Materials 15 - More on the CEMP vs. CUSUM criteria and optimal Taylor rule coefficients

Laura Gáti

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Overview

1 Model summary

$$x_t = -\sigma i_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t}\big((1-\beta)x_{T+1} - \sigma(\beta i_{T+1} - \pi_{T+1}) + \sigma r_T^n\big) \tag{1}$$

$$\pi_t = \kappa x_t + \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}\big(\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} + u_T\big) \tag{2}$$

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \bar{i}_t \tag{3}$$

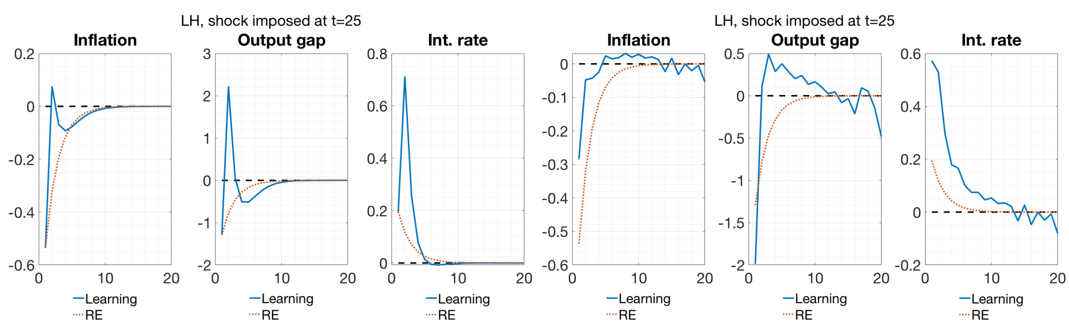
$$\hat{\mathbb{E}}_t z_{t+h} = \bar{z}_{t-1} + b h_x^{h-1} s_t \quad \forall h \geq 1 \qquad b = g_x\, h_x \qquad \text{PLM} \tag{4}$$

$$\bar{z}_t = \bar{z}_{t-1} + k_t^{-1} \underbrace{\big(z_t - (\bar{z}_{t-1} + b s_{t-1})\big)}_{\text{fcst error using (4)}} \tag{5}$$

(Vector learning. For scalar learning, $\bar{z} = \begin{pmatrix} \bar{\pi} & 0 & 0 \end{pmatrix}'$. I'm also not writing the case where the slope b is also learned.)

$$k_t = \begin{cases} k_{t-1} + 1 & \text{for decreasing gain learning} \\ \bar{g}^{-1} & \text{for constant gain learning.} \end{cases} \tag{6}$$

Figure 1: Reference: baseline model



(a) Learning constant only (b) Learning slope and constant

2 The CEMP vs. the CUSUM criterion

CEMP's criterion

$$\theta_t = |\hat{\mathbb{E}}_{t-1}\pi_t - \mathbb{E}_{t-1}\,\pi_t|/(\mathrm{Var}(\text{shocks})) \tag{7}$$

$$\text{i.e.} \quad \text{PLM- E[ALM], scaled by shocks} \tag{8}$$

For my version of CEMP's criterion, I rewrite the ALM

$$z_t = A_a f_a + A_b f_b + A_s s_t \tag{9}$$

$$\text{as} \quad z_t = F + G s_t \tag{10}$$

$$\Leftrightarrow \quad z_t = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} 1 \\ s_t \end{bmatrix} \tag{11}$$

Then, since the PLM is $z_t = \phi \begin{bmatrix} 1 \\ s_t \end{bmatrix}$, the generalized CEMP criterion becomes

$$\theta_t = \max |\Sigma^{-1}(\phi - \begin{bmatrix} F & G \end{bmatrix})| \tag{12}$$

where Σ is the VC matrix of shocks. As for the CUSUM criterion, what I did in Materials 5 was

$$\omega_t = \omega_{t-1} + \kappa k_{t-1}^{-1}(FE_t^2 - \omega_{t-1}) \tag{13}$$

$$\theta_t = \theta_{t-1} + \kappa k_{t-1}^{-1}(FE_t^2/\omega_t - \theta_{t-1}) \tag{14}$$

where FE_t is the most recent short-run forecast error ($ny \times 1$), and ω_t is the agents' estimate of the forecast error variance ($ny \times ny$). To take into account that these are now matrices, I now write

$$\omega_t = \omega_{t-1} + \kappa k_{t-1}^{-1}(f_t f_t' - \omega_{t-1}) \tag{15}$$

