Tests for Serial Correlation in Regression Analysis Based on the Periodogram of Least-Squares Residuals

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Tests for serial correlation in regression analysis based on the periodogram of least-squares residuals

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Summary

A well-known procedure for testing for serial correlation is to plot out the sample path of the cumulated periodogram and to compare the resulting graph with the Kolmogorov–Smirnov limits. The paper considers small-sample aspects of this procedure when the periodogram is calculated from the residuals from least-squares regression. It is shown that for a test against an excess of low-frequency relative to high-frequency variation in the errors of the regression model, a pair of lines can be drawn on the graph such that if the path crosses the upper line the hypothesis of serial independence is definitely rejected, while if the path fails to cross the lower line the hypothesis is definitely accepted. In the intermediate case the test is inconclusive. Similar procedures are given for tests against an excess of high-frequency variation and for two-sided tests. To facilitate the tests a table of significance values of the appropriate modified Kolmogorov–Smirnov statistics is given. A further test based on the mean of the ordinates of the cumulated periodogram is considered. It is shown that bounding significance values are easily obtainable from significance values of the mean of a uniform distribution.

1. Introduction

Least-squares regression analysis of time-series data rests on the assumption that the errors in the regression model are serially uncorrelated. If there is serial correlation present the least-squares estimates of the regression coefficients can be inefficient and their estimates of variance can be biased. Tests of serial correlation therefore form an essential part of the least-squares analysis of time-series data.

A test based on the statistic $d = \Sigma(z_t - z_{t-1})^2/\Sigma z_t^2$, where $z_1, \ldots, z_T$ are the residuals from the fitted regression, was proposed by Durbin & Watson (1950, 1951). An exact test which may be applied when the bounds test proposed in these papers is inconclusive is given by Durbin (1969*b*). Modifications needed when the model is dynamic, e.g. when some of the regressors are lagged dependent variables, are developed by Durbin (1969*a*). Hannan & Terrell (1968) have given a useful review of the field.

Although it was shown by Durbin & Watson (1950) that the d-test has reasonably good power properties against a first-order autoregressive alternative, the investigator may wish to have a more comprehensive picture of the departure from serial independence than is provided by a single statistic such as d. From this standpoint it seems more appropriate to ask what the data tell us about the departure from serial independence. than to set up a particular parametric alternative and seek a test which has high power against it.

Quite apart from the regression aspects of the problem, a suitable technique for studying the general nature of the serial dependence in a stationary series of observations $y_1, \ldots, y_T$ is to plot out the cumulated periodogram

$$s_j = \sum_{r=1}^{j} p_r \Big/ \sum_{r=1}^{m} p_r, \tag{1}$$

where $r = 1, \ldots, m$ are the periodogram ordinates,

$$p_r = \frac{2}{T}\left|\sum_{t=1}^{T} y_t\, e^{(2\pi i r t)/T}\right|^2.$$

The resulting diagram presents an illuminating picture of the pattern of serial dependence in the data. It was shown by Bartlett (1954, 1966, p. 318) that when $y_1, \ldots, y_T$ are independent the sample path behaves asymptotically like that of the sample distribution function in the theory of order statistics. Bartlett therefore suggested that the ordinary Kolmogorov–Smirnov limits should be applied to the sample path to provide a test of serial independence.

Small-sample aspects of this procedure are considered by Durbin (1967), who showed that for tests on the cumulated periodogram slight modifications of the usual Kolmogorov–Smirnov statistics are desirable. Significance points of these statistics are given in Table 1 below and for a test against positive serial correlation are used in the following way. Let c_0 be the significance value obtained from Table 1. If the sample path of s_j crosses the line $y = c_0 + j/m$ the hypothesis of serial independence is rejected; otherwise it is accepted.

The present paper considers the modifications to this procedure which are necessary when the periodogram is computed from least-squares residuals. That modifications are necessary follows from the fact that even when the errors in a regression model are serially uncorrelated the calculated residuals are correlated in virtue of the singularity of their distribution. In place of the single line $y = c_0 + j/m$, two lines are necessary and these lead to a bounds test analogous to that for the d test. When the path of s_j crosses the upper line the hypothesis of serial independence of the errors is rejected and when it fails to cross the lower line the hypothesis is accepted. In the intermediate case the test is inconclusive.

The horizontal distance between the two lines turns out to be the ratio of the number of degrees of freedom attributable to regression to the total number of degrees of freedom. This indicates rather neatly the way in which the degree of indeterminacy in the bounds procedure depends on both the number of regressors and the sample size.

Similar procedures are given for tests against negative serial correlation and for two-sided tests.

It is worth stressing that a single table of significance values suffices for all these tests. In this respect, the situation is better than that for d, where different sets of values are required for different numbers of regressors in the model.

A further statistic

$$\bar{s} = \frac{1}{m-1}\sum_{j=1}^{m-1} s_j$$

was suggested by Durbin (1967). This has the attractive property that when calculated from $T = 2m+1$ independent normally distributed observations, it is distributed exactly as the mean of $m-1$ independent uniform (0, 1) variables. In § 6, the distribution of $\bar{s}$ when calculated from least-squares residuals is considered. Because $\bar{s}$ has the general form considered by Durbin & Watson (1950) we can apply their results directly to give the inequalities $\bar{s}_L \leqslant \bar{s} \leqslant \bar{s}_U$, where $\bar{s}_L$ and $\bar{s}_U$ are found to have significance values which are simply obtainable from those of the mean of a rectangular distribution. We therefore have an alternative bounds test in which, for a test against positive serial correlation, the null hypothesis is rejected when $\bar{s}$ is greater than the significance value of $\bar{s}_U$ and accepted when $\bar{s}$ is less than the significance value of $\bar{s}_L$, the remaining case being indeterminate.

The tests are illustrated by applying them to two sets of published data.

2. The cumulated periodogram

In this section we consider briefly some basic properties of the cumulated periodogram when calculated from a series $y_1, \ldots, y_T$ of independent $N(\mu, \sigma^2)$ variables, referring the reader to Durbin (1967) for a fuller treatment. Consideration of regression complications is deferred to the next section.

The periodogram p_j is defined by

$$a_j = \sqrt{\frac{2}{T}} \sum_{t=1}^{T} y_t \cos\left(\frac{2\pi jt}{T}\right), \quad b_j = \sqrt{\frac{2}{T}} \sum_{t=1}^{T} y_t \sin\left(\frac{2\pi jt}{T}\right), \quad p_j = a_j^2 + b_j^2 \; (j = 1, \ldots, [\tfrac{1}{2}T]), \tag{2}$$

where $[\frac{1}{2}T] = \frac{1}{2}T$ for T even and $\frac{1}{2}T - \frac{1}{2}$ for T odd. For simplicity we assume that T is odd, $T = 2m+1$, referring again to the case of T even in § 5.

