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Geometry and Spectrum of typical hyperbolic surfaces

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Abstract

The aim of thesis is to prove new results on the geometry and spectrum of typical compact hyperbolic surfaces, in the high-genus limit. We focus on statements that are true with probability going to one as the genus $g \to +\infty$ for the Weil–Petersson probability measure, hence excluding a small set of atypical surfaces.

We first study geometric properties at the large scale $\log g$. We prove *Benjamini-Schramm convergence* and define a new notion of *tangle-freeness*. Several geometric consequences are derived from this definition, amongst which an improved collar lemma and the fact that all closed geodesics of length $\leq a \log g$ are simple if $a < 1$.

Then, we prove that the spectral density of typical hyperbolic surfaces converges to the spectral density of the hyperbolic plane in the high-genus limit. This implies a uniform Weyl-law and improved bounds on the multiplicity of eigenvalues.

Finally, we suggest a method to prove that typical hyperbolic surfaces have an optimal spectral gap. The method relies on an asymptotic expansion of Weil–Petersson volume polynomials in powers of $1/g$ and a new integration-by-parts argument.

Résumé

L'objectif de cette thèse est de démontrer de nouveaux résultats sur la géometrie et le spectre des surfaces hyperboliques compactes typiques, dans la limite où le genre g tend vers l'infini. Nous prouvons des propriétés vraies avec un probabilité de Weil–Petersson tendant vers 1 quand $g \to +\infty$, et donc sauf sur un petit ensemble de surfaces atypiques.

Nous étudions dans un premier temps des propriétés géometriques à l'échelle $\log g$. Nous prouvons la convergence au sens de *Benjamini-Schramm* et définissons une nouvelle notion de surface « *tangle-free* ». Nous en déduisons plusieurs résultats, dont un lemme du collier amélioré et le fait que, pour $a < 1$, toutes les géodésiques fermées de longueur $\leq a \log g$ sont simples.

Nous prouvons ensuite que la densité spectrale des surfaces hyperboliques typiques converge vers la densité spectrale du plan hyperbolique quand $g \to +\infty$. Ceci implique une loi de Weyl uniforme et une amélioration des bornes sur les multiplicités.

Enfin, nous suggérons une méthode pour démontrer que les surfaces hyperboliques typiques ont un trou spectral optimal. Cette méthode est fondée sur un développement asymptotique des volumes de Weil–Petersson en puissances de $1/g$, et un nouvel argument utilisant des intégrations par parties.

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Contents

Notations

$\mathbb{N}$	set of positive integers $\{1, 2, \ldots\}$.
$\mathbb{N}_0$	set of non-negative integers $\{0, 1, 2, \ldots\}$.
$\lvert\mathbf{x}\rvert$, $\lvert\mathbf{x}\rvert_\infty$	ℓ^1 and ℓ^∞ norms on $\mathbb{R}^n$.
$\langle \mathbf{x} \rangle$	Japanese bracket $\sqrt{1+\lvert\mathbf{x}\rvert^2}$.
X, $\mathcal{H}$	compact/bordered hyperbolic surface, hyperbolic plane.
$\mathrm{InjRad}_X(z)$	injectivity radius of the surface X at the point z.
InjRad_X	(global) injectivity radius of the surface X.
$X^-(L)$, $X^+(L)$	L-thin and L-thick parts of the surface X.
$\mathcal{M}_g$	moduli space of compact hyperbolic surfaces of genus g.
$\mathrm{Vol}_g^{\mathrm{WP}}$	the Weil-Petersson volume form on $\mathcal{M}_g$.
V_g	the total volume of $\mathcal{M}_g$ for $\mathrm{Vol}_g^{\mathrm{WP}}$.
$\mathbb{P}_g^{\mathrm{WP}}$	the Weil-Petersson probability measure on $\mathcal{M}_g$.
$\mathbb{E}_g^{\mathrm{WP}}$	expectation w.r.t. the Weil-Petersson probability measure.
$\mathcal{M}_{g,n}(\mathbf{x})$	moduli space of bordered hyperbolic surfaces of signature (g, n) with boundary components of lengths $\mathbf{x} = (x_1, \ldots, x_n)$.
$\mathcal{T}_{g,n}(\mathbf{x})$	Teichmüller space of bordered hyperbolic surfaces [...].
$S_{g,n}$, $\mathrm{MCG}_{g,n}$	fixed base surface for $\mathcal{T}_{g,n}(\mathbf{x})$ and its mapping class group.
$V_{g,n}(\mathbf{x})$	total volume of $\mathcal{M}_{g,n}(\mathbf{x})$ for the Weil-Petersson volume.
$V_{g,n}$	value of $V_{g,n}(\mathbf{x})$ for $\mathbf{x} = (0, \ldots, 0)$.

Chapter 1

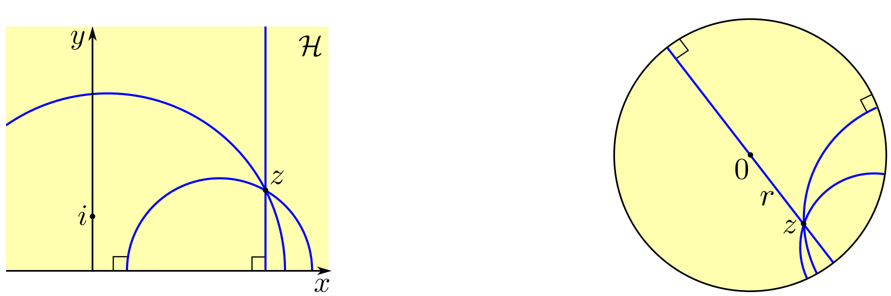
Introduction (in English)

The aim of this thesis is to provide information on *typical* compact hyperbolic surfaces. We focus on two aspects: *geometric* properties, such as the lengths of closed geodesics on the surface, and *spectral* properties, related to the distribution of the eigenvalues of the Laplacian. This is achieved thanks to a *new probabilistic approach*, using a powerful tool-set developed by Mirzakhani [Mir13] that allows us to sample random surfaces, and hence remove a small set of pathological surfaces from our consideration.

We start by a description of the objects studied in this thesis and the questions we wish to address. My contributions are then presented in Section 1.3, and highlighted by a vertical line in the left margin, like this paragraph is.

1.1 The objects at play in this thesis

Compact hyperbolic surfaces Throughout this thesis, a *compact hyperbolic surface* X is a closed (i.e. compact without boundary), connected, oriented surface equipped with a *Riemannian metric* of constant curvature equal to -1. The topology of the surface is entirely defined by its *genus* $g \geq 2$. There exists compact hyperbolic surfaces of every genus $g \geq 2$.



(a) The Poincaré half-plane. (b) The Poincaré disk.

Figure 1.1: Two models for the hyperbolic plane and their geodesics.

Compact hyperbolic surfaces are locally isometric to the *hyperbolic plane*. We use two models for the hyperbolic plane, both represented in Figure 1.1:

(a) the Poincaré half-plane model

$$\mathcal{H} := \{x + iy \ : \ y > 0\} \quad \text{with the metric} \quad \mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

(b) or the Poincaré disk model

$$\{x + iy \ : \ r^2 := x^2 + y^2 < 1\} \quad \text{with the metric} \quad \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - r^2)^2}.$$

An oriented Riemannian manifold possesses a natural measure form; on the Poincaré half-plane, it can be expressed in coordinates as $\mathrm{dVol}_{\mathcal{H}} = \frac{\mathrm{d}x \, \mathrm{d}y}{y^2}$. By the Gauss-Bonnet formula, the area of a compact hyperbolic surface X of genus g, for its standard volume form dVol_X, is equal to $2\pi(2g - 2)$.

Throughout this thesis, most of the results will be stated in the *large genus limit*, i.e. as $g \to +\infty$. This limit can be interpreted in two ways: we describe (typical) compact hyperbolic surfaces with a rich topology and large area.

The length spectrum The collection of all closed geodesics on a compact hyperbolic surface X is countable. Indeed, each *free-homotopy class* (see Section 3.2.1.1) on a compact hyperbolic surface contains exactly one closed geodesic, which is the unique minimiser of the length in the free-homotopy class.

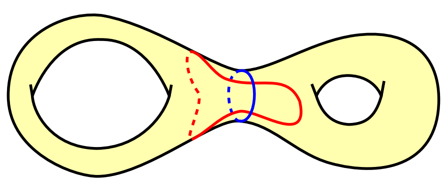


Figure 1.2: Two closed curves freely homotopic to one another on a genus two surface. The shortest curve (in blue) is the geodesic representative of the free-homotopy class.

The uniqueness of the minimiser is linked to the fact that the geodesic flow on a compact hyperbolic surface is *chaotic*. While there are parallel lines on the flat plane, it is not the case on the hyperbolic plane: as one might observe on Figure 1.1, a small perturbation of a geodesic on $\mathcal{H}$ is ultimately very far away from the initial geodesic. As a consequence, situations such as cylinders of closed geodesics on the flat torus do not occur on compact hyperbolic surfaces.

A closed geodesic is called *primitive* if it is not obtained by travelling along a single closed geodesic $k \geq 2$ times. The *(primitive) length spectrum* of the surface X is the ordered list of the lengths of the *(primitive) closed geodesics* on X. Let

$$0 < \ell_1 \leq \ell_2 \leq \ldots \leq \ell_i \underset{i \to +\infty}{\longrightarrow} +\infty$$

denote the primitive length spectrum. Any closed geodesic can be written uniquely as an iterate of a primitive closed geodesic. As a consequence the lengths of all closed geodesics are the $k\ell_i$ for $k, i \geq 1$, and the primitive length spectrum determines the length spectrum.

The first length ℓ_1 is the length of the shortest closed geodesic, called the *systole*. It is equal to twice the *injectivity radius* InjRad_X of the surface X.

Spectrum of the Laplacian For any compact hyperbolic surface X, the *Laplace-Beltrami operator* Δ_X is an unbounded operator on $L^2(X)$, which acts as a second-order differential operator on smooth functions and is invariant by isometries. Throughout this thesis, we will call this operator the *Laplacian*. On the Poincaré half-plane $\mathcal{H}$, the Laplacian can be expressed in coordinates as

$$\Delta_{\mathcal{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

An *eigenvalue* of the Laplacian is a real number λ such that the equation $\Delta_X f = \lambda f$ has a non-zero solution. Since the Laplacian appears in many important partial differential equations, such as Laplace, heat, wave and Schrödinger equations [Eva98], its eigenvalues can be interpreted physically, as the frequencies of a drum or quantum energy levels for instance.

Thanks to the compactness of the surface X, by the spectral theorem, the spectrum of the Laplacian Δ_X is a *discrete* sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \underset{j \to +\infty}{\longrightarrow} +\infty.$$

The zero eigenvalue corresponds to constant eigenfunctions and is called the *trivial eigenvalue*. It is simple because we assumed that the surface is connected. The smallest non-zero eigenvalue λ_1 is called the *spectral gap*. It is linked to the mixing rate of the geodesic flow [Rat87] and Brownian motion [GK19], as well as the *Cheeger constant* [Che70, Bus82]. The spectral gap is at the center of Selberg's famous conjecture, which states that $\lambda_1 \geq \frac{1}{4}$ for a certain class of surfaces (congruence surfaces), and has important implications in arithmetics [Sel56].

The special part that the value $\frac{1}{4}$ plays in the following discussion is connected to the fact that it is the bottom of the (continuous) spectrum of the Laplacian on the hyperbolic plane $\mathcal{H}$ [McK70]. Non-zero eigenvalues under $\frac{1}{4}$ are called *small eigenvalues*.

1.2 Deterministic state of the art

What can be said of the distribution of the lengths $(\ell_i)_i$ and eigenvalues $(\lambda_j)_j$? This question has been at the center of a very active and prolific research field throughout the second part of the 20th century. A great understanding had been reached by the 1990's, and Buser's book '*Geometry and Spectra of Compact Riemann Surfaces*' [Bus92] is the primary reference of the field to this day. This very complete exposition leaves little room for further enquiries, because many of the results that are presented are optimal, and the remaining open questions are very ambitious or vague.

The aim of this thesis is to move past this obstacle and prove new results, that are true for *most* surfaces rather than every single one of them. In order to motivate this idea, let us overview the known results on the distribution of the families $(\ell_i)_i$ and $(\lambda_j)_j$, with a particular focus on the examples that prevent us to improve them, in order to question their typicality.

1.2.1 Small geodesics and eigenvalues

Question 1. *For any $\varepsilon > 0$, $g \geq 2$, does there exist a genus g surface such that $\ell_1 < \varepsilon$? If so, what is the maximal number I such that $\ell_1, \ldots, \ell_I < \varepsilon$? Same questions for the eigenvalues of the Laplacian.*

The answer to the first question is yes: it is possible to *pinch* one closed geodesic on a surface, reducing its length as much as one might want. Note that, by the *Collar Lemma* [Bus92, Theorem 4.1.1], when doing so, the neighbourhood of the closed geodesic becomes a long and narrow collar, as represented on Figure 1.3.

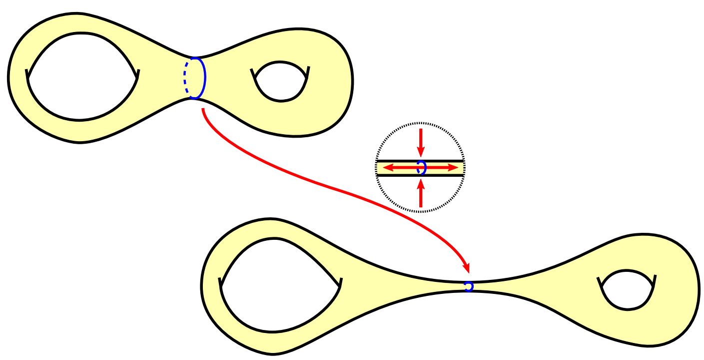
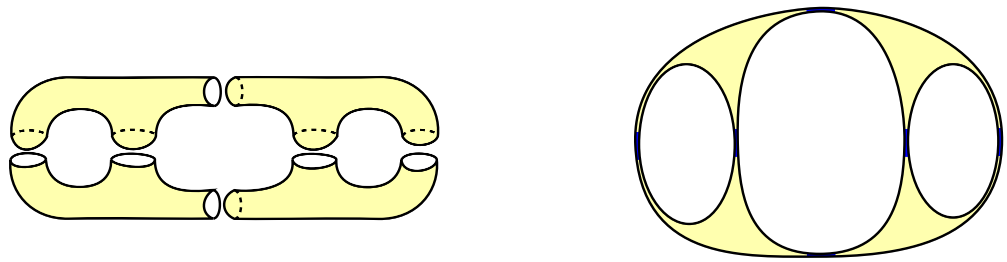


Figure 1.3: Pinched surface along one closed geodesic.

In order to pinch more curves at once, we can cut the surface into *pairs of pants*, i.e. surfaces of genus 0 with 3 boundary components. A surface of genus g can be cut into $2g-2$ pairs of pants (see Figure 1.4a). The fundamental property of pairs of pants is that, for any prescribed lengths x_1, x_2, x_3, their is a unique pair of pants with boundaries of lengths x_1, x_2, x_3. It is therefore possible to simultaneously pinch the $3g-3$ closed curves delimiting the pairs of pants, and as a result obtain $3g-3$ arbitrarily small curves (see Figure 1.4b). Reciprocally, the number of closed geodesics shorter than $2 \operatorname{arcsinh} 1$ is always $\leq 3g-3$ [Bus92, Theorem 4.1.6], so this is a maximum.

These surfaces pinched along a pair of pants decomposition also provide an answer to the eigenvalue part of Question 1. Indeed, they consist of $2g-2$ pieces connected by long and narrow cylinders which act as bottle-necks and almost disconnect the surface. Using the min-max principle, this implies that the first $2g-2$ eigenvalues go to zero



(a) A genus 3 surface cut into 4 pairs of pants. (b) The corresponding pinched surface.

Figure 1.4: Pinching a pair of pants decomposition.

as we pinch all of the curves [Bus92, Theorem 8.1.3]. This is once again a maximum, because the number of eigenvalues smaller than $\frac{1}{4}$ is always at most $2g - 2$ by a result of Otal and Rosas [OR09].

Conversely, the first eigenvalue λ_1 can be quite large: for instance, a compact hyperbolic surface of genus 2 such that $\lambda_1 \approx 3.8$ has been constructed in [Jen84]. However, Cheng proved upper bounds on eigenvalues in [Che75], which imply that in the large-genus limit, $\lambda_1 \leq \frac{1}{4} + o(1)$. The question of the existence of surfaces of large genus such that $\lambda_1 \geq \frac{1}{4}$ is a long-lasting conjecture [BBD88] that remains open in spite of active research [BM01, Mon15].

1.2.2 Estimates of counting functions

For real numbers $0 \leq a \leq b$, let us define the *counting functions*

$$\mathrm{N}_X^{\ell}(a,b) := \#\{i \;:\; a \leq \ell_i(X) \leq b\}$$
$$\mathrm{N}_X^{\Delta}(a,b) := \#\{j \;:\; a \leq \lambda_j(X) \leq b\}.$$

Question 2. *What can we say of the counting functions for a window* $[a,b]$*?*

We have just discussed some optimal bounds for low windows. While it is possible to obtain rough bounds for the length counting function (see [Bus92, Lemma 6.6.4] for instance), it is impossible to bound $\mathrm{N}_X^{\Delta}(0,b)$ in terms of g and b only as soon as $b > \frac{1}{4}$. Indeed, for $\varepsilon > 0$, the number of eigenvalues between $\frac{1}{4}$ and $\frac{1}{4} + \varepsilon$ goes to infinity as we pinch one closed geodesic on a surface [Bus92, Theorem 8.1.2].

It is hard to say much more for a fixed window $[a,b]$. However, we can study the asymptotics of counting functions in the large window limit, and obtain:

- the Weyl law [Bér77, Ran78]

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} = \frac{b}{4\pi} + \mathcal{O}_X\left(\frac{\sqrt{b}}{\log b}\right) \qquad \text{as } b \to +\infty \tag{1.1}$$

- the prime geodesic theorem (with error terms) [Hub59]

$$\frac{\mathrm{N}_X^\ell(0,\log b)}{\mathrm{Vol}_X(X)} = \mathrm{li}(b) + \sum_{\substack{j:\lambda_j=s_j(1-s_j)\\ \text{with } s_j\in(\frac{3}{4},1)}} \mathrm{li}(b^{s_j}) + \mathcal{O}_X\left(\frac{b^{\frac{3}{4}}}{\log b}\right) \qquad \text{as } b\to+\infty \tag{1.2}$$

  where $\mathrm{li}(b) = \int_2^b \frac{\mathrm{d}t}{\log t} \sim_{+\infty} \frac{b}{\log b}$ is the integral logarithm.

The implied constants in both statements depend on the surface.

Question 3. *Can the Weyl law and prime geodesic theorem be made* uniform *in terms of the surface?*

The error term in the prime geodesic theorem depends on the small eigenvalues $\lambda \in (0, \frac{3}{16})$. The reason behind this appearance of eigenvalues is that the two spectra $(\ell_i)_i$ and $(\lambda_j)_j$ are intertwined in the *Selberg trace formula* [Sel56], one of the main tools used to study the geometry and spectrum of compact hyperbolic surfaces. This powerful formula establishes a correspondence between the two following sets:

$$\{\lambda_j,\ j\geq 0\} \qquad \leftrightarrow \qquad \{k\ell_i,\ k,i\geq 1\}.$$

These links are not one-to-one but rather intricate. Notably, the Selberg trace formula involves a Fourier transform, which creates interactions between:

- *small eigenvalues* and *long geodesics*, as in equation (1.2)
- *short geodesics* and *large eigenvalues*, as we will have several opportunities to notice throughout this thesis (see Sections 5.1 and 5.3.1.3 and Chapter 6).

As a consequence, we expect pinched surfaces from Figure 1.4b to behave differently from non-pinched surfaces, since they have many small geodesics and eigenvalues. This is a strong limitation to any uniform estimate.

1.2.3 Multiplicities

The *multiplicity* of a length ℓ or an eigenvalue λ is defined as

$$m_X^\Delta(\lambda) = \#\{j \ :\ \lambda_j = \lambda\}$$
$$m_X^\ell(\ell) = \#\{i \ :\ \ell_i = \ell\}.$$

Question 4. *Can we bound the multiplicities m_X^Δ and m_X^ℓ of a surface?*

By [Bes80], the multiplicity of the j-th eigenvalue λ_j is smaller or equal to $4g+2j-1$. We do not know if this bound is optimal; surfaces of genus g such that the multiplicity of λ_1 is $\lfloor\frac{1+\sqrt{8g+1}}{2}\rfloor$ are constructed in [CC88]. These examples are obtained, once again, by pinching a family of curves on the surface.

The prime geodesic theorem provides a rough bound on the multiplicity of a length ℓ, behaving exponentially in ℓ. Examples of surface with exponential multiplicities are known: often, they come from families of surfaces with *symmetries*, arithmetic surfaces, such as the regular octagon represented in Figure 1.5 [Mar06]. Some numerical simulations seem to suggest that there also exist non-arithmetic surfaces with exponential multiplicities [BGGS97, Section 10].

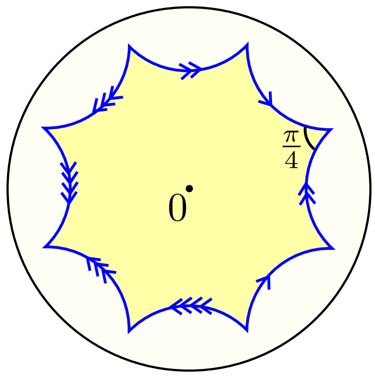


Figure 1.5: A compact arithmetic surface obtained by gluing a regular octagon.

It could be seen as a surprise that there would be so many multiplicities. What are the odds for two eigenvalues λ_j and $\lambda_{j'}$, or two lengths ℓ_i and $\ell_{i'}$, to be precisely equal?

Randol proved in [Ran80] using a result of Horowitz [Hor72] that the length spectrum always has unbounded multiplicities, i.e. for *any* compact hyperbolic surface X and any N, there is a ℓ such that $m_X^\ell(\ell) \geq N$. This is proved using 'rigid' families of geodesics in pairs of pants, which always have the same lengths.

The situation for the *simple* length spectrum is different. One can easily construct families of $3g-3$ simple geodesics with identical lengths using a pair of pants decomposition. But McShane and Parlier proved in [MP08] that the set of surfaces that contain two simple geodesics of equal length is meagre in the set of surface: it is a countable union of analytic sub-manifolds of real co-dimension 1. In other words, multiplicities in the simple length spectrum are exceptional and do not occur for typical surfaces.

1.3 Typical surfaces: literature and contributions

All of the examples exhibited in the previous section are somehow peculiar: they are either obtained by pinching curves, and in a sense 'limit-cases', or discrete families of surfaces that have particular symmetries or algebraic properties. One of the main aims of this thesis is to improve our knowledge of compact hyperbolic surfaces by setting aside a small set of 'pathological' surfaces, and focusing on typical surfaces only.

This idea can be quite challenging to implement, because it requires to decide on a definition of typicality, which is both meaningful and convenient to use.

1.3.1 Different notions of typical surfaces

There is no canonical way to study typical surfaces, and many different approaches have been used successfully to do so; a few examples of models are presented in Table 1.1 together with the reference in which they are introduced.

Probabilistic	Random Belyĭ surfaces	[BM04]
	Weil–Petersson probability measure	[Mir13]
	Combinatorial pair of pants gluings	[BCP19]
	Random covers	[MNP20]
Topological or geometric	'Small' complement (meagre...)	[Wol77] or [MP08]
	'Large' set (unbounded...)	[Mon15]

Table 1.1: Overview of different notions of typicality that can be found in literature.

The bulk of this thesis focuses on *the Weil–Petersson model*. The starting point is very natural: for an integer $g \geq 2$, we equip the moduli space

$$\mathcal{M}_g = \{\text{compact hyperbolic surfaces of genus } g\} \diagup \text{isometry}$$

with a probability measure. We will then say a property is *typical* if occurs *with high probability*, that is to say if

$$\text{Prob}(X \in \mathcal{M}_g \text{ satisfies the property}) \underset{g \to +\infty}{\longrightarrow} 1.$$

A priori, there exist many probability measures on the moduli space $\mathcal{M}_g$. Fortunately, one excellent candidate stands out as a canonical choice: the probability measure $\mathbb{P}_g^{\text{WP}}$ induced by the Weil–Petersson symplectic structure [Wei58]. One strong advantage of this measure is that it has an elementary expression in a set of local coordinates of the moduli space $\mathcal{M}_g$ called *Fenchel–Nielsen coordinates* [Wol81]. Building on this formula, Mirzakhani developed in [Mir07a, Mir13] a powerful tool-set allowing to compute and estimate certain probabilities, hence laying the ground work for a now very active field.

1.3.2 New formulation of our questions

Let us reformulate the previous questions in our new probabilistic setting.

Question 1* *How large are ℓ_1 and λ_1 typically? How many small geodesics and small eigenvalues does a typical surface have?*

Question 2* *What are the statistics of the counting functions in the limit $g \to +\infty$?*

Question 3* *Can the Weyl law and prime geodesic theorem be made uniform if we restrict ourselves to typical surfaces?*

Question 4* *Can multiplicity bounds be improved for typical surfaces?*

The recent apparition of several new models and tools contributed to the increase of popularity of this idea in the last few years. We will try and capture this fast-moving picture.

Because the Weil–Petersson model has strong links with the geometry of the random surfaces, it is generally more direct to obtain information on the geometry of typical surfaces, and spectral information comes as a consequence. For this reason, we organise our exposition in this order. We will keep track of the different questions that we address by indicating their number in the titles of the different paragraphs.

1.3.3 Geometric results

First results ($1^\star$) Mirzakhani presented in [Mir13] a methodology and tools that allow us to study the geometry of random surfaces, as well as the first bounds for many important geometric quantities.

- The length of the systole cannot be bounded away from zero typically:
$$\forall \varepsilon > 0, \quad \mathbb{P}_g^{\mathrm{WP}}(\ell_1 \geq \varepsilon) \underset{g \to +\infty}{\not\to} 1.$$
- The *Cheeger constant*
$$h(X) = \inf \left\{ \frac{\ell(\partial A)}{\mathrm{Vol}_X(A)} \text{ where } X = A \sqcup B \text{ and } \mathrm{Vol}_X(A) \leq \mathrm{Vol}_X(B) \right\},$$
that measures how hard it is to disconnect the surface, is always smaller than $1 + o(1)$ in the large-genus limit [Che75]. Typically, $h(X) \geq \frac{\log 2}{2\pi + \log 2} \approx 0.099$.
- The *diameter* of a typical surface is smaller than $40 \log g$, which is close to the lower bound $\mathrm{diam}(X) \geq \log(4g-2)$.

Besides the fact that the length of the systole cannot be bounded below, all of the results proven in [Mir13] seem to indicate that typical surfaces of large genus are very ‘well-connected’ or ‘tightly packed’ – this contrasts with examples such as Figure 1.6.

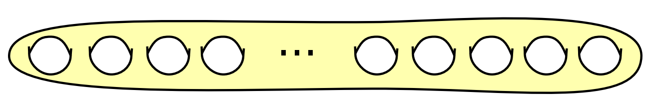


Figure 1.6: An atypical surface of large genus: its Cheeger constant goes to zero as $g \to +\infty$ and its diameter grows linearly as a function of g.

Similar qualitative results have been obtained in the random Belyĭ setting in [BM04], the only known difference so far being the existence of a lower bound for ℓ_1.

Statistic of counting functions for fixed windows ($2^\star$) Mirzakhani and Petri proved in [MP19] that for any fixed window $[a, b]$, the counting function $\mathrm{N}_X^\ell(a, b)$ converges to a Poisson law of parameter

$$\lambda(a, b) = \int_a^b \frac{e^t + e^{-t} - 2}{2t} \, \mathrm{d}t$$

as the genus g approaches infinity. We know that Poisson laws are used to model rare events. Therefore, the presence of small geodesics can be seen as a somehow 'rare' yet not atypical event.

They furthermore proved the independence of counting functions on disjoint windows, and computed the asymptotic law of the systole ℓ_1.

Distribution of 'short' geodesics on the surface ($1^\star$) Chapter 4 is dedicated to pushing the description of 'short' geodesics on typical surfaces further: rather than looking at fixed windows as in [MP19], we study geodesics of length proportional to $\log g$.

Note that $\log g$ is a long distance on a typical surface, comparable with its diameter. Working in this length scale is motivated by the fact that, the same way eigenvalues $\lambda < \frac{3}{16}$ appear in the prime geodesic theorem [Hub59], we expect for geodesics of logarithmic size to intervene when proving spectral asymptotics.

We prove that, though closed geodesics of length proportional to $\log g$ typically do exists, they are *rare* and can be described quite precisely. This is achieved by adopting two complementary viewpoints, Benjamini–Schramm convergence (Section 4.1) and tangle-freeness (Section 4.2). Both notions have significant consequences on the spectrum of typical surfaces, which are presented in Chapters 5 and 6.

For a real number $L > 0$, the L-thin part of a surface X is defined as

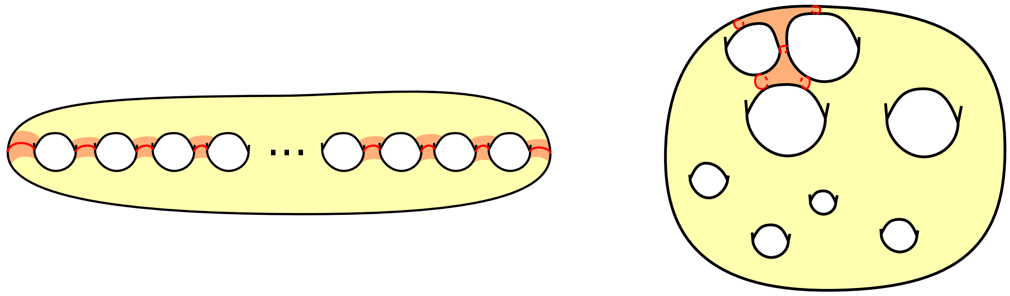
$$X^-(L) = \{z \in X \ : \ \mathrm{InjRad}_X(z) < L\}$$

where $\mathrm{InjRad}_X(z)$ is the radius of the largest embedded disk centered at z.

We prove the following result, which can be interpreted as *Benjamini–Schramm* convergence of typical surfaces to the hyperbolic plane $\mathcal{H}$.

Theorem (Theorem 4.1)**.** *For $a < \frac{1}{3}$, the area of the $(a \log g)$-thin part of the typical surface is negligible compared to the area of the whole surface.*

The proof relies on the fact that the thin part is concentrated around small geodesics (see Figure 1.7a) and a bound on their number. The notion of Benjamini-Schramm converge was adapted from graph theory [BS01] to a continuous setting in [ABB+11], which is known to have applications to spectral theory.



(a) A surface with too many short geodesics and hence a significant thin part.

(b) A surface with a cluster of short geodesics, and hence small pairs of pants.

Figure 1.7: Examples of atypical surfaces.

While Benjamini–Schramm convergence tells us that 'short' geodesics make up for a small proportion of the surface, it says nothing about their distribution. Can these geodesics be clustered together, or do they live in different parts of the surface? We provided answers to this question in [MT21], a collaboration with Joe Thomas.

Inspired by the fact that the presence of clusters of short geodesics imply the existence of small embedded sub-surfaces, as shown in Figure 1.7b, we define a notion of 'tangle-free surface' as follows.

Definition (Definition 4)**.** For a real number $L > 0$, we say that a surface is *L-tangle-free* if it contains no embedded pair of pants or one-hole torus of total boundary length smaller than $2L$.

We prove that typical surfaces are $((1-\varepsilon)\log g)$-tangle-free for any $\varepsilon > 0$, while surfaces of large genus g are always $(4\log g + \mathcal{O}(1))$-tangled. We deduce significant geometric implications, amongst which the following.

Theorem (Theorem 4.9, Proposition 4.14 and Corollary 4.15)**.** *For $L = (1-\varepsilon)\log g$, typically,*

- *all closed geodesics shorter that L are* simple
- *two closed geodesics such that $\ell(\gamma_1) + \ell(\gamma_2) < L$ do not intersect*
- *the neighbourhood of width $(L-\ell)/2$ around a closed geodesic of length $\ell < L$ is an embedded cylinder (much wider than the one granted by the collar Lemma)*
- *the topology of a ball of radius $L/8$ is a ball or a cylinder.*

Nie, Wu and Xue proved in [NWX20] that the length of the separating systole is typically $2\log g$, and other results at the scale $\log g$ of a similar flavour.

Multiplicities (4*) Let $\varepsilon \in (0,1)$. Our results [MT21] together with [MP08] imply that, typically, their are no multiplicities in the length spectrum up to the length $(1-\varepsilon)\log g$:

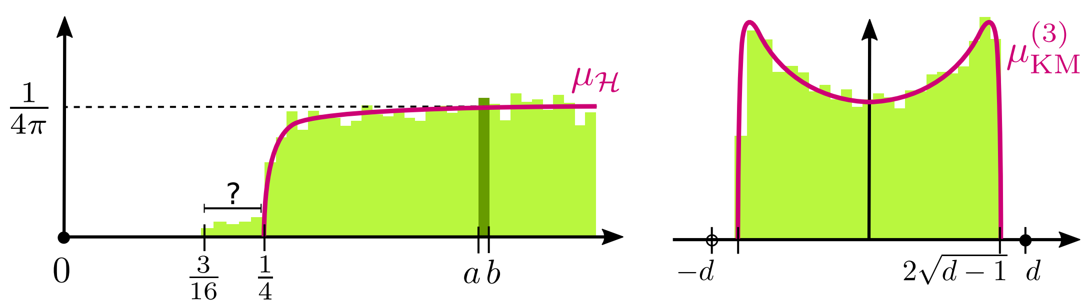
$$\forall i \neq j \text{ such that } \ell_i, \ell_j \leq (1-\varepsilon)\log g, \quad \ell_i \neq \ell_j.$$

In particular, the lengths of the examples of large multiplicity found in [Hor72] are typically $> (1-\varepsilon)\log g$.

1.3.4 Spectral results

A highlight of this thesis are the new results published in [Mon21], one of the first articles describing the spectrum of typical compact hyperbolic surfaces. We have proved that the density $\mu_{\mathcal{H}}$ represented in Figure 1.8a is typically a good approximation of the spectrum of a surface of high genus.

Two independent teams have recently proved that there is no eigenvalues below $\frac{3}{16}$ [WX21, LW21]. The information that we currently possess is summarized in Figure 1.8a. These results are natural adaptations of similar results for typical regular graphs, as suggested in Figure 1.8 – we detail these links in Section 1.4.



(a) Typical compact hyperbolic surface. (b) Typical 3-regular graph.

Figure 1.8: Illustration of what we know the histogram of the spectrum (divided by the area or number of vertices) to look like for a typical hyperbolic surface or graph.

Let us present these results in more details, and our current understanding of some open questions.

Number of small eigenvalues (1*) The first spectral result that appears in the literature is the lower bound

$$\lambda_1 \geq \frac{1}{4}\left(\frac{\log 2}{2\pi + \log 2}\right)^2 \approx 0.002 \quad \text{typically}$$

which comes as a direct consequence of Mirzakhani's bound on Cheeger's constant [Mir13] together with Cheeger's inequality [Che70].

We prove a bound on the number of small eigenvalues in Section 5.2.

Theorem (Theorem 5.2)**.** *For a typical surface X,*

$$\forall b \geq 0, \quad \frac{\mathrm{N}_X^\Delta(0,b)}{\mathrm{Vol}_X(X)} \leq 32\, g^{-\frac{1}{27}(\frac{1}{4}-b)} (\log g)^{-\frac{3}{2}}. \tag{1.3}$$

The quality of this estimate depends on the position of b with respect to $\frac{1}{4}$.

- Taking $b = \frac{1}{4}$, we obtain that the number of small eigenvalues of a typical surface is $\mathcal{O}\left(g(\log g)^{-\frac{3}{2}}\right)$, which is an improvement by a logarithmic correction of the optimal bound $2g-2$ from [OR09].
- For $b < \frac{1}{4}$, we obtain an additional correction by a negative power of g, greater the further away we are from the interval $[\frac{1}{4}, +\infty)$.
- When $b > \frac{1}{4}$, the bound is still valid, but quite weak.

Neither of these results are true when we pinch a pair of pants decomposition: we recall that in this case, for any $\varepsilon > 0$, there is up to $2g-2$ eigenvalues in $[0, \varepsilon]$ and an arbitrarily high number in $[\frac{1}{4}, \frac{1}{4}+\varepsilon]$. Extremely pinched surfaces are, indeed, atypical, in the Weil–Petersson probability setting, and excluding them allows us to prove better results.

Counting functions: upper bounds and equivalents ($2^\star$)

In [Mon21], we study the counting function $\mathrm{N}_X^\Delta(a,b)$ using Benjamini–Schramm convergence and the Selberg trace formula and prove the following results. The precise statements of the results and their proofs can be found in Section 5.3.

Theorem (Theorems 5.8 and 5.9)**.** *For a typical surface X,*

- *for any $0 \leq a \leq b$,*

$$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(b - a + \sqrt{\frac{b+1}{\log g}}\right). \tag{1.4}$$

- *for any $0 \leq a \leq b$ such that $b - a \gg \sqrt{\frac{b+1}{\log g}}$,*

$$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} \sim \mu_{\mathcal{H}}(a,b) := \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathbb{1}_{[a,b]}(\lambda)\, \mathrm{d}\lambda \tag{1.5}$$

as b and/or g approaches infinity.

All the constants are universal constants, independent of X, g, a and b.

The quantity $\mu_{\mathcal{H}}(a,b)$ is the spectral density of the hyperbolic plane, represented in Figure 1.8a. It is supported on $[\frac{1}{4}, +\infty)$.

Taking a fixed window $[a,b]$, our results provide a first answer to Question $2^\star$. The next step would be to estimate correlations between the counting functions on two disjoint windows, or improve the rates in the estimates.

Applications The results proven in [Mon21] have two significant corollaries.

- Since $\mu_{\mathcal{H}}(0,b)$ is equivalent to $\frac{b}{4\pi}$ as b approaches $+\infty$, we can deduce from equation (1.5) a uniform Weyl law

$$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} = \frac{b}{4\pi} + \mathcal{O}_g\left(\sqrt{b \log b}\right),$$

  hence answering to Question $3^\star$.

- The upper bound (1.4), applied to a shrinking window around an eigenvalue, leads to a bound on the multiplicity of any eigenvalue λ in term of λ and g. Using equation (1.5) to estimate the typical size the λ_j for any j, we prove in Section 5.4 the following improvement of the deterministic bound from [Bes80] (Question $4^\star$).

  Theorem (Corollaries 5.28 and 5.29)**.** *For a typical surface X,*

$$\forall j \geq 1, \quad m_X(\lambda_j) = \mathcal{O}\left(g\sqrt{\frac{1+\frac{j}{g}}{\log g}}\right).$$

These estimates have also been used in [LMS20] to prove quantum ergodicity of typical surfaces of large genus.

Spectral gap ($1^\star$) Can we improve Mirzakhani's lower bound on λ_1? Proving the following conjecture, first stated in [Wri20], is currently one of the most awaited results in the spectral theory of compact hyperbolic surfaces.

Conjecture. *For any $\varepsilon > 0$,*

$$\lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{1}{4} - \varepsilon\right) = 1.$$

By [Che75], the value $\frac{1}{4}$ is the greatest number for which this statement can be true. Mondal's results [Mon15], in genus 2, are encouraging. He proved that the set of genus 2 surfaces such that $\lambda_1 \geq \frac{1}{4}$ is unbounded and disconnects the moduli space $\mathcal{M}_2$, and therefore many genus 2 surfaces satisfy the conjecture. The result would imply the existence of surfaces of arbitrarily large genus such that $\lambda_1 \geq \frac{1}{4} - \varepsilon$, and provide a positive answer to the Buser–Burger–Dodziuk conjecture [BBD88].

The introduction of the notion of tangle-free surfaces in [MT21] was motivated by research on this conjecture in collaboration with Nalini Anantharaman. We present a few elements related to this conjecture in Chapter 6.

A weaker version of the conjecture have been obtained very recently by two independent teams [LW21, WX21], with $\frac{3}{16}$ in place of $\frac{1}{4}$. The same bound has been established in the random cover model [MNP20]. Though there are some key differences in the method we suggest and the ones they have used, some of which we will present in Chapter 6, many elements are similar. Notably, some definitions and results from [MT21] have been used independently in [LW21].

The value $\frac{3}{16}$ appears because computations in both articles are done at a precision $1/g$. In order to obtain the result for $\frac{1}{4}$, we know that we need to be able to do computations with arbitrarily high precision, i.e. errors of size $1/g^N$ for $N \gg 1$.

In order to make this possible, we have:

- computed a high-genus asymptotic expansion of the Weil–Petersson volume polynomials, that appear when computing probabilities in the Weil–Petersson setting (**Theorems** 3.17 and 3.18);
- generalised the tangle-free hypothesis to a weaker notion, adding a parameter so that the probability of not being tangle-free can be $\mathcal{O}\left(g^{-N}\right)$ for an arbitrarily large N (**Theorems** 4.19 and 4.20).

1.4 Comparison with regular graphs

To conclude this introduction, let us quickly outline the correspondence between regular graphs and compact hyperbolic surfaces, which is the inspiration for many of the results presented in this thesis.

Let $d \geq 3$ be an integer. In the following, a *d-regular graph* is a graph such that every vertex has d neighbours. Let n denote the number of vertices of such a graph.

Regular graphs bear surprising resemblances to compact hyperbolic surfaces on many levels, both geometrically and spectrally. Here are two facts that contribute to these similarities:

- the neighbourhood of any point in the object is identical (a small hyperbolic disk or exactly d vertices)
- the size (area or number of vertices) of a ball grows exponentially with its radius.

	compact hyp surface	**d-regular graph**
asymptotic regime:	genus $g \to +\infty$	number of vertices $n \to +\infty$
probabilistic model:	Weil–Petersson law $\mathbb{P}_g^{\mathrm{WP}}$	uniform law $\mathbb{P}_n^{(d)}$
spectrum included in:	$[0, +\infty)$	$[-d, d]$
'trivial' eigenvalues:	0	d, and maybe $-d$ ($\Leftrightarrow$ bipartite)
universal cover:	hyperbolic plane $\mathcal{H}$	infinite d-regular tree
and its spectrum:	$[\frac{1}{4}, +\infty)$	$[-2\sqrt{d-1}, 2\sqrt{d-1}]$
typical limit of	spectral density of $\mathcal{H}$	Kesten–McKay law
the spectral density:	[Mon21]	[Kes59, McK81]
spectral gap:	λ_1	$d - \lambda_+$
		where $\lambda_+ := \max\{\lvert\lambda_i\rvert : \lvert\lambda_i\rvert < d\}$
deterministic bound:	$\lambda_1 \leq \frac{1}{4} + o(1)$ [Che75]	$\lambda^+ \geq 2\sqrt{d-1} - o(1)$ [Nil91]
probabilistic bound:	$\lambda_1 \geq \frac{3}{16} - \varepsilon$ [WX21, LW21]	$\lambda^+ \leq 2\sqrt{d-1} + o(1)$ [Fri03]

Table 1.2: Comparison between the properties of the eigenvalues $(\lambda_j)_{j\geq 0}$ of the Laplacian Δ on a compact hyperbolic surface and the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of the adjacency matrix $A = dI_n - \Delta$ of a d-regular graph.

All of the definitions and results presented in this thesis have counterparts in the world of regular-graphs, as summarized in Table 1.2. Results are often (if not always) proven significantly before for graphs, because they are simpler to study (due to their finiteness), but also because they are very fundamental objects which appear in countless parts of mathematics and science.

1.5 Organisation of this thesis

The organisation of the different chapters of this thesis and the way they depend on one another is summarized in Figure 1.9. We have high-lighted the new results by putting them in solid-line boxes.

Here is a short outline of the different chapters.

- In Chapter 3, we provide an introduction to the Weil–Petersson probabilistic setting. The presentation is somewhat informal, but precise references and motivations for the important ideas and tools of this field are given. The last section of this chapter contains new estimates announced in [AM20].

- Chapters 4 and 5 constitute the bulk of this thesis. Most of the new results have been extracted from [Mon21, MT21] and significantly rearranged: the geometric results in Chapter 4 and the results on spectral density in Chapter 5.

- Chapter 6 presents advances towards proving that typical surfaces have an optimal spectral gap.
- In Chapter A, we generalise some of the main results of this thesis to another probabilistic model, the model of random Belyĭ surfaces introduced by Brooks and Makover [BM04].

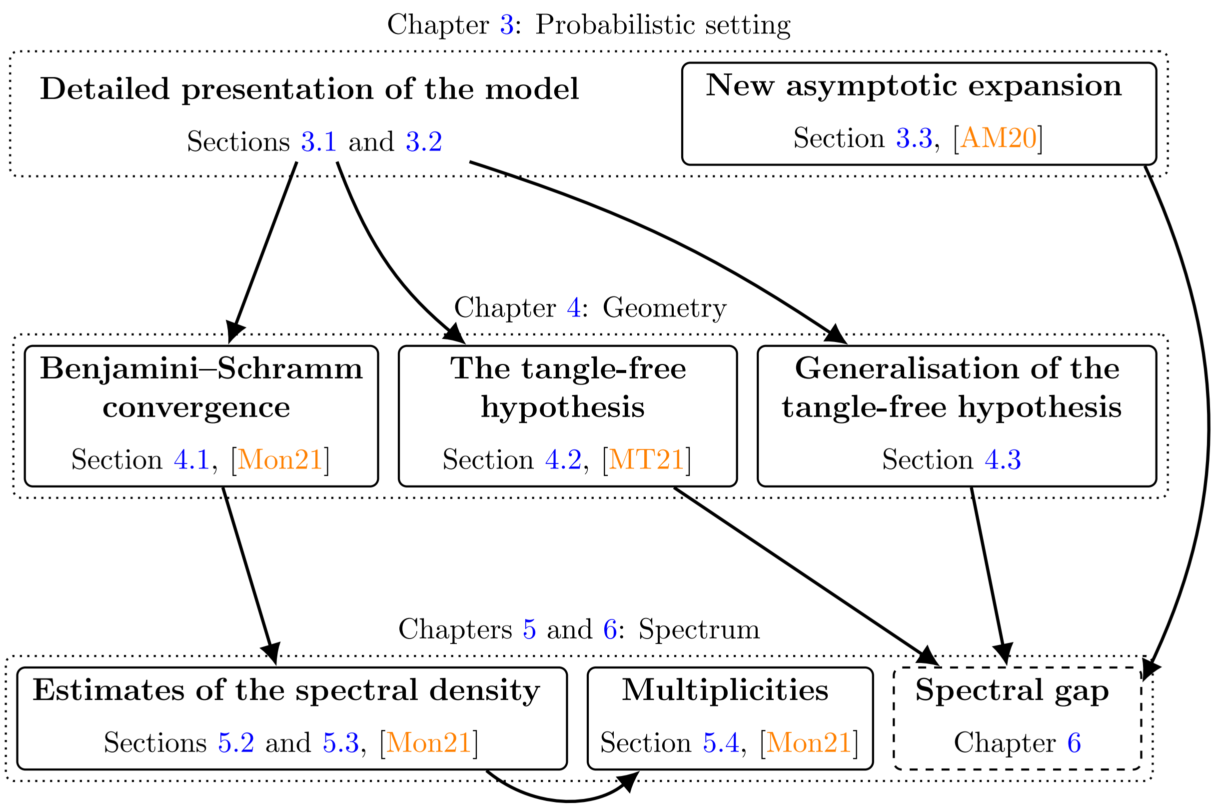


Figure 1.9: Organisation of the chapters of this thesis.

Chapitre 2

Introduction (en français)

L'objectif de cette thèse est d'améliorer notre connaissance des surfaces hyperboliques compactes *typiques*. Nous nous concentrons sur deux types de propriétés en particulier : les propriétés *géométriques*, liées (par exemple) aux longueurs des géodésiques fermées, et *spectrales*, concernant la répartition des valeurs propres du laplacien. Notre approche est basée sur de nouvelles méthodes *probabilistes* développées par Mirzakhani [Mir13], qui nous permettent de considérer des surfaces aléatoires, et ainsi de retirer un petit ensemble de surfaces pathologiques de notre étude.

Nous commençons par introduire les objets et enjeux de cette thèse. Mes contributions sont ensuite présentées dans la section 2.3 et mises en valeur par une ligne verticale dans la marge, à gauche, comme ce paragraphe.

2.1 Les objets étudiés dans cette thèse

Les surfaces hyperboliques compactes Tout au long de cette thèse, une *surface hyperbolique compacte* X est une surface compacte, sans bord, connexe, orientée, équipée d'une métrique riemannienne de courbure constante égale à -1. La topologie de la surface est entièrement décrite par son *genre* $g \geq 2$. Il existe des surfaces hyperboliques compactes de tout genre $g \geq 2$.

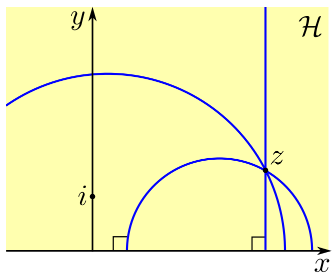
Les surfaces hyperboliques compactes sont localement isométriques au *plan hyperbolique*. Nous utilisons deux modèles pour le plan hyperbolique, représentées dans la figure 2.1 :

(a) le modèle du demi-plan de Poincaré

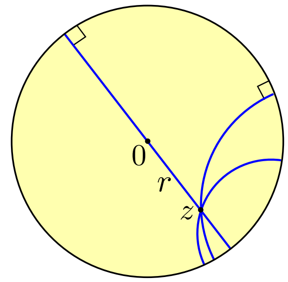
$$\mathcal{H} := \{x + iy \ : \ y > 0\} \quad \text{muni de la métrique} \quad \mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

(b) ou le modèle du disque de Poincaré

$$\{x + iy \ : \ r^2 := x^2 + y^2 < 1\} \quad \text{muni de la métrique} \quad \frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1 - r^2)^2}.$$



(a) Le demi-plan de Poincaré.



(b) Le disque de Poincaré.

FIGURE 2.1 : Deux modèles pour le plan hyperbolique, et leurs géodésiques.

Toute variété riemannienne orientée admet une forme volume naturelle. Sur le demi-plan de Poincaré, son expression en coordonnées est $\mathrm{dVol}_{\mathcal{H}} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$. Par la formule de Gauss–Bonnet, l'aire d'une surface hyperbolique compacte X de genre g pour sa forme volume standard dVol_X est égale à $2\pi(2g-2)$.

Dans cette thèse, la plupart des résultats démontrés le sont dans la *limite de grand genre*, c'est-à-dire quand $g \to +\infty$. Nous décrivons donc des surfaces hyperboliques compactes (typiques) qui ont une topologie riche et une grande aire.

Le spectre des longueurs L'ensemble des géodésiques fermées sur une surface hyperbolique compacte X est dénombrable. En effet, chaque *classe d'homotopie libre* (voir section 3.2.1.1) sur X contient exactement une géodésique fermée. Il s'agit de l'unique minimum de la fonction longueur sur la classe d'homotopie libre.

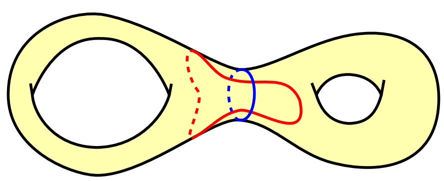


FIGURE 2.2 : Deux courbes fermées librement homotopes sur une surface de genre 2. La courbe la plus courte (en bleu) est le représentant géodésique de cette classe d'homotopie libre.

L'unicité du minimum est liée au caractère *chaotique* du flot géodésique sur les surfaces hyperboliques compactes. Comme on peut l'observer sur la figure 2.1, la géodésique obtenue après une petite perturbation d'une géodésique γ sur $\mathcal{H}$ est à une distance non bornée de γ en temps longs. Par conséquent, il ne peut pas y avoir de cylindres de géodésiques fermées sur les surfaces hyperboliques compactes, contrairement à ce qui est observé sur le tore plat.

Une géodésique fermée est dite *primitive* si elle ne peut être obtenue en parcourant une géodésique fermée $k \geq 2$ fois. Le *spectre des longueurs (primitives)* de la surface X

est la liste ordonnée des longueurs des *géodésiques fermées (primitives)* sur X. Notons

$$0 < \ell_1 \leq \ell_2 \leq \ldots \leq \ell_i \underset{i \to +\infty}{\longrightarrow} +\infty$$

le spectre des longueurs primitives. Toute géodésique fermée peut être obtenue de manière unique comme itérée d'une géodésique fermée primitive. Par conséquent, les longueurs des géodésiques fermées sont exactement les $k\ell_i$ pour $k, i \geq 1$, et le spectre des longueurs primitives détermine le spectre des longueurs.

La première longueur ℓ_1 est la longueur de la géodésique fermée la plus courte, appelée *systole*. Elle vaut le double du *rayon d'injectivité* InjRad_X de la surface X.

Spectre du laplacien Pour une surface hyperbolique compacte X, l'*opérateur de Laplace–Beltrami* Δ_X est un opérateur non borné sur $L^2(X)$, qui agit comme un opérateur différentiel d'ordre deux sur les fonctions lisses, et est invariant par isométrie. Dans cette thèse, nous appellerons cet opérateur le *laplacien*. Sur le demi-plan de Poincaré $\mathcal{H}$, l'expression du laplacien en coordonnées est

$$\Delta_{\mathcal{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Nous appelons *valeur propre* du laplacien tout nombre réel λ tel que $\Delta_X f = \lambda f$ admette une solution non nulle. L'apparition du laplacien dans de nombreuses équations aux dérivées partielles importantes, comme l'équation de Laplace, de la chaleur, des ondes et de Schrödinger [Eva98], donne diverses interprétations physiques à ses valeurs propres. Elles peuvent notamment être vues comme les fréquences d'un tambour ou des niveaux d'énergie quantique.

Par le théorème spectral pour les surfaces compactes, le spectre du laplacien Δ_X est une famille *discrète* de valeurs propres

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \underset{j \to +\infty}{\longrightarrow} +\infty.$$

La valeur propre nulle correspond aux fonctions propres constantes et est appelée la *valeur propre triviale*. Elle est simple car nous avons supposé que la surface est connexe. La plus petite valeur propre non nulle λ_1 est appelée le *trou spectral*. Cette quantité est liée au temps de mélange du flot géodésique [Rat87] et du mouvement brownien [GK19], ainsi qu'à la *constante de Cheeger* [Che70, Bus82]. Elle est au cœur de la célèbre conjecture de Selberg, qui affirme que $\lambda_1 \geq \frac{1}{4}$ pour une certaine classe de surfaces (les surfaces de congruence), et a des conséquences importantes en arithmétique [Sel56].

L'apparition du nombre $\frac{1}{4}$ dans cette discussion est liée au fait qu'il s'agit du bas du spectre (continu) du laplacien sur le plan hyperbolique $\mathcal{H}$ [McK70]. Les valeurs propres non nulles strictement inférieures à $\frac{1}{4}$ sont appelées *petites valeurs propres*.

2.2 État de l'art déterministe

Quelles informations sur la répartition des longueurs $(\ell_i)_i$ et valeurs propres $(\lambda_j)_j$ est-il possible d'obtenir ? Cette question fut au cœur d'un champ de recherche très actif et prolifique tout au cours de la seconde partie du 20ème siècle. Un très bon niveau de compréhension a été atteint dans les années 90, et le livre de Buser '*Geometry and Spectra of Compact Riemann Surfaces*' [Bus92] est encore à ce jour la référence du domaine. Cette exposition très complète n'invite que peu à continuer l'exploration de ces objets, car beaucoup des résultats présentés sont optimaux, et les questions qui restent ouvertes sont très ambitieuses ou vagues.

L'objectif de cette thèse est de dépasser cette difficulté et démontrer de nouveaux résultats, qui sont vrais pour *la plupart* des surfaces, plutôt que toutes. Afin de motiver cette idée, nous commençons par exposer les résultats classiques concernant la répartition des familles $(\ell_i)_i$ et $(\lambda_j)_j$. Nous insisterons sur les exemples qui prouvent leur optimalité, afin de pouvoir remettre en question leur caractère typique plus tard.

2.2.1 Petites géodésiques et valeurs propres

Question 1. *Pour un $\varepsilon > 0$, $g \geq 2$, existe-t-il une surface de genre g telle que $\ell_1 < \varepsilon$? Si oui, quel est de plus grand indice I tel que l'on puisse avoir $\ell_1, \ldots, \ell_I < \varepsilon$? Mêmes questions pour les valeurs propres du laplacien.*

La réponse à la première question est oui : il est possible de *pincer* une géodésique fermée sur une surface, et ainsi de réduire sa longueur à souhait. Le *lemme du collier* [Bus92, Théorème 4.1.1] implique que, ce faisant, le voisinage de la géodésique fermée devient un cylindre long et fin, comme celui représenté sur la figure 2.3.

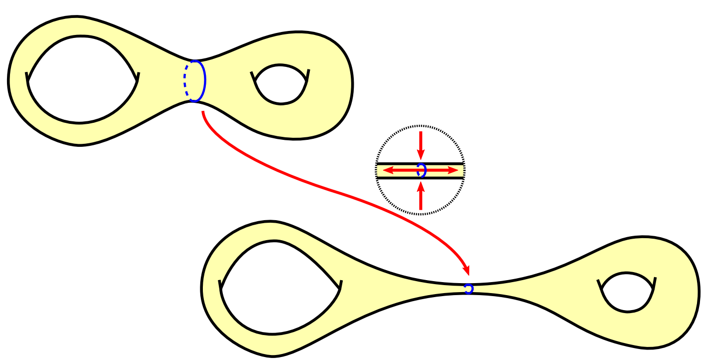
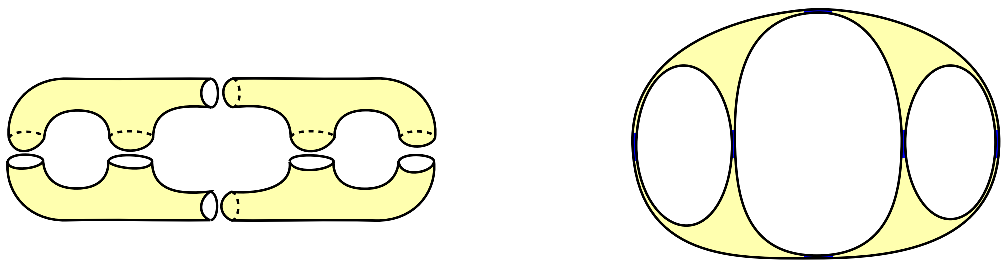


FIGURE 2.3 : Surface pincée le long d'une géodésique fermée.

Afin de pincer plus de courbes simultanément, il est possible de couper la surface en *pantalons*, c'est-à-dire en surfaces de genre 0 avec 3 composantes de bord. Une surface de genre g peut être décomposée en $2g-2$ pantalons, comme indiqué sur la figure 2.4a. Les pantalons vérifient une propriété fondamentale : pour toutes longueurs fixées x_1, x_2, x_3, il existe un unique pantalon hyperbolique de bords de longueurs

x_1, x_2, x_3. Il est par conséquent possible de pincer les $3g-3$ géodésiques délimitant les pantalons en même temps, et ainsi d'obtenir $3g-3$ géodésiques de longueurs arbitrairement petites (voir Figure 2.4b). Réciproquement, le nombre de géodésiques fermées plus courtes que $2 \operatorname{arcsinh} 1$ est toujours $\leq 3g-3$ [Bus92, Théorème 4.1.6] ; ce nombre est donc un maximum.



(a) Une surface de genre 3 coupée en 4 pantalons. (b) La surface pincée associée.

FIGURE 2.4 : Pincer une décomposition en pantalons.

Ces surfaces pincées le long d'une décomposition en pantalons répondent également à la question 1 du point de vue des valeurs propres. En effet, elles sont composées de $2g-2$ morceaux liés par des cylindres longs et fins, qui jouent le rôle de goulots d'étranglement. En utilisant le principe du min-max, il est possible de montrer que cela implique que les $2g-2$ premières valeurs propres tendent vers zéro lorsque l'on pince toutes ces géodésiques [Bus92, Théorème 8.1.3]. Il s'agit, ici aussi, d'un maximum : Otal et Rosas ont montré dans [OR09] que le nombre de valeurs propres plus petites que $\frac{1}{4}$ est toujours majoré par $2g-2$.

Inversement, la première valeur propre λ_1 peut être relativement grande : par exemple, [Jen84] construit une surface hyperbolique compacte de genre 2 telle que $\lambda_1 \approx 3.8$. Néanmoins, Cheng a établi des bornes supérieures sur les valeurs propres, qui impliquent que, dans la limite de grand genre, $\lambda_1 \leq \frac{1}{4} + o(1)$ [Che75]. La question de l'existence de surfaces de grand genre telles que $\lambda_1 \geq \frac{1}{4}$ est ancienne [BBD88], et reste ouverte en dépit de l'intérêt qu'elle suscite [BM01, Mon15].

2.2.2 Estimées des fonctions de comptage

Pour tous nombres réels $0 \leq a \leq b$, définissons les *fonctions de comptage*

$$\mathrm{N}_X^{\ell}(a,b) := \#\{i \;:\; a \leq \ell_i(X) \leq b\}$$
$$\mathrm{N}_X^{\Delta}(a,b) := \#\{j \;:\; a \leq \lambda_j(X) \leq b\}.$$

Question 2. *Que peut-on dire des fonctions de comptage pour une fenêtre* $[a,b]$ *?*

Nous avons présenté jusqu'à présent des bornes optimales pour des petites valeurs de a et b. Alors qu'il est toujours possible d'obtenir des bornes pour la fonction de comptage des longueurs (voir [Bus92, Lemme 6.6.4] par exemple), il n'est pas possible de borner $\mathrm{N}_X^{\Delta}(0,b)$ en fonction de g et b dès que $b > \frac{1}{4}$. En effet, pour tout $\varepsilon > 0$, le nombre de valeur propres entre $\frac{1}{4}$ et $\frac{1}{4}+\varepsilon$ tend vers l'infini lorsque l'on pince une géodésique fermée sur une surface [Bus92, Théorème 8.1.2].

Il est difficile de dire plus pour une fenêtre fixée $[a,b]$. Cependant, il est intéressant d'étudier le comportement asymptotique des fonctions de comptage quand la fenêtre devient grande :

- par la loi de Weyl [Bér77, Ran78],

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} = \frac{b}{4\pi} + \mathcal{O}_X\left(\frac{\sqrt{b}}{\log b}\right) \qquad \text{quand } b \to +\infty \tag{2.1}$$

- par le théorème des géodésique primitives (avec termes d'erreur) [Hub59],

$$\frac{\mathrm{N}_X^{\ell}(0,\log b)}{\mathrm{Vol}_X(X)} = \mathrm{li}(b) + \sum_{\substack{j:\lambda_j = s_j(1-s_j) \\ \text{with } s_j \in (\frac{3}{4},1)}} \mathrm{li}(b^{s_j}) + \mathcal{O}_X\left(\frac{b^{\frac{3}{4}}}{\log b}\right) \qquad \text{quand } b \to +\infty \tag{2.2}$$

  où $\mathrm{li}(b) = \int_2^b \frac{\mathrm{d}t}{\log t} \sim_{+\infty} \frac{b}{\log b}$ est le logarithme intégral.

Les constantes implicites dans ces deux résultats dépendent de la surface.

Question 3. *Est-il possible de démontrer une version de la loi de Weyl ou du théorème des géodésique primitives* uniforme *en terme de la surface ?*

Le terme d'erreur dans le théorème des géodésiques primitives dépend des petites valeurs propres $\lambda \in (0, \frac{3}{16})$. Le fait que les petites valeurs propres interviennent dans cette formule provient de la *formule des traces de Selberg* [Sel56], qui décrit les interactions entre les deux spectres $(\ell_i)_i$ et $(\lambda_j)_j$. Cette formule est un des principaux outils utilisés pour étudier la géométrie et le spectre des surfaces hyperboliques compactes, et établit une correspondance entre les deux ensembles suivants :

$$\{\lambda_j,\, j \geq 0\} \qquad \leftrightarrow \qquad \{k\ell_i,\, k, i \geq 1\}.$$

Il ne s'agit pas d'une relation directe entre ces deux quantités, mais plutôt d'interactions complexes. Notamment, la formule des traces de Selberg fait intervenir une transformée de Fourier, qui crée des interactions entre :

- les *petites valeur propres* et *longues géodésiques*, comme dans l'équation (2.2)
- les *géodésiques courtes* et *grandes valeur propres*, comme nous pourrons l'observer à plusieurs occasions au cours de cette thèse (voir sections 5.1, 5.3.1.3 et 6).

Par conséquent, les surfaces pincées de la figure 2.4b ont probablement un comportement particulier, car elles possèdent de nombreuses petites géodésiques et valeur propres. Ceci est un obstacle significatif à l'établissement de bornes uniformes.

2.2.3 Multiplicités

La *multiplicité* d'une longueur ℓ ou d'une valeur propre λ est définie comme étant

$$m_X^\Delta(\lambda) = \#\{j \ : \ \lambda_j = \lambda\}$$
$$m_X^\ell(\ell) = \#\{i \ : \ \ell_i = \ell\}.$$

Question 4. *Est-il possible de borner les multiplicités m_X^Δ et m_X^ℓ ?*

D'après [Bes80], la multiplicité de la j-ème valeur propre λ_j est inférieure ou égale à $4g+2j-1$. Nous ne savons pas si cette borne est optimale ; des surfaces de genre g telles que la multiplicité de λ_1 est $\lfloor \frac{1+\sqrt{8g+1}}{2} \rfloor$ ont été construites dans [CC88]. Ces exemples sont obtenus, encore une fois, en pinçant une famille de géodésiques sur une surface.

Le théorème des géodésique primitives implique une borne sur la multiplicité d'une longueur ℓ, de taille exponentielle en ℓ. Des exemples de surfaces avec des multiplicités exponentielles sont connus : souvent, ils proviennent de familles de surfaces avec des *symétries*, les surfaces arithmétiques, comme l'octogone régulier représenté en Figure 2.5 [Mar06]. Des simulations numériques semblent indiquer qu'il existe également des surfaces non arithmétiques avec des multiplicités exponentielles [BGGS97, Section 10].

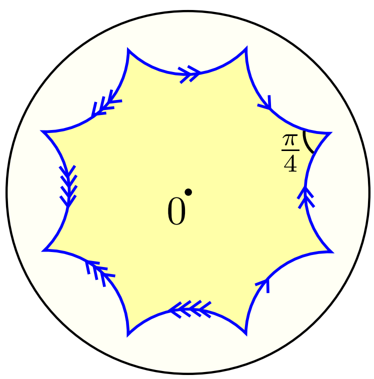


FIGURE 2.5 : Surface arithmétique compacte obtenue en collant un octogone régulier.

L'existence de si grandes multiplicités pourrait être considérée comme surprenante. Est-il réellement fréquent que deux valeurs propres λ_j et $\lambda_{j'}$, ou deux longueurs ℓ_i et $\ell_{i'}$, soient précisément égales ?

Randol a démontré dans [Ran80], à l'aide d'un résultat de Horowitz [Hor72], que le spectre des longueurs admet toujours des multiplicités de degré arbitrairement grand. Plus précisément, pour *toute* surface hyperbolique compacte X et tout N, il existe un nombre ℓ tel que $m_X^\ell(\ell) \geq N$. Cet énoncé est prouvé en utilisant des familles de géodésiques « rigides » dans des pantalons, qui ont toujours des longueurs identiques.

La situation pour le spectre des longueurs *simples* est différente. Il est facile de construire des familles de $3g-3$ géodésiques simples de longueurs identiques, à l'aide d'une décomposition en pantalons. Mais McShane et Parlier ont prouvé dans [MP08] que l'ensemble des surfaces qui contiennent deux géodésiques simples de même longueur est maigre dans l'ensemble des surfaces : il s'agit d'une union dénombrable de sous-variétés analytiques de codimension réelle 1. En d'autres termes, les multiplicités dans le spectre des longueurs simples sont exceptionnelles et ne se produisent pas pour des surfaces typiques.