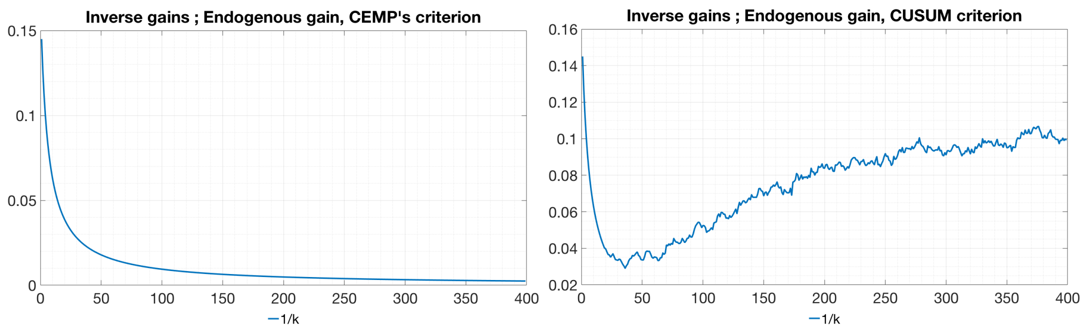
$$\theta_t = \theta_{t-1} + \kappa k_{t-1}^{-1}(f_t' \omega_t^{-1} f_t - \theta_{t-1}) \tag{16}$$

(Note: I'm using Lütkepohl's *Introduction to Multiple Time Series Analysis*, p. 160 to reformulate the CUSUM criterion as a statistic that has a χ^2 distribution.)

3 Investigating the behavior of CEMP and CUSUM criteria

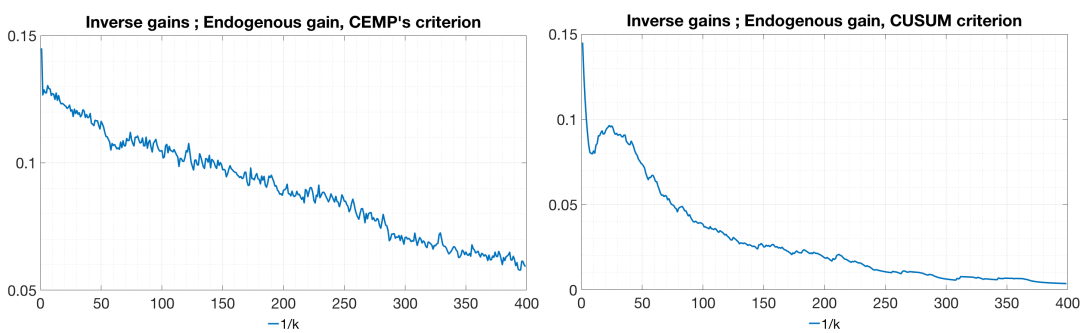
3.1 Anchoring as a function of ψ_π, fixing $\psi_x = 0, \bar{\theta} = 4, \tilde{\theta} = 2.5$

Figure 2: Inverse gains, $\psi_\pi = 1.01$



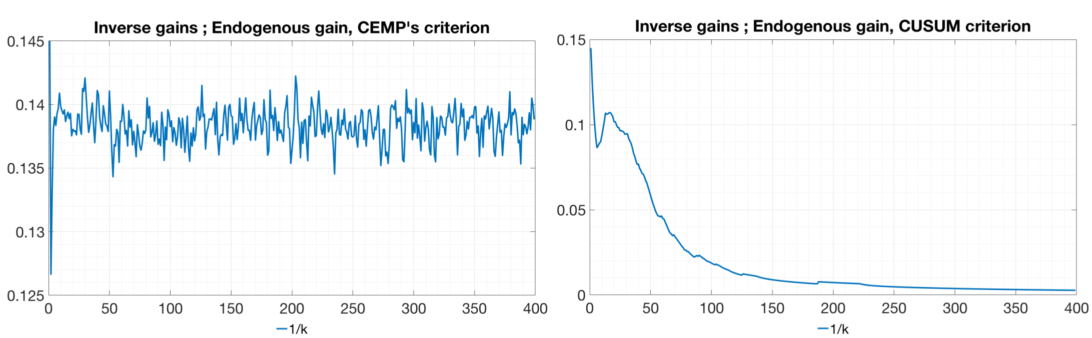
(a) CEMP's criterion (b) CUSUM criterion