It is well known that the graph of p_j against j presents a highly unstable appearance which makes it of doubtful value for visual inspection. A better method of presenting the information in the p_j's is to plot the cumulated periodogram (1). We shall suppose that this is plotted against j/m so that the sample path lies entirely inside the unit square.

It is known that when $y_1, \ldots, y_T$ are independently normally distributed, $s_1, \ldots, s_{m-1}$ are distributed like the order statistics from a sample of $m-1$ independent observations from a uniform $(0, 1)$ distribution. Bartlett's (1954, 1966, p. 319) suggestion was to test for serial independence by testing the maximum discrepancy between s_j and its expectation, i.e. j/m. For a test against an excess of low-frequency relative to high-frequency variation, as would be expected in the presence of positive serial correlation, this approach leads to the statistic

$$c^+ = \max_j \left(s_j - \frac{j}{m}\right), \tag{3}$$

while for a test against an excess of high-frequency variation the appropriate statistic is

$$c^- = \max_j \left(\frac{j}{m} - s_j\right). \tag{4}$$

The corresponding two-sided statistic is

$$c = \max_j \left| s_j - \frac{j}{m} \right| = \max(c^+, c^-). \tag{5}$$

These statistics are closely related to the Kolmogorov–Smirnov statistics D_n^+, D_n^-, D_n and to their modified forms C_n^+, C_n^-, C_n considered by Pyke (1959) and Brunk (1962). For example, $D_n^- = \max_j \{s_j - (j-1)/(m-1)\}$ and $C_n^- = c^+$. The relation between these various statistics and the reasons for using the forms stated are considered in detail by Durbin (1967).

Significance values for the statistics (3)–(5) are given in Table 1. The procedure suggested for using these values is as follows. Suppose that we wish to test against excess low-frequency relative to high-frequency variation. Let c_0 be the appropriate significance value of c^+

obtained from Table 1. Draw on a diagram the line $y = c_0 + j/m$ and the sample path obtained by joining up the points $(j/m, s_j)$. If the sample path crosses the line, reject the hypothesis of serial independence. Similarly, for a test against excess high-frequency relative

Table 1. *Significance values for* c^+ *and* c (*also for* C_n^+ *and* C)

$\mathrm{pr}(c^+ > \text{tabled value}) = \mathrm{pr}(C_n^+ > \text{tabled value}) = \alpha$; $\mathrm{pr}(c > \text{tabled value}) = \mathrm{pr}(C > \text{tabled value}) \simeq 2\alpha$.

	α						α				
n	0·10	0·05	0·025	0·01	0·005	n	0·10	0·05	0·025	0·01	0·005
1	0·40000	0·45000	0·47500	0·49000	0·49500	41	0·14916	0·17215	0·19254	0·21667	·233310
2	·35044	·44306	·50855	·56667	·59596	42	·14761	·17034	·19050	·21436	·23081
3	·35477	·41811	·46702	·53456	·57900	43	·14611	·16858	·18852	·21212	·22839
4	·33435	·39075	·44641	·50495	·54210	44	·14466	·16688	·18661	·20995	·22605
5	·31556	·37359	·42174	·47692	·51576	45	·14325	·16524	·18475	·20785	·22377
6	·30244	·35522	·40045	·45440	·48988	46	·14188	·16364	·18295	·20581	·22157
7	·28991	·33905	·38294	·43337	·46761	47	·14055	·16208	·18120	·20383	·21943
8	·27828	·32538	·36697	·41522	·44819	48	·13926	·16058	·17950	·20190	·21735
9	·26794	·31325	·35277	·39922	·43071	49	·13800	·15911	·17785	·20003	·21534
10	·25884	·30221	·34022	·38481	·41517	50	·13678	·15769	·17624	·19822	·21337
11	·25071	·29227	·32894	·37187	·40122	51	·13559	·15630	·17468	·19645	·21146
12	·24325	·28330	·31869	·36019	·38856	52	·13443	·15495	·17316	·19473	·20961
13	·23639	·27515	·30935	·34954	·37703	53	·13330	·15363	·17168	·19305	·20780
14	·23010	·26767	·30081	·33980	·36649	54	·13221	·15235	·17024	·19142	·20604
15	·22430	·26077	·29296	·33083	·35679	55	·13113	·15110	·16884	·18983	·20432
16	·21895	·25439	·28570	·32256	·34784	56	·13009	·14989	·16746	·18828	·20265
17	·21397	·24847	·27897	·31489	·33953	57	·12907	·14870	·16613	·18677	·20101
18	·20933	·24296	·27270	·30775	·33181	58	·12807	·14754	·16482	·18529	·19942
19	·20498	·23781	·26685	·30108	·32459	59	·12710	·14641	·16355	·18385	·19786
20	·20089	·23298	·26137	·29484	·31784	60	·12615	·14530	·16230	·18245	·19635
21	·19705	·22844	·25622	·28898	·31149	62	·12431	·14316	·15990	·17973	·19341
22	·19343	·22416	·25136	·28346	·30552	64	·12255	·14112	·15760	·17713	·19061
23	·19001	·22012	·24679	·27825	·29989	66	·12087	·13916	·15540	·17464	·18792
24	·18677	·21630	·24245	·27333	·29456	68	·11926	·13728	·15329	·17226	·18535
25	·18370	·21268	·23835	·26866	·28951	70	·11771	·13548	·15127	·16997	·18288
26	·18077	·20924	·23445	·26423	·28472	72	·11622	·13375	·14932	·16777	·18051
27	·17799	·20596	·23074	·26001	·28016	74	·11479	·13208	·14745	·16566	·17823
28	·17533	·20283	·22721	·25600	·27582	76	·11341	·13048	·14565	·16363	·17604
29	·17280	·19985	·22383	·25217	·27168	78	·11208	·12894	·14392	·16167	·17392
30	·17037	·19700	·22061	·24851	·26772	80	·11079	·12745	·14224	·15978	·17188
31	·16805	·19427	·21752	·24501	·26393	82	·10955	·12601	·14063	·15795	·16992
32	·16582	·19166	·21457	·24165	·26030	84	·10835	·12462	·13907	·15619	·16802
33	·16368	·18915	·21173	·23843	·25683	86	·10719	·12327	·13756	·15449	·16618
34	·16162	·18674	·20901	·23534	·25348	88	·10607	·12197	·13610	·15284	·16440
35	·15964	·18442	·20639	·23237	·25027	90	·10499	·12071	·13468	·15124	·16268
36	·15774	·18218	·20387	·22951	·24718	92	·10393	·11949	·13331	·14970	·16101
37	·15590	·18003	·20144	·22676	·24421	94	·10291	·11831	·13198	·14820	·15940
38	·15413	·17796	·19910	·22410	·24134	96	·10192	·11716	·13070	·14674	·15783
39	·15242	·17595	·19684	·22154	·23857	98	·10096	·11604	·12944	·14533	·15631
40	·15076	·17402	·19465	·21906	·23589	100	·10002	·11496	·12823	·14396	·15483