2.3 Surfaces typiques : littérature et contributions

Tous les exemples présentés dans la section précédente sont, d'une certaine manière, particuliers. Ils sont obtenus soit en pinçant des géodésiques (et sont donc des « cas limites »), ou bien comme des familles discrètes de surfaces avec des symétries ou propriétés algébriques. Un des objectifs centraux de cette thèse est d'améliorer notre connaissance des surfaces hyperboliques compactes, en mettant de côté un petit ensemble de surfaces « pathologiques », et en se concentrant ainsi uniquement sur les surfaces typiques.

La réalisation de cette idée peut être délicate, car elle nécessite de choisir une notion de typicalité, qui soit à la fois pertinente et pratique d'utilisation.

2.3.1 Différentes notions de surfaces typiques

Il n'y a pas de manière d'étudier les surfaces typiques qui soit canonique, et plusieurs approches différentes ont déjà été utilisées avec succès ; quelques exemples de modèles sont présentés dans la Table 2.1, avec les références dans lesquelles elles sont introduites.

Probabiliste	Surfaces de Belyĭ aléatoires	[BM04]
	Mesure de probabilité de Weil–Petersson	[Mir13]
	Collages combinatoires de pantalons	[BCP19]
	Revêtements aléatoires	[MNP20]
Topologique ou géométrique	« Petit » complément (maigre...)	[Wol77] ou [MP08]
	« Grand » ensemble (non borné...)	[Mon15]

Table 2.1 : Présentation de différentes notions de typicalité dans la littérature.

La majeure partie de cette thèse porte sur un modèle en particulier, le modèle de *Weil–Petersson*. Son point de départ est très naturel : pour un entier $g \geq 2$, nous munissons l'espace des modules

$$\mathcal{M}_g = \{\text{surfaces hyperboliques compactes de genre } g\} / \text{isométrie}$$

d'une mesure de probabilité. Une propriété sera dite *typique* si elle se produit *avec haute probabilité*, c'est-à-dire si

$$\mathrm{Prob}(X \in \mathcal{M}_g \text{ vérifie la propriété}) \underset{g \to +\infty}{\longrightarrow} 1.$$

A priori, il existe de nombreuses mesures de probabilité sur l'espace des modules $\mathcal{M}_g$. Heureusement, un excellent candidat se démarque : la mesure de probabilité $\mathbb{P}_g^{\mathrm{WP}}$ induite par la structure symplectique de Weil–Petersson [Wei58]. Un avantage significatif de cette mesure est son expression élémentaire dans un système de coordonnées locales de l'espace des modules $\mathcal{M}_g$, appelé les *coordonnées de Fenchel–Nielsen* [Wol81]. À l'aide de cette expression, Mirzakhani a développé dans [Mir07a, Mir13] des outils puissants permettant de calculer et d'estimer certaines probabilités. Elle a ainsi mis en place les fondations d'un sujet de recherche maintenant extrêmement actif.

2.3.2 Nouvelle formulation de nos questions

Reformulons les questions précédentes dans notre nouveau cadre probabiliste.

Question 1⋆ *Que valent ℓ_1 et λ_1 typiquement ? Combien de géodésiques courtes et de petites valeur propres une surface typique possède-t-elle ?*

Question 2⋆ *Quelles sont les propriétés statistiques des fonctions de comptage dans la limite $g \to +\infty$?*

Question 3⋆ *La loi de Weyl et le théorème des géodésique primitives peuvent-ils être rendus uniformes s'il sont restreints à un ensemble de surfaces typiques ?*

Question 4⋆ *Est-il possible d'améliorer les bornes de multiplicité, typiquement ?*

L'apparition récente de plusieurs nouveaux modèles et outils a contribué à la popularité croissante de cette idée au cours des dernières années. Nous tentons de fournir une description de ce domaine en évolution rapide.

Du fait des liens profonds entre le modèle de Weil–Petersson et la géométrie des surfaces aléatoires, obtenir des informations sur la géométrie des surfaces typiques est en général plus direct, et les informations spectrales en découlent ensuite. Ainsi, nous présentons les résultats dans cet ordre. Nous indiquons les numéros des différentes questions auxquelles nous donnons des éléments de réponse dans les titres des paragraphes.

2.3.3 Résultats géométriques

Premiers résultats (1⋆) Mirzakhani a présenté une méthodologie et des outils permettant l'étude de la géométrie des surfaces aléatoires dans [Mir13], ainsi que des premières bornes sur de nombreuses quantités géométriques importantes.

- Il n'existe pas de borne inférieure typique pour la longueur de la systole :
$$\forall \varepsilon > 0, \quad \mathbb{P}_g^{\mathrm{WP}}\left(\ell_1 \geq \varepsilon\right) \underset{g \to +\infty}{\not\to} 1.$$
- La *constante de Cheeger*
$$h(X) = \inf \left\{ \frac{\ell(\partial A)}{\mathrm{Vol}_X(A)} \text{ où } X = A \sqcup B \text{ et } \mathrm{Vol}_X(A) \leq \mathrm{Vol}_X(B) \right\},$$

qui quantifie à quel point il est difficile de déconnecter la surface, est toujours plus petite que $1 + o(1)$ dans la limite de grand genre [Che75]. Typiquement, $h(X) \geq \frac{\log 2}{2\pi + \log 2} \approx 0.099$.

- Le *diamètre* d'une surface typique est inférieur à $40 \log g$, ce qui est proche de la borne inférieure $\mathrm{diam}(X) \geq \log(4g-2)$.

Mis à part le fait que la longueur de la systole ne peut être bornée inférieurement, tous les résultats prouvés dans [Mir13] semblent indiquer que les surfaces typiques de grand genre sont très « connectées » ou « compressées » – au contraire de l'exemple représenté en Figure 2.6.

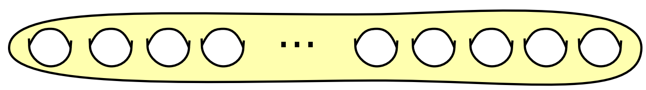


FIGURE 2.6 : Une surface atypique de grand genre : sa constante de Cheeger tend vers zéro quand $g \to +\infty$ et son diamètre croît linéairement en fonction de g.

Des résultats qualitativement similaires ont été obtenus dans le modèle des surfaces de Belyĭ aléatoires [BM04]. Le seule différence significative connue à ce jour est l'existence d'une borne inférieure pour ℓ_1.

Statistique des fonctions de comptage pour une fenêtre fixée ($2^\star$) Mirzakhani et Petri ont prouvé dans [MP19] que pour toute fenêtre $[a, b]$ fixée, la fonction de comptage $\mathrm{N}_X^\ell(a, b)$ converge vers une loi de Poisson de paramètre

$$\lambda(a, b) = \int_a^b \frac{e^t + e^{-t} - 2}{2t} \, \mathrm{d}t$$

quand le genre g tend vers l'infini. Les lois de Poisson sont utilisées pour modéliser des événements rares. Ainsi, la présence de petites géodésiques peut être vue comme un événement « rare » mais non atypique.

Dans [MP19], les auteurs ont également établi l'indépendance des fonctions de comptage sur des fenêtres disjointes, et calculé la loi limite de la systole ℓ_1.

Répartition des géodésiques « courtes » sur la surface ($1^\star$) Le chapitre 4 de cette thèse est dédié à l'amélioration de la description des géodésiques « courtes » sur les surfaces typiques : plutôt que d'étudier des fenêtres fixées comme dans [MP19], nous étudions des géodésiques de longueur proportionnelle à $\log g$.

Remarquons que $\log g$ est une longue distance sur une surface typique, comparable à son diamètre. Le choix de cette échelle est motivé par le fait que, au même titre que les valeur propres $\lambda < \frac{3}{16}$ apparaissent dans le théorème des géodésiques premières [Hub59], nous nous attendons à rencontrer les géodésiques de longueur logarithmique en démontrant des estimées spectrales.

Nous montrons que, même si, typiquement, les géodésiques fermées de longueur proportionnelle à $\log g$ existent, elles sont *rares* et peuvent être décrites assez précisément. Cette affirmation s'appuie sur deux propriétés, adaptées de la théorie des graphes : la convergence de Benjamini–Schramm (section 4.1) et l'hypothèse « tangle-free » (section 4.2). Ces deux notions ont des conséquences significatives sur le spectre des surfaces typiques, qui sont présentées dans les chapitres 5 et 6.

Pour un nombre réel $L > 0$, la partie L-fine d'une surface X est définie comme étant

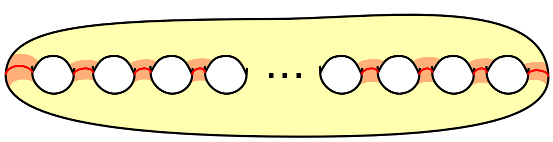
$$X^{-}(L) = \{z \in X \ : \ \mathrm{InjRad}_X(z) < L\}$$

où $\mathrm{InjRad}_X(z)$ est le rayon du plus grand disque plongé de centre z.

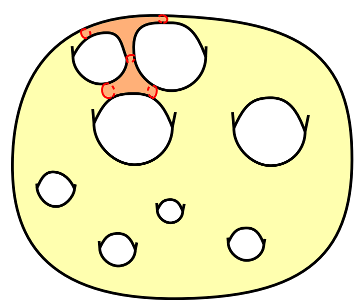
Nous prouvons le résultat suivant, qui peut être vu comme un énoncé de convergence au sens de *Benjamini–Schramm* des surfaces typiques vers le plan hyperbolique $\mathcal{H}$.

Théorème (Théorème 4.1)**.** *Pour $a < \frac{1}{3}$, l'aire de la partie $(a \log g)$-fine d'une surface typique est négligeable devant l'aire de la surface entière.*

La preuve est fondée sur le fait que la partie fine se situe autour de géodésiques courtes (voir Figure 2.7a), et une borne sur leur nombre. Il est bien connu que cette notion, adaptée de la théorie des graphes [BS01] à un cadre continu dans [ABB+11], a des applications en théorie spectrale.



(a) Une surface avec trop de géodésiques courtes, et donc une partie fine d'aire trop importante.



(b) Une surface avec un amas de géodésiques courtes, et donc des petits pantalons.

Figure 2.7 : Exemples de surfaces atypiques.

Si la convergence de Benjamini–Schramm nous indique que les géodésiques courtes constituent une petite proportion de la surface, elle ne donne pas d'information sur leur répartition. Ces géodésiques courtes peuvent-elles être regroupées toutes ensemble, ou doivent-elles être situées sur différentes parties de la surface ? Nous avons répondu à cette question dans [MT21], une collaboration avec Joe Thomas.

Inspirés par le fait que la présence d'amas de géodésiques courtes implique l'existence de petites surfaces plongées, comme représenté en Figure 2.7b, nous avons défini une notion de surface « tangle-free » (ou non emmêlées) comme suit.

Définition (Définition 4). Pour un nombre réel $L > 0$, nous disons qu'une surface est L-*tangle-free* si elle ne contient pas de pantalons ou tore à un trou de bord de longueur totale inférieure à $2L$.

Nous montrons que les surfaces typiques sont $((1-\varepsilon)\log g)$-tangle-free pour tout $\varepsilon > 0$, tandis que toutes les surfaces de grand genre g sont $(4 \log g + \mathcal{O}(1))$-tangled. Nous en déduisons des conséquences géométriques significatives, dont les résultats suivants.

Theorem (Théorème 4.9, Proposition 4.14 et Corollaire 4.15). *Pour* $L = (1-\varepsilon)\log g$, *typiquement,*

- *toutes les géodésiques fermées de longueur* $\leq L$ *sont* simples
- *deux géodésiques fermées telles que* $\ell(\gamma_1) + \ell(\gamma_2) < L$ *ne s'intersectent pas*
- *le* $\left(\frac{L-\ell}{2}\right)$*-voisinage d'une géodésique fermée de longueur* $\ell < L$ *est un cylindre plongé (beaucoup plus large que celui garanti par le lemme du collier)*
- *la topologie d'une boule de rayon* $L/8$ *est celle d'une boule ou d'un cylindre.*

Nie, Wu et Xue ont prouvé dans [NWX20] que la longueur de la systole séparante est typiquement $2 \log g$, et d'autres résultats dans le même esprit à la même échelle $\log g$.

Multiplicités ($4^\star$) Soit $\varepsilon \in (0,1)$. Nos résultats [MT21], couplés avec [MP08], permettent d'affirmer que, typiquement, il n'y a pas de multiplicités dans le spectre des longueurs avant la longueur $(1-\varepsilon)\log g$:

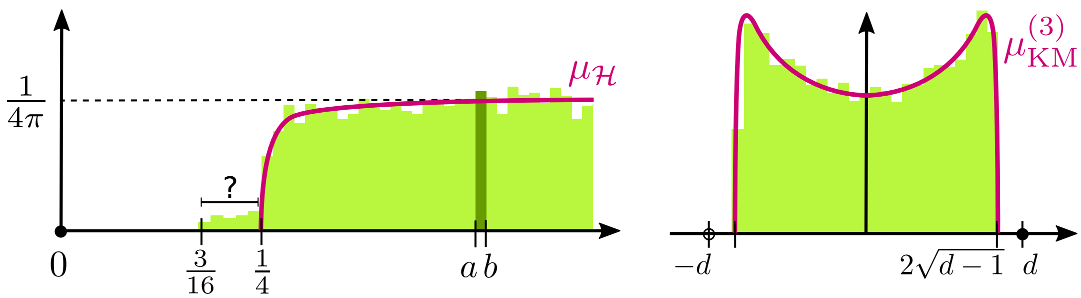
$$\forall i \neq j \text{ tel que } \ell_i, \ell_j \leq (1-\varepsilon)\log g, \quad \ell_i \neq \ell_j.$$

En particulier, les longueurs de haute multiplicité construites dans [Hor72] sont typiquement supérieures à $(1-\varepsilon)\log g$.

2.3.4 Résultats spectraux

Les nouveaux résultats publiés dans [Mon21] sont un des points forts de cette thèse : il s'agit d'un des premiers articles décrivant le spectre des surfaces hyperboliques compactes typiques. Nous avons montré que la densité $\mu_{\mathcal{H}}$ représentée figure 1.8a est typiquement une bonne approximation de la densité du spectre d'une surface de grand genre.

Deux équipes indépendantes ont récemment prouvé qu'il n'y a pas de valeur propre en dessous de $\frac{3}{16}$ [WX21, LW21]. Les informations que nous possédons à ce jour sont résumées sur la figure 2.8a. Ces résultats sont des adaptations naturelles de résultats similaires pour les graphes réguliers typiques, comme suggéré dans la figure 2.8 – nous détaillons ces liens dans la section 2.4.



(a) Surface hyperbolique compacte typique.

(b) Graphe 3-régulier typique.

FIGURE 2.8 : Illustration des informations que nous possédons sur l'histogramme du spectre pour une surface hyperbolique ou un graphe typique (après normalisation par l'aire ou le nombre de sommets).

Présentons maintenant ces résultats plus en détail, ainsi que quelques questions ouvertes.

Nombre de petites valeurs propres (1⋆) Le premier résultat spectral dans la littérature est la borne inférieure

$$\lambda_1 \geq \frac{1}{4}\left(\frac{\log 2}{2\pi + \log 2}\right)^2 \approx 0.002 \quad \text{typiquement}$$

qui est une conséquence immédiate de la borne sur la constante de Cheeger prouvée par Mirzakhani dans [Mir13], grâce à l'inégalité de Cheeger [Che70].

Nous prouvons une borne sur le nombre de petites valeurs propres dans la section 5.2.

Théorème (Théorème 5.2)**.** *Pour une surface typique X,*

$$\forall b \geq 0, \quad \frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq 32\, g^{-\frac{1}{27}\left(\frac{1}{4}-b\right)} (\log g)^{-\frac{3}{2}}. \tag{2.3}$$

La qualité de cette majoration dépend de la position relative de b et $\frac{1}{4}$.

- En prenant $b = \frac{1}{4}$, nous obtenons que le nombre de petites valeurs propres d'une surface typique est un $\mathcal{O}\left(g(\log g)^{-\frac{3}{2}}\right)$, ce qui est une amélioration de la borne optimale $2g-2$ de [OR09] par une correction logarithmique.

- Si $b < \frac{1}{4}$, nous obtenons une correction additionnelle par une puissance de g négative, qui s'améliore quand b s'éloigne de l'intervalle $[\frac{1}{4}, +\infty)$.
- Quand $b > \frac{1}{4}$, la borne est toujours vraie, mais moins intéressante.

Rappelons qu'aucun de ces résultats n'est vérifié lorsque l'on pince une décomposition en pantalons : dans ce cas, pour tout $\varepsilon > 0$, le nombre de valeurs propres dans $[0, \varepsilon]$ atteint $2g-2$, et le nombre de valeurs propres dans $[\frac{1}{4}, \frac{1}{4}+\varepsilon]$ tend vers l'infini. Ces surfaces extrêmement pincées sont donc, en effet, atypiques pour le modèle de Weil–Petersson, et les exclure de l'étude permet d'améliorer les résultats connus.

Fonctions de comptage : borne supérieure et équivalent ($2^\star$)

Dans [Mon21], nous étudions la fonction de comptage $\mathrm{N}_X^\Delta(a,b)$ à l'aide de la convergence de Benjamini–Schramm et de la formule des traces de Selberg. Nous prouvons les résultats suivants, dont les énoncés précis et preuves se situent dans la section 5.3.

Théorème (Théorèmes 5.8 et 5.9)**.** *Pour une surface typique X,*

- *pour tout $0 \leq a \leq b$,*

$$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(b - a + \sqrt{\frac{b+1}{\log g}}\right). \tag{2.4}$$

- *pour tout $0 \leq a \leq b$ tels que $b - a \gg \sqrt{\frac{b+1}{\log g}}$,*

$$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} \sim \mu_{\mathcal{H}}(a,b) := \frac{1}{4\pi}\int_{\frac{1}{4}}^{+\infty} \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathbb{1}_{[a,b]}(\lambda)\,\mathrm{d}\lambda \tag{2.5}$$

  quand b et/ou g tend vers l'infini.

Toutes les constantes sont des constantes numériques, indépendantes de X, g, a et b.

La quantité $\mu_{\mathcal{H}}(a,b)$ est la densité spectrale du plan hyperbolique, représentée en Figure 2.8a. Elle est supportée sur $[\frac{1}{4}, +\infty)$.

En prenant une fenêtre fixée $[a,b]$, nos résultats donnent une première réponse à la question $2^\star$. L'étape suivante consisterait à étudier les corrélations entre les fonctions de comptage de deux fenêtres disjointes, ou améliorer les vitesses de convergence des bornes.

Applications Les résultats prouvés dans [Mon21] ont deux corollaires importants.

- Comme $\mu_{\mathcal{H}}(0,b)$ est équivalente à $\frac{b}{4\pi}$ quand b tend vers l'infini, nous pouvons déduire de l'équation (2.5) une loi de Weyl uniforme

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} = \frac{b}{4\pi} + \mathcal{O}_g\left(\sqrt{b \log b}\right),$$

et ainsi répondre à la question $3^\star$.

- La borne supérieure (2.4), appliquée à une fenêtre qui rétrécit autour d'une valeur propre, implique une borne sur la multiplicité de toute valeur propre λ en fonction de λ et g. En utilisant l'équation (2.5) pour estimer la taille typique de λ_j pour tout j, nous prouvons une amélioration de la borne déterministe linéaire prouvée dans [Bes80] (Question $4^\star$) dans la section 5.4.

Théorème (Corollaires 5.28 et 5.29)**.** *Pour une surface typique X,*

$$\forall j \geq 1, \quad m_X(\lambda_j) = \mathcal{O}\left(g\sqrt{\frac{1+\frac{j}{g}}{\log g}}\right).$$

Ces estimées ont également été utilisées dans [LMS20] afin de démontrer un résultat d'ergodicité quantique pour les surfaces typiques de grand genre.

Trou spectral ($1^\star$) Est-il possible d'améliorer la borne inférieure de λ_1 prouvée par Mirzakhani ? L'un des résultats les plus attendus dans le domaine de la théorie spectrale des surfaces hyperboliques compactes est une preuve de la conjecture suivante, formulée dans [Wri20].

Conjecture. *Pour tout $\varepsilon > 0$,*

$$\lim_{g \to +\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{1}{4} - \varepsilon\right) = 1.$$

D'après [Che75], la valeur $\frac{1}{4}$ est le plus grand nombre pour lequel cette affirmation peut être vérifiée. Les résultats de Mondal [Mon15], en genre 2, sont encourageants. Il est démontré que l'ensemble des surfaces de genre 2 telles que $\lambda_1 \geq \frac{1}{4}$ est non borné et déconnecte l'espace des modules $\mathcal{M}_2$. En particulier, de nombreuses surfaces de genre 2 vérifient la conjecture. Le résultat impliquerait l'existence de surfaces de genre arbitrairement grand telles que $\lambda_1 \geq \frac{1}{4} - \varepsilon$, répondant ainsi positivement à la conjecture de Buser–Burger–Dodziuk [BBD88].

L'introduction de la notion de surface tangle-free dans [MT21] est motivée par nos recherches sur cette conjecture, en collaboration avec Nalini Anantharaman. Nous présentons quelques éléments autour de cette conjecture dans le chapitre 6.

Une version plus faible de la conjecture, avec $\frac{3}{16}$ au lieu de $\frac{1}{4}$, a été obtenue très récemment par deux équipes indépendantes [LW21, WX21]. La même borne a été établie dans le modèle des revêtements aléatoires [MNP20]. Même s'il existe quelques différences significatives entre la méthode que nous suggérons et celles qui ont été utilisées (nous en décrirons certaines dans le chapitre 6), beaucoup d'éléments sont similaires. Notamment, des définitions et résultats de [MT21] ont été utilisés indépendamment dans [LW21].

La valeur $\frac{3}{16}$ apparaît parce que les calculs dans les deux articles sont effectués à une précision $1/g$. Afin d'obtenir la conjecture pour $\frac{1}{4}$, nous savons qu'il est nécessaire d'être capable de faire des calculs à une précision arbitrairement grande, i.e. à des erreurs de taille $1/g^N$ près, pour $N \gg 1$.

Afin de rendre cela possible, nous avons :

- calculé un développement asymptotique en grand genre des volumes de Weil–Petersson qui apparaissent naturellement dans les calculs de probabilité dans le modèle de Weil–Petersson (**Théorèmes** 3.17 et 3.18)
- généralisé l'hypothèse tangle-free, ajoutant un paramètre de sorte que la probabilité de ne pas être tangle-free puisse être prise comme étant $\mathcal{O}\left(1/g^N\right)$ pour un N arbitrairement grand (**Théorèmes** 4.19 et 4.20).

2.4 Comparaison avec les graphes réguliers

Pour conclure cette introduction, nous exposons succinctement la correspondance entre les graphes réguliers et les surfaces hyperboliques compactes, qui est la motivation au cœur de beaucoup des résultats présentés dans cette thèse.

Soit d un entier supérieur à 3. Dans la suite, nous appelons *graphe d-régulier* tout graphe tel que chaque sommet a exactement d voisins. Soit n le nombre de sommets d'un tel graphe.

Les graphes réguliers et les surfaces hyperboliques compactes sont des objets aux ressemblances surprenantes, d'un point de vue géométrique et spectral. Voici deux faits qui contribuent à ces similarités :

- le voisinage de tous les points sur ces objets est le même (un petit disque hyperbolique ou exactement d sommets)
- la taille (aire ou nombre de sommets) d'une boule croît exponentiellement avec son rayon.

	surface hyp compacte	**graphe d-régulier**
limite considérée :	genre $g \to +\infty$	nombre de sommets $n \to +\infty$
modèle probabiliste :	loi de Weil–Petersson $\mathbb{P}_g^{\mathrm{WP}}$	loi uniforme $\mathbb{P}_n^{(d)}$
spectre inclus dans :	$[0, +\infty)$	$[-d, d]$
valeurs propres triviales :	0	d, et parfois $-d$ ($\Leftrightarrow$ bipartite)
revêtement universel :	plan hyperbolique $\mathcal{H}$	arbre infini d-régulier
et son spectre :	$[\frac{1}{4}, +\infty)$	$[-2\sqrt{d-1}, 2\sqrt{d-1}]$
limite typique de de la densité spectrale :	densité spectrale de $\mathcal{H}$ [Mon21]	loi de Kesten–McKay [Kes59, McK81]
trou spectral :	λ_1	$d - \lambda_+$ où $\lambda_+ := \max\{\lvert\lambda_i\rvert : \lvert\lambda_i\rvert < d\}$
borne déterministe :	$\lambda_1 \leq \frac{1}{4} + o(1)$ [Che75]	$\lambda^+ \geq 2\sqrt{d-1} - o(1)$ [Nil91]
borne probabiliste :	$\lambda_1 \geq \frac{3}{16} - \varepsilon$ [WX21, LW21]	$\lambda^+ \leq 2\sqrt{d-1} + o(1)$ [Fri03]

Table 2.2 : Comparaison entre les propriétés des valeurs propres $(\lambda_j)_{j\geq 0}$ du laplacien Δ sur une surface hyperbolique compacte et des valeurs propres $\lambda_1 \geq \ldots \geq \lambda_n$ de la matrice d'adjacence $A = dI_n - \Delta$ d'un graphe d-régulier.

Toutes les définitions et résultats présentés dans cette thèse ont des homologues dans le monde des graphes réguliers, qui sont énumérées dans la Table 2.2. Les résultats sont souvent (voire toujours) démontrés bien avant pour les graphes, d'une part parce qu'ils sont plus simples à étudier (car ce sont des modèles finis), mais aussi car ce sont des objets fondamentaux qui apparaissent dans d'innombrables domaines des mathématiques et des sciences.

2.5 Organisation de cette thèse

L'organisation des différents chapitres de cette thèses et la manière dont ils dépendent les uns des autres sont présentés dans la figure 2.9. Nous avons mis en valeur les nouveaux résultats en les encadrant avec une ligne continue.

Voici un résumé court des différents chapitres.

- Dans le chapitre 3, nous introduisons le modèle probabiliste de Weil–Petersson. Le discours se veut assez informel, mais nous indiquons des références précises, et motivons les idées et outils essentiels dans ce domaine. La dernière section de ce chapitre contient un nouveau développement asymptotique, annoncé dans [AM20].
- Les chapitres 4 et 5 constituent le cœur de cette thèse. La plupart des nouveaux résultats ont été extraits de [Mon21, MT21] et significativement réorganisés :

les résultats géométriques dans le chapitre 4 et les résultats spectraux dans le chapitre 5.

- Le chapitre 6 présente des avancées récentes qui pourraient permettre de prouver que les surfaces typiques ont un trou spectral optimal.
- Dans l'annexe A, nous généralisons certains des résultats principaux de cette thèse à un autre modèle probabiliste, le modèle des surfaces de Belyĭ aléatoires introduit par Brooks et Makover [BM04].

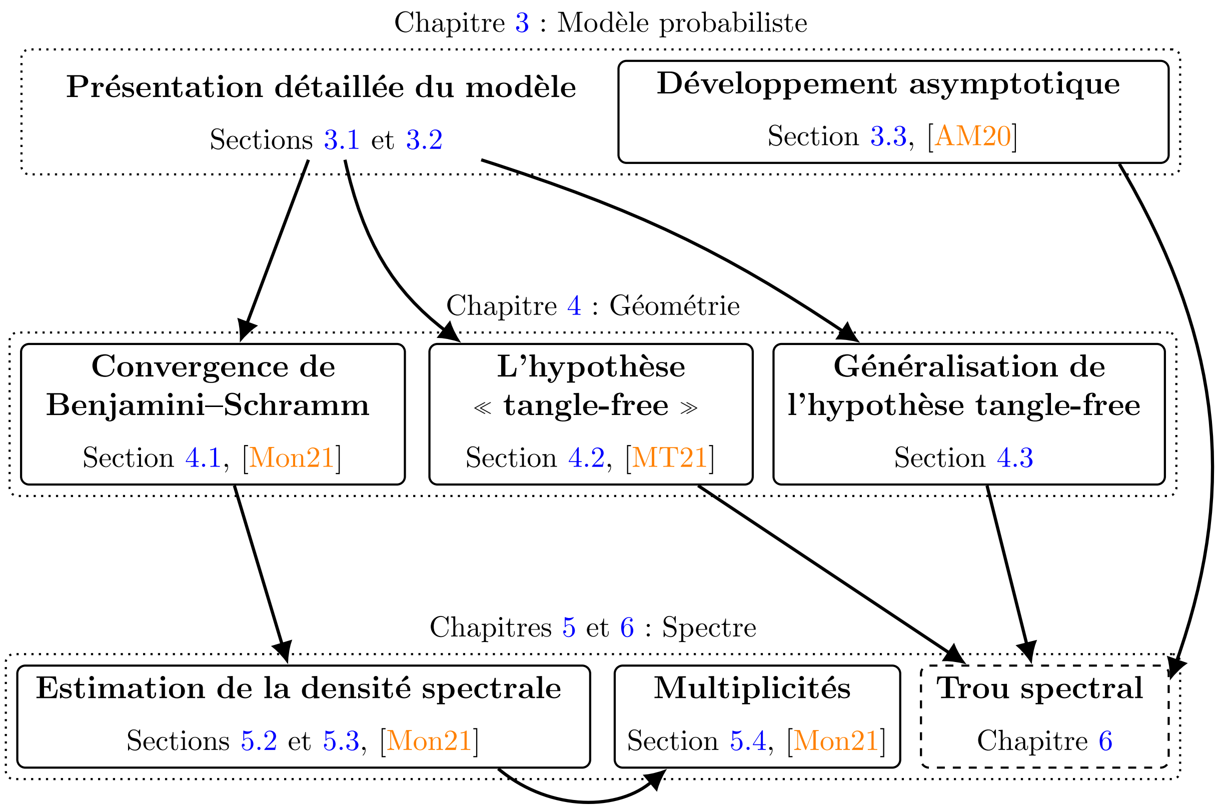


FIGURE 2.9 : Organisation des chapitres de cette thèse.

Chapter 3

The Weil-Petersson probabilistic model

The aim of this chapter is to introduce all the definitions and tools that are necessary to sample *random surfaces* in the Weil–Petersson model. This is done in two steps:

- we define and describe the sample space, the 'set of compact hyperbolic surfaces', and equip it with a probability measure in Section 3.1;
- we present various tools used to study random surfaces in Section 3.2.

The last section presents new asymptotic expansions for volume polynomials and can be omitted at first read; it will only be used in Chapter 6.

3.1 Description of the set of compact hyperbolic surfaces

In order to define a notion of random compact hyperbolic surface, let us introduce a *sample space*, the 'set of compact hyperbolic surfaces' or *moduli space*, and equip it with local coordinates, the *Fenchel–Nielsen coordinates*. We will explain how the standard volume form in these coordinates can be renormalised to equip the set of surfaces with a natural *probability measure*.

3.1.1 Prerequisite: hyperbolic pairs of pants

A key character that will appear in many instances in this manuscript is the *hyperbolic pair of pants*. A pair of pants is a surface of genus 0 with 3 boundary components, or a three-holed sphere. Pairs of pants are very important in hyperbolic geometry because they satisfy the following property.

Theorem 3.1 ([Bus92, Theorem 3.1.7]). *For any lengths x_1, x_2, x_3, there is a unique hyperbolic pair of pants with boundary geodesics γ_1, γ_2, γ_3 such that*

$$\forall i \in \{1, 2, 3\}, \ell(\gamma_i) = x_i.$$

Any compact hyperbolic surface can be cut into hyperbolic pairs of pants. In order to do so, we pick a simple closed geodesic on the surface and cut along it. We obtain one or two hyperbolic surfaces with boundary geodesics, depending on the curve we started with. We repeat this operation, cutting along more and more simple closed geodesic, as represented in Figure 3.1. The process stops when the surface has been entirely cut into pairs of pants, because they do not contain any simple closed geodesic.

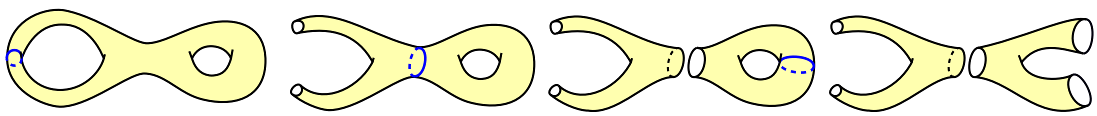


Figure 3.1: Cutting a genus 2 surface along 3 curves to obtain 2 pairs of pants.

The Euler characteristic of a genus g surface is $-2g + 2$ and that of a pair of pants is -1. By additivity of the Euler characteristic, there is always $2g - 2$ pairs of pants in a decomposition, delimited by $3g - 3$ simple closed geodesics.

Note that pairs of pants are not compact hyperbolic surfaces but rather *bordered hyperbolic surfaces* (see Section 3.1.2.3). Bordered hyperbolic surfaces will appear throughout this thesis for the same reason that they appear here: as byproducts of cutting compact hyperbolic surfaces along closed geodesics.

3.1.2 The set of hyperbolic surfaces

3.1.2.1 The moduli space $\mathcal{M}_g$: definition and first observations

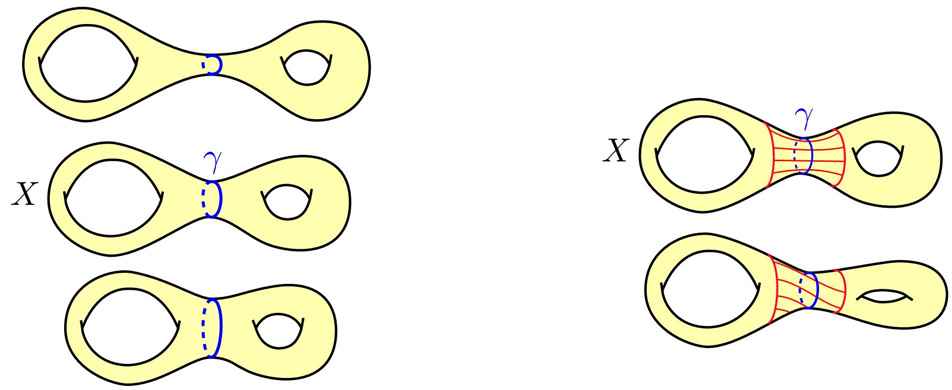
For a genus $g \geq 2$, let

$$\mathcal{M}_g := \{\text{compact hyperbolic surfaces of genus } g\} \big/ \{\text{isometries}\}$$

denote the *moduli space* of genus g surfaces. Throughout this thesis, when taking random genus g surfaces, the set $\mathcal{M}_g$ will be the sample space.

What does the set $\mathcal{M}_g$ look like? First, it is not not empty, because there exists compact hyperbolic surfaces of any genus $g \geq 2$. Let us take one element $X \in \mathcal{M}_g$ and explore its neighbourhood in $\mathcal{M}_g$. There are two simple ways to move around the point $X \in \mathcal{M}_g$ continuously, represented in Figure 3.2.

- It is possible to pinch or expand a simple closed geodesic.
- It is possible to twist the surface along a simple closed geodesic.



(a) Change of the length of γ. (b) Twist along γ.

Figure 3.2: Two deformations of a compact hyperbolic surface X.

This seems to indicate that the moduli space is a continuous object, like a manifold.

If so, what is its dimension? If we admit that the operations described in Figure 3.2 generate all of the continuous deformations of X, then this question becomes: how many of these operations can we do independently? Thanks to the fundamental property of pairs of pants, Theorem 3.1, we can perform these two operations on all of the $3g-3$ curves cutting a surface into a family of pairs of pants. This is the reason why the moduli space is an object of dimension $2 \times (3g-3) = 6g-6$. We will make this intuition precise in Section 3.1.3, when we introduce a set of parameters describing the moduli space.

Unfortunately, the moduli space $\mathcal{M}_g$ is not a smooth manifold but rather an *orbifold* (a generalisation of manifolds which can contain certain types of singularities). These singularities can be explained by the fact that some compact hyperbolic surfaces have exceptional isometries, and they correspond to singular points, as shown in Figure 3.3.

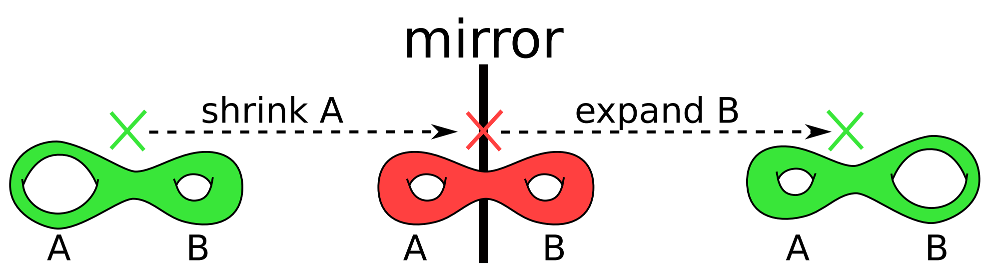


Figure 3.3: A path on the moduli space $\mathcal{M}_2$. The left and right surfaces are isometric and therefore correspond to the same point. The middle surface is a singular point, due to its exceptional symmetry, linked to the equality of the lengths on the sides A and B. Surfaces satisfying this equality are a sort of 'mirror' in the moduli space $\mathcal{M}_2$.

3.1.2.2 The universal cover of $\mathcal{M}_g$

To understand the quotient by isometries better, we will benefit from working in the much nicer universal cover of $\mathcal{M}_g$, the *Teichmüller space* $\mathcal{T}_g$. In this set, the two isometric surfaces from Figure 3.3 will be two different points, hence removing the 'mirror'-singularity that we observed.

In order to define $\mathcal{T}_g$, let us set S_g to be a *fixed* genus g topological surface called the *base surface*. The Teichmüller space is a set of *marked* surfaces (X, ϕ): the *marking* ϕ tells us how to equip the fixed base surface S_g with the hyperbolic structure X. More precisely,

$$\mathcal{T}_g = \left\{ (X, \phi) \ : \ \begin{array}{l} X \text{ compact hyperbolic surface of genus } g \\ \phi : S_g \to X \text{ homeomorphism} \end{array} \right\} \diagup \sim$$

where the equivalence relation $\sim$ is defined by:

$$(X, \phi) \sim (Y, \psi) \Leftrightarrow \exists \text{ isometry } h : X \to Y \text{ s.t. } \psi^{-1} \circ h \circ \phi \text{ is isotopic to } \mathrm{id}_{S_g}.$$

Figure 3.4 illustrates the difference between the moduli space and Teichmüller space: for one fixed compact hyperbolic surface X of genus g, corresponding to one point on the moduli space $\mathcal{M}_g$, there are many distinct markings $\phi : S_g \to X$, each leading to a different point (X, ϕ) in the Teichmüller space $\mathcal{T}_g$.

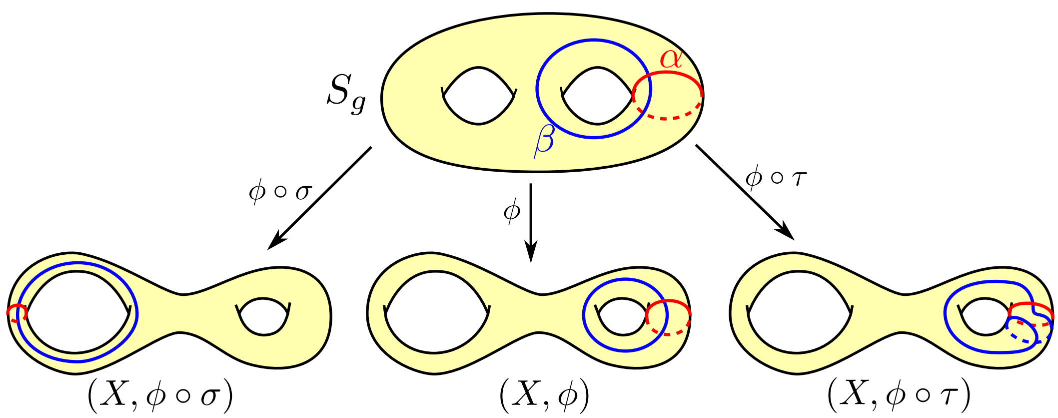


Figure 3.4: Three different markings for one compact hyperbolic surface of genus 2. They are obtained by precomposition of one marking ϕ by two homeomorphisms of S_g: a positive involution σ switching the two 'holes' of S_g and a *Dehn twist* τ around the curve α (see [FM12, Chapter 3]).

The moduli space $\mathcal{M}_g$ can be obtained from the Teichmüller space by 'forgetting the marking'. More precisely, the *mapping class group* of S_g

$$\mathrm{MCG}_g := \{\text{homeomorphisms } S_g \to S_g\} \diagup \text{isotopy}$$

acts on the Teichmüller space $\mathcal{T}_g$ by precomposition of the marking:

$$\forall h \in \mathrm{MCG}_g, \forall (X, \phi) \in \mathcal{T}_g, \quad h \cdot (X, \phi) := (X, \phi \circ h).$$

Then, the moduli space is the quotient of the Teichmüller space for this action:

$$\mathcal{M}_g = \mathcal{T}_g \diagup \mathrm{MCG}_g.$$

We observe that the examples of Figure 3.4 have been constructed by exploring an orbit of the action of MCG_g on $\mathcal{T}_g$: starting with an element (X, ϕ), we precomposed the marking ϕ by elements $\sigma, \tau \in \mathrm{MCG}_g$.

3.1.2.3 Extension to bordered hyperbolic surfaces

Let us now extend the previous definitions to *bordered* hyperbolic surfaces, i.e. hyperbolic surfaces of *finite type* with *geodesic boundary*. Note that, by finite type, we mean that the fundamental group is finitely generated.

The topology of such a surface depends on its genus g, its number of cusps c and its number of boundary geodesics b. Topologically, cusps and boundary components play the same role, and we will refer to cusps as boundary geodesics of length 0. Then, the topology of a surface is entirely determined by its *signature* (g, n), with $n = c + b$ the number of boundary components (geodesics or cusps).

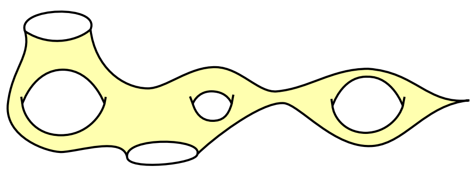


Figure 3.5: A bordered hyperbolic surface of genus 3 with one cusp and two geodesic boundary components – its signature is $(3, 3)$.

Why talk about bordered surfaces when our main focus is *compact* surfaces? This will actually be absolutely necessary: in order to describe random compact surfaces, we will need to *cut them into smaller surfaces*, hence obtaining pieces with boundary components. This idea is central in several of Mirzakhani's breakthroughs in the theory of random surfaces, such as Mirzakhani's integration formula (Theorem 3.7) and topological recursion (see Section 3.2.2).

Let $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 0$, which implies that bordered hyperbolic surfaces of signature (g, n) exist. For a length vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, we define the *moduli space* $\mathcal{M}_{g,n}(\mathbf{x})$ to be the set of bordered hyperbolic surfaces of genus g with n geodesic boundary components $\beta_1, \ldots, \beta_n$ such that $\forall i, \ell(\beta_i) = x_i$, quotiented by the isometries preserving the set β_i set-wise for all i.

Example 1. *For $(g, n) = (0, 3)$, the elements of $\mathcal{M}_{0,3}(\mathbf{x})$ are pairs of pants. By Theorem 3.1, there exists exactly one pair of pants with prescribed boundary lengths x_1, x_2, x_3. As a consequence, the moduli space $\mathcal{M}_{0,3}(\mathbf{x})$ is reduced to exactly one element.*

Inspired by the compact case, we set $S_{g,n}$ to be a fixed topological surface of signature (g, n) with n boundary components $\beta_1, \ldots, \beta_n$. Then, the moduli space can be seen as the quotient

$$\mathcal{M}_{g,n}(\mathbf{x}) = \mathcal{T}_{g,n}(\mathbf{x}) \diagup \mathrm{MCG}_{g,n}$$

where $\mathcal{T}_{g,n}(\mathbf{x})$ is the *Teichmüller space* defined by

$$\mathcal{T}_{g,n}(\mathbf{x}) := \left\{ (X,\phi) : \begin{array}{l} X \text{ bordered hyperbolic surface of signature } (g,n) \\ \phi : S_{g,n} \to X \text{ homeomorphism} \\ \forall i, \phi(\beta_i) \text{ is a boundary geodesic of length } x_i \end{array} \right\} \diagup \sim,$$

the equivalence relation $\sim$ is the one defined in the compact case (replacing S_g by $S_{g,n}$), and $\mathrm{MCG}_{g,n}$ the *mapping class group* of $S_{g,n}$

$$\mathrm{MCG}_{g,n} := \left\{ \begin{array}{l} \text{homeomorphisms } h : S_{g,n} \to S_{g,n} \\ \text{such that } h(\beta_i) = \beta_i \text{ for all } i \end{array} \right\} \diagup \text{isotopy}$$

acting on $\mathcal{T}_{g,n}(\mathbf{x})$ by precomposition of the marking.

Whenever $\mathbf{x} = (0, \dots, 0)$, and therefore all the boundary components of the surfaces are cusps, in order to simplify notation we set

$$\mathcal{M}_{g,n} := \mathcal{M}_{g,n}(0, \dots, 0) \qquad \text{and} \qquad \mathcal{T}_{g,n} := \mathcal{T}_{g,n}(0, \dots, 0).$$

We observe that the definitions are compatible with the compact case, so that $\mathcal{T}_g$ and $\mathcal{M}_g$ correspond to $\mathcal{T}_{g,0}(\mathbf{x})$ and $\mathcal{M}_{g,0}(\mathbf{x})$ respectively, where $\mathbf{x} \in \mathbb{R}^0_{\geq 0}$ is the 'empty vector'.

3.1.3 Fenchel–Nielsen coordinates

Let $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 0$ and $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ be a length vector. Let us describe the Teichmüller space $\mathcal{T}_{g,n}(\mathbf{x})$ by introducing some global coordinates, *Fenchel–Nielsen coordinates*. We only present an intuition for the construction of these coordinates. A detailed construction can be found in [Bus92, Sections 3.6, 6.2 and 6.3], and precise references for the main technical steps are provided in the following.

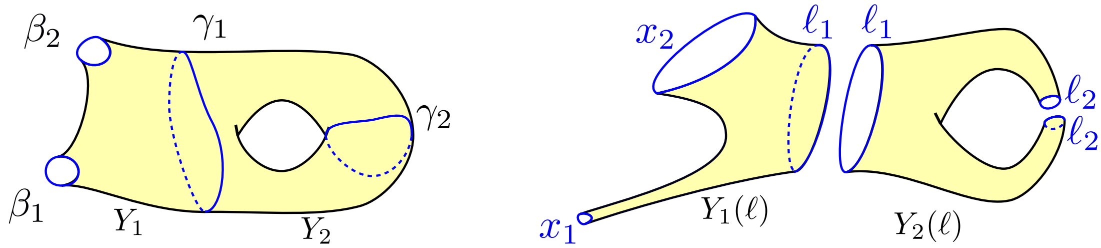
The idea is to find a bijective map of the form

$$\begin{aligned} \mathbb{R}^{\cdots} &\to \mathcal{T}_{g,n}(\mathbf{x}) \\ \text{real coordinates} &\mapsto (X, \phi) \bmod \sim . \end{aligned}$$

In other words, we want to be able to describe entirely a marked hyperbolic surface with a set of real numbers. This will be achieved thanks to Theorem 3.2.

3.1.3.1 How to build a bordered hyperbolic surface

We first focus on describing any bordered hyperbolic surface X using a set of parameters. This will be achieved by building the surface starting with some small building blocks. We want to be able to describe each building block *and* the way to assemble them, with a set of real coordinates. For this reason, pairs of pants are a great set of



(a) Pair of pants decomposition $S_{1,2}$. (b) The hyperbolic pairs of pants $(Y_j(\ell))_j$.

Figure 3.6: Cutting into building blocks and defining each of them individually.

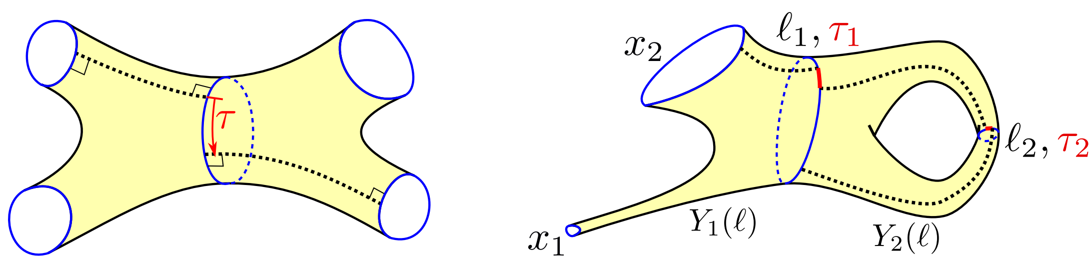
building blocks: by Theorem 3.1, their geometry is entirely described by the list of the lengths of their boundary components.

As a consequence, the first step of the construction is to cut the base surface $S_{g,n}$ into pairs of pants, as represented on Figure 3.6a. More precisely, we pick a family $\Gamma = (\gamma_1, \ldots, \gamma_N)$ of disjoint simple closed curves such that the (disconnected) surface $S_{g,n} \setminus \Gamma$ obtained by cutting $S_{g,n}$ along the curves of Γ is a family of pairs of pants $(Y_j)_j$.

The Euler characteristic of a surface of signature (g, n) is $\chi_{g,n} = -2g + 2 - n$. In particular, the Euler characteristic of a pair of pants is $\chi_{0,3} = -1$. By additivity, the number of pairs of pants in the decomposition is exactly $2g - 2 + n$. As a consequence, the pairs of pants are delimited by $6g - 6 + 3n = 2N + n$ boundary components, and therefore, $N = 3g - 3 + n$.

For every j, the topological pair of pant Y_j on $S_{g,n}$ is delimited by 3 closed curves, which are either boundary components of $S_{g,n}$ or curves of Γ. In order to equip the pairs of pants with a hyperbolic metric, we need to pick a length for each of these boundary curves, like in Figure 3.6b. We take a length vector $\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{R}_{>0}^N$ and define a family of hyperbolic pairs of pants $(Y_j(\ell))_j$ such that:

- the lengths of the boundary component corresponding to $\beta_i \subset \partial S_{g,n}$ is x_i
- the lengths of the two boundary components corresponding to $\gamma_i \in \Gamma$ is ℓ_i.



(a) The twist angle when gluing two pants. (b) The hyperbolic surface $X(\ell, \tau)$.

Figure 3.7: Gluing the building blocks to obtain a surface in $\mathcal{M}_{g,n}(\mathbf{x})$.

We now wish to glue the pairs of pants $(Y_j(\ell))_j$ to form a bordered hyperbolic surface. When gluing hyperbolic pairs of pants along two boundary components, there is a degree of freedom, the *twist angle* $\tau \in \mathbb{R}$. It corresponds to the distance between the bases of the common perpendiculars on the two pairs of pants, as represented on Figure 3.7a (see [Bus92, Section 3.3] for more details).

We pick a family of twist angles $\tau = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N$. For all $i \in \{1, \ldots, N\}$, we glue the two boundary components associated to γ_i with the twist angle τ_i. This defines a bordered hyperbolic surface $X(\ell, \tau)$, represented on Figure 3.7b.

The fact that we choose the twist angle τ_i as an element of $\mathbb{R}$ rather than the circle of length ℓ_i might surprise the reader. Indeed, replacing τ_i by $\tau_i + \ell_i$ does not change anything to Figure 3.7a. The difference between these two points will appear in the next step of the construction, when picking the marking: the two markings will differ by precomposition by a Dehn-twist along γ_i, as in Figure 3.4.

Another way to make sense of this is to remember that the Teichmüller space $\mathcal{T}_{g,n}(\mathbf{x})$ is the universal cover of the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$. While the twist parameters live in the torus $\mathbb{R}/\ell_1\mathbb{Z} \times \ldots \times \mathbb{R}/\ell_N\mathbb{Z}$ when describing the moduli space, they live in its universal covering $\mathbb{R}^N$ for the Teichmüller space.

3.1.3.2 Definition of the marking

So far, we have constructed a map which associates to a set of Fenchel–Nielsen coordinates (ℓ, τ) a bordered hyperbolic surface $X(\ell, \tau)$. This map is a surjection onto the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$ (see [Bus92, Theorem 3.6.4] when $n = 0$). A lot of these surfaces are isometric: for instance, a Dehn twist around a component γ_i transforms $X(\ell, \tau)$ into the isometric surface $X(\ell, (\tau_1, \ldots, \tau_i + \ell_i, \ldots, \tau_N))$. But as we hinted in Figure 3.4, these surfaces will be distinguished in the Teichmüller space by their *markings*

$$\phi(\ell, \tau) : S_{g,n} \to X(\ell, \tau)$$

which we now introduce.

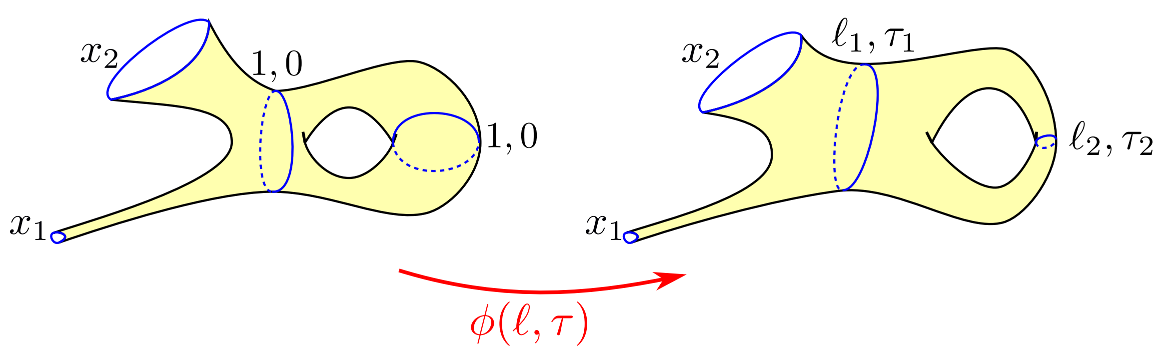


Figure 3.8: The marking $\phi(\ell, \tau) : S_{g,n} \to X(\ell, \tau)$ can be constructed by equipping $S_{g,n}$ with a hyperbolic metric, and seeing $X(\ell, \tau)$ as a deformation of $S_{g,n}$.

Let us identify the base surface $S_{g,n}$ with the bordered hyperbolic surface $X(\mathbf{1}, \mathbf{0})$, with all lengths parameters equal to one and all twist angles equal to zero (see Figure 3.8). Then, we define $\phi(\ell, \tau)$ as the composition $\mathrm{tw}_{\ell,\tau} \circ \mathrm{str}_\ell$, where[1]

- $\mathrm{str}_\ell : X(\mathbf{1}, \mathbf{0}) \to X(\ell, \mathbf{0})$ 'stretches' each curve γ_i so that its length goes from 1 to ℓ_i, while keeping all the twist angles to zero;
- $\mathrm{tw}_{\ell,\tau} : X(\ell, \mathbf{0}) \to X(\ell, \tau)$ 'twists' the surface in a collar neighbourhood of each γ_i so that the twist angle goes from 0 to τ_i.

This concludes the construction of the parametrisation of the Teichmüller space, thanks to the following theorem.

Theorem 3.2 ([Bus92, Theorem 6.2.7])**.** *For any $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 0$, any fixed pair of pants decomposition Γ of the base surface $S_{g,n}$, and any length vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, the map*

$$\begin{aligned} \mathbb{R}^{3g-3+n}_{>0} \times \mathbb{R}^{3g-3+n} &\to \mathcal{T}_{g,n}(\mathbf{x}) \\ (\ell, \tau) &\mapsto [(X(\ell, \tau), \phi(\ell, \tau))]_\sim \end{aligned}$$

is a bijection. The parameters (ℓ, τ) are called Fenchel–Nielsen coordinates*.*

This result allows us (with some additional work!) to define a topology, and even a real analytic structure on the Teichmüller spaces (see [Bus92, Section 6.3]).

3.1.4 The Weil–Petersson symplectic structure

Now that we have introduced and described the set of surfaces, we now proceed to the last step of the construction of a probabilistic setting: the choice of a measure. In order to do so, we add more structure to the Teichmüller and moduli spaces, by equipping them with a symplectic form.

3.1.4.1 Symplectic forms

We recall that a *symplectic form* ω on a manifold[2] M of dimension $2N$ is a *closed* and *non-degenerate* 2-form. The canonical symplectic form on the Euclidean space $\mathbb{R}^N \times \mathbb{R}^N$ of coordinates $(x_1, \dots, x_N, y_1, \dots, y_N)$ is

$$\omega = \sum_{i=1}^{N} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

A symplectic form induces a *volume form* on the manifold M, defined by

$$\mathrm{Vol}_\omega = \frac{\omega^{\wedge N}}{N!}.$$

On the Euclidean space with the canonical symplectic form, this volume form corresponds to the Lebesgue measure $\mathrm{d}x_1 \dots \mathrm{d}x_N \, \mathrm{d}y_1 \dots \mathrm{d}y_N$.

[1] The construction of these maps can be found in [Bus92, Definition 3.2.4 and eq (6.2.3)].

[2] Let us recall that, actually, the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$ is not a manifold but an orbifold, but this is not an issue and symplectic forms can be defined the same way on orbifolds.

3.1.4.2 The Weil–Peterson symplectic form

Throughout the rest of this section, we set $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 0$ and $\mathbf{x}$ a length vector in $\mathbb{R}^n_{\geq 0}$. Let $N := 3g - 3 + n$ denote the dimension of the Teichmüller space $\mathcal{T}_{g,n}(\mathbf{x})$. Our aim is to equip it with a 'good' symplectic form, and the associated volume form.

Theorem 3.3 ([Wei58, Wol81]). *There exists a symplectic form $\omega^{\mathrm{WP}}_{g,n,\mathbf{x}}$ on $\mathcal{T}_{g,n}(\mathbf{x})$, called the* Weil–Petersson form*, which:*

- *can be written in any set of Fenchel–Nielsen coordinates $(\ell, \tau) \in \mathbb{R}^N_{>0} \times \mathbb{R}^N$ as*

$$\omega^{\mathrm{WP}}_{g,n,\mathbf{x}} = \sum_{i=1}^{N} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i \tag{3.1}$$

- *is invariant by the action of the mapping class group $\mathrm{MCG}_{g,n}$ and therefore descends to the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$.*

There is no trivial reason to believe in the existence of such a symplectic form. Indeed, we can easily construct one symplectic form by picking a pair of pants decomposition of the base surface and using equation (3.1) as a definition. However, there is no reason to expect that this form would be independent of the choice of the pair of pants decomposition or invariant by the action of $\mathrm{MCG}_{g,n}$.

Weil equipped Teichmüller spaces with a symplectic form in [Wei58], but he did so using a different viewpoint. Thanks to the uniformisation theorem [Gra94], there is a correspondence between hyperbolic surfaces and Riemann surfaces. When we view $\mathcal{T}_{g,n}(\mathbf{x})$ as a set of Riemann surfaces, it is possible to describe its cotangent space in an intrinsic way and to define a symplectic form on it (using the Petersson inner product). The advantage of this very natural definition is that it automatically grants the invariance of the form by the action of $\mathrm{MCG}_{g,n}$. But since it is expressed in a completely different setting, its relation to Fenchel–Nielsen coordinates is unclear.

In [Wol81], Wolpert proved that the form defined by Weil can be expressed in any set Fenchel–Nielsen coordinates as is it in equation (3.1). This result is often called Wolpert's magic formula and is the fundamental tool on which all the theory of Weil–Petersson random surfaces is built.

3.1.4.3 Weil–Petersson volumes

As explained at the beginning of his section, the symplectic form $\omega^{\mathrm{WP}}_{g,n,x}$ induces a volume form $\mathrm{Vol}^{\mathrm{WP}}_{g,n,\mathbf{x}}$ on the Teichmüller and moduli spaces. In any set of Fenchel–Nielsen coordinates $(\ell, \tau) \in \mathbb{R}^N_{>0} \times \mathbb{R}^N$,

$$\mathrm{Vol}^{\mathrm{WP}}_{g,n,\mathbf{x}} = \mathrm{d}\ell_1 \wedge \ldots \wedge \mathrm{d}\ell_N \wedge \mathrm{d}\tau_1 \wedge \ldots \wedge \mathrm{d}\tau_N.$$

As a consequence, the total mass of the Teichmüller space $\mathcal{T}_{g,n}(\mathbf{x}) = \mathbb{R}^N_{>0} \times \mathbb{R}^N$ is infinite. This is not surprising because the Teichmüller space is very 'large': it contains many isometric surfaces, which are identified on the much smaller moduli space. Let us prove the following statement.

Proposition 3.4. *The total mass* $\mathrm{Vol}^{\mathrm{WP}}_{g,n,\mathbf{x}}(\mathcal{M}_{g,n}(\mathbf{x}))$ *of the Weil–Petersson volume on the moduli space is finite.*

Proof. By Bers' theorem [Ber85], there is a constant $B_{g,n,\mathbf{x}}$ such that any bordered hyperbolic surface X of signature (g, n) with boundary of length $\mathbf{x}$ admits a pair of pants decomposition $\Gamma = (\gamma_1, \ldots, \gamma_N)$ such that for all $1 \leq i \leq N$, $\ell_X(\gamma_i) \leq B_{g,n,\mathbf{x}}$.

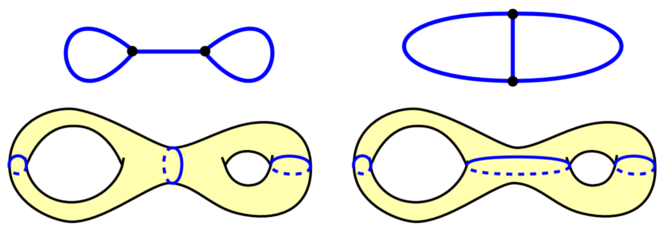


Figure 3.9: The two different topological pairs of pants decompositions of a genus 2 surface, and the corresponding graphs.

The pair of pants decomposition provided by Bers' theorem will a priori depend on the surface X. However, one can associate to a pair of pants decomposition Γ a graph $G(\Gamma)$: the vertices correspond to the pairs of pants, and two vertices are connected by an edge when their pairs of pants are glued together – see Figure 3.9. Then, for two pairs of pants decomposition Γ and Γ' of $S_{g,n}$, there exists an element of the mapping class group $\mathrm{MCG}_{g,n}$ sending Γ on Γ' if and only if the graphs $G(\Gamma)$ and $G(\Gamma')$ are isomorphic. Since there is a finite number of possible graphs, we can pick a finite family of pairs of pants $\mathcal{P}$ such that every element of the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$ admits a representative in $\bigcup_{\Gamma \in \mathcal{P}} \{(\ell^\Gamma, \tau^\Gamma) \ : \ \forall i, 0 \leq \ell_i^\Gamma \leq B_{g,n,\mathbf{x}}\}$, where ℓ^Γ and τ^Γ are the Fenchel–Nielsen coordinates associated to Γ.

Remains the question of the twist parameters. We recall that for any i, the Dehn twist around the i-th curve of the pair of pant decomposition sends the point of Fenchel–Nielsen coordinates (ℓ, τ) to the translated point $(\ell, (\tau_1, \ldots, \tau_i + \ell_i, \ldots, \tau_N))$, which are therefore identified in the moduli space. As a consequence, up to precomposition by an element of $\mathrm{MCG}_{g,n}$, we can always assume that all of the twist angles satisfy $0 \leq \tau_i \leq \ell_i$.

Then, the set

$$\mathcal{D} = \bigcup_{\Gamma \in \mathcal{P}} \{(\ell^\Gamma, \tau^\Gamma) \ : \ \forall i, 0 \leq \tau_i^\Gamma \leq \ell_i^\Gamma \leq B_{g,n,\mathbf{x}}\}$$

contains one representative of each element of $\mathcal{M}_{g,n}(\mathbf{x})$. $\mathcal{D}$ is a finite union of sets of finite volume, and the volume of the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$ is therefore finite. $\square$

Thanks to Proposition 3.4, we can set the following notations.

Notation 1. For any $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 1$ and any $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, we set

$$V_{g,n}(\mathbf{x}) := \mathrm{Vol}^{\mathrm{WP}}_{g,n,\mathbf{x}}(\mathcal{M}_{g,n}(\mathbf{x})).$$

Additionally, we define

$$\forall \mathbf{x} \in \mathbb{R}^3_{\geq 0}, V_{0,3}(\mathbf{x}) := 1$$
$$\forall x \in \mathbb{R}_{\geq 0}, V_{1,1}(x) := \frac{1}{2} \mathrm{Vol}^{\mathrm{WP}}_{1,1,x}(\mathcal{M}_{1,1}(x))$$

We set $V_{g,n} := V_{g,n}(0)$ and $V_g := V_{g,0}$ for $g \geq 2$.

The definition for $(g, n) = (0, 3)$ corresponds to the fact that for any $\mathbf{x}$, the moduli space $\mathcal{M}_{0,3}(\mathbf{x})$ is reduced to exactly one element, the unique pair of pants of boundary lengths $\mathbf{x}$ (see Theorem 3.1). The convention of dividing $V_{1,1}(x)$ by a factor two unfortunately varies throughout literature. We will provide more details on the reasons behind this choice when explaining Mirzakhani's integration formula in Section 3.2.1.

3.1.4.4 Definition of the probabilistic setting

We can now define our probability space. We focus on compact surfaces of high genus, but similar notions can be defined for surfaces with cusps or boundary geodesics.

Definition 1. Let $g \geq 2$ be an integer. A *random surface* of genus g is an element of the moduli space $\mathcal{M}_g$ sampled with the probability measure

$$\mathbb{P}^{\mathrm{WP}}_g := \frac{1}{V_g} \mathrm{Vol}^{\mathrm{WP}}_g.$$

The *expectation* of a positive measurable random variable $F : \mathcal{M}_g \to \mathbb{R}_{\geq 0}$ is defined by

$$\mathbb{E}^{\mathrm{WP}}_g[F] := \frac{1}{V_g} \int_{\mathcal{M}_g} F(X) \, \mathrm{dVol}^{\mathrm{WP}}_g(X).$$

We say that a sequence of measurable events $\mathcal{A}_g \subset \mathcal{M}_g$ has *high probability* if

$$\lim_{g \to +\infty} \mathbb{P}^{\mathrm{WP}}_g(\mathcal{A}_g) = 1.$$

Thanks to this notion of 'high probability event', we will be able to prove properties true for *typical surfaces* of high genus. Note that we could have decided to focus on *almost-sure* events, that is to say events $\mathcal{A}_g \subset \mathcal{M}_g$ such that

$$\mathbb{P}^{\mathrm{WP}}_g(\mathcal{A}_g) = 1,$$

but this stronger definition is too restrictive. Our definition was adapted from random graph theory: in this case, we say an event occurs with *high probability* if its probability goes to one as the parameter (e.g. the number of vertices) goes to infinity – see [ER60, Bol01] for various examples. The motivation is clear: since there is only a finite number of graphs with a given number of vertices, every single graph a priori has a non-zero probability of appearing and as a consequence one single counter-example will cause the probability to be strictly smaller than one. The situation is different for surfaces, since the probability space contains no atoms, but we will see in Chapters 4 and 5 that random surfaces are extremely similar to random regular graphs. In particular, unlikely counter-examples encountered for graphs are also events of small yet non-zero probability for surfaces (see [Mir13, Theorem 4.2] and Theorems 4.1, 4.8 for instance).

Each result we prove with high probability comes with a surprising consequence: an *existence* result. Indeed, if a sequence of events $\mathcal{A}_g \subset \mathcal{M}_g$ occurs with high probability, then there is an integer g_0 such that for any $g \geq g_0$, $\mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_g) > 0$ and in particular $\mathcal{A}_g$ is not empty. This remark might seem quite naive, but thanks to the methods presented in Section 3.2, it will frequently be easier to prove that a property is typical rather than exhibit an example of a surface of arbitrarily high genus satisfying it.