Figure 3: Inverse gains, $\psi_\pi = 1.5$



(a) CEMP's criterion (b) CUSUM criterion

Figure 4: Inverse gains, $\psi_\pi = 2$



(a) CEMP's criterion (b) CUSUM criterion

3.2 Why do the two criteria behave opposite ways?

A restatement of the two criteria: you get unanchored expectations if:

$$\theta_t^{CEMP} = \max(\Sigma^{-1}|(\phi - \begin{bmatrix} F & G \end{bmatrix})|) > \bar{\theta} \qquad \text{vs.} \qquad \theta_t^{CUSUM} = f'\omega^{-1}f > \tilde{\theta} \tag{17}$$

where ϕ is the agents' estimated matrix, F, G are the ALM matrices, f is the one-period ahead forecast error and ω is the estimated forecast error variance matrix.

Sloppily, we can write the two criteria as:

$$\frac{\mathbb{E}(FE)}{\text{Var(noise)}} \qquad \text{vs.} \qquad \frac{FE^2}{F\hat{E}V} \tag{18}$$

- The big difference in the two is the way they treat noise: CEMP integrates it out, while CUSUM tries to cancel it out by dividing by the estimated FEV.
- Squaring the numerator in the CEMP criterion doesn't change its behavior.
- Trying various ways of not squaring or taking the square root of the CUSUM criterion doesn't change its behavior.
- Inputting Var(noise) instead of $F\hat{E}V$ in the CUSUM denominator, or simply using the CUSUM numerator only changes its behavior! You obtain the CEMP-criterion behavior: little anchoring for high ψ_π, much anchoring for low ψ_π.

→ Is the difference due to using the FEV instead of Var(noise) or does it come from using an *estimated* FEV?

- Decompose forecast errors as FE = part due to learning + noise. CEMP's criterion is able to distinguish between those two and responds only to the part due to learning. The CUSUM criterion cannot. To see this, write the CUSUM criterion as a function of the CEMP criterion:

$$\begin{aligned}
FE &= \phi s_{t-1} - [F,G]s_t \\
&= \phi s_{t-1} - [F,G]s_{t-1} - [F,G]\varepsilon_t \\
&= (\phi - F,G)s_{t-1} - [F,G]\varepsilon_t \qquad |\text{ignoring } \Sigma^{-1} \text{ and } s_{t-1}\text{, this is} \\
&= \theta^{CEMP} - [F,G]\varepsilon_t \qquad |\text{Nb.: 2nd component conflates learning \& noise} \qquad (19) \\
FE^2 &= (\theta^{CEMP})^2 + [F,G]^2\varepsilon_t^2 - 2\theta^{CEMP}[F,G]\varepsilon_t \qquad (20) \\
FEV &= \mathbb{E}(FE^2) = (\theta^{CEMP})^2 + [F,G]^2\Sigma
\end{aligned}$$

$$\begin{aligned}
&\rightarrow F\hat{E}V = \mathbb{E}(FE^2) = (\theta^{CEMP})^2 + [F,G]^2\hat{\Sigma} && (21)\\
&\Rightarrow \theta^{CUSUM} = \frac{(\theta^{CEMP})^2 + [F,G]^2\varepsilon_t^2 - 2\theta^{CEMP}[F,G]\varepsilon_t}{(\theta^{CEMP})^2 + [F,G]^2\hat{\Sigma}} && |\text{Supposing that } \hat{\Sigma} \approx \varepsilon_t^2\\
&\theta^{CUSUM} = 1 - 2\frac{\theta^{CEMP}[F,G]\varepsilon_t}{(\theta^{CEMP})^2 + [F,G]^2\hat{\Sigma}} && (22)
\end{aligned}$$

$\Rightarrow$ Suggests that the difference comes from dividing the CUSUM-criterion by the FEV (doesn't matter if it's estimated or not). The reason this division is not innocent is because the CUSUM-criterion doesn't integrate out shocks - it relies on forecast errors, *not expected forecast errors*, as the CEMP criterion does. Thus the cross-term $-2\theta^{CEMP}[F,G]\varepsilon_t$ is not cancelled out, introducing movement in the opposite direction as the CEMP-criterion.

What's the intuition?