Values for odd N greater than 60 are available from the author on request.

to low-frequency reject if the path crosses the line $y = -c_0 + j/m$. For a two-sided test, reject if the path crosses either line. For the purpose of drawing these lines it is convenient to regard j as varying continuously over the interval $(0, m)$ although strictly speaking only the values for $j = 1, \ldots, m$ are required.

3. The cumulated periodogram based on least-squares residuals

In this section we consider the modifications to these plotting and testing procedures which are necessary when the periodogram is computed from least-squares residuals. Suppose we have the regression model

$$y = X\beta + u, \tag{6}$$

where y is a vector of T observations of the dependent variable, X is a matrix of T observations on k regressors and u is the error vector. One of the columns of X is taken to be a vector of constants, so that the number of regressors in addition to the constant term is $k-1$. Then X is assumed to be constant and to have rank k. For the reasons given by Durbin (1969*a*) none of the tests proposed in this paper is valid, even asymptotically, when any of the regressors are lagged y's.

Suppose that a least-squares analysis has been carried out and that we wish to examine the residuals $z = y - Xb$, where $b = (X'X)^{-1}X'y$, for evidence of serial correlation. We calculate the statistics considered in the last section from the elements $z_1, \ldots, z_T$ of z, i.e. take for $j = 1, \ldots, m$,

$$\left.\begin{aligned} a_j &= \sqrt{\frac{2}{T}} \sum_{t=1}^{T} z_t \cos\left(\frac{2\pi jt}{T}\right), \quad p_j = a_j^2 + b_j^2, \\ b_j &= \sqrt{\frac{2}{T}} \sum_{t=1}^{T} z_t \sin\left(\frac{2\pi jt}{T}\right), \quad s_j = \frac{\sum_{r=1}^{j} p_r}{\sum_{r=1}^{m} p_r}. \end{aligned}\right\} \tag{7}$$

Of course in practice the factor $\sqrt{(2/T)}$ can be dropped without affecting $s_1, \ldots, s_m$.

The basic proposal of this paper is that serial independence in the errors u should be assessed by inspecting the graph of s_j against j/m. We therefore have to consider how the distribution of the s_j's is affected by the fact that $z_1, \ldots, z_T$ are residuals from regression. On the basis of the results of Durbin & Watson (1950) we are led to conjecture that the following procedures will give valid and suitable bounds tests.

Assume that T and k are odd and let $m' = \frac{1}{2}(T-k)$. On the graph of s_j against j/m draw two parallel lines, an upper line $y = c_0 + j/m'$ $(0 \leqslant j \leqslant m')$ and a lower line $y = c_0 + \{j - \frac{1}{2}(k-1)\}/m'$ for $\frac{1}{2}k - \frac{1}{2} \leqslant j \leqslant m$; here c_0 is the significance value of c^+ at significance level α obtained by entering Table 1 with m' in place of m. For a test against excess low-frequency relative to high-frequency variation, if the sample path crosses the upper line we regard the result as significant and if it fails to cross the lower line we regard the result as non-significant. The intermediate case is regarded as inconclusive.

Similarly, for a test against excess high-frequency relative to low-frequency, draw an upper line $y = -c_0 + j/m'$ $(0 \leqslant j \leqslant m')$ and a lower line

$$y = -c_0 + \{j - \tfrac{1}{2}(k-1)\}/m' \quad \text{for} \quad \tfrac{1}{2}k - \tfrac{1}{2} \leqslant j \leqslant m.$$

If the path crosses the lower line, the result is regarded as significant, if it fails to cross the upper line, it is regarded as non-significant and in the remaining case it is treated as inconclusive.

A two-sided test is obtained by applying both one-sided tests; for high α this may require a slight adjustment of the significance level of the test as indicated in the last paragraph of the paper.

In the next section it is shown that these are valid bounds tests in the sense that for a test at significance level α the probability of rejection does not exceed α and the probability of acceptance does not exceed $1-\alpha$. The bounds are best in the sense that for suitable sets of regressors the rejection and acceptance probabilities can be made to equal α and $1-\alpha$ exactly. The cases of even T and k are considered in § 5.

4. Proofs of validity of tests

To establish the validity of these tests consider the one-sided test against excess low frequency. Let $G_U(x)$ be the distribution function of $\max(s_j - j/m')$ and let $C(x)$ be the distribution function of $\max(s_{jU} - j/m')$, both maxima being taken over the range $j = 1, \ldots, m'-1$. Here

$$s_{jU} = \frac{\sum_{i=1}^{j} \xi_i^2}{\sum_{i=1}^{T-k} \xi_i^2} \quad (j = 1, \ldots, m),$$

$\xi_1, \ldots, \xi_{T-k}$ being independent $N(0, \sigma^2)$ variables. Put $p'_r = \xi_{2r-1}^2 + \xi_{2r}^2$, giving

$$s_{jU} = \frac{\sum_{r=1}^{j} p'_r}{\sum_{r=1}^{m'} p'_r},$$

from which we deduce that the s_{jU} are distributed as the s_j in (1) with m replaced by m'. Consequently $C(x)$ is the distribution function of c^+, as considered for the non-regression case considered in § 2, with m replaced by m'. The validity of the proposed test therefore requires:

LEMMA 1. *For all* $x \geqslant 0$,

$$G_U(x) \geqslant C(x). \tag{8}$$

PROOF. Now

$$G_U(x) = \mathrm{pr}(s_j \leqslant x + j/m', j = 1, \ldots, m'-1)$$

and

$$C(x) = \mathrm{pr}(s_{jU} \leqslant x + j/m', j = 1, \ldots, m'-1).$$

It is well known that ratios of the form s_j and s_{jU} are distributed independently of their denominators (Pitman, 1937). Consequently, their conditional distributions given

$$\sum_{t=1}^{T} z_t^2 = \sum_{i=1}^{T-k} \xi_i^2 = 1$$

are the same as their unconditional distributions. Let $w = Fz$, where F is the Fourier matrix whose $(2j-1)$th and $(2j)$th rows are $\sqrt{(2/T)}\cos(2\pi jt/T)$ and $\sqrt{(2/T)}\sin(2\pi jt/T)$, $t = 1, \ldots, T$; $j = 1, \ldots, m$, and whose Tth row is $[T^{-\frac{1}{2}}, \ldots, T^{-\frac{1}{2}}]$. Since $p_j = w_{2j-1}^2 + w_{2j}^2$,

$$s_j = \sum_{i=1}^{2j} w_i^2 \Big/ \sum_{i=1}^{2m} w_i^2; \quad \text{also} \quad \sum_{t=1}^{T} z_t^2 = \sum_{i=1}^{T} w_i^2 \quad \text{and} \quad w_T = 0.$$

It is therefore sufficient to prove that

$$\operatorname{pr}\left(\sum_{i=1}^{2j} w_i^2 \leqslant a_j^2, j = 1, \ldots, m'-1\right) \geqslant \operatorname{pr}\left(\sum_{i=1}^{2j} \zeta_i^2 \leqslant a_j^2, j = 1, \ldots, m'-1\right) \tag{9}$$

conditional on

$$\sum_{i=1}^{T-1} w_i^2 = \sum_{i=1}^{T-k} \zeta_i^2 = 1,$$

where $a_j^2 = x + j/m'$.

We have $w = Fz = FMU$, where $M = I - X(X'X)^{-1}X'$. The variance matrix of the vector $[w_1, \ldots, w_{T-k}, 0, \ldots, 0]'$ is $\sigma^2 QFM^2F'Q$, where Q is a matrix with unity in the first $T-k$ positions in the leading diagonal and zeros elsewhere. Transform orthogonally to a set of $T-k$ uncorrelated variables $\eta_1, \ldots, \eta_{T-k}$ together with k zeros. Then $\eta_1, \ldots, \eta_{T-k}$ have variances $\sigma^2\nu_1, \ldots, \sigma^2\nu_{T-k}$, where $\nu_1, \ldots, \nu_{T-k}$ together with k zeros are the eigenvalues of $QFM^2F'Q$, i.e. of $MF'QF$, since the eigenvalues of the product of two square matrices are independent of the order of multiplication and Q and M are idempotent. Durbin & Watson (1950, p. 415) have shown that $\lambda_i \leqslant \nu_i \leqslant \lambda_{i+k}$, $i = 1, \ldots, T-k$, where $\lambda_1, \ldots, \lambda_T$ are the eigenvalues of $F'QF$. Since F is orthogonal these are the same as those of Q, i.e. $T-k$ ones and k zeros. Thus $0 \leqslant \nu_i \leqslant 1, i = 1, \ldots, k$ and $\nu_i = 1, i = k+1, \ldots, T-k$.

For simplicity, suppose that every $\nu_i > 0$. This is the usual situation and the fact that the results hold for the remaining cases follows by continuity arguments on letting the appropriate ν_i's $\to 0$. Let $\zeta_i = \eta_i/\sqrt{\nu_i}, i = 1, \ldots, T-k$. Then $\zeta_1, \ldots, \zeta_{T-k}$ are independent $N(0, \sigma^2)$ variables.

Let $v = Hz$, where H is a $T-k \times T$ matrix whose rows are orthonormal and orthogonal to the columns of X. Then

$$\sum_{i=1}^{T-k} v_i^2 = \sum_{t=1}^{T} z_t^2 = \sum_{i=1}^{T-1} w_i^2 \quad \text{and} \quad E(vv') = \sigma^2 H\{I - X(X'X)^{-1}X'\}H' = \sigma^2 I.$$

Since $\zeta_1, \ldots, \zeta_{T-k}$ are obtained by a linear transformation of the w_i's they must be obtainable from the v_i's by a linear transformation $\zeta = Kv$ say, where K is $T-k \times T-k$. We have $\sigma^2 I = E(\zeta\zeta') = E(Kvv'K') = \sigma^2 KK'$. Thus K is orthogonal, so that

$$\sum_{i=1}^{T-k} \zeta_i^2 = \sum_{i=1}^{T-k} v_i^2 = \sum_{i=1}^{T-1} w_i^2. \tag{10}$$

It follows that the conditional distribution of $\zeta_1, \ldots, \zeta_{T-k}$ given $w_1^2 + \ldots + w_{T-1}^2 = 1$ is that of equi-probable points on the sphere $\zeta_1^2 + \ldots + \zeta_{T-k}^2 = 1$.

Let R be the region $w_1^2 + \ldots + w_{2j}^2 \leqslant a_j^2$, $j = 1, \ldots, m'-1$ and let R' be the transformed region corresponding to R in ζ space. The left-hand probability in (9) is the fraction of the surface volume of the $(T-k)$-dimensional unit sphere cut off by R'. Let S be the region $\zeta_1^2 + \ldots + \zeta_{2j}^2 \leqslant a_j^2$, $j = 1, \ldots, m'-1$. The right-hand probability in (9) is the fraction of the surface volume of the unit sphere cut off by S. We prove (9) by showing that by means of a suitable choice of co-ordinate axes S can be made to lie inside R'.

Consider the region $w_1^2+\ldots+w_{2j}^2 \leqslant a_j^2$. This is a cylinder R_j whose $(T-k-2j)$-dimensional generators $w_i =$ constant, $i = 1, \ldots, 2j, w_1^2+\ldots+w_{2j}^2 = a_j^2$ touch the sphere $w_1^2+\ldots+w_{T-k}^2 = a_j^2$ at the intersection of the cylinder $w_1^2+\ldots+w_{2j}^2 = a_j^2$ with the region $w_i = 0$, $i = 2j+1, \ldots, T-k$. Transform from $w_1, \ldots, w_{T-k}$ first to $\eta_1, \ldots, \eta_{T-k}$ and then to $\zeta_1, \ldots, \zeta_{T-k}$. Since the transformation from the w's to the η's is orthogonal, the sphere $w_1^2+\ldots+w_{T-k}^2 = a_j^2$ transforms first to $\eta_1^2+\ldots+\eta_{T-k}^2 = a_j^2$ and then to the ellipsoid $\nu_1\zeta_1^2+\ldots+\nu_{T-k}\zeta_{T-k}^2 = a_j^2$. Also R_j transforms to a cylinder R_j' with ellipsoidal cross-section whose generators touch the ellipsoid $\nu_1\zeta_1^2+\ldots+\nu_{T-k}\zeta_{T-k}^2 = a_j^2$. Since $\nu_i \leqslant 1$ this ellipsoid encloses the sphere