3.2 A tool-kit to describe random surfaces

In this section, we present the main tools used to estimate probabilities and expectations in the Weil–Petersson probability model.

The first tool, presented in Section 3.2.1, is the integration formula proved by Mirzakhani in [Mir07a]. Using Mirzakhani's integration formula will allow us to transform questions about the properties of random surfaces into questions about the behaviour of Weil–Petersson volumes. As a consequence, in order to reach a good understanding of random surfaces, we need to compute and estimate Weil–Petersson volumes. This is the aim of Sections 3.2.2 and 3.2.3, where we explain Mirzakhani's recursion formula [Mir07b] and provide various asymptotics in the high-genus limit [Mir13].

3.2.1 Mirzakhani's integration formula

Let $g \geq 2$ and $F : \mathcal{M}_g \to \mathbb{R}_{\geq 0}$ be a random variable[3]. How can one compute the expectation of F for the Weil–Petersson probability measure?

The formula proved by Mirzakhani in [Mir07a] is an expression for $\mathbb{E}_g^{\mathrm{WP}}[F]$ when F belongs to a certain class of functions, *geometric functions*. This formula relies on the symplectic structure on $\mathcal{M}_g$ and Wolpert's magic theorem (Theorem 3.3).

Note that we will state the integration formula for any moduli space of bordered surfaces $\mathcal{M}_{g,n}(\mathbf{x})$ for integers $g, n \in \mathbb{N}_0$ such that $2g - 2 + n > 0$ and any length vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, because they will be useful. However, since we are mostly interested in the probability measure $\mathbb{P}_g^{\mathrm{WP}}$ on $\mathcal{M}_g$, the examples and illustrations will be set in the compact setting.

[3]We will restrict ourselves to positive random variables for this discussion, but the results and definitions can be extended to real or complex-valued *integrable* random variables, by the usual method.

3.2.1.1 Multi-curves and free-homotopy

As suggested by the name, geometric functions $\mathcal{M}_{g,n}(\mathbf{x}) \to \mathbb{R}$ depend on the geometry of the elements of $\mathcal{M}_{g,n}(\mathbf{x})$ and more precisely of closed geodesic multi-curves on them. Let us recall a few usual definitions and notations.

A *closed curve* on a surface S is a continuous map $c : [0,1] \to S$ such that $c(0) = c(1)$. We say the curve is *simple* if it does not self-intersect.

Two closed curves c_0 and c_1 on a surface S are said to be *freely homotopic* if there is a continuous map $h : [0,1]^2 \to S$ such that $h_{|\{0\}\times[0,1]} = c_0$ and $h_{|\{1\}\times[0,1]} = c_1$. This is an equivalence relation. We will often consider curves up to free-homotopy and abusively refer to a free-homotopy class $[c]$ by one of its representative c.

A closed curve c on S is called *essential* if it is non-contractible and freely homotopic to none of the boundary components (or cusps) of S. If X is a bordered hyperbolic surface, and c is an essential closed curve on X, then there is a unique closed geodesic in its free-homotopy class [Bus92, Theorem 1.6.6]. We define $\ell_X([c]) = \ell_X(c)$ to be the length of this geodesic representative.

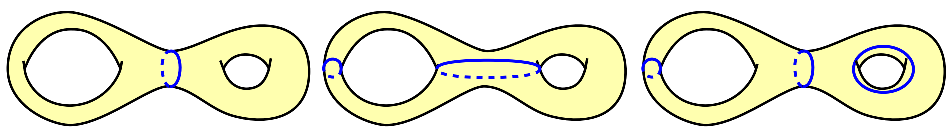


Figure 3.10: Examples of multi-curves on a compact surface of genus 2.

A *multi-curve* Γ on a surface is a family of disjoint essential simple closed curves $(\gamma_1, \ldots, \gamma_k)$, so that for any indices $i \neq j$, γ_i is freely homotopic to neither γ_j nor γ_j^{-1}. As for closed curves, we will often consider multi-curves up to free-homotopy.

By [Bus92, Theorem 1.6.6-7], any free-homotopy class of multi-curves $\Gamma = (\gamma_1, \ldots, \gamma_k)$ has a representative which is a *multi-geodesic*. This means that it is possible to 'tighten' each curve γ_i and slide it to its geodesic representative, while keeping all of the curves $(\gamma_i)_i$ simple and disjoint. We can therefore define the *total length* $\ell_X(\Gamma)$ and *length vector* $\vec{\ell}_X(\gamma)$ of a multi-curve Γ as:

$$\ell_X(\Gamma) := \sum_{i=1}^{k} \ell_X(\gamma_i) \qquad \text{and} \qquad \vec{\ell}_X(\Gamma) := (\ell_X(\gamma_1), \ldots, \ell_X(\gamma_k)).$$

3.2.1.2 Multi-curves on elements of $\mathcal{T}_{g,n}(\mathbf{x})$

Let Γ be a multi-curve on $S_{g,n}$. For any marked hyperbolic surface (X, ϕ), the marking homeomorphism $\phi : S_{g,n} \to X$ sends Γ onto a multi-curve $\phi(\Gamma)$ on the bordered hyperbolic surface X, as illustrated on Figure 3.4 with two multi-curves α and β.

Proposition 3.5. *Let Γ be a multi-curve with k components on the base surface $S_{g,n}$.*

The length function

$$\begin{aligned} \mathcal{T}_{g,n}(\mathbf{x}) &\to \mathbb{R}^k_{\geq 0} \\ (X,\phi) &\mapsto \vec{\ell}_{(X,\phi)}(\Gamma) := \vec{\ell}_X(\phi(\Gamma)) \end{aligned}$$

is a well-defined map.

Proof. We simply need to prove that the quantity $\vec{\ell}_X(\phi(\Gamma))$ is invariant for the equivalence relation $\sim$. For any marked surface (Y,ψ) such that $(X,\phi) \sim (Y,\psi)$, by definition, there exists an isometry $h : X \to Y$ such that $\psi^{-1} \circ h \circ \phi$ is isotopic to the identity on $S_{g,n}$. This implies that $h \circ \phi(\Gamma)$ and $\psi(\Gamma)$ are freely homotopic on Y and in particular

$$\vec{\ell}_Y(\psi(\Gamma)) = \vec{\ell}_Y(h \circ \phi(\Gamma)) = \vec{\ell}_X(\phi(\Gamma))$$

because h is an isometry. $\square$

In other words, there is a one-to-one correspondence between free homotopy classes on the base surface $S_{g,n}$ and on an element of the Teichmüller space $\mathcal{T}_{g,n}(\mathbf{x})$.

3.2.1.3 Geometric functions on $\mathcal{M}_{g,n}(\mathbf{x})$

As we saw in Figure 3.4, one free homotopy class on the base surface $S_{g,n}$ can be sent on many different free homotopy classes on one given surface X, by precomposition of the marking by an element of the mapping class group. The lengths of the geodesics in these free homotopy classes will a priori be different, so the length function that we considered on the Teichmüller space is no longer well-defined on the moduli space. In order to create a function on $\mathcal{M}_{g,n}(\mathbf{x})$, we will organise multi-curves on $S_{g,n}$ by *orbit* for the action of the mapping class group.

Definition 2. The mapping class group $\mathrm{MCG}_{g,n}$ acts by composition on the set of multi-curves on the base surface $S_{g,n}$. For a multi-curve Γ on the base surface $S_{g,n}$, we set

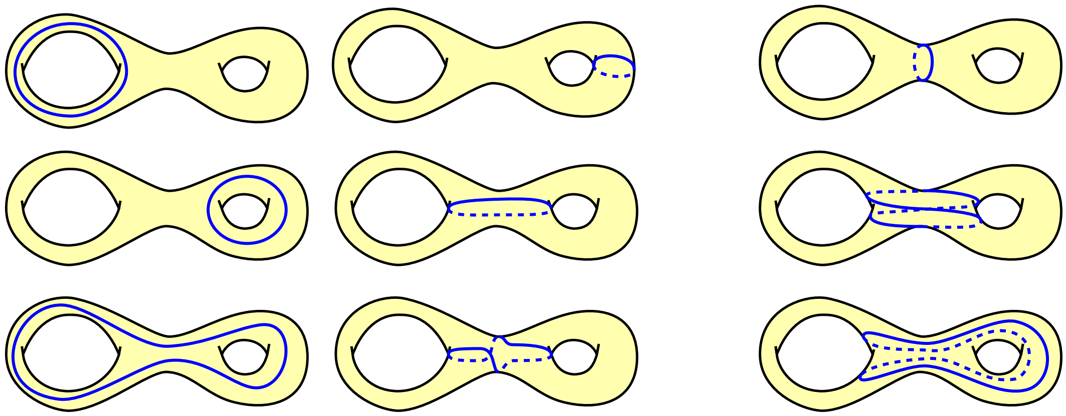
$$\mathrm{Orb}(\Gamma) = \{h(\Gamma)\,,\, h \in \mathrm{MCG}_{g,n}\}$$

to be the orbit of Γ for this action. We say that two multi-curves have the same *topological type* if they belong in the same orbit.

Let us describe these topological types in two extreme cases, for a multi-curve with only one component (i.e. one simple closed curve) and a pair of pants decomposition.

Proposition 3.6. *The topological type of a simple closed curve β on S_g is entirely determined by the topology of the surface $S_g \setminus \beta$ obtained by cutting along it. More precisely, there are exactly $\lfloor \frac{g}{2} \rfloor + 1$ orbits of simple curves on the compact surface S_g:*

- *the* non-separating *curve γ_0, which cuts S_g into a surface of signature* $(g-1,2)$



(a) Non-separating curves in $\mathrm{Orb}(\gamma_0)$. (b) Separating curves in $\mathrm{Orb}(\gamma_1)$.

Figure 3.11: Simple closed curves on a genus 2 compact hyperbolic surface. They are organised by topological type, i.e. by orbit for the action of the mapping class group.

- *for* $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$*, the* i*-th separating curve* γ_i *which separates* $S_{g,n}$ *into two components, of respective signatures* $(i, 1)$ *and* $(g-i, 1)$*.*

In particular, when $g = 2$, there are two orbits, represented in Figure 3.11.

Elements of proof. Let β, β' be two non-separating simple closed curves on S_g. Then, the surfaces $S_g \setminus \beta$ and $S_g \setminus \beta'$ both are of signature $(g-1, 2)$. By the classification of surfaces, there is a homeomorphism $h : S_g \setminus \beta \to S_g \setminus \beta'$. This allows us to construct a homeomorphism $\tilde{h} : S_g \to S_g$ sending β on β', which imply that β and β' belong in the same orbit for the action of MCG_g.

The proof for separating curves is the same. □

Example 2. *We saw in the proof of Proposition 3.4 that there is a finite number of topological types of pairs of pants decomposition of* S_g*, and they correspond to the isomorphic classes of* 3*-regular multi-graphs with* $2g-2$ *vertices and* $3g-3$ *edges. The two different classes for* $g = 2$ *are represented in Figure 3.9.*

It is then possible to define functions from $\mathcal{M}_{g,n}(\mathbf{x})$ to $\mathbb{R}$, by summing over the orbit of a multi-curve.

Definition 3. A function $F : \mathcal{M}_{g,n}(\mathbf{x}) \to \mathbb{R}_{\geq 0}$ is called a *geometric function* if there exists an integer $k \geq 1$, a multi-curve $\Gamma = (\gamma_1, \ldots, \gamma_k)$ on $S_{g,n}$ and a function $f : \mathbb{R}^k \to \mathbb{R}$ such that

$$\forall X \in \mathcal{M}_{g,n}(\mathbf{x}), F(X) = f^{\Gamma}(X) := \sum_{\Gamma' \in \mathrm{Orb}(\Gamma)} f(\vec{\ell}_X(\Gamma')).$$

Though a fixed term of the sum in the previous definition only really makes sense for an element (X, ϕ) of the Teichmüller space, the summation over the orbit makes it invariant under the action of the mapping class group, and hence a well-defined function on the moduli space $\mathcal{M}_{g,n}(\mathbf{x})$.

Example 3. *For $0 \leq a \leq b$, the* simple *counting function defined for $X \in \mathcal{M}_g$ by*

$$\mathrm{N}_X^{\ell,s}(a,b) = \#\{\textit{primitive simple geodesics } \gamma \textit{ s.t. } a \leq \ell_X(\gamma) \leq b\}$$

is not *a geometric function, but it can be expressed as*

$$\mathrm{N}_X^{\ell,s}(a,b) = \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \mathrm{N}_X^{\ell,s,i}(a,b)$$

where for each $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$, $\mathrm{N}_X^{\ell,s,i}(a,b)$ is the geometric function

$$\begin{aligned}\mathrm{N}_X^{\ell,s,i}(a,b) &= \#\{\textit{primitive simple geodesics } \gamma \in \mathrm{Orb}(\gamma_i) \textit{ s.t. } a \leq \ell_X(\gamma) \leq b\} \\ &= \sum_{\gamma \in \mathrm{Orb}(\gamma_i)} \mathbb{1}_{[a,b]}(\ell_X(\gamma)).\end{aligned}$$

Unfortunately, the full counting function $\mathrm{N}_X^{\ell}(a,b) = \#\{$geodesics γ s.t. $a \leq \ell_X(\gamma) \leq b\}$ cannot be expressed in a similar fashion, because it contains non-simple geodesics, which are not multi-curves.

Example 4. *The function counting the number of pairs of pants decompositions of total boundary length smaller than L,*

$$p_X(L) := \#\{\Gamma \textit{ pair of pants decomposition of } X \ : \ \ell_X(\Gamma) \leq L\},$$

can also be written as a sum of geometric functions, indexed by the topological types of the pairs of pants decompositions.

3.2.1.4 The integration formula

Mirzakhani provided an expression for the integral of any geometric function as an integral over a power of $\mathbb{R}_{\geq 0}$ [Mir07a]. In order to write the formula, we must set notations to describe the surface obtained by cutting $S_{g,n}$ along a multi-curve Γ.

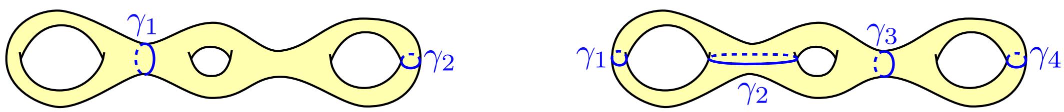
Notation 2 (see Figure 3.12)**.** Let Γ be a multi-curve on $S_{g,n}$. The cut surface $S_{g,n} \setminus \Gamma$ can be written as the disjoint union $\bigsqcup_{i=1}^{q} S_{g_i,n_i}$ of its connected pieces, and the k curves of Γ form $2k$ boundary components of the cut surface.

If we equip $S_{g,n}$ with a hyperbolic metric such that the length of its boundary components are $\mathbf{x}$, then the lengths of the components of the multi-curve Γ form a length vector $\mathbf{y} \in \mathbb{R}_{>0}^{k}$. These lengths become new boundary lengths of the surface $S_{g,n} \setminus \Gamma$. We can therefore consider that each component S_{g_i,n_i} has boundaries of fixed lengths, defined by a length vector $(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}) \in \mathbb{R}_{>0}^{n_i}$, made up of old and new lengths. We then set

$$V_{g,n}(\mathbf{x}; \Gamma, \mathbf{y}) := \prod_{i=1}^{q} V_{g_i,n_i}(\mathbf{x}^{(i)}, \mathbf{y}^{(i)}).$$

When $n = 0$, in order to simplify notations, we define

$$V_g(\Gamma, \mathbf{y}) := \prod_{i=1}^{q} V_{g_i,n_i}(\mathbf{y}^{(i)}).$$



(a) $V_{1,1}(y_1)\, V_{2,3}(y_1, y_2, y_2)$. (b) $V_{0,5}(y_1, y_1, y_2, y_2, y_3)\, V_{0,3}(y_3, y_4, y_4)$.

Figure 3.12: Value of $V_3(\Gamma, \mathbf{y})$ for two examples of multi-curves on the base surface S_3.

Mirzakhani's integration formula can then be written as follows.

Theorem 3.7 ([Mir07a, Theorem 8.1])**.** *Given a multi-curve Γ with k components and a positive measurable function $f : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$,*

$$\int_{\mathcal{M}_{g,n}(\mathbf{x})} f^\Gamma(X)\, \mathrm{dVol}^{\mathrm{WP}}_{g,n,\mathbf{x}}(X) = \int_{\mathbb{R}^k_{\geq 0}} f(\mathbf{x})\, V_{g,n}(\mathbf{x}; \Gamma, \mathbf{y})\, y_1 \cdots y_k \, \mathrm{d}\mathbf{y}.$$

The formula proved in [Mir07a] has an additional factor $2^{-M(\Gamma)}$ where $M(\Gamma)$ is the number of one-holed tori separated by Γ, because every surface of signature $(1, 1)$ with one boundary component of length x admits one involution isometry[4]. It does not appear here thanks to our convention for the volume $V_{1,1}(x)$.

Example 5. *Thanks to the integration formula and the geometric expression in Example 3, we can write the expectation of the simple counting function $\mathrm{N}^{\ell,s}_X(a, b)$ as*

$$\mathbb{E}^{\mathrm{WP}}_g \left[\mathrm{N}^{\ell,s}_X(a, b)\right] = \frac{1}{V_g} \int_a^b V_{g-1,2}(x, x)\, x \, \mathrm{d}x + \frac{1}{V_g} \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \int_a^b V_{i,1}(x) V_{g-i,1}(x)\, x \, \mathrm{d}x.$$

In order to actually compute the expectation of $\mathrm{N}^{\ell,s}_X(a, b)$, we see that we need to compute $V_{g-1,2}(x, x)$ and $V_{g',1}(x)$ for $x \in \mathbb{R}_{\geq 0}$ and $g' \leq g$.

Example 6. *The expectation of the function $p_X(L)$ introduced in Example 4 can be computed using Mirzakhani's integration formula. The expression is very simple: contrarily to Example 5, all of the volumes that appear in the integral are volumes $V_{0,3}(\,\cdot\,)$ which are always equal to one.*

[4]The involution corresponds to a rotation of angle π of the boundary geodesic or cusp, represented in [FM12, Figure 3.8]. Note that some exceptional isometries also occur for $(g, n) = (2, 0)$ or $(1, 2)$ if $x_1 = x_2$. This has no incidence in the formula because only surfaces with a boundary intervene in the integration formula, and because the condition $x_1 = x_2$ is only verified on a set of measure 0.

3.2.2 Mirzakhani's topological recursion formula

Let $g, n \in \mathbb{N}_0$ such that $2g-2+n>0$. The aim of this section is to provide a method to compute the Weil–Petersson volume $V_{g,n}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$.

3.2.2.1 Polynomial expression

Mirzakhani proved in [Mir07a, Theorem 6.1] the following result.

Theorem 3.8. *The function $\mathbf{x} \mapsto V_{g,n}(\mathbf{x})$ is a polynomial function that can be written as*

$$V_{g,n}(\mathbf{x}) = \sum_{|\alpha| \leq 3g-3+n} c_{g,n}(\alpha) \prod_{j=1}^{n} \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}.$$

Interestingly, the coefficients $c_{g,n}(\alpha)$ can be interpreted as intersection numbers [Mir07b]. The choice of the normalisation by $2^{2\alpha_j}(2\alpha_j+1)!$ is partly motivated by this interpretation, but also simplifies the recursion formulas linking the coefficients. Note that the polynomial $V_{g,n}(\mathbf{x})$ is symmetric in the variables $\mathbf{x}$, and therefore the coefficient $c_{g,n}(\alpha)$ is invariant by permutation of the multi-index α.

Example 7. *We have already seen that $V_{0,3}(\mathbf{x})$ is the constant polynomial equal to 1. Näätänen and Nakanishi proved in [NN98] that for all $x \geq 0$,*

$$V_{1,1}(x) = \frac{\pi^2}{12} + \frac{x^2}{48}.$$

Elements of proof of Theorem 3.8. McShane's identity [McS91] states that for any punctured torus X (i.e. element of $\mathcal{M}_{1,1}(0) = \mathcal{M}_{1,1}$),

$$\sum_{\substack{\gamma \text{ simple} \\ \text{closed geodesic}}} \frac{1}{1+\exp(\ell_X(\gamma))} = \frac{1}{2}.$$

We observe that there is only one topological type of simple essential closed curve on a surface of signature $(1,1)$: one that cuts the surface into a pair of pants. Let γ be such a curve on $S_{1,1}$. The previous equation can be rewritten as:

$$\forall X \in \mathcal{M}_{1,1}, \quad f^\gamma(X) = \frac{1}{2}$$

where f is the function $x \mapsto \frac{1}{1+\exp(x)}$. In order words, f^γ is a *constant geometric function* on the moduli space $\mathcal{M}_{1,1}$. On the one hand, since it is a geometric function, it can be integrated thanks to Mirzakhani's integration formula:

$$\int_{\mathcal{M}_{1,1}} f^\gamma(X) \, \mathrm{dVol}^{\mathrm{WP}}_{1,1}(X) = \int_0^{+\infty} \frac{x V_{0,3}(x,x,0) \, \mathrm{d}x}{1+\exp(x)} = \frac{\pi^2}{12}.$$

On the other hand, it is constant so

$$\int_{\mathcal{M}_{1,1}} f^\gamma(X)\,\mathrm{dVol}_{1,1}^{\mathrm{WP}}(X) = \frac{\mathrm{Vol}_{1,1}^{\mathrm{WP}}(\mathcal{M}_{1,1})}{2} = V_{1,1}.$$

And therefore $V_{1,1} = \pi^2/12$.

Mirzakhani generalised McShane's identity in [Mir07a]. For any (g,n) such that $2g-2+n > 0$ and $n \neq 0$ and any length vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, she found a family of multi-curves $\Gamma_1, \ldots, \Gamma_N$ on the base surface $S_{g,n}$ and functions $f_{1,\mathbf{x}}, \ldots, f_{N,\mathbf{x}}$, such that:

$$\forall X \in \mathcal{M}_{g,n}(\mathbf{x}), \quad \sum_{i=1}^N f_{i,\mathbf{x}}^{\Gamma_i}(X) = 1.$$

As a consequence,

$$\mathrm{Vol}_{g,n}^{\mathrm{WP}}(\mathcal{M}_{g,n}(\mathbf{x})) = \sum_{i=1}^N \int_{\mathcal{M}_{g,n}(\mathbf{x})} f_{i,\mathbf{x}}^{\Gamma_i}(X)\,\mathrm{dVol}_{g,n,\mathbf{x}}^{\mathrm{WP}}(X).$$

These integrals are integrals of geometric functions, and can therefore be expressed using Theorem 3.7. The expression will contain some volume polynomials $V_{g',n'}(\mathbf{x}')$ for g', n' such that the Euler characteristic $|\chi'| = 2g'-2+n' < |\chi| = 2g-2+n$, because the surface $S_{g,n}$ has been *cut* along a multi-curve. By an induction on the quantity $2g-2+n$, we find that all the volumes $V_{g,n}(\mathbf{x})$ are polynomials in $\mathbf{x}$, and that their coefficients satisfy a recursion formula. □

3.2.2.2 Recursion formula

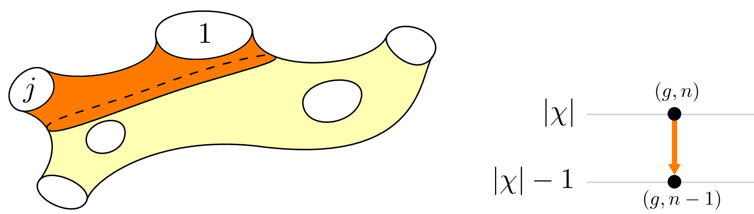
Whenever the number of boundary component n is different from 0, the volume polynomial $V_{g,n}(\mathbf{x})$, and therefore its coefficients $c_{g,n}(\alpha)$, can be computed using a topological recursion formula proved by Mirzakhani [Mir07a].

More precisely, the coefficients of the volume $V_{g,n}(\mathbf{x})$ can be expressed as a linear combination of the coefficients of certain volumes $V_{g',n'}(\mathbf{x})$, corresponding to Euler characteristics $|\chi'| = 2g'-2+n'$ strictly smaller than $|\chi| = 2g-2+n$. This ultimately allows the computation of all the volume polynomials $V_{g,n}(\mathbf{x})$ (with non-zero n) starting only with the base cases for which $|\chi| = 1$, the pair of pant and the one-holed torus for which the volumes are known (see Example 7).

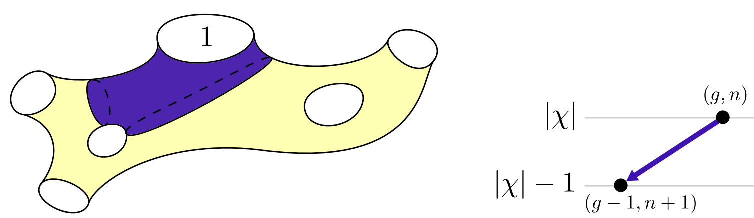
In order to state the recursion formula proved by Mirzakhani in [Mir07a], and the numerous terms it contains, let us first explain its topological interpretation. Let us consider a bordered hyperbolic surface $X \in \mathcal{M}_{g,n}(\mathbf{x})$. Our objective is to 'construct' X using smaller pieces. One way to do so is the following: we focus on one boundary component of X, the first one for instance. We will try to remove from the surface X a pair of pants containing the boundary component β_1. Since the Euler characteristic of the pair of pants is -1, the Euler characteristic obtained after removing the pair of pants will decrease in absolute value.

There are many topological types of embedded pairs of pants bounded by β_1. They can be arranged in three categories.

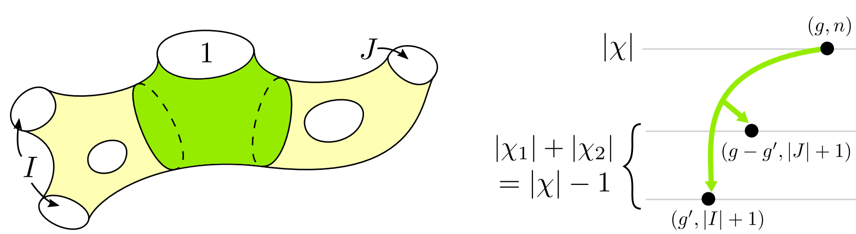
(A) Pairs of pants with two boundary components from ∂X, the component β_1 and β_j for a $j \in \{2, \ldots, n\}$. Then, the signature of the surface obtained when removing this pair of pants is $(g, n-1)$, with $n - 1 > 0$.



(B) *Non-separating* pairs of pants, that is to say a pair of pants delimited by the boundary component β_1 and two inner curves, and such that the surface obtained when removing the pair of pants is still connected (and therefore of signature equal to $(g-1, n+1)$).



(C) *Separating* pair of pants, that is to say a pair of pants delimited by the boundary component β_1 and two inner curves, but which separates the surface into two connected components. The topological situation can be entirely described by the genus g' of one of the components (the other genus being $g - g'$), and a partition (I, J) of the boundary components $\{2, \ldots, n\}$ of X. Note that the only cases which will appear are those for which $2g' - 2 + |I| + 1 > 0$ and $2(g - g') - 2 + |J| + 1 > 0$. Let $\mathcal{I}_{g,n}$ denote the set of all these topological possibilities.



The coefficients of the volume $V_{g,n}(\mathbf{x})$ can be expressed as a linear combination of the coefficients of all the embedded surfaces we encountered in this enumeration.

Theorem 3.9 ([Mir07a])**.** *The coefficients of the volume polynomial $V_{g,n}(\mathbf{x})$ can be written as a sum of three contribution, corresponding to the cases (A-C):*

$$c_{g,n}(\alpha) = \sum_{j=2}^{n} \mathcal{A}_{g,n}^{(j)}(\alpha) + \mathcal{B}_{g,n}(\alpha) + \sum_{\iota \in \mathcal{I}_{g,n}} \mathcal{C}_{g,n}^{(\iota)}(\alpha). \tag{3.2}$$

Each of these terms is a combination of coefficients of the volumes of the corresponding embedded surfaces:

$$\mathcal{A}_{g,n}^{(j)}(\alpha) = 8\,(2\alpha_j+1) \sum_{i=0}^{+\infty} u_i\, c_{g,n-1}(i+\alpha_1+\alpha_j-1, \alpha_2, \ldots, \hat{\alpha}_j, \ldots, \alpha_n) \tag{3.3}$$

$$\mathcal{B}_{g,n}(\alpha) = 16 \sum_{i=0}^{+\infty} \sum_{k_1+k_2=i+\alpha_1-2} u_i\, c_{g-1,n+1}(k_1, k_2, \alpha_2, \ldots, \alpha_n) \tag{3.4}$$

$$\mathcal{C}_{g,n}^{(\iota)}(\alpha) = 16 \sum_{i=0}^{+\infty} \sum_{k_1+k_2=i+\alpha_1-2} u_i\, c_{g',|I|+1}(k_1, \alpha_I)\, c_{g-g',|J|+1}(k_2, \alpha_J), \tag{3.5}$$

where for any $i \geq 0$,

$$u_i = \begin{cases} \zeta(2i)(1-2^{1-2i}) & \text{when } i > 0 \\ \frac{1}{2} & \text{when } i = 0. \end{cases}$$

Note that all the sums are finite because the coefficient $c_{g,n}(\beta)$ is equal to zero as soon as $|\beta| > 3g-3+n$, and therefore the non-zero terms satisfy $i \leq 3g-1+n-|\alpha|$.

Example 8. *The coefficients that intervene when computing $V_{g,n}(\mathbf{x})$ for each (g,n) such that $|\chi| \leq 3$ are represented by the arrows in Figure 3.13.*

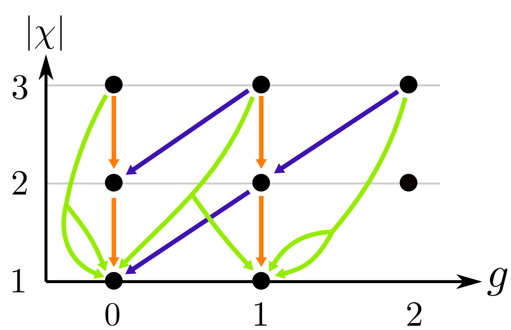


Figure 3.13: Dependency of the coefficients of the volume polynomials $V_{g,n}(\mathbf{x})$ when $|\chi| = 2g-2+n \leq 3$. Note that all the coefficients for which $n \neq 0$ can therefore be computed thanks to the base coefficients for which $|\chi| = 1$.

3.2.2.3 Sequence $(u_i)_i$ and first estimates

In order to use this recursion formula, we need some information of the sequence $(u_i)_i$.

Lemma 3.10 ([Mir13, Lemma 3.1]). *The sequence $(u_i)_i$ is increasing, converges to 1 as i approaches infinity, and there exists a constant $C > 0$ such that*

$$\forall i, u_{i+1} - u_i \leq \frac{C}{4^i}. \tag{3.6}$$

We can deduce from the monotonicity of the sequence $(u_i)_i$ the fact that the coefficients $c_{g,n}(\alpha)$ are decreasing functions of α in the following sense.

Lemma 3.11. *We define the following partial order on multi-indices in $\mathbb{N}^n$:*

$$\alpha \leq \tilde{\alpha} \quad \Leftrightarrow \quad \forall j \in \{1, \ldots, n\}, \alpha_j \leq \tilde{\alpha}_j.$$

Then, the coefficients $c_{g,n}(\alpha)$ of the volume polynomial $V_{g,n}(\mathbf{x})$ decrease with the multi-index α. In particular,

$$\forall \alpha, \quad 0 \leq c_{g,n}(\alpha) \leq V_{g,n}. \tag{3.7}$$

Proof. By symmetry of the coefficients, we can reduce the problem to proving that for any multi-indices α and $\tilde{\alpha} = (\tilde{\alpha}_1, \alpha_2, \ldots, \alpha_n)$ such that $\tilde{\alpha}_1 \geq \alpha_1$, $c_{g,n}(\tilde{\alpha}) \leq c_{g,n}(\alpha)$. More precisely, we will show that every single term in equation (3.2) is smaller for the index $\tilde{\alpha}$ than it is for α. The method being the same for every contribution, so we only detail the proof of the fact that $\mathcal{B}_{g,n}(\tilde{\alpha}) \leq \mathcal{B}_{g,n}(\alpha)$. By equation (3.4),

$$\mathcal{B}_{g,n}(\alpha) - \mathcal{B}_{g,n}(\tilde{\alpha}) = 16 \sum_{k_1, k_2} (u_{k_1+k_2+2-\alpha_1} - u_{k_1+k_2+2-\tilde{\alpha}_1})\, c_{g-1,n+1}(k_1, k_2, \alpha_2, \ldots, \alpha_n)$$

with the convention $u_i = 0$ when $i < 0$. Then, the difference $u_{k_1+k_2+2-\alpha_1} - u_{k_1+k_2+2-\tilde{\alpha}_1}$ is always positive because the sequence $(u_i)_{i \in \mathbb{Z}}$ is increasing, so the sum $\mathcal{B}_{g,n}(\alpha) - \mathcal{B}_{g,n}(\tilde{\alpha})$ is too. □

Due to the expression of $V_{g,n}(\mathbf{x})$ in terms of the coefficients $c_{g,n}(\alpha)$, Lemma 3.11 directly implies the following inequality on the volume polynomials.

Corollary 3.12. *For any length vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$,*

$$x_1 \ldots x_n V_{g,n}(x_1, \ldots, x_n) \leq 2^n V_{g,n} \prod_{j=1}^n \sinh\left(\frac{x_j}{2}\right) \leq V_{g,n} \exp\left(\frac{x_1 + \ldots + x_n}{2}\right). \tag{3.8}$$

Proof. By definition of the coefficients $c_{g,n}(\alpha)$,

$$\begin{aligned} x_1 \ldots x_n V_{g,n}(\mathbf{x}) &= \sum_{|\alpha| \leq 3g-3+n} c_{g,n}(\alpha) \prod_{j=1}^n \frac{x_j^{2\alpha_j+1}}{2^{2\alpha_j}(2\alpha_j+1)!} \\ &\leq V_{g,n} \sum_{\alpha \in \mathbb{N}^n} \prod_{j=1}^n \frac{x_j^{2\alpha_j+1}}{2^{2\alpha_j}(2\alpha_j+1)!} = V_{g,n} \prod_{j=1}^n 2 \sinh\left(\frac{x_j}{2}\right) \end{aligned}$$

which leads to the conclusion because $2\sinh(x) \leq \exp(x)$. □

3.2.3 High-genus estimates and applications to geometry

In this section, we will:

- provide the last tool we need to study random hyperbolic surfaces: estimates on the Weil–Petersson volumes as $g \to +\infty$
- describe a methodology to prove geometric properties true for typical surfaces, and give a few examples.

3.2.3.1 Estimates of ratios of Weil–Petersson volumes

Let us present known estimates on ratios of Weil–Petersson volumes in the large-genus limit. These properties have been established in [Mir13] using several recursion formulas for Weil–Petersson volume [Mir07a, DN09, LX09].

Same Euler characteristic Since two surfaces with the same Euler characteristic are at the same height in the recursion formula, one could expect that the volumes $V_{g,n}$ and $V_{g-1,n+2}$ are of similar sizes. This is indeed the case: by [Mir13, Theorem 3.5], for all $n \geq 1$, there is a constant $C_n > 0$ such that for any $g \in \mathbb{N}_0$ satisfying $2g-2+n > 0$,

$$\left| \frac{V_{g-1,n+2}}{V_{g,n}} - 1 \right| \leq \frac{C_n}{g+1}. \tag{3.9}$$

Note that this result has been extended to cases where $n = n(g)$ under the assumption that $n(g) = o(\sqrt{g})$ in [MZ15, Lemma 5.1].

Actually, one might perhaps expect that all of the $g+1$ volumes

$$V_{g,n}, V_{g-1,n+2}, V_{g-2,n+4}, \ldots, V_{1,n+2g-2}, V_{0,n+2g}$$

are of similar size. While we can compare $V_{g,n}$ to $V_{g-k,n+2k}$ for a fixed k by iterating equation (3.9), the bound becomes bad for a large k. By [Mir13, equation (3.20)], the first item of this list dominates all of the others: there exists a constant $c > 0$ such that for any $g, n \in \mathbb{N}_0$ satisfying $2g-2+n > 0$,

$$\forall k \leq g, \quad V_{g-k,n+2k} \leq c\, V_{g,n}. \tag{3.10}$$

However, there is no similar bound the other way: as a consequence of the asymptotic expansion of $V_{g,n}$ for fixed g and large n proved in [MZ00], we have that

$$V_{0,n+2g} = \mathcal{O}_n\left(\frac{V_{1,n+2g-2}}{\sqrt{g}}\right).$$

Cutting into two smaller surfaces Similarly, since we can cut surfaces of signature (g,n) into two surfaces of respective signatures (g_1, n_1+1) and (g_2, n_2+1) with $g_1+g_2 = g$ and $n_1 + n_2 = n$, one could expect the product $V_{g_1,n_1+1} \times V_{g_2,n_2+1}$ to be of similar size as $V_{g,n}$. Actually, these quantities are much smaller: by [Mir13, Lemma 3.3], for any $n \geq 0$, there exists a constant $C_n > 0$ such that for all g satisfying $2g-2+n > 0$,

$$\sum_{\substack{g_1+g_2=g \\ n_1+n_2=n \\ 2g_i+n_i>1}} V_{g_1,n_1+1} V_{g_2,n_2+1} \leq C_n \frac{V_{g,n}}{g+1}. \tag{3.11}$$

As we will see in the applications in Section 3.2.3.2, the presence of this decay in $1/(g+1)$ is linked to the fact that typical surfaces of large genus are very well-connected, and therefore quite difficult to cut into smaller pieces.

Adding a cusp We can furthermore compare $V_{g,n}$ and $V_{g,n+1}$ using [Mir13, Lemma 3.2]: for any $g, n \in \mathbb{N}_0$ such that $2g-2+n > 0$,

$$\frac{1}{12}\left(1 - \frac{\pi^2}{10}\right) < \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} < \frac{\cosh(\pi)}{\pi^2}. \tag{3.12}$$

The fact that $V_{g,n+1}$ grows roughly like $(2g-2+n)V_{g,n}$ can be interpreted the following way: in order to sample a surface of signature $(g, n+1)$, we can start by sampling a surface of signature (g,n). We then need to decide where to add a cusp, by picking a point on the surface, of area proportional to $2g-2+n$.

3.2.3.2 Geometry of typical compact hyperbolic surfaces

We are now (finally!) equipped with all of the tools necessary to study typical surfaces.

As a general rule, geometric estimates on typical compact hyperbolic surfaces are always established following the same methodology:

- we express the question in terms of expectations of geometric functions
- we use Mirzakhani's integration formula to express the expectations of these geometric functions
- we use estimates, such as Corollary 3.12 and the high-genus estimates from Section 3.2.3.1, to conclude.

All of the results proven in [Mir13, Section 4], [MP19], and the two main geometric results from this thesis (Theorem 4.1 and 4.8) are proved using the same method. Let us prove the following proposition, adapted from [Mir13, Section 4], in order to illustrate the method.

Proposition 3.13. *For all $a \in (0,1)$ and any genus $g \geq 2$,*

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sys}(X) \leq a\right) = \mathcal{O}\left(a^2\right) \tag{3.13}$$

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{SimSepSys}(X) \leq a \log g\right) = \mathcal{O}\left(\frac{1}{g^{1-a}}\right) \tag{3.14}$$

where $\mathrm{Sys}(X)$ *denotes the length of the shortest closed geodesic on X (which is always simple), and* $\mathrm{SimSepSys}(X)$ *the length of the shortest simple closed geodesic that separates the surface X.*

In other words, typically,

$$\mathrm{Sys}(X) \geq \frac{1}{\log g} \quad \text{and} \quad \mathrm{SimSepSys}(X) \geq a \log g.$$

We observe that the length of the simple separating systole seems to be much longer than the length of the systole. This is an indicator of the fact that typical surface are very well-connected: they might have short geodesics, but cutting along them does not disconnect the surface. The reason behind this scale-difference is the additional factor $1/g$ in the estimate (3.11) compared to (3.9), as we will see in the proof.

Proof of Proposition 3.13. The first step of the proof is to express the probabilities in terms of the geometric functions introduced in Example 3. We observe that:

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sys}(X) \leq a\right) = \mathbb{P}_g^{\mathrm{WP}}\left(\exists i \geq 0 : \mathrm{N}_X^{\ell,s,i}(0,a) \geq 1\right)$$
$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{SimSepSys}(X) \leq a \log g\right) = \mathbb{P}_g^{\mathrm{WP}}\left(\exists i > 0 : \mathrm{N}_X^{\ell,s,i}(0,a \log g) \geq 1\right).$$

We use Markov's inequality in order to replace these probabilities by *expectations*:

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sys}(X) \leq a\right) \leq \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} \mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,i}(0,a)\right]$$
$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{SimSepSys}(X) \leq a \log g\right) \leq \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,i}(0,a \log g)\right].$$

We now need to estimate these expectations, which is the aim of the following lemma.

Lemma 3.14. *For any integer $g \geq 2$ and any real numbers $0 \leq a \leq b$,*

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,0}(a,b)\right] = \mathcal{O}\left(\int_a^b \frac{\sinh^2\left(\frac{x}{2}\right)}{x}\, \mathrm{d}x\right) = \mathcal{O}\left(e^b\right)$$
$$\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,i}(a,b)\right] = \mathcal{O}\left(\frac{1}{g}\int_a^b \frac{\sinh^2\left(\frac{x}{2}\right)}{x}\, \mathrm{d}x\right) = \mathcal{O}\left(\frac{e^b}{g}\right).$$

Proof. We have expressed these expectations using Mirzakhani's integration formula in Example 5:

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,0}(a,b)\right] = \frac{1}{V_g}\int_a^b V_{g-1,2}(x,x)\,x\,\mathrm{d}x$$
$$\forall i>0, \mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^{\ell,s,i}(a,b)\right] = \frac{1}{V_g}\int_a^b V_{i,1}(x)V_{g-i,1}(x)\,x\,\mathrm{d}x.$$

By Corollary 3.12, for all $x>0$ and $i>0$,

$$V_{g-1,2}(x,x) \leq \frac{4\sinh^2\left(\frac{x}{2}\right)}{x^2}\,V_{g-1,2}$$
$$V_{i,1}(x)V_{g-i,1}(x) \leq \frac{4\sinh^2\left(\frac{x}{2}\right)}{x^2}\,V_{i,1}V_{g-i,1}.$$

We conclude using equations (3.9) and (3.11). □

As a consequence, on the one hand,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sys}(X)\leq a\right) = \mathcal{O}\left(\int_0^a \frac{\sinh^2\left(\frac{x}{2}\right)}{x}\,\mathrm{d}x\right) = \mathcal{O}\left(\int_0^a x\,\mathrm{d}x\right) = \mathcal{O}\left(a^2\right).$$

because there exists a constant C such that for all $x\in[0,1]$, $\sinh^2\left(\frac{x}{2}\right)\leq Cx^2$. On the other hand,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{SimSepSys}(X)\leq a\log g\right) = \mathcal{O}\left(\frac{e^{a\log g}}{g}\right) = \mathcal{O}\left(\frac{1}{g^{1-a}}\right).$$

□

This proof is an example of a situation where the first step of the method, namely relating the event to expectations of geometric functions, is straightforward. In general, it can be quite hard to do so. Here are a few examples of difficulties, and the way they are addressed in literature.

- Not every quantities we are interested in is related to lengths of multi-curves. For instance, how is it possible to study the diameter with this method? Or the lengths of non-simple curves?
  - In [Mir13, Section 4], many of the estimates are obtained from inequalities between different quantities. For instance, Mirzakhani proved that, typically, $\mathrm{diam}(X)\leq 40\log g$ thanks to the bound

$$\mathrm{diam}(X) \leq \mathrm{Sys}(X) + \frac{2}{h(X)}\log\left(\frac{\pi(2g-2)}{\cosh\left(\frac{\mathrm{Sys}(X)}{2}\right)-1}\right)$$

from [Bro92], together with bounds on $\mathrm{Sys}(X)$ and the Cheeger constant $h(X)$ [Mir13, Theorem 4.10]. This is a good estimate: we know that the diameter is always at least logarithmic in the genus. However, the value 40 is (a priori) not optimal at all. Improving the estimate would require a betterment of the bound on the Cheeger constant or a new approach.

– In Section 4.2, we adapt the notion of 'tangle-free' graphs to surfaces. We define the notion in terms of geometric functions, so that its probability is easy to study (see Theorem 4.8). We then obtain information on other quantities, harder to express in terms of geometric functions (such as the length of the shortest non-simple geodesic or the width of the collar surrounding a closed geodesic) using tools from hyperbolic geometry.

- For any integer-valued random variable $F : \mathcal{M}_g \to \mathbb{N}_0$, such as the counting function $\mathrm{N}_X^{\ell,s}(a,b)$ or $\mathrm{N}_X^{\ell,s,i}(a,b)$, Markov's inequality
$$\mathbb{P}_g^{\mathrm{WP}}\left(F(X) \geq 1\right) \leq \mathbb{E}_g^{\mathrm{WP}}\left[F(X)\right]$$
can only give us upper bounds on probabilities. In order to prove lower bounds, we need to use other tricks, such as writing
$$\mathbb{E}_g^{\mathrm{WP}}\left[F(X)\right] = \sum_{k \geq 1} k \, \mathbb{P}_g^{\mathrm{WP}}\left(F(X) = k\right) = \sum_{k \geq 1} \mathbb{P}_g^{\mathrm{WP}}\left(F(X) \geq k\right). \tag{3.15}$$
This is the method that Mirzakhani used to prove [Mir13, Theorem 4.2], which states that there exists a constant $C > 0$ such that
$$\forall a \in (0,1), \mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sys}(X) \leq a\right) \geq C a^2.$$

- Can we be more precise and compute asymptotics as $g \to +\infty$, rather than upper and lower bounds? This was achieved in [MP19], where Mirzakhani and Petri proved that the length-counting function $\mathrm{N}_X^{\ell}(a,b)$ converges to a Poisson-law. They used the method of factorial moments: we deduce the convergence in distribution of a sequence of random variables $(F_i)_i$ from the convergence of its factorial moments $\mathbb{E}[F_i(F_i-1)\dots(F_i-r+1)]$ (see [Bol01, Theorem 1.23]).

3.2.3.3 Some generalisations

In this thesis, we will need two generalisations of equation (3.11) that will be useful to the following, but can be skipped at first read. The first generalisation is a new version, when we cut the surface into more than two pieces.

Lemma 3.15. *For any $k \geq 1$, $n \geq 0$, there exists a constant $C_{n,k} > 0$ such that, for all g, all integers $n_1, \dots, n_k$ such that $n_1 + \dots + n_k = n$,*
$$\sum_{\substack{g_1, \dots, g_k \geq 0 \\ g_1 + \dots + g_k = g}} \prod_{i=1}^{k} V_{g_i, n_i} \leq C_{n,k} \frac{V_{g,n}}{g^{3(k-1)}}.$$

It is obtained by a straightforward iteration of equation (3.11). The second generalisation is a new version of equation (3.11) with an additional power of the genus in the sum.

Lemma 3.16. *Let $N_1, N_2 \geq 0$, $n \geq 0$. There exists a constant $C_{n,N_1,N_2} > 0$ such that for any g satisfying $2g-2+n>0$, any n_1, n_2 such that $n_1+n_2=n$,*

$$\sum_{\substack{g_1+g_2=g \\ 2g_i+n_i>N_i+1}} \frac{V_{g_1,n_1+1}V_{g_2,n_2+1}}{(g_1+1)^{N_1}(g_2+1)^{N_2}} \leq C_{n,N_1,N_2} \frac{V_{g,n}}{(g+1)^{N_1+N_2+1}}. \tag{3.16}$$

We insist on the fact that the sum is only taken over the set of indices such that $2g_i+n_i>N_i+1$. As we will see in the following proof, this is necessary and the result is false if we add the terms such that $1<2g_i+n_i\leq N_i+1$.

Proof. The proof is an induction on the integer $N := N_1+N_2$, the case $N=0$ being equation (3.11).

Let $N_1, N_2 \geq 0$ such that $N>0$. We assume the property at the rank $N-1$. By symmetry, we can assume that $N_1 \geq N_2$, and in particular $N_1>0$. Then, for any n_1, n_2 such that $n_1+n_2=n$, the left hand side of equation (3.16) restricted to the terms where $g_1>0$ (which only exist if $g>0$) satisfies

$$\sum_{\substack{g_1+g_2=g \\ 2g_i+n_i>N_i+1 \\ g_1>0}} \frac{V_{g_1,n_1+1}V_{g_2,n_2+1}}{(g_1+1)^{N_1}(g_2+1)^{N_2}} = \mathcal{O}_{n_1}\left(\sum_{\substack{g_1'+g_2=g-1 \\ 2g_1'+n_1'>N_1 \\ 2g_2+n_2>N_2+1}} \frac{V_{g_1',n_1'+1}V_{g_2,n_2+1}}{(g_1+1)^{N_1-1}(g_2+1)^{N_2}} \right)$$

since $V_{g_1,n_1+1}/(g_1+1) = \mathcal{O}_{n_1}(V_{g_1-1,n_1+2})$ by equations (3.9) and (3.12), and by the change of indices $g_1' = g_1-1$, $n_1' = n_1+1$. By the induction hypothesis, this sum is

$$\mathcal{O}_{n+1,N_1-1,N_2}\left(\frac{V_{g-1,n+1}}{g^N}\right) = \mathcal{O}_{n,N_1,N_2}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\right)$$

by equations equations (3.9) and (3.12) again. As a consequence, we are left to bound the term for which $g_1=0$. If such a term is present in the sum, then the integer $n_1 = 2g_1+n_1$ satisfies $n_1>N_1+1$, and hence $n-n_2-1 \geq N_1+1$. Then, the term of the sum is

$$\frac{V_{0,n_1+1}V_{g,n_2+1}}{(0+1)^{N_1}(g+1)^{N_2}} = \mathcal{O}_n\left(\frac{V_{g,n}}{(g+1)^{N_2+n-n_2-1}}\right) = \mathcal{O}_n\left(\frac{V_{g,n}}{(g+1)^{N+1}}\right)$$

by equation (3.12) applied $n-n_2-1$ times. □

3.3 High-genus asymptotic expansion

The contents from this section will be presented in a new version of [AM20].

Mirzakhani and Zograf provided in [MZ15] an asymptotic expansion of the values at zero $V_{g,n} = V_{g,n}(0)$ of the Weil–Petersson volume polynomials for large genus g and fixed n:

$$V_{g,n} = C\,\frac{(2g-3+n)!(4\pi^2)^{2g-3+n}}{\sqrt{g}}\left(1+\frac{c_n^{(1)}}{g}+\cdots+\frac{c_n^{(N)}}{g^N}+\mathcal{O}_{n,N}\left(\frac{1}{g^{N+1}}\right)\right).$$

The form of this expansion had been conjectured by Zograf in [Zog08] following numerical experiments. The value of the universal constant C is unknown to this day; the data collected by Zograf seem to suggest that $C = 1/\sqrt{\pi}$.

The best known approximation of $V_{g,n}(\mathbf{x})$ for general $\mathbf{x}$ is the following, proved by Mirzakhani and Petri [MP19, Proposition 3.1]: for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\frac{V_{g,n}(\mathbf{x})}{V_{g,n}} = \prod_{j=1}^{n} \operatorname{sinhc}\left(\frac{x_j}{2}\right) + \mathcal{O}_n\left(\frac{\exp\left(\frac{x_1+\ldots+x_n}{2}\right)}{g+1}\right) \tag{3.17}$$

where sinhc is the function defined by

$$\operatorname{sinhc}(x) = \begin{cases} \frac{\sinh(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

This first-order approximation plays an essential part in the proof of the convergence of the counting function $\mathrm{N}_X^{\ell}(a,b)$ to a Poisson-law as $g \to +\infty$ [Mir13], and the two proofs of the fact that $\lambda_1 \geq \frac{3}{16} - \varepsilon$ typically [WX21, LW21].

In this section, we provide an asymptotic expansion in decreasing powers of g of the volume $V_{g,n}(\mathbf{x})$ for any $\mathbf{x}$.

Theorem 3.17. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g-2+n>0$, there exists a family of polynomial functions $(P_{g,n}^{(N,\varepsilon)})_{N\geq 0, \varepsilon\in\{-1,0,1\}^n}$ of n variables, such that for all $N \in \mathbb{N}_0$ and all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$,*

$$\frac{x_1 \ldots x_n V_{g,n}(\mathbf{x})}{V_{g,n}} = \sum_{\varepsilon\in\{-1,0,1\}^n} P_{g,n}^{(N,\varepsilon)}(\mathbf{x}) \exp\left(\frac{\varepsilon\cdot\mathbf{x}}{2}\right) + \mathcal{O}_{N,n}\left(\frac{\langle \mathbf{x}\rangle^{3N+2}}{(g+1)^{N+1}} \exp\left(\frac{x_1+\ldots+x_n}{2}\right)\right). \tag{3.18}$$

Furthermore, for all N, $\varepsilon \in \{-1,0,1\}^n$, the degree of $P_{g,n}^{(N,\varepsilon)}$ is $\mathcal{O}_N(1)$ and its coefficients are $\mathcal{O}_{N,n}(1)$.

This expansion was encountered in an ongoing project in collaboration with Nalini Anantharaman, aiming at proving that $\lambda_1 \geq \frac{1}{4} - \varepsilon$ typically. We explain the key role this expansion plays in this important conjecture in Section 6.2.

The proof does not provide an expression of the coefficients, and they are not uniquely defined. However, we provide an explicit expression for a second-order approximation.

Theorem 3.18. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g-2+n>0$, and any $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$,*

$$\frac{V_{g,n}(\mathbf{x})}{V_{g,n}} = f^{(1)}_{g,n}(\mathbf{x}) \prod_{i=1}^{n} \mathrm{sinhc}\left(\frac{x_i}{2}\right) + \mathcal{O}_n\left(\frac{\langle \mathbf{x} \rangle^3 \exp\left(\frac{x_1+\ldots+x_n}{2}\right)}{(g+1)^2}\right)$$

where $f^{(1)}_{g,n} : \mathbb{R}^n \to \mathbb{R}$ is the function defined by:

$$\begin{aligned} f^{(1)}_{g,n}(\mathbf{x}) = 1 &+ 8\frac{V_{g-1,n+1}}{V_{g,n}} \sum_{i=1}^{n} \left(-\frac{x_i^2}{16} - 2 + \frac{\cosh\left(\frac{x_i}{2}\right)+1}{\mathrm{sinhc}\left(\frac{x_i}{2}\right)}\right) \mathbb{1}_{g\geq 1} \\ &+ 4\frac{V_{g,n-1}}{V_{g,n}} \sum_{1 \leq i<j \leq n} \left(2 - \frac{\cosh\left(\frac{x_i}{2}\right)\cosh\left(\frac{x_j}{2}\right)+1}{\mathrm{sinhc}\left(\frac{x_i}{2}\right)\mathrm{sinhc}\left(\frac{x_j}{2}\right)}\right) \mathbb{1}_{n\geq 2}. \end{aligned}$$

This second-order estimate could be useful to improve bounds on the spectral gap of typical surfaces, as explained in Section 6.2.2.

Remark. Using Theorem 1.4 of Mirzakhani and Zograf [MZ15], we have that for $g \geq 1$,

$$\begin{aligned} \frac{V_{g,n-1}}{V_{g,n}} &= \frac{1}{8\pi^2 g} + \mathcal{O}_n\left(\frac{1}{g^2}\right) \\ \frac{V_{g-1,n+1}}{V_{g,n}} &= \frac{V_{g-1,n+1}}{V_{g,n-1}} \frac{V_{g,n-1}}{V_{g,n}} = \frac{1}{8\pi^2 g} + \mathcal{O}_n\left(\frac{1}{g^2}\right) \end{aligned}$$

and therefore the previous result still holds if we substitute $f^{(1)}_{g,n}$ by

$$1 + \frac{1}{\pi^2 g} \sum_{i=1}^{n} \left[-\frac{x_i^2}{16} - 2 + \frac{\cosh\left(\frac{x_i}{2}\right)+1}{\mathrm{sinhc}\left(\frac{x_i}{2}\right)} + \sum_{j\neq i}\left(\frac{1}{2} - \frac{\cosh\left(\frac{x_i}{2}\right)\cosh\left(\frac{x_j}{2}\right)+1}{4\,\mathrm{sinhc}\left(\frac{x_i}{2}\right)\mathrm{sinhc}\left(\frac{x_j}{2}\right)}\right)\right]. \tag{3.19}$$

A similar expansion at arbitrarily high precision, such that all of the coefficients are of the form $\frac{c_{n,k}}{g^k}$, can be deduced from [MZ15, Theorem 4.1, Lemma 4.7]. The expansion that we prove here is simpler, because we only focus on the behaviour in terms of $\mathbf{x}$ of the polynomial function $\mathbf{x} \mapsto V_{g,n}(\mathbf{x})$, while the authors of [MZ15] are focused mainly on the dependency on g. Notably, the second-order approximation equation (3.19) is new and does not directly follows from the method of [MZ15].

Example 9. *For $n=1$, we obtain that for any $g \geq 1$,*

$$\begin{aligned} \frac{V_{g,1}(x)}{V_{g,1}} &= \left[1 - \left(\frac{x^2}{2}+16\right)\frac{V_{g-1,2}}{V_{g,1}}\right]\mathrm{sinhc}\left(\frac{x}{2}\right) + 8\frac{V_{g-1,2}}{V_{g,1}}\left(\cosh\left(\frac{x}{2}\right)+1\right) + \mathcal{O}\left(\frac{\langle x\rangle^3 e^{\frac{x}{2}}}{g^2}\right) \\ &= \mathrm{sinhc}\left(\frac{x}{2}\right) + \frac{1}{\pi^2 g}\left[-\left(\frac{x^2}{16}+2\right)\mathrm{sinhc}\left(\frac{x}{2}\right) + \cosh\left(\frac{x}{2}\right)+1\right] + \mathcal{O}\left(\frac{\langle x\rangle^3 e^{\frac{x}{2}}}{g^2}\right). \end{aligned}$$

3.3.1 The leading term

Let us first start this section by a detailed proof of an estimate similar to the Mirzakhani–Petri first-term estimate, equation (3.17). This will allow us to present a few ideas that will be used in the general case.

More precisely, we prove the following.

Proposition 3.19. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g-2+n>0$, and any length vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$,*

$$\frac{V_{g,n}(\mathbf{x})}{V_{g,n}} = \prod_{j=1}^n \mathrm{sinhc}\Big(\frac{x_j}{2}\Big) + \mathcal{O}_n\left(\frac{|\mathbf{x}| \exp\big(\frac{x_1+\ldots+x_n}{2}\big)}{g+1}\right).$$

3.3.1.1 The coefficient estimate

Proposition 3.19 comes as a consequence of the expression of the volume polynomials,

$$V_{g,n}(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^n} c_{g,n}(\alpha) \prod_{j=1}^n \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}$$

together with the following first-order estimate of the coefficients $(c_{g,n}(\alpha))_\alpha$.

Lemma 3.20. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g-2+n>0$, and any multi-index $\alpha \in \mathbb{N}_0^n$,*

$$\frac{c_{g,n}(\alpha)}{V_{g,n}} = 1 + \mathcal{O}_n\left(\frac{|\alpha|^2}{g+1}\right).$$

Remark. We insist on the fact that this estimate is true *for any* α and not only for multi-indices α such that $|\alpha| \leq 3g-3+n$. Indeed, if $|\alpha| > 3g-3+n$, then the bound is trivial, because $c_{g,n}(\alpha)=0$ and $\frac{|\alpha|^2}{g+1} \gg 1$.

Let us first prove that Lemma 3.20 implies Proposition 3.19.

Proof. Using the expression of sinhc as a power series, we can write

$$V_{g,n}(\mathbf{x}) - V_{g,n} \prod_{j=1}^n \mathrm{sinhc}\Big(\frac{x_j}{2}\Big) = \sum_{\alpha \in \mathbb{N}^n} (c_{g,n}(\alpha) - V_{g,n}) \prod_{j=1}^n \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}.$$

As a consequence, by the triangle inequality and Lemma 3.20,

$$\left| V_{g,n}(\mathbf{x}) - V_{g,n} \prod_{j=1}^n \mathrm{sinhc}\Big(\frac{x_j}{2}\Big) \right| \leq C_n \frac{V_{g,n}}{g+1} \sum_{\alpha \in \mathbb{N}^n} |\alpha|_\infty^2 \prod_{j=1}^n \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}. \tag{3.20}$$

We cut the sum over α in equation (3.20) depending on the index j for which $|\alpha|_\infty = \alpha_j$. Since $\alpha_j^2 \leq (2\alpha_j+1)(2\alpha_j)/4$,

$$\sum_{\alpha_j=1}^{+\infty} \frac{\alpha_j^2 \, x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!} \leq \sum_{k=0}^{+\infty} \frac{x_j^{2k+2}}{2^{2k}(2k+1)!} = 2x_j \sinh\Big(\frac{x_j}{2}\Big) \leq |\mathbf{x}| \exp\Big(\frac{x_j}{2}\Big).$$

Also, for any i,

$$\sum_{\alpha_i=0}^{+\infty} \frac{x_i^{2\alpha_i}}{2^{2\alpha_i}(2\alpha_i+1)!} \leq \sum_{k=0}^{+\infty} \frac{x_i^k}{2^k k!} \leq \exp\left(\frac{x_i}{2}\right).$$

This allows us to conclude. □

3.3.1.2 Estimate of the discrete derivative

Lemma 3.20 states that the coefficients $c_{g,n}(\alpha)$ are almost constant, equal to the value $V_{g,n} = c_{g,n}(0)$. We will prove this by estimating the *discrete derivatives* of the coefficients.

Notation 3. Let $n \geq 1$ be an integer. For any $i \in \{1, \ldots, n\}$, we set δ_i to be the discrete derivative with respect to the i-th coordinate, defined for function $v : \mathbb{N}^n \to \mathbb{R}$ by:

$$\forall \alpha \in \mathbb{N}^n, \delta_i v(\alpha) := v(\alpha) - v(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n).$$

The technical step to prove Lemma 3.20 is the following discrete derivative estimate.

Lemma 3.21. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g - 2 + n > 0$, and any multi-index $\alpha \in \mathbb{N}^n$,*

$$\delta_1 c_{g,n}(\alpha) = \mathcal{O}_n\left(\frac{\langle \alpha \rangle}{g+1}\right).$$

Proof. The result is trivially true when $2g - 2 + n = 1$, so we can assume that it is not the case and apply Mirzakhani's topological recursion formula, Theorem 3.9.

$$\delta_1 c_{g,n}(\alpha) = \sum_{j=2}^{n} \delta_1 \mathcal{A}_{g,n}^{(j)}(\alpha) + \delta_1 \mathcal{B}_{g,n}(\alpha) + \sum_{\iota \in \mathcal{I}_{g,n}} \delta_1 \mathcal{C}_{g,n}^{(\iota)}(\alpha).$$

We prove that each of these three term is $\mathcal{O}_n\left(\langle \alpha \rangle V_{g,n}/(g+1)\right)$ separately thanks to their respective expressions (equations (3.3) to (3.5)).

Let us begin by the first sum. Let $j \geq 2$. We write equation (3.3) for $\mathcal{A}_{g,n}^{(j)}(\alpha)$ and $\mathcal{A}_{g,n}^{(j)}(\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$, isolating the term $i = 0$ in the first sum and using a change of index on the sum over $i \geq 1$. We obtain

$$\begin{aligned}\delta_1 \mathcal{A}_{g,n}^{(j)}(\alpha) = 4\ (2\alpha_j + 1)\ c_{g,n-1}(\alpha_1 + \alpha_j - 1, \alpha_2, \ldots, \hat{\alpha}_j, \ldots, \alpha_n) \\ + 8\ (2\alpha_j + 1) \sum_{i=0}^{+\infty} (u_{i+1} - u_i)\ c_{g,n-1}(i + \alpha_1 + \alpha_j, \alpha_2, \ldots, \hat{\alpha}_j, \ldots, \alpha_n).\end{aligned}$$

But we know by Lemma 3.11 that for any multi-index $\beta \in \mathbb{N}^{n-1}$,

$$0 \leq c_{g,n-1}(\beta) \leq V_{g,n-1}.$$

Then,

$$0 \leq \delta_1 \mathcal{A}^{(j)}_{g,n}(\alpha) \leq 8(2\alpha_j+1)V_{g,n-1} = \mathcal{O}_n\left(\frac{\langle\alpha_j\rangle V_{g,n}}{g}\right)$$

because $\sum_{i=0}^{+\infty}(u_{i+1}-u_i) = \lim u - u_0 = 1-\frac{1}{2} = \frac{1}{2}$ (see Lemma 3.10), and by equation (3.12). Since there are $n-1 = \mathcal{O}_n(1)$ possible values for j,

$$\sum_{j=2}^{n} \delta_1 \mathcal{A}^{(j)}_{g,n}(\alpha) = \mathcal{O}_n\left(\frac{\langle\alpha\rangle V_{g,n}}{g}\right). \tag{3.21}$$

We now look at the non-separating term $\delta_1\mathcal{B}_{g,n}(\alpha)$. Note that this term only appears whenever $g \geq 1$. By the same method, this time applied to equation (3.4),

$$\begin{aligned}\delta_1\mathcal{B}_{g,n}(\alpha) =& 8 \sum_{k_1+k_2=\alpha_1-2} c_{g-1,n+1}(k_1,k_2,\alpha_2,\dots,\alpha_n)\\ &+16\sum_{i=0}^{+\infty}\sum_{k_1+k_2=i+\alpha_1-1}(u_{i+1}-u_i)\, c_{g-1,n+1}(k_1,k_2,\alpha_2,\dots,\alpha_n)\cdot\end{aligned}$$

By Lemma 3.11, for any multi-index $\beta \in \mathbb{N}^{n+1}$,

$$0 \leq c_{g-1,n+1}(\beta) \leq V_{g-1,n+1} = \mathcal{O}_n\left(\frac{V_{g,n}}{g}\right)$$

thanks to equations (3.9) and (3.12). Then,

$$\delta_1\mathcal{B}_{g,n}(\alpha) = \mathcal{O}_n\left(\alpha_1\frac{V_{g,n}}{g} + \sum_{i=0}^{+\infty}(i+\alpha_1)(u_{i+1}-u_i)\frac{V_{g,n}}{g}\right) = \mathcal{O}_n\left(\frac{\langle\alpha_1\rangle V_{g,n}}{g+1}\right) \tag{3.22}$$

because $\frac{1}{g} \leq \frac{2}{g+1}$, and the series $\sum_i(u_{i+1}-u_i)$ and $\sum_i i(u_{i+1}-u_i)$ converge.

Finally, for any configuration $\iota = (g', I, J) \in \mathcal{I}_{g,n}$,

$$\begin{aligned}\delta_1\mathcal{C}^{(\iota)}_{g,n}(\alpha) =& 8\sum_{k_1+k_2=\alpha_1-2} c_{g',|I|+1}(k_1,\alpha_I)\, c_{g-g',|J|+1}(k_2,\alpha_J)\\ &+16\sum_{i=0}^{+\infty}\sum_{k_1+k_2=i+\alpha_1-1}(u_{i+1}-u_i)\, c_{g',|I|+1}(k_1,\alpha_I)\, c_{g-g',|J|+1}(k_2,\alpha_J)\\ =& \mathcal{O}_n\left(\langle\alpha_1\rangle V_{g',|I|+1}V_{g-g',|J|+1}\right).\end{aligned}$$

As a consequence,

$$\sum_{\iota\in\mathcal{I}_{g,n}} \delta_1\mathcal{C}^{(\iota)}_{g,n} = \mathcal{O}_n\left(\langle\alpha_1\rangle \sum_{\substack{g_1+g_2=g\\ n_1+n_2=n-1\\ 2g_i+n_i>1}} V_{g_1,n_1+1}V_{g_2,n_2+1}\right)$$

and therefore
$$\sum_{\iota \in \mathcal{I}_{g,n}} \delta_1 \mathcal{C}^{(\iota)}_{g,n} = \mathcal{O}_n\left(\frac{\langle \alpha_1 \rangle V_{g,n-1}}{g+1}\right) = \mathcal{O}_n\left(\frac{\langle \alpha_1 \rangle V_{g,n}}{(g+1)^2}\right) \tag{3.23}$$
by equations (3.11) and (3.12). The conclusion follows from adding equations (3.21) to (3.23). $\square$

3.3.1.3 Discrete integration

In order to go from the estimate of the discrete derivative to an estimate on the actual coefficients, we use the following discrete integration lemma.

