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Monetary policy, learning and the speed of convergence

Giuseppe Ferrero*

Bank of Italy, Economic Research Department, via Nazionale 91, 00184 Rome, Italy

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Abstract

Under the assumption of bounded rationality, economic agents learn from their past mistaken predictions by combining new and old information to form new beliefs. The purpose of this paper is to investigate how the policy-maker, by affecting private agents' learning process, determines the speed at which the economy converges to the rational expectation equilibrium. I find that by reacting strongly to private agents' expected inflation, a central bank increases the speed of convergence and shortens the length of the transition to the rational expectation equilibrium. I use speed of convergence as an additional criterion for evaluating alternative monetary policies. I find that a fast convergence is not always desirable.

JEL classification: E52; C62; D83

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1. Introduction

The recent literature on monetary policy has emphasized that while rational expectations is an important and useful benchmark, a policy-maker should consider the robustness of any equilibrium reached under a particular monetary policy to

*Tel.: +39 06 47922103; fax: +39 06 47923720.
E-mail address: giuseppe.ferrero@bancaditalia.it.

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deviations from rational expectations. In the presence of structural changes, in fact, agents in the economy may need time in order to learn about the new environment[1]: in the early stages of this process, previously held beliefs could lead to biased predictions. Thus, a relevant issue concerns the conditions under which learning agents end up forming rational expectations.

A common way to carry out this topic is to employ the least-squared learning approach of Marcet and Sargent (1989a,b) and Evans and Honkapohja (2001), assuming that agents in the model form expectations via econometric forecasts. In this environment, Evans and Honkapohja (2003a,b) and Bullard and Mitra (2002) suggest that economic policies should be designed to be conducive to long-run convergence of private expectations to rational expectations. Failing to do so gives rise to an equilibrium which is not robust to small expectational errors. In accordance with this literature, 'good' policies are those that induce a determinate and learnable rational expectations equilibrium (REE) (see Bullard and Mitra, 2002).

Another small but growing body of research is concerned with the properties of the convergence along the learning process. Evans and Honkapohja (1993), Timmerman (1996), Sargent (1999) and Marcet and Nicolini (2003) make use of learning models not only to study the asymptotic properties of the equilibrium attainable under learning, but as a compelling alternative to study economic behaviors in the short and medium-run. The works of Giannitsarou (2003), Aoki and Nikolov (2006) and Orphanides and Williams (2004) analyze the transition to the REE in the context of policy decisions, addressing the question of whether all policies that produce learnability and determinacy are equally good from a learning perspective.

I take up this point by adapting theoretical results of Benveniste et al. (1990) and Marcet and Sargent (1995). I first examine how the policy-maker, by affecting the private agents' learning process, can influence the transition to the REE (i.e. the speed of learning). I show that by reacting strongly to expected inflation, a central bank can shorten the length of the transition and increase the speed of convergence to the REE. Next, I consider the case where the central bank has a stabilization objective on inflation and output gap and I focus on the optimal discretionary policy described in Evans and Honkapohja (2003a), *EH policy.* I show that this policy, even though it meets all of the objectives listed above (determinacy and stability under learning) and is optimal under rational expectations, is not suitable from the perspective of the speed of learning, as it implies a very slow transition. Therefore, I show how a policy-maker who wants to reach in the long run the same REE determined as under the EH policy, can manipulate the speed of learning of the private sector. Finally, I analyze the welfare implications of converging to a given REE at different speeds. My main conclusion is that fast learning is not always desirable. In the absence of an inflation bias, fast learning always increases social

[1]'It (the ECB) argues that forecasts are too inaccurate [. . .], especially in the early years of the euro when financial markets adjust and, more generally, when the broad public learns how to operate in a radically new environment', Favero et al. (2001).

welfare. In the presence of such a bias, however, the relation between speed of convergence and welfare is not as straightforward. If the initial expected inflation is higher than in the REE, the policy-maker by inducing a fast learning has room to substantially increase social welfare. If, instead, perceived inflation is initially lower, a slow transition might be preferable, as inflation would remain closer to the first best for a longer period of time.

The paper is organized as follows. Section 2 presents the monetary policy problem and describes the learning dynamics under a restricted set of expectations-based policy rules. Section 3 considers the optimal policy under discretion described in Evans and Honkapohja (2003a), as a specific policy in this set. The section ends showing that the *EH policy* determines a very slow convergence to the REE. In Section 4, I study policies that allow the central bank to shorten (or extend) the transition period without affecting the long-run equilibrium and I analyze how social welfare is influenced under these policies. Section 5 presents robustness checks. Section 6 describes some points that deserve further analysis. Section 7 summarizes and concludes.

2. The baseline model

Much of the recent theoretical analysis on monetary policy has been conducted under the 'New Phillips curve' paradigm reviewed in Clarida et al. (1999) and Woodford (2003). The baseline framework is a dynamic general equilibrium model with money and temporary nominal price rigidities. I consider the linearized reduced form of the economy with monopolistically competitive firms, staggered prices and private agents that maximize intertemporal utility. Linearizing private agents' optimality conditions we obtain the following intertemporal *IS curve*[2]:

$$x_t = \mathrm{E}_t^* x_{t+1} - \varphi(i_t - \mathrm{E}_t^* \pi_{t+1}) + g_t. \tag{1}$$

The aggregate supply curve (AS) is modeled by an expectations-augmented *Phillips curve*[3]:

$$\pi_t = \alpha x_t + \beta \mathrm{E}_t^* \pi_{t+1}, \tag{2}$$

where x_t is the output gap, measured as the log deviation of actual output from potential output; π_t is actual inflation at time t; $\mathrm{E}_t^* \pi_{t+1}$ is the level of inflation expected by private agents for period $t+1$, given the information at time t. Similarly, $\mathrm{E}_t^* x_{t+1}$ is the level of the output gap that private agents expect for period $t+1$, given the information at time t. I use the notation E_t^* to indicate that expectations need not

[2]The IS relationship approximates the Euler equation characterizing optimal aggregate consumption choices and the parameter φ can be interpreted as the rate of intertemporal substitution.

[3]The AS relation approximates aggregate pricing emerging from monopolistically competitive firms' optimal behavior in Calvo's model of staggered prices. Here I am not considering cost-push shocks. Introducing cost-push shocks, would not change substantially the results on speed of convergence and the role of policy decisions along the transition. In Section 5, the welfare analysis is handled also in the presence of cost-push shocks.

be rational (E_t without $*$ denotes rational expectations); i_t is the short-term nominal interest rate and is taken to be the instrument for monetary policy; g_t denotes an observable demand shock following a first order autoregressive process, $g_t = \rho_g g_{t-1} + \varepsilon_{gt}$, where $0 \leqslant |\rho_g| < 1$ and $\varepsilon_{gt} \sim \text{i.i.d.}\, N(0, \sigma_g^2)$.

In order to complete the model, it is necessary to specify how the interest rate is settled and how agents form beliefs. I consider the nominal interest rate as the policy instrument and model it by means of a reaction function. Thus, the policy rule takes the form of a functional relationship between the interest rate, the dependent variable, some endogenous variables – expected inflation and output gap – and exogenous variables – shocks.

I analyze two cases. First, I consider a restricted set of expectations-based policies that allows me to introduce in a simple and intuitive way the concept of speed of convergence. Then I focus on a specific element of this set of rules, the optimal policy under discretion derived in Evans and Honkapohja (2003a).[4]

Concerning beliefs, I start each analysis by considering the rational expectations hypothesis in order to study subsequently the implications of bounded rationality.

2.1. Simple expectations-based reaction functions

It has long been recognized that monetary policy needs a forward-looking dimension.[5] I assume that the central bank, in order to set the current interest rate, uses simple policy rules that feed back from expected values of future inflation and output gap and the actual values of the exogenous shock[6]:

$$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1} + \gamma_g g_t. \tag{3}$$

expectations-based TR

The restricted class of *expectations-based reaction functions* that I consider here requires that $\gamma_x = 1/\varphi$ and $\gamma_g = 0$, in order to simplify the interaction between actual and expected variables. Under (3) the economy evolves according to the following system of equations:

$$Y_t = Q + F E_t^* Y_{t+1} + S g_t, \tag{4}$$

[4] I leave for future research a general study of the transition along the learning process for monetary policy problems under commitment.

thank you

[5] In a recent paper that analyzes monetary policy decisions in the US in the last two decades, Greenspan (2004), Chairman of the Board of Governors of the Federal Reserve System, writes '*In recognition of the lag in monetary policy's impact on economic activity, a preemptive response to potential for building inflationary pressures was made an important feature of policy. As a consequence, this approach elevated forecasting to an even more prominent place in policy deliberations*'.

[6] One theme in the literature on rules of this type is that they tend to induce large regions of indeterminacy of the REE and are therefore undesirable (see, for example, Bernanke and Woodford, 1997; Bullard and Mitra, 2002). In this paper, however, I focus on policies that, as a basic requirement, imply a determinate REE and I will consider learnability and speed of learning as additional constraints in this set of policies.

where

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \quad Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad F = \begin{bmatrix} \beta + \alpha\varphi(1-\gamma_\pi) & 0 \\ \varphi(1-\gamma_\pi) & 0 \end{bmatrix}, \quad S = \begin{bmatrix} \alpha \\ 1 \end{bmatrix},$$

and neither the IS nor the AS are affected by expectations on output gap.[7]

Under rational expectations (i.e. $\mathrm{E}_t^* x_{t+1} = \mathrm{E}_t x_{t+1}$ and $\mathrm{E}_t^* \pi_{t+1} = \mathrm{E}_t \pi_{t+1}$), the necessary and sufficient condition for the dynamic system defined by (4) to have a unique non-explosive equilibrium is that[8]

$$\gamma_\pi \in S_1 = \left\{ \gamma_\pi : 1 - \left(\frac{1-\beta}{\alpha\varphi} \right) < \gamma_\pi < 1 + \left(\frac{1+\beta}{\alpha\varphi} \right) \right\}. \tag{5}$$

Assuming, for simplicity, that $\rho_g = 0$, the REE can be written as a linear function of a constant and the shock[9]

$$\pi_t = \bar{a}_\pi + \alpha g_t \quad \text{and} \quad x_t = \bar{a}_x + g_t, \tag{6}$$

and agents' forecasts are constant

$$\mathrm{E}_t \pi_{t+1} = \bar{a}_\pi \quad \text{and} \quad \mathrm{E}_t x_{t+1} = \bar{a}_x. \tag{7}$$

2.2. Adaptive learning

Let us assume that private agents form expectations by learning from past experiences and update their forecasts through recursive least squares estimates.[10] Since, under the *simple expectations-based reaction function* (3) with $\gamma_x = 1/\varphi$, neither the IS nor the AS relations depend on expected output gap, the system under learning can be described by focusing directly on beliefs regarding expected inflation.

First of all, I am interested in studying, when $\gamma_\pi \in S_1$, whether the economy might converge to the determinate equilibrium (6). I assume that agents do not know the effective value of $\bar{a}_\pi$, but estimate it using past information. In this case, private agents' expected inflation is given by

$$\mathrm{E}_t^* \pi_{t+1} = a_{\pi,t}, \tag{8}$$

where $a_{\pi,t}$ is a statistic inferred recursively from past data according to

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}(\pi_{t-1} - a_{\pi,t-1}). \tag{9}$$

[7] For a more general class of expectations-based policy rules without restrictions on γ_x, see Ferrero (2004).

[8] See Appendix A.

[9] Considering an i.i.d stochastic process instead of an AR(1) does not affect the results on speed of convergence. As the literature usually consider AR(1) shocks, the welfare analysis in Sections 5 and 6 is obtained assuming persistent demand shocks.

[10] See Marcet and Sargent (1989a,b) or Evans and Honkapohja (2001) for a detailed analysis of least squares learning.