$$\zeta_1^2+\ldots+\zeta_{T-k}^2 = a_j^2.$$

Let S_j be the cylinder with generators parallel to those of R_j' and touching $\zeta_1^2+\ldots+\zeta_{T-k}^2 = a_j^2$. Then S_j lies inside R_j'. The co-ordinate axes of $\zeta_1, \ldots, \zeta_{2j}$ are chosen to be orthogonal axes spanning the intersection of S_j and the sphere $\zeta_1^2+\ldots+\zeta_{T-k}^2 = a_j^2$, $j = 1, \ldots, m'-1$. Since R' is the intersection of R_j and S is the intersection of S_j, $j = 1, \ldots, m'-1$, it follows that S lies inside R' which proves (9).

For tests against excess low-frequency we need to consider the distribution function of $\max(j/m'-s_j)$, say $H_U(x)$, maximized over $j = 1, \ldots, m'-1$. The distribution function of $\max(j/m'-s_{jU})$ is $C(x)$ since c^+ and c^- have the same distribution.

LEMMA 2. *For* $x \geqslant 0$, $H_U(x) \leqslant C(x)$.

PROOF. Now

$$H_U(x) = \mathrm{pr}\,(s_j \geqslant -x+j/m', j = 1, \ldots, m'-1)$$

and

$$C(x) = \mathrm{pr}\,(s_{jU} \geqslant -x+j/m', j = 1, \ldots, m'-1).$$

As for Lemma 1, it is sufficient to prove that

$$\mathrm{pr}\left(\sum_{i=1}^{2j} w_i^2 \geqslant a_j^2, j = 1, \ldots, m'-1\right) \leqslant \mathrm{pr}\left(\sum_{i=1}^{2j} \xi_i^2 \geqslant a_j^2, j = 1, \ldots, m'-1\right) \tag{11}$$

conditional on

$$\sum_{i=1}^{T-1} w_i^2 = \sum_{i=1}^{T-k} \xi_i^2 = 1,$$

where now $a_j^2 = -x+j/m'$ for $x < j/m'$ and $a_j^2 = 0$ for $x \geqslant j/m'$.

With the same notation as for Lemma 1, let $\bar{R}_j'$ denote the region in ζ space corresponding to the region $w_1^2+\ldots+w_{2j}^2 > a_j^2$ in w space, i.e. $\bar{R}_j'$ is the region complementary to R_j'. Similarly, let $\bar{S}_j$ denote the region complementary to S_j. Since R_j' contains S_j, $\bar{S}_j$ contains $\bar{R}_j'$. Thus the intersection $\bar{S}$ of the $\bar{S}_j$ contains the intersection $\bar{R}'$ of the $\bar{R}_j'$, $j = 1, \ldots, m'-1$. But the left-hand and right-hand probabilities of (11) are the fractions of the surface volume cut off the $(T-k)$-dimensional unit sphere by $\bar{R}'$ and $\bar{S}$ respectively. Hence (11) follows. Difficulties arising from degeneracy where $a_j^2 = 0$ are avoided by taking $a_j^2 > 0$, letting $a_j \to 0$ and using a continuity argument.

To obtain the corresponding results for the lower line let $G_L(x)$ be the distribution function of $\max[s_j-\{j-\tfrac{1}{2}(k-1)\}/m']$, maximized over $j = \tfrac{1}{2}(k+1), \ldots, m-1$.

LEMMA 3. *For all* $x \geqslant 0$, $G_L(x) \leqslant C(x)$.

PROOF. Let $w'_i = w_{T-i}$, $i = 1, \ldots, T-1$ and let

$$s'_{j'} = \frac{\sum_{i=1}^{2j'} w'^2_i}{\sum_{i=1}^{T-1} w^2_i} \quad (j' = 1, \ldots, m-1).$$

Then $s_j - \{j - \frac{1}{2}(k-1)\}/m' = 1 - s'_{j'} - (1 - j'/m') = j'/m' - s'_{j'}$, where $j' = m-j$. Thus

$$\max_{j=\frac{1}{2}(k+1), \ldots, m-1} [s_j - \{j - \tfrac{1}{2}(k-1)\}/m'] = \max_{j'=1, \ldots, m'-1} (j'/m' - s'_{j'}).$$

Denote the distribution function of the latter by $H'_U(x)$. The only essential property of $w_1, \ldots, w_{T-1}$ used in the proof of Lemma 2 is that it is the vector formed by the projection of a $(T-1)$-dimensional vector of independent $N(0, \sigma^2)$ variables on a $(T-k)$-dimensional subspace. But this is also true of $w'_1, \ldots, w'_{T-1}$. Consequently, Lemma 2 applies to s'_j, i.e. $H'_U(x) \leqslant C(x)$. Since $H'_U(x) = G_L(x)$ the result follows.

Similarly, let $H_L(x)$ be the distribution function of

$$\max_{j=\frac{1}{2}(k+1), \ldots, m-1} [\{j - \tfrac{1}{2}(k-1)\}/m' - s_j].$$

LEMMA 4. *For all* $x \geqslant 0$, $H_L(x) \geqslant C(x)$.

PROOF. Now $\{j - \frac{1}{2}(k-1)\}/m' - s_j = 1 - j'/m' - (1 - s'_{j'}) = s'_{j'} - j'/m'$. Hence

$$\max_{j=\frac{1}{2}(k+1), \ldots, m-1} [\{j - \tfrac{1}{2}(k-1)\}/m' - s_j] = \max_{j'=1, \ldots, m'-1} (s'_{j'} - j'/m').$$

Denote the distribution function of the latter by $G'_U(x)$. Since Lemma 1 applies to this we have $H_L(x) = G'_U(x) \geqslant C(x)$.

To apply these results to the c^+-test we note that the path of s_j crosses the upper line $y = c_0 + j/m'$ when max $(s_j - j/m') > c_0$, maximized over $j = 1, \ldots, m'-1$, the probability of which is $1 - G_U(c_0)$. By Lemma 1, $1 - G_U(c_0) \leqslant 1 - C(c_0)$ which equals $1 - (1-\alpha) = \alpha$ for a test at significance level α. Thus the probability of crossing the upper line $\leqslant \alpha$; in other words, if the upper line is crossed we can definitely conclude that significance has been established at significance level α.