In absence of knowing the model, CUSUM-agents aren't able to evaluate the expectations $\mathbb{E}(FE)$ and $\mathbb{E}(FE^2) = FEV$. Thus they are not able to distinguish between forecast errors coming from learning or from noise. In particular, to evaluate the forecast error variance, ideally they'd want to compute it as

$$FEV = \text{variation coming from learning} + \text{variation coming from noise} - 2\text{cov(learn, noise)} \tag{23}$$

But since they are not able to distinguish between these components, the cross-term cov(learn, noise) contaminates their FE^2 expression, leading to a discrepancy between FE^2 and their estimated $F\hat{E}V$.

In a sense this is like omitted variable bias, which biases their estimate θ^{CUSUM}. In particular, I'd suggest that errors from learning and noise are likely to go in the same direction, which means that θ^{CUSUM} will move in the opposite direction as θ^{CEMP}.

A small side-note:

- *the CUSUM-criterion is very sensitive to initialization:*
  Especially the choice of θ_0 matters a lot for initial dynamics which is why on these graphs you initially have anchoring b/c I initialize θ_0 as zero. If, on the contrary, I initialize it as $\tilde{\theta}$ or higher, you don't get the initial anchored period. The same effect is achieved by setting a burn-in period of 100 or so.

4 Analytical expressions for optimal Taylor rule coefficients

Following Woodford's *Interest and Prices*, here's a procedure to obtain optimal Taylor rule coefficients. In Woodford's terminology, this consists of solving for the *optimal noninertial plan* (oni) for the endogenous variables, and then doing coefficient comparison between the Taylor rule and the *oni*.

4.1 In-a-nutshell algorithm for optimal Taylor rule coefficients

1. Postulate conjectures $z_t = \bar{z} + f_j u_t + g_j \hat{r}^n_t, \quad j = \pi, x, i \quad z = (\pi, x, i)'$ for the model consisting of an NKPC and NKIS relation and the AR(1) shocks $u, \hat{r}^n$, where $\hat{r}^n_t = r^n_t - \bar{r}$ (so that the natural rate has a drift, but $\hat{r}^n$ is detrended).

2. Plug the conjectures into the two model equations to derive 2 constraints on the 3 deterministic components $(\bar{\pi}, \bar{x}, \bar{i})$ and 4 constraints on the 6 coefficients on disturbances, $f_j, g_j, \quad j = \pi, x, i$. (Use the known LOMs of shocks to write everything in terms of time t shocks.)

3. Solve 3 sets of optimizations

   (a) $(\bar{\pi}, \bar{x}, \bar{i}) = \arg\min L^{det} \quad s.t.$ the 2 constraints on deterministic components

   (b) $f_j = \arg\min L^{stab,u} \quad s.t.$ 2 out of 4 constraints on shock-coefficients, $j = \pi, x, i$.

   (c) $g_j = \arg\min L^{stab,r} \quad s.t.$ the last 2 out of 4 constraints on shock-coefficients, $j = \pi, x, i$.

4. Compare coefficients of Taylor rule to *oni*-solution of i_t.

4.2 Details

1. The optimal noninertial plan (oni)

   A purely forward-looking set of optimal policies that specifies a LOM for each endogenous variable as the sum of a deterministic component (a long-run average, denoted above by "bar") and a state-contingent component with an optimal response to state t disturbances (the f_j and g_j above). Moreover, the deterministic components are optimal from a timeless perspective (i.e. they minimize L^{det}), and the state-contingent components minimize fluctuations coming from shocks ($L^{stab,u}, L^{stab,r}$).

2. The loss function of the monetary authority and its decomposition into deterministic and shock-contingent parts

Following Woodford, I augment my loss function with some concern for interest rate stabilization:

$$L^{CB} = \mathbb{E}_t \sum_{T=t}^{\infty} \{\pi_T^2 + \lambda_x (x_T - x^*)^2 + \lambda_i (i_T - i^*)\} \tag{24}$$

Woodford decomposes this as $L^{CB} = L^{det} + L^{stab}$ where the former only depends on the "deterministic component of the equilibrium paths of the target variables," while the latter "depends only on the equilibrium responses to unexpected shocks" (p. 509). In particular:

$$L^{det} = \sum_{T=t}^{\infty} \beta^{T-t} \{\mathbb{E}_t\, \pi_T^2 + \lambda_x (\mathbb{E}_t\, x_T - x^*)^2 + \lambda_i (\mathbb{E}_t\, i_T - i^*)^2\} \tag{25}$$

$$L^{stab} = \sum_{T=t}^{\infty} \beta^{T-t} \{\mathrm{var}_t(\pi_T) + \lambda_x \mathrm{var}_t(x_T) + \lambda_i \mathrm{var}_t(i_T)\} \tag{26}$$

Woodford then further decomposes L^{stab} into an element conditional on each shock, but this is only for algebraic convenience.