Forecasts are updated by a term that depends on the last prediction error[11] weighted by the *gain sequence*, t^{-1}. It is well known that in this case the adaptive procedure is the result of a least squares regression of inflation on a constant, and perceived inflation is just equal to the sample mean of past inflations:

$$a_{\pi,t} = \frac{1}{t}\sum_{i=1}^{t} \pi_{i-1}. \tag{10}$$

this is the sense in which CEMP's inflation drift is also mean inflation

In order to study whether the recursive least-squares estimates, $a_{\pi,t}$, converge to the REE, $\bar{a}_\pi$, I refer to the concept of expectation stability (E-stability) described in Evans and Honkapohja (2001). It is known for learning problems of the type described here that, under fairly general assumptions, convergence to REE obtains if and only if E-stability conditions are satisfied.

The issue of stability under learning of a particular equilibrium is addressed by studying the mapping from the estimated parameters – the perceived law of motion, PLM – to the true data generating process – the actual law of motion, ALM. When expectations in system (4) evolve according to expression (8), the inflation's ALM is

$$\pi_t = T(a_{\pi,t}) + \alpha g_t, \tag{11}$$

where

$$T(a_{\pi,t}) = -\alpha\varphi\gamma + [\beta + \alpha\varphi(1-\gamma_\pi)]a_{\pi,t} \tag{12}$$

is the mapping from PLM to ALM of inflation.

As shown in Marcet and Sargent (1989a,b) and Evans and Honkapohja (2001), it turns out that the dynamic system described by Eqs. (9), (11) and (12) can be studied directly in terms of the associated *ordinary differential equation* (ODE)[12]

$$\frac{\mathrm{d}a_\pi}{\mathrm{d}\tau} = h(a_\pi) = \mathrm{E}[T(a_\pi) + \alpha g_t - a_\pi], \tag{13}$$

where τ denotes 'notional' or 'artificial' time and $h(a_\pi)$ is the asymptotic mean prediction error (i.e. the mean distance between the ALM and the PLM).

E-stability conditions are readily obtained by computing the derivative of the ODE with respect to a_π and checking whether it is smaller than zero. Note that this condition translates into the slope of the $T(a_\pi)$ mapping being smaller than 1. Given the parameters α, φ, β, E-stability is obtained for $\gamma_\pi > 1-((1-\beta)/\alpha\varphi)$.[13]

[11]This formula implies that private agents do not use today's inflation to formulate their forecasts. The assumption is made purely for convenience and it is often made in models of learning as it simplifies solving the model. The dynamics of the model are unlikely to change.

[12]In general, a REE is locally stable under real time learning if the relevant real time learning algorithm converges to it. The REE is E-stable if it is locally asymptotically stable under the associated ODE. In our example, since regularities conditions described in chapter 6 of Evans and Honkapohja (2001) are satisfied, E-stability implies stability under real time learning, and therefore local stability of the REE under learning can be studied directly in terms of the ODE.

[13]In this model determinacy implies E-stability of the REE, but not vice versa.

2.3. Speed of convergence

In the remaining part of this section I will focus only on policies that imply a determinate and E-stable REE (i.e. $\gamma_\pi \in S_1$) and I will show that policy decisions concerning γ_π are important not only to describe asymptotic properties of the equilibrium under learning, but also to determine the speed at which the distance between PLM and ALM shortens over time.

Fig. 1 plots the mapping from PLM to ALM (12) and shows how private agents' estimates affect actual inflation along the transition to the REE. When the slope of the mapping is smaller than 1 and the economy starts from a perceived inflation $a_L < \bar{a}_\pi$ or $a_H > \bar{a}_\pi$, the mean of the prediction error, $\mathrm{E}[T(a_{\pi,t}) + \alpha g_t - a_{\pi,t}]$, decreases over time and asymptotically converges to zero. In this case the determinate REE (6) is E-stable.

so well put

Is there any difference between a policy that results in the slope of $T(.)$ equal to 0.01 and one with the slope equal to 0.99? The recent literature on monetary policy and learning (Evans and Honkapohja, 2003a,b; Bullard and Mitra, 2002), by focusing on asymptotic properties, does not provide an answer to this question. Since in both cases the REE is determinate and E-stable, both policies are 'good'. However, by looking at Fig. 1, it is clear that a policy that implies a slope close to 1 would determine a different transition than under a policy that implies a slope close to 0.

The concept of speed of convergence can be used in order to refine the set of 'good' policies, S_1. If convergence is rapid, then we may think to focus on asymptotic behavior, because the economy would typically be close to the REE. Conversely, if convergence is slow, then the economy would be far from its REE, and, hence, its behavior would be dominated by the transitional dynamics.

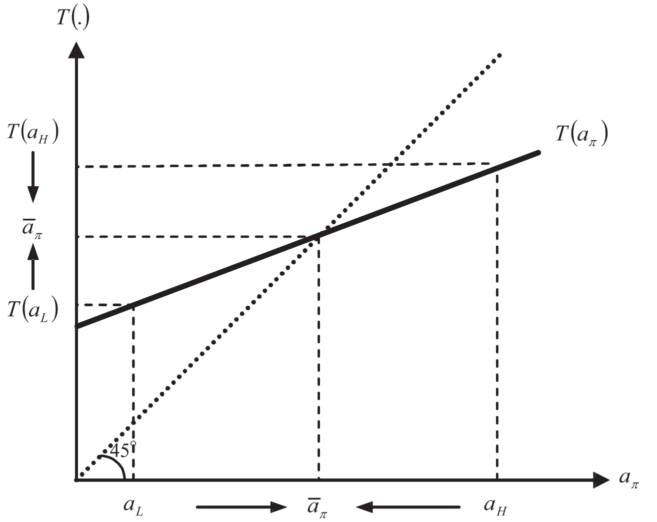

Fig. 1. Mapping from PLM to ALM.

In the literature, the problem of the speed of convergence of recursive least square learning algorithms in stochastic models has been analyzed mainly through numerical procedures and simulations. The few analytical results on the transition to the REE environment are obtained by using a theorem of Benveniste et al. (1990) that relates the speed of convergence of the learning process to the derivative of the associated ODE at the fixed point. In the present case, the ODE to be analyzed is the one described in expression (13).

Let us define

$$S_2 = \left\{ \gamma_\pi : 1 - \left(\frac{0.5 - \beta}{\alpha\varphi} \right) < \gamma_\pi < 1 + \left(\frac{1 + \beta}{\alpha\varphi} \right) \right\}$$

the set of policies under which the derivative of the ODE (13) is smaller than -0.5. The following proposition, adapting arguments from Marcet and Sargent (1995), shows that by choosing γ_π the policy-maker determines the speed at which the distance between perceived and actual inflation narrows over time.

Proposition 1. *Under the simple expectations-based reaction function* (3), *if* $\gamma_\pi \in S_2$, *then*

$$\sqrt{t}(a_{\pi,t} - \overline{a}_\pi) \xrightarrow{\mathrm{D}} \mathrm{N}(0, \sigma_a^2)$$

with

$$\sigma_a^2 = \frac{\alpha^2 \sigma_g^2}{[1 - \beta - \alpha\varphi(1 - \gamma_\pi)]}. \tag{14}$$

Proof. See Appendix B.

Proposition 1 indicates the sufficient condition under which the estimates converge at *root-t* speed to a normal distribution with mean equal to the determinate REE, $\overline{a}_\pi$. Root-t is the speed at which, in classical econometrics, the mean of the distribution of the least square estimates converges to the true value of the parameters estimated. The fact that the formula for the variance of the estimator a_π is modified with respect to the classical case where $\sigma_a^2 = \alpha^2\sigma_g^2$ by a term that depends on the policy parameter γ_π, allows to state the following corollary.

Corollary 2. *Under a simple expectations-based reaction function* (3), *if* $\gamma_\pi \in S_2$, *the weaker the response to expected inflation* (*i.e. the smaller* γ_π), *the greater the asymptotic variance of the limiting distribution,* σ_a^2.

Proof. See Appendix C.

While Proposition 1 describes sufficient conditions under which the mean of the distribution of the least square estimates converges to the REE at root-t speed, the Corollary states that, given root-t convergence, the weaker the response to changes in expected inflation, the higher is the probability that a shock will drive the estimates far away from the REE and, therefore, the longer the period of time that agents will need in order to learn it back.

It is important to notice that the condition to apply the theorem of Benveniste et al. (i.e., the derivative of the ODE being smaller than −0.5) translates into the slope of the $T(\cdot)$ mapping (12) being smaller than 0.5.

Let us define

$$S_3 = \left\{ \gamma_\pi\colon 1 - \left(\frac{1-\beta}{\alpha\varphi}\right) < \gamma_\pi \leqslant 1 - \left(\frac{0.5-\beta}{\alpha\varphi}\right) \right\}$$

the set of policies which lead to a slope of the $T(.)$ mapping smaller than 1 but bigger or equal to 0.5. Proposition 1 and its Corollary do not apply when $\gamma_\pi \in S_3$. However, It can be shown by Monte Carlo calculations that the estimates, $a_{\pi,t}$, converge to the REE, $\overline{a}_\pi$, at a speed different from root-t. In this case, the effects of initial conditions fail to die out at an exponential rate (as it is needed for *root-t convergence*) and agents' beliefs converge to rational expectations at a rate slower than root-t. In particular, also when $\gamma_\pi \in S_3$, the link between the derivative of the ODE, the slope of the $T(\cdot)$ mapping and the speed of convergence holds.

Marcet and Sargent (1995) suggest a numerical procedure to obtain an estimate of the rate of convergence when $\gamma_\pi \in S_3$. In this case it is possible to define the rate of convergence, δ, for which

$$t^\delta (a_{\pi,t} - \overline{a}_\pi) \xrightarrow{\mathrm{D}} F \tag{15}$$

for some non-degenerate well-defined distribution F with mean zero and variance σ_F^2.

Expression (15) can be used to obtain an approximation of the rate of convergence[14] for large t. Since $\mathrm{E}[t^\delta(a_{\pi,t} - \overline{a}_\pi)]^2 = \sigma_F^2$ as $t \to \infty$, we can write

$$\frac{\mathrm{E}[t^\delta(a_{\pi,t} - \overline{a}_\pi)]^2}{\mathrm{E}[(tk)^\delta(a_{\pi,tk} - \overline{a}_\pi)]^2} \to 1 \quad \text{or} \quad \delta = \frac{1}{2\log k} \log \frac{\mathrm{E}[t(a_{\pi,t} - \overline{a}_\pi)]^2}{\mathrm{E}[(tk)^\delta(a_{\pi,tk} - \overline{a}_\pi)]^2}.$$

The expectations can be approximated by simulating a large number of independent realizations of length t and tk, and calculating the mean square across realizations.[15]

Table 1 reports the rate of convergence and the slope of the $T(.)$ mapping for different values of the policy parameter $\gamma_\pi \in S_3$, under Clarida et al. (CGG, 2000) calibration.[16]

Calculations show that the numerical rate of convergence when $\gamma_\pi = 2.63$ and the slope of the $T(\cdot)$ mapping is equal to 0.5, is very close to root-t (when the length of the observation goes from 9000 to 10,000), but it is much smaller for lower values of γ_π. In general, Table 1 shows that for $\gamma_\pi \in S_3$, the rate is smaller the weaker the reaction to expected inflation.

[14] The calculation of the rate of convergence is based on the assumption that such a δ exists.

[15] In all simulations I calculated the rate of convergence with 1000 independent realizations.

[16] I have also derived δ under Woodford (2003) parameters. Results on speed of convergence are not affected. CGG (2000) derives from regressions on US data, $\varphi = 4$, $\alpha = 0.075$, $\beta = 0.99$; Woodford (2003) finds $\varphi = (0.157)^{-1}$, $\alpha = 0.024$, $\beta = 0.99$. I use quarterly interest rates and I measure inflation as quarterly changes in the log of prices, as in Woodford (2003). Therefore, my CGG calibration divides by 4 the α and multiplies by 4 the φ reported by CGG (see also Honkapohja and Mitra, 2004).