Similarly, the path of s_j will fail to cross the lower line $y = c_0 + \{j - \frac{1}{2}(k-1)\}/m$ when

$$\max_{j=\frac{1}{2}(k+1), \ldots, m-1} [s_j - \{j - \tfrac{1}{2}(k-1)\}/m'] \leqslant c_0,$$

the probability of which is $G_L(c_0)$. By Lemma 3, $G_L(c_0) \leqslant C(c_0) = 1 - \alpha$. Thus the probability of not crossing the lower line $\leqslant 1 - \alpha$; in other words, if the lower line is not crossed we can definitely regard the result as non-significant at significance level α. If the lower line is crossed but not the upper we regard the result of the test as inconclusive at the given significance level.

Similarly, for the c^--test we note that the path of s_j crosses the lower line

$$y = -c_0 + \{j - \tfrac{1}{2}(k-1)\}/m'$$

when

$$\max_{j=\frac{1}{2}(k+1), \ldots, m-1} [\{j - \tfrac{1}{2}(k-1)\}/m' - s_j] > c_0,$$

the probability of which is $1 - H_L(c_0)$. By Lemma 4, $1 - H_L(c_0) \leqslant 1 - C(c_0) = \alpha$. We therefore regard the result of the test as definitely significant. In the same way, we deduce from Lemma 3 that the probability of not crossing the upper line $y = -c_0 + j/m'$ is $\leqslant 1-\alpha$, indicating definite non-significance. If the path crosses the upper line but not the lower we treat the result as non-significant.

The validity of the two-sided test follows immediately from the validity of the two one-sided tests.

The question of what should be done when these tests give inconclusive results will not be pursued in this paper except for the following brief remarks. In connection with the original d-test, Hannan (1957) and Theil & Nagar (1961) pointed out essentially that in many applications the regressors are 'slowly-changing' and that the significance points of the upper bounding variable d_U then give a good approximation to the true significance points of d. In the present context, if the regressors were exact linear combinations of the low-frequency Fourier vectors $\cos(2\pi jt/T)$ and $\sin(2\pi jt/T)$, $t = 1, \ldots, T$ and $j = 1, \ldots, \frac{1}{2}(k-1)$, then the test based on the lower line would be exact, i.e. the probability of crossing the line would be α and the probability of not crossing it would be $1-\alpha$. In the more usual situation in which the regressors are not exactly related to these vectors but are 'slowly changing' and can therefore be expected to be fairly highly correlated with them, the test based on the lower line can be expected to give a good approximation.

An alternative and more accurate approximation would be to compute

$$\max_{j=1,\ldots,m-1} \{s_j - E(s_j)\} \quad \text{and} \quad \max_{j=1,\ldots,m-1} \{E(s_j) - s_j\},$$

where $E(s_j)$ can be obtained from formulae (6) and (8) of Durbin & Watson (1950), and to treat these as values of c^+ and c^- calculated from a sample of $T-k+1$ independent normally distributed variables.

5. Procedures for T and k even

The above theory has been developed on the assumption that T and k are odd and we must consider what should be done when this assumption does not hold. My personal suggestion would be to use the same method in all cases, i.e. to define s_j by (2) with $m = \frac{1}{2}T$ when T is even and to take as the upper and lower lines $y = \pm c_0 + j/m'$ for the c^+-test and $y = \pm c_0 + \{j - \frac{1}{2}(k-1)\}/m'$ for the c^--test, where $m' = \frac{1}{2}(T-k)$ for all T and k. For fractional m', I would suggest interpolating linearly in Table 1 to obtain the significance values c_0.

Whereas the tests suggested in § 3 are exact when T and k are odd, they are approximate when either T or k both are even. However, the amount of approximation involved is slight and should be negligible in practice unless m' is very small.

6. Bounds test for the mean of the s_j

For situations where the sample path of s_j lies above the line $y = j/m$, § 2 suggests the use of the statistic $\max(s_j - j/m)$ as a measure of the overall discrepancy between the path and the line. An alternative is the statistic

$$\bar{s} = \frac{1}{m-1} \sum_{j=1}^{m-1} s_j, \tag{12}$$

which can be expected to have high values in the presence of positive serial correlation and low values in the presence of negative serial correlation.

This statistic was proposed by Durbin (1967) and, for a sample of $2m+1$ independent $N(\mu, \sigma^2)$ variables, its distribution was shown to be that of the mean of a sample of $m-1$ independent observations from the uniform $(0, 1)$ distribution, significance values of which have been tabulated by Stephens (1966). The distribution converges rapidly to normality with mean $\frac{1}{2}$ and variance $1/\{12(m-1)\}$. A substantial improvement on the normal approximation can be obtained by using the Cornish–Fisher series (Kendall & Stuart, 1958, § 6·56). For instance, truncating at the fourth cumulant, we obtain the following approximate expression for the upper percentage points of $\bar{s}$:

$$\tfrac{1}{2} + \frac{1}{\sqrt{(12m)}}\xi - \frac{1}{40m\sqrt{(3m)}}(\xi^3 - 3\xi),$$

where ξ is the normal deviate at level α. This may be compared with the value obtained from the normal approximation, i.e. $\frac{1}{2} + \xi/(12m)$. Power can be expected to be good against a stationary alternative for which the reciprocal of the spectral density is approximately a linear function of frequency.