Notation 4. For an integer i, we set

$$\mathbf{0}^i := \underbrace{0, \dots, 0}_{i \text{ zeroes}}.$$

Lemma 3.22. *Let $n \geq 1$ be an integer. For any $v : \mathbb{N}^n \to \mathbb{R}$,*

$$v(\alpha) = v(\mathbf{0}^n) - \sum_{i=1}^{n} \sum_{k=0}^{\alpha_i - 1} \delta_i v(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n).$$

Proof. We observe that for any index i, the sum over k is a telescopic sum:

$$\begin{aligned} S_i &:= \sum_{k=0}^{\alpha_i - 1} \delta_i v(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n) \\ &= \sum_{k=0}^{\alpha_i - 1} \left[v(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n) - v(\mathbf{0}^{i-1}, k+1, \alpha_{i+1}, \dots, \alpha_n) \right] \\ &= v(\mathbf{0}^{i}, \alpha_{i+1}, \dots, \alpha_n) - v(\mathbf{0}^{i-1}, \alpha_i, \alpha_{i+1}, \dots, \alpha_n). \end{aligned}$$

As a consequence, $\sum_{i=1}^{n} S_i = v(\mathbf{0}^n) - v(\alpha)$, which what was claimed. $\square$

Lemma 3.20 directly follows from this formula and the estimate on the discrete derivatives, Lemma 3.21.

3.3.2 The second term

Let us now compute the second term of the asymptotic expansion of $V_{g,n}(\mathbf{x})$ in powers of $1/(g+1)$, prove Theorem 3.18. We start by estimating the volume coefficients $(c_{g,n}(\alpha))_\alpha$ up to errors of size $V_{g,n}/(g+1)^2$.

Proposition 3.23. *For any $n \in \mathbb{N}$, $g \in \mathbb{N}_0$, such that $2g - 2 + n > 0$,*

$$\forall \alpha \in \mathbb{N}_0^n, \quad c_{g,n}(\alpha) = \phi^{(1)}_{g,n}(\alpha) + \mathcal{O}_n\left(\frac{|\alpha|^4 V_{g,n}}{(g+1)^2}\right)$$

where $\phi^{(1)}_{g,n} : \mathbb{N}_0^n \to \mathbb{R}$ *is the function defined by:*

$$\phi^{(1)}_{g,n}(\alpha) = V_{g,n} + 8V_{g-1,n+1} \sum_{i=1}^{n} \left(-\frac{p_2(\alpha_i)}{4} - 2 + p_1(\alpha_i) + \mathbb{1}_{\alpha_i=0} \right) + 4V_{g,n-1} \sum_{1 \leq i < j \leq n} \left(2 - p_1(\alpha_i)p_1(\alpha_j) - \mathbb{1}_{\alpha_i=\alpha_j=0} \right)$$

and $p_1(X) := 2X+1$*,* $p_2(X) := (2X+1)(2X)$*.*

3.3.2.1 Estimate of the discrete derivative

The expansion of $(c_{g,n}(\alpha))_{\alpha \in \mathbb{N}_0^n}$ is obtained thanks to the following estimate on the discrete derivatives $\delta_1 c_{g,n}(\alpha)$ obtained using Mirzakhani's integration formula.

Lemma 3.24. *For any* $n \in \mathbb{N}$*,* $g \in \mathbb{N}_0$*, such that* $2g-2+n>0$*,*

$$\forall \alpha \in \mathbb{N}_0^n, \quad \delta_1 c_{g,n}(\alpha) = \psi^{(1)}_{g,n}(\alpha) + \mathcal{O}_n\left(\langle \alpha \rangle^3 \frac{V_{g,n}}{(g+1)^2} \right)$$

where $\psi_{g,n} : \mathbb{N}_0^n \to \mathbb{R}$ *is the function defined by:*

$$\psi_{g,n}(\alpha) = 4(4\alpha_1 - 1 + 2\,\mathbb{1}_{\alpha_1=0})V_{g-1,n+1}\mathbb{1}_{g\geq 1} + 4\sum_{j=2}^{n}(4\alpha_j + 2 - \mathbb{1}_{\alpha_1=\alpha_j=0})V_{g,n-1}\mathbb{1}_{n\geq 2}.$$

Proof. We follow the same proof as for Lemma 3.21, this time replacing all the coefficients appearing in Mirzakhani's recursion formula by their first order approximation. Indeed, by Theorem 3.9,

$$\delta_1 c_{g,n}(\alpha) = \sum_{j=2}^{n} \delta_1 \mathcal{A}^{(j)}_{g,n}(\alpha) + \delta_1 \mathcal{B}_{g,n}(\alpha) + \sum_{\iota \in \mathcal{I}_{g,n}} \delta_1 \mathcal{C}^{(\iota)}_{g,n}(\alpha).$$

We estimate each term up to errors of size $V_{g,n}/(g+1)^2$.

We notice that the first term is zero if $n=1$. Let us assume otherwise, and take $j \in \{2, \ldots, n\}$. As before, we can write

$$\begin{aligned}\delta_1 \mathcal{A}^{(j)}_{g,n}(\alpha) &= 4(2\alpha_j+1)\, c_{g,n-1}(\alpha_1+\alpha_j-1, \alpha_2, \ldots, \hat{\alpha}_j, \ldots, \alpha_n) \\ &\quad + 8(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_{i+1}-u_i)\, c_{g,n-1}(i+\alpha_1+\alpha_j, \alpha_2, \ldots, \hat{\alpha}_j, \ldots, \alpha_n) \\ &=: T_1 + T_2.\end{aligned}$$

- Term T_1:
  - If $\alpha_1 = \alpha_j = 0$, then $\alpha_1 + \alpha_j - 1 < 0$ and therefore $T_1 = 0$.

– Otherwise, by Lemma 3.20 applied to the coefficient appearing in T_1,

$$T_1 = 4(2\alpha_j+1)V_{g,n-1} + \mathcal{O}_n\left((2\alpha_j+1)|\alpha|^2\frac{V_{g,n-1}}{g+1}\right) = 4(2\alpha_j+1)V_{g,n-1} + \mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right)$$

by equation (3.12).

- Term T_2:

$$\begin{aligned} T_2 &= 8(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_{i+1}-u_i)\; c_{g,n-1}(i+\alpha_1+\alpha_j,\alpha_2,\ldots,\hat{\alpha}_j,\ldots,\alpha_n) \\ &= 8(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_{i+1}-u_i)\left(V_{g,n-1}+\mathcal{O}_n\left((i+|\alpha|)^2\frac{V_{g,n-1}}{g+1}\right)\right) \end{aligned}$$

by Lemma 3.20. But $\sum_{i=0}^{+\infty}(u_{i+1}-u_i)=\frac{1}{2}$ and $\sum_{i=0}^{+\infty} i(u_{i+1}-u_i)$ converges by Lemma 3.10. Therefore, by equation (3.12) again,

$$T_2 = 4(2\alpha_j+1)V_{g,n-1} + \mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right).$$

As a conclusion, we have proved that

$$\delta_1\mathcal{A}^{(j)}_{g,n}(\alpha) = \begin{cases} 4V_{g,n-1}+\mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right) & \text{if } \alpha_1=\alpha_j=0 \\ 8(2\alpha_j+1)V_{g,n-1}+\mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right) & \text{otherwise.}\end{cases}$$

We rewrite this equation as

$$\delta_1\mathcal{A}^{(j)}_{g,n}(\alpha) = 4(4\alpha_j+2-\mathbb{1}_{\alpha_1=\alpha_j=0})V_{g,n-1} + \mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right).$$

By the same process, we prove that, when $g\geq 1$,

$$\delta_1\mathcal{B}_{g,n}(\alpha) = 4(4\alpha_1-1+2\,\mathbb{1}_{\alpha_1=0})V_{g-1,n+1} + \mathcal{O}_n\left(\langle\alpha\rangle^3\frac{V_{g,n}}{(g+1)^2}\right).$$

Indeed,

$$\begin{aligned} \delta_1\mathcal{B}_{g,n}(\alpha) =& 8\sum_{k_1+k_2=\alpha_1-2} c_{g-1,n+1}(k_1,k_2,\alpha_2,\ldots,\alpha_n) \\ &+16\sum_{i=0}^{+\infty}\sum_{k_1+k_2=i+\alpha_1-1}(u_{i+1}-u_i)\; c_{g-1,n+1}(k_1,k_2,\alpha_2,\ldots,\alpha_n) \\ =& T_1+T_2. \end{aligned}$$

- On the one hand, T_1 is equal to zero if $\alpha_1 = 0$, and otherwise,

$$T_1 = 8(\alpha_1 - 1)V_{g-1,n+1} + \mathcal{O}_n\left(\langle\alpha\rangle^3 \frac{V_{g,n}}{(g+1)^2}\right).$$

- On the other hand,

$$\begin{aligned} T_2 &= 16\sum_{i=0}^{+\infty}(\alpha_1 + i)(u_{i+1} - u_i)V_{g-1,n+1} + \mathcal{O}_n\left(\langle\alpha\rangle^3 \frac{V_{g,n}}{g^2}\right) \\ &= 4(2\alpha_1 + 1)V_{g-1,n+1} + \mathcal{O}_n\left(\langle\alpha\rangle^3 \frac{V_{g,n}}{(g+1)^2}\right) \end{aligned}$$

  because $\sum_{i=0}^{+\infty}(u_{i+1} - u_i) = \frac{1}{2}$ and $\sum_{i=0}^{+\infty} i(u_{i+1} - u_i) = \frac{1}{4}$ [MZ15, Lemma 2.1].

Finally, we observe that we have proved, when computing the first order term, that

$$\sum_{\iota \in \mathcal{I}_{g,n}} \delta_1 \mathcal{C}_{g,n}^{(\iota)}(\alpha) = \mathcal{O}_n\left(\langle\alpha\rangle \frac{V_{g,n}}{(g+1)^2}\right)$$

so the separating term does not contribute to the second-order approximation.

Summing the different terms $\delta_1 \mathcal{A}_{g,n}^{(j)}$ for $j \in \{2, \ldots, n\}$ and $\delta_1 \mathcal{B}_{g,n}(\alpha)$ leads to the claim. □

3.3.2.2 Discrete integration

We can now prove Proposition 3.23 using Lemma 3.24 and discrete integration.

Proof. By Lemma 3.22, 3.24 and the symmetry of the coefficients, we can write

$$\begin{aligned} c_{g,n}(\alpha) &= V_{g,n} - \sum_{i=1}^{n}\sum_{k=0}^{\alpha_i - 1} \delta_i c_{g,n}(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \ldots, \alpha_n) \\ &= V_{g,n} - \sum_{i=1}^{n}\sum_{k=0}^{\alpha_i - 1} \psi_{g,n}^{(1)}(k, \mathbf{0}^{i-1}, \alpha_{i+1}, \ldots, \alpha_n) + \mathcal{O}_n\left(\langle\alpha\rangle^4 \frac{V_{g,n}}{(g+1)^2}\right) \\ &= V_{g,n} - 4V_{g-1,n+1}T_1 - 4V_{g,n-1}T_2 + \mathcal{O}_n\left(\langle\alpha\rangle^4 \frac{V_{g,n}}{(g+1)^2}\right) \end{aligned}$$

where

$$\begin{aligned} T_1 &:= \sum_{i=1}^{n}\sum_{k=0}^{\alpha_i - 1}(4k - 1 + 2\,\mathbb{1}_{k=0}) \\ \text{and } T_2 &:= \sum_{i=1}^{n}\sum_{k=0}^{\alpha_i - 1}\left[(i-1)(2 - \mathbb{1}_{k=0}) + \sum_{j=i+1}^{n}(4\alpha_j + 2 - \mathbb{1}_{k=\alpha_j=0})\right]. \end{aligned}$$

- On the one hand, we observe that the term $\mathbb{1}_{k=0}$ contributes to the sum if and only if $\alpha_i > 0$, and hence

$$\begin{aligned} T_1 &= \sum_{i=1}^{n} \left(2\alpha_i^2 - 3\alpha_i + 2\,\mathbb{1}_{\alpha_i>0}\right) \\ &= \sum_{i=1}^{n} \left(\frac{(2\alpha_i+1)(2\alpha_i)}{2} + 4 - 2(2\alpha_i+1) - 2\,\mathbb{1}_{\alpha_i=0} \right). \end{aligned}$$

- On the other hand,

$$\begin{aligned} T_2 &= \sum_{i=1}^{n} \left[(i-1)(2\alpha_i - \mathbb{1}_{\alpha_i>0}) + \sum_{j=i+1}^{n} (4\alpha_i\alpha_j + 2\alpha_i - \mathbb{1}_{\alpha_i>0}\mathbb{1}_{\alpha_j=0}) \right] \\ &= 4\sum_{i<j} \alpha_i\alpha_j + 2(n-1)\sum_{i=1}^{n}\alpha_i + \sum_{i>j}(\mathbb{1}_{\alpha_i=0} - 1) + \sum_{i<j}(\mathbb{1}_{\alpha_i=0} - 1)\mathbb{1}_{\alpha_j=0} \\ &= \sum_{i<j}\left((2\alpha_i+1)(2\alpha_j+1) - 2 + \mathbb{1}_{\alpha_i=\alpha_j=0}\right). \end{aligned}$$

□

3.3.2.3 Proof of the volume estimate

In order to conclude to the proof of Theorem 3.18, we need to compute

$$\sum_{\alpha\in\mathbb{N}_0^n} \phi_{g,n}^{(1)}(\alpha) \prod_{j=1}^{n} \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}$$

where $\phi_{g,n}^{(1)}(\alpha)$ is the approximation of the coefficient $c_{g,n}(\alpha)$ from Proposition 3.23. We have expressed $\phi_{g,n}^{(1)}$ in terms of polynomials $p_1(X) = 2X+1$ and $p_2(X) = (2X+1)(2X)$ in order to make this computation easier.

Since this will be useful for the general case, let us set some notations.

Notation 5. For any integer $k \geq 0$, we set

$$p_k(X) = \prod_{j=0}^{k-1}(2X+1-j) = (2X+1)(2X)(2X-1)\dots(2X+2-k),$$

with the convention that the empty product is equal to one so that $p_0(X) = 1$.

Then, we can prove the following.

Lemma 3.25. *Let $k \in \mathbb{N}_0$. For any $x \in \mathbb{R}$,*

$$\sum_{\alpha=0}^{+\infty} \frac{p_k(\alpha)\,x^{2\alpha}}{2^{2\alpha}(2\alpha+1)!} = \begin{cases} \frac{x^k}{2^k}\,\mathrm{sinhc}\left(\frac{x}{2}\right) & \text{if } k \text{ is even} \\ \frac{x^{k-1}}{2^{k-1}}\cosh\left(\frac{x}{2}\right) & \text{if } k \text{ is odd.} \end{cases}$$

Since the polynomials $(p_k)_{k\geq 0}$ are a basis of the set of polynomials $\mathbb{R}[X]$, we will be able to express any polynomial function of α in terms of them.

We can now finish the proof of Theorem 3.18.

Proof. By Proposition 3.23 and the expression of $V_{g,n}(\mathbf{x})$ in terms of $(c_{g,n}(\alpha))_\alpha$,

$$\frac{V_{g,n}(\mathbf{x})}{V_{g,n}} = f^{(1)}_{g,n}(\mathbf{x}) \prod_{j=1}^{n} \operatorname{sinhc}\left(\frac{x_j}{2}\right) + \mathcal{O}_n\left(\frac{V_{g,n}}{(g+1)^2} \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \alpha\neq \mathbf{0}^n}} |\alpha|_\infty^4 \prod_{j=1}^{n} \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}\right)$$

where

$$f^{(1)}_{g,n}(\mathbf{x}) = \frac{1}{\prod_{j=1}^n \operatorname{sinhc}\left(\frac{x_j}{2}\right)} \sum_{\alpha\in\mathbb{N}_0^n} \frac{\phi^{(1)}_{g,n}(\alpha)}{V_{g,n}} \prod_{j=1}^{n} \frac{x_j^{2\alpha_j}}{2^{2\alpha_j}(2\alpha_j+1)!}.$$

We replace $\phi^{(1)}_{g,n}$ by its expression from Proposition 3.23, and find the claimed expression by Lemma 3.25. The remainder is

$$\mathcal{O}_n\left(\frac{V_{g,n}}{(g+1)^2}\langle \mathbf{x}\rangle^3 \exp\left(\frac{x_1+\ldots+x_n}{2}\right)\right)$$

because for all $y\geq 0$,

$$\sum_{k=1}^{+\infty} \frac{k^4 y^{2k}}{2^{2k}(2k+1)!} \leq y^2 + \sum_{k=2}^{+\infty} \frac{p_4(k) y^{2k}}{2^{2k}(2k+1)!} = y^2 + \frac{y^4}{2^4}\operatorname{sinhc}\left(\frac{y}{2}\right) \leq \langle y\rangle^3 \exp\left(\frac{y}{2}\right).$$

□

3.3.3 Proof of the asymptotic expansion

We now focus on the proof of an asymptotic expansion of $V_{g,n}(\mathbf{x})$, at higher orders, Theorem 3.17. The proof goes at follows.

- In Section 3.3.3.1, we prove a technical estimate on discrete derivatives of the volume coefficients, of the form

$$\delta^{\mathbf{m}} c_{g,n}(\alpha) = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^N}\langle\alpha\rangle^N\right).$$

  for large enough $\mathbf{m}$ and α (depending on N) – see Theorem 3.26.

- We then conclude quite straightforwardly in Section 3.3.3.2 using discrete Taylor expansions (Lemma 3.30).

3.3.3.1 Estimate of discrete derivatives

Let us first start by proving an estimate of the iterated derivatives on the volume coefficients $(c_{g,n}(\alpha))_\alpha$.

Theorem 3.26. *There exists an increasing sequence $(a_N)_{N\geq 0}$ of integers such that, for any $n \in \mathbb{N}$, $N \in \mathbb{N}_0$, there exists a constant $C_{n,N} > 0$ satisfying the following. For any integer g such that $2g-2+n>0$, any multi-index $\mathbf{m}$ of norm $|\mathbf{m}| \in \{2N-1, 2N\}$ and any multi-index $\alpha \in \mathbb{N}_0^n$ such that $\forall i, (m_i \neq 0 \Rightarrow \alpha_i \geq a_N)$,*

$$|\delta^{\mathbf{m}} c_{g,n}(\alpha)| \leq C_{n,N} \frac{V_{g,n} \langle \alpha \rangle^N}{(g+1)^N}.$$

The proof uses the topological recursion formula (3.2). In order to be able to apply the discrete differential operator δ on its terms (3.4) and (3.5), we will use the following lemma.

Lemma 3.27. *Let us consider a sequence $(v_k)_{k\geq 0}$ of the form*

$$v_k = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} c_{k_1,k_2}$$

where $(c_{k_1,k_2})_{k_1,k_2\geq 0}$ is a family of real numbers. Then, for any integers $m \geq 1$ and $k \geq 0$,

$$\delta^m v_k = \sum_{\substack{k_1+k_2=k \\ k_1\geq k_2}} \delta_1^m c_{k_1,k_2} + \sum_{\substack{k_1+k_2=k \\ k_1<k_2}} \delta_2^m c_{k_1,k_2} - \sum_{\substack{m_1+m_2=m-1 \\ m_1,m_2\geq 0}} \delta_1^{m_1} \delta_2^{m_2} c_{\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor +1}.$$

Proof. We prove the formula by induction on the integer m. The initialisation at $m=0$ is trivial, since the last sum is empty in this case. For $m \geq 0$, let us assume the property at the rank m.

Let k be an integer. We assume that $k = 2p+1$ is an odd number. By definition of the operator δ and thanks to the induction hypothesis,

$$\begin{aligned}
\delta^{m+1} v_k =& \delta^m v_k - \delta^m v_{k+1} \\
=& \sum_{\substack{k_1+k_2=2p+1 \\ k_1\geq k_2}} \delta_1^m c_{k_1,k_2} - \sum_{\substack{k_1+k_2=2p+2 \\ k_1\geq k_2}} \delta_1^m c_{k_1,k_2} \\
&+ \sum_{\substack{k_1+k_2=2p+1 \\ k_1<k_2}} \delta_2^m c_{k_1,k_2} - \sum_{\substack{k_1+k_2=2p+2 \\ k_1<k_2}} \delta_2^m c_{k_1,k_2} \\
&- \sum_{m_1+m_2=m-1} \left(\delta_1^{m_1} \delta_2^{m_2} c_{p+1,p+1} - \delta_1^{m_1} \delta_2^{m_2} c_{p+1,p+2} \right) \\
=:& S_1 - S_2 + S_3 - S_4 - S_5.
\end{aligned}$$

Let us perform a change of indices $k_1' = k_1 - 1$ in the sum S_2, singling out the term of S_2 for which $k_1 = k_2 = p+1$, so that we sum over the same set of indices as S_1:

$$S_1 - S_2 = \sum_{\substack{k_1+k_2=2p+1 \\ k_1 \geq k_2}} \delta_1^{m+1} c_{k_1,k_2} - \delta_1^m c_{p+1,p+1}.$$

There is no boundary term when we do the same to S_3 and S_4, now changing the index k_2:

$$S_3 - S_4 = \sum_{\substack{k_1+k_2=2p+1 \\ k_1 < k_2}} \delta_2^{m+1} c_{k_1,k_2}.$$

We then observe that $\delta_1^m c_{p+1,p+1} + S_5$ is equal to

$$\delta_1^m c_{p+1,p+1} + \sum_{m_1+m_2=m-1} \delta_1^{m_1} \delta_2^{m_2+1} c_{p+1,p+1} = \sum_{m_1+m_2=m} \delta_1^{m_1} \delta_2^{m_2} c_{p+1,p+1}$$

which leads to the claimed expression for $\delta^{m+1} v_k$. The proof when k is even is the same. □

We can now proceed to the proof of Theorem 3.26.

Proof. The proof is an induction on the integer N. The case $N = 0$ is trivial: indeed, by Lemma 3.11,

$$\forall \alpha \in \mathbb{N}_0^n, |c_\alpha^{(g,n)}| \leq V_{g,n}.$$

In order to be able to use Mirzakhani's recursion formula, we observe that the result is trivial when $2g - 2 + n = 1$, for any $N > 0$. Indeed,

- if $(g,n) = (0,3)$, then $\delta^{\mathbf{m}} c_{0,3}(\alpha_1, \alpha_2, \alpha_3) = 0$ for any $\mathbf{m}$ and any $\alpha \neq \mathbf{0}^3$
- if $(g,n) = (1,1)$, then $\delta^m c_{1,1}(\alpha) = 0$ for any m and any $\alpha \geq 2$.

As a consequence, provided that $a_N \geq 2$ for $N \geq 1$, the result is automatic.

For an integer $N \geq 0$, let us assume the result to hold at the rank N, and prove it at the rank $N+1$. Let us consider integers g, n such that $2g-2+n > 1$. Let $\mathbf{m}, \alpha \in \mathbb{N}_0^n$ be multi-indices, such that:

- $|\mathbf{m}| = m_1 + \ldots + m_n = 2N + 1$
- $\forall i, (m_i \neq 0 \Rightarrow \alpha_i \geq a_{N+1})$,

where a_{n+1} is an integer that will be determined during the proof. By symmetry of the volume coefficients, we can assume that $m_1 > 0$.

Let us write the coefficient $\delta^{\mathbf{m}} c_\alpha^{(g,n)}$ using Mirzakhani's topological recursion formula, equation (3.2).

$$\begin{aligned}|\delta^{\mathbf{m}} c_{g,n}(\alpha)| &\leq \sum_{j=2}^{n} |\delta^{\mathbf{m}} \mathcal{A}_{g,n}^{(j)}(\alpha)| + |\delta^{\mathbf{m}} \mathcal{B}_{g,n}(\alpha)| + \sum_{\iota \in \mathcal{I}_{g,n}} |\delta^{\mathbf{m}} \mathcal{C}_{g,n}^{(\iota)}(\alpha)| \\ &=: (A) + (B) + (C).\end{aligned}$$

We shall estimate these different contributions successively, keeping in mind that the aim is to establish a decay for each term at the rate $V_{g,n}\langle\alpha\rangle^{N+1}/(g+1)^{N+1}$.

Estimate of the term (A). The term (A) is equal to zero if $n=1$, and then there is nothing to be proved. Otherwise, let $j \in \{2,\dots,n\}$. By equation (3.3),

$$\mathcal{A}^{(j)}_{g,n}(\alpha) = 8\,(2\alpha_j+1)\sum_{i=0}^{+\infty} u_i\; c_{g,n-1}(\tilde{\alpha}^{(i)})$$

where $\tilde{\alpha}^{(i)} := (i+\alpha_1+\alpha_j-1, \alpha_2, \dots, \hat{\alpha}_j, \dots, \alpha_n)$. By a change of variable in the sum, if we set $u_{-1}=0$, then

$$\delta_1\mathcal{A}^{(j)}_{g,n}(\alpha) = 8\,(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_i-u_{i-1})\; c_{g,n-1}(\tilde{\alpha}^{(i)}).$$

- Let us first treat the case when $m_j=0$. By applying the discrete derivatives $\delta_1^{m_1-1}$ and $\delta_i^{m_i}$ for $i \notin \{1,j\}$, we observe that

  $$\delta^{\mathbf{m}}\mathcal{A}^{(j)}_{g,n}(\alpha) = 8\,(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_i-u_{i-1})\;\delta^{\tilde{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)})$$

  for $\tilde{\mathbf{m}} = (m_1-1, m_2, \dots, \hat{m}_j, \dots, m_n)$. Then, the bound on u_i-u_{i-1} obtained in Lemma 3.10 implies the existence of a universal constant $C>0$ such that

  $$|\delta^{\mathbf{m}}\mathcal{A}^{(j)}_{g,n}(\alpha)| \le C\langle\alpha\rangle\sum_{i=0}^{+\infty}4^{-i}\;|\delta^{\tilde{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)})|. \tag{3.24}$$

  We now want to use the induction hypothesis to bound $\delta^{\tilde{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)})$, for every $i\in\mathbb{N}_0$. We observe that $|\tilde{\mathbf{m}}|=2N$. We decide to chose the parameter a_{N+1} so that $a_{n+1}>a_N$. Then, $i+\alpha_1+\alpha_j-1 \ge \alpha_1-1 \ge a_N$, and the multi-indices $\tilde{\mathbf{m}}$, $\tilde{\alpha}^{(i)}$ satisfy the hypotheses at the rank N. Hence,

  $$|\delta^{\tilde{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)})| \le C_{n-1,N}\frac{V_{g,n-1}}{(g+1)^N}\langle\tilde{\alpha}^{(i)}\rangle^N = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^N\langle i\rangle^N\right)$$

  by equation (3.12). Then, if $m_j=0$,

  $$\begin{aligned}|\delta^{\mathbf{m}}\mathcal{A}^{(j)}_{g,n}(\alpha)| &= \mathcal{O}_{n,N}\left(\langle\alpha\rangle\sum_{i=0}^{+\infty}4^{-i}\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^N\langle i\rangle^N\right)\\ &= \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^{N+1}\right),\end{aligned}$$

  which is precisely our claim.

- Now, if $m_j > 0$, we need to be more careful when applying the derivative δ_j because of the presence of the term α_j in $\mathcal{A}^{(j)}_{g,n}(\alpha)$. We prove that

$$\delta^{\mathbf{m}}\mathcal{A}^{(j)}_{g,n}(\alpha) = 8\,(2\alpha_j+1)\sum_{i=0}^{+\infty}(u_i-u_{i-1})\,\delta^{\tilde{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)}) - 16\,m_j\sum_{i=0}^{+\infty}(u_i-u_{i-1})\,\delta^{\hat{\mathbf{m}}}c_{g,n-1}(\tilde{\alpha}^{(i)})$$

  where $\hat{\mathbf{m}} = (m_1+m_j-2, m_2, \dots, \hat{m}_j, \dots, m_n)$. We observe that $|\hat{\mathbf{m}}| = 2N-1$, and this allows us to apply the induction hypothesis to this additional term. The same computation as in the case $m_j = 0$ leads to the same bound, since $m_j = \mathcal{O}_N(1)$.

We add the $n-2 = \mathcal{O}_n(1)$ contributions for $j \in \{2, \dots, n\}$ and conclude that

$$\sum_{j=2}^{n}|\delta^{\mathbf{m}}\mathcal{A}^{(j)}_{g,n}(\alpha)| = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^{N+1}\right). \tag{3.25}$$

◇

Estimate of the term (B). Let us first observe that this term only appears whenever $g \geq 1$. As in the case (A), we start by observing that by equation (3.4),

$$\delta_1\mathcal{B}_{g,n}(\alpha) = 16\sum_{i=0}^{+\infty}\sum_{k_1+k_2=i+\alpha_1-2}(u_i-u_{i-1})\,c_{g-1,n+1}(\tilde{\alpha}^{(k_1,k_2)})$$

where $\tilde{\alpha}^{(k_1,k_2)} = (k_1, k_2, \alpha_2, \dots, \alpha_n)$. However, for this term, the dependency on α_1 is more complex, and we need to use Lemma 3.27 to apply the operator $\delta_1^{m_1-1}$ to the previous equation. We obtain:

$$|\delta^{\mathbf{m}}\mathcal{B}^{(g,n)}_{\alpha}| \leq C\sum_{i=0}^{+\infty}\sum_{\substack{k_1+k_2=i+\alpha_1-2\\ k_1\geq k_2}}4^{-i}\,|\delta_1^{m_1-1}\delta^{\tilde{\mathbf{m}}}c_{g-1,n+1}(\tilde{\alpha}^{(k_1,k_2)})| \tag{3.26}$$

$$+ C\sum_{i=0}^{+\infty}\sum_{\substack{k_1+k_2=i+\alpha_1-2\\ k_1<k_2}}4^{-i}\,|\delta_2^{m_1-1}\delta^{\tilde{\mathbf{m}}}c_{g-1,n+1}(\tilde{\alpha}^{(k_1,k_2)})| \tag{3.27}$$

$$+ C\sum_{\mu_1+\mu_2=m_1-2}\sum_{i=0}^{+\infty}4^{-i}|\delta_1^{\mu_1}\delta_2^{\mu_2}\delta^{\tilde{\mathbf{m}}}c_{g-1,n+1}(\tilde{\alpha}^{(\lfloor\frac{i+\alpha_1-1}{2}\rfloor,\lfloor\frac{i+\alpha_1}{2}\rfloor)})|, \tag{3.28}$$

where $\tilde{\mathbf{m}} = (0, 0, m_2, \dots, m_n) \in \mathbb{N}_0^{n+1}$, and the universal constant $C > 0$ comes once again from equation (3.6). We estimate each term successively, using the induction hypothesis.

- Let us assume that the parameter a_{N+1} is $\geq 2a_N + 2$. Then, by hypothesis, $\alpha_1 \geq 2a_N + 2$, and therefore for any $i \geq 0$ and any k_1, k_2 in the i-th term of the sum (3.26),
$$k_1 \geq \frac{k_1 + k_2}{2} = \frac{i + \alpha_1 - 2}{2} \geq a_N.$$
We can then apply the induction hypothesis to the multi-indices $(m_1 - 1, \mathbf{0}^n) + \tilde{\mathbf{m}}$ of norm $2N$ and $\tilde{\alpha}^{(k_1,k_2)}$, which yields
$$\begin{aligned}|\delta_1^{m_1-1}\delta^{\tilde{\mathbf{m}}} c_{g-1,n+1}(\tilde{\alpha}^{(k_1,k_2)})| &\leq C_{n+1,N}\frac{V_{g-1,n+1}}{g^N}\langle\tilde{\alpha}^{(k_1,k_2)}\rangle^N \\ &= \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\tilde{\alpha}^{(k_1,k_2)}\rangle^N\right)\end{aligned}$$
since $g \geq 1$, and by equations (3.9) and (3.12). We then use the fact that
$$\sum_{i=0}^{+\infty} \sum_{\substack{k_1+k_2=i+\alpha_1-2 \\ k_1 \geq k_2}} 4^{-i}\langle\tilde{\alpha}^{(k_1,k_2)}\rangle^N = \mathcal{O}_N\left(\langle\alpha\rangle^{N+1}\right), \tag{3.29}$$
to conclude that if $a_{N+1} \geq 2a_N + 2$, then the term (3.26) is $\mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^{N+1}\right)$.

- By symmetry of the coefficients, the term (3.27) is equal to
$$\sum_{i=0}^{+\infty} \sum_{\substack{k_1+k_2=i+\alpha_1-2 \\ k_2 > k_1}} 4^{-i}\, |\delta_1^{m_1-1} c_{g-1,n+1}(\tilde{\alpha}^{(k_2,k_1)})|$$
is therefore smaller than the term (3.26).

- For the term (3.28), we observe that since $\alpha_1 \geq 2a_N + 2$, for all $i \geq 0$,
$$\left\lfloor \frac{i+\alpha_1}{2} \right\rfloor \geq \left\lfloor \frac{i+\alpha_1-1}{2} \right\rfloor \geq \frac{\alpha_1 - 2}{2} \geq a_N.$$
Furthermore, for any integers such that $\mu_1 + \mu_2 = m_1 - 1$, the norm of $(\mu_1, \mu_2, \mathbf{0}^{n-1}) + \tilde{\mathbf{m}}$ is equal to $2N - 1$, and therefore, by the induction hypothesis,
$$|\delta_1^{\mu_1}\delta_2^{\mu_2}\delta^{\tilde{\mathbf{m}}} c_{g-1,n+1}(\tilde{\alpha}^{(\lfloor \frac{i+\alpha_1-1}{2}\rfloor, \lfloor \frac{i+\alpha_1}{2}\rfloor)})| \leq C_{n+1,N}\frac{V_{g-1,n+1}}{g^N}\langle\tilde{\alpha}^{(\lfloor \frac{i+\alpha_1-1}{2}\rfloor, \lfloor \frac{i+\alpha_1}{2}\rfloor)}\rangle^N.$$
and therefore we can prove that the term (3.28) satisfies the same bound as the other terms.

As a conclusion, provided that $a_{N+1} \geq 2a_N + 2$,
$$(B) = |\delta^{\mathbf{m}}\mathcal{B}_{g,n}(\alpha)| = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^{N+1}\right).$$

$\diamond$

Estimate of the term (C). For the term (C), similarly, by equation (3.5), for every $\iota = (g_1, I, J)$, where $g_1 + g_2 = g$, if we denote $n_1 = |I|$ and $n_2 = |J|$,

$$\delta_1 \mathcal{C}_{g,n}^{(\iota)}(\alpha) = 16 \sum_{i=0}^{+\infty} \sum_{k_1+k_2=i+\alpha_1-2} (u_i - u_{i-1})\, c_{g_1,n_1+1}(\tilde{\alpha}_I^{(k_1)})\, c_{g_2,n_2+1}(\tilde{\alpha}_J^{(k_2)})$$

where $\tilde{\alpha}_I^{(k_1)} = (k_1, \alpha_I)$ and $\tilde{\alpha}_J^{(k_2)} = (k_2, \alpha_J)$. As before, we prove that

$$|\delta^{\mathbf{m}} \mathcal{C}_{g,n}^{(\iota)}(\alpha)| \tag{3.30}$$

$$\leq C \sum_{i=0}^{+\infty} \sum_{\substack{k_1+k_2=i+\alpha_1-2 \\ k_1 \geq k_2}} 4^{-i} |\delta^{(m_1-1,\mathbf{m}_I)} c_{g_1,n_1+1}(\tilde{\alpha}_I^{(k_1)})||\delta^{(0,\mathbf{m}_J)} c_{g_2,n_2+1}(\tilde{\alpha}_J^{(k_2)})| \tag{3.31}$$

$$+ C \sum_{i=0}^{+\infty} \sum_{\substack{k_1+k_2=i+\alpha_1-2 \\ k_1 < k_2}} 4^{-i} |\delta^{(0,\mathbf{m}_I)} c_{g_1,n_1+1}(\tilde{\alpha}_I^{(k_1)})||\delta^{(m_1-1,\mathbf{m}_J)} c_{g_2,n_2+1}(\tilde{\alpha}_J^{(k_2)})| \tag{3.32}$$

$$+ C \sum_{\mu_1+\mu_2=m_1-2} \sum_{i=0}^{+\infty} 4^{-i} |\delta^{(\mu_1,\mathbf{m}_I)} c_{g_1,n_1+1}(\tilde{\alpha}_I^{(\lfloor \frac{i+\alpha_1-1}{2} \rfloor)})||\delta^{(\mu_2,\mathbf{m}_J)} c_{g_2,n_2+1}(\tilde{\alpha}_I^{(\lfloor \frac{i+\alpha_1}{2} \rfloor)})|. \tag{3.33}$$

We now estimate the term (3.31) using the induction hypothesis on the two terms $\delta^{(m_1-1,\mathbf{m}_I)} c_{g_1,n_1+1}(\tilde{\alpha}_I^{(k_1)})$ and $\delta^{(0,\mathbf{m}_J)} c_{g_2,n_2+1}(\tilde{\alpha}_J^{(k_2)})$. Let us set

$$N_1 := \left\lfloor \frac{m_1 + |\mathbf{m}_I|}{2} \right\rfloor \quad \text{and} \quad N_2 := \left\lfloor \frac{|\mathbf{m}_J| + 1}{2} \right\rfloor$$

so that $m_1 - 1 + |\mathbf{m}_I| \in \{2N_1 - 1, 2N_1\}$ and $|\mathbf{m}_J| \in \{2N_2 - 1, 2N_2\}$. Then, we observe that under the hypothesis $a_{N+1} \geq 2a_N + 2$, for any term in equation (3.31), $k_1 \geq a_N \geq a_{N_1}$. We can therefore apply the induction hypothesis at the rank N_1 and obtain

$$|\delta^{(m_1-1,\mathbf{m}_I)} c_{g_1,n_1+1}(\tilde{\alpha}_I^{(k_1)})| = \mathcal{O}_{n_1,N_1}\left(\frac{V_{g_1,n_1+1}}{(g_1+1)^{N_1}} \langle \tilde{\alpha}_I^{(k_1)} \rangle^{N_1} \right).$$

We also have that

$$|\delta^{(0,\mathbf{m}_J)} c_{g_2,n_2+1}(\tilde{\alpha}_J^{(k_2)})| = \mathcal{O}_{n_2,N_2}\left(\frac{V_{g_2,n_2+1}}{(g_2+1)^{N_2}} \langle \tilde{\alpha}_J^{(k_2)} \rangle^{N_2} \right)$$

(note that there is no condition on the index k_2 because there is no derivative w.r.t. the first variable in $\delta^{(0,\mathbf{m}_J)}$). We obtain by the same method as before that the term (3.31) is

$$\mathcal{O}_{n,N_1,N_2}\left(\frac{V_{g_1,n_1+1} V_{g_2,n_2+1}}{(g_1+1)^{N_1}(g_2+1)^{N_2}} \langle \alpha \rangle^{N_1+N_2+1} \right). \tag{3.34}$$

We then wish to apply Lemma 3.16 in order to bound the sum over all configurations. More precisely, this lemma states that

$$\sum_{\iota : 2g_i + n_i > N_i + 1} \frac{V_{g_1,n_1+1} V_{g_2,n_2+1}}{(g_1+1)^{N_1}(g_2+1)^{N_2}} = \mathcal{O}_{n,N_1,N_2}\left(\frac{V_{g,n_1+n_2}}{g^{N_1+N_2+1}} \right) = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{g^{N+2}} \right)$$

since $n_1 + n_2 = n - 1$, by equation (3.12), and because

$$N_1 + N_2 = \left\lfloor \frac{m_1 + |\mathbf{m}_I|}{2} \right\rfloor + \left\lfloor \frac{|\mathbf{m}_J| + 1}{2} \right\rfloor \in \{N, N+1\}.$$

Then, we observe that we need to be able to say that $2g_i + n_i > N_i + 1$ for $i \in \{1, 2\}$. This is achieved by adding a new constraint on the parameter a_{N+1}. Indeed,

$$k_1 + |\alpha_I| \geq \frac{k_1 + k_2}{2} + |\alpha_I| \geq \frac{\alpha_1 + |\alpha_I|}{2} - 1 \geq \frac{a_{N+1}}{2} \#\{i \in \{1\} \cup I \,:\, m_i \neq 0\} - 1$$

by hypothesis on α, and

$$\#\{i \in \{1\} \cup I \,:\, m_i \neq 0\} \geq \frac{m_1 + |\mathbf{m}_I|}{|\mathbf{m}|_\infty} \geq \frac{2N_1}{2N+1}.$$

As a consequence, if we assume that $a_{N+1} \geq 3(2N+1)$, then for any configuration such that $2g_1 + n_1 \leq N_1 + 1$,

$$k_1 + |\alpha_I| \geq \frac{a_{N+1} N_1}{2N+1} - 1 \geq 3N_1 - 1 \geq 6g_1 + 3n_1 - 4 > 3g_1 - 3 + (n_1 + 1),$$

because $3g_1 + 2n_1 > 2$. The latter quantity is the degree of the polynomial $V_{g_1, n_1+1}(\mathbf{x})$, and therefore the previous inequality implies that $\delta^{(m_1 - 1, \mathbf{m}_I)} c_{g_1, n_1+1}(k_1, \alpha_I) = 0$. Similarly, we prove that

$$k_2 + |\alpha_J| \geq \frac{a_{N+1} |\mathbf{m}_J|}{|\mathbf{m}|_\infty} \geq 3(2N_2 - 1)$$

and therefore if $2g_2 + n_2 \leq N_2 + 1$, then

$$k_2 + |\alpha_J| \geq 12g_2 + 6n_2 - 9 > 3g_2 - 3 + (n_2 + 1)$$

and hence $\delta^{(0, \mathbf{m}_J)} c_{g_2, n_2+1}(k_2, \alpha_J) = 0$. This allows us to conclude for the term (3.31): we have proved that it is

$$\mathcal{O}_{n,N}\left(\frac{V_{g,n}}{g^{N+2}} \langle \alpha \rangle^{N+2} \right) = \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{g^{N+1}} \langle \alpha \rangle^{N+1} \right)$$

because it is equal to zero unless $|\alpha| \leq 3g - 3 + n$.

The estimate of the term (3.33) is the same: we apply the induction hypothesis to $\delta^{(\mu_1, \mathbf{m}_I)} c_{g_1, n_1+1}(\tilde{\alpha}_I^{(\lfloor \frac{i + \alpha_1 - 1}{2} \rfloor)})$ and $\delta^{(\mu_2, \mathbf{m}_J)} c_{g_2, n_2+1}(\tilde{\alpha}_I^{(\lfloor \frac{i + \alpha_1}{2} \rfloor)})$, at the admissible ranks

$$N_1 := \left\lfloor \frac{\mu_1 + |\mathbf{m}_I| + 1}{2} \right\rfloor \quad \text{and} \quad N_2 := \left\lfloor \frac{\mu_2 + |\mathbf{m}_J| + 1}{2} \right\rfloor.$$

We observe that $N_1 + N_2 = N$, and this therefore yields the claimed result. $\diamond$

As a conclusion, we have proved that under the hypotheses $a_{N+1} \geq 2a_N + 2$ and $a_{N+1} \geq 3(2N+1)$, for any multi-index $\mathbf{m}$ of norm $|\mathbf{m}| = 2N+1$ and any multi-index α such that $\forall i, (m_i \neq 0 \Rightarrow \alpha_i \geq a_{N+1})$,

$$|\delta^{\mathbf{m}} c_{g,n}(\alpha)| \leq (A) + (B) + (C) \leq C_{n,N+1} \frac{V_{g,n}}{(g+1)^{N+1}} \langle \alpha \rangle^{N+1}.$$

This implies the result for any multi-index $\mathbf{m}$ of norm $2N+2$ too, simply because for any sequence $(v(\alpha))_\alpha$, if $m_1 > 0$ for instance, then for all α,

$$|\delta^{\mathbf{m}} v(\alpha)| \leq |\delta^{(m_1-1,m_2,\dots,m_n)} v(\alpha)| + |\delta^{(m_1-1,m_2,\dots,m_n)} v(\alpha_1+1, \alpha_2, \dots, \alpha_n)|.$$

This concludes the induction. □

3.3.3.2 Function ultimately polynomial in each variable

We observe that Theorem 3.26 states that the function $\alpha \mapsto c_{g,n}(\alpha)$ has small derivatives for large enough values of α. Had we proved that the iterated derivatives are small for any α, we could have used a discrete version of the Taylor-Lagrange formula, such as the one that follows, to conclude that $\alpha \mapsto c_{g,n}(\alpha)$ is well-approximated by polynomial functions.

Lemma 3.28 (Discrete Taylor-Lagrange formula)**.** *Let $n \in \mathbb{N}$ and $f : \mathbb{N}_0^n \to \mathbb{R}$. We assume that there exists numbers $M > 0$ and $K, d \in \mathbb{N}_0$ such that, for any multi-index $\mathbf{m}$ of norm $|\mathbf{m}| = K+1$,*

$$\forall \alpha \in \mathbb{N}_0^n, \quad |\delta^{\mathbf{m}} f(\alpha)| \leq M \langle \alpha \rangle^d.$$

Then, there exists a polynomial function $\tilde{f}^{(K)} : \mathbb{N}_0^n \to \mathbb{R}$ of degree at most K such that

$$\forall \alpha \in \mathbb{N}_0^n, \quad |f(\alpha) - \tilde{f}^{(K)}(\alpha)| \leq M n^{K+1} \langle \alpha \rangle^{d+K+1}.$$

Furthermore, the coefficients of the polynomial function $\tilde{f}^{(K)}$ can be expressed as a linear combination of the derivatives $\delta^{\mathbf{m}} f(\mathbf{0}^n)$ for $\mathbf{m} \in \mathbb{N}_0^n$ of norm $|\mathbf{m}| \leq K$.

Proof. We proceed by induction on the integer K.

For $K = 0$, we observe that by Lemma 3.22, for all α,

$$|f(\alpha) - f(\mathbf{0}^n)| \leq \sum_{i=1}^{n} \sum_{k=0}^{\alpha_i - 1} |\delta_i f(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n)| \leq M n \langle \alpha \rangle^{d+1},$$

so the result holds if we take $\tilde{f}^{(0)}$ to be the constant function equal to $f(\mathbf{0}^n)$.

Let us now assume the result at a rank $K-1$ for a $K \geq 1$, and deduce the result at the rank K. For any integer $i \in \{1, \dots, n\}$, the function $\delta_i f$ satisfies the induction hypothesis at the rank $K-1$. Hence, there exists a polynomial function $\tilde{f}_i^{(K-1)}$ of degree at most $K-1$, and coefficients that can be expressed as linear combinations of the $\delta^{\mathbf{m}}\delta_i f(\mathbf{0}^n)$ for $|\mathbf{m}| \le K-1$, such that

$$\forall \alpha \in \mathbb{N}_0^n, \quad |\delta_i f(\alpha) - \tilde{f}_i^{(K-1)}(\alpha)| \le M n^K \langle \alpha \rangle^{d+K}.$$

We define

$$\tilde{f}^{(K)}(\alpha) := f(\mathbf{0}^n) - \sum_{i=1}^{n} \sum_{k=0}^{\alpha_i - 1} \tilde{f}_i^{(K-1)}(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n).$$

We notice that $\tilde{f}^{(K)}$ is a polynomial of degree at most K, and its coefficients are linear combinations of $f(\mathbf{0}^n)$ and the coefficients of $(\tilde{f}_i)_i$, and therefore linear combinations of the $\delta^{\mathbf{m}} f(\mathbf{0}^n)$ for $|\mathbf{m}| \le K$. By Lemma 3.22, for all $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned} |f(\alpha) - \tilde{f}^{(K)}(\alpha)| &\le \sum_{i=1}^{n} \sum_{k=0}^{\alpha_i - 1} |\delta_i f(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n) - \tilde{f}_i^{(K-1)}(\mathbf{0}^{i-1}, k, \alpha_{i+1}, \dots, \alpha_n)| \\ &\le M n^{K+1} \langle \alpha \rangle^{d+K+1}, \end{aligned}$$

and the conclusion follows. $\square$

However, we can expect from the second-order approximation, Proposition 3.23, that the function $\alpha \mapsto c_{g,n}(\alpha)$ is not well-approximated by polynomial functions, but rather by a combination of polynomial functions and indicator functions, correcting the small values. In order to make the description of such functions more systematic, we introduce the following definition.

Lemma 3.29 (and Definition)**.** *For any $n \in \mathbb{N}$ and $K, a \in \mathbb{N}_0$, the two following families of functions from $\mathbb{N}_0^n$ to $\mathbb{R}$,*

- *functions of the form*
$$\alpha \mapsto \prod_{i \in I} \alpha_i^{k_i} \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i}$$
where $I \subseteq \{1, \dots, n\}$, $\mathbf{k} = (k_i)_{i \in I}$ is a multi-index of norm $|\mathbf{k}| \le K$, and $\beta = (\beta_i)_{i \notin I}$ is such that $|\beta|_\infty < a$;

- *functions of the form*
$$\alpha \mapsto \prod_{i \in I} \alpha_i^{k_i} \mathbb{1}_{\alpha_i \ge a} \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i}$$
where $I, \mathbf{k}$ and β are defined the same way as in the first point;

generate the same linear subspace of the space of functions $\mathbb{N}_0^n \to \mathbb{R}$. We denote this space as $\mathcal{P}_{n,K,a}$, and call its elements polynomials (of degree at most K) in each variable greater than a.

Proof. The equivalence of these two definitions comes from the simple observation that for any $a, \alpha \in \mathbb{N}_0$,

$$1 = \mathbb{1}_{\alpha \geq a} + \sum_{\beta=0}^{a-1} \mathbb{1}_{\alpha=\beta}.$$

□

Then, elements of $\mathcal{P}_{n,K,a}$ are exactly the kind of functions we imagine the coefficients $\alpha \mapsto c_{g,n}(\alpha)$ to be well-approximated by: since the derivatives vanish for large enough α, beyond a few small values, the functions are approximated by polynomials. More precisely, let us use the following shifted Taylor lemma.

Lemma 3.30. *Let $n \in \mathbb{N}$ and $f : \mathbb{N}_0^n \to \mathbb{R}$. We assume that there exists numbers $M > 0$ and $K, a, d \in \mathbb{N}_0$ satisfying the following. For any multi-index $\mathbf{m} \in \mathbb{N}_0^n$ of norm $|\mathbf{m}| = K+1$, any $\alpha \in \mathbb{N}_0^n$ such that $\forall i, (m_i \neq 0 \Rightarrow \alpha_i \geq a)$, we have*

$$|\delta^{\mathbf{m}} f(\alpha)| \leq M \langle \alpha \rangle^d.$$

Then, there exists a function $\tilde{f}^{(K)} \in \mathcal{P}_{n,K,a}$ such that

$$\forall \alpha \in \mathbb{N}_0^n, \quad |f(\alpha) - \tilde{f}^{(K)}(\alpha)| \leq C_{n,a,d,K} M \langle \alpha \rangle^{d+K+1}$$

where $C_{n,a,d,K} = 2^{\frac{d}{2}+n} \langle 2na \rangle^d a^n n^{K+1}$.

The coefficients of $\tilde{f}^{(K)}$ can be expressed as linear combinations of the values $\delta^{\mathbf{m}} f(\alpha)$ for $\alpha \in \mathbb{N}_0^n$ such that $|\alpha|_\infty \leq a$ and $\mathbf{m} \in \mathbb{N}_0^n$ such that $|\mathbf{m}| \leq K$.

Proof. The idea is to decompose $\mathbb{N}_0^n$ into subsets on which all of the variables are greater than a. More precisely, we notice that

$$1 = \sum_{I \subset \{1,\dots,n\}} \sum_{\substack{(\beta_i)_{i \notin I} \\ |\beta|_\infty < a}} \prod_{i \in I} \mathbb{1}_{\alpha_i \geq a} \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i}.$$

Then, we can rewrite the function f as

$$f(\alpha) = \sum_{\substack{I \subset \{1,\dots,n\} \\ = \{i_1 < \dots < i_r\}}} \sum_{\substack{(\beta_i)_{i \notin I} \\ |\beta|_\infty < a}} g_{I,\beta}(\alpha_{i_1} - a, \dots, \alpha_{i_r} - a) \prod_{i \in I} \mathbb{1}_{\alpha_i \geq a} \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i} \tag{3.35}$$

where $g_{I,\beta} : \mathbb{N}_0^r \to \mathbb{R}$ is defined by $g_{I,\beta}(\hat{\alpha}) := f(\alpha)$ for any $\hat{\alpha} \in \mathbb{N}_0^r$, where

$$\forall i, \alpha_i := \begin{cases} \hat{\alpha}_k + a & \text{if } i = i_k \text{ for a } k \in \{1, \dots, r\} \\ \beta_i & \text{if } i \notin I. \end{cases}$$

We wish to apply Lemma 3.28 to the function $g_{I,\beta}$. In order to do so, we observe that, for any multi-index $\hat{\mathbf{m}} \in \mathbb{N}_0^r$ of norm $K+1$, if we can define $\mathbf{m} \in \mathbb{N}_0^n$ of norm $K+1$ so that

$$\forall i, m_i := \begin{cases} \hat{m}_k & \text{if } i = i_k \text{ for a } k \in \{1, \dots, r\} \\ 0 & \text{if } i \notin I. \end{cases}$$

Then, for any $\hat{\alpha} \in \mathbb{N}_0^r$, $\alpha_i \geq a$ as soon as $m_i \neq 0$, and therefore, by hypothesis on f,

$$|\delta^{\hat{\mathbf{m}}} g_{I,\beta}(\hat{\alpha})| = |\delta^{\mathbf{m}} f(\alpha)| \leq M\langle\alpha\rangle^d \leq M2^{\frac{d}{2}}\langle 2na\rangle^d \langle\hat{\alpha}\rangle^d$$

because for any x, y, $\langle x+y\rangle \leq \sqrt{2}\langle x\rangle\langle y\rangle$, and $|\alpha| = |\hat{\alpha}| + ra + |\beta| \leq 2na + |\hat{\alpha}|$. Then, by Lemma 3.28, there exists a polynomial $\tilde{g}_{I,\beta}^{(K)}$ in r variables, of degree at most K, such that

$$\forall \hat{\alpha} \in \mathbb{N}_0^r, \quad |g_{I,\beta}(\hat{\alpha}) - \tilde{g}_{I,\beta}^{(K)}(\hat{\alpha})| \leq M2^{\frac{d}{2}}\langle 2na\rangle^d n^{K+1}\langle\hat{\alpha}\rangle^{d+K+1}. \tag{3.36}$$

Let us now define an element $\tilde{f}^{(K)}$ of $\mathcal{P}_{n,K,a}$ by the formula

$$\tilde{f}^{(K)}(\alpha) := \sum_{\substack{I \subset \{1,\ldots,n\} \\ = \{i_1 < \ldots < i_r\}}} \sum_{\substack{(\beta_i)_{i \notin I} \\ |\beta|_\infty < a}} \tilde{g}_{I,\beta}^{(K)}(\alpha_{i_1} - a, \ldots, \alpha_{i_r} - a) \prod_{i \in I} \mathbb{1}_{\alpha_i \geq a} \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i}. \tag{3.37}$$

By equations (3.35) and (3.37) together with (3.36), for any $\alpha \in \mathbb{N}_0^n$,

$$|f(\alpha) - \tilde{f}^{(K)}(\alpha)| \leq M2^{\frac{d}{2}+n}\langle 2na\rangle^d a^n n^{K+1}\langle\alpha\rangle^{d+K+1}$$

because there are 2^n terms in the sum over the $I \subseteq \{1, \ldots, n\}$, and always less than a^n possible choices for β. This is the claimed inequality.

The coefficients of $\tilde{f}^{(K)}$ are linear combinations of the coefficients of the $\tilde{g}_{I,\beta}^{(K)}$. By Lemma 3.28, these are themselves linear combinations of the values $\delta^{\hat{\mathbf{m}}} g_{I,\beta}(\mathbf{0}^{\#I})$ for multi-indices $\hat{\mathbf{m}}$ of norm $|\hat{\mathbf{m}}| \leq K$. By definition of $g_{I,\beta}$, these derivatives are derivatives of the form $\delta^{\mathbf{m}} f(\alpha)$ for multi-indices $\mathbf{m}, \alpha$ such that $|\alpha|_\infty \leq a$ and $|\mathbf{m}| \leq K$. □

We can now conclude with the proof of the asymptotic expansion, Theorem 3.17.

Proof. Let g, n be integers such that $2g - 2 + n > 0$, and $N \in \mathbb{N}_0$ be a fixed order. By Theorem 3.26, there exists constants $C_{n,N}, a_{N+1}$ such that

$$|\delta^{\mathbf{m}} c_{g,n}(\alpha)| \leq C_{n,N} \frac{V_{g,n}}{(g+1)^{N+1}} \langle\alpha\rangle^{N+1}$$

for any multi-index $\mathbf{m}$ for norm $2N + 1$ and α such that $\forall i, (m_i \neq 0 \Rightarrow \alpha_i \geq a_{N+1})$. This is exactly the hypothesis of Lemma 3.30, for the parameters $K = 2N$, $d = N + 1$, $a = a_{N+1}$ and $M = C_{n,N} V_{g,n}/(g+1)^{N+1}$. As a consequence, there exists an element $\tilde{c}_{g,n}^{(K)}$ of $\mathcal{P}_{n,K,a}$ such that for all $\alpha \in \mathbb{N}_0^n$,

$$|c_{g,n}(\alpha) - \tilde{c}_{g,n}^{(K)}(\alpha)| \leq MC_{n,a,d,K}\langle\alpha\rangle^{d+K+1}$$

or, in other words,

$$c_{g,n}(\alpha) = \tilde{c}_{g,n}^{(K)}(\alpha) + \mathcal{O}_{n,N}\left(\frac{V_{g,n}}{(g+1)^{N+1}}\langle\alpha\rangle^{3N+2}\right). \tag{3.38}$$

Let us now define, for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, a good candidate for the approximating function,

$$f^{(N)}_{g,n}(\mathbf{x}) := \frac{1}{V_{g,n}} \sum_{\alpha \in \mathbb{N}_0^n} \tilde{c}^{(2N)}_{g,n}(\alpha) \prod_{i=1}^n \frac{x_i^{2\alpha_i+1}}{2^{2\alpha_i}(2\alpha_i+1)!}.$$

Then, by equation (3.38) and the definition of $V_{g,n}(\mathbf{x})$ and $f^{(N)}_{g,n}(\mathbf{x})$,

$$\begin{aligned}\frac{x_1 \ldots x_n V_{g,n}(\mathbf{x})}{V_{g,n}} &= f^{(N)}_{g,n}(\mathbf{x}) + \mathcal{O}_{n,N}\left(\frac{1}{(g+1)^{N+1}} \sum_{\alpha \in \mathbb{N}_0^n} \langle \alpha \rangle^{3N+2} \prod_{i=1}^n \frac{x_i^{2\alpha_i+1}}{2^{2\alpha_i}(2\alpha_i+1)!}\right) \\ &= f^{(N)}_{g,n}(\mathbf{x}) + \mathcal{O}_{n,N}\left(\frac{1}{(g+1)^{N+1}} \langle \mathbf{x} \rangle^{3N+2} \exp\left(\frac{x_1 + \ldots + x_n}{2}\right)\right).\end{aligned}$$

As a consequence, the approximating function $f^{(N)}_{g,n}$ satisfies the claimed estimate.

Let us now prove that $\mathbf{x} \mapsto f^{(N)}_{g,n}(\mathbf{x})$ has the claimed form. Note that by definition of the set $\mathcal{P}_{n,K,a}$, and because the polynomials $(p_i)_{0 \leq i \leq K}$ introduced in Notation 5 are a basis of the set of polynomial of degree K, we can express the function $\alpha \mapsto \tilde{c}^{(K)}_{g,n}(\alpha)$ as a linear combination of functions of the form

$$g_{I,\beta,\mathbf{k}}(\alpha) = \prod_{i \in I} p_{k_i}(\alpha_i) \prod_{i \notin I} \mathbb{1}_{\alpha_i = \beta_i},$$

where $|\mathbf{k}| \leq K$ and $|\beta|_\infty < a_{N+1}$. By Lemma 3.25,

$$\begin{aligned}&\sum_{\alpha \in \mathbb{N}_0^n} g_{I,\beta,\mathbf{k}}(\alpha) \prod_{i=1}^n \frac{x_i^{2\alpha_i+1}}{2^{2\alpha_i}(2\alpha_i+1)!} \\ &= \prod_{\substack{i \in I \\ k_i \text{ even}}} \frac{x_i^{k_i}}{2^{k_i-1}} \sinh\left(\frac{x_i}{2}\right) \prod_{\substack{i \in I \\ k_i \text{ odd}}} \frac{x_i^{k_i}}{2^{k_i-1}} \cosh\left(\frac{x_i}{2}\right) \prod_{i \notin I} \frac{x_i^{2\beta_i+1}}{2^{2\beta_i}(2\beta_i+1)!}.\end{aligned}$$

By replacing sinh and cosh by their definitions, we deduce that $f^{(N)}_{g,n}$ is a linear combination of functions of the form

$$\mathbf{x} \mapsto \mathbf{x}^{\mathbf{k}} \prod_{i \in I_+} \exp\left(\frac{+x_i}{2}\right) \prod_{i \in I_-} \exp\left(\frac{-x_i}{2}\right)$$

where I_+ and I_- are disjoint subsets of $\{1, \ldots, n\}$, and $\mathbf{k}$ is a multi-index such that

$$\sum_{i \in I_+} k_i + \sum_{i \in I_-} k_i \leq K \quad \text{and} \quad \forall i \notin I_+ \cup I_-, k_i < 2a_{N+1}.$$

We set $\varepsilon_i = 1$ if $i \in I_+$, -1 if $i \in I_-$ and 0 otherwise, and find the claimed expression.

The coefficients are linear combinations of the derivatives $\frac{1}{V_{g,n}}\delta^{\mathbf{m}}c_{g,n}(\alpha)$ for $|\mathbf{m}| \leq K$ and $|\alpha|_\infty \leq a$. By Lemma 3.11,

$$\frac{1}{V_{g,n}}\delta^{\mathbf{m}}c_{g,n}(\alpha) \leq 2^{|\mathbf{m}|} \leq 2^{2N+1}$$

and, therefore, the coefficients of the polynomials are bounded by a constant depending only on N and n (the number n appears because of the linear combinations). $\square$

Chapter 4

Large-scale geometry

The objective of this chapter is to study the geometry of typical surfaces of high genus g at a scale $\log g$. We aim to demonstrate that, even though geodesics shorter than $\log g$ exist, they are rare. This is achieved by adapting two concepts from the theory of random graphs.

- In Section 4.1, we prove that typical surfaces converge in the sense of Benjamini–Schramm to the hyperbolic plane. In other words, most points of a typical surface have a large injectivity radius.
- In Sections 4.2 and 4.3, we prove that the few short geodesics are far away from one another rather than clustered. In order to do so, we introduce a notion of 'tangle-freeness' and study its geometric implications.

4.1 Benjamini–Schramm convergence

The contents of this section are adapted from a part of the article [Mon21], to appear in Analysis & PDE. Their generalisation to the non-compact case will appear in a new version of [LMS20].

The notion of Benjamini–Schramm convergence has first been introduced by Benjamini and Schramm in the context of sequences of graphs [BS01], but can naturally be extended to a continuous setting (see [ABB$^+$11, ABB$^+$17, Bow15]). There is a general definition of Benjamini–Schramm convergence for a *deterministic* sequence of hyperbolic surfaces $(X_g)_g$. In the special case when the limit of $(X_g)_g$ is the hyperbolic plane $\mathcal{H}$, it is equivalent to the following simpler property:

$$\forall L > 0, \lim_{g \to +\infty} \frac{\mathrm{Vol}_{X_g}(\{z \in X_g \ : \ \mathrm{InjRad}_z(X_g) < L\})}{\mathrm{Vol}_{X_g}(X_g)} = 0 \tag{4.1}$$

which we will therefore use as a definition.

The idea behind this characterisation is the following. We consider a distance $L > 0$. On the (fixed) surface X_g, one can pick a point z at random, using the normalised measure $\frac{1}{\mathrm{Vol}_{X_g}(X_g)} \mathrm{Vol}_{X_g}$. Equation (4.1) means that the probability for the ball of center z and radius L to be isometric to a ball in the hyperbolic plane goes to one as $g \to +\infty$.

The case where $\mathrm{InjRad}(X_g) \to +\infty$ as $g \to +\infty$ is a simple situation which implies Benjamini–Schramm convergence towards $\mathcal{H}$. However, [Mir13, Theorem 4.2] proves that this does not occur with high probability for our probabilistic model.

The main result of this section is the following, which is a quantitative estimate of the Benjamini–Schramm speed of convergence of typical surfaces to the hyperbolic plane in the high-genus limit.

Theorem 4.1. *For any $g \geq 2$ and any $L, M > 0$, there exists a set $\mathcal{A}_{g,L,M} \subset \mathcal{M}_g$ such that for any hyperbolic surface $X \in \mathcal{A}_{g,L,M}$,*

$$\mathrm{Vol}_X(\{z \in X \ : \ \mathrm{InjRad}_z(X) < L\}) \leq e^L M, \tag{4.2}$$

and $1 - \mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_{g,L,M}) = \mathcal{O}\left(\frac{e^{2L}}{M}\right)$.

We recall that the implied constant is independent of the genus g and the parameters L, M. Since the total area of a compact hyperbolic surface of genus g is $2\pi(2g-2)$, this result will only be interesting for $L \leq \log g$ and $M \leq g$. Specifying the parameters to be $L = \frac{1}{6}\log g$, $M = g^{\frac{1}{2}}$ and $r_g = g^{-\frac{1}{24}}(\log g)^{\frac{9}{16}}$, Mirzakhani's result on the injectivity radius [Mir13, Theorem 4.2] and Theorem 4.1 together lead to the following corollary.

Corollary 4.2 (Geometric assumptions)**.** *For large enough g, there exists a subset $\mathcal{A}_g \subset \mathcal{M}_g$ such that, for any hyperbolic surface $X \in \mathcal{A}_g$,*

$$\mathrm{InjRad}(X) \geq g^{-\frac{1}{24}}(\log g)^{\frac{9}{16}} \tag{4.3}$$

$$\frac{\mathrm{Vol}_X\left(\{z \in X \ : \ \mathrm{InjRad}_z(X) < \frac{1}{6}\log g\}\right)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(g^{-\frac{1}{3}}\right) \tag{4.4}$$

and $1 - \mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_g) = \mathcal{O}\left(g^{-\frac{1}{12}}(\log g)^{\frac{9}{8}}\right)$.

The estimate (5.14) is similar to the one proved in [Mir13, Section 4.4]. The proof is the same, though an incorrect argument has been modified here.

This result was stated this way in order to prove the main result of [Mon21]: an estimate the number of eigenvalues of a typical surface in an spectral window $[a, b]$ – see Chapter 5. It has also been used as stated by LeMasson and Sahlsten in [LMS20], in order to prove quantum ergodicity of eigenfunctions of the Laplacian for random high-genus surfaces.

In Section 4.1.2, we extend this result for non-compact surfaces of high genus g, under the hypothesis that the number of cusps is $o(\sqrt{g})$ – see Theorem 4.3.