Table 1
Rate of convergence when $\gamma_\pi \in S_3$

γ_π	$\frac{\partial T(a_\pi)}{\partial a_\pi}$	δ ($t = 2000$–3000)	δ ($t = 9000$–$10,000$)
2.63	0.5	0.4655	0.4999
2.3	0.6	0.4120	0.3735
1.97	0.7	0.2992	0.3109
1.63	0.8	0.1959	0.1960
1.3	0.9	0.0974	0.1000
1	0.99	0.0090	0.0091

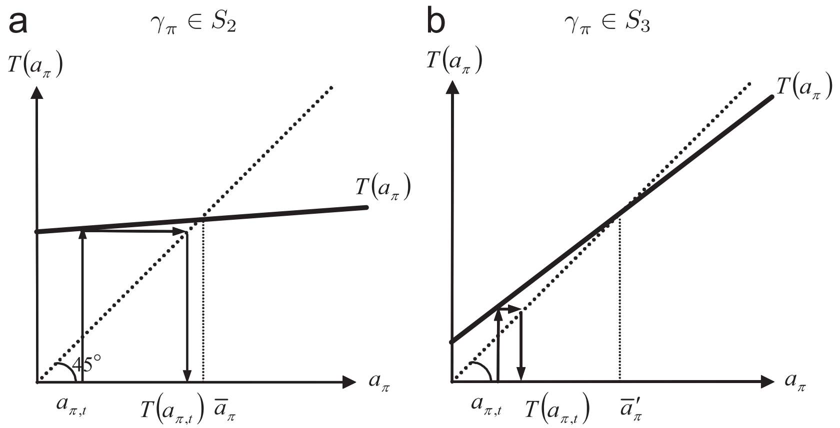

Fig. 2. Mapping from PLM to ALM and the speed of convergence. (a) $\gamma_\pi \in S_2$ and (b) $\gamma_\pi \in S_3$.

Fig. 2 depicts examples for the cases where $\gamma_\pi \in S_2$ and $\gamma_\pi \in S_3$. As the least squares algorithm adjusts each parameter towards the truth when new information is received, the new belief $a_{\pi,t+1}$ will be an average of the previous beliefs $a_{\pi,t}$ and the actual value $T(a_{\pi,t})$ plus an error. When the reaction of the policy-maker to expected inflation is *strong* ($\gamma_\pi \in S_2$), the derivative of $T(.)$ is smaller than 0.5 and $T(a_{\pi,t})$ is close to $\bar{a}_\pi$; when the reaction is *weak* ($\gamma_\pi \in S_3$), the derivative of $T(.)$ is larger than (or equal to) 0.5 and $T(a_{\pi,t})$ is close to $a_{\pi,t}$ instead of being close to $\bar{a}_\pi$, so the average can stay far from the REE for a long time.

It is worth noting that even though the transition is quite different in the two cases analyzed here, the learning process may end up converging to the same REE and, according to policy-maker preferences, the speed of convergence could become a relevant variable in the policy decision problem.

3. The optimal monetary policy under discretion

The reason why I started the analysis with the *simple expectations-based reaction functions* (3) is that they simplify the study of the dynamics under learning and, at the same time, they allow to introduce a general approach to evaluate policies in terms of the associated speed of convergence. I now consider a specific element of this set of rules, the optimal monetary policy without commitment (discretionary policy). In deriving the optimal discretionary policy, I follow Evans and Honkapohja (2003a), assuming that the policy-maker cannot manipulate private agent's beliefs. This assumption implies that the optimality conditions derived under learning are equivalent to the ones obtained under RE.

The policy problem consists in choosing the time path for the instrument i_t to engineer a contingent plan for the target variables π_t and $(x_t - \overline{x})$ that maximizes the objective function[17]

$$\operatorname*{Max}_{x_t,\pi_t} -\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t),$$

where

$$L(\pi_t, x_t) = \tfrac{1}{2}[\pi_t^2 + \lambda(x_t - \overline{x})^2]$$

subject to the constraints (1) and (2) and $\mathrm{E}_t^*\pi_{t+1}$, $\mathrm{E}_t^* x_{t+1}$ given.

The solution of this problem, as derived in Evans and Honkapohja (2003a), yields a reaction function that relates the policy instrument i_t to the current state of the economy and private agents' expectations[18]:

$$i_t = \gamma^\star + \gamma_x^\star \mathrm{E}_t^* x_{t+1} + \gamma_\pi^\star \mathrm{E}_t^* \pi_{t+1} + \gamma_g^\star g_t, \tag{16}$$

where $\gamma^\star = -(\lambda/(\lambda+\alpha^2)\varphi)\overline{x}$, $\gamma_x^\star = \gamma_g^\star = 1/\varphi$ and $\gamma_\pi^\star = 1 + \alpha\beta/(\lambda+\alpha^2)\varphi$.

Since interest rate rule (16) states that the policy-maker should react to expected inflation and expected output gap, it is sometimes called the *optimal expectations-based reaction function* (Evans and Honkapohja, 2003a). However, to stress the fact that this policy is optimal under rational expectations but is not necessarily optimal under learning, one might call this the *RE-optimal expectations-based reaction function* (in order to avoid notational clutter, I call it *Evans and Honkapohja policy*, or *EH* policy).

Under rational expectations, the equilibrium has the form

$$\pi_t = \mathrm{E}_t\pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad x_t = \mathrm{E}_t x_{t+1} = \overline{a}_x. \tag{17}$$

Assuming that private agents do not know $\overline{a}_\pi$ and $\overline{a}_x$ but estimate them recursively, the expected inflation and output gap evolve as described in the previous

[17] I consider the relative weight to output gap, $\lambda \in (0, \infty)$, as an exogenous policy parameter, as it is often done in the literature. An alternative approach is to obtain λ as the result of a general equilibrium problem. In this case λ would depend on representative consumer preferences and firms' price setting rules.

[18] Evans and Honkapohja (2003a) consider a system that also has an exogenous cost-push shock, u_t; consequently their interest rate rule also depends on u_t. In Section 5.2 I consider this case and I show that the main conclusions do not change substantially.

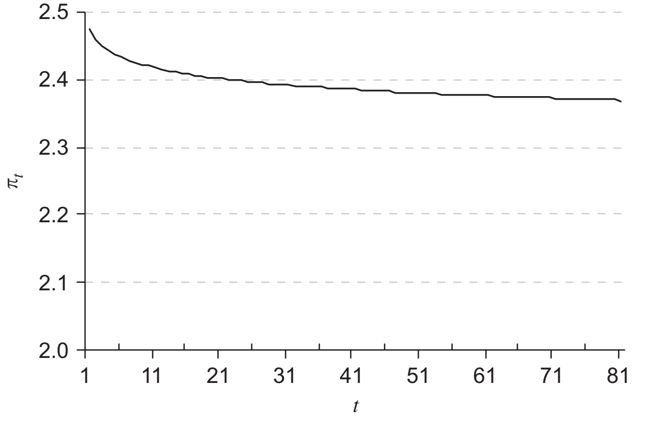

Fig. 3. Deviation of actual inflation from the REE ($\pi^{RE} = 2\%$; $a_{\pi,0=2.5\%}$; $\lambda = 0.1$; annualized data).

section, while the mapping from PLM to ALM is now given by

$$T(a_{\pi,t}, a_{x,t}) = (\Phi^* + \Gamma^* a_{\pi,t}, \Phi^* \alpha^{-1} - (\beta - \Gamma^*)\alpha^{-1} a_{\pi,t}), \tag{18}$$

where

$$\Gamma^* = \frac{\lambda\beta}{(\lambda + \alpha^2)}, \quad \Phi^* = \frac{\lambda\alpha}{(\lambda + \alpha^2)}\overline{x}.$$

Notice that, since the right-hand side of (18) does not depend on $a_{x,t}$, properties of the equilibrium under learning can be described simply by focusing on the mapping from perceived inflation to actual inflation.

As $0 < \Gamma^* < 1$ for all relevant values of parameters α, β and λ, the necessary and sufficient conditions for determinacy (i.e. $-1 < \Gamma^* < 1$) and for E-stability (i.e. $\Gamma^* < 1$) are satisfied under the *EH* policy. Therefore, Evans and Honkapohja (2003a) conclude that the policy derived as the optimal solution of the problem under discretion and rational expectations is also 'good' under learning.

However, if we simulate the model under the *EH-policy* and Clarida et al. (2000) or Woodford (2003) calibrations, it turns out that the distance between the actual inflation and the REE would be significantly different from zero for many periods. Fig. 3 shows the evolution of perceived inflation under learning.[19] Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$, $\overline{x} = 0.0044$[20] and using CGG (2000) calibration,[21] the REE for inflation is around 0.5% (inflation here is measured as quarterly changes in the log of prices, therefore the annualized inflation in the REE is around 2%). I consider an initial expected inflation 0.5 percentage points higher (in annualized terms) than the REE. After 20 years ($t = 80$) perceived inflation is still 0.37 percentage points higher than the REE.

[19]In Eq. (9), an initial t very small would imply a much higher weight to the present than to the past. In all the simulations, I give very low initial weight to past data ($t_0 = 2$).

[20]I choose this value for the output gap target to match the annualized inflation rate of 2%.

[21]Similar results were obtained under Woodford (2003) calibration.

In order to study how policy decisions (in this case, the value of λ) affect the speed of convergence of the learning process, we may think of using similar arguments to those used in Section 2. However, it should be noticed that, under the *EH-policy*, the ALM and the PLM are both deterministic. In this case, the relation between policy decisions, the slope of the $T(.)$ mapping and the speed of convergence can be studied analytically, for all values of λ and t.

When $a_{\pi,t}$ is estimated recursively, the dynamic system described by expressions (17) and (18) can be analyzed in terms of the deterministic difference equation[22]

$$a_{\pi,t} - \overline{a}_{\pi} = ((1 - t^{-1}) + t^{-1}\Gamma^{*})(a_{\pi,t-1} - \overline{a}_{\pi}). \tag{19}$$

Expression (19), describes how the gap between agents' estimates and the REE vanishes over time.

In general, in comparing economies with the same deterministic ALM and PLM as the one described by expressions (17)–(19), but that differ for the value of the parameters (α, β, λ), we may define the speed of convergence in the following way:

Definition 3. Let (α, β, λ) and $(\alpha', \beta', \lambda')$ be the sets of parameters that characterize two economies with the same deterministic ALM and PLM as the one described by expressions (17)–(19), with the REE, $\overline{a}_{\pi}$, $\overline{a}'_{\pi}$, and the agents' estimates at time t, $a_{\pi,t}$, $a'_{\pi,t}$. We say that the learning process of the agents in the economy with parameters (α, β, λ) converges at a faster speed to the REE than in the economy with $(\alpha', \beta', \lambda')$, if

$$\left|\frac{a_{\pi,t} - \overline{a}_{\pi}}{a_{\pi,t-1} - \overline{a}_{\pi}}\right| < \left|\frac{a'_{\pi,t} - \overline{a}'_{\pi}}{a'_{\pi,t-1} - \overline{a}'_{\pi}}\right| \quad \text{for } t = 1, 2, \ldots$$

Based on this definition, the following lemma connects the speed of convergence to the mapping from PLM to ALM.

Lemma 4. *Let* (α, β, λ) *index the relevant set of parameters for the economy described by Eqs.* (17)–(19), *and let* $T_{\alpha,\beta,\lambda}(a_{\pi,t})$ *be the mapping from* PLM *to the* ALM *associated to parameters* (α, β, λ). *If*

$$\left|\frac{\partial T_{\alpha,\beta,\lambda}(a_{\pi,t})}{\partial a_{\pi,t}}\right| < \left|\frac{\partial T_{\alpha',\beta',\lambda'}(a'_{\pi,t})}{\partial a'_{\pi,t}}\right|,$$

then the set of parameters (α, β, λ) *induces faster convergence than the set of parameters* $(\alpha', \beta', \lambda')$.

Proof. See Appendix E.

Finally, for a given set of parameters in the economy (α, β), the next proposition focuses on the role of the relative weight that the policy-maker gives to output gap in determining the speed of convergence.