In this section we consider the properties of $\bar{s}$ when the s_j's are calculated from regression residuals. As in the proof of Lemma 1, let $w = Fz$, where F is the Fourier matrix. We obtain

$$\bar{s} = \frac{\sum_{j=1}^{m-1} (m-j)(w_{2j-1}^2 + w_{2j}^2)}{(m-1)\sum_{i=1}^{T} w_i^2}, \tag{13}$$

which is of the general form r considered by Durbin & Watson (1950, 1951). For T and k odd it follows from their results (particularly 1951, p. 177) that $\bar{s}_L \leqslant \bar{s} \leqslant \bar{s}_U$, where

$$\bar{s}_L = \frac{\sum_{i=1}^{T-k} \lambda_i \zeta_i^2}{(m-1)\sum_{i=1}^{T-k} \zeta_i^2} \quad \text{and} \quad \bar{s}_U = \frac{\sum_{i=1}^{T-k} \lambda_{i+k-1} \zeta_i^2}{(m-1)\sum_{i=1}^{T-k} \zeta_i^2},$$

in which $\lambda_1 \leqslant \ldots \leqslant \lambda_{T-1}$ is the set of values $0, 0, 1, 1, \ldots, m-1, m-1$ and $\zeta_1, \ldots, \xi_{T-k}$ are independent $N(0, \sigma^2)$. Thus

$$\bar{s}_L = \frac{\sum_{j=1}^{m'-1} (m'-j)(\zeta_{2j-1}^2 + \zeta_{2j}^2)}{(m-1)\sum_{i=1}^{T-k} \zeta_i^2} \quad \text{and} \quad \bar{s}_U = \frac{\sum_{j=1}^{m'} \{\frac{1}{2}(k-1) + m' - j\}(\zeta_{2j-1}^2 + \zeta_{2j}^2)}{(m-1)\sum_{i=1}^{T-k} \zeta_i^2} = \frac{1}{2}\frac{k-1}{m-1} + \bar{s}_L, \tag{14}$$

where $m' = \frac{1}{2}(T-k)$. Comparing the expression for $\bar{s}_L$ in (14) with (13), we see that $(m-1)\bar{s}_L/(m'-1)$ has the same distribution as a value of $\bar{s}$ calculated from a sample of $T-k+1$ independent normally distributed variables, i.e. is distributed as the mean of $m'-1$ independent uniform $(0, 1)$ variables.

The bounds test for $\bar{s}$ therefore takes a particularly simple form. Let s_0 be the upper significance point at significance level α for the mean of $m'-1$ observations from a uniform $(0, 1)$ distribution. This can be obtained either exactly from Table 1 of Stephens (1966) or approximately either from the $N(\frac{1}{2}, 1/\{12(m'-1)\})$ distribution or from the Cornish–Fisher approximation mentioned above. For a test against positive serial correlation reject if

$\bar{s} > \{\frac{1}{2}(k-1)+(m'-1)s_0\}/(m-1)$, accept if $\bar{s} \leqslant (m'-1)s_0/(m-1)$ and treat as inconclusive in the intermediate case. Similarly, for a test against negative serial correlation reject if $\bar{s} < (m'-1)(1-s_0)/(m-1)$, accept if $\bar{s} \geqslant \{\frac{1}{2}(k-1)+(m'-1)(1-s_0)\}/(m-1)$ and treat as inconclusive in the intermediate case. A two-sided test is obtained by carrying out both one-sided tests.

In comparison with d, $\bar{s}$ is harder to compute but does not require special tabulations to obtain bounding significance points as d does. The statistic $\bar{s}$ should be particularly useful when the periodogram is already available or for values of T and k outside the range covered in existing tables of bounding significance points of d. The powers of the two tests may be taken as approximately comparable for alternatives of interest.

The above tests are exact for T and k odd. For even T, I would suggest computing s_j from (2) with $m = \frac{1}{2}T$ and proceeding for both odd and even T as above, where $T-k$ is even, interpolating linearly between significance points for $m'-\frac{1}{2}$ and $m'+\frac{1}{2}$ for $T-k$ odd. This leads to the correct distribution for $\bar{s}_U$, where T and k are both even and to approximations which should be good, unless m' is very small, in the remaining cases.

When the result of the test is inconclusive, it should be noted that, where the regressors are slowly changing, the distribution of $\bar{s}_L$ provides a fairly good approximation to the true distribution of $\bar{s}$. A more accurate approximation for moderate sample sizes based on the exact first two moments of $\bar{s}$ could be developed by a technique described for d by Durbin & Watson (1951, pp. 163–6).

7. EXAMPLES

The tests will be illustrated by applying them to two sets of published data. The first was used by Durbin & Watson (1951) to illustrate the d-test and refers to the regression of log consumption of spirits per head on log real income per head and log relative price in the U.K., 1870–1938. The data are given in the paper cited but note that the value of x_1 for 1938 should read 2·1182.

The cumulated periodogram is plotted in Fig. 1 together with the upper and lower lines for a test at the 1 % level against positive serial correlation. Since $T = 69$ and $k = 3$, $m' = 33$. The 1 % significance value for this value of m' is obtained from Table 1 as $c_0 = 0{\cdot}24165$.

The value of $\bar{s}$ is 0·914 with bounding 1 % significance values $\bar{s}_L = 0{\cdot}598$ and $\bar{s}_U = 0{\cdot}628$. These may be compared with the value of $d = 0{\cdot}249$ with 1 % significance values $d_L = 1{\cdot}40$ and $d_U = 1{\cdot}52$.

The result of all tests is significance at the 1 % level. The rapid rise of the path of s_j is characteristic of a situation in which there is a large amount of low-frequency variation present.

The second set of data relates to the regression of consumption on profits and wages as fitted experimentally by least squares by Klein (1950, pp. 74–5), using data given on his p. 135. Since $T = 21$ and $k = 3$, $m' = 9$. From Table 1 we find the 5 % significance value is $c_0 = 0{\cdot}32538$. The path of s_j, together with the upper and lower lines for a test against positive serial correlation, are given in Fig. 2. The other statistics are $\bar{s} = 0{\cdot}640$, compared with 5 % significance values of $\bar{s}_L = 0{\cdot}594$, $\bar{s}_U = 0{\cdot}705$, and $d = 1{\cdot}277$, compared with $d_L = 1{\cdot}13$, $d_U = 1{\cdot}54$.

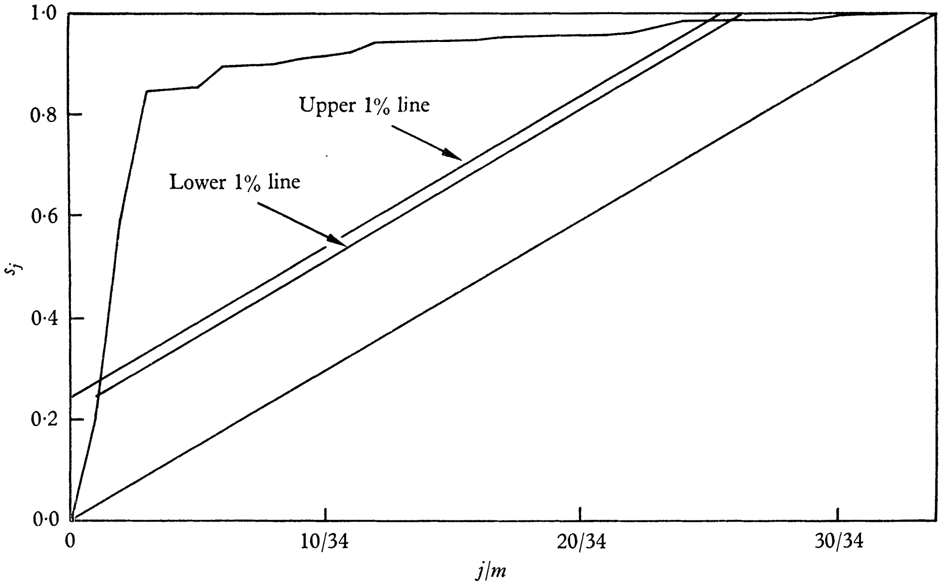

Fig. 1. Cumulated periodogram s_j of residuals from regression of log spirits consumption on log income and log price for the U.K. 1870–1938 (69 observations).

