Problem A: Game of Ecology

Laura Nolt, Ben Peterson, Chad Russell, Rachel Slark

Colorado State University, Math 435

Abstract

The purpose of this paper is to explore different factors about dragons. Due to the fictional aspect of these animals, we will be utilizing many assumptions to examine characteristics of the dragons, as well as how they will affect different species in other environments. Research about similar animals and differing climates will aid in our attempts to model the effects of introducing dragons to the world.

1. Introduction & Motivation

This problem was introduced to us by The Consortium for Mathematics and Its Applications (COMAP), under the Mathematical Contest in Modeling problems (MCM). Our group chose Problem A: Game of Ecology. The general idea behind this problem is the ecology behind the three dragons in the popular television show, *Game of Thrones*. The problem gives some basic knowledge on what we can assume about the dragons to help us later answer the questions the problem dives into. Our group was attracted to this problem because we all share a similar interest in sustainability, conservation, and populations. This problem seemed to combine all our interests and would result in an interesting outcome. The questions we aim to answer about the dragons include analyzing the ecological impact of introducing them to different climates, their caloric intake and energy required to sustain life, the area needed to support them, and what is required to grow or sustain the three dragons. This is important to analyze because they are considered apex predators. These are predators “at the top of a food chain that are not preyed upon by any other animal” [12]. If an apex predator was ever introduced to an ecosystem, understanding the effects it causes and how to sustain other populations concurrently will be valuable.

With research and new information, we then created a predator-prey model in MATLAB using differential equations. Specifically, the equations that we used are called the Lotka-Volterra equations, which are widely known as the predator-prey equations. They are a pair of first-order, nonlinear differential equations, which are frequently used to describe the dynamics of biological systems in which two species interact. One species is the predator, and the other the prey. In other words, we will see what happens when dragons are introduced to different populations.

Questions we aim to answer:

- Analyze overall characteristics, behavior, habits, diet, and interaction with the environment.
- What is the ecological impact of the three dragons?

- What are the energy expenditures of the dragons, along with what is the caloric intake requirements? How many calories are needed to sustain the dragons? How many calories to help the dragons grow?
- How much area is required to support the three dragons?
- How important are climate conditions? Would moving a dragon between an arid region, a warm temperate region, and an arctic region make a big difference in the resources required to sustain the dragons?

1.1. Assumptions

For the sake of this project, we will assume that there are only the three original *Game of Thrones* dragons. The dragons will begin in their natural habitat, being the continents of Westeros and Essos. Because these are fictional settings, we will assume they have qualities most similar to Great Britain. The far north experiences harsh winters and extremely cold temperatures, whereas the southern parts experience warmer weather and humid air [6]. The dragons were born in the temperate grasslands of Essos, which lie closer inland with more moderate temperatures [5]. Since our project is based on information from *Game of Thrones*, the setting will most closely resemble the Medieval time period.

Other necessary assumptions for the dragons include their physical characteristics. Provided they have enough food and freedom to roam, the dragons will never stop growing. Their body mass will continue to increase and their scales will grow stronger and thicker, both contributing to the overall protection of the dragon's outer physique. We will assume the dragons need to maintain their own high internal body temperature, thus requiring them to produce their own heat. This is created by fire that burns inside the dragon, requiring them to burn enough energy to keep warm. They will have a total of six limbs: four legs and two wings. The dragons will have a durable outer body structure with strong jaws, sharp teeth and claws, leathery wings, long necks and tails, and horns on their heads [4]. Overall, the dragons will be able to fly great distances, breathe hot fire, and resist tremendous trauma.

The diet of the dragons is also assumed for our project. They are best characterized as obligate carnivores. Also known as "true" carnivores, these are animals whose diet requires nutrients found only in meat and animal flesh [8]. Although they do have the ability to ingest plants and other plant matter, these animals do not have the right ability to digest it for the energy they need. For the sake of this report, we are also going to assume that the dragons leave marine life untouched.

Finally, we will assume the dragons are not hunting for themselves until they are at least 4 years old. They will continue growing throughout their lifetimes, but hit massive growth in the first few years of their life. Utilizing the information given in the MCM prompt, we know the dragons are born at about 10 kg and grow to 30-40 kg in the first year. Given this information, we calculated that a 4 year old dragon will be about 2.5 tons (2,560 kg). So, the dragons will begin to explore different parts of the world for food when they are about 4 years old.