Proposition 5. *For a given set of parameters* (α, β), *under the EH policy the speed of convergence of the learning process depends on the relative weight that the policy-*

[22] See Appendix D.

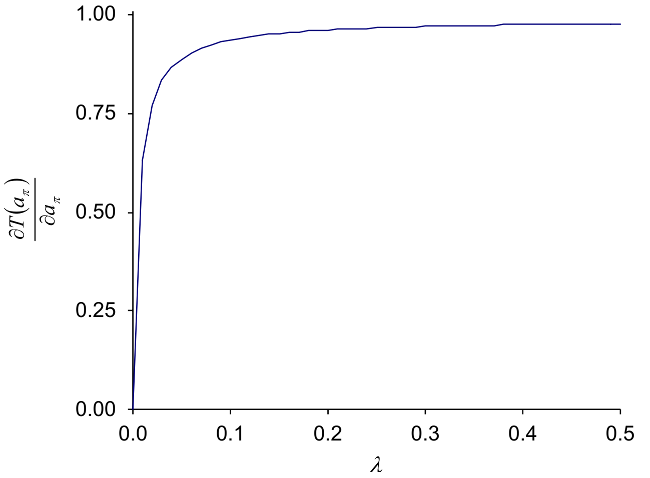

Fig. 4. Slope of the mapping from PLM to ALM.

Table 2
Reduction of initial prediction error after T periods (percent)

λ	$T=4$	$T=20$	$T=40$	$T=80$	$T=200$	$T=400$	$T=800$
0.05	11.5	25.2	30.6	35.6	41.8	46.1	50.0
0.1	6.6	15.1	18.7	22.1	26.4	29.5	32.6
0.3	3.0	7.1	8.8	10.6	12.9	14.5	16.2
0.5	2.3	5.3	6.7	8.0	9.7	11.1	12.3
1	1.7	4.0	5.0	6.0	7.3	8.3	9.3

maker gives to output gap, λ. In particular, the greater the weight to output gap, the slower the learning process.

Proof. See Appendix F.

Fig. 4 shows how the slope of the mapping from the PLM to ALM of inflation changes as λ increases.[23]

Even if the policy-maker cares relatively little about the output gap, the slope of the $T(.)$ mapping can be quite close to 1 (for example when $\lambda = 0.1$, the slope is 0.94). Notice that also in the case where λ is computed as the result of a general equilibrium problem (Woodford, 2003, suggests in this case a value for λ close to 0.05) the slope is close to 0.9.

Table 2 shows how long does it takes to reduce initial errors in perceived inflation under different values of the policy parameter λ.

[23]Simulation are obtained under CGG (2000) calibration.

For a very small weight on output gap, $\lambda = 0.05$, after 200 years ($T = 800$) initial bias in expected inflation is only halved. Things are much worse if we consider higher values for the policy parameter.

The fact that the learning speed could be very slow (or very fast) depending on policy decisions, suggests that when they consider the monetary policy problem under learning, policy-makers should take into account the transition to the REE. E-stability and determinacy are not sufficient to characterize policies in the context of adaptive learning.

4. Adjusting the speed of learning

In the previous section we have seen that, in order to identify *EH policy* (16) as the optimal policy under discretion and learning, the crucial assumption was that '*the policy-maker does not make active use of learning behavior on the part of agents*' (Evans and Honkapohja, 2003a). Under rational expectations, the problem of optimal 'discretionary policy' implies, by definition, that policy-makers cannot affect private agents' expectations. However, under learning, a rational policy-maker with full information should take transition into account. In fact, if private agents' expectations are the result of estimations that depend on past values of the policy instrument, the policy-maker's decisions will affect future estimates and, consequently, the private agents' learning process.

Therefore, *EH policy* is not necessarily optimal under learning but could be defined as *asymptotically optimal* in the sense that it is 'consistent with the optimal (rational expectations) equilibrium' (Svensson and Woodford, 2003). However, if private agents' PLM is well specified, not only the EH-policy is consistent with the optimal REE under discretion, but there is a continuum of *expectations-based policy rules* that result in the same REE.

Let us consider the set of expectations-based reaction functions

$$\Omega = \left\{ \begin{array}{l} \gamma_\pi, \gamma, \gamma_g, \gamma_x \colon \gamma_g = \dfrac{1}{\varphi}, \gamma = \dfrac{\lambda \bar{x}}{((\beta-1)\lambda - \alpha^2)}(\gamma_x(1-\beta) + (\gamma_\pi - 1)\alpha), \\ \gamma_x < \dfrac{1+\beta}{\varphi\beta}, 1 - \dfrac{(1-\beta)}{\alpha}\gamma_x < \gamma_\pi < 1 + 2\dfrac{(1+\beta)}{\varphi\alpha} - \dfrac{(1+\beta)}{\alpha}\gamma_x \end{array} \right.$$

The following lemma characterizes these policies.

Lemma 6. *All policies* $(\gamma_\pi, \gamma, \gamma_g, \gamma_x) \in \Omega$ *result, under rational expectations, in the optimal REE described in* (17) *and yield stability under recursive learning.*

Proof. See Appendix G.

Since all the policies in Ω result, in the long run, in the same optimal allocation, but determine different transitions to the REE, the concept of speed of convergence can be used to discriminate between them.

4.1. More on the mapping from PLM to ALM

In Section 3 we have seen that even though, under the EH policy, the REE meets the objectives of determinacy and stability under learning, the transition is very slow. Now I show how a policy-maker who wants to reach in the long run the same REE determined as under the EH policy, can adjust in a simple and intuitive way the speed of learning of the private sector.

Let us consider the set of expectations-based reaction functions

$$\Omega^{\mathrm{ALS}} = \left\{ \begin{array}{c} \gamma_\pi^{\mathrm{ALS}}, \gamma^{\mathrm{ALS}}, \gamma_g^{\mathrm{ALS}}, \gamma_x^{\mathrm{ALS}}: \gamma_g^{\mathrm{ALS}} = \gamma_x^{\mathrm{ALS}} = \frac{1}{\varphi}, \\ \gamma^{\mathrm{ALS}} = -\frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha\varphi}, \gamma_\pi^{\mathrm{ALS}} = \left(1 + \frac{\beta - \Gamma'}{\alpha\varphi}\right) \end{array} \right\}. \tag{20}$$

This set includes policies that are consistent with the optimal REE and allow to offset demand shocks and expected output gap movements, as under the EH policy.

Lemma 7. *The set of Adjusted Learning Speed-Γ' (ALS-Γ') policy rules, Ω^{ALS}, is a subset of Ω, where $1 < \Gamma' < 0$ is the slope of the mapping from perceived inflation to actual inflation under an ALS-Γ'policy*

$$T_{\mathrm{ALS}-\Gamma'}(a_{\pi,t}) = \frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_{\pi,t}. \tag{21}$$

Proof. See Appendix H.

Note that, as under the EH policy, the ALS-Γ' policy leads to a mapping from PLM to ALM

$$T_{\mathrm{ALS}-\Gamma'}(a_{\pi,t}, a_{x,t}) = \left(\frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_\pi, \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} - \frac{(\beta-\Gamma')}{\alpha} a_{\pi,t} \right), \tag{22}$$

that does not depend on perceived output gap. Therefore, in order to study convergence to the REE, the analysis can focus on the mapping from perceived to actual inflation.[24]

Since the ALM and the PLM are both deterministic and the $T_{\mathrm{ALS}-\Gamma'}(.)$ mapping is linear in $a_{\pi,t}$, we can extend the definition of speed of convergence used in the previous section to this set of policy rules.

The next proposition shows that the policy-maker by choosing appropriately the combinations of $\gamma_\pi^{\mathrm{ALS}}$ and γ^{ALS}, not only can determine in the long-run the same inflation and output gap as under the EH policy, but is also able to choose the Γ'

[24]Here I am not considering the complete set of policies in Ω that allow to offset demand shocks and expected output gap. This should include also policies under which $-1 < \Gamma' < 0$. This choice is made for two reasons: first, for comparability with the EH policy, under which the slope of the $T(.)$ mapping ranges between 1 and 0, depending on the policy parameter λ; second, because the reaction to expected inflation has to be unrealistically high, $4.3 < \gamma_\pi < 7.6$, in order to have $-1 < \Gamma' < 0$.

and, therefore, the speed of convergence that he prefers:

Proposition 8. *The stronger the response to expected inflation* (*i.e. the bigger the* $\gamma_\pi^{\mathrm{ALS}}$) *the flatter the slope of the mapping* $T_{\mathrm{ALS}-\Gamma'}(.)$ *and the faster the convergence to the* REE.

Proof. See Appendix I.

Since the *EH policy* is an element of Ω^{ALS} (with $\Gamma' = \Gamma^*$), the following corollary is directly derived from Proposition 8.

Corollary 9. *Assume that private agents form expectations through recursive least squares learning and that initial perceived inflation is the same under both* ALS-Γ' *and EH policies but different from the* REE. *If the reaction to expected inflation is stronger under* ALS-Γ' *than under EH policy, i.e.* $\gamma_\pi^{\mathrm{ALS}} > \gamma_\pi^*$, *then, along the transition, perceived and actual inflation will be closer to the* REE *under* ALS-Γ' *than under the EH policy. The opposite is true when* $\gamma_\pi^{\mathrm{ALS}} < \gamma_\pi^*$.

The intuition is the following: if the policy-maker reacts strongly to a change in expected inflation, the difference between private agents' expectations and actual inflation will be greater and the prediction error will be initially bigger; if private agents make larger errors they will adapt their estimates faster and both expected and actual inflation will move closer to the REE. In other words, the stronger the policy-maker's response to a change in private agents' expectations, the faster private agents learn and the shorter the transition to the REE (though of course this comes at the cost of larger initial private agents' prediction errors).

Fig. 5 shows the new mapping $T_{\mathrm{ALS}-\Gamma'}(a_{\pi,t})$ under the ALS-Γ'*policy*. This mapping has the same fixed point, $\overline{a}_\pi$, as under the *EH policy*, but the intercept and

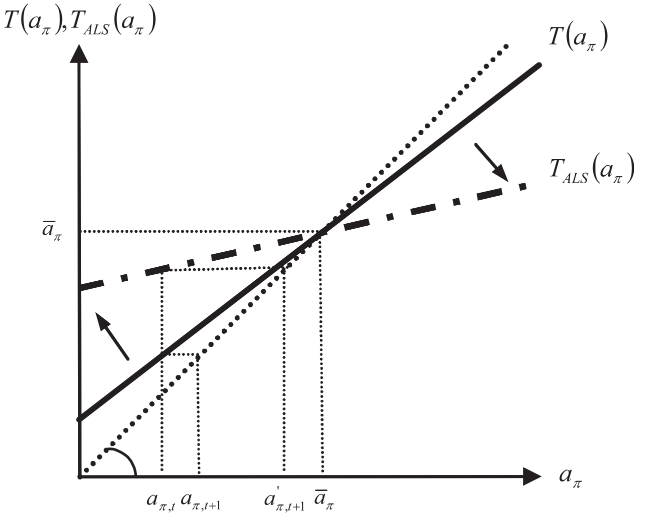

Fig. 5. The mapping from PLM to ALM under the ALS-Γ' policy ($\Gamma' < \Gamma^*$).

the slope are different. The policy-maker, in order to speed up (slow down) the transition to the REE can follow an *expectations-based reaction function* that induces a rotation of the mapping from PLM to ALM around the fixed-point (i.e., the REE), with a flatter (steeper) slope Γ' than under the EH policy.

The following proposition formalize an important difference between ALS and EH policies along the transition.

Proposition 10. *Under* ALS-Γ' *policy* (20) *the speed of convergence does not depend on the relative weight on output gap.*

Proof. See Appendix J.

Taking parameters α, φ, β as given, under the EH policy, the speed of convergence relies entirely on the relative weight to output gap: by choosing λ the policy-maker is also choosing the slope of the $T(.)$ mapping (in the previous example, with $\lambda = 0.1$, the slope was equal to 0.94) and, therefore, he determines the speed of convergence. Proposition 10, instead, shows that under ALS-Γ' *policy*, the policy-maker chooses independently the relative weight on output gap and the speed at which agents learn.