3. The Taylor rule

   Woodford postulates a Taylor rule of the form

$$i_t = \bar{i} + \phi_\pi (\pi_t - \bar{\pi}) + \phi_x (x_t - \bar{x})/4 \tag{27}$$

(He divides by 4 to make the output gap quarterly.) Substituting in the conjectured and solved-for *oni*-solutions for the endogenous variables, one obtains:

$$i_t = \bar{i} + \phi_\pi (f_\pi u_t + g_\pi \hat{r}_t^n) + \phi_x (f_x u_t + g_x \hat{r}_t^n)/4 \tag{28}$$

$$i_t = \bar{i} + f_i u_t + g_i \hat{r}_t^n \tag{29}$$

allowing one to solve for (ϕ_π^*, ϕ_x^*) as the solution to

$$f_i = \phi_\pi f_\pi + \phi_x f_x \tag{30}$$

$$g_i = \phi_\pi g_\pi + \phi_x g_x \tag{31}$$

4.3 Optimal Taylor rule coefficients for RE model, with the simplifying assumption $\rho_u = \rho_r = \rho$

$$\phi_\pi^* = \frac{\kappa\sigma}{\lambda_i(\rho-1)(\beta\rho-1) - \kappa\lambda_i\rho\sigma} \tag{32}$$

$$\phi_x^* = \frac{\lambda_x\sigma(1-\beta\rho)}{\lambda_i(\rho-1)(\beta\rho-1) - \kappa\lambda_x\rho\sigma} \tag{33}$$

which is - fabulously enough - exactly what Woodford obtains.

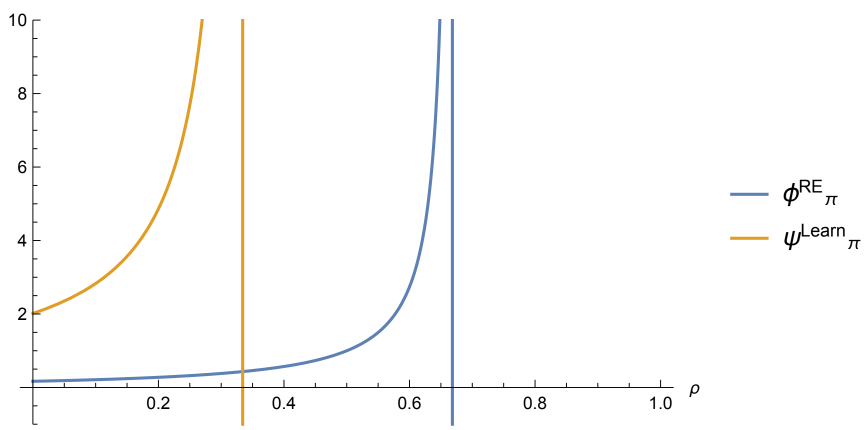
4.4 Optimal Taylor rule coefficients for the learning model, with the simplifying assumption $\rho_u = \rho_r = \rho$

$$\psi_\pi^* = \frac{\kappa\sigma(\beta(\rho-1)-1)(\alpha\beta(\rho-1)-1)}{\kappa\lambda_i\sigma(\alpha\beta(\rho-1)-1) + \beta\lambda_i(\rho-1)(\alpha\beta(\rho-1)+\beta-1)} \tag{34}$$

$$\psi_x^* = \frac{\lambda_x\sigma(\beta(\rho-1)-1)(\alpha\beta(\rho-1)+\beta-1)}{\kappa\lambda_i\sigma(\alpha\beta(\rho-1)-1) + \beta\lambda_i(\rho-1)(\alpha\beta(\rho-1)+\beta-1)} \tag{35}$$

For a simple calibration of $\lambda_i = 1$ and $\lambda_x = 0$ and $\{\beta \to 0.99, \sigma \to 1, \kappa \to 0.16, \rho \to 0.3, \alpha \to 0.5\}$, I get $\phi_\pi^* = 0.360279$ and $\psi_\pi^* = 18.7714$.

Figure 5: Optimal Taylor rule coefficients for $\lambda_i = 1$ as a function of ρ



Like Woodford observes, equations (32)-(35) impose certain bounds on ρ: these bounds are higher for RE than for learning. Another point to note is that decreasing λ_i just scales the optimal coefficients as a function of ρ up, but preserves the functional form.