4.1.1 Proof in the compact case

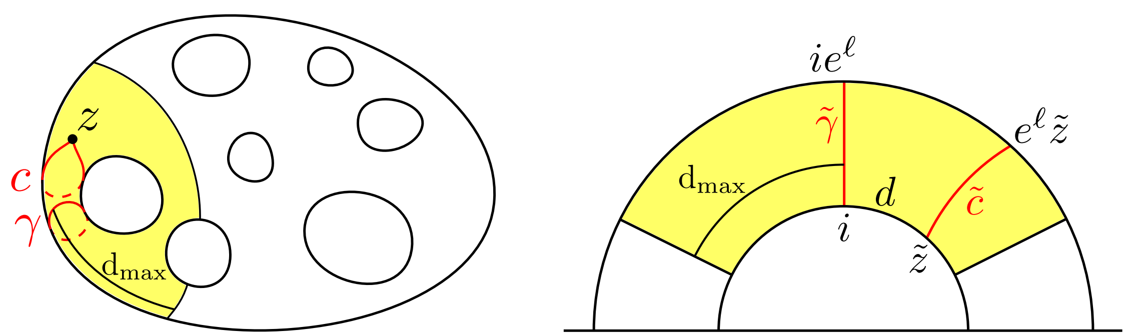
Let us prove Theorem 4.1, that is to say Benjamini–Schramm convergence of typical surfaces to the hyperbolic plane in the high-genus limit. We follow the approach developed by Mirzakhani in [Mir13, Section 4.4].

Proof of Theorem 4.1. Let X be a compact hyperbolic surface of genus g. In order to estimate the volume of

$$X^-(L) = \{z \in X \ : \ \mathrm{InjRad}_z(X) < L\},$$

we establish a link between this volume and the number of small geodesics on X.

Let z be a point in X of radius of injectivity $r < L$. There is a simple geodesic arc c in X of length $2r$ based at z, which is freely homotopic to a closed geodesic γ of length $\ell \leq 2r$. Let us bound the distance between z and γ; this way, we will be able to say z belongs in a neighbourhood of γ of small volume.



(a) On the surface X (b) On the universal cover $\mathcal{H}$

Figure 4.1: Illustration of the geometric construction in the proof of Theorem 4.1.

By lifting z, c and γ to the hyperbolic plane and applying an isometry, we reduce the problem to the situation represented in Figure 4.1: the geodesic γ is lifted to the geodesic segment between i and $e^\ell i$, and c to the segment between a point $\tilde{z} = x + iy$ of modulus 1, and $e^\ell \tilde{z}$. Let us bound the distance d between $\tilde{z}$ and $\tilde{\gamma}$. By usual expressions for the hyperbolic distance in the Poincaré half-plane model (see [Kat92, Theorem 1.2.6] for instance),

$$\cosh(d) = \cosh(\mathrm{d}_{\mathcal{H}}(\tilde{z}, i)) = 1 + \frac{|\tilde{z} - i|^2}{2y} = 1 + \frac{x^2 + (y-1)^2}{2y} = \frac{1}{y}$$

and

$$\sinh(r) = \sinh\left(\frac{\mathrm{d}_{\mathcal{H}}(\tilde{z}, e^\ell \tilde{z})}{2}\right) = \frac{1}{2}\frac{|\tilde{z}e^\ell - \tilde{z}|}{\sqrt{y^2 e^\ell}} = \frac{\sinh\left(\frac{\ell}{2}\right)}{y}.$$

As a consequence,

$$\cosh(d) = \frac{\sinh(r)}{\sinh\left(\frac{\ell}{2}\right)} \leq \frac{e^L}{2\sinh\left(\frac{\ell}{2}\right)} =: \cosh(\mathrm{d}_{\max}(\gamma, L)).$$

Then z belongs to the $\mathrm{d}_{\max}(\gamma, L)$-neighbourhood of the closed geodesic γ in X.

The volume of this neighbourhood is less than the volume of the corresponding collar in the cylinder of central geodesic γ, which can be computed using the Fermi coordinates:

$$\int_0^\ell \int_{-\mathrm{d}_{\max}(\gamma,L)}^{\mathrm{d}_{\max}(\gamma,L)} \cosh(\rho)\,\mathrm{d}\rho\,\mathrm{d}t = 2\ell \sinh(\mathrm{d}_{\max}(\gamma, L))$$
$$\leq 2\ell \cosh(\mathrm{d}_{\max}(\gamma, L)) = \frac{\ell\, e^L}{\sinh\left(\frac{\ell}{2}\right)}.$$

Since $x \leq \sinh x$, the volume of the $\mathrm{d}_{\max}(\gamma, L)$-neighbourhood of γ in X is smaller than $2e^L$. As a consequence, any point $z \in X^-(L)$ is in a neighbourhood of volume less than $2e^L$ around a simple closed geodesic of length at most $2r \leq 2L$. This implies:

$$\mathrm{Vol}_X(X^-(L)) \leq 2e^L\, \mathrm{N}_X^\ell(0, 2L),$$

where, for any positive L, $\mathrm{N}_X^\ell(0, L)$ is the number of simple closed geodesics with length at most L on the hyperbolic surface X. Then, on the set

$$\mathcal{A}_{g,L,M} = \left\{ X \in \mathcal{M}_g \ :\ \mathrm{N}_X^\ell(0, 2L) \leq \frac{M}{2} \right\}$$

equation (4.2) is proved.

Let us estimate the Weil–Petersson probability of this event using Markov's inequality:

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{M}_g \smallsetminus \mathcal{A}_{g,L,M}\right) = \frac{1}{V_g} \int_{\mathcal{M}_g} \mathbb{1}_{\left\{\mathrm{N}_X^\ell(0,2L) > \frac{M}{2}\right\}} \mathrm{dVol}_g^{\mathrm{WP}}(X) \leq \frac{2}{M}\, \mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^\ell(0, 2L)\right].$$

The expectation of the length counting function $\mathrm{N}_X^\ell(a, b)$ is estimated in Lemma 3.14. We sum up the two equations in this lemma, with the length $b = 2L$, to prove that, as $g \to +\infty$:

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^\ell(0, 2L)\right] = \mathcal{O}\left(e^{2L} + \frac{e^{2L}}{g}\right) = \mathcal{O}\left(e^{2L}\right).$$

□

4.1.2 Extension to the non-compact case

The Benjamini–Schramm convergence result of Section 4.1.1 can be extended to the non-compact case under the hypothesis that the number of cusps is $o(\sqrt{g})$.

Theorem 4.3. *Let us fix sequences $(n(g))_g$, $(L_g)_g$ and $(M_g)_g$ such that $L_g \to +\infty$, $M_g \to +\infty$ and $n(g) = o(\sqrt{g})$ as g approaches infinity. Then, there exists a sequence of subsets $\mathcal{A}_g \subset \mathcal{M}_{g,n(g)}$ such that:*

- *for any* $X \in \mathcal{A}_g$,

$$\frac{\mathrm{Vol}(\{z \in X \ : \ \mathrm{InjRad}_z(X) < L_g\})}{\mathrm{Vol}(X)} \leq (M_g + n(g))\, e^{L_g}.$$

- $1 - \mathbb{P}^{\mathrm{WP}}_{g,n(g)}(\mathcal{A}_g) = \mathcal{O}\left(\frac{1}{M_g}\, e^{2L_g}\right)$.

Taking $M_g = \frac{1}{2}\sqrt{g}$ and $L_g = \frac{1}{6}\log g$, we obtain the following result, a natural generalisation of Theorem 4.1 to the non-compact case.

Corollary 4.4. *Let* $(n(g))_g$ *be a sequence such that* $n(g) = o(\sqrt{g})$ *as* g *approaches infinity. Then,*

$$\mathbb{P}^{\mathrm{WP}}_{g,n(g)}\left(\frac{\mathrm{Vol}(\{z \in X \ : \ \mathrm{InjRad}_z(X) < \frac{1}{6}\log g)\}}{\mathrm{Vol}(X)} \leq g^{-\frac{1}{3}}\right) = 1 - \mathcal{O}\left(g^{-\frac{1}{6}}\right).$$

The proof is similar to the compact case, with two main differences.

- Some points of the surface can have a small injectivity radius because they are close to a cusp. We estimate the volume of this set of points straightforwardly in terms of the number of cusps $n(g)$.
- The probabilistic estimates used in the compact case need to be generalised to the non-compact case. This will require to assume $n(g) = o(\sqrt{g})$ in order to use [MZ15, Theorem 1.8] and prove the following lemma.

Lemma 4.5. *There exists a constant* $C > 0$ *such that for any sequence* $(n(g))_g$ *such that* $n(g) = o(\sqrt{g})$, *for all large enough* g,

$$\frac{1}{V_{g,n(g)}} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n(g)}} V_{g_1,n_1+1}V_{g_2,n_2+1} \leq C\, \frac{n(g)^2}{g}.$$

Proof. By equation (3.10), for large enough g,

$$\begin{aligned}
&\frac{1}{V_{g,n(g)}} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n(g)}} V_{g_1,n_1+1}V_{g_2,n_2+1} \\
&\leq \frac{c}{V_{g,n(g)}} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n(g)}} V_{g_1+\lfloor\frac{n_1+1}{2}\rfloor, n_1+1-2\lfloor\frac{n_1+1}{2}\rfloor} V_{g_2+\lfloor\frac{n_2+1}{2}\rfloor, n_2+1-2\lfloor\frac{n_2+1}{2}\rfloor} \\
&\leq c\, n(g)^2 \sum_{\substack{2G_1-3+r_1+2G_2-3+r_2=K \\ r_1,r_2\in\{0,1\}}} \frac{V_{G_1,r_1}V_{G_2,r_2}}{V_{g,n(g)}}
\end{aligned}$$

by putting together identical terms, where $K = 2g - 4 + n(g)$.

We now use the equivalent of $V_{g,n(g)}$ proved in [MZ15, Theorem 1.8]:

$$V_{g,n(g)} \sim c_1 \frac{(2g-3+n(g))!(4\pi^2)^{2g-3+n(g)}}{\sqrt{g}} \geq c_2 \frac{K^{K+1}}{\exp(K)}(4\pi^2)^K$$

by Stirling's formula.

We observe that we can use [MZ15, Theorem 1.2] to find an upper bound on $V_{G_1,r_1}V_{G_2,r_2}$ because $r_1, r_2 \leq 1$. Then there exists a constant c_3 such that for any G_1, G_2 and any $r_1, r_2 \in \{0,1\}$ satisfying $2G_1 - 3 + r_1 + 2G_2 - 3 + r_2 = K$,

$$V_{G_1,r_1}V_{G_2,r_2} \leq c_3 \frac{(2G_1-3+r_1)^{2G_1-3+r_1}(2G_2-3+r_2)^{2G_2-3+r_2}}{\exp(K)}(4\pi^2)^K.$$

Then the quantity we want to estimate is smaller than

$$\frac{c\,c_3}{c_2} n(g)^2 \sum_{\substack{2G_1-3+r_1+2G_2-3+r_2=K \\ r_1,r_2\in\{0,1\}}} \frac{(2G_1-3+r_1)^{2G_1-3+r_1}(2G_2-3+r_2)^{2G_2-3+r_2}}{K^{K+1}}$$
$$\leq \frac{4\,c\,c_3}{c_2} \frac{n(g)^2}{K} \sum_{k=0}^{K} \frac{k^k(K-k)^{K-k}}{K^K} \leq \frac{4\,c\,c_3}{c_2} \frac{n(g)^2}{K} \sum_{k=0}^{K} \left(1 - \frac{k}{K}\right)^K.$$

This leads to the conclusion because $K \sim 2g$ and

$$\sum_{k=0}^{K} \left(1 - \frac{k}{K}\right)^K \leq \sum_{k=0}^{K} \exp(-k) \leq \frac{1}{1-e^{-1}}.$$

□

We can now prove Theorem 4.3.

Proof. Let X be a hyperbolic surface of genus g with $n(g)$ cusps. Let

$$X^-(L_g) := \{z \in X \ : \ \mathrm{InjRad}_z(X) < L_g\}.$$

and z be an element of $X^-(L_g)$. Then, the shortest geodesic arc through z is freely homotopic to either a closed geodesic on X, like in the compact case, or a cusp. Hence, the set $X^-(L_g)$ can be covered by the union of:

- the $(\mathrm{d}_{\max}(\gamma, L_g))$-neighbourhoods of the closed geodesics γ of length $\leq 2L_g$ constructed in the compact case
- neighbourhoods of the cusps of X.

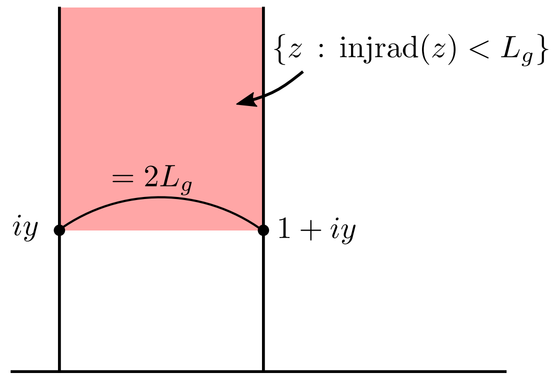


Figure 4.2: The thin part of the cusp represented in a fundamental domain in $\mathcal{H}$.

We know by the compact case that the volume of the neighbourhoods of the closed geodesics are smaller than $2e^{L_g}$. Let us now bound the volume of the cusp neighbourhood. In order to do so, we compare it with the cuspidal domain

$$\{z = x + iy \,|\, x \in [0,1], y > 0\} \big/ \{z \sim z + 1\}$$

as represented in Figure 4.2. Let $z = x + iy$ be a point of this set. Then, the length of the geodesic arc based at z freely homotopic to the puncture is the distance between z and $z + 1$

$$\mathrm{d}_{\mathcal{H}}(z, z+1) = 2 \operatorname{arcsinh}\left(\frac{1}{2y}\right).$$

This length is smaller than $2L_g$ if and only if $y \geq 1/(2\sinh(L_g))$, so the volume of the neighbourhood of the cusp is smaller than

$$\int_0^1 \int_{\frac{1}{2\sinh(L_g)}} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = 2\sinh(L_g) \leq e^{L_g}.$$

As a consequence,

$$\mathrm{Vol}(X^-(L_g)) \leq (2\mathrm{N}_X^\ell(0, 2L_g) + n(g))e^{L_g}$$

where $\mathrm{N}_X^\ell(0, 2L_g)$ is the number of closed geodesics on X of length $\leq 2L_g$. Then, the claim is true on the set

$$\mathcal{A}_g = \left\{ X \in \mathcal{M}_{g,n(g)} \,:\, \mathrm{N}_X^\ell(0, 2L_g) \leq \frac{M_g}{2} \right\}.$$

Let us bound the probability of the complement of this set using Markov's inequality:

$$1 - \mathbb{P}_g^{\mathrm{WP}}(A_g) = \mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{N}_X^\ell(0, 2L_g) > \frac{M_g}{2}\right) \leq \frac{2}{M_g}\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{N}_X^\ell(0, 2L_g)\right].$$

This expectation can be computed using Mirzakhani's integration formula. In order to do so, we need to distinguish the various topological situations.

- The expectation of the number of non-separating closed geodesics of length $\leq 2L_g$ is equal to

$$\frac{1}{V_{g,n(g)}} \int_0^{2L_g} V_{g-1,n(g)+2}(0,\dots,0,x,x)\, x \,\mathrm{d}x = \mathcal{O}\left(\frac{V_{g-1,n(g)+2}}{V_{g,n(g)}} e^{2L_g}\right) = \mathcal{O}\left(e^{2L_g}\right)$$

  by equation (3.8) and equation (3.10).

- If the curve is separating, then it separates the surface in two components of respective signatures (g_1, n_1+1) and (g_2, n_2+1), such that $g_1 + g_2 = g$ and $n_1 + n_2 = n(g)$. Then, the sum of the expectations of the counting functions for all these cases is equal to

$$\frac{1}{V_{g,n(g)}} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \int_0^{2L_g} V_{g_1,n_1+1}(0,\dots,0,x) V_{g_2,n_2+1}(0,\dots,0,x) x \,\mathrm{d}x$$
$$= \mathcal{O}\left(e^{2L_g} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \frac{V_{g_1,n_1+1} V_{g_2,n_2+1}}{V_{g,n(g)}} \right) = \mathcal{O}\left(\frac{n(g)^2}{g} e^{2L_g}\right).$$

  by Lemma 4.5.

□

4.2 The tangle-free hypothesis

The contents of this section are adapted from the article [MT21], a collaboration with Joe Thomas (University of Manchester), to appear in International Mathematics Research Notices.

In this section, we introduce the tangle-free hypothesis on compact (connected, oriented) hyperbolic surfaces (without boundary), and explore some of its geometric implications, with a special emphasis on random surfaces, which we show are almost optimally tangle-free.

This work follows several recent articles aimed at adapting results on random regular graphs in both geometry and spectral theory to the setting of random hyperbolic surfaces – see [Mir13, MP19, GLMST21, Mon21, Tho20, MNP20] for instance. Though the initial motivation was to provide some useful tools for spectral theory, the results and techniques developed here are purely geometric. Several of our results are significant improvements of useful properties of geodesics on compact hyperbolic surfaces, allowed by the random setting: the length scale at which they apply goes from constant to logarithmic in the genus.

A key innovation of this article is finding an elementary geometric condition which is simultaneously easy to prove for random surfaces, and has far-reaching consequences on their geometry (notably their geodesics) at a large scale. Similar geometric assumptions have been made recently by Mirzakhani and Petri [MP19, Proposition 4.5] and Gilmore, Le Masson, Sahlsten and Thomas [GLMST21]. The use of the tangle-free hypothesis simplifies and improves the result in [GLMST21], and generalises one consequence of [MP19, Proposition 4.5] to a larger scale.

The tangle-free hypothesis for hyperbolic surfaces

Let us first define what we mean by tangle-free and contrast it with existing concepts in the graph theoretic and hyperbolic surface literature. Heuristically speaking, we shall say that a surface is tangle-free if it does not contain embedded pairs of pants or one-holed tori with 'short' boundaries. More precisely, we make the following definition.

Definition 4. Let X be a compact hyperbolic surface and $L > 0$. Then, X is said to be *L-tangle-free* if all embedded pairs of pants and one-holed tori in X have total boundary length larger than $2L$. Otherwise, X is *L-tangled*.

To be precise, we emphasise that a pair of pants and a one-holed torus are respectively surfaces of signature $(0, 3)$ and $(1, 1)$, and the embedded surfaces we consider have totally geodesic boundary. The total boundary length is defined as the sum of the length of all the boundary geodesics. One should note that we could also have defined the notion of tangle-free using the maximum boundary length (the length of the longest boundary geodesic) and the results of this paper would follow through (up to changes of constants).

It may not be so clear to the reader why we call such a property *tangle-free*. In order to clarify this, we prove that, when a surface is tangled, it contains a non-simple geodesic; that is, a *tangled* geodesic in the literal sense of the word.

Proposition (Proposition 4.7)* *Any L-tangled surface contains a self-intersecting geodesic of length smaller than $2L + 2\pi$.*

Tangle-free graphs

One can motivate the study of this geometric property of surfaces through the medium of regular graphs. Indeed, the naming of this property is inspired by a similar notion Bordenave introduced in [Bor20] in order to prove Friedman's theorem [Fri03] regarding the spectral gap of the Laplacian on large regular graphs. A graph $G = (V, E)$ is said to be *L-tangle-free* if, for any vertex v, the ball for the graph distance d_G

$$\mathcal{B}_L(v) = \{w \in V \,:\, \mathrm{d}_G(v, w) \leq L\},$$

contains at most one cycle. This definition might seem quite different to the surface definition given above, but we shall prove that balls on tangle-free surfaces contain at most one 'cycle' in the following sense.

Proposition (Proposition 4.14)* *If a surface X is L-tangle-free, then for any point $z \in X$, the ball*

$$\mathcal{B}_{\frac{L}{8}}(z) = \left\{ w \in X \,:\, \mathrm{d}_X(z,w) < \frac{L}{8} \right\}$$

is isometric to a ball in the hyperbolic plane or a hyperbolic cylinder.

It is worth noting that in the original proof by Friedman [Fri03], there is also a notion of 'supercritical tangle' in a graph, which are small subgraphs with many cycles. In a sense, pairs of pants or one-holed tori with small total boundary lengths can be seen as analogues of these bad tangles for surfaces.

Admissible values of L

Let us now discuss typical values that L can take in Definition 4 both for being tangle-free and tangled. Throughout, we shall use the notation $A = \mathcal{O}(B)$ to indicate that there is a constant $C > 0$ such that $|A| \leq C|B|$ with C independent of all other variables such as the genus.

It is clear that a surface of injectivity radius r is r-tangle-free, for it has no closed geodesic of length smaller than $2r$. In a deterministic setting, it is hard to say much more than this.

On the other hand, we know that a hyperbolic surface of genus g admits a pants decomposition with all boundary components smaller than the *Bers constant* $\mathcal{B}_g$ – see [Bus92, Chapter 5]. We know that $\mathcal{B}_g \geq \sqrt{6g} - 2$ [Bus92, Theorem 5.1.3], and the best known upper bounds on $\mathcal{B}_g$ are linear in g [BS92, Par14]. All surfaces of genus g are $\frac{3}{2}\mathcal{B}_g$-tangled. This bound however is rather loose, since it follows from cutting *all* of the surface into pairs of pants rather than isolating a single short pair of pants. In light of this, we in fact prove the following, using a method based on Parlier's work [Par14].

Proposition (Proposition 4.16)* *Any hyperbolic surface of genus g is L-tangled for $L = 4 \log g + \mathcal{O}(1)$.*

Random graphs and surfaces

How tangle-free can a *typical* surface be? Can L be much larger than the injectivity radius for a large class of surfaces? An instructive method to answer these questions is to consider the setting of random surfaces, and to find an L for which most surfaces are L-tangle-free.

For d-regular graphs with n vertices, sampled with the uniform probability measure $\mathbb{P}_n^{(d)}$, Bordenave proved [Bor20] that for any real number $0 < a < \frac{1}{4}$,

$$\mathbb{P}_n^{(d)}(G \text{ is } (a \log_{d-1}(n))\text{-tangle-free}) \underset{n \to +\infty}{\longrightarrow} 1.$$

This is a key ingredient in Bordenave's proof of Friedman's theorem [Bor20].

We prove that a similar result holds for the Weil–Petersson probability model.

Theorem (Theorem 4.8)* *For any real number $0 < a < 1$,*

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \text{ is } (a\log g)\text{-tangle-free}\right) = 1 - \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

Since any surface of genus g is $(4\log g + \mathcal{O}(1))$-tangled, random surfaces are almost as tangle-free as possible. We recover the scale $\log g$ as in the Benjamini–Schramm result, which we recall is approximately the value of the diameter of a typical surface [Mir13]. Therefore, the geometric results in this sections will describe the geometry of a typical genus g surface at the large scale $\log g$, as aimed in this chapter.

We can note that the tangle-free hypothesis does not exclude the presence of small closed geodesics (which occur with non-zero probability [Mir13]), but it implies that these geodesics can never bound a pair of pants or separate a one-holed torus.

Geometric implications of the tangle-free hypothesis

The L-tangle-free hypothesis has various consequences on the local geometry of the surface at a scale (roughly) L, which we explore in Section 4.2.3. This will be particularly interesting when L is large; in the case of random surfaces notably, where $L = a\log g$ for $a < 1$. All the results are stated for any L-tangle-free surface, with a general L and no other assumption, so that they can be directly applied to another setting in which a tangle-free hypothesis is established.

First and foremost, we analyse the embedded cylinders around simple closed geodesics. In a hyperbolic surface with no further geometric assumptions to it, the standard collar theorem [Bus92, Theorem 4.1.1] proves that the collar of width $\operatorname{arcsinh}\left(\sinh(\ell/2)^{-1}\right)$ around a simple closed geodesic of length ℓ is an embedded cylinder; moreover, at this width, disjoint simple closed geodesics have disjoint collars. The width of this deterministic collar is optimal and very satisfying for small ℓ. For larger values of ℓ however, it becomes very poor. Under the tangle-free hypothesis, we are able to obtain significant improvements to the collar theorem that remedy this issue at larger scales.

Theorem (Theorem 4.9)* *On a L-tangle-free hyperbolic surface, the collar of width $\frac{L-\ell}{2}$ around a closed geodesic of length $\ell < L$ is isometric to a cylinder.*

This implies that we can find wide collars around geodesics of size $a\log g$, $a < 1$, on random surfaces; as a comparison, the width of the deterministic collar around such a geodesic decreases like $g^{-\frac{a}{2}}$. By a volume argument, Theorem 4.9 is optimal up to multiplication of the width by a factor two.

The methodology to prove this result is to examine the topology of an expanding neighbourhood of the geodesic. Since the two simplest hyperbolic subsurfaces (namely the pair of pants and one-holed torus) cannot be encountered up to a scale $\sim L$ due to the tangle-free hypothesis, the neighbourhood remains a cylinder.

An immediate consequence of this improved collar theorem is a bound on the number of intersections of a closed geodesic of length $\ell < L$ and any other geodesic of length ℓ'. We prove in Corollary 4.10 that two such geodesics intersect at most $1 + \frac{\ell'}{L-\ell}$ times (and we can remove the 1 if the two geodesics are closed). Therefore, two closed geodesics of length $< \frac{L}{2}$ do not intersect; Proposition 4.11 furthermore states that the collars of width $\frac{L}{2} - \ell$ around two such geodesics are disjoint.

As well as the neighbourhood of geodesics, one can look at the geometric consequences that the tangle-free hypothesis has on the neighbourhood of points. To this end, we explore the set of geodesic loops based at a point on the surface on length scales up to L. As has already been mentioned above in Proposition 4.14, which establishes a link between our tangle-free definition and that of graphs, on an L-tangle-free surface, balls of radius $\frac{L}{8}$ are isometric to balls in either the hyperbolic plane or a hyperbolic cylinder. There are several ways to prove this property, some of which are similar to the proof of the improved collar theorem. In order to present different methods, we rather deduce it from the following slightly more general result.

Theorem (Theorem 4.12)* *If z is a point on a L-tangle-free surface, and δ_z is the shortest geodesic loop based at z, then any other loop β based at z such that $\ell(\delta_z) + \ell(\beta) < L$ is homotopic to a power of δ_z.*

Another consequence of Theorem 4.12 is Corollary 4.15, which states that any closed geodesic of length $< L$ on a L-tangle-free surface is simple. Put together, these observations imply the following corollary.

Corollary* *On a L-tangle-free hyperbolic surface,*

1. *all closed geodesics of length $< L$ are simple;*
2. *all closed geodesics of length $< \frac{L}{2}$ are pairwise disjoint;*
3. *all closed geodesics of length $< \frac{L}{4}$ are embedded in pairwise disjoint cylinders of width $\geq \frac{L}{4}$.*

In the random case, this result is an improvement of the very useful collar theorem II [Bus92, Theorem 4.1.6], which states that all closed geodesics of length $< 2 \operatorname{arcsinh} 1$ on a hyperbolic surface are simple and do not intersect.

Short closed geodesics in random hyperbolic surfaces have been studied by Mirzakhani and Petri [Mir13, MP19]. One can deduce from [MP19, Proposition 4.5] and Markov's inequality that, for any fixed M,

$$1 - \mathbb{P}_g^{\mathrm{WP}} \left(\text{all closed geodesics of length } < M \text{ are simple}\right) \leq \frac{C_M}{g}$$

for a constant $C_M > 0$, when we prove that, for any real number $0 < a < 1$,

$$1 - \mathbb{P}_g^{\mathrm{WP}} \left(\text{all closed geodesics of length } < a \log g \text{ are simple}\right) \leq C \frac{(\log g)^2}{g^{1-a}}$$

for a constant $C > 0$. In order to push the proof in [MP19] to a scale $\log g$, one would need to use strong properties of the Weil–Petersson volumes and deal with technical estimates, while our approach is quite elementary in both the geometric and probabilistic sense.

As illustrated in Section 3.2.1, the tools used to study random surfaces in the Weil–Petersson setting require to reduce problems to the study of multicurves. Knowing that all closed geodesics of length $< \frac{a}{2} \log g$ form a multicurve can be useful to the understanding of other properties of random surfaces.

Furthermore, McShane and Parlier proved in [MP08] that for any $g \geq 2$,

$$\mathbb{P}_g^{\mathrm{WP}} \left(\text{the } \textit{simple} \text{ length spectrum has no multiplicities}\right) = 1$$

where the simple length spectrum of a surface is the list of all the lengths of its simple closed geodesics. Corollary 4.15 then implies the following.

Corollary 4.6. *For any $a \in (0,1)$, if $\mathcal{L}(X)$ denotes the length spectrum of X, then*

$$\mathbb{P}_g^{\mathrm{WP}} \left(\mathcal{L}(X) \cap [0, a \log g] \textit{ has no multiplicities}\right) = 1 - \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

This could be surprising since, by the work of Horowitz and Randol, for any compact hyperbolic surface, the length spectrum has unbounded multiplicity [Bus92, Theorem 3.7.1]. However, these high multiplicities are constructed in embedded pairs of pants, and therefore it is natural that their lengths are large for tangle-free surfaces.

Motivations in spectral theory

To conclude this introduction we will outline the connection between the geometry of hyperbolic surfaces and their spectral theory and in particular discuss how the tangle-free hypothesis and its implications on the geometry of surfaces on $\log g$ scales, which is a crucial scale in spectral theory, could be used to tackle some open problems in this area. As promised, let us first return to the relation of the tangle-free hypothesis with spectral graph theory and contrast this with that of surfaces.

In spectral theory, when studying large-scale limits ($n \to +\infty$ for a graph, $g \to +\infty$ for a surface), it is important to know that the small-scale geometry of the object will not affect the spectrum. Often, a simple assumption to avoid this is to assume the injectivity radius to be large.

Unfortunately, random regular graphs and surfaces have an asymptotically non-zero probability of having a small injectivity radius (see [Wor81b] and [Mir13, Theorem 4.2]). As a consequence, in both cases, if we want to prove results true with probability approaching 1 in the large-scale limit, one needs to impose weaker and more typical geometric conditions.

For instance, Brooks and Lindenstrauss [BL13] and Brooks and Le Masson [BLM20] studied eigenfunctions on regular graphs of size $n \to +\infty$, under assumptions on the number of cycles up to a certain length L. This parameter L can always be taken to be the injectivity radius, but in the case of random graphs, it can be increased to be of order $\log n$. In a recent article of Gilmore, Le Masson, Sahlsten and Thomas [GLMST21], a similar geometric hypothesis on the number of geodesic loops shorter than a scale L based at each point is made, in order to control the L^p-norms of eigenfunctions of the Laplacian on hyperbolic surfaces. The authors prove it holds for random surfaces of high genus g at a scale $L = c \log g$, but the proof provides no explicit value of the constant $c > 0$.

This limitation could be seen as originating from the methodology used to study the geometry of the surfaces. In essence, the authors prove that the loop condition is implied by a geometric condition, which is typical. This condition however is quite complex, and both the proof of its sufficiency and typicality are rather technical, leaving the local geometry of the random surfaces that are selected to remain quite opaque.

It follows from Corollary 4.13 that the constant c in [GLMST21] can be taken to be any value $< \frac{1}{4}$. In turn, this improves (and makes precise) the rate of convergence of the probability for which the L^p-norm estimates in [GLMST21] hold. This is rather demonstrative of the capabilities of the tangle-free geometric condition allowing for a firm grasp over $\log g$-scale geometries for spectral theoretic purposes.

Outline of the section

The section is organised as follows:

- Section 4.2.1: tangled surfaces have tangled geodesics.
- Section 4.2.2: random surfaces are $(a \log g)$-tangle-free for any $a < 1$.
- Section 4.2.3: geometric consequences of the tangle-free hypothesis.
- Section 4.2.4: any surface of genus g is $(4 \log g + \mathcal{O}(1))$-tangled.

4.2.1 Tangled surfaces have tangled geodesics

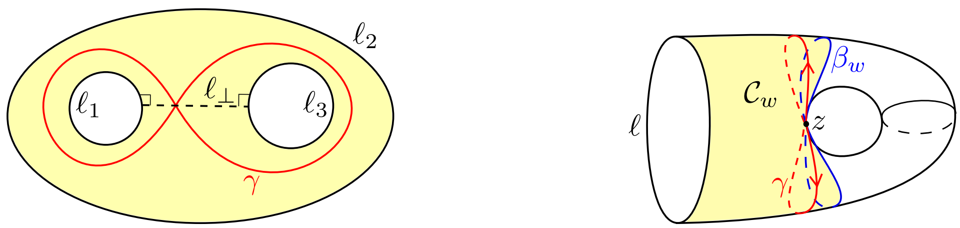
The aim in this section is to prove that being tangled implies having a tangled geodesic - that is to say a *non-simple* closed geodesic of length $\leq 2L + \mathcal{O}(1)$.

Proposition 4.7. *Let X be a compact hyperbolic surface and $L > 0$. Assume that X is L-tangled. Then, there exists a closed geodesic γ in X of length smaller than $2L + 2\pi$ with one self-intersection.*

The geodesic we construct is what is called a *figure eight*. Any non-simple geodesic on a hyperbolic surface has length greater than $4 \operatorname{arcsinh} 1 \approx 3.52\ldots$, and this result is sharp (see [Bus92, Theorem 4.2.2]).

Proof. It suffices to prove that there is such a geodesic in any pair of pants or one-holed torus of total boundary length smaller than $2L$.

Let us first consider a hyperbolic pair of pants of boundary lengths ℓ_1, ℓ_2, ℓ_3, such that $\ell_1 + \ell_2 + \ell_3 < 2L$. We construct a closed curve with one self-intersection as



(a) for a pair of pants (b) for a one-holed torus

Figure 4.3: Construction of a short self-intersecting geodesic.

represented in Figure 4.3a. By [Bus92, Formula 4.2.3], which can be proven using the common perpendicular $\ell_\perp$,

$$\cosh\left(\frac{\ell(\gamma)}{2}\right) = 2\cosh\left(\frac{\ell_1}{2}\right)\cosh\left(\frac{\ell_3}{2}\right) + \cosh\left(\frac{\ell_2}{2}\right) \leq 3\,e^L.$$

Since $\cosh x \geq \frac{e^x}{2}$, we deduce that the length of γ is smaller than $2L + 2\log 6$.

We use a different proof in the one-holed torus case, because we do not have access to several small geodesics straight away. Let us study a one-holed torus T of boundary length $\ell \leq 2L$. Let $w > 0$, and $\mathcal{C}_w$ be the w-neighbourhood of the boundary geodesic

$$\mathcal{C}_w = \{z \in T \,:\, \mathrm{d}(z, \partial T) < w\}.$$

By the collar theorem [Bus92, Theorem 4.1.1], when w is small enough, $\mathcal{C}_w$ is a half-cylinder with Fermi coordinates (ρ, t), in which the hyperbolic metric is $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \cosh^2\rho\,\mathrm{d}t^2$. This isometry has to break down at some point, because the area of the one-holed torus is 2π, and as long as the isometry holds

$$\mathrm{Vol}(\mathcal{C}_w) = \int_0^\ell \int_0^w \cosh\rho\,\mathrm{d}\rho\,\mathrm{d}t = \ell\sinh w \leq 2\pi. \tag{4.5}$$

We pick w to be the supremum of the widths for which the isometry holds. By continuity, w satisfies inequality (4.5). Up until w, the length $\ell(\beta_{w'})$ of the inside boundary $\beta_{w'} = \partial\mathcal{C}_{w'} \setminus \partial T$ of the half-collar satisfies

$$\ell(\beta_{w'}) = \ell\cosh w', \tag{4.6}$$

which by continuity still holds for the supremum width w. Put together, equations (4.5) and (4.6) imply that

$$\ell(\beta_w) = \ell\sqrt{1 + \sinh^2(w)} \leq \ell\sqrt{1 + \frac{4\pi^2}{\ell^2}} \leq 2L + 2\pi.$$

The reason why the isometry ceases at w has to be that the boundary β_w self-intersects. Let z be a self-intersection point. This intersection point allows us to create a pants decomposition of the handle (see Figure 4.3b), as well as find a closed geodesic γ which is freely homotopic to the closed curve β_w. We can then conclude by observing that the length of γ is less than $\ell(\beta_w) \leq 2L + 2\pi$ and γ has one self-intersection because it is the figure eight geodesic constructed in the pair of pants case projected to the one-holed torus. □

4.2.2 Random surfaces are $(a \log g)$-tangle-free

In this section, we will show that, for any $0 < a < 1$, typical surfaces of genus g are $(a \log g)$-tangle-free. By typical we mean in the probabilistic sense for the Weil–Petersson model of random surfaces.

The proof relies on Mirzakhani's interation formula (Theorem 3.7) together with the volume estimates listed in Section 3.2.3.

Theorem 4.8. *For any real number $0 < a < 1$,*

$$\mathbb{P}_g^{\mathrm{WP}}\left(X \text{ is } (a \log g)\text{-tangle-free}\right) = 1 - \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

Proof. Let us list all the topological types of embedded one-holed tori or pair of pants in a genus g surface (see Figure 4.4):

(i) a curve separating a one-holed torus;

(ii) three curves cutting S_g into a pair of pants and a component $S_{g-2,3}$;

(iii) three curves cutting S_g into a pair of pants and two components $S_{g_1,1}$ and $S_{g_2,2}$ such that $g_1 + g_2 = g - 1$;

(iv) three curves cutting S_g into a pair of pants and three connected components $S_{g_1,1}$, $S_{g_2,1}$ and $S_{g_3,1}$ with $1 \leq g_1 \leq g_2 \leq g_3$ and $g_1 + g_2 + g_3 = g$.

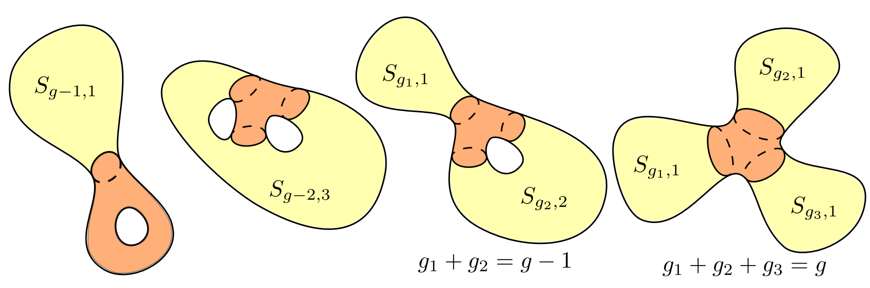


Figure 4.4: The different topological ways to embed a one-holed torus or pair of pants in a surface of genus g.

For any topological situation, we will consider a multicurve α on the base surface S_g realising the topological configuration and study the counting function

$$N_L^\alpha(X) = \#\{\beta \in \mathcal{O}(\alpha) \ : \ \ell_X(\beta) \leq 2L\},$$

where the length of a multi-curve is defined as the sum of its components. Then, the probability of finding a component in the topological situation α of total boundary length $\leq 2L$ can be bounded by Markov's inequality:

$$\mathbb{P}_g^{\mathrm{WP}}\left(N_L^\alpha(X) \geq 1\right) \leq \mathbb{E}_g^{\mathrm{WP}}\left[N_L^\alpha(X)\right].$$

We observe that $N_L^\alpha(X)$ is a geometric function, and its expectation can therefore be computed using Mirzakhani's integration formula (3.7). This reduces the problem to estimating integrals with Weil–Petersson volumes, which we will now detail.

In case (i), the integral that appears is

$$\int_0^{2L} V_{1,1}(\ell) V_{g-1,1}(\ell)\, \ell \,\mathrm{d}\ell.$$

From [NN98], it is known that $V_{1,1}(\ell) = \frac{\ell^2}{24} + \frac{\pi^2}{6}$. Moreover, by equation (3.8),

$$\ell V_{g-1,1}(\ell) \leq 2 e^{\frac{\ell}{2}} V_{g-1,1}.$$

It follows that the probability is smaller than

$$\frac{V_{g-1,1}}{V_g} \int_0^{2L} 2\left(\frac{\ell^2}{24} + \frac{\pi^2}{6}\right) e^{\frac{\ell}{2}} \,\mathrm{d}\ell = \mathcal{O}\left(\frac{V_{g-1,1}}{V_g} L^2 e^L\right) = \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}}\right)$$

where the last bound is deduced from equations (3.10) and (3.12) and taking $L = a \log g$.

In case (ii), the integral that appears is

$$\frac{1}{V_g} \iiint_{0 \leq \ell_1+\ell_2+\ell_3 \leq 2L} V_{0,3}(\ell_1,\ell_2,\ell_3) V_{g-2,3}(\ell_1,\ell_2,\ell_3)\, \ell_1\ell_2\ell_3 \,\mathrm{d}\ell_1\,\mathrm{d}\ell_2\,\mathrm{d}\ell_3.$$

Due to the fact that $V_{0,3}(\ell_1,\ell_2,\ell_3) = 1$ and by equation (3.8), we need to estimate

$$\frac{V_{g-2,3}}{V_g} \iiint_{0 \leq \ell_1+\ell_2+\ell_3 \leq 2L} \exp\left(\frac{\ell_1+\ell_2+\ell_3}{2}\right) \mathrm{d}\ell_1\,\mathrm{d}\ell_2\,\mathrm{d}\ell_3 = \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}}\right)$$

by equations (3.10) and (3.12).

Let us now bound the sum of all the topological situations of case (iii). By the same manipulations, we obtain that the probability is

$$\mathcal{O}\left(\frac{L^2 e^L}{V_g} \sum_{g_1+g_2=g-1} V_{g_1,1} V_{g_2,2}\right) = \mathcal{O}\left(\frac{(\log g)^2}{g^{1-a}} \frac{V_{g-1,1}}{V_g}\right) = \mathcal{O}\left(\frac{(\log g)^2}{g^{2-a}}\right)$$

by equation (3.11) and then equations (3.10) and (3.12).

Finally, in the last case we have to estimate

$$\begin{aligned}
&\sum_{\substack{g_1+g_2+g_3=g \\ 1 \leq g_1 \leq g_2 \leq g_3}} V_{g_1,1} V_{g_2,1} V_{g_3,1} \\
&= \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} V_{g_1,1} \sum_{g_2+g_3=g-g_1} V_{g_2,1} V_{g_3,1} \leq C_0 \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} \frac{V_{g_1,1} V_{g-g_1,0}}{g-g_1}
\end{aligned}$$

where C_0 is the constant from equation (3.11). We observe that $g - g_1 \geq \frac{2}{3}g$ and use equation (3.12) to conclude that the probability is

$$\mathcal{O}\left(\frac{(\log g)^2}{V_g g^{2-a}} \sum_{g_1=1}^{\lfloor \frac{g-2}{3} \rfloor} V_{g_1,1} V_{g-g_1,1}\right) = \mathcal{O}\left(\frac{(\log g)^2}{g^{3-a}}\right)$$

by equation (3.11). □

Remark. In the cases (i), (iii) and (iv), there is a separating geodesic of length $\leq 2a \log g$. Therefore, we could have bounded these probabilities by the probability of having a separating geodesic of length $\leq 2a \log g$, which has been estimated by Mirzakhani in [Mir13, Theorem 4.4]. This approach yields the same end result, but the authors decided to detail the four cases for the sake of self-containment. Furthermore, this more detailed study allows us to see that the most likely cases are cases (i.) and (ii.), and therefore we expect the first length at which the surface is tangled to be obtained by one of these two topological situations.

4.2.3 Geometry of tangle-free surfaces

The aim of this section is to provide information about geodesics and neighbourhoods of points on tangle-free surfaces. The results will be expressed in terms of an arbitrary L-tangle-free surface X, but can also been seen as result that are true with high probability for $L = a \log g$, $a < 1$ due to Theorem 4.8.

An improved collar theorem

Theorem 4.9. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Let γ be a simple closed geodesic of length $\ell < L$. Then, for $w := \frac{L-\ell}{2}$, the neighbourhood*

$$\mathcal{C}_w(\gamma) = \{z \in X \,:\, \mathrm{d}(z, \gamma) < w\}$$

is isometric to a cylinder.

The collar theorem [Bus92] is a similar result, with the width $\operatorname{arcsinh}\big(\sinh(\ell/2)^{-1}\big)$.

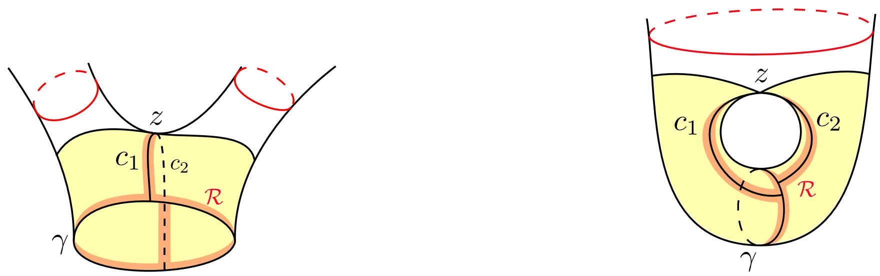
We recall that, in the random case, for $a < 1$, with high probability, we can take $L = a \log g$. This result therefore is a significant improvement for geodesics of length $b \log g$, $0 < b < a$. We obtain a collar of width $w = \frac{a-b}{2} \log g$, which is expanding with the genus, as opposed to the deterministic collar, of width $\simeq g^{-\frac{b}{2}}$.

For very short geodesics, the width of this new collar is $\simeq \frac{a}{2} \log g$. It might seem less good than the deterministic collar, which is of width $\simeq -\log(\ell)$. However, by Theorem 4.2 in [Mir13], the injectivity radius of a random surface is greater than $g^{-\frac{a}{2}}$ with probability $1 - \mathcal{O}(g^{-a})$. Under this additional probabilistic assumption, the two collars are of similar sizes.

Proof. For small enough w, the neighbourhood $\mathcal{C}_w(\gamma)$ is a cylinder, with two boundary components $\gamma_w^{\pm}$. Let us assume that, for a certain w, the topology of the neighbourhood changes. There are two ways for this to happen (and both can happen simultaneously) – see Figure 4.5.

(A) One boundary component, γ_w^+ or γ_w^-, self-intersects.

(B) The two boundary components γ_w^+ and γ_w^- intersect one another.



(a) one side self-intersects (b) the two sides intersect one another

Figure 4.5: Illustration of the ways the isometry breaks down when expanding a cylinder around the geodesic γ.

In both cases, let $z \in X$ denote one intersection point. Since the distance between z and γ is w, there are two distinct geodesic arcs c_1, c_2 of length w, going from z to points of γ, and intersecting γ perpendicularly. Both c_1 and c_2 are orthogonal to the boundaries of the cylinder and the two boundaries are tangent to one another by minimality of the width w. As a consequence, the curve $c = c_1^{-1}c_2$ is a geodesic arc.

The regular neighbourhood of the curves γ and c has Euler characteristic -1. There are two possible topologies for this neighbourhood.

- If it is a pair of pants, then it has three boundary components. Neither of them is contractible on the surface X. Indeed, one component is freely homotopic to γ, and the two others to c and a portion of γ, which are geodesic bigons. Therefore, when we replace the boundary components of the regular neighbourhood by the closed geodesic in their free homotopy classes, we obtain a pair of pants or a one-holed torus (if two of the boundary components are freely homotopic to one another), of total boundary length smaller than $2\ell + 4w$.

- Otherwise, it is a one-holed torus. Its boundary component is not contractible, because there is no hyperbolic surface of signature $(1, 0)$. Therefore, the closed geodesic in its free homotopy class separates a one-holed torus with boundary length smaller than $2\ell + 4w$ from X.

In both cases, by the tangle-free hypothesis, $2L < 2\ell + 4w$, which allows us to conclude. $\square$

Remark. Let $\mathcal{A}_g \subset \mathcal{M}_g$ be the event "the surface has a simple closed geodesic of length between 1 and 2". By work of Mirzakhani and Petri [MP19],

$$\mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{A}_g\right) \underset{g\to+\infty}{\longrightarrow} 1 - e^{-\lambda} \quad \text{with} \quad \lambda = \int_1^2 \frac{e^t + e^{-t} - 2}{t}\,\mathrm{d}t > 0,$$

so this event has asymptotically non-zero probability.

Let X be an element of $\mathcal{A}_g$ which is also $(a \log g)$-tangle-free, and let γ be a closed geodesic on X of length $\ell \in [1, 2]$. Then, the collar $\mathcal{C}_w(\gamma)$ given by Theorem 4.9 has volume

$$\mathrm{Vol}(\mathcal{C}_w(\gamma)) = 2\ell \sinh w \geq 2 \sinh\left(\frac{a}{2} \log g - 1\right) \sim g^{\frac{a}{2}} \quad \text{as } g \to +\infty.$$

However, $\mathrm{Vol}(\mathcal{C}_w(\gamma)) \leq \mathrm{Vol}\, X = 2\pi(2g-2)$. This leads to a contradiction for g approaching $+\infty$ as soon as $a > 2$. Hence, for large g, the elements of $\mathcal{A}_g$ are $(a \log g)$-tangled for $a > 2$:

$$\limsup_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(X \text{ is } (a\log g)\text{-tangled}\right) \geq \lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{A}_g\right) > 0.$$

Therefore, for all $a > 2$, random surfaces do *not* have high probability of being $(a \log g)$-tangle-free.

By taking a close to but larger than 1, this same line of reasoning and the fact that we know surfaces to be $(a \log g)$-tangle-free with high probability implies that the improved collar cannot be much larger than $L - \ell$. As a consequence, our result is optimal up to multiplication by 2.

Number of intersections of geodesics

A consequence of this improved collar theorem is a bound on the number of intersections of a short closed geodesic with any other geodesic.

Corollary 4.10. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface.*

Let γ be a simple closed geodesic of length $< L$ on X. Then, for any geodesic γ' transverse to γ, the number of intersections $i(\gamma, \gamma')$ between γ and γ' satisfies

$$i(\gamma, \gamma') \leq \frac{\ell(\gamma')}{L - \ell(\gamma)} + 1.$$

In the case where γ' is also closed, then

$$i(\gamma, \gamma') \leq \frac{\ell(\gamma')}{L - \ell(\gamma)}.$$

In particular, if $\ell(\gamma) + \ell(\gamma') < L$, then γ and γ' do not intersect.

Proof. By Theorem 4.9, γ is embedded in an open cylinder $\mathcal{C}$ of width $w = \frac{L-\ell(\gamma)}{2}$.

Let us parametrise the geodesic $\gamma' : [0,1] \to X$. The set of times when γ' visits the cylinder can be decomposed as

$$\bigsqcup_{i=1}^{k} (t_i^-, t_i^+), \qquad 0 \leq t_1^- < t_1^+ \leq \ldots \leq t_k^- < t_k^+ \leq 1,$$

as respresented in Figure 4.6. The restriction c_i of γ' between t_i^- and t_i^+ is a geodesic in

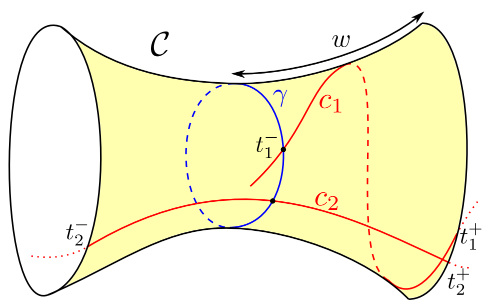


Figure 4.6: Illustration of the proof of Corollary 4.10.

the cylinder $\mathcal{C}$, transverse to the central geodesic γ. Therefore, if c_i intersects γ, then it does at most once. Let $I \subset \{1, \ldots, k\}$ be the set of i such that c_i intersect γ. We have that $i(\gamma, \gamma') = \#I \leq k$.

We assume that $\#I \geq 2$ (otherwise their is nothing to prove). Any geodesic intersecting the central geodesic transversally travels through the entire cylinder, and is therefore of length greater than $2w$. As a consequence, for any $i \in I$ different from 1 and k, $\ell(c_i) \geq 2w$. Also, if $i = 1$ or k belongs in I, then $\ell(c_i) \geq w$. This leads to our claim, because

$$(i(\gamma, \gamma') - 1)(L - \ell(\gamma)) = (\#I - 1) \cdot 2w \leq \sum_{i \in I} \ell(c_i) \leq \ell(\gamma').$$

The case when the curve γ' is closed can be obtained observing that, in this case, $\ell(c_1)$ and $\ell(c_k)$ also are greater than $2w$ (when 1 or k belongs in I). $\square$

Like the collars from the usual collar theorem, the collars of two small enough distinct geodesics are disjoint.

Proposition 4.11. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Let γ, γ' be two distinct simple closed geodesics such that $\ell(\gamma) + \ell(\gamma') < L$. Then, the distance between γ and γ' is greater than $L - \ell(\gamma) - \ell(\gamma')$.*

In particular, if $\ell(\gamma), \ell(\gamma') < \frac{L}{2}$, then the collars of width $\frac{L}{2} - \ell(\gamma)$ around γ and $\frac{L}{2} - \ell(\gamma')$ around γ' are two disjoint embedded cylinders.

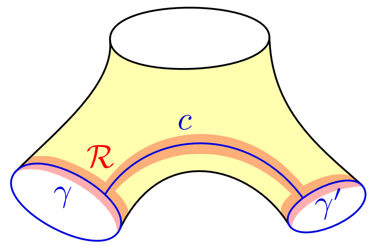


Figure 4.7: Illustration of the proof of Proposition 4.11

Proof. We already know, owed to Corollary 4.10, that γ and γ' do not intersect. Let c be a length-minimising curve with one endpoint on γ and the other on γ' (see Figure 4.7). Then, by minimality, c is simple and only intersects γ and γ' at is endpoints. The regular neighbourhood $\mathcal{R}$ of γ, γ' and c is a topological pair of pants of total boundary length less than $2(\ell(\gamma)+\ell(\gamma')+\ell(c))$. Since γ and γ' are non-contractible and not freely homotopic to one another, the third boundary component is not contractible and $\mathcal{R}$ corresponds to an embedded pair of pants or one-holed torus on X. By the tangle-free hypothesis, $\ell(\gamma) + \ell(\gamma') + \ell(c) \geq L$, and therefore the distance between γ and γ' is greater than $L - \ell(\gamma) - \ell(\gamma')$. This implies our claim. □

Short loops based at a point

Let us now study short loops based at a point on a tangle-free surface.

Theorem 4.12. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Let $z \in X$, and let δ_z be the shortest geodesic loop based at z.*

Let β be a (non necessarily geodesic) loop based at z, such that $\ell(\beta) + \ell(\delta_z) < L$. Then β is homotopic with fixed endpoints to a power of δ_z.

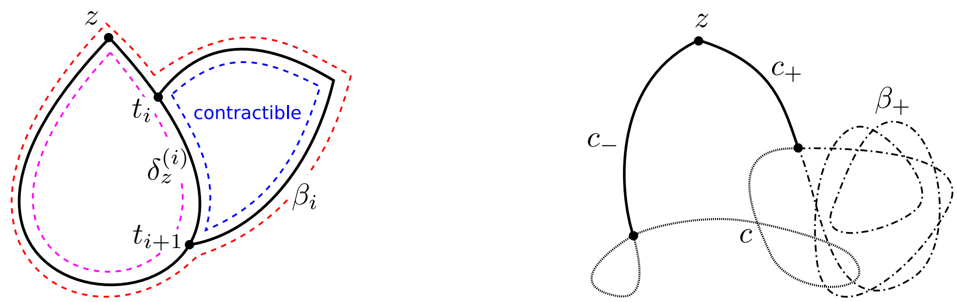
The result is empty if the injectivity radius of the point z is greater than $\frac{L}{2}$. The "shortest geodesic loop" δ_z is not necessarily unique. It will be as soon as the injectivity radius at z is smaller than $\frac{L}{4}$. More precisely, we directly deduce from Theorem 4.12 the following corollary, which was used in [GLMST21] for random surfaces (with a length $L = a \log g$, but the value of a was not explicit). Note the similarity of this result to the classical Margulis lemma [Rat19]. In particular, we obtain an explicit constant for the Margulis lemma in the case of tangle-free surfaces in the same way that the classical collar theorem provides.

Corollary 4.13. *Let $L > 0$, and $X = {}^{\mathcal{H}}\!/_{\Gamma}$ be an L-tangle-free hyperbolic surface. Then, for any $z \in \mathcal{H}$, the set $\{T \in \Gamma \,:\, \mathrm{d}_{\mathcal{H}}(z, T \cdot z) < \frac{L}{2}\}$ is:*

- *reduced to the identity element (when the injectivity radius at z is $\geq \frac{L}{4}$),*
- *or included in the subgroup $\langle T_0 \rangle$ generated by the element $T_0 \in \Gamma$ corresponding to the shortest geodesic loop through z.*

We recall that any compact hyperbolic surface is isometric to a quotient of the hyperbolic plane $\mathcal{H}$ by a Fuchsian co-compact group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ – see [Kat92] for more details.

We could prove Theorem 4.12 using the same method as we used for Theorem 4.9 and Corollary 4.10, expanding a cylinder around δ_z. However, our initial proof used a different method, which we decided to present here, in order to expose different ways to use the tangle-free hypothesis.



(a) Case $k = 0$. (b) Case $k > 0$.

Figure 4.8: Illustrations of the proof of Theorem 4.12.

Proof of Theorem 4.12. By replacing β by a new curve in its homotopy class, we can assume that β has a finite number of self-intersections, and of intersections with δ_z, while still satisfying the length condition.

We now prove this result by induction on the number of self-intersections $k \geq 0$ of β. We start with the base case of $k = 0$ so that β is simple. We parametrise $\beta : [0, 1] \to X$. Let $0 = t_0 < t_1 < \ldots < t_I = 1$ be the times when β meets δ_z.

Let $0 \leq i < I$, and β_i be the restriction of β to $[t_i, t_{i+1}]$ – see Figure 4.8a. Then, the regular neighbourhood $\mathcal{R}$ of δ_z and β_i has Euler characteristic -1, and total boundary length $\leq 2(\ell(\delta_z) + \ell(\beta_i)) < 2L$. If $\mathcal{R}$ is a topological one-holed torus, then by the tangle-free hypothesis, its boundary component is contractible, which is impossible for there is no hyperbolic surface of signature $(1, 0)$.

Therefore, $\mathcal{R}$ is a topological pair of pants. By the tangle-free hypothesis, one of its boundary components is contractible. It can not be the component corresponding to δ_z, so it is another one. Hence, β_i is homotopic with fixed endpoints to a portion $\delta_z^{(i)}$ of δ_z.

As a consequence, $\beta = \beta_0 \ldots \beta_{I-1}$ is homotopic with fixed endpoints to the product

$$c = \delta_z^{(0)} \delta_z^{(1)} \ldots \delta_z^{(I-1)}.$$

c goes from z to z following only portions of δ_z. Therefore, c is homotopic with fixed endpoints to a power δ_z^j of δ_z.

We now move forward to the case $k > 0$. We assume the result to hold for any smaller k. The idea is to find a way to cut β into smaller loops on which to apply the induction hypothesis; the construction is represented in Figure 4.8b.

Let $\ell = \ell(\beta)$. We pick a length parametrisation of $\beta : \mathbb{R}/\ell\mathbb{Z} \to X$ such that $\beta(0) = z$. We look for the first intersection point of β, starting a 0, but looking in both directions:

$$\ell_+ = \min\{t \geq 0 \,:\, \exists s \in (t, \ell) \text{ such that } \beta(s) = \beta(t)\}$$
$$\ell_- = \min\{t \geq 0 \,:\, \exists s \in (t, \ell) \text{ such that } \beta(-s) = \beta(-t)\}.$$

Up to a change of orientation of β, we can assume that $\ell_+ \leq \ell_-$. Then, we set

$$t = \max\{s \in (\ell_+, \ell) \,:\, \beta(s) = \beta(\ell_+)\}$$

to be the last time at which β visits $\beta(\ell_+)$, so that the restriction of β to $[\ell_+, t]$ is a loop β_+. The curve has no self-intersection between $\ell - \ell_-$ and ℓ, so $t \leq \ell - \ell_-$. Then, if we denote by c_+, c and c_- the respective restrictions of β to $[0, \ell_+]$, $[t, \ell - \ell_-]$ and $[\ell - \ell_-, \ell]$, we can write $\beta = c_+ \, \beta_+ \, c \, c_-$, which is homotopic with fixed endpoints to $(c_+ \, \beta_+ \, c_+^{-1}) \, (c_+ \, c \, c_-)$.

Let us apply the induction hypothesis to the two loops $c_+ \, \beta_+ \, c_+^{-1}$ and $c_+ \, c \, c_-$. It will follow that they, and hence β, are homotopic with fixed endpoints to a power of δ_z.

β_+ is a sub-loop of β. As a consequence, $c_+ \, c \, c_-$ has less self-intersections than β, and hence strictly less than k. Furthermore, it is shorter, so it satisfies the length hypothesis $\ell(c_+ \, c \, c_-) + \ell(\delta_z) < L$. So we can apply the induction hypothesis.

c_+ is simple and does not intersect β_+ (except at its endpoint). As a consequence, we can find a curve b homotopic to $c_+ \, \beta_+ \, c_+^{-1}$ with as many self-intersections as β_+. β_+ is a strict sub-loop of β, so this intersection number is strictly smaller than k. The length of b can be taken as close as desired to that of $c_+ \, \beta_+ \, c_+^{-1}$. Moreover,

$$\ell(c_+ \, \beta_+ \, c_+^{-1}) = 2\ell_+ + \ell(\beta_+) \leq \ell_+ + \ell_- + \ell(\beta_+) \leq \ell(\beta)$$

so b can be chosen to satisfy the length hypothesis $\ell(\delta_z) + \ell(b) < L$, and we can apply the induction hypothesis to it. $\square$

Neighbourhood of a point and graph definition

Now that we know about short loops based at a point, we can understand the geometry (and topology) of balls on a tangle-free surface.

Proposition 4.14. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. For a point z in X, let $\mathcal{B}_{\frac{L}{8}}(z) := \left\{w \in X \,:\, \mathrm{d}_X(z, w) < \frac{L}{8}\right\}$. Then, $\mathcal{B}_{\frac{L}{8}}(z)$ is isometric to a ball in either the hyperbolic plane (whenever the injectivity radius at z is $\geq \frac{L}{8}$) or a hyperbolic cylinder.*

In the second case, since the injectivity radius at z is greater than $\frac{L}{8}$, the ball $\mathcal{B}_{\frac{L}{8}}(z)$ is not contractible on X; it is therefore homeomorphic to a cylinder (see Figure 4.9).

In a sense, this corollary proves that our notion of tangle-free implies the natural translation of the notion of tangle-free for graphs. Indeed, the ball $\mathcal{B}_{\frac{L}{8}}(z)$ has either

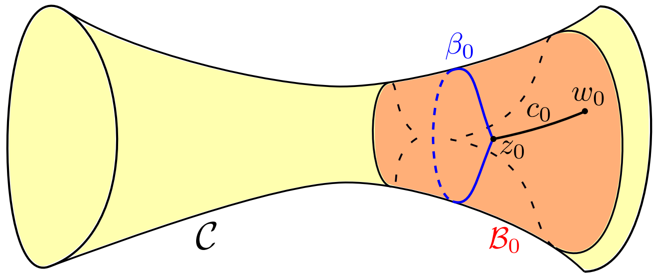


Figure 4.9: Illustration of the proof of Proposition 4.14 in the cylinder $\mathcal{C}$. Neighbourhoods of points of small injectivity radius on a tangle-free surface are isometric to balls in cylinders, like $\mathcal{B}_0$.

no non-contractible geodesic loop, or only one (and its iterates). We could have picked Proposition 4.14 to be a definition for tangle-free, but we consider the pair of pants definition to be both convenient to use and natural in the context of hyperbolic geometry and the Weil–Petersson model.

Proof. In order to prove this result, we will work in the universal cover $\mathcal{H}$ of X. Let us write $X = {}^{\mathcal{H}}\!/_{\Gamma}$, for a co-compact Fuchsian group Γ.

Let z be a point on X of injectivity radius smaller than $\frac{L}{8}$ (otherwise, the conclusion is immediate). Then, the shortest geodesic loop β based at z satisfies $\ell(\beta) < \frac{L}{4}$.

Let $\tilde{z} \in \mathcal{H}$ be a lift of z, $\tilde{\beta}$ be a lift of β starting at $\tilde{z}$, and $\tilde{\mathcal{B}}$ be the ball of radius $\frac{L}{8}$ around $\tilde{z}$ in $\mathcal{H}$. Let $T_\beta \in \Gamma$ be the covering transformation corresponding to β. The quotient $\mathcal{C} = {}^{\mathcal{H}}\!/_{\langle T_\beta \rangle}$ is a hyperbolic cylinder. The ball $\tilde{\mathcal{B}}$ is projected on a ball $\mathcal{B}_0$ on $\mathcal{C}$. Let us prove that the projection from $\mathcal{B}_0$ on $\mathcal{C}$ to $\mathcal{B}$ on X is an isometry.

In order to do so, we shall establish that for any $\tilde{w} \in \tilde{\mathcal{B}}$, the set of transformations $T \in \Gamma$ such that $T \cdot \tilde{w} \in \tilde{\mathcal{B}}$ is included in $\langle T_\beta \rangle$. Since any two points in $\tilde{\mathcal{B}}$ are at a distance at most $\frac{L}{4} < \frac{L}{2}$, this will follow from proving

$$\Gamma_L(\tilde{w}) := \left\{ T \in \Gamma \ : \ \mathrm{d}_{\mathcal{H}}(\tilde{w}, T \cdot \tilde{w}) < \frac{L}{2} \right\} \subset \langle T_\beta \rangle.$$

Let c be the shortest path from $\tilde{w}$ to $\tilde{z}$. The path $c\ \tilde{\beta}\ (T_\beta \circ c^{-1})$ is a path from $\tilde{w}$ to $T_\beta \cdot \tilde{w}$. Its length is $2\ell(c) + \ell(\beta) < 2 \times \frac{L}{8} + \frac{L}{4} = \frac{L}{2}$. As a consequence, T_β belongs in $\Gamma_L(\tilde{w})$. Then, $\Gamma_L(\tilde{w})$ is not reduced to $\{\mathrm{id}\}$. By Corollary 4.13, it is included in a cyclic subgroup $\langle T_0 \rangle$. T_β hence is a power of T_0, but T_β is primitive. Therefore, $T_\beta = T_0^{\pm 1}$, and the conclusion follows. □

Short geodesics are simple

Corollary 4.15. *Let $L > 0$, and X be a L-tangle-free hyperbolic surface. Any primitive closed geodesic on X of length $< L$ is simple.*

This consequence of Theorem 4.12 can also be deduced from the fact that the shortest non-simple primitive closed geodesic on a compact hyperbolic surface is a figure eight geodesic [Bus92, Theorem 4.2.4], which is embedded in a pair of pants or one-holed torus.

Proof. Let us assume by contradiction that γ is not simple; we can then pick an intersection point z. This allows us to write γ as the product of two geodesic loops γ_1, γ_2 based at z. Since $\ell(\gamma_1) + \ell(\gamma_2) < L$, one of them is $< L/2$. Up to a change of notation, we take it to be γ_1.

Let δ_z be the shortest geodesic loop based at z. By definition, $\ell(\delta_z) \leq \ell(\gamma_1)$. So γ_1 and γ_2 both satisfy the length hypothesis of Theorem 4.12:

$$\begin{aligned} \ell(\gamma_1) + \ell(\delta_z) &\leq 2\ell(\gamma_1) < L \\ \ell(\gamma_2) + \ell(\delta_z) &\leq \ell(\gamma) < L. \end{aligned}$$

Therefore, they are both homotopic with fixed endpoints to powers of δ_z, which implies γ is too. So γ is freely homotopic to a power j of the simple closed geodesic γ_0 in the free homotopy class of δ_z. By uniqueness, $\gamma = \gamma_0^j$. γ is primitive, so $j = 0$ or 1. But γ is not contractible (so $j \neq 0$) and not simple (so $j \neq 1$): we reach a contradiction, which allows us to conclude. □

Remark. Put together, Corollary 4.15 and 4.10 imply that all primitive closed geodesics of length $< \frac{L}{2}$ are simple and disjoint. Any such family of curves has cardinality at most $2g-2$. But we know that the number of primitive closed geodesics of length $< \frac{L}{2}$ on a fixed surface is equivalent to $\frac{2}{L}e^{\frac{L}{2}}$ as $L \to +\infty$ [Hub59, Bus92]. This can be seen as another indicator that, if X is L-tangle-free of large genus, then we expect L to be at most logarithmic in g.