The fact that under the ALS policy, for every $0 < t < \infty$, the distance from the REE could be smaller (greater) than under the EH policy, brings us to the following question: how long does it take under the two policies to get ε-close to the REE, i.e., starting from the same distance from the REE, $|a_{\pi,0} - \overline{a}_\pi| > \varepsilon$, how many periods are needed under the two policies in order to get $|\pi_t - \overline{a}_\pi| < \varepsilon$?

Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$ and $\overline{x} = 0.0044$, and using CGG (2000) calibration, Fig. 6 compares the results of a simulation under the EH policy and under an ALS-Γ' policy with $\gamma_\pi^{\text{ALS}} = 2.6$ (that implies $\Gamma' = 0.5$). Given that the REE for annual inflation is around 2.0%, I consider an initial expected annualized inflation 0.5 percentage points higher than in the REE.

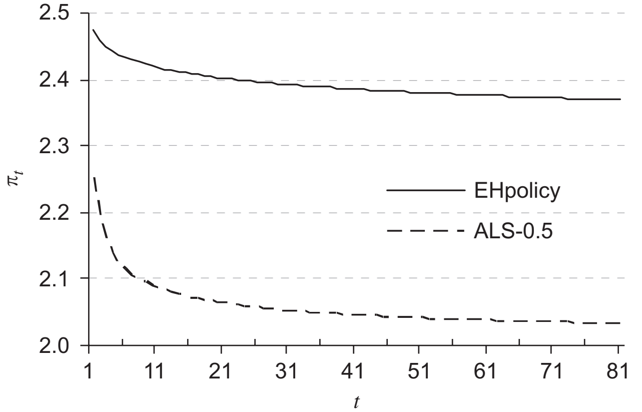

Fig. 6. Deviation of actual inflation from the REE ($\pi^{\text{RE}} = 2\%$; $a_{\pi,0} = 2.5\%$; $\lambda = 0.1$).

Table 3
Transition under the ALS-Γ' policy ($\pi^{\mathrm{RE}} = 2\%$; $a_{\pi,0} = 2.5\%$; $\lambda = 0.1$)

Quarters needed in order to have $(\pi_t - \overline{\pi}_{\mathrm{REE}})$ smaller than					
$\gamma_\pi^{\mathrm{ALS}}$	Γ'	0.4%	0.3%	0.2%	0.1%
3.6	0.2	1	1	1	2
3.0	0.4	1	1	2	4
2.6	0.5	1	1	2	8
2.3	0.6	1	2	4	22
1.6	0.8	2	7	50	>1000
1.3	0.9	6	98	>1000	>1000
1.2[a]	0.94	24	>1000	>1000	>1000
1.03	0.98	>1000	>1000	>1000	>1000

[a]ALS policy with $\gamma_\pi^{\mathrm{ALS}} = 1.2$ coincides with EH-policy.

After 1 quarter, under the ALS-Γ' policy, the initial error is already halved, after 1 year ($t = 4$) is below 0.2 percentage points and after 5 years ($t = 20$) the distance from the REE is smaller than 0.1 points. On the contrary, under EH policy, after 1 quarter inflation is still 0.5 percentage points higher than the REE and after 20 years is still 0.3 points higher.

Table 3 compares the transition to the REE for different ALS-Γ' policies.

Let us consider the ALS-Γ' with $\gamma_\pi^{\mathrm{ALS}} = 2.6$ ($\Gamma' = 0.5$). Inflation can be reduced by more than 0.3 percentage points, in half of the time needed under the ALS-Γ' with $\gamma_\pi^{\mathrm{ALS}} = 2.3$ ($\Gamma' = 0.6$), approximately $\frac{1}{25}$ of the time needed under the ALS-Γ' with $\gamma_\pi^{\mathrm{ALS}} = 1.6$ ($\Gamma' = 0.8$) and more than $\frac{1}{1000}$ of the time needed under the EH-policy.

In this section I have studied the role of policy decisions in determining the speed of convergence under learning, focusing on the mapping from perceived inflation to actual inflation. So far, I have used the concept of speed of learning to refine further the set of 'good' policies: grounded on the set of policies that imply determinate and learnable rational expectations equilibria, I have considered speed of learning as a criterion to characterize its elements. Now I will conduct a welfare analysis under the EH and the ALS policies, in order to address some issues of interest: is the *EH policy* still optimal under learning? Are policies that speed up the learning process always better than policies that involve a slow transition to the REE? Are initial beliefs important in order to evaluate policy decisions?

4.2. Welfare analysis

In January 1999, with the start of stage 3 of the Economic and Monetary Union, monetary competencies were transferred from each country of the European Union to the European Central Bank. Before that date people were accustomed to take into account the monetary policy of their own country when making economic decisions. After the start of stage 3, they faced a new policy-maker (and a new monetary policy) and inflation and output gap equilibria determined under the new policy regime were, in some cases, different from the ones implied by the previous policies. In countries like Italy or Spain, whose rates of inflation are historically higher than in other member states, expected inflation at the start of the EMU was higher than the long run equilibrium determined by the new monetary regime. In this case, it is clear that the transition to the REE could play an important role in the analysis of monetary policy decisions based on welfare measures.[25] Questions like the ones rose at the end of the previous section show up spontaneously.

To answer to those questions I consider separately the two cases where initial expected inflation is higher than the REE and where it is lower. The reason why I proceed in this way is twofold. First, when the policy-maker chooses the policy, he has many instruments to infer private agents' expectations and to gather the sign of the initial prediction bias. Second, the welfare implications may differ in the two cases. In the literature it is well known that under the loss function described in Section 3 the first best plan would be to have inflation and output gap at their target levels, i.e., $\pi_t^{\mathrm{FB}} = 0$ and $x_t^{\mathrm{FB}} = \overline{x}$. As many works have shown, under no commitment, the first best solution is not feasible if $\overline{x} \neq 0$. The optimal (time-consistent) policy in this case leads to a REE with inflation higher than the first best and output gap lower:

$$\pi_t^{\mathrm{REE}} = \frac{\lambda\alpha}{(\lambda + \alpha^2) - \lambda\beta}\overline{x} > \pi_t^{\mathrm{FB}}, \quad x_t^{\mathrm{REE}} = \frac{\lambda(1-\beta)}{(\lambda + \alpha^2) - \lambda\beta}\overline{x} < x_t^{\mathrm{FB}} \quad \text{for all } t.$$

In the previous section, however, I have shown that under learning inflation and output gap could remain far from the REE for a long time. Therefore, if initial perceived inflation is higher than the REE along the transition actual inflation will be higher and output gap lower than the REE. In this case, a policy-maker who grounds his decision on the loss function described in Section 3 would prefer policies that make inflation fall and output gap rise quickly to the REE. On the contrary, if initial perceived inflation is lower than the REE, the policy-maker would prefer policies that make inflation rising and output gap falling slowly to the REE.

Since *EH policy* is not taking into account the transition, I claim that there are ALS-Γ' *policies* that will make our economy better off. In order to verify this claim, I compute the welfare gains (or costs) of adopting alternative ALS-Γ' policies.

The aim of this section is to offer a robustness analysis in the presence of small deviations of private sector expectations from rational expectations, seeking to design policies that are optimal in the long run, but perform also well along the transition. I do not offer a normative analysis of optimal monetary policies under discretion when the private sector is learning. In this case, the true optimal policy under discretion would take into account the fact that the policy-maker could make active use of private agents' learning behavior; since the private agents' perceived law

[25]An important issue that is not addressed in this paper, is related to the notion of welfare used to evaluate policy decisions. In general, the literature on monetary policy and learning focuses on a 'government criterion' that, under rational expectations, may be obtained as a quadratic approximation to households' welfare. Under learning, however, government and households criterions may not coincide. While agents' objective function is affected by the speed of learning since bigger errors (faster learning) may, intuitively, determine a lower welfare, the government's objective function, being based on rational expectations, does not respond directly to the size of the forecast errors. Assessing policy decisions in terms of welfare of the representative agent is an important issue to be addressed in future research.

of motion of the main variables in the economy is time-dependent, also the optimal policy would be time-dependent.[26]

4.2.1. Percentage loss in total welfare

The social loss associated with *EH policy* is defined as

$$L_0^{\mathrm{EH}} = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t(i^{\mathrm{EH}}), x_t(i^{\mathrm{EH}})),$$

where $L(\pi_t(i^{\mathrm{EH}}), x_t(i^{\mathrm{EH}}))$ is the period t loss function defined above and $\pi_t(i^{\mathrm{EH}}), x_t(i^{\mathrm{EH}})$ denote the contingent plans for inflation and output gap under the *EH policy*. Similarly, the social loss associated with ALS-Γ' *policies* is defined as

$$L_0^{\mathrm{ALS}}(\Gamma') = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t(i^{\mathrm{ALS}}(\Gamma')), x_t(i^{\mathrm{ALS}}(\Gamma'))).$$

I measure the welfare cost (or gain) of adopting policy ALS-Γ' instead of the reference *EH policy* as the percentage increase (decrease) in the social loss of moving from EH to ALS-Γ'*policy*:

$$\omega(L_0^{\mathrm{ALS}}(\Gamma')) = \left(\frac{L_0^{\mathrm{ALS}}(\Gamma') - L_0^{\mathrm{EH}}}{L_0^{\mathrm{EH}}}\right) * 100.$$

Note that for values of $\omega(L_0^{\mathrm{ALS}}(\Gamma'))<0$ there is a welfare gain in adopting ALS-Γ' policy instead of EH, while for $\omega(L_0^{\mathrm{ALS}}(\Gamma'))>0$ there is a welfare loss.

I run simulations[27] for 10,000 periods, assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$, the output gap target is $\overline{x} = 0.0044$ and parameters are derived from CGG (2000). The annualized inflation in the REE is 2.0% and the initial expected inflation is 0.5 percentage points higher (in annualized terms) than the REE. I compute social losses under the EH and ALS-Γ' policies for different values of $\gamma_\pi^{\mathrm{ALS}}$ (i.e., different Γ').

Fig. 7 reports the welfare cost (gain) of adopting an ALS-Γ' policy instead of an EH policy.

ALS-Γ' policies with $\gamma_\pi^{\mathrm{ALS}} > \gamma_\pi^{\mathrm{EH}}$, by inducing a flatter slope of the mapping from perceived to actual inflation and therefore a faster convergence, reduce the social loss up to 25% relative to EH policy. A central bank that follows an ALS-Γ' policy with $\gamma_\pi^{\mathrm{ALS}} = 2.6$ can lower the social loss by 20%, relative to the EH policy.

Policies with $\gamma_\pi^{\mathrm{ALS}} < \gamma_\pi^{\mathrm{EH}}$, on the contrary, by inducing a steeper slope of the mapping and a slower convergence, increase the social loss by up to 10%.

In order to analyze how the percentage increase (decrease) in the social loss evolves along the transition, simulations are also run for $T < 10,000$. Table 4 shows the results, pointing out that most of the gain from using an ALS-Γ' policy with fast transition is concentrated in the first 10 years ($T = 40$).

[26] Further analysis in this direction is required and will be left for future research.

[27] In the following simulations the demand shock follows an AR(1) process, $g_t = \rho_g g_{t-1} + \varepsilon_{gt}$, with $\rho_g = 0.95$ and $\varepsilon_{gt} \sim$i.i.d. $N(0, 0.005)$.