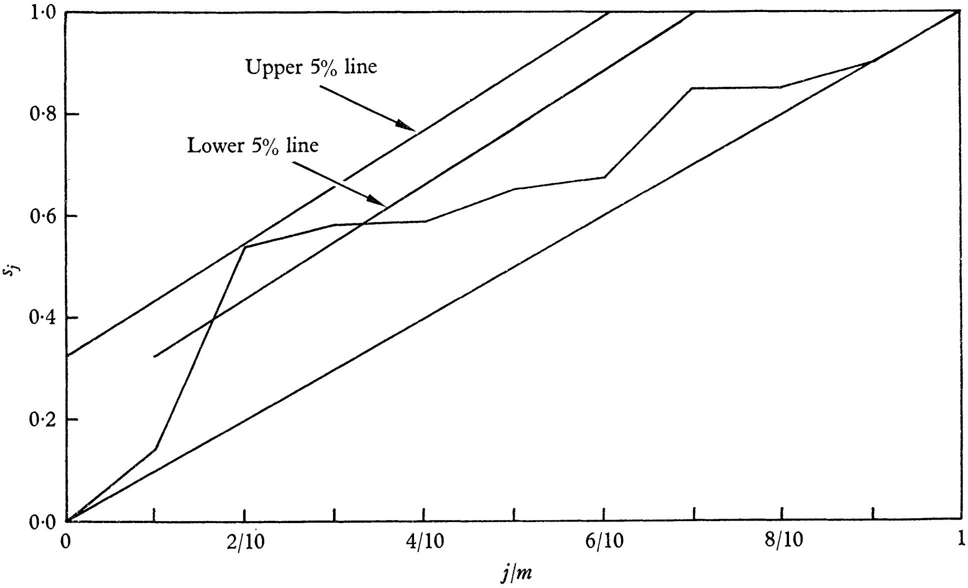

Fig. 2. Cumulated periodogram s_j of residuals from regression of consumption on wages and profits for U.S.A. 1921–41 (21 observations).

8. Table of significance values of c^+ and c

Significance values for one-sided tests based on c^+ or c^- and for two-sided tests based on c are given in Table 1. Since c^+, c^-, C_n^+ and C_n^- all have the same distribution these values are also suitable for tests based on Pyke's modified Kolmogorov–Smirnov statistics C_n^+, C_n^- and C. This table was computed by my colleague, Mr C. E. Rogers, who both wrote the programs and also developed a variety of checking procedures intended to detect and avoid the excessive accumulation of rounding errors. I am extremely grateful for this assistance.

The basic formula used for the computations was (30′) of Durbin (1968), due to Dempster (1959), which in the present notation reads

$$\mathrm{pr}\left(c^+ > \frac{a}{m}\right) = \frac{a+1}{m^{m-1}} \sum_{j=[a]+1}^{m-1} \binom{m-1}{j} (j-a)^j (m+a-j)^{m-2-j}. \tag{15}$$

Rapid convergence to the significance values was obtained by Newton's method. Calculations were performed to sixteen significant figures on the IBM 360–65 of University College London. It is believed that the values are correct to the fifth decimal place.

The following checks were carried out:

(i) The probabilities were calculated from the alternative form (30) of Durbin (1968), namely

$$\mathrm{pr}\left(c^+ > \frac{a}{m}\right) = 1 - \frac{a+1}{m^{m-1}} \sum_{j=0}^{[a]} \binom{m-1}{j} (j-a)^j (m+a-j)^{m-2-j}. \tag{16}$$

For large values of m, rounding errors are difficult to control with this form owing to the fact that, unlike those of (15), the terms of (16) alternate in sign. Except for large m the two sets of values agreed.

(ii) The same method was used to recompute Owen's (1962, p. 424) table of significance values of D_n^+ and exact agreement was obtained.

The two-sided probabilities $\mathrm{pr}(c > a/m)$ were then independently computed by two different methods, first by the matrix difference equation (1) of Durbin (1968) and secondly by the difference equation (19) with initial conditions (21) of Durbin (1968). For $\alpha = 0{\cdot}005$, $0{\cdot}01$ and $0{\cdot}025$ the resulting probabilities were exactly 2α to five places of decimals for all n, while for $\alpha = 0{\cdot}05$ the probability was $0{\cdot}10000$ for $n = 1, \ldots, 9$ and $0{\cdot}09999$ for $n = 10, \ldots, 100$. For $\alpha = 0{\cdot}10$ the probability was $0{\cdot}19981$ for $n = 49, \ldots, 100$ and was between $0{\cdot}19981$ and $0{\cdot}20000$ for $n = 1, \ldots, 48$. The tabulated values are therefore exact, to the accuracy indicated, for two-sided tests at significance levels 0·01, 0·02 and 0·05, and they are almost exact at significance levels 0·10 and 0·20. The fact that the one-sided and two-sided calculations agree to this extent can be regarded as a further check on the computing. It should be noted that the difference-equation method of calculating the two-sided probabilities turned out to be less accurate than the matrix difference equation for the same reason as was found in comparing (15) and (16), i.e. the alternation of signs in successive terms of the difference equation led to the build-up of larger rounding errors for large m.

The ideas which led to this paper arose from discussions with Professor G. S. Watson about the desirability of extending our previous results on the testing of serial correlation in regression analysis to the frequency domain. I am grateful for this stimulus and for the facilities at Johns Hopkins University provided by Professor Watson which enabled me to carry out the work. I am grateful to Rudy J. Beran for carrying out the calculations for the two examples. The research was supported by the U.S. Air Force Office of Scientific Research.

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