4.2.4 Any surface of genus g is $(4 \log g + \mathcal{O}(1))$-tangled

We recall that any surface is L-tangled for $L = \frac{3}{2}\mathcal{B}_g$, the Bers constant, because it can be entirely decomposed in pairs of pants of maximal boundary length smaller than $\mathcal{B}_g$. The best known estimates on the Bers constant $\mathcal{B}_g$ are linear in the genus g [BS92, Par14], which is pretty far off the $c \log g$ we obtained for random surfaces. This is not a surprise, because in order to prove that a surface is tangled, we only need to find one embedded pair of pants or one-holed torus. In Buser and Parlier's estimates on $\mathcal{B}_g$ [Bus92, Par14], the pair of pants decomposition is constructed by successively exhibiting short curves on the surface; the first ones are of length $\simeq \log g$, but as the construction goes on, and we find $2g-2$ curves to entirely cut the surface, a linear factor appears.

In our case, we only need to stop the construction as soon as we manage to separate a pair of pants. Following Parlier's approach in [Par14] to bound the Bers constant, we prove the following.

Proposition 4.16. *There exists a constant $C > 0$ such that, for any $g \geq 2$, any compact hyperbolic surface of genus g is X is L-tangled for $L = 4 \log g + C$.*

This goes to prove that random hyperbolic surfaces are almost optimally tangle-free, despite the possibility of having a small injectivity radius.

The proof relies on the following two Lemmas, which are all used by Parlier [Par14]. Lemma 4.17, due to Bavard [Bav96], allows us to find a small geodesic loop on our surface.

Lemma 4.17. *Let X be a hyperbolic surface of genus g. For any $z \in X$, the length of the shortest geodesic loop through z is smaller than*

$$2\operatorname{arccosh}\left(\frac{1}{2\sin\frac{\pi}{12g-6}}\right) = 2\log g + \mathcal{O}(1).$$

Some problems will arise in the proof if the geodesic loop we obtain using this result is too small. These difficulties can be solved by assuming a lower bound on the injectivity radius of the surface; for instance, for random surfaces, with high probability, one can assume that the injectivity radius is bounded below by $g^{-\varepsilon}$ for a $\varepsilon > 0$ [Mir13]. However, such an assumption makes the final inequality weaker.

Another way to fix this issue, used in [Par14], is to expand all the small geodesics, and by this process obtain a new surface, with an injectivity radius bounded below, and in which the lengths of all the curves are longer. For our purposes, we only need to expand one curve. This is achieved by the following Lemma.

Lemma 4.18 (Theorem 3.2 in [Par05])**.** *Let $S_{g,n}$ be a base surface with $n > 0$ boundary components. Let $(X, f) \in \mathcal{T}_{g,n}(\ell_1, \ldots, \ell_n)$ and $\varepsilon_1, \ldots, \varepsilon_n \geq 0$. Then, there exists a marked hyperbolic surface $(\tilde{X}, \tilde{f})$ in $\mathcal{T}_{g,n}(\ell_1 + \varepsilon_1, \ldots, \ell_n + \varepsilon_n)$ such that, for any closed curve c on the base surface $S_{g,n}$, $\ell_X(c) \leq \ell_{\tilde{X}}(c)$.*

We are now able to prove the result.

Proof. Let γ be the systole of X which is necessarily simple. We cut the surface X along this curve, and obtain a (possibly disconnected) hyperbolic surface X_{cut} with two boundary components. By the extension Lemma (applied to both components separately if need be), there exists a surface X_{cut}^+ such that:

- the boundary components β_1, β_2 in X_{cut}^+ are of length $1 \leq \ell \leq 2\log g + \mathcal{O}(1)$.
- for any closed curve c not intersecting γ, $\ell_{X_{\mathrm{cut}}}(c) \leq \ell_{X_{\mathrm{cut}}^+}(c)$.

We shall find a pair of pants in X_{cut}^+, and use the relationship between lengths in X and X_{cut}^+ to conclude.

For $w > 0$, let us consider the w-neighbourhood of one component β_1 of the boundary of X_{cut}^+

$$\mathcal{C}_w(\beta_1) = \{z \in X_{\mathrm{cut}}^+ \,:\, \mathrm{d}(z, \beta_1) < w\}.$$

For small enough w, $\mathcal{C}_w(\beta_1)$ is a half-cylinder. However, there is a w at which this isometry stops. This w can be bounded by a volume argument: as long as $\mathcal{C}_w$ is a half-cylinder,

$$\mathrm{Vol}(\mathcal{C}_w(\beta_1)) = \ell \sinh w \leq \mathrm{Vol}\, X = 2\pi(2g-2).$$

However, $\ell \geq 1$. It follows that $w \leq \log g + \mathcal{O}(1)$.

There are two reasons for this isometry to stop.

- The half-cylinder self-intersects inside the surface (see Figure 4.5a). Then, one can construct an embedded pair of pant on X_{cut}^+, of total boundary length $\leq 2\ell + 4w$. This pair of pant will also be one on X, with shorter boundary components.
- The half-cylinder reaches the boundary of X_{cut}^+. It can only do so by intersecting the component β_2. Then, one can construct an embedded pair of pant on X_{cut}^+ of boundaries shorter than ℓ, ℓ, and $2\ell + 2w$, which corresponds to a one-holed torus on X, of boundary shorter than $2\ell + 2w$ (see Figure 4.5b, but expanding only a half-cylinder).

We can conclude that the surface X is L-tangled, for $L = \ell + 2w \leq 4 \log g + \mathcal{O}(1)$. □

4.3 Generalisation of the tangle-free hypothesis

In this section, we introduce a new notion of tangle-freeness, which generalises the one presented in Section 4.2. Our main objective is to define a weaker and more typical notion, of probability converging arbitrarily fast to one in the high-genus limit.

4.3.1 Motivation, definition and results

Probabilistic motivation The notion of tangle-free hyperbolic surface has proven to be very useful in a geometric viewpoint, allowing to improve many classic tools of hyperbolic geometry such as the collar lemmas. However, the fact that the probability of being $(a \log g)$-tangle-free converges rather slowly to one (at a rate $1/g^{1-a} > 1/g$) can be a significant limitation when trying to use it.

Indeed, if one was to try and estimate the expectation of a positive random variable $F_g : \mathcal{M}_g \to \mathbb{R}$ using a geometric hypothesis $\mathcal{A}_g \subset \mathcal{M}_g$ of high probability, they could for instance write

$$\mathbb{E}_g^{\mathrm{WP}}\left[F_g(X)\right] \leq \mathbb{E}_g^{\mathrm{WP}}\left[F_g(X) \mathbb{1}_{\mathcal{A}_g}(X)\right] + \|F_g\|_\infty (1 - \mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_g)).$$

If the random variable F_g takes large values on a set of small probability, then in order for this estimate to provide satisfying results, one needs to be able to pick a set $\mathcal{A}_g$ of probability very close to one. These issues have been encountered when trying to use the tangle-free hypothesis to recover the result [MP19, Proposition 4.5] and when trying to prove that $\lambda_1 \geq \frac{1}{4} - \varepsilon$ typically (see Section 6.3).

Definition and probabilistic estimate In order to remedy these issues, let us introduce a new notion of tangle-free surface, which is weaker than the one presented in Section 4.2 and hence has higher probability.

Definition 5. Let $N \geq 1$ be an integer and L be a positive real number.

A compact hyperbolic surface X is (N, L)-*tangled* if there exists an embedded connected surface $S \subset X$ with geodesic boundary such that:

- the Euler characteristic of S is $-N$
- the total boundary length of S is smaller than $(N+1)L$.

Otherwise, X is (N, L)-*tangle-free*.

The definition of tangle-freeness from Section 4.2 corresponds to the case $N = 1$. Let us estimate the probability of being (N, L)-tangle-free for the large scale $L = \log g$ and a fixed N.

Theorem 4.19. *For any integer $N > 1$, there exists a constant $C_N > 0$ such that*

$$1 - \mathbb{P}_g^{\mathrm{WP}} \left(X \textit{ is } (N, \log g)\textit{-tangle-free} \right) \leq C_N \frac{(\log g)^{3N}}{g^{\frac{N-1}{2}}} \cdot$$

The power $g^{-\frac{N-1}{2}}$ in this estimate can solve the issues that were raised at the beginning of the section: the new parameter N provides us with an additional degree of freedom, allowing us to pick a probabilistic assumption with an arbitrarily good polynomial rate of convergence to 1.

Geometric consequences Let us now describe the geometry of (N, L)-tangle-free surfaces. Unfortunately, this new hypothesis is significantly weaker, and its geometric implications are less striking.

For instance, one can prove that the length of the figure-eight geodesic in a pair of pants with three boundaries of length $\varepsilon \to 0^+$ goes to $2 \operatorname{arccosh}(3) \approx 3.52...$ (see [Bus92, Theorem 4.2.2]). As a consequence, a $(2, \log g)$-tangle-free surface can contain non-simple closed geodesics of constant length. Furthermore, by going around one of the sides of the figure eight repeatedly, we can even find short closed geodesics with many self-intersections in some $(2, \log g)$-tangle-free surfaces.

However, our aim when introducing this generalisation of the tangle-free hypothesis was to use it to prove Friedman's theorem for random surfaces. We do not necessarily expect to need very precise geometric information in order to do so, but mostly to be able to describe *long geodesics*, of length $A \log g$ for $A \gg 1$ (the motivation behind this choice of regime is detailed in Section 6.1.2). This will be achieved as a consequence of the following result, which in a sense bounds the topology of the surface filled by a ball of radius $L/4$ and hence generalises Proposition 4.14 to this new notion of tangle-freeness.

Theorem 4.20. *Let $N \geq 1$ and $L > 0$. Let X be a compact (N, L)-tangle-free surface of fundamental group Γ. Then, for any point $z \in X$, the set*

$$\Gamma_{\frac{L}{2}}(z) := \left\{ T \in \Gamma \setminus \{\mathrm{id}\} \ : \ell(\gamma_T(z)) < \frac{L}{2} \right\},$$

where $\gamma_T(z)$ is the geodesic loop based at z in the homotopy class T, is included in a subgroup of Γ of rank at most N.

We say a curve γ *fills* a surface S if all the connected components of $S \setminus \gamma$ are either contractible or homeomorphic to cylinders. The tangle-free hypothesis has the following consequence on the Euler characteristic of the surface filled by a closed geodesic.

Proposition 4.21. *There exists a constant $C > 0$ satisfying the following. For any $N \geq 1$, $L > 0$ and any compact (N, L)-tangle-free hyperbolic surface X, for any closed geodesic γ of length $\ell > 0$, the surface $S \subset X$ filled by γ satisfies*

$$|\chi_S| \leq C \frac{\ell}{L}\left(\frac{\ell}{L} + N\right).$$

In particular, for fixed N and $L = A \log g$, A large but fixed, the Euler characteristic of the surface filled by a closed geodesic of length $A \log g$ on a (N, L)-tangle-free surface is bounded.

4.3.2 Probabilistic estimate

Let us prove Theorem 4.19, the estimate on the probability for a surface of high genus g to be $(N, \log g)$-tangle-free. Here is a sketch of the proof, following the classic method developed by Mirzakhani [Mir13] and used in the proof of Theorem 4.8 for $N = 1$.

- We use Markov's inequality to bound the probability of being $(N, \log g)$-tangle-free by an expectation.
- We write this expectation as a sum of expectations of geometric functions, over all the different topological configurations for an embedded surface (see Figure 4.10 for $N = 2$, which is the counter-part of Figure 4.4 for $N = 1$).
- These expectations can then be computed using Mirzakhani's integration formula (Theorem 3.7) and estimated using volume estimates (presented in Section 3.2.3).

Proof. Each topological configurations for an embedded subsurface S of Euler characteristic $-N$ is given by:

- a signature (g_S, n_S) for S such that $2g_S - 2 + n_S = N$, $n_S > 0$
- a number of components $k \in \{1, \dots, n_S\}$ to attach to S
- a partition $(I_i)_{1 \leq i \leq k}$ of the n_s components of ∂S
- a genus g_i for each component, so that if $n_i = \# I_i$, $2g_i - 2 + n_i > 0$, and

$$\sum_{i=1}^{k} g_i = g - g_S - (n_S - k) \tag{4.7}$$

  by additivity of the Euler characteristic.

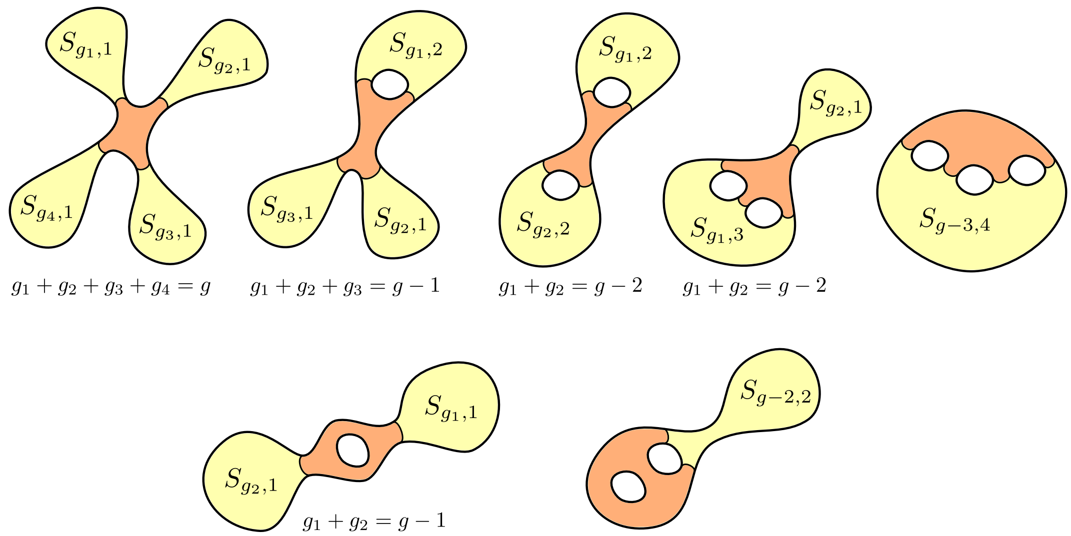


Figure 4.10: The different topological cases for $N = 2$. The first line corresponds to the cases where the signature of S is $(0,4)$ and the second $(1,2)$. The cases are organised from left to right in decreasing number k of components of $X \setminus S$.

Let us use Mirzakhani's integration formula to estimate the term corresponding a certain configuration $(\star)$. By Markov's inequality,

$$\begin{aligned}
&\mathbb{P}_g^{\mathrm{WP}}\left(\exists S \text{ embedded subsurface in config } (\star) \text{ s.t. } \ell^T(\partial S) < (N+1)\log g\right)\\
&\leq \mathbb{E}_g^{\mathrm{WP}}\left[\# \text{ embedded subsurfaces in config } (\star) \text{ s.t. } \ell^T(\partial S) < (N+1)\log g\right]\\
&\leq \frac{1}{V_g}\int_{T_\star} V_{g_S,n_S}(\ell)\prod_{i=1}^{k} V_{g_i,n_i}(\ell_{I_i})\ \ell_1\ldots\ell_{n_S}\,\mathrm{d}\ell_1\ldots\,\mathrm{d}\ell_{n_s}.
\end{aligned}$$

where

$$T_\star = \left\{(\ell_1,\ldots,\ell_{n_S}) \in \mathbb{R}_+^{n_S} \ : \ \sum_{i=1}^{n_S} \ell_i < (N+1)\log g\right\}.$$

$V_{g_S,n_S}(\ell)$ is a polynomial of degree $3g_S - 3 + n_S$ in ℓ_i^2, with all coefficients smaller than V_{g_S,n_S} by Lemma 3.11. Therefore, for any g_S, n_S, $\ell \in T_\star$,

$$V_{g_S,n_S}(\ell) = \mathcal{O}\left(V_{g_S,n_S}((N+1)\log g)^{6g_S-6+2n_S}\right) = \mathcal{O}_N\left((\log g)^{6g_S-6+2n_S}\right)$$

where the implied constant is $(N+1)^{3N}\max\{V_{\tilde{g},\tilde{n}} \,|\, 2\tilde{g}-2+\tilde{n} = N\}$. We can now use equation (3.8) to control the other volumes:

$$\begin{aligned}
&\int_{T_\star} \prod_{i=1}^{k} V_{g_i,n_i}(\ell_{I_i})\, \ell_1 \dots \ell_{n_S} \, \mathrm{d}\ell_1 \dots \mathrm{d}\ell_{n_S} \\
&\leq \prod_{i=1}^{k} V_{g_i,n_i} \int_{T_\star} \exp\left(\frac{1}{2}\sum_{i=1}^{n_S} \ell_i\right) \mathrm{d}\ell_1 \dots \mathrm{d}\ell_{n_s} \\
&= \mathcal{O}_N\left((\log g)^{n_S} g^{\frac{N+1}{2}} \prod_{i=1}^{k} V_{g_i,n_i}\right).
\end{aligned}$$

It follows the term corresponding to any given topological configuration ($\star$) is

$$\mathcal{O}_N\left(\log g^{3N} g^{\frac{N+1}{2}} \frac{1}{V_g} \prod_{i=1}^{k} V_{g_i,n_i}\right).$$

For a fixed N, there are a finite number of possibilities for the parameters (g_S, n_S), k, (I_i). Therefore, we simply need to pay attention to the summation over the $(g_i)_i$. By Lemma 3.15,

$$\begin{aligned}
\sum_{\substack{g_1+\ldots+g_k \\ = g-g_S-n_S+k}} \prod_{i=1}^{k} V_{g_i,n_i} &= \mathcal{O}_{n_s,k}\left(\frac{V_{g-g_S-n_S+k,n_S}}{(g-g_S-n_S+k)^{3(k-1)}}\right) \\
&= \mathcal{O}_N\left(\frac{V_g}{g^{3(k-1)+2(g_S+n_S-k)-n_S}}\right) \\
&= \mathcal{O}_N\left(\frac{V_g}{g^{N+k-1}}\right) = \mathcal{O}_N\left(\frac{V_g}{g^N}\right)
\end{aligned}$$

by equations (3.10) and (3.12). This leads to our claim. □

4.3.3 Geometry of (N, L)-tangle-free surfaces

Topology of balls Let us prove Theorem 4.20, which generalises Proposition 4.14 to $N > 1$ and describes balls of radius $\frac{L}{4}$ on (N, L)-tangle-free surfaces.

Proof. Let z be a point of X. Let us denote by $(\gamma_j)_{0 \leq j \leq J}$ the family of geodesic loops based at z of length $< \frac{L}{2}$, ordered by non-decreasing lengths.

Lemma 4.22. *We can construct an increasing family of graphs $(G_k)_{0 \leq k \leq K}$ on the surface X such that*

(A) for all k such that $0 \leq k \leq K$,

- *the edges of G_k are simple geodesic segments or loops, included in $\cup_j \gamma_j$, which do not intersect each other except at their endpoints*

– *there exists a regular neighbourhood $\mathcal{R}_k$ of G_k on X such that, if $\mathcal{R}_k^+$ is the subset of X obtained by adding to $\mathcal{R}_k$ all the components homeomorphic to disks or cylinders in its complement, then $\chi_{\mathcal{R}_k^+} = -k$ and $\ell(\partial \mathcal{R}_k^+) < (k+1)L$.*

(B) $\gamma_0, \ldots, \gamma_J$ are homotopic with fixed endpoints to loops on the graph G_K.

Let us first prove that this construction allows us to conclude.

For any integer k such that $1 \leq k \leq K$, when we replace the boundary components of $\mathcal{R}_k^+$ by the closed geodesics in their free homotopy class, we obtain an embedded surface S_k of Euler characteristic $-k$ with total boundary length $< (k+1)L$. But by the (N, L)-tangle-free hypothesis, X does not contain any embedded surface of Euler characteristic $-N$ with total boundary length smaller than $(N+1)L$. Hence, $K < N$.

For any $j \in \{0, \ldots, J\}$, γ_j is homotopic with fixed endpoints to a loop on the graph $G_K \subset \mathcal{R}_K^+$, and therefore $\Gamma_{\frac{L}{2}}(z)$ can be seen as a subset of the fundamental group of $\mathcal{R}_K^+$. This allows us to conclude, because the Euler characteristic of $\mathcal{R}_K$ is $-K$ so the fundamental group of $\mathcal{R}_K^+$ is generated by $K+1 \leq N$ elements.

Let us now prove the lemma.

Construction of the family of graphs. We initiate the construction with the graph G_0 with one vertex, z, and one edge, γ_0. By minimality, γ_0 is simple. Furthermore, $\ell(\gamma_0) < L/2$, so we can pick a regular neighbourhood $\mathcal{R}_0$ of G_0 such that $\ell(\partial \mathcal{R}_0) < L$. Such a regular neighbourhood is a cylinder of Euler characteristic 0, and with two boundary components both freely homotopic to γ_0, which is not contractible. Therefore, $\mathcal{R}_0^+ = \mathcal{R}_0$ and $(G_k)_{k=0}$ satisfies the hypotheses (A) and the hypothesis (B) for $j = 0$.

Let $j \geq 1$. We assume that we have constructed a family $(G_k)_{0 \leq k \leq K}$ satisfying the hypotheses (A) and (B) for γ_0, …, γ_{j-1}. Let us now take into account the loop γ_j, by adding (if necessary) vertices and edges on G_K.

Let us first prove that if γ_j is *not simple*, then it is already homotopic with fixed endpoints to a loop on G_K, so hypothesis (B) is already valid for γ_j. Indeed, if we pick a parametrisation $\gamma_j : [0,1] \to X$ and two parameters $0 \leq t < s < 1$ such that $\gamma(t) = \gamma(s)$, then we can write $\gamma_j = c_1 \beta c_2$, where c_1 is the restriction of γ_j to $[0, t]$, and c_2 its restriction to $[s, 1]$. We can assume without loss of generality that $\ell(c_1) \leq \ell(c_2)$ (by changing the orientation of γ_j). Then, γ_j is homotopic to the product of the two loops $c_1 \beta c_1^{-1}$ and $c_1 c_2$.

- We observe that $c_1 c_2$ is a loop based at z of length $< \ell(\gamma_j)$ (because we removed the loop β), and therefore there exists $i < j$ such that $c_1 c_2$ is homotopic to γ_i, and hence to a loop on the graph G_K.

- Let us replace the loop $c_1 \beta c_1^{-1}$ by the geodesic loop $\tilde{\beta}$ in its homotopy class. Since $c_1 \beta c_1^{-1}$ is not a geodesic loop (because of the angle 0 at the base-point),

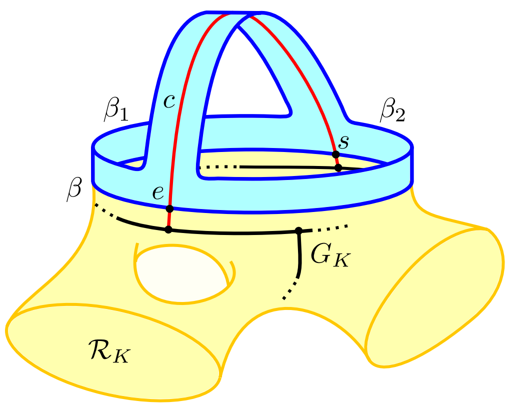
$$\ell(\tilde{\beta}) < \ell(c_1 \beta c_1^{-1}) = \ell(\beta) + 2\ell(c_1) \leq \ell(\beta) + \ell(c_1) + \ell(c_2) = \ell(\gamma_j)$$

  which implies as before that $\tilde{\beta}$ (and hence $c_1 \beta c_1^{-1}$) is homotopic to a loop on G_K.

We now assume that γ_j is *simple*. Its intersections with the graph G_K are transverse, because otherwise γ_j would coincide on an interval with a γ_i for a $i < j$. It would therefore contain a sub-loop $\gamma_{i'}$ of γ_i based at z, for a $i' \leq i$. This would imply that $\gamma_j = \gamma_{i'}$ (because γ_j is simple and therefore only visits z once), which is a contradiction.

We can then write $\gamma_j = c_1 \dots c_n$ where $(c_i)_i$ are disjoint simple geodesic segments or loops, which meet G_K at their endpoints and only there. By reducing the regular neighbourhood $\mathcal{R}_K$ if necessary, we can assume that each c_i meets the boundary of $\mathcal{R}_K$ exactly twice, at two points s_i and e_i. Let $\tilde{c}_i$ denote the portion of c_i in the complement of $\mathcal{R}_K$, i.e. the restriction of c_i to $[s_i, e_i]$.

Let us first consider the segment $c = c_1$. There are several topological configurations for this segment, represented in Figure 4.11.



(a) c connects on component to itself.

(b) c connects two different components.

Figure 4.11: Illustration of the topological situations encountered in the proof of Lemma 4.22, when adding a segment c to the graph G_K.

(a) The exit points s and e of c belong to the same connected component β of $\partial\mathcal{R}_K$, like in Figure 4.11a. Then, since the portion $\tilde{c}$ of the path c between s and e is entirely included the complement of $\mathcal{R}_K$, the path $\tilde{c}$ is on the same side of β just after s and just before e. The regular neighbourhood of β and $\tilde{c}$ is therefore a pair of pants, with three boundary components β' (freely homotopic to β), β_1 and β_2 (see Figure 4.11).

- If at least one of β_1 or β_2 is contractible, then $\tilde{c}$ is homotopic with fixed endpoints to a path on β. As a consequence, c is homotopic to a path on G_K.
- Otherwise, we define G_{K+1} by adding the path c and its endpoints to the graph G_K. G_{K+1} automatically satisfies the first point of (A). For the second point, we observe that a regular neighbourhood $\mathcal{R}_{K+1}$ of G_{K+1} can be obtained by gluing the pair of pants P delimited by β, β_1 and β_2 to the regular neighbourhood $\mathcal{R}_K$ along its boundary component β. Neither β_1 nor β_2 is

contractible, so when we add the disks and cylinders to $\mathcal{R}_{K+1}$ to construct $\mathcal{R}_{K+1}^+$, we obtain $\mathcal{R}_K^+$ together with an additional pair of pant (when β_1 and β_2 are in two different free homotopy classes) or handle (when β_1 and β_2 are freely homotopic to one another, and hence bound a cylinder). In both cases, by additivity of the Euler characteristic, $\chi_{\mathcal{R}_{K+1}^+} = -(K+1)$ since both the pair of pants and handle have Euler characteristic equal to -1. Furthermore, the boundary of $\mathcal{R}_{K+1}^+$ is obtained by taking the boundary of $\mathcal{R}_K^+$ and either removing the component β and adding β_1 and β_2 in the pair of pants case, or just removing β in the handle case. In both cases,

$$\ell(\partial \mathcal{R}_{K+1}^+) \leq \ell(\partial \mathcal{R}_K^+) - \ell(\beta) + \ell(\beta_1) + \ell(\beta_2).$$

By hypothesis on $\mathcal{R}_K^+$, its boundary length is $< (K+1)L$. Furthermore, because the pair of pants delimited by β, β_1 and β_2 is filled by the family of curves β and c, of total length $< \ell(\beta) + L/2$, we can assume that

$$\ell(\partial P) = \ell(\beta) + \ell(\beta_1) + \ell(\beta_2) < 2\ell(\beta) + L.$$

As a consequence, $\ell(\partial \mathcal{R}_{K+1}^+) < (K+2)L$.

(b) Or the exit points s and e belong to two different connected components β_e and β_s of $\partial \mathcal{R}_K$, like in Figure 4.11b. Then there is once again two possibilities.

- If β_e and β_s are in the same free homotopy class, then they bound a cylinder $\mathcal{C}$. Let us modify the definition of G_K and replace it by the graph G_K' obtained by adding the path c and its endpoints to G_K. When constructing $\mathcal{R}_K^+$ we would initially add the cylinder $\mathcal{C}$ to $\mathcal{R}_K$. Since $\tilde{c}$ is a simple curve connecting the two boundary components of $\mathcal{C}$, $\mathcal{C} \setminus \tilde{c}$ is contractible, and therefore added when constructing the new set $(\mathcal{R}_K^+)'$. Hence, the set $\mathcal{R}_K^+$ is not modified by the addition of the edge c to G_K, and therefore it still satisfies the hypotheses (A).
- Otherwise, the regular neighbourhood of β_e, β_s and c is a pair of pants P of boundary components β_s', β_e' and β, where β is not contractible. We add the edge c and its endpoints to G_K and obtain a graph G_{K+1}. The regular neighbourhood $\mathcal{R}_{K+1}$ is then obtained by gluing the pair of pants P on $\mathcal{R}_K$, and $\mathcal{R}_{K+1}^+$ is obtained from $\mathcal{R}_K^+$ by the same process since β is not contractible. Hence, the Euler characteristic of $\mathcal{R}_{K+1}^+$ is $-(K+1)$, and

$$\ell(\partial \mathcal{R}_{K+1}^+) = \ell(\partial \mathcal{R}_K^+) - \ell(\beta_s) - \ell(\beta_e) + \ell(\beta)$$

can be assumed to be $< (K+2)L$ because P is filled by the family of curves β_s, β_e and c of total length $< \ell(\beta_s) + \ell(\beta_e) + L/2$, and therefore we can assume that

$$\ell(\partial P) = \ell(\beta_s) + \ell(\beta_e) + \ell(\beta) < 2\ell(\beta_s) + 2\ell(\beta_e) + L.$$

In all these cases, we obtain a new family of graphs $(G_k)_{0\leq k\leq K'}$, with $K' = K$ or $K+1$, satisfying the hoptheses (A) and (B) for $\gamma_0, \ldots, \gamma_{j-1}$, and such that the path $c = c_1$ is homotopic with fixed endpoints to a path on $G_{K'}$.

We apply this construction successively to $c = c_i$ for $i \in \{1, \ldots, n\}$. At the end, we obtain a family of graphs $(G_k)_{0\leq k\leq K''}$, $K \leq K'' \leq K+n$, satisfying the hypotheses (A) and (B) for the loops $\gamma_0, \ldots, \gamma_j$, which concludes this step of the iterative construction. $\square$

$\square$

Surface filled by a long geodesics Proposition 4.21 is an immediate consequence of the following lemma.

Lemma 4.23. *Let $N \geq 1$ and $L > 0$. Let $X = {}^{\mathcal{H}}\!/_{\Gamma}$ be a (N, L)-tangle-free hyperbolic surface. Let γ be a geodesic of length ℓ on $\mathcal{H}$. Then, there exists a neighbourhood $\mathcal{V}$ of γ in $\mathcal{H}$ and a subgroup $\Gamma_{\mathcal{V}}$ of Γ of rank $\mathcal{O}\left(\frac{\ell}{L}\left(\frac{\ell}{L}+N\right)\right)$ such that*

$$\forall z \in \mathcal{V}, \{T \in \Gamma \,:\, T\cdot z \in \mathcal{V}\} \subset \Gamma_{\mathcal{V}}.$$

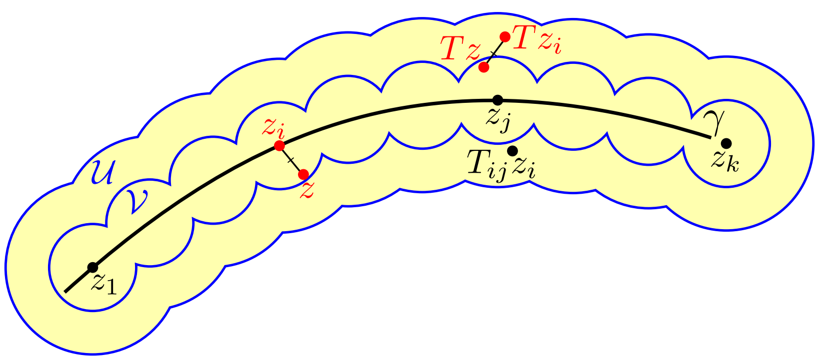


Figure 4.12: Illustration of the proof of Lemma 4.23.

Proof. Let us cover the geodesic γ by $k = \mathcal{O}\left(\frac{\ell}{L}\right)$ balls of radius $\frac{L}{8}$, centered at points denoted by $z_1, \ldots, z_k$. Then, the open sets

$$\mathcal{V} = \bigcup_{i=1}^{k} \mathcal{B}_{\frac{L}{8}}(z_i) \qquad \text{and} \qquad \mathcal{U} = \bigcup_{i=1}^{k} \mathcal{B}_{\frac{L}{4}}(z_i).$$

are neighbourhoods of the curve γ – see Figure 4.12.

For each index $i \in \{1, \ldots, k\}$, we set $\Gamma_i(\mathcal{U}) := \{T \in \Gamma \,:\, T\cdot z_i \in \mathcal{U}\}$. We claim that

$$\forall z \in \mathcal{V},\ \{T \in \Gamma \,:\, T\cdot z \in \mathcal{V}\} \subset \bigcup_{i=1}^{k} \Gamma_i(\mathcal{U}).$$

Indeed, any element z of $\mathcal{V}$ belongs in $\mathcal{B}_{\frac{L}{8}}(z_i)$ for an integer i. For an element $T \in \Gamma$, if $T \cdot z \in \mathcal{V}$, then there is an index j such that $T \cdot z \in \mathcal{B}_{\frac{L}{8}}(z_j)$. Then,

$$\mathrm{d}_{\mathcal{H}}(z_j, T \cdot z_i) \leq \mathrm{d}_{\mathcal{H}}(z_j, T \cdot z) + \mathrm{d}_{\mathcal{H}}(T \cdot z, T \cdot z_i) < \frac{L}{8} + \frac{L}{8} = \frac{L}{4}$$

and therefore $T \cdot z_i \in \mathcal{B}_{\frac{L}{4}}(z_j) \subset \mathcal{U}$, which implies that $T \in \Gamma_i(\mathcal{U})$.

Therefore, we only need to prove that for all $i \in \{1, \ldots, k\}$, the set $\Gamma_i(\mathcal{U})$ is included in a subgroup of Γ of rank $\mathcal{O}\left(\frac{\ell}{L} + N\right)$, using the (N, L)-tangle-free hypothesis. This will lead to the conclusion because $k = \mathcal{O}\left(\frac{\ell}{L}\right)$.

For each $j \neq i$, let us pick a $T_{i,j} \in \Gamma$ such that $T_{i,j} \cdot z_i \in \mathcal{B}_{\frac{L}{4}}(z_j)$, provided such a transformation exists. Then, if $T \in \Gamma_i(\mathcal{U})$, $T \cdot z_i \in \mathcal{U}$ so one of the following occurs:

- either $T \cdot z_i \in \mathcal{B}_{\frac{L}{4}}(z_i)$, which implies that $T \in \Gamma_{\frac{L}{4}}(z_i) \subset \Gamma_{\frac{L}{2}}(z_i)$;
- or $T \cdot z_i \in \mathcal{B}_{\frac{L}{4}}(z_j)$ for a $j \neq i$, and hence

$$\mathrm{d}_{\mathcal{H}}(T_{i,j} \cdot z_i, T \cdot z_i) \leq \mathrm{d}_{\mathcal{H}}(T_{i,j} \cdot z_i, z_j) + \mathrm{d}_{\mathcal{H}}(z_j, T \cdot z_i) < \frac{L}{4} + \frac{L}{4} = \frac{L}{2}$$

  so $T_{i,j}^{-1} T$ is an element of $\Gamma_{\frac{L}{2}}(z_i)$.

As a consequence, every element of $\Gamma_i(\mathcal{U})$ can be written as a product of the $(T_{i,j})_{j \neq i}$ and the elements of $\Gamma_{\frac{L}{2}}(z_i)$. The latter is included in a sub-group of Γ of rank $\leq N$ by the tangle-free hypothesis and Theorem 4.20, while the former contains $k-1$ elements. Therefore, $\Gamma_i(\mathcal{U})$ is included in a sub-group of rank $\leq N + k - 1$, which allows us to conclude. $\square$

Chapter 5

Eigenvalue distribution

In this chapter, we present new results on the spectrum of typical hyperbolic surfaces for the Weil–Petersson probability setting. We start by introducing our main tool to study the spectrum of compact hyperbolic surfaces, the Selberg trace formula, in Section 5.1. The following sections are organised as follows.

- In Section 5.2, we prove an upper bound on the number of small eigenvalues.
- We extend this upper bound to any spectral window in Section 5.3 and prove an equivalent of counting functions:

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} \sim \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathbb{1}_{[a,b]}(\lambda) \, \mathrm{d}\lambda$$

  as soon as b and/or $g \to +\infty$, provided the spectral window $b-a$ is not too small.
- These estimates will allow us to improve the deterministic bounds on the multiplicity of the j-th eigenvalue, and provide an equivalent for the value of λ_j for large j, in Section 5.4.

5.1 The Selberg trace formula

Our main tool in the next two chapters is the Selberg trace formula, a generalisation of the Poisson summation formula to hyperbolic surfaces. It is a summation formula, linking the values of a test function evaluated at the *spectrum* of the surface and the values of the Fourier transform of the test function at the *lengths of closed geodesics*. This formula is therefore a very powerful way to relate the geometry and spectrum of hyperbolic surfaces, though also quite subtle because it relies on good choices of test functions rather than a one-to-one correspondence of the eigenvalues and lengths.

Some of the (numerous) notable applications of the Selberg trace formula are the Weyl law [Ber16, Theorem 5.23], Huber's theorem [Hub59] stating the equivalence of the length and eigenvalue spectra as geometric quantities [Bus92, Theorem 9.2.9], the prime number theorem for the length of geodesics [Bus92, Theorem 9.6.1] [Hub59], the existence of hyperbolic surfaces with small eigenvalues [Ran74].

Let $X = \mathcal{H}/\Gamma$ be a compact hyperbolic surface, where Γ is a Fuchsian group acting on the hyperbolic plane $\mathcal{H}$. We pick a fundamental domain D for this action.

Let $(\lambda_j)_j$ denote the non-decreasing sequence of eigenvalues of X, with multiplicity, and for any index j let us pick a complex number r_j in $\mathbb{R} \cup i\left[-\frac{1}{2}, \frac{1}{2}\right]$ such that $\lambda_j = \frac{1}{4} + r_j^2$.

Let us introduce a class of test functions.

Definition 6. A function $h : \mathbb{C} \to \mathbb{C}$ is an *admissible test function* if it satisfies the following hypotheses:

1. h is even: $h(-r) = h(r)$ for any $r \in \mathbb{C}$;
2. h is analytic in the strip $\left\{r : |\Im r| \leq \frac{1}{2} + \varepsilon\right\}$ for some $\varepsilon > 0$;
3. for all r in that strip,
$$h(r) = \mathcal{O}\left(\frac{1}{(1 + |r|^2)^{1+\varepsilon}}\right). \tag{5.1}$$

Then, the inverse Fourier transform of h is

$$\check{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, e^{iru}\, \mathrm{d}r = \frac{1}{\pi} \int_{0}^{+\infty} h(r)\, \cos(ru)\, \mathrm{d}r$$

and we say that $(h, \check{h})$ is an *admissible transform pair.*

The Selberg trace formula can be stated as follows.

Theorem 5.1 (Selberg trace formula [Sel56])**.** *For any admissible transform pair $(h, \check{h})$, the following formula holds (with every term well-defined and converging):*

$$\sum_{j=0}^{+\infty} h(r_j) = \frac{\mathrm{Vol}_X(X)}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r)\, \mathrm{d}r + \int_D \sum_{T \in \Gamma \setminus \{\mathrm{id}\}} K(z, T \cdot z)\, \mathrm{d}\,\mathrm{Vol}_{\mathcal{H}}(z) \tag{5.2}$$

$$= \frac{\mathrm{Vol}_X(X)}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r)\, \mathrm{d}r + \sum_{\gamma} \sum_{n=1}^{+\infty} \frac{\ell(\gamma)}{2 \sinh\left(\frac{n\ell(\gamma)}{2}\right)} \check{h}(n\ell(\gamma)) \tag{5.3}$$

where:

- *$K(z, w) = K(\mathrm{d}_{\mathcal{H}}(z, w))$ is the kernel associated to h, which has the following expression:*
$$K(\rho) = -\frac{1}{\sqrt{2}\,\pi} \int_{\rho}^{+\infty} \frac{\check{h}'(u)}{\sqrt{\cosh u - \cosh \rho}}\, \mathrm{d}u \tag{5.4}$$
- *the final sum is taken over all non-oriented primitive closed geodesics on X, and $\ell(\gamma)$ is the length of the geodesic γ.*

We note that the trace formula has three contributions.

The left-hand side term is a *spectral average* with density h. In the following, we will pick h so that it captures the parts of the spectrum we are interested in. For instance, h can be taken to be a smooth approximation of a step function, so that the spectral term is close to the number of eigenvalues in a spectral window $[a, b]$.

The first term on the right hand side is called the *topological* term, because it does not depend on the metric on X but only on its topology (we recall that the volume of any hyperbolic surface of genus g is $2\pi(2g-2)$). If we take a test function h such that the spectral term is approximately the number of eigenvalues in a segment $[a, b]$, then it will be close to the spectral density of the hyperbolic plane

$$\mu(a,b) = \frac{1}{4\pi}\int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda)\tanh\left(\pi\sqrt{\lambda-\frac{1}{4}}\right)\mathrm{d}\lambda.$$

The last term is a *geometric term.* Even though both formulations will be useful for the purpose of this chapter, equation (5.3) is the usual way to write it because the geometric interpretation of the term is more striking. It is obtained by rearranging the elements of Γ in equation (5.2) by conjugacy class and computing the integration over the fundamental domain D.

Understanding the behaviour of the geometric term of the Selberg trace formula is the main challenge of the next two chapters. By the uncertainty principle, we know that the more concentrated the test function, the more spread out the Fourier transform. This means that as we increase the precision of our spectral approximations, the contributions of longer geodesics become significant in the geometric term. We will therefore apply the large-scale geometry results from Chapter 4, such as Benjamini–Schramm convergence and the tangle-free hypothesis, in order to bound the geometric term.

5.2 An upper bound at the bottom of the spectrum

Let us first use the Selberg trace formula and Benjamini–Schramm convergence to prove an upper bound on the number of eigenvalues in a segment of the form $[0, b]$ (this bound being interesting only whenever $b \leq \frac{1}{4}$, that is to say for small eigenvalues).

Theorem 5.2. *For any large enough g, there exists a subset $\mathcal{A}_g \subset \mathcal{M}_g$ such that:*

- *For any $X \in \mathcal{A}_g$ and any $b \geq 0$,*

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq 32\,\frac{g^{-\frac{1}{27}\left(\frac{1}{4}-b\right)}}{(\log g)^{\frac{3}{2}}}. \tag{5.5}$$

- $1 - \mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{A}_g\right) = \mathcal{O}\left(g^{-d}\right)$ *for a* $d \in \left(0, \frac{1}{27}\right)$.

When taking $b = \frac{1}{4}$, we obtain the following corollary, which improves the (optimal) deterministic estimate $\mathrm{N}_X^{\Delta}\left(0, \frac{1}{4}\right) \leq 2g-2$ by Otal and Rosas [OR09].

Corollary 5.3. *With high probability, the number of small eigenvalues of a random hyperbolic surface of genus g is $\mathcal{O}\left(g(\log g)^{-\frac{3}{2}}\right) = o(g)$.*

Furthermore, this bound becomes better the further b is from the bulk spectrum $\left[\frac{1}{4}, +\infty\right)$. As a consequence, the examples of surfaces with $2g-2$ eigenvalues in $[0, \varepsilon]$ (for arbitrarily small ε) are not typical.

Remark. A similar estimate is proved in [Mon21, Theorem 5], leading to the bound

$$\frac{\mathrm{N}_X^\Delta(0,b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(\frac{g^{-2^{-15}\left(\frac{1}{4}-b\right)^2}}{(\log g)^{\frac{3}{4}}}\right).$$

Theorem 5.2 was added to this manuscript because the result we prove here is better when $b = \frac{1}{4}$, and the proof is more elementary (we use a simpler test function in the Selberg trace formula).

The proof relies on the following result, a bound on the number $\mathrm{N}_X^\Delta(0,b)$ for a *given* surface X in terms of *geometric quantities* only.

Proposition 5.4. *Let $r \in (0,1]$, $L \geq 1$. There exists a constant $C > 0$ such that for any hyperbolic surface X of injectivity radius greater than r,*

$$\forall b \geq 0, \frac{\mathrm{N}_X^\Delta(0,b)}{\mathrm{Vol}_X(X)} \leq \frac{e^{-(\frac{1}{4}-b)t}}{t^{\frac{3}{2}}}\left[\frac{\sqrt{\pi}}{8} + C\frac{t^4 e^t}{r^4}\left(\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)} + Le^{L-\frac{L^2}{4t}}\right)\right]. \tag{5.6}$$

We recall that for a hyperbolic surface X and a real number $L > 0$, the set $X^-(L)$ is defined by

$$X^-(L) = \{z \in X \ : \ \mathrm{InjRad}_X(z) < L\}$$

where $\mathrm{InjRad}_X(z)$ is the injectivity radius of the surface X at the point z, that is to say twice the length of the shortest geodesic arc based at z.

In order to prove the result, we will need the following classic lemma, which is a bound on the number of points in the intersection of the lattice $\Gamma \cdot z$ and a ball.

Lemma 5.5. *Let $r > 0$ and $X = {}^{\mathcal{H}}\!/_{\Gamma}$ be a compact hyperbolic surface of injectivity radius greater than r. Then, for any $z \in \mathcal{H}$ and any $M > 0$,*

$$\#\{T \in \Gamma \ : \ \mathrm{d}_{\mathcal{H}}(z, T\cdot z) \leq M\} \leq \frac{8\exp\left(M + \frac{r}{2}\right)}{r^2}. \tag{5.7}$$

Proof. By definition of the injectivity radius, the balls B_T of center $T \cdot z$ and radius $\frac{r}{2}$, for $T \in \Gamma$, are disjoint. If T is such that $\mathrm{d}_{\mathcal{H}}(z, T\cdot z) \leq M$, then B_T is included in the ball of center z and radius $M + \frac{r}{2}$. Since the volume of a hyperbolic ball of radius R is $\cosh(R) - 1$, the number of such T is smaller than

$$\frac{\cosh\left(M+\frac{r}{2}\right)-1}{\cosh\left(\frac{r}{2}\right)-1} \leq \frac{\cosh\left(M+\frac{r}{2}\right)}{\frac{1}{2}\left(\frac{r}{2}\right)^2} \leq \frac{8\exp\left(M+\frac{r}{2}\right)}{r^2}.$$

$\square$

We can now proceed to the proof of Proposition 5.4, using the Selberg trace formula.

Proof. We can write X as a quotient $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$ by a Fuchsian group Γ. Let D be a fundamental domain of this action.

Let $(\lambda_j)_{j\geq 0}$ be the ordered sequence of eigenvalues of the Laplacian on X with multiplicities, and for any integer $j \geq 0$, let r_j be one solution of $\lambda_j = \frac{1}{4} + r_j^2$.

For a parameter $t \geq 1$, let us apply the Selberg trace formula (Theorem 5.1) to the even analytic function $h_t : \mathbb{C} \to \mathbb{C}$ defined by

$$h_t : r \mapsto \exp\left(-t\left(\frac{1}{4} + r^2\right)\right).$$

This can be done because it is a quickly decreasing function: for any $r = x + iy$ such that $|y| \leq 1$,

$$|h_t(r)| = \exp\left(-t\,\Re\left(\frac{1}{4} + r^2\right)\right) = \exp\left(-t\left(\frac{1}{4} + x^2 - y^2\right)\right) \leq \exp\bigl(2t - t|r|^2\bigr).$$

We obtain

$$\sum_{j=0}^{+\infty} e^{-t\lambda_j} = \frac{\mathrm{Vol}_X(X)}{4\pi} \int_{\mathbb{R}} e^{-t\left(\frac{1}{4}+r^2\right)} \tanh(\pi r) r \,\mathrm{d}r + \int_D \sum_{T \in \Gamma \setminus \{\mathrm{id}\}} K_t(z, T\cdot z)\,\mathrm{dVol}_{\mathcal{H}}(z). \tag{5.8}$$

Since all terms of the left-hand side of equation (5.8) are positive and decreasing with respect to λ_j, one has

$$\mathrm{N}_X^{\Delta}(0,b)\; e^{-tb} \leq \sum_{j=0}^{+\infty} e^{-t\lambda_j}.$$

Therefore,

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq e^{tb}(A_t + B_t(X)), \tag{5.9}$$

where

$$A_t = \frac{1}{4\pi} \int_{\mathbb{R}} e^{-t\left(\frac{1}{4}+r^2\right)} \tanh(\pi r) r \,\mathrm{d}r,$$

$$B_t(X) = \frac{1}{\mathrm{Vol}_X(X)} \int_D \sum_{T \in \Gamma \setminus \{\mathrm{id}\}} K_t(z, T\cdot z)\,\mathrm{dVol}_{\mathcal{H}}(z).$$

It is straightforward to estimate A_t, using the change of variable $u = r\sqrt{t}$, and the fact that $\tanh(v) \leq v$ for all $v \geq 0$:

$$A_t = \frac{1}{2\pi} \int_0^{+\infty} e^{-t\left(\frac{1}{4}+r^2\right)} \tanh(\pi r) r \,\mathrm{d}r \leq \frac{e^{-\frac{t}{4}}}{2\,t^{\frac{3}{2}}} \int_0^{+\infty} e^{-u^2} u^2 \,\mathrm{d}u = \frac{\sqrt{\pi}}{8} \frac{e^{-\frac{t}{4}}}{t^{\frac{3}{2}}}. \tag{5.10}$$

To estimate $B_t(X)$, we shall use the following lemma.

Lemma 5.6. *For any $t > 0$ and $\rho > 0$,*

$$0 \le K_t(\rho) \le \frac{2\rho}{(2\pi t)^{\frac{3}{2}}}\left(1+\frac{t^2}{\rho^2}\right)\exp\left(-\frac{t}{4}-\frac{\rho^2}{4t}\right). \tag{5.11}$$

Proof. We compute the Fourier transform of the Gaussian and obtain:

$$\check{h}_t(u) = \frac{e^{-\frac{t}{4}}}{2\pi}\int_{-\infty}^{+\infty} e^{-tr^2+iru}\,\mathrm{d}r = \frac{e^{-\frac{t}{4}-\frac{u^2}{4t}}}{2\sqrt{\pi t}}.$$

Then, by definition of the kernel K_t, equation (5.4),

$$K_t(\rho) = \frac{e^{-\frac{t}{4}}}{2(2\pi t)^{\frac{3}{2}}}\int_{\rho}^{+\infty}\frac{u\exp(-\frac{u^2}{4t})}{\sqrt{\cosh(u)-\cosh(\rho)}}\,\mathrm{d}u.$$

Let us estimate this integral by cutting it at 2ρ.

- For $u \ge \rho$, $\cosh u - \cosh\rho \ge (u-\rho)\sinh(\rho) \ge (u-\rho)\rho$ so

$$\int_{\rho}^{2\rho}\frac{u\exp\left(-\frac{u^2}{4t^2}\right)}{\sqrt{\cosh u-\cosh\rho}}\,\mathrm{d}u \le 2\rho\exp\left(-\frac{\rho^2}{4t^2}\right)\int_{\rho}^{2\rho}\frac{\mathrm{d}u}{\sqrt{(u-\rho)\rho}} = 4\rho\exp\left(-\frac{\rho^2}{4t^2}\right).$$

- For $u \ge 2\rho$, $\cosh u - \cosh\rho \ge \frac{1}{2}(u-\rho)^2 \ge \frac{1}{2}\rho^2$ so

$$\int_{2\rho}^{+\infty}\frac{u\exp\left(-\frac{u^2}{4t^2}\right)}{\sqrt{\cosh u-\cosh\rho}}\,\mathrm{d}u \le \frac{\sqrt{2}}{\rho}\int_{2\rho}^{+\infty} u\exp\left(-\frac{u^2}{4t^2}\right)\,\mathrm{d}u = 2\sqrt{2}\,\frac{t^2}{\rho}\,\exp\left(-\frac{\rho^2}{t^2}\right).$$

As a consequence,

$$K_t(\rho) \le \frac{e^{-\frac{t}{4}}}{2(2\pi t)^{\frac{3}{2}}}\left(4\rho+2\sqrt{2}\,\frac{t^2}{\rho}\right)\exp\left(-\frac{\rho^2}{4t^2}\right)$$

which leads to the conclusion. ◻

In the rest of the proof, we shall denote $f_1 = \mathcal{O}(f_2)$ if there is a numerical constant $C > 0$ such that $f_1 \le Cf_2$.

When we use equation (5.11) in the expression of B_t, we obtain

$$B_t(X) = \mathcal{O}\left(\frac{\sqrt{t}}{r^2}\frac{e^{-\frac{t}{4}}}{\mathrm{Vol}_X(X)}\int_D\sum_{T\in\Gamma\setminus\{\mathrm{id}\}}\mathrm{d}_{\mathcal{H}}(z,T\cdot z)\,e^{-\frac{\mathrm{d}_{\mathcal{H}}(z,T\cdot z)^2}{4t}}\,\mathrm{dVol}_{\mathcal{H}}(z)\right)$$

since the distance between z and $T\cdot z$ is always greater than r, and by hypotheses on the parameters, $1+\frac{t^2}{r^2}\le\frac{2t^2}{r^2}$. Let us now regroup the sum over Γ according to the distance between z and $T\cdot z$

$$B_t(X) = \mathcal{O}\left(\frac{\sqrt{t}}{r^2}\frac{e^{-\frac{t}{4}}}{\mathrm{Vol}_X(X)}\int_D\sum_{k=0}^{+\infty}ke^{-\frac{k^2}{4t}}\;n_\Gamma(z,k)\,\mathrm{dVol}_{\mathcal{H}}(z)\right)$$

where for $z \in D$,

$$n_\Gamma(z,k) = \#\{T \in \Gamma \setminus \{\mathrm{id}\} \,:\, k \leq \mathrm{d}_{\mathcal{H}}(z, T \cdot z) < k+1\}.$$

For $z \in D$, let $k(z)$ be the integer part of the injectivity radius of X at the point z. Whenever $k < k(z)$, $n_\Gamma(z,k) = 0$, since for all $T \neq \mathrm{id}$, the distance from z to $T \cdot z$ is greater than $k(z)$. For the other terms, we use Lemma 5.5, that states that for any k,

$$n_\Gamma(z,k) \leq \frac{8\, e^{k+1+\frac{r}{2}}}{r^2} = \mathcal{O}\left(\frac{e^k}{r^2}\right)$$

and therefore

$$B_t(X) = \mathcal{O}\left(\frac{\sqrt{t}}{r^4} \, \frac{e^{-\frac{t}{4}}}{\mathrm{Vol}_X(X)} \int_D \sum_{k=k(z)}^{+\infty} k e^{k - \frac{k^2}{4t}} \,\mathrm{dVol}_{\mathcal{H}}(z)\right). \tag{5.12}$$

We now cut the integral over the fundamental domain D into two parts, depending on a positive parameter L:

$$\begin{aligned} D^-(L) &= \{z \in D \,:\, \mathrm{InjRad}_z(X) \leq L\} \\ D^+(L) &= D \setminus D^-(L). \end{aligned}$$

The integration over $D^+(L)$ corresponds to the places where the integrand is small, because the integer $k(z)$ is large. Indeed, if $z \in D^+(L)$,

$$\sum_{k=k(z)}^{+\infty} k e^{k-\frac{k^2}{4t}} \leq \sum_{k=\lfloor L \rfloor}^{+\infty} k e^{k-\frac{k^2}{4t}} \leq L e^{L+1} e^{-\frac{L^2}{4t}} \sum_{k=0}^{+\infty} (k+2) e^{k-\frac{k^2}{4t}}.$$

The argument $k - \frac{k^2}{4t}$ of the exponential is always smaller than t, and becomes smaller than $-k$ as soon as $k \geq 8t$. Hence,

$$\sum_{k=0}^{+\infty} (k+2) e^{k-\frac{k^2}{4t}} \leq \sum_{k=0}^{\lfloor 8t \rfloor} (k+2) e^t + \sum_{k=\lfloor 8t \rfloor + 1}^{+\infty} (k+2) e^{-k} = \mathcal{O}\left(t^2 e^t\right).$$

Whenever $z \in D^+(L)$, we simply use:

$$\sum_{k=k(z)}^{+\infty} e^{k-\frac{k^2}{4t}} \leq \sum_{k=0}^{+\infty} e^{k-\frac{k^2}{4t}} = \mathcal{O}\left(t^2 e^t\right).$$

Then, equation (5.12) implies that

$$B_t(X) = \mathcal{O}\left(\frac{\sqrt{t}}{r^4} \, e^{-\frac{t}{4}} \left[\frac{\mathrm{Vol}_{\mathcal{H}}(D^-(L))}{\mathrm{Vol}_X(X)} + \frac{\mathrm{Vol}_{\mathcal{H}}(D^+(L))}{\mathrm{Vol}_X(X)} L e^{L - \frac{L^2}{4t}}\right] t^2 e^t\right).$$

This together with equations (5.9) and (5.10) allows us to conclude. □

We can now prove Theorem 5.2.

Proof. For a genus $g \geq 2$, let us pick the parameters:

$$L = \frac{2}{9} \log g \qquad M = g^{\frac{2}{3}}$$
$$t = \frac{1}{27} \log g \qquad r = g^{-\frac{d}{2}} \text{ for a } d \in \left(0, \frac{1}{27}\right).$$

Then, for large enough g, $L \geq 1$, $t \geq 1$, $r \leq 1$. Let $\mathcal{A}_g$ be the set of compact hyperbolic surfaces of genus g such that:

- $\mathrm{InjRad}(X) \geq r = g^{-\frac{d}{2}}$.
- $\mathrm{Vol}_X(X^-(L)) \leq Me^L = g^{\frac{8}{9}}$.

Then, by the estimate on the injectivity radius [Mir13, Theorem 4.2] and Theorem 4.1, the probability of $\mathcal{A}_g$ satisfies

$$1 - \mathbb{P}_g^{\mathrm{WP}}(\mathcal{A}_g) = \mathcal{O}\left(r^2 + \frac{e^{2L}}{M}\right) = \mathcal{O}\left(g^{-d} + g^{-\frac{2}{9}}\right) = \mathcal{O}\left(g^{-d}\right).$$

Let X be an element of $\mathcal{A}_g$. It satisfies the hypotheses of Proposition 5.4, and therefore for any $b \geq 0$,

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq \frac{e^{-(\frac{1}{4}-b)t}}{t^{\frac{3}{2}}} \left[\frac{\sqrt{\pi}}{8} + C\frac{t^4 e^t}{r^4}\left(\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)} + Le^{L-\frac{L^2}{4t}}\right)\right].$$

We observe that $L - \frac{L^2}{4t} = -\frac{1}{9}\log g$ and therefore

$$\frac{t^4 e^t}{r^4}\left(\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)} + Le^{L-\frac{L^2}{4t}}\right) = \mathcal{O}\left((\log g)^5 g^{\frac{1}{27}+2d-\frac{1}{9}}\right) = \mathcal{O}\left((\log g)^5 g^{-2(\frac{1}{27}-d)}\right)$$

which goes to zero as $g \to +\infty$ because $d < \frac{1}{27}$. Since $\frac{27^{\frac{3}{2}}\sqrt{\pi}}{8} < 32$, we obtain that for large enough g,

$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq 32\frac{e^{-(\frac{1}{4}-b)t}}{(\log g)^{\frac{3}{2}}}$$

which is what was claimed. $\square$

5.3 Number of eigenvalues in a window

The contents of the following two sections have been adapted from the article 'Benjamini–Schramm convergence and spectrum of random hyperbolic surfaces of high genus' [Mon21], to appear in Analysis & PDE.

The aim of this section is to describe the behaviour of the number of eigenvalues in any window $[a,b]$ for a typical surface. This will require a similar geometric assumption to the one used in Section 5.2, which was stated in Chapter 4 as Corollary 4.2 and we recall here.

Corollary 5.7 (Geometric assumptions)**.** *For large enough g, there exists a subset $\mathcal{A}_g \subset \mathcal{M}_g$ such that, for any hyperbolic surface $X \in \mathcal{A}_g$,*

$$\mathrm{InjRad}(X) \geq g^{-\frac{1}{24}} (\log g)^{\frac{9}{16}} \tag{5.13}$$

$$\frac{\mathrm{Vol}_X\left(\{z \in X \ : \ \mathrm{InjRad}_z(X) < \frac{1}{6}\log g\}\right)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(g^{-\frac{1}{3}}\right) \tag{5.14}$$

and $1 - \mathbb{P}_g^{\mathrm{WP}}\left(\mathcal{A}_g\right) = \mathcal{O}\left(g^{-\frac{1}{12}} (\log g)^{\frac{9}{8}}\right)$.

Statement of the results We will prove the two following results, true for the surfaces in the set $\mathcal{A}_g$.

Theorem 5.8. *For any large enough g, any $0 \leq a \leq b$ and any surface $X \in \mathcal{A}_g$ from Corollary 5.7,*

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(b - a + \sqrt{\frac{b+1}{\log g}}\right). \tag{5.15}$$

This theorem provides us with an upper bound on the number of eigenvalues in an interval $[a,b]$. In equation (5.15), the term $\sqrt{\frac{b+1}{\log g}}$ corresponds to a minimum scale, below which we can have no additional information by shrinking the spectral window.

The second result is a more precise approximation of the counting functions.

Theorem 5.9. *There exists a universal constant $C > 0$ such that, for any large enough g, any $0 \leq a < b$ and any hyperbolic surface $X \in \mathcal{A}_g$ from Corollary 5.7, one can write the counting function $\mathrm{N}_X^{\Delta}(a,b)$ as*

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + (b-a) R(X,a,b)$$

where the remainder $R(X,a,b)$ satisfies

$$-\frac{C}{M_{a,b,g}} \leq R(X,a,b) \leq \frac{C}{M_{a,b,g}} \sqrt{\log(2 + M_{a,b,g})}.$$

for $M_{a,b,g} = (b-a)\sqrt{\frac{\log g}{b+1}}$.

The remainder $R(X,a,b)$ is negligible compared to the main term as soon as the size $b-a$ of the spectral window is much larger than the minimal spacing $\sqrt{\frac{b+1}{\log g}}$, that is to say $M_{a,b,g} \gg 1$. There are several limits that can be interesting to study.

- When $[a,b]$ is fixed and $g \to +\infty$, our result shows that the spectrum of Δ_X approaches the continuous spectrum of the Laplacian on $\mathcal{H}$. This is the analogous of the fact that, if a sequence of d-regular graphs converges in the sense of Benjamini–Schramm to the d-regular tree, then their spectral measure converges to the Kesten-McKay law [Ana17].

- By taking $a = 0$ and b going to infinity, one can recover a uniform Weyl law, with remainder of order $\mathcal{O}_g\left(\sqrt{b \log b}\right)$. The constant is independent of the surface, and explicit in terms of g. We could probably have obtained a better remainder (for instance, $\mathcal{O}_g\left(\frac{\sqrt{b}}{\log b}\right)$ as in [Bér77, Ran78]) had we allowed the probability set $\mathcal{A}_g$ to depend on the parameter b, which we did not do here in order to make the discussion simpler.

- One can also consider mixed regimes, where both b and g go to infinity.

Our approach in proving the two theorems is inspired by [LMS17, Part 9], where Le Masson and Sahlsten prove the convergence of $\frac{\mathrm{N}^{\Delta}_{X_g}(a,b)}{\mathrm{Vol}_{X_g}(X_g)}$ to the spectral density of the hyperbolic plane $\frac{1}{4\pi}\int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda$ as $g \to +\infty$, for a uniformly discrete sequence of compact hyperbolic surfaces $(X_g)_g$ converging to $\mathcal{H}$ in the sense of Benjamini–Schramm. Here, we do not consider a sequence but a fixed surface of high genus. Furthermore, we estimate precisely the error term, which leads us to considering different kernels in the trace formula.

Organisation of the proof The proof is organised as follows.

The proof of Theorems 5.8 and 5.9 then spans over Section 5.3.1 and 5.3.2, which corresponds respectively to the case where $0 \leq a \leq b \leq 1$ (bottom of the spectrum) and $\frac{1}{2} \leq a \leq b$ (away from small eigenvalues). The way the different parts depend on one another is explained in Figure 5.1 for clarity. We finish by proving some consequences on the eigenvalue multiplicity in Section 5.4.

$$
\begin{tikzcd}
\text{Sections } *.1.1 \text{ to } *.1.4 \arrow[dd] \arrow[dr] & & \text{Sections } *.2.1 \text{ to } *.2.4 \arrow[dd] \arrow[dl] \\
& \text{Theorem } 5.8 \arrow[dl] \arrow[dr] & \\
\text{Section } *.1.5 \arrow[dr] & & \text{Section } *.2.5 \arrow[dl] \\
& \text{Theorem } 5.9 &
\end{tikzcd}
$$

Figure 5.1: The steps of the proofs of Theorems 5.8 and 5.9, and the way they depend on one another. The symbol $*$ stands for Section 5.3. The left part corresponds to the case $b \leq 1$, and the right part to the case $a \geq \frac{1}{2}$.

5.3.1 Proof at the bottom of the spectrum

Let us prove Theorems 5.8 and 5.9 in the case where the spectral window is bounded above, more precisely $0 \leq a \leq b \leq 1$. The reason why we make the assumption $b \leq 1$ is that our choice of test function behaves poorly for large values of b. The value 1 is arbitrary, and could be replaced by any fixed value larger than $\frac{1}{2}$ (because the proof in the case away from small eigenvalues only works for $a \geq \frac{1}{2}$).

5.3.1.1 Trace formula, test function and sketch of the proof

The test function Let $0 \leq a \leq b \leq 1$. The test function we are going to use in this section is defined by

$$h_t(r) = f_t\left(\frac{1}{4} + r^2\right),$$

where, for $\lambda \geq 0$,

$$f_t(\lambda) = (\mathbb{1}_{[a,b]} \star v_t)(\lambda) = \frac{t}{\sqrt{\pi}} \int_a^b \exp\left(-t^2(\lambda - \mu)^2\right) \mathrm{d}\mu.$$

Here, $t > 0$ is a parameter that will grow like $\sqrt{\log g}$, and $v_t(x) = \frac{t}{\sqrt{\pi}} \exp(-t^2\rho^2)$ is the centered normalised Gaussian of variance $\frac{1}{t}$. As a consequence, as we observe on Figure 5.2, f_t is a smooth (point-wise) approximation of the function $\tilde{\mathbb{1}}_{[a,b]}$ defined by

$$\tilde{\mathbb{1}}_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ \frac{1}{2} & \text{if } x = a \text{ or } x = b \\ 0 & \text{otherwise.} \end{cases}$$

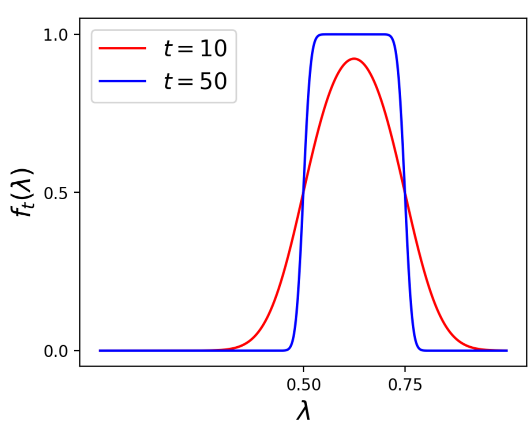


Figure 5.2: Plot of the test function f_t for the parameters $a = 0.5$ and $b = 0.75$.

The trace formula It is clear that $h_t : \mathbb{C} \to \mathbb{C}$ is analytic and even. In order to apply the trace formula, we need to prove an estimate like the one in equation (5.1), which is the aim of the following lemma.