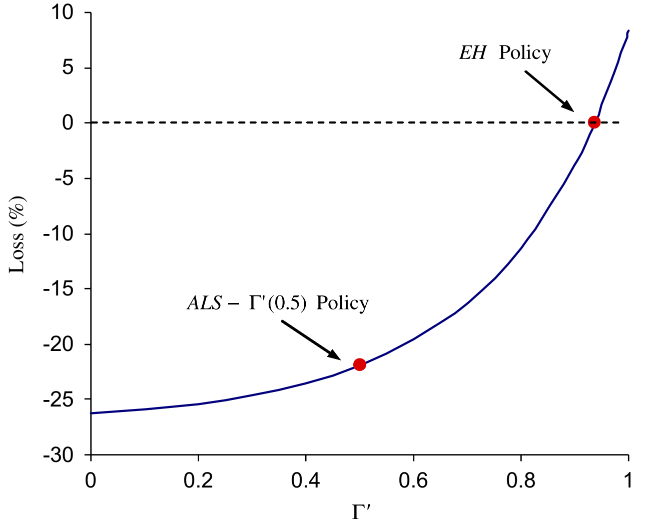

Fig. 7. Percentage loss in total welfare ($\pi_0 > \pi^{RE}$).

Table 4
Percentage loss in total welfare after T quarters ($\pi_0 > \pi^{RE}$)

γ_π^{ALS}	Γ'	$T = 10$	$T = 20$	$T = 40$	$T = 100$	$T = 10,000$
3.6	0.2	−11	−18	−22	−24.5	−25.4
3.0	0.4	−10	−16	−19	−22.5	−23.6
2.6	0.5	−9	−14	−17	−21	−21.9
2.3	0.6	−8	−12	−15	−18	−19.6
1.3	0.9	−1.5	−2	−3	−3.5	−3.8
1.03	0.98	3	4	5	6	6.9

The fact that efficient policies that take account of private learning call for more aggressive responses to inflation than it would be optimal under perfect knowledge is a result also described by Orphanides and Williams (2004). In their model of perpetual learning, endogenous fluctuations determine repeated and persistent deviations of inflation expectations from those implied by rational expectations.[28] Their analysis is mainly focused on numerical simulations, while here the strength of the response to expected inflation is interpreted in the light of the concept of speed of convergence to the REE.[29]

[28]The authors evaluate the performance of different policy rules in this environment and conclude that '*a strategy emphasizing tight inflation control can yield superior economic performance*, *in terms of both inflation and output stability*, *than can policies that appear efficient under rational expectations*' (Orphanides and Williams, 2004).

[29]In this sense this paper could be considered a theoretical justification for Orphanides and Williams work.

Fig. 8 and Table 5 show that under the assumption of an initial expected inflation 0.5 percentage points lower than the REE, by inducing a slower convergence, the policy-maker could sensibly reduce the welfare loss. A policy-maker who speeds up the transition with $\gamma_\pi^{ALS} = 2.6$, would increase the value of the loss function by approximately 40% relative to the EH policy! In this case, in fact, since initial inflation is lower than the REE, it is better to keep inflation expectations below the long-run equilibrium as long as possible.

Also in this case the *EH policy* is not optimal. A central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 1.03$, by increasing the slope of the mapping from perceived inflation to actual inflation to $\Gamma' = 0.98$, can slow down the transition and lower the value of the loss function by approximately 10% relative to the *EH policy*.

Table 5 shows that most of the loss from using an ALS-Γ' *policy* with fast transition is concentrated in the first 5 years.

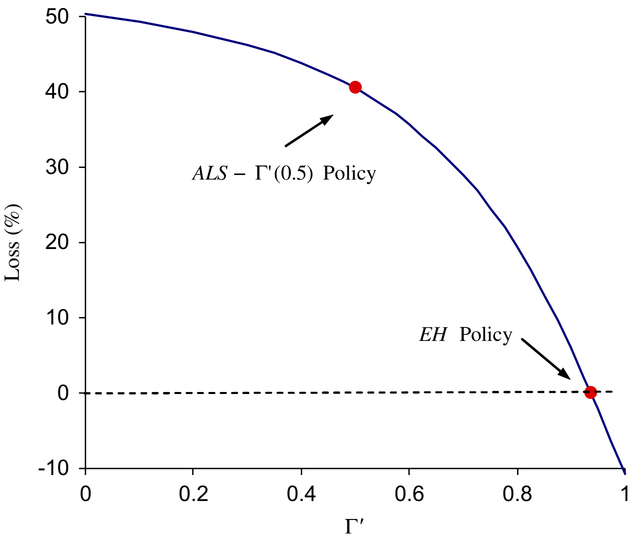

Fig. 8. Percentage loss in total welfare ($\pi_0 < \pi^{RE}$).

Table 5
Percentage loss in total welfare after T quarters ($\pi_0 < \pi^{RE}$)

γ_π^{ALS}	Γ'	$T = 10$	$T = 20$	$T = 40$	$T = 100$	$T = 10,000$
3.6	0.2	56	53	51	49.1	47.9
3.0	0.4	43	43	44	43.9	43.8
2.6	0.5	36	38	39	40.2	40.4
2.3	0.6	28	30	33	35.1	35.7
1.3	0.9	3	4	5	5.5	6.0
1.03	0.98	−3	−5	−6	−8.1	−9.0

Table 6
Inflation equivalent, $100 * \omega_{\pi}^{\text{ALS}}(\Gamma')$ ($\bar{x} = 0.004,\ T = 10,000$)

$\gamma_{\pi}^{\text{ALS}}$	Γ'	$\pi_0 > \pi^{\text{RE}}$	$\pi_0 < \pi^{\text{RE}}$
3.6	0.2	−15.4	22.5
3.0	0.4	−14.6	21.3
2.6	0.5	−13.8	20.2
2.3	0.6	−12.6	18.4
1.3	0.9	−2.5	3.7
1.03	0.98	3.5	−5.1

As it is clear from Figs. 7 and 8, the largest increase of social welfare in the economy is obtained for extreme values of $\gamma_{\pi}^{\text{ALS}} \in \Omega_{\text{ALS}}$. When the economy starts from an expected inflation lower than the REE, the policy-maker should pick up the smallest $\gamma_{\pi}^{\text{ALS}}$ (largest Γ') possible; when initial beliefs are higher than the REE, the response to expected inflation should be the largest (smallest Γ') possible.[30]

4.2.2. *The inflation equivalent*

An alternative measure of welfare loss, denoted by $\omega_{\pi}^{\text{ALS}}(\Gamma')$, is computed as the fraction of inflation under *EH policy* that a central bank is willing to accept above $\pi_t(i^{\text{EH}})$ to be as well off under policy ALS-Γ' as under policy EH (Table 6). Formally the 'inflation equivalent' is implicitly defined by

$$L_0^{\text{ALS}}(\Gamma') = \text{E}_0 \sum_{t=0}^{\infty} \beta^t L((1 + \omega_{\pi}^{\text{ALS}}(\Gamma'))\pi_t(i^{\text{EH}}), x_t(i^{\text{EH}})).$$

Note that a negative value for $\omega_{\pi}^{\text{ALS}}(\Gamma')$ means a welfare gain in adopting ALS-Γ' *policy* instead of EH, while for a positive value there is a welfare loss.

Solving for $\omega_{\pi}^{\text{ALS}}(\Gamma')$, I obtain similar qualitative results as before (Tables 4 and 5). However, by computing the 'inflation equivalent' I can say that when initial expected inflation is 0.5 percentage points higher than in the REE, an ALS-Γ' *policy* with $\gamma_{\pi}^{\text{ALS}} = 2.6$ increases welfare by an amount equivalent to a permanent reduction of inflation of about 14.0% below the $\pi_t(i^{\text{EH}})$ level. On the contrary, when initial expected inflation is lower than in the REE, an ALS-Γ' *policy* with $\gamma_{\pi}^{\text{ALS}} = 1.03$ increases welfare by an amount equivalent to a permanent reduction of inflation of 5.0% below the $\pi_t(i^{\text{EH}})$ level.

5. Robustness

In the previous section I have studied speed of convergence and welfare by running simulations with $\lambda = 0.1$ and $\bar{x} = 0.0044$. Changing these parameters would

[30] If we include in Ω_{ALS} also policies under which $-1 < \Gamma' \leqslant 0$, when initial perceived inflation is higher than the REE, the policy-maker could reduce further social loss by choosing a γ_{π} that induces a slightly negative slope of the T_{ALS} mapping.

not change the result that *EH policy* is not optimal when the central bank makes active use of learning and that, when initial perceived inflation is higher than in the REE, the central bank could increase welfare by inducing a faster transition. However, in the extreme case where the inflation bias is absent, i.e. $\overline{x} = 0$, if initial expected inflation is lower than in the REE, the finding that a slower convergence increases welfare does not hold anymore. In fact, when $\overline{x} = 0$, the optimal policy under discretion results in a REE with inflation and output gap equal to the first best and, in this case, a faster transition is always preferable (Table 7).

An interesting difference with the case with inflation bias is that, for $\overline{x} = 0$, the largest increase of social welfare is not obtained for an extreme value of $\gamma_\pi^{\mathrm{ALS}} \in \Omega_{\mathrm{ALS}}$, but for an 'interior' policy ($\gamma_\pi^{\mathrm{ALS}} = 3.1$ and $\Gamma' = 0.37$; Fig. 9).

In Section 3 I have shown that lowering the relative weight on output gap, the central bank reduces the slope of the $T(.)$ mapping and increases the speed of convergence. Table 8 reports the results of a simulation with $\overline{x} = 0.0044$ and $\lambda = 0.05$ (i.e. the value obtained as the result of a general equilibrium problem in Woodford,

Table 7
Percentage loss in total welfare, when $\overline{x} = 0$ ($\omega(L_0^{\mathrm{ALS}}(\Gamma'))$; $T = 10{,}000$)

$\gamma_\pi^{\mathrm{ALS}}$	Γ'	$\pi_0 > \pi^{\mathrm{RE}}$	$\pi_0 < \pi^{\mathrm{RE}}$
3.6	0.2	−57	−57
3.0	0.4	−59	−59
2.6	0.5	−58	−58
2.3	0.6	−56	−56
1.3	0.9	−20	−20
1.03	0.98	69	69

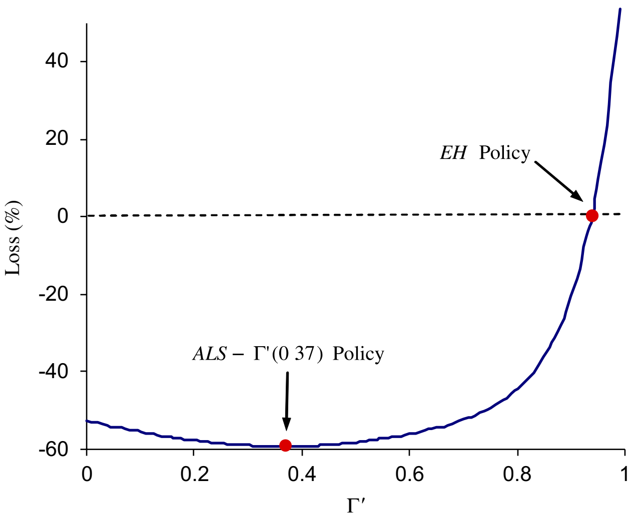

Fig. 9. Percentage loss in total welfare ($\bar{x} = 0$).

Table 8
Percentage loss in total welfare, when $\lambda = 0.05$ ($\omega(L_0^{\mathrm{ALS}}(\Gamma'))$; $\bar{x} = 0.004$, $T = 10{,}000$)

$\gamma_\pi^{\mathrm{ALS}}$	Γ'	$\pi_0 > \pi^{\mathrm{RE}}$	$\pi_0 < \pi^{\mathrm{RE}}$
3.6	0.2	−21	34.5
3.0	0.4	−19	31
2.6	0.5	−18	28
2.3	0.6	−15	24
1.3	0.9	1	−1
1.03	0.98	14	−15.5

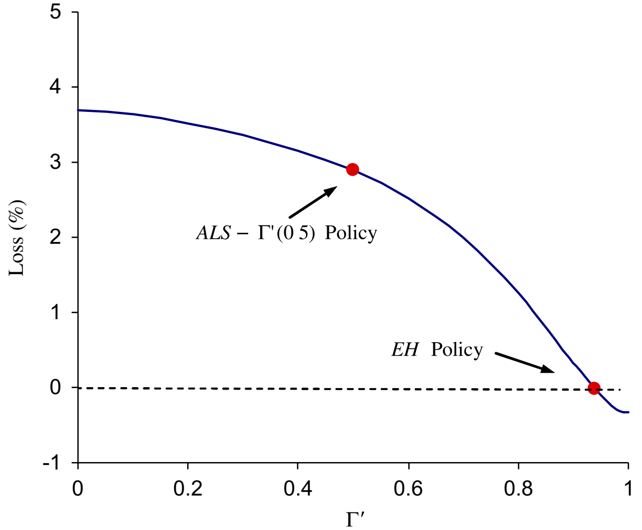

Fig. 10. Percentage loss in total welfare (π_0 symmetrically distributed around π^{RE}).

