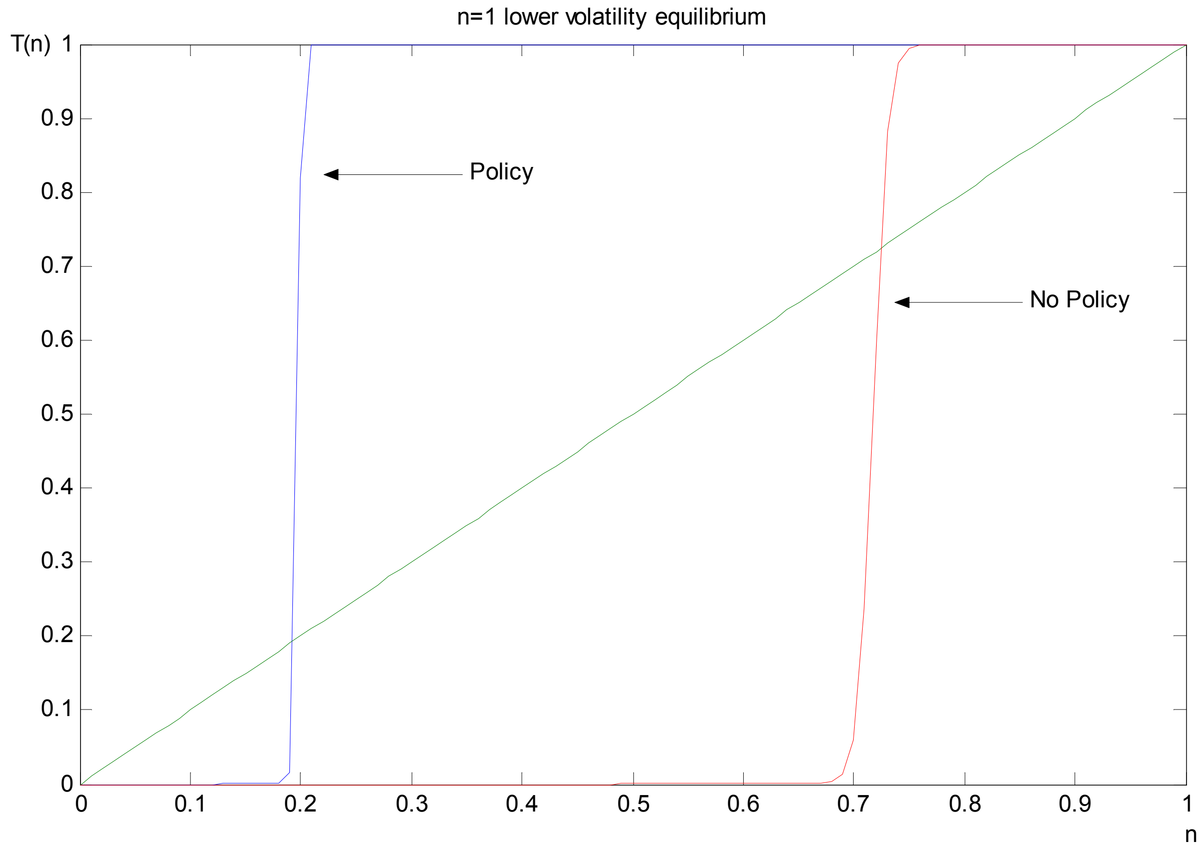

Figure 4. Multiple Misspecification Equilibria with and without optimal policy. The *n=.7* ME is when there is no policy, *n=.2*, is for optimal policy.