Lemma 5.10. *Let $0 \leq a \leq b$ and $t > 0$. There exists a constant $C = C(a,b,t) > 0$ such that*

$$\forall x \in \mathbb{R}, \forall y \in [-1,1], |h_t(x+iy)| \leq C \exp(-t^2x^4 + 2t^2(3+b)x^2).$$

Proof. Let us write

$$h_t(r) = \frac{t}{\sqrt{\pi}} \int_{a-\frac{1}{4}}^{b-\frac{1}{4}} \exp\left(-t^2\left(r^2-\mu\right)^2\right) \mathrm{d}\mu.$$

The modulus of the integrand for a $r = x+iy$, $-1 \leq y \leq 1$ is

$$\begin{aligned}\left|\exp\left(-t^2\left(r^2-\mu\right)^2\right)\right| &= \exp\left(-t^2\left(x^4+y^4+\mu^2-6x^2y^2-2\mu x^2+2\mu y^2\right)\right) \\ &\leq \exp\left(-t^2x^4+6t^2x^2+2t^2\left(b-\frac{1}{4}\right)x^2+2t^2\left|\frac{1}{4}-a\right|\right).\end{aligned}$$

We then integrate this inequality between $a-\frac{1}{4}$ and $b-\frac{1}{4}$. $\square$

This exponential decay guarantees a polynomial decay in the strip $\{|\Im r| \leq 1\}$ like the one required in the trace formula. Therefore, one can apply it to h_t, and rewrite equation (5.2) as:

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} f_t(\lambda_j) = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda-\frac{1}{4}}\right) \mathrm{d}\lambda + \mathcal{R}_I(t,a,b) + \mathcal{R}_K(X,t,a,b) \tag{5.16}$$

where

$$\mathcal{R}_I(t,a,b) = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} (f_t(\lambda) - \mathbb{1}_{[a,b]}(\lambda)) \tanh\left(\pi\sqrt{\lambda-\frac{1}{4}}\right) \mathrm{d}\lambda, \tag{5.17}$$

$$\mathcal{R}_K(X,t,a,b) = \frac{1}{\mathrm{Vol}_X(X)} \int_D \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} K_t(z, \gamma \cdot z) \, \mathrm{dVol}_{\mathcal{H}}(z), \tag{5.18}$$

K_t is the kernel associated to h_t and D is a fundamental domain of $X = {}^{\mathcal{H}}\!/_{\Gamma}$.

Sketch of the proof There are four steps in the proof of Theorems 5.8 and 5.9 at the bottom of the spectrum.

- Control the integral term $\mathcal{R}_I(t,a,b)$ (Section 5.3.1.2).

- Estimate the geometric term $\mathcal{R}_K(X,t,a,b)$ using the uniform discreteness and the Benjamini–Schramm convergence assumptions (Section 5.3.1.3).
- Control $\mathrm{N}_X^\Delta(a,b)$ with $\sum_{j=0}^{+\infty} f_t(\lambda_j)$, and deduce Theorem 5.8 (Section 5.3.1.4).
- Compare more precisely the sum $\sum_{j=0}^{+\infty} f_t(\lambda_j)$ to $\mathrm{N}_X^\Delta(a,b)$, and conclude (Section 5.3.1.5).

5.3.1.2 The integral term

In order to control $\mathcal{R}_I(t,a,b)$, we need to know more about the speed of convergence of f_t towards $\tilde{\mathbb{1}}_{[a,b]}$ as t goes to infinity, which is the aim of the following lemma.

Lemma 5.11. *Let $0 \leq a \leq b$. For any $t > 0$ and $\lambda \in [0,+\infty)$,*

$$|f_t(\lambda) - \tilde{\mathbb{1}}_{[a,b]}(\lambda)| \leq \begin{cases} s(t|\lambda - a|) & \text{if } \lambda \in [0,a) \cup \{b\} \\ s(t|\lambda - a|) + s(t|\lambda - b|) & \text{if } \lambda \in (a,b) \\ s(t|\lambda - b|) & \text{if } \lambda \in \{a\} \cup (b,+\infty) \end{cases} \tag{5.19}$$

where $s : (0,+\infty) \to \mathbb{R}$ is the (decreasing) function defined by $s(\rho) = \frac{e^{-\rho^2}}{2\sqrt{\pi}\rho}$.

Proof. Let us assume that $\lambda \in (b,+\infty)$. In that case, $\tilde{\mathbb{1}}_{[a,b]}(\lambda) = 0$, and

$$|f_t(\lambda) - \tilde{\mathbb{1}}_{[a,b]}(\lambda)| = f_t(\lambda) \leq \frac{1}{\sqrt{\pi}} \int_{t(\lambda-b)}^{+\infty} e^{-\rho^2}\, \mathrm{d}\rho \leq \frac{e^{-t^2(\lambda-b)^2}}{2\sqrt{\pi}\, t(\lambda-b)}$$

since for $\rho > t(\lambda-b)$, $1 < \frac{2\rho}{2t(\lambda-b)}$. All the other cases can be proved in the same way, using, when $\lambda \in [a,b]$, the fact that the Gaussian we used in the definition of f_t is normalised. □

We can now prove the following estimate.

Proposition 5.12. *Let $0 \leq a \leq b$. For any $t > 0$,*

$$\mathcal{R}_I(t,a,b) = \mathcal{O}\left(\frac{1}{t}\right).$$

Proof. Let us replace $\mathbb{1}_{[a,b]}$ by the limit $\tilde{\mathbb{1}}_{[a,b]}$ of f_t as $t \to +\infty$, for they differ on a set of measure zero. We observe that the right hand side of equation (5.19) blows up around a and b, while the left hand side remains bounded. We shall therefore cut small intervals around a and b, and only apply Lemma 5.11 outside them.

Let C_ε be the set

$$C_\varepsilon = \left\{ \lambda \in \left[\frac{1}{4}, +\infty\right) \;:\; |\lambda - a| < \varepsilon \text{ or } |\lambda - b| < \varepsilon \right\}.$$

C_ε has at most two connected components, each of them of length at most 2ε. Since $|f_t - \tilde{\mathbb{1}}_{[a,b]}| \leq 1$,

$$\int_{C_\varepsilon} |f_t(\lambda) - \tilde{\mathbb{1}}_{[a,b]}(\lambda)| \,\mathrm{d}\lambda \leq 4\varepsilon.$$

There are at most three connected components in $\left[\frac{1}{4}, +\infty\right) \setminus C_\varepsilon$, and the estimate in every case is the same so we limit ourselves to the study of $[b+\varepsilon, +\infty)$. Lemma 5.11 implies that

$$\int_{b+\varepsilon}^{+\infty} |f_t(\lambda) - \tilde{\mathbb{1}}_{[a,b]}(\lambda)| \,\mathrm{d}\lambda = \mathcal{O}\left(\int_{b+\varepsilon}^{+\infty} \frac{\exp(t^2(\lambda-b)^2)}{t(\lambda-b)} \,\mathrm{d}\lambda\right) = \mathcal{O}\left(\frac{1}{\varepsilon t^2} \int_{\varepsilon t}^{+\infty} e^{-\rho^2} \,\mathrm{d}\rho\right).$$

Putting the two contributions together, we obtain

$$\mathcal{R}_I(t,a,b) = \mathcal{O}\left(\varepsilon + \frac{1}{\varepsilon t^2} \int_{\varepsilon t}^{+\infty} e^{-\rho^2} \,\mathrm{d}\rho\right),$$

which leads to our claim if we set $\varepsilon = \frac{1}{t}$. □

5.3.1.3 The geometric term

Let us now control the geometric term

$$\mathcal{R}_K(X,t,a,b) = \frac{1}{\mathrm{Vol}_X(X)} \int_D \sum_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} K_t(z, \gamma \cdot z) \,\mathrm{dVol}_{\mathcal{H}}(z)$$

for any compact hyperbolic surface $X \in \mathcal{A}_g$ from Corollary 5.7. In order to do so, we will estimate the kernel function K_t. We will then regroup the terms in the sum according to the distance between z and $\gamma \cdot z$. This is where we will use the Benjamini–Schramm hypothesis. Indeed, if $z \in D$ has a large injectivity radius, then the decay of K_t will cause the sum to be small. Otherwise, the sum might not be small, but the volume of the set of such z will.

Fourier estimate In order to estimate the kernel K_t, we first need to know about the derivative of $\check{h}_t$, the inverse Fourier transform of the test function h_t.

Lemma 5.13. *Let $0 \leq a \leq b \leq 1$ and $r \in (0,3)$. For any $u \geq r$, $t \geq \frac{1}{200}$,*

$$\check{h}_t'(u) = \mathcal{O}\left(r^{-\frac{2}{3}} \exp\left(-\frac{7}{32} u^{\frac{4}{3}} t^{-\frac{2}{3}} + \frac{3}{16} u^{\frac{2}{3}} t^{\frac{2}{3}}\right)\right). \tag{5.20}$$

Proof. By definition of $\check{h}_t$,

$$\check{h}_t(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h_t(r) e^{iru} \,\mathrm{d}r = \frac{t}{2\pi^{\frac{3}{2}}} \int_{-\infty}^{+\infty} \int_a^b e^{-t^2\left(\frac{1}{4}+r^2-\mu\right)^2} e^{iru} \,\mathrm{d}\mu \,\mathrm{d}r.$$

By the change of variables $\tilde{\mu} = t\big(\mu - \frac{1}{4}\big)$ and $\tilde{r} = \sqrt{t}\, r$ and Fubini's theorem, one can rewrite this integral as

$$\check{h}_t(u) = \frac{1}{2\pi^{\frac{3}{2}}\sqrt{t}} \int_{t\left(a-\frac{1}{4}\right)}^{t\left(b-\frac{1}{4}\right)} \int_{-\infty}^{+\infty} e^{-\left(r^2-\mu\right)^2} e^{i\frac{ru}{\sqrt{t}}} \,\mathrm{d}r\,\mathrm{d}\mu.$$

As a consequence, the derivative of $\check{h}_t$ is

$$\check{h}_t'(u) = \frac{i}{2\pi^{\frac{3}{2}}t} \int_{t\left(a-\frac{1}{4}\right)}^{t\left(b-\frac{1}{4}\right)} F_{\frac{u}{\sqrt{t}}}(\mu)\,\mathrm{d}\mu \tag{5.21}$$

where for $u > 0$ and $\mu \in \mathbb{R}$,

$$F_u(\mu) = \int_{-\infty}^{+\infty} r\, e^{-\left(r^2-\mu\right)^2+iru}\,\mathrm{d}r.$$

Let us estimate this integral using a change of contour. Let $R > 0$ and $R' > 0$ be two real parameters. The function $z \in \mathbb{C} \mapsto z\, e^{-(z^2-\mu)^2+izu}$ is holomorphic, so its contour integral on the rectangle of vertices R', $R' + iR$, $-R' + iR$, $-R'$ is equal to zero. We compute the modulus of the integrand for a complex number $z = x + iy$:

$$|z\, e^{-(z^2-\mu)^2+izu}| = \sqrt{x^2+y^2}\,\exp\big(-x^4 - y^4 + 6x^2y^2 - \mu^2 + 2x^2\mu - 2y^2\mu - yu\big). \tag{5.22}$$

It follows directly from this inequality that the integrals on the vertical sides of the rectangle go to zero as R' approaches infinity. As a consequence,

$$F_u(\mu) = \int_{\mathbb{R}+iR} z\, e^{-\left(z^2-\mu\right)^2+izu}\,\mathrm{d}z.$$

We use the triangle inequality and equation (5.22) to deduce that

$$|F_u(\mu)| \leq 2\exp\big(-R^4 - \mu^2 - 2R^2\mu - uR\big)\int_0^{+\infty}(x+R)\exp\big(-x^4 + 6x^2R^2 + 2x^2\mu\big)\,\mathrm{d}x.$$

We now study two distinct cases, depending on the sign of μ.

- If $\mu \geq 0$, then

  $$|F_u(\mu)| \leq 2\ \exp\big(8R^4 + 4R^2\mu - uR\big)\int_0^{+\infty}(x+R)\ e^{-(x^2-x_0^2)^2}\,\mathrm{d}x$$

  where $x_0 = \sqrt{3R^2+\mu} > 0$. We observe that

  $$\begin{aligned}\int_0^{+\infty}(x+R)\ e^{-(x^2-x_0^2)^2}\,\mathrm{d}x &\leq \int_{-x_0}^{+\infty}(|y| + x_0 + R)\ e^{-y^4}\ \underbrace{e^{-4y^2x_0(y+x_0)}}_{\leq 1}\,\mathrm{d}y \\ &= \mathcal{O}\left(1 + x_0 + R\right),\end{aligned}$$

  and therefore

  $$F_u(\mu) = \mathcal{O}\left(\left(1 + R + \mu^{\frac{1}{2}}\right)\ \exp\big(8R^4 + 4R^2\mu - uR\big)\right).$$

- If $\mu < 0$, we do the same with $x_0 = \sqrt{3}R$.

$$\begin{aligned}|F_u(\mu)| &\leq 2\exp\bigl(-\mu^2 + 8R^4 - 2R^2\mu - uR\bigr)\int_0^{+\infty}(x+R)\,e^{-(x^2-3R^2)^2}\,\mathrm{d}x\\ &= \mathcal{O}\left((1+R)\exp\bigl(-\mu^2 + 8R^4 + 2R^2|\mu| - uR\bigr)\right).\end{aligned}$$

As a conclusion, for any $\mu \in \mathbb{R}$,

$$F_u(\mu) = \mathcal{O}\left((1 + R + |\mu|^{\frac{1}{2}})\exp\bigl(-\mu_-^2 + 8R^4 + 4R^2|\mu| - uR\bigr)\right)$$

where $\mu_- = \min(\mu, 0)$. We take $R = \frac{1}{4}\,u^{\frac{1}{3}}$ and obtain that, for any $u > 0$ and $\mu \in \mathbb{R}$

$$F_u(\mu) = \mathcal{O}\left((1 + u^{\frac{1}{3}} + |\mu|^{\frac{1}{2}})\,\exp\left(-\mu_-^2 - \frac{7}{32}u^{\frac{4}{3}} + \frac{1}{4}|\mu|u^{\frac{2}{3}}\right)\right). \tag{5.23}$$

We then integrate the upper bound (5.23) in equation (5.21).

$$\begin{aligned}\check{h}_t'(u) &= \mathcal{O}\left(\frac{1}{t}\int_{t\left(a-\frac{1}{4}\right)}^{t\left(b-\frac{1}{4}\right)}|F_{\frac{u}{\sqrt{t}}}(\mu)|\,\mathrm{d}\mu\right)\\ &= \mathcal{O}\left(\frac{1 + u^{\frac{1}{3}}t^{-\frac{1}{6}} + t^{\frac{1}{2}}}{t}\exp\left(-\frac{7}{32}u^{\frac{4}{3}}t^{-\frac{2}{3}}\right)\int_{-\frac{3}{4}t}^{\frac{3}{4}t}\exp\left(\frac{1}{4}|\mu|u^{\frac{2}{3}}t^{-\frac{1}{3}}\right)\mathrm{d}\mu\right)\end{aligned}$$

because $|\mu| \leq \frac{3}{4}t$ for any $\mu \in \left[t\bigl(a-\frac{1}{4}\bigr), t\bigl(b-\frac{1}{4}\bigr)\right]$. Replacing the integral by its value and using the hypotheses about t, u and r concludes the proof. □

Estimate of the kernel function Let us now estimate the kernel function K_t, using its expression in terms of $\check{h}_t'$, equation (5.4).

Lemma 5.14. *Let $0 \leq a \leq b \leq 1$ and $r \in (0, 3)$. For any $\rho \geq r$, $t \geq \frac{1}{200}$,*

$$K_t(\rho) = \begin{cases}\mathcal{O}\left(\frac{t}{r^2}\exp\left(-\frac{\rho^{\frac{4}{3}}}{8t^{\frac{2}{3}}}\right)\right) & \text{if } \rho \geq 6t^2\\ \mathcal{O}\left(\frac{t}{r^2}\exp\left(t^2 - \frac{\rho^{\frac{4}{3}}}{8t^{\frac{2}{3}}}\right)\right) & \text{if } \rho \leq 6t^2.\end{cases} \tag{5.24}$$

In other words, the kernel K_t is exponentially decaying as $\rho \to +\infty$, and that the exponential envelope of the kernel spreads as t increases, as can be seen in Figure 5.3.

Proof. By definition of the kernel associated to h_t,

$$K_t(\rho) = -\frac{1}{\sqrt{2}\pi}\int_\rho^{+\infty}\frac{\check{h}_t'(u)}{\sqrt{\cosh u - \cosh\rho}}\,\mathrm{d}u.$$

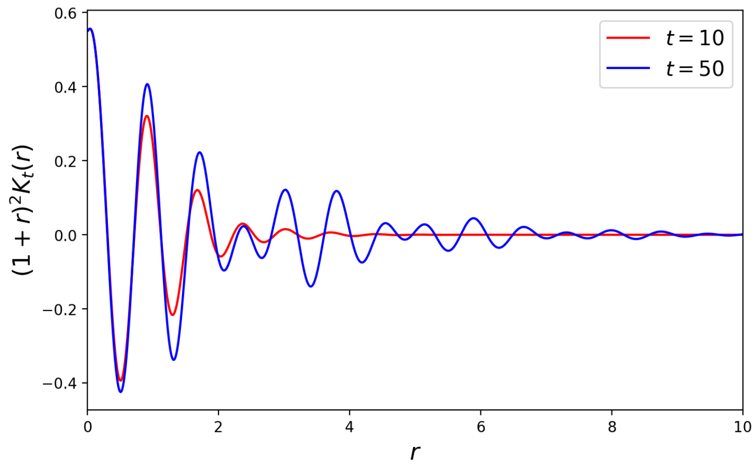


Figure 5.3: Plot of the kernel K_t for the parameters $a = 0.5$ and $b = 0.75$.

Using equation (5.20), we obtain

$$K_t(\rho) = \mathcal{O}\left(r^{-\frac{2}{3}} \int_{\rho}^{+\infty} \underbrace{\frac{\exp\left(-\frac{7}{32}u^{\frac{4}{3}}t^{-\frac{2}{3}} + \frac{3}{16}u^{\frac{2}{3}}t^{\frac{2}{3}}\right)}{\sqrt{\cosh u - \cosh \rho}}}_{(\star)} \,\mathrm{d}u \right). \tag{5.25}$$

Let us cut this integral into two contributions $I_1 = \int_{\rho}^{\rho_{\mathrm{cut}}}(\star)$ and $I_2 = \int_{\rho_{\mathrm{cut}}}^{+\infty}(\star)$ where

$$\rho_{\mathrm{cut}} = \max(2\rho, 12t^2).$$

This choice of ρ_{cut} allows us deal with the cancellation of the denominator in I_1 only, and to be in the asymptotic regime where the decreasing part of the exponential term is predominant everywhere in I_2.

In the second integral, since $u \geq \rho_{\mathrm{cut}} \geq 12t^2$, $\frac{3}{16}u^{\frac{2}{3}}t^{\frac{2}{3}} < \frac{2}{32}u^{\frac{4}{3}}t^{-\frac{2}{3}}$. Hence, the quantity in the exponential function is less than $-\frac{5}{32}u^{\frac{4}{3}}t^{-\frac{2}{3}}$. We deal with the denominator by observing that $\cosh u - \cosh \rho \geq \frac{1}{2}(u-\rho)^2 \geq \frac{1}{2}\rho^2$ since $u \geq \rho_{\mathrm{cut}} \geq 2\rho$. As a consequence,

$$I_2 = \mathcal{O}\left(\frac{1}{\rho}\int_{\rho_{\mathrm{cut}}}^{\infty} \exp\left(-\frac{5}{32}u^{\frac{4}{3}}t^{-\frac{2}{3}}\right)\mathrm{d}u\right).$$

This integral can be controlled by observing that $1 \leq u^{\frac{1}{3}}\rho_{\mathrm{cut}}^{-\frac{1}{3}}$, and then by explicit integration:

$$I_2 = \mathcal{O}\left(\frac{t^{\frac{2}{3}}}{\rho\rho_{\mathrm{cut}}^{\frac{1}{3}}} \exp\left(-\frac{5}{32}\rho_{\mathrm{cut}}^{\frac{4}{3}}t^{-\frac{2}{3}}\right)\right) = \mathcal{O}\left(\frac{t^{\frac{2}{3}}}{r^{\frac{4}{3}}} \exp\left(-\frac{5}{16}\rho^{\frac{4}{3}}t^{-\frac{2}{3}}\right)\right). \tag{5.26}$$

Now, in the first integral we use the inequality $\cosh u - \cosh\rho \geq (u-\rho)\sinh\rho \geq (u-\rho)\rho$.

$$\begin{aligned} I_1 &\leq \frac{\exp\left(-\frac{7}{32}\rho^{\frac{4}{3}}t^{-\frac{2}{3}} + \frac{3}{16}\rho_{\mathrm{cut}}^{\frac{2}{3}}t^{\frac{2}{3}}\right)}{\sqrt{\rho}} \int_{\rho}^{\rho_{\mathrm{cut}}} \frac{\mathrm{d}u}{\sqrt{u-\rho}} \\ &= \frac{1}{2}\sqrt{\frac{\rho_{\mathrm{cut}}-\rho}{\rho}} \exp\left(-\frac{7}{32}\rho^{\frac{4}{3}}t^{-\frac{2}{3}} + \frac{3}{16}\rho_{\mathrm{cut}}^{\frac{2}{3}}t^{\frac{2}{3}}\right). \end{aligned}$$

- When $\rho \leq 6t^2$, $\rho_{\mathrm{cut}} = 12t^2$ so

$$I_1 = \mathcal{O}\left(\frac{t}{r^{\frac{1}{2}}} \exp\left(-\frac{7}{32}\rho^{\frac{4}{3}}t^{-\frac{2}{3}} + t^2\right)\right). \tag{5.27}$$

- Otherwise, $\rho_{\mathrm{cut}} = 2\rho$ so

$$I_1 = \mathcal{O}\left(\exp\left(-\frac{7}{32}\rho^{\frac{4}{3}}t^{-\frac{2}{3}} + \frac{3}{16}2^{\frac{2}{3}}\rho^{\frac{2}{3}}t^{\frac{2}{3}}\right)\right) = \mathcal{O}\left(\exp\left(-\frac{1}{8}\rho^{\frac{4}{3}}t^{-\frac{2}{3}}\right)\right) \tag{5.28}$$

  because the fact that $\rho \geq 6t^2$ implies that $\frac{3}{16}2^{\frac{2}{3}}\rho^{\frac{2}{3}}t^{\frac{2}{3}} < \frac{3}{32}\rho^{\frac{4}{3}}t^{-\frac{2}{3}}$.

Putting together (5.25), (5.26), (5.27) and (5.28) leads to what was claimed. □

Kernel summation We can now proceed to the estimate of the geometric term. In order to do so, we will arrange the terms in the sum depending on the distance between z and $\gamma \cdot z$, as was done in Section 5.2, and use Lemma 5.14.

Lemma 5.15. *Let $0 \leq a \leq b \leq 1$. Let $r \in (0,3)$ and X be a compact hyperbolic surface of injectivity radius larger than r. For any $t \geq \frac{1}{200}$, $L \geq 2^{12}t^2$,*

$$\mathcal{R}_K(X,t,a,b) = \mathcal{O}\left(\frac{t^3}{r^4}\left(\exp(-L) + \frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)}\exp(L)\right)\right) \tag{5.29}$$

where $X^-(L)$ is the set of points in X of injectivity radius smaller than L.

Proof. Let us write a fundamental domain D of $X = {\mathcal{H}}/{\Gamma}$ as a disjoint union of $D^+(L)$ and $D^-(L)$, respectively the set of points in D of injectivity radius larger and smaller than L. We cut according to this partition of D the integral in the definition of

$$\mathcal{R}_K(X,t,a,b) = \frac{1}{\mathrm{Vol}_X(X)} \int_D \sum_{\gamma \neq \mathrm{id}} K_t(z, \gamma\cdot z)\,\mathrm{dVol}_{\mathcal{H}}(z)$$

into two contributions, $\mathcal{R}_K^+(X,t,a,b,L)$ and $\mathcal{R}_K^-(X,t,a,b,L)$.

Let us first estimate the term $\mathcal{R}_K^+(X,t,a,b,L)$. Lemma 5.14 allows us to control $K_t(z,\gamma\cdot z)$ in terms of the distance between z and $\gamma\cdot z$. In order to use it, we regroup the terms of the sum according to this quantity:

$$\mathcal{R}_K^+(X,t,a,b,L) = \frac{1}{\mathrm{Vol}_X(X)} \int_{D^+(L)} \sum_{j\geq L} \sum_{\substack{\gamma\neq \mathrm{id} \\ j\leq \mathrm{d}_{\mathcal{H}}(z,\gamma\cdot z)<j+1}} K_t(z,\gamma\cdot z)\,\mathrm{dVol}_{\mathcal{H}}(z).$$

One should notice that the sum only runs over integers larger than or equal to L as a consequence of the definition of $D^+(L)$. For any $z\in D^+(L)$, $j\geq L$ and $\gamma\in\Gamma\setminus\{\mathrm{id}\}$ such that $j\leq \mathrm{d}_{\mathcal{H}}(z,\gamma\cdot z)<j+1$, by Lemma 5.14 and since $\mathrm{d}_{\mathcal{H}}(z,\gamma\cdot z)\geq L>6t^2$,

$$K_t(z,\gamma\cdot z) = \mathcal{O}\left(\frac{t}{r^2}\exp\left(-\frac{j^{\frac{4}{3}}}{8t^{\frac{2}{3}}}\right)\right).$$

We then apply Lemma 5.5, which proves that

$$\#\{\gamma\in\Gamma \,:\, \mathrm{d}_{\mathcal{H}}(z,\gamma\cdot z)\leq j+1\} = \mathcal{O}\left(\frac{e^j}{r^2}\right).$$

Therefore, and because $\mathrm{Vol}_{\mathcal{H}}(D^+(L))\leq \mathrm{Vol}_X(X)$,

$$\mathcal{R}_K^+(X,t,a,b,L) = \mathcal{O}\left(\frac{t}{r^4}\,S(t,L)\right)$$

where $S(t,L)$ is defined as the sum

$$S(t,L) := \sum_{j\geq L}\exp\left(j-\frac{j^{\frac{4}{3}}}{8t^{\frac{2}{3}}}\right).$$

The fact that $L\geq 2^{12}t^2$ implies that, for any $j\geq L$, $j\leq \frac{j^{\frac{4}{3}}}{16t^{\frac{2}{3}}}$. As a consequence,

$$S(t,L)\leq \sum_{j\geq L}\exp\left(-\frac{j^{\frac{4}{3}}}{16t^{\frac{2}{3}}}\right)$$

which can be estimated by comparison with an integral:

$$S(t,L)\leq \left(1+\frac{12t^{\frac{2}{3}}}{L^{\frac{1}{3}}}\right)\exp\left(-\frac{L^{\frac{4}{3}}}{16t^{\frac{2}{3}}}\right) = \mathcal{O}\left(\exp(-L)\right) \quad \text{since } L^{\frac{1}{3}}\geq 16t^{\frac{2}{3}}.$$

Therefore,

$$\mathcal{R}_K^+(X,t,a,b,L) = \mathcal{O}\left(\frac{t}{r^4}\,\exp(-L)\right). \tag{5.30}$$

The same method, applied to $\mathcal{R}_K^-(X,t,a,b,L)$, leads to

$$\begin{aligned}&\mathcal{R}_K^-(X,t,a,b,L)\\&=\mathcal{O}\left(\frac{t}{r^4}\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)}\sum_{j\geq 0}(1+\mathbb{1}_{[0,6t^2]}(j)\,e^{t^2})\exp\left(j-\frac{j^{\frac{4}{3}}}{8t^{\frac{2}{3}}}\right)\right).\end{aligned}$$

We cut the sum at $j_{\mathrm{cut}}^{(1)}=\lfloor 6t^2\rfloor+1$ and $j_{\mathrm{cut}}^{(2)}=\lfloor 2^{12}t^2\rfloor+1$. The term where $j\geq j_{\mathrm{cut}}^{(2)}$ satisfies the same estimate as before since $j_{\mathrm{cut}}^{(2)}\geq 2^{12}t^2$, and therefore is

$$\mathcal{O}\left(\exp(-j_{\mathrm{cut}}^{(2)})\right)=\mathcal{O}\left(1\right).$$

We control naively the two other terms, which are $\mathcal{O}\left(t^2\exp(2^{12}t^2)\right)$. As a consequence,

$$\mathcal{R}_K^-(X,t,a,b,L)=\mathcal{O}\left(\frac{t^3}{r^4}\;\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)}\;\exp(2^{12}t^2)\right). \tag{5.31}$$

Our claim follows directly from equations (5.30) and (5.31). □

Geometric estimate We are now ready to use the geometric properties of random hyperbolic surfaces, and obtain an estimate of the geometric term in the trace formula true *with high probability*.

Proposition 5.16. *For any large enough g, any $0\leq a\leq b\leq 1$ and any hyperbolic surface $X\in\mathcal{A}_g$ defined in Corollary 5.7, if we set $t=\frac{\sqrt{\log g}}{64\sqrt{6}}$, then*

$$\mathcal{R}_K(X,t,a,b)=\mathcal{O}\left((\log g)^{-\frac{3}{4}}\right). \tag{5.32}$$

Proof. It is a direct consequence of Lemma 5.15 and the properties of the elements of $\mathcal{A}_g$, namely that if X is an element of $\mathcal{A}_g$ and $L=\frac{1}{6}\log g=2^{12}t^2$, then

- the injectivity radius of X is greater than $r=g^{-\frac{1}{24}}(\log g)^{\frac{9}{16}}$;
- $\dfrac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)}=\mathcal{O}\left(g^{-\frac{1}{3}}\right)$.

Since $L\geq 2^{12}t^2$, $r<3$ and $t>\frac{1}{200}$, we can apply Lemma 5.15:

$$\begin{aligned}\mathcal{R}_K(X,t,a,b)&=\mathcal{O}\left(\frac{t^3}{r^4}\left(\exp(-L)+\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)}\;\exp(L)\right)\right)\\&=\mathcal{O}\left(\frac{(\log g)^{\frac{3}{2}}}{g^{-\frac{1}{6}}(\log g)^{\frac{9}{4}}}\left(g^{-\frac{1}{6}}+g^{-\frac{1}{3}+\frac{1}{6}}\right)\right).\end{aligned}$$

□

5.3.1.4 Proof of Theorem 5.8 at the bottom of the spectrum

When we put together equation (5.16), Proposition 5.12 and 5.16, we obtain directly the following statement, which is an estimate of the trace formula. Theorems 5.8 and 5.9 (when $b \leq 1$) will then follow, since the spectral sum $\sum_{j=0}^{+\infty} f_t(\lambda_j)$ approaches $\mathrm{N}_X^\Delta(a,b)$ as $t \to +\infty$.

Corollary 5.17. *For any large enough g, any $0 \leq a \leq b \leq 1$ and any hyperbolic surface $X \in \mathcal{A}_g$ defined in Corollary 5.7, if we set $t = \frac{\sqrt{\log g}}{64\sqrt{6}}$, then*

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} f_t(\lambda_j) = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + \mathcal{O}\left(\frac{1}{\sqrt{\log g}}\right). \quad (5.33)$$

Proof of Theorem 5.8 when $b \leq 1$. Let $t = \frac{\sqrt{\log g}}{64\sqrt{6}}$. Let us distinguish two cases.

- Whenever $t(b-a) \geq \frac{1}{\sqrt{3}}$, since the function f_t only takes positive values,

  $$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} \times \inf_{[a,b]} f_t \leq \frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} f_t(\lambda_j).$$

  It follows directly from equation (5.33) that the right hand term is

  $$\mathcal{O}\left(b - a + \frac{1}{\sqrt{\log g}}\right).$$

  In order to deal with the infimum, we use Lemma 5.11 that states that

  $$\inf_{[a,b]} f_t \geq \frac{1}{2} - \frac{e^{-t^2(b-a)^2}}{2\sqrt{\pi}t(b-a)} \geq \frac{1}{2} - \frac{\sqrt{3}\, e^{-\frac{1}{3}}}{2\sqrt{\pi}} \geq \frac{1}{10}$$

  since we assumed $t(b-a) \geq \frac{1}{\sqrt{3}}$. Therefore, $\left(\inf_{[a,b]} f_t\right)^{-1} = \mathcal{O}(1)$.

- Otherwise, the fact that a and b are very close together prevents the test function f_t from being a good approximation of the indicator function of $[a,b]$. We therefore let a' be $b - \frac{1}{\sqrt{3}\,t}$, so that a' and b satisfy the spacing hypothesis $t(b-a') \geq \frac{1}{\sqrt{3}}$, and we can apply the first point to them:

  $$\frac{\mathrm{N}_X^\Delta(a,b)}{\mathrm{Vol}_X(X)} \leq \frac{\mathrm{N}_X^\Delta(a',b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(b - a' + \frac{1}{\sqrt{\log g}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\log g}}\right).$$

  The issue with this fix is that, when b is small, a' takes negative values. However, throughout this section, the only place where the positivity of a' was used is at the end of Lemma 5.13, when saying that $|\mu| \leq \frac{3}{4}t$ for any $\mu \in \left[t\left(a' - \frac{1}{4}\right), t\left(b - \frac{1}{4}\right)\right]$. This remains true as soon as $a' \geq -\frac{1}{2}$, which will be the case if t is large enough.

□

5.3.1.5 Proof of Theorem 5.9 at the bottom of the spectrum

Let us now proceed to the proof of Theorem 5.9 when $b \leq 1$. Beware that, in the proof of the lower bound, we will need to use Theorem 5.8 *for any* $0 \leq a \leq b$. This is not an issue, as was shown in Figure 5.1.

Proof of the upper bound of Theorem 5.9 when $b \leq 1$. Let $t = \frac{\sqrt{\log g}}{64\sqrt{6}}$.

If $t(b-a) \leq \sqrt{2e}$, then the integral term is $\mathcal{O}(b-a) = \mathcal{O}\left(\frac{1}{\sqrt{\log g}}\right)$, so the result follows directly from Theorem 5.8.

Let us assume $t(b-a) \geq \sqrt{2e}$. The issue in the previous estimate was that the convergence of f_t is slow around a and b, and noticeably $f_t(a)$ and $f_t(b)$ go to $\frac{1}{2}$ and not 1 as $t \to +\infty$. In order to deal with this, we cut a small segment around a and b. Let $\frac{1}{t} \leq \varepsilon \leq \frac{b-a}{2}$, then

$$\mathrm{N}_X^{\Delta}(a,b) = \mathrm{N}_X^{\Delta}(a,a+\varepsilon) + \mathrm{N}_X^{\Delta}(a+\varepsilon,b-\varepsilon) + \mathrm{N}_X^{\Delta}(b-\varepsilon,b).$$

By Theorem 5.8,

$$\frac{\mathrm{N}_X^{\Delta}(a,a+\varepsilon) + \mathrm{N}_X^{\Delta}(b-\varepsilon,b)}{\mathrm{Vol}_X(X)} = \mathcal{O}\left(\varepsilon + \sqrt{\frac{b+1}{\log g}}\right) = \mathcal{O}(\varepsilon).$$

We use the same method as before to control the middle term:

$$\begin{aligned}\frac{\mathrm{N}_X^{\Delta}(a+\varepsilon,b-\varepsilon)}{\mathrm{Vol}_X(X)} \times \inf_{[a+\varepsilon,b-\varepsilon]} f_t &\leq \frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} f_t(\lambda_j) \\ &\leq \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + \frac{C'}{\sqrt{\log g}}\end{aligned}$$

for a constant $C' > 0$, given by Corollary 5.17. By Lemma 5.11, and because $\varepsilon t \geq 1$,

$$\inf_{[a+\varepsilon,b-\varepsilon]} f_t \geq 1 - \frac{e^{-t^2\varepsilon^2}}{2\sqrt{\pi}\varepsilon t} \geq \frac{1}{1+e^{-\varepsilon^2 t^2}}.$$

Putting all the contributions together, there exists a constant $C'' > 0$ such that

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} \leq \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + C'' \left(\varepsilon + (b-a)\, e^{-t^2\varepsilon^2}\right).$$

We can now set

$$\varepsilon = \frac{1}{t}\sqrt{\log\left(\frac{\sqrt{e}\, t(b-a)}{\sqrt{2}}\right)}.$$

The hypothesis $t(b-a) \geq \sqrt{2e}$ directly implies that $\varepsilon t \geq 1$. Furthermore, the fact that for any $x \geq 1$, $\sqrt{\log x} \leq \frac{x}{\sqrt{2e}}$ implies that $\varepsilon \leq \frac{b-a}{2}$. Direct substitution of ε and t by their values in the previous estimate leads to our claim. □

Proof of the lower bound of Theorem 5.9 when $b \leq 1$. Let $t = \frac{\sqrt{\log g}}{64\sqrt{6}}$. Since $0 \leq f_t \leq 1$,

$$\mathrm{N}_X^{\Delta}(a,b) \geq \sum_{a \leq \lambda_j \leq b} f_t(\lambda_j) = \sum_{j=0}^{+\infty} f_t(\lambda_j) - \sum_{0 \leq \lambda_j < a} f_t(\lambda_j) - \sum_{\lambda_j > b} f_t(\lambda_j).$$

By Corollary 5.17, there exists a $C' > 0$ such that

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} f_t(\lambda_j) \geq \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda - \frac{C'}{\sqrt{\log g}},$$

so it suffices to prove that the two remaining terms are $\mathcal{O}\left(\frac{\mathrm{Vol}_X(X)}{\sqrt{\log g}}\right)$.

Both the terms behave the same way, so we only detail the sum over b. Let us divide $(b, +\infty)$ using a subdivision $b_k = b + \frac{k}{t}$, $k \geq 0$. We regroup the terms of the sum according to these numbers.

$$\begin{aligned}
\frac{1}{\mathrm{Vol}_X(X)} \sum_{\lambda_j > b} f_t(\lambda_j) &= \sum_{k=0}^{+\infty} \frac{1}{\mathrm{Vol}_X(X)} \sum_{b_k < \lambda_j \leq b_{k+1}} f_t(\lambda_j) \\
&\leq \sum_{k=0}^{+\infty} \frac{\mathrm{N}_X^{\Delta}(b_k, b_{k+1})}{\mathrm{Vol}_X(X)} \times \sup_{[b_k, b_{k+1}]} f_t \\
&= \mathcal{O}\left(\sum_{k=0}^{+\infty} \left(b_{k+1} - b_k + \sqrt{\frac{b_{k+1}+1}{\log g}}\right) \times \sup_{[b_k, b_{k+1}]} f_t\right)
\end{aligned}$$

by Theorem 5.8. As a consequence,

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{\lambda_j > b} f_t(\lambda_j) = \mathcal{O}\left(\frac{1}{\sqrt{\log g}} + \frac{1}{\sqrt{\log g}} \sum_{k=1}^{+\infty} \sqrt{k} \times \sup_{[b_k, b_{k+1}]} f_t\right).$$

By Lemma 5.11,

$$\sum_{k=1}^{+\infty} \sqrt{k} \times \sup_{[b_k, b_{k+1}]} f_t = \mathcal{O}\left(\sum_{k=1}^{+\infty} \frac{\exp(-k^2)}{\sqrt{k}}\right) = \mathcal{O}(1).$$

□

5.3.2 Proof away from small eigenvalues

We now proceed to the proof of Theorems 5.8 and 5.9 in the case when $\frac{1}{2} \leq a \leq b$. It is easy to see that it suffices to prove the results for these two situations, for we can then apply them to $a, \frac{3}{4}$ and $\frac{3}{4}, b$ and add up the two contributions if $a < \frac{1}{2}$ and $b > 1$.

The proof here is very similar to the previous proof, apart from the fact that the test function we use is different. We will not give all the details, and mostly highlight the differences between the two proofs.

The reason why we need to assume $a \geq \frac{1}{2}$ is that the test function we use behaves poorly for small eigenvalues. We will use the fact that there are at most $2g-2$ of them by work of Otal and Rosas [OR09], and that the spectral window is far enough from them, to deal with this situation (see Section 5.3.2.4).

5.3.2.1 Trace formula, test function and sketch of the proof

We will use once again the Selberg trace formula, but with a different test function this time.

The test function Let $a = \frac{1}{4} + \alpha^2$ and $b = \frac{1}{4} + \beta^2$, for some $0 \leq \alpha \leq \beta$. Let us consider the function

$$h_t(r) = (\mathbb{1}_{[\alpha,\beta]} \star v_t)(r) = \frac{t}{\sqrt{\pi}} \int_\alpha^\beta \exp\left(-t^2(r-\rho)^2\right) \mathrm{d}\rho = \frac{1}{\sqrt{\pi}} \int_{t(\alpha-r)}^{t(\beta-r)} \exp\left(-\rho^2\right) \mathrm{d}\rho,$$

where t still grows like $\sqrt{\log g}$. h_t now is a smooth approximation of the function $\tilde{\mathbb{1}}_{[\alpha,\beta]}$. We make h_t into an even test function by setting $H_t(r) = h_t(r) + h_t(-r)$. It is clear that $H_t : \mathbb{C} \to \mathbb{C}$ is analytic and even. The following lemma is an estimate on h_t aimed at applying the trace formula, but we make it a bit more precise than necessary for later use.

Lemma 5.18. *Let $0 \leq \alpha \leq \beta$, $a = \frac{1}{4} + \alpha^2$ and $b = \frac{1}{4} + \beta^2$. For any $t > 0$,*

$$\forall r = x + iy, \quad |h_t(r)| \leq \frac{1}{2\sqrt{\pi}\,\alpha t} \exp\left(t^2(y^2 - x^2 + 2\beta x - \alpha^2)\right).$$

Proof. Let $r = x + iy$. The modulus of the integrand in the definition of $h_t(r)$ is

$$|\exp\left(-t^2(r-\rho)^2\right)| = \exp\left(-t^2(x-\rho)^2 + t^2y^2\right).$$

As a consequence,

$$|h_t(r)| \leq \frac{t}{\sqrt{\pi}} \exp\left(t^2(y^2 - x^2 + 2\beta x)\right) \int_\alpha^{+\infty} \exp\left(-t^2\rho^2\right) \mathrm{d}\rho$$

which allows us to conclude, using the Gaussian tail estimate. □

Therefore, one can apply the trace formula to H_t:

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{j=0}^{+\infty} H_t(r_j) = \frac{1}{4\pi} \int_a^b \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + \mathcal{R}_I(t,a,b) + \mathcal{R}_K(X,t,a,b) \quad (5.34)$$

where

$$\mathcal{R}_I(t,a,b) = \frac{1}{4\pi}\int_0^{+\infty} (h_t(r) + h_t(-r) - \mathbb{1}_{[\alpha,\beta]}(r))\, r\tanh(\pi r)\,\mathrm{d}r, \tag{5.35}$$

$$\mathcal{R}_K(X,t,a,b) = \frac{1}{\mathrm{Vol}_X(X)}\int_D \sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} K_t(z,\gamma\cdot z)\,\mathrm{dVol}_{\mathcal{H}}(z), \tag{5.36}$$

K_t is the kernel associated to H_t and D is a fundamental domain of $X = {}^{\mathcal{H}}\!/_{\Gamma}$.

Sketch of the proof The steps of the proof are exactly the same as before, and are organised the same way. The only additional step is dealing with the contributions of the small eigenvalues to the sum $\sum_{j=0}^{+\infty}(h_t(r_j)+h_r(-r_j))$, and can be found in Section 5.3.2.4. This is necessary here and was not before because the function h_t is no longer real valued and small on the imaginary axis. This complication is the reason why this test function does not work whenever $a < \frac{1}{2}$.

5.3.2.2 The integral term

The integral estimate is the following.

Proposition 5.19. *Let $\frac{1}{4} \leq a \leq b$. For any $t \geq \frac{1}{10}$,*

$$\mathcal{R}_I(t,a,b) = \mathcal{O}\left(\frac{\sqrt{b}}{t}\right).$$

The proof uses the same method as before, and the following lemma to control the speed of convergence of h_t towards $\tilde{\mathbb{1}}_{[\alpha,\beta]}$ as t goes to infinity.

Lemma 5.20. *Let $0 \leq \alpha \leq \beta$. For any $t > 0$ and $r \in \mathbb{R}$,*

$$|h_t(r) - \tilde{\mathbb{1}}_{[\alpha,\beta]}(r)| \leq \begin{cases} s(t|r-\alpha|) & \text{if } r \in (-\infty,\alpha)\cup\{\beta\} \\ s(t|r-\alpha|) + s(t|r-\beta|) & \text{if } r \in (\alpha,\beta) \\ s(t|r-\beta|) & \text{if } r \in \{\alpha\}\cup(\beta,+\infty) \end{cases} \tag{5.37}$$

where $s : (0,+\infty) \to \mathbb{R}$ is the (decreasing) function defined in Lemma 5.11.

5.3.2.3 The geometric term

The control of the geometric term is simpler in this case, because the test function H_t is a convolution of two functions with simple Fourier transforms (a Gaussian and a step function). Therefore, its Fourier transform has a simple expression.

Lemma 5.21. *Let $\frac{1}{4} \leq a \leq b$ and $r \in (0,3)$. For any $t \geq \frac{1}{10}$, $u > r$,*

$$\check{h}_t'(u) = \mathcal{O}\left(\frac{\sqrt{b}}{r}\exp\left(-\frac{u^2}{4t^2}\right)\right). \tag{5.38}$$

Proof. Let us write $a = \frac{1}{4} + \alpha^2$ and $b = \frac{1}{4} + \beta^2$, with $0 \leq \alpha \leq \beta$.

We can compute $\check{h}_t$ explicitly, knowing the Fourier transform of a Gaussian and a step function:

$$\check{h}_t(u) = \frac{\beta \operatorname{sinc}(\beta u) - \alpha \operatorname{sinc}(\alpha u)}{\pi} \exp\left(-\frac{u^2}{4t^2}\right)$$

where $\operatorname{sinc}(x) = \frac{\sin x}{x}$. Therefore, the derivative of $\check{h}_t$ is

$$\check{h}_t'(u) = -\frac{u}{2t^2}\, \check{h}_t(u) + \frac{\beta^2 \operatorname{sinc}'(\beta u) - \alpha^2 \operatorname{sinc}'(\alpha u)}{\pi} \exp\left(-\frac{u^2}{4t^2}\right).$$

We use the fact that $|\operatorname{sinc}(x)| \leq 1$ and $|x \operatorname{sinc}'(x)| = |\cos x - \operatorname{sinc} x| \leq 2$ to conclude. □

This leads directly to an estimate on the kernel function, by cutting the integral (5.4) expressing K_t in terms of $\check{h}_t$ at 2ρ and using the same inequalities as before for the denominator.

Lemma 5.22. *Let $\frac{1}{4} \leq a \leq b$ and $r \in (0,3)$. For any $\rho \geq r$, $t \geq \frac{1}{10}$,*

$$K_t(\rho) = \mathcal{O}\left(\frac{t\sqrt{b}}{r^2} \exp\left(-\frac{\rho^2}{4t^2}\right)\right). \tag{5.39}$$

Then, the same summation process leads to the following lemma.

Lemma 5.23. *Let $\frac{1}{4} \leq a \leq b$ and $r \in (0,3)$. Let $X = \mathcal{H}/\Gamma$ be a compact hyperbolic surface of injectivity radius larger than r. For any $t \geq \frac{1}{10}$, $L \geq 8t^2$,*

$$\mathcal{R}_K(X,t,a,b) = \mathcal{O}\left(\frac{t^3\sqrt{b}}{r^4}\left[\exp(-L) + \frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)} \exp(L)\right]\right) \tag{5.40}$$

where $X^-(L)$ is the set of points in X of injectivity radius smaller than L.

We can then conclude using the geometric properties of random surfaces.

Proposition 5.24. *For any large enough g, any $\frac{1}{4} \leq a \leq b$ and any hyperbolic surface $X \in \mathcal{A}_g$ defined in Corollary 5.7, if we set $t = \frac{\sqrt{\log g}}{4\sqrt{3}}$, then*

$$\mathcal{R}_K(X,t,a,b) = \mathcal{O}\left(\sqrt{\frac{b}{\log g}}\right). \tag{5.41}$$

5.3.2.4 Small eigenvalues term, and proof of Theorem 5.8 away from them

The behaviour of the function h_t is different on the imaginary and real axes. Noticeably, the function h_t is positive on the real axis, but it is not real valued on the imaginary axis. This will cause some of the inequalities from the previous part to fail. Also, when a is close to $\frac{1}{4}$, the modulus of h_t on the segment $\left[-\frac{i}{2}, \frac{i}{2}\right]$ becomes too large, and the remainder we will obtain will be unsatisfactory. This is the reason why this test function is only suitable for values of a greater than $\frac{1}{2}$.

We shall now deal with the small eigenvalues, so that they do not intervene anymore afterwards.[1]

Lemma 5.25. *Let $\frac{1}{2} \leq a \leq b$. For any compact hyperbolic surface X and any $t > 0$,*

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{r_j \notin \mathbb{R}} (h_t(r_j) + h_t(-r_j)) = \mathcal{O}\left(\frac{1}{t}\right). \tag{5.42}$$

Proof. Let $\frac{1}{2} \leq \alpha \leq \beta$ such that $a = \frac{1}{4} + \alpha^2$ and $b = \frac{1}{4} + \beta^2$. If $r_j \notin \mathbb{R}$, then $r_j = iy_j$ with $y_j \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. By Lemma 5.18,

$$|h_t(\pm r_j)| \leq \frac{1}{2\sqrt{\pi}\,\alpha t} \exp\bigl(t^2(y_j^2 - \alpha^2)\bigr) = \mathcal{O}\left(\frac{1}{t}\right) \quad \text{since } \alpha \geq \frac{1}{2}.$$

The number of such terms is $\leq 2g - 2 = \mathcal{O}\left(\mathrm{Vol}_X(X)\right)$ by [OR09]. □

When we put together equation (5.34), Proposition 5.19, 5.24 and Lemma 5.25, we obtain directly the following statement.

Corollary 5.26. *For any large enough g, any $\frac{1}{2} \leq a \leq b$ and any hyperbolic surface $X \in \mathcal{A}_g$ defined in Corollary 5.7, if we set $t = \frac{\sqrt{\log g}}{4\sqrt{3}}$, then*

$$\frac{1}{\mathrm{Vol}_X(X)} \sum_{r_j \in \mathbb{R}} (h_t(r_j) + h_t(-r_j)) = \frac{1}{4\pi} \int_a^b \tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + \mathcal{O}\left(\sqrt{\frac{b}{\log g}}\right). \tag{5.43}$$

It is straightforward to deduce Theorem 5.8 from this result as was done before.

5.3.2.5 Proof of Theorem 5.9 away from small eigenvalues

A version of Theorem 5.9 in terms of α and β follows directly from the method of Section 5.3.1.

Theorem 5.27 (Theorem 5.9 away from small eigenvalues)**.** *There exists a universal constant $C > 0$ such that, for any large enough g, any $\frac{1}{2} \leq \alpha \leq \beta$ and any hyperbolic*

[1] Note that we could also have used Theorem 5.2, but this does not lead to a better estimate.

surface $X \in \mathcal{A}_g$ from Corollary 5.7, if we set $a = \frac{1}{4} + \alpha^2$ and $b = \frac{1}{4} + \beta^2$, then one can write the counting function $\mathrm{N}_X^{\Delta}(a, b)$ as

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} = \frac{1}{4\pi} \int_a^b \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + R(X,a,b) \tag{5.44}$$

where

$$-C \sqrt{\frac{b}{\log g}} \leq R(X,a,b) \leq C \sqrt{\frac{b}{\log g}} \, \log\left(2 + (\beta - \alpha)\sqrt{\log g}\right)^{\frac{1}{2}}.$$

We translate this statement in terms of a and b thanks to the fact that $\beta - \alpha = \frac{b-a}{\beta+\alpha}$, and therefore, as soon as $b \geq \frac{1}{2}$,

$$\beta - \alpha \leq \frac{b-a}{\sqrt{b - \frac{1}{4}}} \leq \sqrt{2}\, \frac{b-a}{\sqrt{b}}.$$

5.4 Consequences on the multiplicity of eigenvalues

For a compact hyperbolic surface $X \in \mathcal{M}_g$, and a real number $\lambda > 0$, let $m_X(\lambda)$ denote the multiplicity of the eigenvalue λ of Δ_X. We can estimate $m_X(\lambda)$ with high probability, using Theorem 5.8 and 5.2 and a shrinking spectral window around the eigenvalue λ.

Corollary 5.28. *There exists a universal constant $C > 0$ such that, for any large enough g, any $\lambda \geq 0$ and any hyperbolic surface $X \in \mathcal{A}_g$ from Corollary 5.7,*

$$\frac{m_X(\lambda)}{g} \leq C \sqrt{\frac{1+\lambda}{\log g}}.$$

If furthermore $\lambda \leq \frac{1}{4} - \varepsilon$, then

$$\frac{m_X(\lambda)}{g} \leq 41 \frac{g^{-\frac{\varepsilon}{32}}}{(\log g)^{\frac{3}{2}}}.$$

Another probabilistic upper bound $\frac{m_X(\lambda)}{g} = \mathcal{O}_\lambda\left(\frac{1}{\log g}\right)$ and $\frac{m_X(\lambda)}{g} = \mathcal{O}\left(g^{-c\sqrt{\varepsilon}}\right)$ when $\lambda \leq \frac{1}{4} - \varepsilon$ has been proved recently in [GLMST21]. This was achieved by estimating the L^p-norms of eigenfunctions on random hyperbolic surfaces of high genus. Though the behaviour in terms of g of the bound $\frac{1}{\log g}$ is better than our $\frac{1}{\sqrt{\log g}}$, the implied constant depends on λ.

One can also deduce from Theorem 5.9 an estimate on the j-th eigenvalue $\lambda_j(X)$ of X, in terms of j and g, true with high probability.

Corollary 5.29. *There exists a universal constant $C > 0$ such that, for any large enough g, any $j \geq 0$ and any hyperbolic surface $X \in \mathcal{A}_g$ from Corollary 5.7,*

$$\left|\lambda_j(X) - \frac{j}{g}\right| \leq C\left(1 + \sqrt{\frac{j}{g}\log\left(2 + \frac{j}{g}\right)}\right).$$

There are two interesting regimes in which one can apply this corollary:

- If $j \leq Ag$ for a $A \geq 1$, then $\lambda_j(X) = \mathcal{O}(A)$.
- If $j \gg g$, then $\lambda_j(X) \sim \frac{j}{g}$ uniformly in X.

As a consequence, the multiplicity of the j-th eigenvalue $\lambda_j(X)$ of a typical compact hyperbolic surface $X \in \mathcal{A}_g$ satisfies

$$\frac{m_X(\lambda_j(X))}{g} = \mathcal{O}\left(\sqrt{\frac{1 + \frac{j}{g}}{\log g}}\right), \tag{5.45}$$

which is an improvement of the deterministic estimate $m_X(\lambda_j(X)) \leq 4g + 2j + 1$ from [Bes80].

Proof of Corollary 5.29. Let g be large enough for Theorems 5.8 and 5.9 to apply, and $X \in \mathcal{A}_g$. Let $j \geq 0$.

If $\lambda_j(X) \leq \frac{1}{4}$, then $j \leq 2g - 2$ by work of Otal and Rosas [OR09]. It follows that both $\lambda_j(X)$ and $\frac{j}{g}$ are $\mathcal{O}(1)$, which leads to our claim.

We can therefore assume $\lambda_j(X) \geq \frac{1}{4}$. By Theorem 5.9 applied between 0 and $\lambda_j(X)$,

$$\begin{aligned}
&\frac{\mathrm{N}_X^{\Delta}(0, \lambda_j(X))}{2\pi(2g-2)} \\
&= \frac{1}{4\pi}\int_{\frac{1}{4}}^{\lambda_j(X)} \mathbb{1}_{[a,b]}(\lambda)\tanh\left(\pi\sqrt{\lambda - \frac{1}{4}}\right)\mathrm{d}\lambda + \mathcal{O}\left(\sqrt{\lambda_j(X)\log(2 + \lambda_j(X))}\right) \\
&= \frac{\lambda_j(X)}{4\pi} + \mathcal{O}\left(1 + \sqrt{\lambda_j(X)\log(2 + \lambda_j(X))}\right).
\end{aligned}$$

But by definition of the j-th eigenvalue $\lambda_j(X)$, we also have

$$\mathrm{N}_X^{\Delta}(0, \lambda_j(X)) = j + \mathcal{O}\left(m_X(\lambda_j(X))\right),$$

which is $j + \mathcal{O}\left(g\sqrt{\lambda_j(X)}\right)$ by Corollary 5.28. As a consequence, there is a constant $C > 0$ such that

$$\left|\lambda_j(X) - \frac{j}{g}\right| \leq C\left(1 + \sqrt{\lambda_j(X)\log(2 + \lambda_j(X))}\right). \tag{5.46}$$

There exists a constant $M > 0$ such that, as soon as $\lambda_j(X) > M$, the right hand term of equation (5.46) is smaller than $\frac{\lambda_j(X)}{2}$. We distinguish two cases.

- If $\lambda_j(X) > M$, then by equation (5.46) and by definition of M, $\lambda_j(X) \leq 2\frac{j}{g}$. Therefore, equation (5.46) leads to our claim.
- Otherwise, by equation (5.46), $\frac{j}{g}$ and $\lambda_j(X)$ are both $\mathcal{O}_M(1)$, and the conclusion still follows.

$\square$

Chapter 6

Spectral gap

Proving the following conjecture on the spectral gap of typical hyperbolic surfaces of high genus is the aim of an on-going collaboration with Nalini Anantharaman.

Conjecture 6.1 ([Wri20])**.** *For any $\varepsilon > 0$,*

$$\lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{1}{4} - \varepsilon\right) = 1. \tag{6.1}$$

We recall that, by [Che75], this result cannot be true if we replace $\frac{1}{4}$ by a larger number. Mirzakhani proved a first probabilistic lower bound in [Mir13],

$$\forall \varepsilon > 0, \quad \lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{1}{4}\left(\frac{\log 2}{2\pi + \log 2}\right)^2 - \varepsilon\right) = 1,$$

using Cheeger's inequality. The value of the numerical constant is very small, ≈ 0.002. A significant step towards Conjecture 6.1 has been achieved very recently by two independent teams, Wu–Xue [WX21] and Lipnowski–Wright [LW21].

Theorem 6.2. *For any $\varepsilon > 0$,*

$$\lim_{g\to+\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \geq \frac{3}{16} - \varepsilon\right) = 1. \tag{6.2}$$

Note that the situation is the same result in the random cover setting: Magee–Naud–Puder proved that $\lambda_1 \geq \frac{3}{16} - \varepsilon$ typically in [MNP20], and it is conjectured that this result can be improved to the optimal spectral gap $\frac{1}{4}$.

Conjecture 6.1 is the natural adaptation to hyperbolic surfaces of Friedman's theorem, conjectured by Alon in [Alo86] and proven by Friedman in [Fri03].

Theorem 6.3. *Let $d \geq 3$ be an integer. For any $\varepsilon > 0$,*

$$\lim_{n\to+\infty} \mathbb{P}_n^{(d)}\Big(\max\{|\lambda_i| \;:\; |\lambda_i| < d\} \leq 2\sqrt{d-1} + \varepsilon\Big) = 1$$

for diverse usual probability measures $\mathbb{P}_n^{(d)}$ on the set of d-regular graphs with n vertices.

Let present some ideas and methods that could be used to prove Conjecture 6.1. We shall compare the approach we suggest to the one used in [WX21, LW21] and the two proofs of Friedman's theorem [Fri03, Bor20]. Note that this is a hard problem, and there is still a lot to be done to reach to a full proof of the statement.

6.1 The trace method

A powerful way to study extremal eigenvalues is the trace method, which we now present. We first describe the trace method for regular graphs, as used in [McK81, BS87, Fri91] for instance. This will allow to present some core ideas of the method we suggest for surfaces, in a simpler setting.

6.1.1 Regular graphs

Let G be a connected, non-bipartite d-regular graphs, with vertices $V = \{1, \ldots, n\}$. Let

$$d = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n > -d$$

denote the eigenvalues of the adjacency matrix A of G.

6.1.1.1 Trace formula

The starting point is to observe that, for any integer k, the trace of the matrix A^k can be written in two different ways.

- We can diagonalise A^k and express the trace in terms of its eigenvalues:

$$\mathrm{Tr}(A^k) = \sum_{j=1}^{n} \lambda_j^k.$$

- Or we can notice that for all i, j, the entry (i, j) of the matrix A^k is the number of paths of length k from i to j of the graph G, and therefore

$$\mathrm{Tr}(A^k) = \#\{\text{closed paths of length } k \text{ on } G\} =: \mathrm{N}_G(k).$$

As a consequence, for any k,

$$\mathrm{N}_G(k) = \sum_{j=1}^{n} \lambda_j^k. \tag{6.3}$$

6.1.1.2 Seeing the spectral gap

The leading term of $\mathrm{N}_G(k)$ in the limit $k \to +\infty$ is always $\lambda_1^k = d^k$. The second order term is linked to the second largest eigenvalues in absolute value, $\lambda_+ = \max(\lambda_2, -\lambda_n)$. For any even integer k, all of the terms in equation (6.3) are non-negative and therefore,

$$\mathrm{N}_G(k) \geq d^k + \lambda_+^k.$$

In the limit we are interested in, namely $n \to +\infty$, this equation is always true for small k, because $d^k + \lambda_+^k \leq 2d^k$ and $\mathrm{N}_G(k)$ is a sum of $n \gg 1$ terms. It therefore gives us no information as to the size of λ_+.

However, if we now take $k = 2\lfloor A \log n \rfloor$ for a large enough A, then this inequality is no longer empty. It might then be interesting to see it as a characterisation of the size of the spectral gap: for a real number $\alpha > 0$, if $\lambda_+ \geq \alpha$, then $\mathrm{N}_G(k) - d^k \geq \alpha^k$. Hence,

$$\mathbb{P}_n^{(d)}(\lambda_+ \geq \alpha) \leq \mathbb{P}_n^{(d)}(\mathrm{N}_G(k) - d^k \geq \alpha^k)$$

for any probability measure $\mathbb{P}_n^{(d)}$ on the set of connected non-bipartite d-regular graphs with n vertices. The advantage of this new form is that we can use Markov's inequality and write

$$\mathbb{P}_n^{(d)}(\lambda_+ \geq \alpha) \leq \mathbb{E}_n^{(d)}\left[\frac{\mathrm{N}_G(k) - d^k}{\alpha^k}\right].$$

This allows us to reduce the problem of proving that $\mathbb{P}_n^{(d)}(\lambda_+ \geq \alpha) \to 0$ to proving an estimate of the form

$$\mathbb{E}_n^{(d)}[\mathrm{N}_G(k)] = d^k + o(\alpha^k) \quad \text{as } n \to +\infty, \text{ for } k = 2\lfloor A \log n \rfloor. \tag{6.4}$$

This is the idea of the trace method: transforming the difficult problem of estimating the eigenvalue λ_+ into a geometric problem, namely studying the asymptotics of the average $\mathbb{E}_n^{(d)}[\mathrm{N}_G(k)]$ of the path-counting function.

Unfortunately, it is very difficult to prove a statement such as equation (6.4), and both proofs of Friedman's theorem, [Fri03] and [Bor20], use different refinements of the trace method, some of which we will mention over the course of this section.

6.1.2 Compact hyperbolic surfaces

The trace method for compact hyperbolic surfaces is very similar to the method we have just described, apart from the fact that we need to replace equation (6.3) by Selberg's trace formula [Sel56].

6.1.2.1 Trace formula

We recall that Selberg's trace formula, as stated in Section 5.1, can be written as:

$$\sum_{j=0}^{+\infty} \widehat{\chi}(r_j) = (g-1)\int_{\mathbb{R}} \widehat{\chi}(r)\tanh(\pi r) r \,\mathrm{d}r + \sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)}{2\sinh\left(\frac{n\ell_X(\gamma)}{2}\right)} \chi(n\ell_X(\gamma))$$

for any admissible pair $(\widehat{\chi}, \chi)$, where $\lambda_j = \frac{1}{4} + r_j^2$. We rewrite this formula as

$$\mathrm{Sel}_X^{\Delta}(\chi) = \mathrm{Sel}_g^{\mathrm{top}}(\chi) + \mathrm{Sel}_X^{\ell}(\chi)$$

where $\mathrm{Sel}_X^{\Delta}(\chi)$ is the spectral term, $\mathrm{Sel}_X^{\ell}(\chi)$ the geometric term and $\mathrm{Sel}_g^{\mathrm{top}}(\chi)$ the topological term, which does not depend on X.

6.1.2.2 Test function

A new difficulty is that we need to pick a test function in the trace formula. In order to simplify the discussion, let us pick a test function χ_L for a (large) real number $L \geq 1$ satisfying the following assumptions[1].

Assumption 1.

- $\widehat{\chi_L} \geq 0$ on $\mathbb{R} \cup i\mathbb{R}$.
- The restriction of χ_L to $\mathbb{R}$ is real-valued and supported on $[-L, L]$.

The real number L will play the role of the integer k in the graph case. Indeed, since $\chi_L(\ell) = 0$ for any $\ell > L$, the geometric term $\mathrm{Sel}_X^{\ell}(\chi_L)$ of the trace formula will only depend on geodesics of length $\leq L$.

Let us furthermore assume, for this discussion, the following.

Assumption 2.

- χ_L is strictly positive on the interval $(-L, L)$.
- χ_L is obtained by dilation: $\chi_L(x) = \chi(x/L)$ for all x.

Then, all of the terms present in Selberg's trace formula are greater or equal to zero, and the dependency of χ_L and $\widehat{\chi}_L$ on L are simple. The approaches used in [WX21, LW21] both use this additional condition, while the new approach that we present in this thesis cannot, as we will see in Section 6.2.

6.1.2.3 Seeing the spectral gap

Let us study how the spectral gap affects the behaviour of the spectral term of the trace formula,

$$\mathrm{Sel}_X^{\Delta}(\chi_L) = \sum_{j=0}^{+\infty} \widehat{\chi}_L(r_j).$$

We prove that, as in the graph case, the leading term of $\mathrm{Sel}_X^{\Delta}(\chi_L)$ is $\widehat{\chi}_L(r_0)$, and the next-order term $\widehat{\chi}_L(r_1)$. Indeed, if $\lambda = \frac{1}{4} + (i\alpha)^2$ for a real number $\alpha \in [0, \frac{1}{2}]$, then by definition of the Fourier transform

$$\widehat{\chi}_L(i\alpha) = \int_{-L}^{L} \chi_L(x) \exp(-i\alpha \cdot ix) \, \mathrm{d}x = \int_{-L}^{L} \chi_L(x) \exp(\alpha x) \, \mathrm{d}x. \tag{6.5}$$

[1] Such a function can be constructed by taking the square convolution $f \star f$ of a positive even function f with compact support, and dilating the result so that the support is equal to $[-L, L]$.

This expression allows us to appreciate the special part played by small eigenvalues in the trace formula: because small eigenvalues are exactly the eigenvalues such that r_j is purely imaginary, they correspond to a different behaviour of $\widehat{\chi}_L(r_j)$.

We can often prove that the quantity $\widehat{\chi}_L(i\alpha)$ is almost of size $\exp(\alpha L)$ as $L \to +\infty$. For instance, thanks to Assumption 2, for any $\varepsilon > 0$,

$$\widehat{\chi}_L(i\alpha) = L \int_{-1}^{1} \chi(t) \exp(\alpha L t) \, \mathrm{d}t \geq L \int_{1-\varepsilon}^{1} \chi(t) \exp(\alpha L t) \, \mathrm{d}t$$

and therefore

$$\widehat{\chi}_L(i\alpha) \geq C_\varepsilon \; L \exp((1-\varepsilon)\alpha L). \tag{6.6}$$

Then, the contribution $\widehat{\chi}_L(r_0) = \widehat{\chi}_L\left(\frac{i}{2}\right)$ is the leading term of $\mathrm{Sel}_X^\Delta(\chi_L)$, of size almost $e^{\frac{L}{2}}$. If λ_1 is a small eigenvalue, then $\widehat{\chi}_L(r_1)$ is the second-order term, and it will be smaller the further away λ_1 is from zero, i.e. the larger the spectral gap.