2003). In this case, the effects on welfare are only slightly different from the ones obtained with $\lambda = 0.1$.

5.1. Initial expected inflation symmetrically distributed around the REE

Under the assumption that initial expected inflation is random and distributed symmetrically[31]around the REE the policy-maker does not know whether initial inflation is higher or lower than the REE. Not surprisingly, in this case, the EH policy would perform much better than in the previous cases, since it would be more difficult to make active use of private agents' learning behavior. Fig. 10 shows that ALS-Γ' policies are in general slightly worst than the EH policy.

[31]Here I assume an uniform distribution between −0.5 and +0.5 percentage points around the REE.

Table 9
Percentage loss in total welfare with cost-push shocks ($\omega(L_0^{\mathrm{ALS}}(\Gamma'))$; $T = 10,000$)

$\gamma_\pi^{\mathrm{ALS}}$	Γ'	$\pi_0 > \pi^{\mathrm{RE}}$	$\pi_0 < \pi^{\mathrm{RE}}$
3.6	0.2	−23	36
3.0	0.4	−22	32
2.6	0.5	−20	29
2.3	0.6	−19	26
1.3	0.9	−4	4.4
1.03	0.98	8.3	−8.1

5.2. An economy with cost-push shocks

The new-Keynesian model analyzed in the previous sections is derived assuming that only one shock hits the economy. Under this assumption the policy-maker neutralizes real effects of the shock whether it follows the *EH policy* or an ALS-Γ' *policy*, i.e., $\gamma_g^* = \gamma_g^{\mathrm{ALS}} = 1/\varphi$. However, when an additional shock hits the economy (for example, a 'cost-push shock', u_t) the policy-maker cannot, in general, neutralize both shocks at the same time.

I consider here an economy with cost-push shocks as in Evan and Honkapohja (2003a). The Phillips curve now is

$$\pi_t = \alpha x_t + \beta \mathrm{E}_t \pi_{t+1} + u_t,$$

where u_t denotes an exogenous cost-push shock following a first order autoregressive process, $u_t = \rho_u u_{t-1} + \varepsilon_{ut}$, with $0 \leqslant |\rho_u| < 1$ and $\varepsilon_{ut} \sim$ i.i.d. $\mathrm{N}(0, \sigma_u^2)$.

In this case the EH policy is an interest reaction function that also depends on u_t, with $\gamma_u^* = (1 - \Psi^*)/\varphi\alpha$ (see Evan and Honkapohja, 2003a) and all the other coefficients are equal to the ones described in Eq. (16). Similarly, ALS policies react also to the cost-push shock and, in order to be consistent in the long run with the REE obtained under EH policy,

$$\gamma_u^{\mathrm{ALS}} = \frac{(1 - \Psi^*(1 - \rho_u \Gamma^*)^{-1}(1 - \rho_u \Gamma'))}{\varphi\alpha}$$

and the other coefficients are the same as in Eq. (20) (see Appendix K).[32]

Since the two policies along the transition to the REE would react differently to u_t, welfare analysis might be affected. Simulations show that the introduction of a cost-push shock affects the results only slightly in the amount of the welfare gain (or loss).

Table 9 shows welfare results when we add an AR(1) shock u_t in the aggregate supply equation,[33] and initial private agents' perceived inflation is 0.5 percentage points higher or lower (in annualized terms) than the REE. In the first case, a central bank that follows an ALS-Γ' policy with $\gamma_\pi^{\mathrm{ALS}} = 2.6$ can lower the value of the loss

[32]As the REE is now a stochastic process that depends linearly also on u_t, agents forecast inflation (and output) using recursive least-squares regressions (based on the well specified PLM) of π_{t+1}(and x_{t+1}) on an intercept and on u_t.

[33]I assume $\lambda = 0.1$, $\bar{x} = 0.0044$, $u_t = \rho_u u_{t-1} + \varepsilon_{u,t}$ with $\rho_u = 0.35$ and $\varepsilon_{u,t} \sim \mathrm{N}(0, 0.005)$.

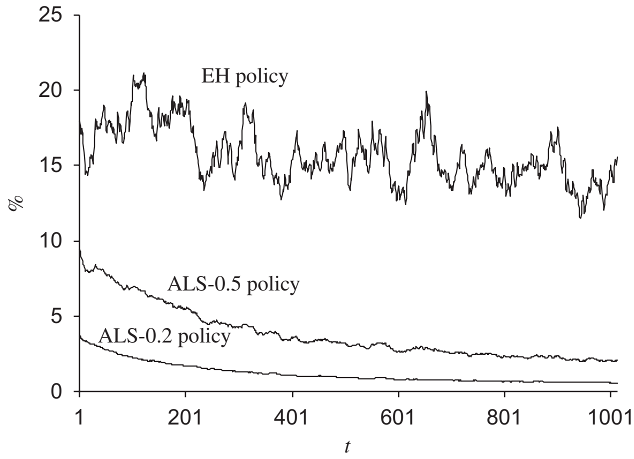

Fig. 11. Differences between standard deviation of inflation under learning and under RE along the transition.

function by approximately 20% relative to the EH policy (22% without cost-push shocks); when initial private agents' perceived inflation is lower than the REE, an ALS-Γ' policy with $\gamma_{\pi}^{\mathrm{ALS}} = 1.03$ reduces losses by approximately 8% (10% without cost-push shocks).

It is worth to notice that increasing the speed of convergence also reduces the variability of inflation along the transition. Fig. 11 shows that the faster the convergence, the smaller would be the difference between the standard deviation of actual inflation under learning and under RE. Under ALS-Γ' *policies* with $\gamma_{\pi}^{\mathrm{ALS}} = 2.6$, at the beginning of the transition to the REE, inflation's standard deviation is 10% higher than under RE, while under EH policy is more than 15%. After 50 years, under ALS-Γ' policy the standard deviation is only 5% higher, while under EH policy is almost unchanged.

6. Further discussions and qualifications

Before concluding I would like to describe three points that to some degree qualify the central argument of this paper and where future research would be useful.

(i) In Sections 3 and 4, because the policy rules under consideration fully offset the g_t shock, the REE is nonstochastic (and the recursion for $a_{\pi,t}$ is deterministic). In such settings, constant gain learning by the private agents is often considered since it is also stable and converges to the RE. For these learning rules the decreasing gain t^{-1} is replaced by a constant gain $0 < \psi < 1$. With $0 < T(.)' < 1$ stability will hold for all $0 < \psi < 1$. Convergence of the EH rule (and of the ALS ones) is now much faster, and always geometric, with a speed that depends on ψ (as well on the policy and other parameters). More in general, since the speed of convergence depends on details of the learning rule used by private agents, this complicates the problem of choosing policy parameters to achieve a desired convergence speed.

(ii) In this paper I focus on the transition properties under the RE optimal discretionary policy. Evans and Honkapohja (2006) discuss at length the properties, under learning, of the REE under their expectations-based rule for implementing the RE optimal monetary policy with commitment. It is well known that when cost-push shocks are present, under rational expectations the optimal discretionary policy is inferior to the optimal policy with commitment, even when the target output gap is zero. An open question is the effect of speed of convergence on expected loss for the EH (2006) rule and for analogously constructed ALS rules.

(iii) The policy-maker loss function used in the paper, $L(\pi_t, x_t) = \frac{1}{2}[\pi_t^2 + \lambda(x_t - \overline{x})^2]$, could be derived as a quadratic approximation to representative agent utility under RE, but this may not be a good approximation under learning. In particular the policy-maker loss function does not take explicit account of the size of the forecast errors made by private agents. An important issue left for future research is the effect of speed of convergence on the welfare of the representative agent.

7. Conclusions

In this paper I have shown that considering learning in a model of monetary policy design is particularly important in order to describe not only the asymptotic properties of the REE to which the economy could converge, but also to describe the dynamics that characterize the transition to this equilibrium.

The central message of the paper is that policy-makers should look at monetary policies which lead to determinate and E-stable equilibria. However, since policies that produce learnable equilibria may imply very different transitions, the policy-maker should also take into account how his decisions affect the speed at which agents' beliefs converge to rational expectations.

Therefore, the aim of this paper is twofold: it helps to explain some facts described in the literature and it shows new results. From the first side, it makes it possible to explain why policies that would be optimal under rational expectations can perform poorly when knowledge is imperfect. Under some policies, the REE is stable under learning, but it could be located near the borderline between stability and instability. In this case the period needed to converge to the REE could be incredibly long. I have shown that by reacting strongly to expected inflation, a central bank can shorten the transition and increase the speed of convergence to the REE. From the other side, I show that a policy-maker who considers his role in determining the dynamics of the agents' learning process could choose a policy rule that induces agents to learn at a given speed, affecting the welfare of society along the transition. In particular, if the policy-maker knows that after a regime change private agents' perceived inflation would be higher than in the REE, by choosing a policy that reacts strongly to expected inflation he would determine a fast convergence and increase social welfare. If, instead, perceived inflation is initially lower than in the REE, a slow transition is preferred only when the output gap target is greater than zero.

This conclusion points out how crucial it is for the design of monetary policies to obtain good data about the state of markets' expectations.

Acknowledgments

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Appendix A. Determinacy of the REE under the simple expectations-based reaction function

Necessary and sufficient condition for determinacy is that the eigenvalues of the F matrix in (4) have real parts inside the unit circle. In the case analyzed here, this reduces to have

$$|\beta + \alpha\varphi(1 - \gamma_\pi)| < 1,$$

that is

$$1 - \left(\frac{1-\beta}{\alpha\varphi}\right) < \gamma_\pi < 1 + \left(\frac{1+\beta}{\alpha\varphi}\right).$$

Appendix B. Proof of Proposition 1

Given the recursive stochastic algorithm

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}(-\alpha\varphi\gamma + [\beta + \alpha\varphi(1-\gamma_\pi)]a_{\pi,t-1} + \alpha g_{t-1} - a_{\pi,t-1}),$$

let

$$h(a_\pi) = \mathrm{E}[-\alpha\varphi\gamma + [\beta + \alpha\varphi(1-\gamma_\pi)]a_\pi + \alpha g_{t-1} - a_\pi],$$

and let a_π be such that $h(a_\pi) = 0$. By the theorem of Benveniste *et al.* (Theorem 3, p. 110), if the derivative of $h(a_\pi)$ is smaller than $-\frac{1}{2}$, then

$$\sqrt{t}(a_{\pi,t} - \overline{a}_\pi) \xrightarrow{\mathrm{D}} \mathrm{N}(0, \sigma_a^2),$$

where σ_a^2 satisfies

$$[h'(\overline{a}_\pi)]\sigma_a^2 + \mathrm{E}[-\alpha\varphi\gamma + [\beta + \alpha\varphi(1-\gamma_\pi)]\overline{a}_\pi - \overline{a}_\pi + \alpha g_t]^2 = 0.$$

Note that the derivative of $\mathrm{E}[-\alpha\varphi\gamma + [\beta + \alpha\varphi(1-\gamma_\pi)]a_\pi - a_\pi]$ being smaller than $-\frac{1}{2}$ coincides with $[\beta + \alpha\varphi(1-\gamma_\pi)]$ being smaller than $\frac{1}{2}$, i.e., γ_π being larger than $1 - (1/2 - \beta)/\alpha\varphi$.