1.2. Climates

We researched the three major climates to help us better understand where the dragons would target different prey and what would happen to them in each region. The three main climates are arid (tropical), temperate, and arctic [16]. These zones can all be further divided into smaller zones to increase accuracy of temperatures. We assumed that the dragons would start out their lives in the temperate climate because this is the environment most similar to where they were born in the *Game of Thrones* setting. It would require the least amount of energy from the dragons, since it is the most moderate of the three zones temperature-wise. Less energy would be needed for the dragons to heat and cool themselves, especially when accounting for maintaining their internal fire temperatures. Assuming the dragons are fully grown, they will eat all of the prey in the temperate zone until nothing is left. They will then move on to the next climate, which would be the arid, tropical environments. This process would repeat until they got to the Arctic. The Arctic climate is last because it is the coldest and would require the most energy consumption from the dragons to keep their internal body temperatures up and to be able to hunt. In the Arctic, there is less available prey and the dragons would have to work extra hard to eat.

Since the dragons are an apex predator and they have to consume so much food to survive, they would quickly deplete their prey in each region. For the sake of this report, we realized analyzing all three zones will look very similar. The dragons would be sustained in the temperate climate the longest due to plentiful prey sources and ideal temperatures. When researching the best places to hunt sheep and deer, the United States was frequently referenced. In fact, Iowa is claimed as the best spot for hunting whitetail deer [19]. A lot of the best places to hunt deer and sheep are in the United States and Canada which are temperate climate zones. Assuming deer and sheep will be a primary food source for the three dragons, this is the ideal region for them. So, we will assume that the dragons will deplete different animal populations at similar rates.

1.3. Similar Animal Research

The approach our group initially took to this problem was trying to find an animal to mimic or resemble a dragon. This is useful to get an idea about eating habits, hunting range/span, and reactions to differing climates. We found the condor bird as a great starting point, for they are the largest flying land birds in the western hemisphere. These giant birds can have a wingspan of over 10 feet, stand up to 3 to 4 feet tall, and fully grown adults can weigh up to 33 pounds [3]. Much like a dragon, condor birds are carnivores and are sometimes referred to as a clean up crew. When traveling and searching for food, condors keep their eyes peeled for already dead prey as an easy meal. When condors go out hunting, they can travel as far as 150 miles in a day. Another thing to note is where condor birds like to live. Due to their weight, condors like to live in windy areas where they can glide on air current with little effort. They are mainly found in mountainous regions, but can be seen near coastlines where there are strong ocean breezes [13]. When doing research on dragons, we saw that dragons also like mountainous or volcanic areas. After collecting data and researching this magnificent bird, we felt we had a strong reference point to begin the modeling process.

1.4. Lotka-Volterra Equations

Utilizing the Lotka-Volterra equations, we were able to create an efficient model for examining what happens when we introduce the dragons to the ecosystems. Lotka-Volterra equations are widely used for studying ecological systems, primarily predator-prey interactions. These predator-prey equations are expressed as first-order, nonlinear differential equations, both expressing instantaneous growth rates for the respective populations:

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{2}$$

where x is the number of prey (i.e. rabbits), y is the number of some predator (i.e. foxes), dx/dt and dy/dt are in respect to time, and α, β, δ, and γ are positive real parameters describing the interaction of the two species populations [11].

2. Data & Methods

2.1. Metabolism

One of the questions we set out to answer was the metabolism of a dragon (i.e. how much food it would take to sustain a dragon). Metabolism refers to the total biochemical reactions taking place inside an organism, or how quickly fuels are broken down to keep the organism's cells running [1]. The basal metabolic rate (BMR) of an animal (measured in energy over time) depends on the size on the animal. Interestingly, the smaller an animal is, the higher the BMR per mass of the animal. However as size increases, overall BMR also increases [1]. Since dragons are flying animals, we assumed that the metabolism of a dragon would be similar to that of a bird. Using some data from a study done by The Cooper Ornithological Society, the BMR of tropical birds increases linearly with mass, as shown in Figure 1.