More precisely, thanks to Assumption 1 and 2, for any $\alpha \in (0, \frac{1}{2})$ and $\varepsilon \in (0,1)$,

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}\right) \leq \mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sel}_X^\Delta(\chi_L) - \widehat{\chi}_L\left(\frac{i}{2}\right) \geq C_\varepsilon L \exp(\alpha L)\right)$$

and then, by Markov's inequality, if we manage to prove that

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^\Delta(\chi_L)\right] = \widehat{\chi}_L\left(\frac{i}{2}\right) + o(L \exp(\alpha L)) \quad \text{as } g \to +\infty, \tag{6.7}$$

then we can conclude that

$$\lim_{g \to +\infty} \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}\right) = 0.$$

6.1.2.4 The topological term

We can now substitute $\mathrm{Sel}_X^\Delta(\chi_L)$ by its expression in Selberg's trace formula, in equation (6.7). This allows us to reduce the problem to a geometric estimate, proving that:

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_g^{\mathrm{top}}(\chi_L) + \mathrm{Sel}_X^\ell(\chi_L)\right] = \widehat{\chi}_L\left(\frac{i}{2}\right) + o(\exp(\alpha L)) \quad \text{as } g \to +\infty. \tag{6.8}$$

Notice that we can take the topological term out of the expectation because it does not depend on the surface. Its behaves linearly in terms of g:

$$0 \leq \mathrm{Sel}_g^{\mathrm{top}}(\chi_L) \leq g \int_{\mathbb{R}} \widehat{\chi}_L(r) r \, \mathrm{d}r = \frac{g}{L} \int_{\mathbb{R}} \widehat{\chi}_1(u) u \, \mathrm{d}u.$$

In order to prove the estimate (6.7), we need to make sure that this linear term is small compared to $L \exp(\alpha L)$. We therefore add an hypothesis on the size of the parameter L.

Assumption 3. $L = A \log g$ for a real number $A \geq \frac{1}{\alpha}$.

Under this assumption, $\mathrm{Sel}_g^{\mathrm{top}} = o(L \exp(\alpha L))$ and equation (6.7) is equivalent to

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi_L)\right] = \widehat{\chi}_L\left(\frac{i}{2}\right) + o(\exp(\alpha L)). \tag{6.9}$$

Assumption 3 appears in the two proofs of the bound $\lambda_1 \geq \frac{3}{16} - \varepsilon$ [WX21, LW21]: in both cases, the authors take $L = 4 \log g$ because $\frac{3}{16} = \frac{1}{4} - \alpha^2$ for $\alpha = \frac{1}{4}$. As we get closer to the optimal estimate $\lambda_1 \geq \frac{1}{4} - \varepsilon$, the parameter α becomes very close to zero, which will require us to let the parameter A take arbitrarily large values.

We recall that $L = A \log g$ is a cut-off in the geometric term $\mathrm{Sel}_X^{\ell}(\chi_L)$ of Selberg's trace formula: only geodesics of length $\ell \leq L$ intervene. As a consequence, proving Conjecture 6.1 requires to inspect larger and larger scales on typical surfaces.

Now that we have sketched the basic ideas of the trace method and how it can be used to prove Conjecture 6.1, let us dive into more precise issues that we have identified and some partial solutions.

6.2 Cancellation of the main term

It is a priori quite difficult to prove an estimate such as

$$\begin{aligned} &\mathbb{E}_n^{(d)}[\mathrm{N}_G(k)] = d^k + o(\alpha^k) \\ \text{or} \quad &\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi_L)\right] = \widehat{\chi}_L\left(\frac{i}{2}\right) + o(L\exp(\alpha L)). \end{aligned}$$

Indeed, it requires to identify a part of $\mathbb{E}_n^{(d)}[\mathrm{N}_G(k)]$ or $\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi_L)\right]$ that is precisely of the right size to correspond to the main term on the r.h.s., and to then prove a very precise estimate on the rest of the terms. Finding a good way to cancel the trivial eigenvalues is therefore an important part of the proof of spectral gap results, for graphs or surfaces.

6.2.1 Methods for graphs

For Friedman's theorem, the two approaches [Fri03, Bor20] differ greatly.

Friedman uses a notion of *Ramanujan functions*. A function f is said to be d-Ramanujan of order α if it satisfies an estimate of the form

$$f(k) = P(k)\, d^k + \mathcal{O}\left(k^m \alpha^k\right)$$

for a polynomial function $P(k)$ and a constant m. Friedman significantly refined the trace method, and after numerous very technical manipulations, managed to prove that all of the terms left to estimate were Ramanujan. We can easily adapt this definition to hyperbolic surfaces, but it it not very enlightening and it would require many additional ideas to use it to prove Conjecture 6.1.

Bordenave composed the matrix in the trace by the projector on the orthogonal of the space of constant vectors, the eigenspace associated to the trivial eigenvalue d. This causes the main term of the trace to disappear, but it makes the combinatorial interpretation of the trace as the number of paths significantly more complicated. This method relies on finite-dimensional linear algebra and combinatorial properties, and we do not expect to be able to adapt it to hyperbolic surfaces.

6.2.2 Methods for surfaces: the first-order method

Wu–Xue and Lipnowski–Wright both use a beautiful and strikingly simple argument to cancel the main term of Selberg's trace formula in [WX21, LW21]. We shall explain the argument and the reason why it is difficult to extend it past $\frac{3}{16}$.

The idea is to identify the leading term of the expectation of the geometric term of the trace formula,

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi_L)\right] = \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)}{2\sinh\left(\frac{n\ell_X(\gamma)}{2}\right)}\, \chi_L(n\ell_X(\gamma))\right].$$

We know that typical surfaces do have short closed geodesics [MP19], and that they are often simple (at least for $L = A\log g$, $A < 1$ [MT21]). Inspired by this observation, we can single out the contribution of $\mathrm{Sel}_X^{\ell}(\chi_L)$ corresponding to *primitive simple closed geodesics*,

$$\mathrm{Sel}_X^{\ell,p,s}(\chi_L) := \sum_{\substack{\gamma \text{ primitive} \\ \text{simple closed geod}}} \frac{\ell_X(\gamma)\chi_L(\ell_X(\gamma))}{2\sinh\left(\frac{\ell_X(\gamma)}{2}\right)}.$$

Miraculously, the leading term of $\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell,p,s}(\chi_L)\right]$ as $g \to +\infty$ is exactly the main term of the trace formula, $\widehat{\chi}_L\left(\frac{i}{2}\right)$.

Proposition 6.4 ([WX21, Proposition 28] or [LW21, Lemma 5.2])**.**

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell,p,s}(\chi_L)\right] = \widehat{\chi}_L\left(\frac{i}{2}\right) + \mathcal{O}\left(\|\chi_L\|_\infty\left(1 + \frac{L^2\exp\left(\frac{L}{2}\right)}{g}\right)\right).$$

The error term in this estimate is of size roughly $e^{\frac{L}{2}}/g = g^{\frac{A}{2}-1}$. We recall that the trace method required to prove estimates with a precision $o(\exp(\alpha L)) = o(g^{\alpha A})$, and that we already had an error term of size g. As a consequence, we need to assume that:

$$1 \leq \alpha A \quad \text{and} \quad \frac{A}{2} - 1 \leq \alpha A.$$

This system of equation has a solution if and only if $\alpha \geq \frac{1}{4}$. The value $\alpha = \frac{1}{4}$ corresponds to $\lambda = \frac{3}{16}$ and $A = 4$. In other words, the value $\frac{3}{16}$ appears in [WX21, LW21] because it is the best precision we can obtain by using estimates at the precision $1/g$.

Proof. Conveniently, the expectation we need to compute is a sum of geometric functions, and we can apply Mirzakhani's integration formula (Theorem 3.7) to compute its expectation.

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell,p,s}(\chi_L)\right] = \frac{1}{V_g}\int_0^L \frac{x^2\chi_L(x)}{2\sinh\left(\frac{x}{2}\right)}\left(V_{g-1,2}(x,x) + \sum_{i=1}^{\lfloor \frac{g}{2}\rfloor} V_{i,1}(x)V_{g-i,1}(x)\right)\mathrm{d}x.$$

Let us replace the Weil–Petersson volumes that appear in this integration formula by their first-order approximation. By [MP19, Proposition 3.1] and equations (3.9) and (3.11)

$$\frac{1}{V_g}\left(x^2V_{g-1,2}(x,x) + \sum_{i=1}^{\lfloor \frac{g}{2}\rfloor} x^2V_{i,1}(x)V_{g-i,1}(x)\right) = 4\sinh^2\left(\frac{x}{2}\right) + \mathcal{O}\left(\frac{x^2\exp(x)}{g}\right).$$

Then, when we replace these volumes by this estimate, we obtain

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell,p,s}(\chi_L)\right] = \int_0^L 2\chi_L(x)\sinh\left(\frac{x}{2}\right)\mathrm{d}x + \mathcal{O}\left(\frac{\|\chi_L\|_\infty}{g}\int_0^L x^2\exp\left(\frac{x}{2}\right)\mathrm{d}x\right).$$

The first term is almost $\widehat{\chi}_L\left(\frac{i}{2}\right)$ (we just need to replace sinh by cosh, which leads to an error of size $\mathcal{O}\left(\|\chi_L\|_\infty\right)$). □

Limitations of the method If we improve Proposition 6.4 and now estimate every term at a precision $1/g^N$ rather than $1/g$, the second conditions on the parameters α and A becomes $\frac{A}{2} - N \leq \alpha A$. As a consequence, the best result we can hope to obtain is $\lambda_1 \geq \frac{1}{4} - \frac{1}{4(N+1)^2} - \varepsilon$.

We observe that, in order to reach a proof of Conjecture 6.1, we need to work at arbitrarily high precision. We therefore expect that high-order asymptotic expansions of the Weil–Petersson volume polynomials, such as the ones presented in Section 3.3, are absolutely necessary to prove Conjecture 6.1 (in the Weil–Petersson model).

However, the fact that the term of Selberg's trace formula corresponding to simple closed geodesics, at the first order, is very close to $\widehat{\chi}_L\left(\frac{i}{2}\right)$, seems like a very fortunate coincidence. It is extremely convenient that we were able to find a meaningful contribution of $\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi_L)\right]$ which cancels the leading term at the first order. If we try and push the method to larger orders of precision, we will have to create other similar cancellations, and it is quite unlikely that they will be as easy to understand, especially as the level of precision increases.

6.2.3 Methods for surfaces: integration by parts

Let us present a different approach that can be used to cancel the trivial eigenvalue in the trace formula. The idea is to modify the test function so that $\widehat{\chi}_L\left(\frac{i}{2}\right) = 0$, and hope that this modification translates into cancellations in the expectation of the geometric term of Selberg's trace formula. The significant advantage of this method is that it works at arbitrarily high order, which is essential to reach the optimal value $\frac{1}{4}$ in Conjecture 6.1.

6.2.3.1 New test function

We start by taking a function χ_L satisfying Assumption 1 to 3 as before. We then pick an integer $m \geq 1$, and define a new function $\psi_{L,m}$ by:

$$\forall r, \quad \widehat{\psi}_{L,m}(r) = \left(\frac{1}{4} + r^2\right)^m \widehat{\chi}_L(r).$$

As a consequence, $\widehat{\psi}_{L,m}$ has a zero of order m at $i/2$. Thanks to the properties of the Fourier transform with respect to multiplication and differentiation,

$$\psi_{L,m} = D^m \chi_L \qquad \text{where} \quad D := \frac{1}{4} - \partial^2$$

and ∂ denotes the usual derivative ($\partial f = f'$).

We notice that the differential operator D precisely cancels the main term of the trace formula. Indeed,

$$\left(\frac{1}{4} - \partial^2\right)\left[x \mapsto \exp\left(\frac{x}{2}\right)\right] = 0.$$

This observation will be a key ingredient of the integration by part argument that we develop in Section 6.2.3.4.

6.2.3.2 Trace method

The new test function $\psi_{L,m}$ is supported on the segment $[-L, L]$ and has a positive Fourier transform; it satisfies Assumption 1. However,

$$\int_{-L}^{L} \psi_{L,m}(x) \exp\left(\frac{x}{2}\right) \mathrm{d}x = \widehat{\psi}_{L,m}\left(\frac{i}{2}\right) = 0$$

and therefore $\psi_{L,m}$ is not positive everywhere on $[-L, L]$, and Assumption 2 is never true in this method.

As a consequence, we need to make a few adjustments to the trace method. We observe that if $\lambda_1 = \frac{1}{4} - \alpha^2$ for a $\alpha \in [0, \frac{1}{2}]$, by positivity of $\widehat{\psi}_{L,m}$ on $\mathbb{R}$,

$$\mathrm{Sel}_X^{\Delta}(\psi_{L,m}) \geq \widehat{\psi}_{L,m}(i\alpha) = \lambda_1^m \, \widehat{\chi}_L(i\alpha) \geq C_\varepsilon \, \lambda_1^m \, L \exp((1-\varepsilon)\alpha L)$$

by equation (6.6) for the function χ_L. This bound is quite loose if λ_1 is very small, which is to be expected because of the cancellation we created at λ_0. But we already know that typical surfaces have a spectral gap, so we can write

$$\begin{aligned}\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}\right) &= \mathbb{P}_g^{\mathrm{WP}}\left(0.01 \leq \lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}\right) + \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 < 0.01\right) \\ &\leq \mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{Sel}_X^{\Delta}(\psi_{L,m}) \geq C_{\varepsilon,m}\, L\exp(\alpha L)\right) + o(1)\end{aligned}$$

for the constant $C_{\varepsilon,m} = C_\varepsilon 10^{-2m}$, and thanks to Mirzakhani's bound on λ_1 from [Mir13].

We apply the same method as before, and obtain that if we prove a estimate of the form

$$\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\psi_{L,m})\right] = o(L\exp(\alpha L)) \quad \text{as } g \to +\infty \tag{6.10}$$

for $L = A\log g$, $A \geq \frac{1}{\alpha}$, then we can deduce that, typically, $\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}$. As a consequence, in this new form of the problem, we no longer need to identify a leading contribution in the expectation of the trace formula; we 'simply' have to prove an upper bound.

6.2.3.3 A useful lemma

Before we provide some arguments towards a proof of the bound (6.10), we make an additional simplification. We prove that we can replace the geometric term of Selberg's trace formula,

$$\mathrm{Sel}_X^{\ell}(\psi_{L,m}) = \sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)\, \psi_{L,m}(n\ell_X(\gamma))}{2\sinh\left(\frac{n\ell_X(\gamma)}{2}\right)}$$

by a regularised version, which does not diverge as the injectivity radius goes to zero:

$$\widetilde{\mathrm{Sel}}_X^{\ell}(\psi_{L,m}) = \sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)\, \psi_{L,m}(n\ell_X(\gamma))}{2\cosh\left(\frac{n\ell_X(\gamma)}{2}\right)}.$$

Lemma 6.5. *There exists a constant $C > 0$ such that for any $g \geq 2$, $L > 0$, and for any admissible pair $(\widehat{\chi}, \chi)$,*

$$\left|\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\chi) - \widetilde{\mathrm{Sel}}_X^{\ell}(\chi)\right]\right| \leq C\|\chi\|_{L^\infty(\mathbb{R})}\, g.$$

Since $\forall x, \partial^k\chi_L(x) = \frac{1}{L^k}\partial^k\chi\left(\frac{x}{L}\right)$ and $L \geq 1$, there is a constant $C_M > 0$ such that

$$\|\psi_{L,m}\|_{L^\infty(\mathbb{R})} \leq C_M\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})} \tag{6.11}$$

where $\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})} := \max\{\|\partial^k\chi\|_{L^\infty(R)}, 0 \leq k \leq 2m\}$. As a consequence, thanks to the hypothesis $A \geq 1/\alpha$,

$$\left|\mathbb{E}_g^{\mathrm{WP}}\left[\mathrm{Sel}_X^{\ell}(\psi_{L,m}) - \widetilde{\mathrm{Sel}}_X^{\ell}(\psi_{L,m})\right]\right| = \mathcal{O}_m\left(\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})}\, g\right) = o(L\exp(\alpha L))$$

and it is equivalent to prove equation (6.10) for Sel or $\widetilde{\mathrm{Sel}}$.

Proof. We observe that

$$\frac{1}{2\sinh\left(\frac{x}{2}\right)} - \frac{1}{2\cosh\left(\frac{x}{2}\right)} = \frac{e^{-\frac{x}{2}}}{\sinh x},$$

and therefore the quantity we want to estimate is smaller than

$$\|\chi\|_{L^\infty(\mathbb{R})}\, \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}} \sum_{n=1}^{\infty} \frac{\ell_X(\gamma)\, e^{-\frac{\ell_X(\gamma)}{2}}}{\sinh(\ell_X(\gamma))} \right]. \tag{6.12}$$

In order to bound this sum, we shall control the number of terms it contains. We do so by using the upper bounds [Bus92, Theorem 4.1.6 and Lemma 6.6.4]:

- there are at most $3g-3$ primitive closed geodesics of length $\leq 2\operatorname{arcsinh} 1$,
- for any $L > 0$, the set

  $$\mathcal{G}_L(X) := \{\gamma \text{ closed geodesic} : \ell_X(\gamma) \leq L\} \setminus \{\gamma^n : \ell_X(\gamma) \leq 2\operatorname{arcsinh} 1 \text{ and } n \geq 1\}$$

  contains at most $(g-1)\, e^{L+6}$ elements.

We cut the sum (6.12) into two contributions, corresponding to these two cases.

- Let $X \in \mathcal{M}_g$. For any primitive closed geodesic of length $\ell = \ell_X(\gamma) \leq 2\operatorname{arcsinh} 1$, the sum over its iterates is

  $$\sum_{n=1}^{+\infty} \frac{\ell e^{-\frac{n\ell}{2}}}{\sinh(n\ell)} \leq \frac{\ell}{\sinh(\ell)} + \int_1^{+\infty} \frac{\ell \,\mathrm{d}x}{\sinh(x\ell)}$$
  $$\leq 1 + \frac{1}{\ell}\int_\ell^{+\infty} \frac{y\,\mathrm{d}y}{\sinh(y)} \leq 1 + \frac{C}{\operatorname{InjRad} X}$$

  for a constant $C > 0$. As a consequence, by [Bus92, Theorem 4.1.6],

  $$\mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod} \\ \ell_X(\gamma) \leq 2\operatorname{arcsinh} 1}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)\, e^{-\frac{\ell_X(\gamma)}{2}}}{\sinh(\ell_X(\gamma))} \right] = \mathcal{O}\left(\mathbb{E}_g^{\mathrm{WP}}\left[g + \frac{g}{\operatorname{InjRad}_X} \right] \right) = \mathcal{O}\left(g\right)$$

  because the expectation of $1/\operatorname{InjRad}_X$ is finite by [Mir13, Corollary 4.3].

- Let $X \in \mathcal{M}_g$. We reorganise the sum depending on the length of the geodesics:

  $$\sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod} \\ \ell_X(\gamma) > 2\operatorname{arcsinh} 1}} \sum_{n=1}^{+\infty} \frac{\ell_X(\gamma)\, e^{-\frac{\ell_X(\gamma)}{2}}}{\sinh(\ell_X(\gamma))} = \sum_{j=1}^{+\infty} \sum_{\gamma \in \mathcal{G}_{j+1}(X) \setminus \mathcal{G}_j(X)} \frac{\ell_X(\gamma)\, e^{-\frac{\ell_X(\gamma)}{2}}}{\sinh(\ell_X(\gamma))}$$
  $$\leq \sum_{j=1}^{+\infty} \frac{(j+1)\, e^{-\frac{j}{2}}}{\sinh(j)} \#\mathcal{G}_{j+1}(X) \leq e^7 (g-1) \sum_{j=1}^{+\infty} \frac{(j+1) e^{\frac{j}{2}}}{\sinh(j)} = \mathcal{O}\left(g\right)$$

  and therefore the expectation of this quantity is also $\mathcal{O}\left(g\right)$.

$\square$

6.2.3.4 The integration by parts argument

Let us estimate the expectation of the contribution $\mathrm{Sel}_X^{\ell,p,s}(\psi_{L,m})$ of primitive simple closed geodesics in $\mathrm{Sel}_X^{\ell}(\psi_{L,m})$.

Theorem 6.6. *Let $A > 0$. There exists integers $m = m(A)$, $M = M(A)$ and a constant $C = C(A)$ satisfying the following. For any large enough g, for $L := A \log g \geq 1$,*

$$\left| \mathbb{E}_g^{\mathrm{WP}} \left[\widetilde{\mathrm{Sel}}_X^{\ell,p,s}(\psi_{L,m}) \right] \right| \leq C \, \|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})} L^M.$$

This result is an improvement of Proposition 6.4, which corresponds to approximations at arbitrarily high precision in g, and therefore can be used in a proof of Conjecture 6.1. Indeed, we can deduce from this statement the following corollary.

Corollary 6.7. *For any $\alpha \in (0, \frac{1}{2})$, if $A := \frac{1}{\alpha}$, $m := m(A)$ from Theorem 6.6, and $L := A \log g$, then*

$$\mathbb{E}_g^{\mathrm{WP}} \left[\widetilde{\mathrm{Sel}}_X^{\ell,p,s}(\psi_{L,m}) \right] = o(L \exp(\alpha L)) \quad \text{as } g \to +\infty.$$

In other words, if we want to prove that $\lambda_1 \geq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}$ for a $\alpha > 0$, then it is always possible to take a large enough $m = m(\alpha)$ so that the estimate (6.10) holds *for the sum of primitive simple closed geodesics*.

The proof relies on a integration by parts for the differential operator D^m, and the fact that

$$\forall k < m, \quad D^m \left(x^k \exp\left(\frac{x}{2}\right) \right) = 0.$$

Lemma 6.8 (Integration by parts)**.** *Let $m \geq 0$. There exists a constant $C_m > 0$ such that, for any $L > 0$, any smooth functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ supported on $[-L, L]$,*

$$\left| \int_0^L D^m f_1(x) f_2(x) \,\mathrm{d}x - \int_0^L f_1(x) D^m f_2(x) \,\mathrm{d}x \right| \leq C_m \max_{0 \leq \mu_1, \mu_2 < 2m} \{ |\partial^{\mu_1} f_1(0)| \cdot |\partial^{\mu_2} f_2(0)| \}.$$

Proof of Lemma 6.8. This is a simple induction, proved by iterating the formula

$$\int_0^L D f_1(x) f_2(x) \,\mathrm{d}x - \int_0^L f_1(x) D f_2(x) \,\mathrm{d}x = f_1'(0) f_2(0) - f_1(0) f_2'(0)$$

true for any smooth functions f_1, f_2 supported on $[-L, L]$. Note we use the fact that all derivatives of f_1 and f_2 vanish at L. □

Proof of Theorem 6.6. We express the sum $\widetilde{\mathrm{Sel}}_X^{\ell,p,s}(\psi_{L,m})$ as a sum of geometric functions. By Mirzakhani's integration formula, the contribution $\widetilde{\mathrm{Sel}}_X^{\ell,p,s,0}$ of non-separating simple closed geodesics can be written as

$$\mathbb{E}_g^{\mathrm{WP}} \left[\widetilde{\mathrm{Sel}}_X^{\ell,p,s,0}(\psi_{L,m}) \right] = \int_0^L \psi_{L,m}(x) F_{g,0}(x) \,\mathrm{d}x \tag{6.13}$$

where

$$F_{g,0}(x) := \frac{1}{V_g} \frac{x^2 V_{g-1,2}(x,x)}{\cosh\left(\frac{x}{2}\right)}.$$

We pick an integer N (to be determined later), and use Theorem 3.17 to find an approximation of the volume polynomial $V_{g-1,2}(x,x)$ at the order $1/g^N$:

$$\begin{aligned} \frac{x^2 V_{g-1,2}(x,x)}{V_g} &= P_{g,2}^{N++}(x,x)\, e^x + P_{g,2}^{N+-}(x,x) + P_{g,2}^{N-+}(x,x) + P_{g,2}^{N--}(x,x)\, e^{-x} \\ &+ P_{g,2}^{N+0}(x,x)\, e^{\frac{x}{2}} + P_{g,2}^{N0+}(x,x)\, e^{\frac{x}{2}} + P_{g,2}^{N-0}(x,x)\, e^{-\frac{x}{2}} + P_{g,2}^{0-}(x,x)\, e^{-\frac{x}{2}} \\ &+ P_{g,2}^{N00}(x,x) + \mathcal{O}_N\left(\frac{x^{3N+2}\exp(x)}{g^N}\right). \end{aligned}$$

By Theorem 3.17, all of these polynomials are of degree and coefficients smaller than a constant $K_N \geq 3N+2$. Hence,

$$\frac{x^2 V_{g-1,2}(x,x)}{V_g} = P_{g,2}^{N++}(x,x)\, \exp(x) + \mathcal{O}_N\left(x^{K_N} \exp\left(\frac{x}{2}\right) + \frac{x^{K_N}\exp(x)}{g^N}\right).$$

Since $\exp(x) = 2\cosh\left(\frac{x}{2}\right) e^{\frac{x}{2}} + \mathcal{O}\left(1\right)$, we obtain that

$$F_{g,0}(x) = \sum_{k=0}^{K_N} a_k(g,N)\, x^k \exp\left(\frac{x}{2}\right) + \mathcal{O}_N\left(x^{K_N} + \frac{x^{K_N}\exp\left(\frac{x}{2}\right)}{g^N}\right) \tag{6.14}$$

for a family of numbers $(a_k(g,N))_k$ such that, $\forall k, a_k(g,N) = \mathcal{O}_N\left(1\right)$.

- For any integer $k \leq K_N$, let us bound the integral

$$\int_0^L \psi_{L,m}(x)\ a_k(g,N)\ x^k \exp\left(\frac{x}{2}\right) \mathrm{d}x = a_k(g,N) \int_0^L D^m f_1(x)\ f_2(x)\, \mathrm{d}x$$

  for $f_1 := \chi_L$ and $f_2 : x \mapsto x^k \exp\left(\frac{x}{2}\right)$, using Lemma 6.8. If $m \geq K_N + 1$, then $D^m f_2$ is identically equal to 0, and the integration by part yields

$$\left| \int_0^L D^m f_1(x)\ f_2(x)\, \mathrm{d}x - 0 \right| \leq C_m \max_{0 \leq \mu_1, \mu_2 < 2m} \{ |\partial^{\mu_1} \chi_L(0)| \cdot |\partial^{\mu_2} f_2(0)| \}$$

  We assume that g is large enough so that $L \geq 1$, and therefore

$$\int_0^L \psi_{L,m}(x)\ a_k(g,N)\ x^k \cosh\left(\frac{x}{2}\right) \mathrm{d}x = \mathcal{O}_{m,N}\left(\|\chi\|_{\mathcal{C}^{2m-1}(\mathbb{R})}\right). \tag{6.15}$$

- We now need to estimate the error term coming from replacing the quantity $F_{g,0}$ by its approximation (6.14) in equation (6.13). By the triangle inequality, this error is

$$\mathcal{O}_N\left(\|\psi_{L,m}\|_{L^\infty(\mathbb{R})}\int_0^L\left(x^{K_N}+\frac{x^{K_N}\exp\left(\frac{x}{2}\right)}{g^N}\right)\mathrm{d}x\right)$$
$$=\mathcal{O}_N\left(\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})}\left(L^{K_N+1}+L^{K_N}\frac{\exp\left(\frac{L}{2}\right)}{g^N}\right)\right).$$

We take $N = N(A) := \lfloor \frac{A}{2} \rfloor + 1$. Then, $\exp\left(\frac{L}{2}\right) \leq g^N$, and therefore the error is

$$\mathcal{O}_N\left(\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})}L^{K_N+1}\right). \tag{6.16}$$

Then, by equations (6.15) and (6.16),

$$\mathbb{E}_g^{\mathrm{WP}}\left[\widetilde{\mathrm{Sel}}_X^{\ell,p,s,0}(\psi_{L,m})\right]=\mathcal{O}_{m,N}\left(\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})}L^{K_N+1}\right)$$

for $N = \lfloor \frac{A}{2} \rfloor + 1$, K_N the associated constant from Theorem 3.17, and any integer $m \geq K_N + 1$.

The proof is similar for the separating terms. For any $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, we let $\widetilde{\mathrm{Sel}}_X^{\ell,p,s,i}(\psi_{L,m})$ denote the part of $\widetilde{\mathrm{Sel}}_X^{\ell,p,s}(\psi_{L,m})$ corresponding to simple closed geodesics cutting the surface into two components of signatures $(i,1)$ and $(g-i,1)$. By Mirzakhani's integration formula, the expectation of the separating term is

$$\mathbb{E}_g^{\mathrm{WP}}\left[\sum_{i=1}^{\lfloor \frac{g}{2}\rfloor}\widetilde{\mathrm{Sel}}_X^{\ell,p,s,i}(\psi_{L,m})\right]=\sum_{i=1}^{\lfloor \frac{g}{2}\rfloor}\frac{1}{V_g}\int_0^L\frac{x^2\psi_{L,m}(x)V_{i,1}(x)V_{g-1,1}(x)}{\cosh\left(\frac{x}{2}\right)}\mathrm{d}x$$
$$=\sum_{i=1}^{\lfloor \frac{g}{2}\rfloor}\frac{V_{i,1}V_{g-i,1}}{V_g}\int_0^L\psi_{L,m}(x)F_{g,i}(x)\,\mathrm{d}x.$$

For each i, we apply Theorem 3.17 to the two volumes in this integral at the order N, which allows us to write

$$F_{g,i}(x)=\sum_{k=0}^{K'_N}b_k^{g,i,N}x^k\exp\left(\frac{x}{2}\right)+\mathcal{O}_N\left(x^{K'_N}+\frac{x^{2K'_N}\exp\left(\frac{x}{2}\right)}{g^N}\right),$$

where all of the coefficients $b_k^{g,i,N}$ are bounded by a constant depending on N only. By the same integration by parts, as soon as $m \geq K'_N + 1$,

$$\forall i\in\left\{1,\ldots,\left\lfloor\frac{g}{2}\right\rfloor\right\},\int_0^L\psi_{L,m}(x)F_{g,i}(x)\,\mathrm{d}x=\mathcal{O}_{N,m}\left(\|\chi\|_{\mathcal{C}^{2m-1}}L^{K'_N+1}\right).$$

We conclude thanks to the fact that, by equation (3.11),

$$\frac{1}{V_g}\sum_{i=1}^{\lfloor \frac{g}{2}\rfloor}V_{i,1}V_{g-i,1}=\mathcal{O}(1).$$

As a conclusion, if we take $N = \lfloor \frac{A}{2} \rfloor + 1$, $K := \max(K_N, K'_N)$, $m := M := K + 1$, then

$$\mathbb{E}_g^{\mathrm{WP}}\left[\widetilde{\mathrm{Sel}}_X^{\ell,p,s}(\psi_{L,m})\right] = \mathcal{O}_A\left(\|\chi\|_{\mathcal{C}^{2m}(\mathbb{R})} L^M\right),$$

which is what we had to prove. □

6.3 Restriction to a set of ‘good’ surfaces

For the sake of simplicity, let us now go back to the ‘simpler’ setting used in [WX21, LW21], where we study a test function χ_L satisfying Assumption 1 to 3.

We observed in Section 6.2 that we can easily use Mirzakhani's integration formula to compute ‘elementary’ terms of the geometric term of Selberg's trace formula, $\mathrm{Sel}_X^{\ell}(\chi_L)$, such as simple primitive geodesics. However, the situation becomes much more complicated when we consider general geodesics, and notably geodesics with many self-intersections, which can be quite wild and numerous in general. This can be observed in both proofs of the fact that $\lambda_1 \geq \frac{3}{16} - \varepsilon$, [WX21, LW21]: most of the work consists in dealing with the very large sum over all closed geodesics. We discuss informally a few ideas that can be used to study more complicated geodesics.

The tangle-free hypothesis Let us make a probabilistic assumption, and restrict ourselves to a set of ‘good’ surfaces. Inspired by the case of graphs, for a fixed $\eta > 0$, we set

$$\mathrm{TF}_g = \{X \in \mathcal{M}_g \ : \ X \text{ is } (\eta \log g)\text{-tangle-free}\},$$

as done in [LW21]. Then, by Theorem 4.8, $\mathbb{P}_g^{\mathrm{WP}}(\mathrm{TF}_g)$ goes to 1 as $g \to +\infty$.

The advantage of this hypothesis is that it allows us to describe quite precisely closed geodesics of length $\leq L = A \log g$, which appear in the geometric contribution of the trace formula. Indeed, as we saw in Section 4.2.3, the tangle-free hypothesis implies that, if we cut a closed geodesic γ in portions of length $a \log g$ for small enough a, then each piece is either simple or an iterated loop. This greatly reduces the possibilities for the topology of γ. Notably, as proven in Proposition 4.21, the surface filled by γ has bounded Euler characteristic.

Using a geometric hypothesis in the trace method In order to introduce our geometric hypothesis, tangle-freeness, in the trace method introduced in Section 6.1.2, we start by noticing that

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2}\right) \leq \mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \frac{\alpha^2}{(1-\varepsilon)^2} \text{ and } \mathrm{TF}_g\right) + \underbrace{\mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{TF}_g^c\right)}_{o(1)}.$$

This allows us reduce the problem to estimating the expectation

$$(\star) := \mathbb{E}_g^{\mathrm{WP}}\left[\widetilde{\mathrm{Sel}}_X^{\ell}(\chi_L)\mathbb{1}_{\mathrm{TF}_g}(X)\right] = \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\substack{\gamma \text{ primitive} \\ \text{closed geod}}}\sum_{n=1}^{+\infty}\frac{\ell_X(\gamma)\chi_L(n\ell_X(\gamma))}{2\cosh\left(\frac{n\ell_X(\gamma)}{2}\right)}\,\mathbb{1}_{\mathrm{TF}_g}(X)\right].$$

Unfortunately, the presence of this indicator function will make it significantly more difficult to apply Mirzakhani's integration formula and estimate the geometric term $\mathrm{Sel}_X^{\ell}(\chi_L)$.

From tangle-free surface to tangle-free curve A trick, inspired from the case of graphs [Bor20], is to use the tangle-free hypothesis to reduce the number of terms in the sum over all closed geodesics. More precisely, we prove that if $X \in \mathrm{TF}_g$, then every closed geodesic of length $\leq L$ on X is in a (much smaller) set of 'tangle-free configurations' $\mathcal{G}_{\mathrm{TF}}(L)$, seen on the base surface S_g.

For instance, by Corollary 4.15, if $L = \eta \log g$, then all of the geodesics of length $\leq L$ are simple, and we can take $\mathcal{G}_{\mathrm{TF}}(L)$ to be the set of all simple closed curves on the base surface S_g. For $L = A \log g$, $\mathcal{G}_{\mathrm{TF}}(L)$ could be a set of closed curves filling a surface of Euler characteristic bounded by a constant $C_{A,\eta}$, and that can be cut into $\leq K_{A,\eta}$ pieces so that each piece tangles around at most one closed geodesic.

We can then transform the sum over all closed geodesics of length $\leq L$ to a smaller sum, thanks to the indicator function:

$$(\star) = \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\gamma\in\mathcal{G}_{\mathrm{TF}}(L)}\sum_{n=1}^{+\infty}\frac{\ell_X(\gamma)\chi_L(n\ell_X(\gamma))}{2\cosh\left(\frac{n\ell_X(\gamma)}{2}\right)}\,\mathbb{1}_{\mathrm{TF}_g}(X)\right].$$

Removal of the indicator function When we remove the indicator function, in the hope to use Mirzakhani's integration formula, we obtain that

$$(\star) = \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\gamma\in\mathcal{G}_{\mathrm{TF}}(L)}\sum_{n=1}^{+\infty}\frac{\ell_X(\gamma)\chi_L(n\ell_X(\gamma))}{2\cosh\left(\frac{n\ell_X(\gamma)}{2}\right)}\right] - \mathrm{R}_g(\chi_L)$$

where

$$\mathrm{R}_g(\chi_L) := \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\gamma\in\mathcal{G}_{\mathrm{TF}}(L)}\sum_{n=1}^{+\infty}\frac{\ell_X(\gamma)\chi_L(n\ell_X(\gamma))}{2\cosh\left(\frac{n\ell_X(\gamma)}{2}\right)}\,\mathbb{1}_{\mathrm{TF}_g^c}(X)\right].$$

The hope it that this new expression of $(\star)$ is much better behaved, because the sum is taken over a much smaller set of topological configurations, which can be described. However, in order to reduce the problem to an estimate on this new sum, we need to check that the remainder $\mathrm{R}_g(\chi_L)$ is small enough. We can prove, using Theorem 4.8 and bounds on the number of closed geodesics $\leq L$ like in the proof of Lemma 6.5,

$$|\mathrm{R}_g(\chi_L)| \leq \|\chi_L\|_\infty \sup_{X \in \mathcal{M}_g} \left(\sum_{\substack{\gamma \text{ closed geod} \\ \ell_X(\gamma) \leq L}} \frac{\ell_X(\gamma)}{\cosh\left(\frac{\ell_X(\gamma)}{2}\right)} \right) \mathbb{P}_g^{\mathrm{WP}}\left(\mathrm{TF}_g^c\right) = \mathcal{O}\left(\frac{L^2 e^{\frac{L}{2}}}{g^{1-\eta}}\right).$$

As a consequence, if we take $L = 4 \log g$ as in the proof of the bound $\lambda_1 \geq \frac{3}{16} - \varepsilon$, then this error term is $\mathcal{O}\left((\log g)^2 g^{1+\eta}\right)$, i.e. just a bit larger than the topological term of the trace formula, $\mathrm{Sel}_g^{\mathrm{top}}$, which we guaranteed to be small enough. By adjusting the parameters, we will be able to make this new error term small enough.

However, if we want to reach the value $\frac{1}{4}$ and prove Conjecture 6.1, then we need the probability $\mathbb{P}_g^{\mathrm{WP}}\left(\mathbb{1}_{\mathrm{TF}_g^c}\right)$ to be much smaller in order to cancel out the term $e^{\frac{L}{2}}$. This is the reason why we have introduced the generalised tangle-free hypothesis in Section 4.3.

Appendices

Appendix A

Random Belyĭ surfaces

The bulk of this thesis is entirely focused on one probabilistic model to sample random surfaces, induced by the Weil–Petersson symplectic structure on the moduli space. In this appendix, we present another model, discrete and more combinatorial, introduced by Brooks and Makover [BM04]. We briefly review the literature on the geometry and spectrum of surfaces sampled this way, and prove new results similar to that of Section 4.1 and Chapter 5, namely Benjamini-Schramm convergence to the hyperbolic plane and convergence of the spectral density.

A.1 The probabilistic model and known results

Let us introduce the random construction of compact hyperbolic surfaces developed by Brooks and Makover [BM04].

The random construction An *ideal triangle* is a triangle with three vertices at infinity, or equivalently three angles equal to zero, like the one represented in Figure A.1, with vertices 0, 1 and ∞. When we glue two of these triangles along an edge there is one degree of freedom: the distance between their middle points, called the *shear*.

We can construct a random surface by gluing an even number $2N$ of ideal triangles with shear 0. The configuration in which the triangles are glued is described by:

- a random 3-regular graph Γ, picked using the uniform probability measure $\mathbb{P}_N^{\mathcal{G}}$ on the set $\mathcal{G}(N)$ of 3-regular graphs with $2N$ vertices
- a random orientation $\vec{o}$ of the graph Γ, picked uniformly amongst all of its possible orientations.

An example of such a gluing is represented in Figure A.1.

The resulting surface is a *non-compact surface* $S_o(\Gamma, \vec{o})$ of finite area $2\pi N$, which is typically connected because the underlying graph is [Bol01, Wor81a]. In order to sample random *compact* surfaces, we consider the conformal compactification $S_c(\Gamma, \vec{o})$ of the surface $S_o(\Gamma, \vec{o})$. We can prove that with high probability, the surface that we obtain

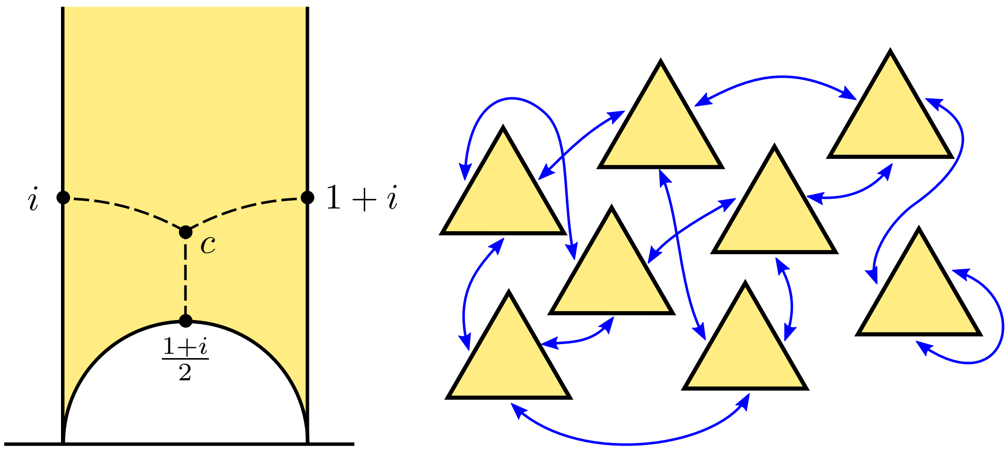


Figure A.1: On the left, an ideal triangle represented in the Poincaré half-plane with its center $c = \frac{1+i\sqrt{3}}{2}$ and the midpoints of its three edges. On the right, an example of a random gluing of $2N = 8$ oriented triangles.

carries a hyperbolic metric, which resembles the metric we would obtain by replacing neighbourhoods of the cusps of $S_o(\Gamma, \vec{o})$ by hyperbolic disks (see [Bro99] for a detailed construction of the compactification and [BM04] for the probabilistic result).

Relevance of the model This combinatorial model is discrete, and therefore one can wonder to what extent random surfaces sampled this way are representative, and how they are distributed in the set of surfaces.

An answer to this question can be found by observing that the compact Riemann surfaces obtained by this construction are exactly the Belyĭ surfaces, that is to say the surfaces S such that there exists a covering $f : S \to \overline{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$ [Gam06]. Belyĭ established in [Bel79] that these surfaces are exactly the Riemann surfaces which can be defined over some number field. In a sense, this indicates that the set of Belyĭ surfaces is 'dense' in the set of Riemann surfaces.

Geometric and spectral results The geometry of random Belyĭ surfaces has first been studied by Brooks and Makover in [BM04], were rough estimates were obtained on various quantities. Some of them (the genus and diameter notably) have been describe more precisely since then. Table A.1 sums up the information we have to this day, with references to the different articles.

As a consequence, random Belyĭ surfaces are quite similar to random Weil–Petersson surfaces: they are well-connected, with a Cheeger constant bounded away from zero and a logarithmic diameter. The only difference known so far is the fact that the injectivity radius is bounded away from zero.

Note that, in most cases (see [BM04] for instance), we prove geometric results on the non-compact surface S_o, which then remain true on its compactification S_c thanks to the results of Brooks [Bro99] comparing the geometry of S_o and S_c. We will follow

Quantity	bound for a typical Belyĭ surface	
spectral gap	$\lambda_1 \geq c_1$	[BM04]
Cheeger constant	$h \geq c_2$	[BM04]
injectivity radius	$\mathrm{InjRad} \geq c_3$	[BM04]
genus	strongly concentrated around $N/2$	[Gam06]
diameter	$\mathrm{diam}/(\log N) \to 2$ in probability	[BCP19]
short curves	Poisson distributed	[PT17]

Table A.1: Summary of the geometric and spectral results known of typical Belyĭ surfaces. All the constants are independent of N.

this approach in the next section.

A.2 Benjamini-Schramm convergence

Our aim in this section is to prove the following statement.

Theorem A.1. *With high probability, a random Belyĭ surface $S_c = S_c(\Gamma, \vec{o})$ satisfies:*

$$\frac{\mathrm{Vol}_{S_c}(S_c^-(\frac{1}{32}\log N))}{\mathrm{Vol}_{S_c}(S_c)} \leq 3N^{-\frac{1}{6}}.$$

The proof is divided into two steps: first, we prove Benjamini-Schramm convergence for the non-compact surfaces $S_o(\Gamma, \vec{o})$ in Section A.2.1, and then we prove that it implies the result for the compactified surface $S_c(\Gamma, \vec{o})$ in Section A.2.2. As in the Weil–Petersson model, this Benjamini-Schramm convergence result will have consequences on the spectral density of the Laplacian, which are detailed in Section A.3.

A.2.1 Non-compact surface

Let us first prove that, with high probability, the non-compact surface $S_o(\Gamma, \vec{o})$ converges to the hyperbolic plane in the sense of Benjamini-Schramm as N approaches infinity.

Proposition A.2. *With probability $1 - o(N^{-\frac{1}{6}})$ as N approaches infinity,*

$$\frac{\mathrm{Vol}_{S_o}\left(\left\{z \in S_o \,:\, \mathrm{InjRad}(z) \leq \frac{1}{30}\log N\right\}\right)}{\mathrm{Vol}_{S_o}(S_o)} \leq 3N^{-\frac{1}{6}}. \tag{A.1}$$

Our main argument to prove Proposition A.2 is the fact that, with high probability, random 3-regular graphs converge in the sense of Benjamini-Schramm to the infinite 3-regular tree. More precisely, we will use the following well-known result, as stated in [BHY19, Proposition 4.1].

Lemma A.3. *Let $\kappa < \frac{5}{24}$. For large enough N, if we set $K_N = \lfloor \kappa \log_2 N \rfloor$, then*

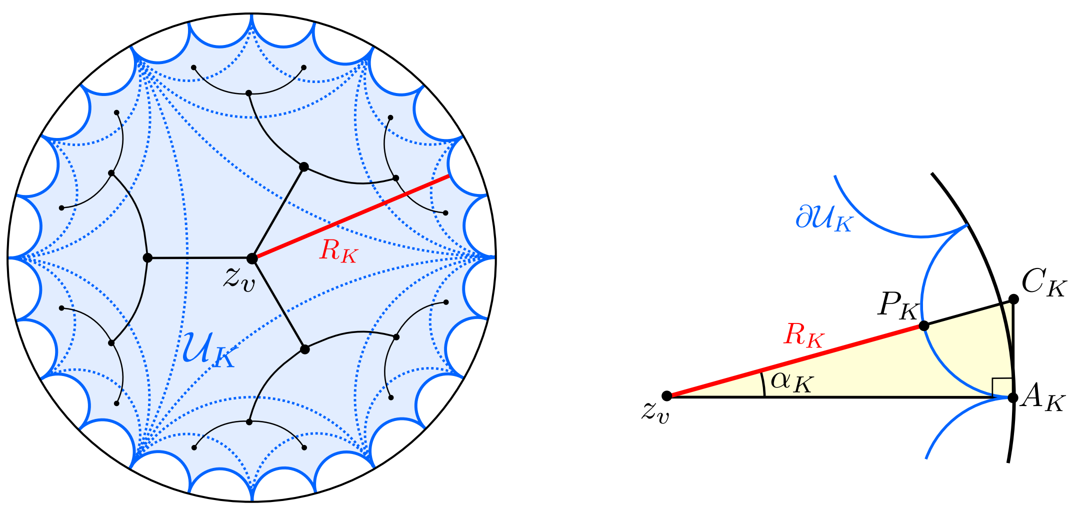
$$\mathbb{P}_N^{\mathcal{G}}\left(\frac{\#\{v \in V \ : \ \mathcal{B}_{K_N}(v) \text{ contains a cycle}\}}{\#V} \leq N^{-\frac{1}{6}} \right) = 1 - o(N^{-\frac{1}{6}})$$

where $\mathbb{P}_N^{\mathcal{G}}$ is the uniform probability measure on the set $\mathcal{G}(N)$ of 3-regular graphs with vertices indexed by $\{1, \dots, 2N\}$.

This lemma allows us to distinguish two types of triangles on the triangulation of a surface $S_o(\Gamma, \vec{o})$, depending on whether the injectivity radius of the corresponding vertex in Γ is larger than K_N or not. Let us prove that a large injectivity radius on the graph translates into a large injectivity radius on the open surface.

Lemma A.4. *Let $\Gamma = (V, E)$ be a 3-regular graph, and $\vec{o}$ be an orientation of Γ. Let v be a vertex of Γ with radius of injectivity $K \geq 0$. Then, the radius of injectivity of the center z_v of T_v in $S_o(\Gamma, \vec{o})$ is larger than*

$$R_K = \log\left(\frac{\cos\left(\frac{\pi}{3 \cdot 2^K}\right) - \sin\left(\frac{\pi}{3 \cdot 2^K}\right) + 1}{\cos\left(\frac{\pi}{3 \cdot 2^K}\right) + \sin\left(\frac{\pi}{3 \cdot 2^K}\right) - 1} \right) \sim_{K \to +\infty} K \log(2).$$



(a) Neighbourhood of a point z_v for a vertex v of injectivity radius $K = 3$ (in the disk model).

(b) Construction to compute R_K.

Figure A.2: Illustrations for the proof of Lemma A.4.

Proof. Since the ball of radius K in the graph Γ is a tree, the union of the corresponding triangles is isometric to the set $\mathcal{U}_K$ represented in Figure A.2 as a subset of the hyperbolic disk. Then, the injectivity radius at z_v is larger than the distance R_K between the origin and $\partial\mathcal{U}_K$.

Let us compute this distance. By the expression of the hyperbolic distance in the disk model,

$$R_K = \log\left(\frac{1+r_K}{1-r_K}\right)$$

where r_K is the Euclidean distance between z_v and a point P_K realising the distance between z_v and $\partial\mathcal{U}_K$. Let C_K be the center of circle containing P_K, and A_K one of its intersection points with the unit disk. Then, the triangle $C_K z_v A_K$ is rectangle because the two circles are orthogonal. Furthermore, the angle $\widehat{C_K z_v A_K}$ is equal to $\alpha_K = \frac{\pi}{3\cdot 2^K}$. Hence,

$$r_K = z_v C_K - P_K C_K = z_v C_K - A_K C_K = \frac{1}{\cos(\alpha_K)} - \tan(\alpha_K) = \frac{1-\sin(\alpha_K)}{\cos(\alpha_K)}$$

which proves that

$$R_K = \log\left(\frac{\cos\left(\frac{\pi}{3\cdot 2^K}\right) + 1 - \sin\left(\frac{\pi}{3\cdot 2^K}\right)}{\cos\left(\frac{\pi}{3\cdot 2^K}\right) - 1 + \sin\left(\frac{\pi}{3\cdot 2^K}\right)}\right) = \log\left(\frac{2+o(1)}{\frac{\pi}{3}\, 2^{-K} + o(2^{-K})}\right) \sim K\log(2).$$

□

We can now prove Proposition A.2.

Proof. Let us set a $\kappa \in \left(\frac{1}{5}, \frac{5}{24}\right)$. By Lemma A.3, with probability $1 - o(N^{-\frac{1}{6}})$, the 3-regular graph $\Gamma = (V, E)$ satisfies the condition

$$\frac{\#V^-(K_N)}{\#V} \leq N^{-\frac{1}{6}} \tag{A.2}$$

where $K_N = \lfloor \kappa \log_2(N) \rfloor$ and $V^-(K_N) = \{v \in V \ : \ \mathrm{InjRad}_\Gamma(v) \leq K_N\}$.

Let $\vec{o}$ be an orientation of Γ, and $X = S_o(\Gamma, \vec{o})$. Let us bound the volume of $X^-(L_N)$ for $L_N = \frac{1}{30}\log N$, by estimating the contribution to each triangle.

$$\frac{\mathrm{Vol}_X(X^-(L_N))}{\mathrm{Vol}_X(X)} = \frac{1}{\#V}\sum_{v\in V}\frac{\mathrm{Vol}_X(X^-(L_N)\cap T_v)}{\mathrm{Vol}_X(T_v)}.$$

We shall cut this sum into two contributions.

- The sum over the vertices $v \in V^-(K_N)$, which is smaller than its number of terms because the volume ratio is smaller than 1. We use equation (A.2) to bound the number of terms by $N^{-\frac{1}{6}}$.

- The sum over the vertices $v \in V^+(K_N) = V \setminus V^-(K_N)$, for which we will bound the volume $\mathrm{Vol}_X(X^-(L_N)\cap T_v)$ using the fact that

$$\mathrm{InjRad}_X(z_v) \geq R_{K_N} \geq \frac{1}{5}\log N \tag{A.3}$$

for large enough N, because $R_{K_N} \sim \log(2)\lfloor \kappa \log_2 N \rfloor \sim \kappa \log N$ as $N \to +\infty$.

Then,

$$\frac{\mathrm{Vol}_X(X^-(L_N))}{\mathrm{Vol}_X(X)} \leq N^{-\frac{1}{6}} + \frac{1}{\pi} \max\left\{\mathrm{Vol}_X(X^-(L_N) \cap T_v) \,:\, v \in V^+(K_N)\right\}. \tag{A.4}$$

Let $v \in V^+(K_N)$. We set $r_N = \frac{1}{6}\log N$. For any $z \in T_v$ such that $\mathrm{d}(z, z_v) < r_N$, the ball of center z and radius $\frac{1}{5}\log N - r_N$ is included in the ball of center z_v and radius $\frac{1}{5}\log N$, which is isometric to a ball in the hyperbolic plane by equation (A.3). Therefore, $\mathrm{InjRad}_X(z)$ is greater than $\frac{1}{5}\log N - r_N = L_N$, and $z \notin X^-(L_N)$. Thus,

$$\mathrm{Vol}_X(X^-(L_N) \cap T_v) \leq \mathrm{Vol}_X(T_v \setminus \mathcal{B}_{r_N}(z_v)). \tag{A.5}$$

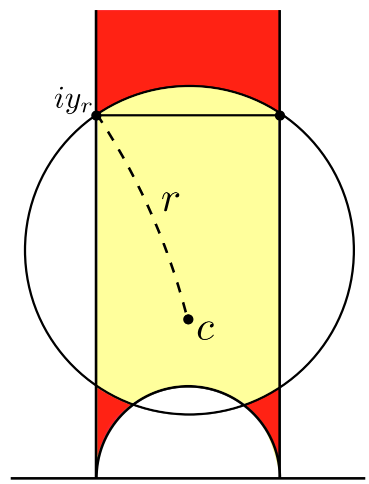


Figure A.3: A ball of radius r and center c in the ideal triangle T of vertices 0, 1 and ∞. Note that $T \setminus \mathcal{B}_r(c)$ has three components; we bound the volume of the top one by observing it is included in $\{z : \Im z \geq y_r\}$.

Let us estimate the volume of $T \setminus \mathcal{B}_r(c)$ in the Poincaré half-plane model, where T is the ideal triangle with vertices 0, 1 and ∞ and $c = \frac{1+i\sqrt{3}}{2}$ is its center – see Figure A.3. For large enough r, the set has three isometric connected components. We will bound the volume of the component corresponding to the vertex ∞. This component is entirely included in the set $\{z \in T \,:\, \Im z \geq y_r\}$, where iy_r is the highest intersection point of the circle $\mathcal{C}_r(c)$ with the geodesic connecting 0 and ∞. Hence,

$$\mathrm{Vol}_{\mathcal{H}}(T \setminus \mathcal{B}_r(c)) \leq 3 \int_0^1 \int_{y_r}^{+\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \frac{3}{y_r} \tag{A.6}$$

where y_r is the largest solution of the equation $\mathrm{d}(iy_r, c) = r$. By the formula for the hyperbolic distance in $\mathcal{H}$ [Bus92, equation 1.1.2], this equation can be rewritten as

$$y_r^2 - \sqrt{3}\cosh(r) y_r + 1 = 0.$$

Hence,

$$y_r = \frac{\sqrt{3}}{2}\left(\cosh(r) + \sqrt{\cosh^2(r) - \frac{4}{3}}\right) \sim \frac{\sqrt{3}}{2}e^r \tag{A.7}$$

as r approaches infinity.

Finally, by putting together equations (A.4) to (A.6),

$$\frac{\mathrm{Vol}_X(X^-(L_N))}{\mathrm{Vol}_X(X)} \leq N^{-\frac{1}{6}} + \frac{3}{\pi y_{r_N}} \sim \left(1 + \frac{2\sqrt{3}}{\pi}\right)N^{-\frac{1}{6}}$$

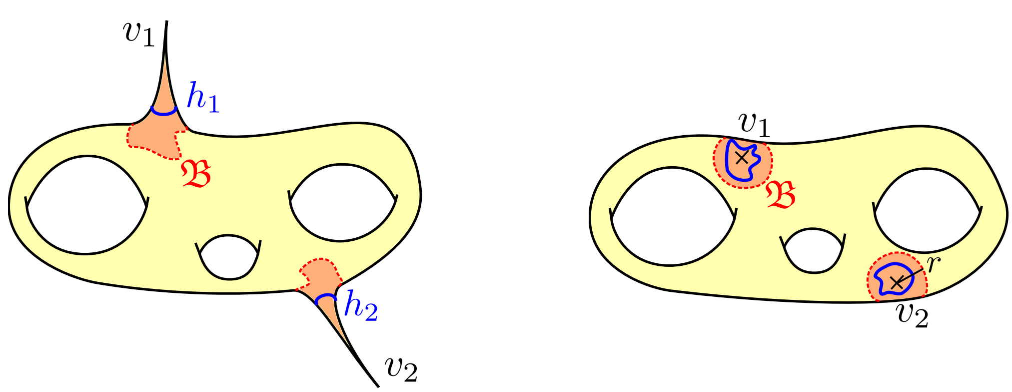
by equation (A.7), so it is smaller than $3N^{-\frac{1}{6}}$ for large enough N. □

A.2.2 Compactified surface

Let $(v_1, \ldots, v_s)$ denote the family of cusps of a non-compact surface S_o. Its compactification is constructed by observing that a small enough neighbourhood of a cusp v_i is conformally equivalent to a punctured unit disk. We can "fill in" this puncture by replacing its neighbourhood by a solid disk. We will identify the center of this new disk and the puncture v_i. As a consequence, setwise, the set S_o can be obtained by removing the points $(v_1, \ldots, v_s)$ from the compactified surface S_c – see Figure A.4.

By the uniformisation theorem, there is a unique metric of constant curvature on S_c. Brooks and Makover proved in [BM04] that with high probability, the genus of the surface is ≥ 2, and the metric is therefore hyperbolic. The metrics on the surface S_o and its compactification S_c have been compared by Brooks in [Bro99], under the *large-cusp assumption*.

Definition 7. Let $\ell > 0$. A hyperbolic surface S_o with n cusps is said to have *cusps of length* $\geq \ell$ if there exists a family $(h_i)_{1\leq i\leq n}$ of disjoint closed curves on X such that for all $i \in \{1, \ldots, n\}$, h_i is a horocycle of length $\geq \ell$ around the i-th cusp of X.



(a) A surface with cusps of length $\geq \ell$. (b) Its compactification.

Figure A.4: The two metrics are very close outside the 'bad' set, highlighted (in red).

Brooks and Makover proved that this condition is true with high probability for any fixed $\ell > 0$. This implies that, outside neighbourhoods of the cusps, the geometry of the compactified surface is arbitrarily close to that of the non-compact surface [Bro99]. We will only need the two following results.

Lemma A.5. *Let $\varepsilon > 0$. There exists positive real numbers $\ell = \ell(\varepsilon)$ and $r = r(\varepsilon)$ such that, for any surface S_o with cusps of length $\geq \ell$, if we define $\mathfrak{B} := \bigcup_{i=1}^{s} \mathcal{B}_r(v_i)$, then the following holds.*

- *For any closed geodesic arc β on S_c based at a point z outside of $\mathfrak{B}$, there is a geodesic arc $\tilde{\beta}$ based at z on S_o, such that $\tilde{\beta}$ is homotopic with fixed endpoints to β on S_c and*
$$\ell_{S_o}(\tilde{\beta}) \leq (1+\varepsilon)\,\ell_{S_c}(\beta).$$
- *For any set A included in the complement of $\mathfrak{B}$ on S_c,*
$$\frac{\mathrm{Vol}_{S_o}(A)}{1+\varepsilon} \leq \mathrm{Vol}_{S_c}(A) \leq (1+\varepsilon)\,\mathrm{Vol}_{S_o}(A).$$

We can now prove Theorem A.1.

Proof. Let $\mathcal{A}_N$ be the event:

- S_c has genus $g_c \geq \frac{N}{3}$
- S_o has $s \leq 2\log N$ cusps and they are of length $\geq \ell$
- S_o satisfies the property of Proposition A.2:
$$\frac{\mathrm{Vol}_{S_o}(S_o^-(\frac{1}{30}\log N))}{\mathrm{Vol}_{S_o}(S_o)} \leq 3N^{-\frac{1}{6}}. \tag{A.8}$$

By the references of Table A.1 together with Proposition A.2,
$$\lim_{N\to+\infty} \mathbb{P}_N^{\mathcal{G}}(\mathcal{A}_N) = 1.$$

Let us prove that these conditions imply that
$$\frac{\mathrm{Vol}_{S_c}(S_c^-(\frac{1}{32}\log N))}{\mathrm{Vol}_{S_c}(S_c)} \leq 3N^{-\frac{1}{6}}.$$

Let $\varepsilon \in (0, 1/15)$. Let $\ell = \ell(\varepsilon)$ and $r = r(\varepsilon)$ be the numbers given by Lemma A.5.

For any point z in $S_c \setminus \mathfrak{B}$, if $z \in S_c^-(\frac{1}{32}\log N)$, then the shortest geodesic arc based at z, denoted by β, is of length $\leq \frac{1}{16}\log N$. By Lemma A.5, there is loop $\tilde{\beta}$ such that $\tilde{\beta}$ is homotopic to β with fixed endpoints on S_c and $\tilde{\beta}$ is a geodesic arc on S_o, of length
$$\ell_{S_o}(\tilde{\beta}) \leq (1+\varepsilon)\ell_{S_c}(\beta) \leq \frac{1+\varepsilon}{16} \leq \frac{1}{15}.$$

$\tilde{\beta}$ is not contractible on S_c because it is homotopic to β. It is therefore not contractible on S_o either. Hence, $z \in S_o^-(\frac{1}{30}\log N)$.

As a consequence,

$$\frac{\mathrm{Vol}_{S_c}(S_c^-(\frac{1}{32}\log N))}{\mathrm{Vol}_{S_c}(S_c)} \leq \frac{\mathrm{Vol}_{S_c}(S_o^-(\frac{1}{30}\log N) \setminus \mathfrak{B})}{\mathrm{Vol}_{S_c}(S_c)} + \frac{\mathrm{Vol}_{S_c}(\mathfrak{B})}{\mathrm{Vol}_{S_c}(S_c)}.$$

Let us look at these two contributions separately.

- On the one hand, by Lemma A.5,
$$\begin{aligned}\frac{\mathrm{Vol}_{S_c}(S_o^-(\frac{1}{30}\log N) \setminus \mathfrak{B})}{\mathrm{Vol}_{S_c}(S_c)} &\leq \frac{(1+\varepsilon)\,\mathrm{Vol}_{S_o}(S_o^-(\frac{1}{30}\log N))}{2\pi(2g_c-2)} \\ &\leq \frac{4N^{-\frac{1}{6}}\,\mathrm{Vol}_{S_o}(S_o)}{5\pi\left(\frac{N}{3}-1\right)} = \frac{12\,N^{\frac{5}{6}}}{5(N-3)}\end{aligned}$$
because $g_c \geq \frac{N}{3}$ and thanks to equation (A.8).
- On the other hand,
$$\frac{\mathrm{Vol}_{S_c}(\mathfrak{B})}{\mathrm{Vol}_{S_c}(S_c)} \leq \frac{s\,(\cosh r - 1)}{2\pi(2g_c-2)} = \mathcal{O}\left(\frac{\log N}{N}\right)$$
since $g_c \geq \frac{N}{3}$ and $s \leq 2\log N$.

Adding the two estimates, we obtain that the quantity is smaller than $3N^{-\frac{1}{6}}$ for large enough N, which concludes the proof. □

A.3 Consequences on the spectral density

Brooks and Makover proved in [BM04] that typical Belyĭ surfaces have a uniform spectral gap using Cheeger's inequality. Now that we furthermore know that these surfaces converge in the sense of Benjamini-Schramm to the hyperbolic plane, we can provide additional information on the distribution of the spectrum of these surfaces, using the results from Chapter 5.

Theorem A.6. *There exists a constant $C > 0$ such that, for any large enough N, with high probability, a compact random Belyĭ surface X constructed with $2N$ triangles satisfies the following.*

- *For any real numbers $0 \leq a \leq b$,*
$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} \leq C\left(b - a + \sqrt{\frac{b+1}{\log N}}\right) \tag{A.9}$$
and as soon as $b \leq \frac{1}{4}$,
$$\frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} \leq C\frac{N^{-\frac{1}{2^8}(\frac{1}{4}-b)}}{(\log N)^{\frac{3}{2}}}. \tag{A.10}$$

- *For any real numbers* $0 \leq a \leq b$*, we can write*

$$\frac{\mathrm{N}_X^{\Delta}(a,b)}{\mathrm{Vol}_X(X)} = \frac{1}{4\pi} \int_{\frac{1}{4}}^{+\infty} \mathbb{1}_{[a,b]}(\lambda) \tanh\left(\pi \sqrt{\lambda - \frac{1}{4}}\right) \mathrm{d}\lambda + (b-a) R_X(a,b)$$

where the remainder $R_X(a,b)$ *satisfies*

$$-\frac{C}{M_{a,b,N}} \leq R_X(a,b) \leq \frac{C}{M_{a,b,N}} \sqrt{\log(2 + M_{a,b,N})} \tag{A.11}$$

where $M_{a,b,N} = (b-a)\sqrt{\frac{\log N}{b+1}}$.

The results of Theorem A.6 have been stated in terms of N because we are set in the regime $N \to +\infty$, but they can be translated in terms of the genus g because g is typically concentrated around the value $\frac{N}{2}$ [Gam06]. As a consequence, the spectral density of typical Belyĭ surfaces and typical Weil–Petersson surfaces are quantitatively similar to the best of our current knowledge.

Proof. We know that with high probability, the surface X satisfies:

- $\mathrm{InjRad}\, X \geq C$ for a certain constant $C > 0$.
- $\frac{\mathrm{Vol}_X(X^-(\frac{1}{32}\log N))}{\mathrm{Vol}_X(X)} \leq 4N^{-\frac{1}{6}}$.

Let us prove the claimed estimates on this set of high probability.

We first obtain the upper bound for small eigenvalues thanks to the results from Section 5.1, and more precisely the deterministic estimate Proposition 5.4. Let us set $L = \frac{1}{32}\log N$, $r = \min(C, 1)$ and $t = \frac{1}{2^8}\log N$. Then, for any $b \geq 0$,

$$\begin{aligned} \frac{\mathrm{N}_X^{\Delta}(0,b)}{\mathrm{Vol}_X(X)} &\leq \frac{e^{-(\frac{1}{4}-b)t}}{t^{\frac{3}{2}}} \left[\frac{\sqrt{\pi}}{8} + C\frac{t^4 e^t}{r^4} \left(\frac{\mathrm{Vol}_X(X^-(L))}{\mathrm{Vol}_X(X)} + L e^{L - \frac{L^2}{4t}} \right) \right] \\ &= \vec{o}\left(\frac{N^{-\frac{1}{2^8}(\frac{1}{4}-b)}}{(\log N)^{\frac{3}{2}}} \left(1 + \underbrace{(\log N)^4 N^{\frac{1}{2^8}} \left(N^{-\frac{1}{6}} + (\log N) N^{-\frac{1}{32}} \right)}_{o(1)} \right) \right) \end{aligned}$$

which our claim.

Then, the other two estimates follow straightforwardly from the method detailed in Section 5.3. The key argument is that the geometric assumptions allow us to bound the geometric term of the Selberg trace formula using the deterministic estimates Lemma 5.15 and 5.23 with the length parameter $L = \frac{1}{32}\log N$, which will cause the parameter t to be (at most) proportional to $\sqrt{\log N}$. ∎

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