Appendix C. Proof of Corollary 2

The formula for the asymptotic variance of the limiting distribution is

$$\sigma_a^2 = \frac{\alpha^2}{[1-\beta-\alpha\varphi(1-\gamma_\pi)]}\sigma_g^2$$

and the derivative is

$$\frac{\partial\sigma_a^2}{\partial\gamma_\pi} = -\frac{\alpha\varphi}{[1-\beta-\alpha\varphi(1-\gamma_\pi)]^2}\alpha^2\sigma_g^2 < 0.$$

Appendix D. Speed of convergence under the EH policy

Under least squares learning

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}(T(a_{\pi,t-1}) - a_{\pi,t-1})$$

and

$$\overline{a}_\pi = \overline{a}_\pi + t^{-1}(T(\overline{a}_\pi) - \overline{a}_\pi).$$

Subtracting the second expression from the first one, we obtain

$$a_{\pi,t} - \overline{a}_\pi = a_{\pi,t-1} - \overline{a}_\pi + t^{-1}(T(a_{\pi,t-1}) - T(\overline{a}_\pi) - (a_{\pi,t-1} - \overline{a}_\pi)).$$

Substituting the value of $T(.)$, we obtain the following expression:

$$a_{\pi,t} - \overline{a}_\pi = a_{\pi,t-1} - \overline{a}_\pi + t^{-1}(\Phi^* + \Gamma^* a_{\pi,t-1} - \Phi^* - \Gamma^*\overline{a}_\pi - (a_{\pi,t-1} - \overline{a}_\pi)),$$

or

$$a_{\pi,t} - \overline{a}_\pi = (1 - t^{-1} + t^{-1}\Gamma^*)(a_{\pi,t-1} - \overline{a}_\pi).$$

Appendix E. Proof of Lemma 4

Let $(\alpha', \beta', \lambda')$ and $(\alpha'', \beta'', \lambda'')$ index the parameters in the economy that induce the REE $\overline{a}'_\pi$ and $\overline{a}''_\pi$, and let $a'_{\pi,t}$ and $a''_{\pi,t}$ be the agents' estimates at time t.

Let

$$|\Gamma^{*\prime}| = \left|\frac{\beta'\alpha'}{(\lambda' + \alpha'^2)}\right| < \left|\frac{\beta''\alpha''}{(\lambda'' + \alpha''^2)}\right| = |\Gamma^{*\prime\prime}|.$$

Notice that since for all relevant values of the parameters (i.e., $\lambda \in (0,\infty)$, $\beta \in (0,1)$), $\beta\alpha/(\lambda + \alpha^2) > 0$, we have $\Gamma^{*\prime} < \Gamma^{*\prime\prime}$.

From (19) we can write

$$\left|\frac{a'_{\pi,t} - \overline{a}'_\pi}{a'_{\pi,t-1} - \overline{a}'_\pi}\right| = \left|(1 - t^{-1}) + t^{-1}\Gamma^{*\prime}\right|$$

and

$$\left|\frac{a''_{\pi,t}-\overline{a}''_{\pi}}{a''_{\pi,t-1}-\overline{a}''_{\pi}}\right| = |(1-t^{-1})+t^{-1}\Gamma^{*\prime\prime}|.$$

Since $(1-t^{-1})\geqslant 0$ for $t=1,2,\ldots$, the term $(1-t^{-1})+t^{-1}\Gamma^{*}$ is never negative. Therefore,

$$|(1-t^{-1})+t^{-1}\Gamma^{*\prime}|<|(1-t^{-1})+t^{-1}\Gamma^{*\prime\prime}|$$

and

$$\left|\frac{a'_{\pi,t}-\overline{a}'_{\pi}}{a'_{\pi,t-1}-\overline{a}'_{\pi}}\right| < \left|\frac{a''_{\pi,t}-\overline{a}''_{\pi}}{a''_{\pi,t-1}-\overline{a}''_{\pi}}\right|.$$

The economy with $(\alpha',\beta',\lambda')$ induces a faster convergence than the one with $(\alpha'',\beta'',\lambda'')$.

Appendix F. Proof of Proposition 5

Let λ' and λ'' index the policy choices inducing the REE $\pi_t=\overline{a}'_{\pi}$ and $\pi_t=\overline{a}''_{\pi}$, and let $a'_{\pi,t}$ and $a''_{\pi,t}$ be the agents' estimate at time t, under the two policies. Let $\lambda'<\lambda''$; since $\partial\Gamma^{*}/\partial\lambda=\beta\alpha^2/(\lambda+\alpha^2)^2>0$, we have $\Gamma^{*\prime}<\Gamma^{*\prime\prime}$. From lemma 3, the EH-policy with λ' induces a faster convergence than EH-policy with λ''.

Appendix G. Proof of Lemma 6

It is easy to show that the REE of the system described by Eqs. (1) and (2) and the expectations-based reaction functions (3) is

$$\pi_t=\frac{-\alpha\gamma}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha}+\alpha(1-\varphi\gamma_g)g_t,$$

$$x_t=\frac{-\gamma(1-\beta)}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha}+(1-\varphi\gamma_g)g_t,$$

and the REE under the optimal discretionary policy (16) is

$$\pi_t=\frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} \quad \text{and} \quad x_t=\frac{\lambda(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x}.$$

In order to impose the same REE, policy parameters can be obtained by solving the system of equations

$$\begin{cases} \frac{-\alpha\gamma}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha}=\frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x}, \\ \frac{-\gamma(1-\beta)}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha}=\frac{\lambda(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x}, \\ \qquad \alpha(1-\varphi\gamma_g)=0. \end{cases}$$

As the second equation is equal to the first one multiplied by $\alpha/(1-\beta)$, we have a system of two equations in four unknowns

$$\begin{cases} \gamma((\beta-1)\lambda-\alpha^2) = (\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha)\lambda\overline{x}, \\ \gamma_g = \frac{1}{\varphi}, \end{cases}$$

with multiple solutions. By imposing determinacy and E-stability of the REE, we could restrict the set of policies (that still include infinite elements). As shown in Bullard and Mitra necessary and sufficient condition for the REE of the dynamic system described by Eqs. (1)–(3) to be determinate and E-stable is

$$\gamma_x < \frac{1+\beta}{\varphi\beta}$$

and

$$1-\frac{(1-\beta)}{\alpha}\gamma_x < \gamma_\pi < 1+2\frac{(1+\beta)}{\varphi\alpha}-\frac{(1+\beta)}{\alpha}\gamma_x.$$

Therefore, $(\gamma_\pi, \gamma, \gamma_g, \gamma_x) \in \Omega$

Appendix H. Proof of Lemma 7

From the definition of ALS-policies (20), we have that:

(i) $\gamma_g^{\mathrm{ALS}} = 1/\varphi = \gamma_g$.
(ii) $\gamma_x^{\mathrm{ALS}} = 1/\varphi < (1+\beta)/\varphi\beta$.
(iii) For $1 < \Gamma' < 1$, $\gamma_\pi^{\mathrm{ALS}} = 1+\beta-\Gamma'/\alpha\varphi < 1+2(1+\beta)/\varphi\alpha-(1+\beta)/\alpha\gamma_x$; in fact, substituting the value of γ_x^{ALS} on it, we have that $1-(1-\beta)/\alpha < 1+(\beta-\Gamma')/\alpha\varphi < 1+(\beta-1)/\alpha\varphi$.
(iv) By substituting the values of γ_x^{ALS} and $\gamma_\pi^{\mathrm{ALS}}$ into $\gamma = \lambda\overline{x}/((\beta-1)\lambda-\alpha^2)(\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha)$ we obtain $\gamma^{\mathrm{ALS}} = -\Phi^*(1-\Gamma')/(1-\Gamma^*)\alpha\varphi = \gamma$.

Therefore all elements $(\gamma_\pi^{\mathrm{ALS}}, \gamma^{\mathrm{ALS}}, \gamma_g^{\mathrm{ALS}}, \gamma_x^{\mathrm{ALS}}) \in \Omega^{\mathrm{ALS}}$ are also in Ω.

Appendix I. Proof of Proposition 8

Let rewrite $\gamma_\pi^{\mathrm{ALS}} = (1+(\beta-\Gamma')/\alpha\varphi)$ as

$$\Gamma' = \alpha\varphi+\beta-\alpha\varphi\gamma_\pi^{\mathrm{ALS}}.$$

Since $\partial T'(a_{\pi,t})/\partial a_{\pi,t} = \Gamma'$, it follows that

$$\frac{\partial\Gamma'}{\partial\gamma_\pi^{\mathrm{ALS}}} = -\alpha\varphi < 0.$$

Appendix J. Proof of Proposition 10

Since under ALS-Γ' policy

$$\Gamma' = (1-\gamma_\pi^{\mathrm{ALS}})\alpha\varphi + \beta,$$

given the parameters α, φ, β, to each value of the policy reaction parameter $\gamma_\pi^{\mathrm{ALS}}$ it corresponds to a slope of the $T(.)$ mapping, Γ', that does not depend on λ.

Appendix K. ALS policies in an economy with cost-push shocks

I want to show that ALS policies $(\gamma_\pi^{\mathrm{ALS}}, \gamma^{\mathrm{ALS}}, \gamma_g^{\mathrm{ALS}}, \gamma_x^{\mathrm{ALS}})$, with

$$\gamma_u^{\mathrm{ALS}} = \frac{(1-\Psi^*(1-\rho_u\Gamma^*)^{-1}(1-\rho_u\Gamma'))}{\varphi\alpha}$$

and all the other coefficients as in (20), are consistent with the REE attainable under the RE optimal discretionary policy.

Under the EH policy described in Evans and Honkapohja (2003a) the REE is

$$\pi_t = a_\pi + b_\pi u_t, \quad x_t = a_x + b_x u_t$$

and

$$\mathrm{E}_t\pi_{t+1} = a_\pi + b_\pi\rho_u u_t, \quad \mathrm{E}_t x_{t+1} = a_x + b_x\rho_u u_t,$$

with

$$a_\pi = \Phi^*(1-\Gamma^*)^{-1}, \quad b_\pi = \Psi^*(1-\rho_u\Gamma^*)^{-1},$$

$$a_x = \frac{(1-\beta)}{\alpha}\frac{\Phi^*}{(1-\Gamma^*)}, \quad b_x = \frac{(1-\rho_u\beta)\Psi^* - (1-\rho_u\Gamma^*)}{\alpha(1-\rho_u\Gamma^*)}.$$

Substituting the new ALS policy rule into the IS and the Phillips curves we obtain a system of dynamic equations that does not depend on g_t and $\mathrm{E}_t x_{t+1}$:

$$\pi_t = \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} + \Gamma'\mathrm{E}_t\pi_{t+1} + \frac{\Psi(1-\rho_u\Gamma')}{(1-\rho_u\Gamma^*)}u_t$$

and

$$x_t = \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} - \frac{(\beta-\Gamma')}{\alpha}\mathrm{E}_t\pi_{t+1} - \frac{(1-\Psi^*(1-\rho_u\Gamma^*)^{-1}(1-\rho_u\Gamma'))}{\alpha}u_t.$$

Using $\mathrm{E}_t\pi_{t+1} = a_\pi + b_\pi\rho_u u_t$ we obtain the ALM for inflation

$$\pi_t = \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} + \Gamma' a_\pi + \left(\Gamma' b_\pi\rho_u + \frac{\Psi(1-\rho_u\Gamma')}{(1-\rho_u\Gamma^*)}\right)u_t$$

and in the REE

$$a_\pi = \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} + \Gamma' a_\pi = \Phi^*(1-\Gamma^*)^{-1}$$

and

$$b_{\pi} = \Gamma' b_{\pi} \rho_u + \frac{\Psi(1 - \rho_u \Gamma')}{(1 - \rho_u \Gamma^*)} = \Psi^*(1 - \rho_u \Gamma^*)^{-1}.$$

Similarly for a_x and b_x.

Therefore, the ALS policies are consistent with the REE attainable under the RE optimal discretionary policy.

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