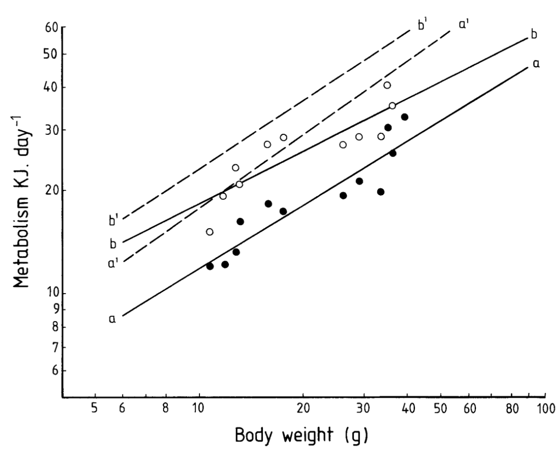

Figure 1: Basal metabolic rates of a range of tropical passerines during the active and resting stages of the circadian cycle. Closed circles (a-a) resting phase: $M = 202.15W^{0.615}$. Open circles (b-b) active phase: $M = 188.2W^{0.506}$. Lines a'-a' and b'-b' are the regression lines of Aschoff and Pohl(l970a) for temperate passerines during the resting and active phases respectively [14].

If we extrapolate from the graph given by this study, we get that $BMR = 0.32 * mass + 7.08$ for the resting phase of the circadian cycle, and $BMR = 0.37 * mass + 12.78$ for the active phase. In these equations, mass is in grams and BMR is in kJ per day [14].

Since body mass is required for these equations, this raises the question about how big the dragons are. According to the MCM prompt, when they are born they are 10 kg. Then after a year, they are 30-40 kg, continuing to grow as long as there is food available to them. Since the equations above were calculated using data from adult birds which have stopped growing, we need to take into account the fact that the dragons never stop growing. Using the equations above, calculating the energy needed to sustain a dragon would be a low estimate. Since dragons are active animals that spend time flying and hunting, it is a safe assumption to use the equation for the active phase. This means that a newborn dragon at 10 kg would require more than 3,713 kJ per day. After 1 year, a 40 kg dragon will require more than 14,812 kJ per day. We can convert this to calories using the fact that 1 kJ is approximately 0.239 Calories (i.e. kilocalories) [17]. So, a newborn dragon would need more than 887 Calories a day, and a year old dragon would need over 3,540 Calories a day.

In a 1997 study of the wing loading in owls, the amount of mass carried per the area of the wings, it was found that the wing loading of owls ranges from 0.211 to 0.545 g/cm^2 [15]. Drogon, one of the dragons in the show *Game of Thrones*, is said to have a wingspan of 44.8 m [4]. Using some very rough estimating, the wing area is approximately 500 m^2 or 5,000,000 cm^2. From this we calculated that Drogon weighs between 1 and 2.5 tons, being 2.5 tons around 4 years old. Using these numbers and the equations noted above, Drogon would need 89,936 to 226,384 Calories a day to maintain his size. If Drogon was going to continue growing, he would need more.

Assuming that a sheep is 130 lbs, it has 2,080 oz of meat. With 83.3 Calories per oz of cooked meat, the total Calories in an average sheep is then roughly 173,264 [18]. Using this number, a newborn dragon would need at least 11 oz of sheep a day, and a year old dragon would need 43 oz of sheep a day. Drogon, assuming he weighs 1 ton, would need 1,080 oz of sheep a day. Now assuming he is older and weighs 2.5 tons, Drogon would need 2,717 oz of sheep a day, which is equivalent to about a little over one sheep per day.

2.2. Energy Usage

Although the dragons need a certain amount to maintain their weight, we also need to account for the fire that burns inside the dragons. The dragons will only eat cooked meat, resulting in using their fire breathing abilities. Based on the *Game of Thrones* show, these three dragons are a bit smaller and we can assume the dragons will breathe fire of about 1,000°C [24]. Since the dragons are using more energy to breathe fire and cook their food, they will require more food to sustain them.

Comparing the dragons' fire to burning charcoal, we can gain a general idea about how much energy it takes to maintain the dragons' 1,000°C internal fire. Charcoal burns at temperatures exceeding 1,100°C and is used to heat substances such as wood and iron [9]. A btu per pound, another measurement of energy, is the equivalent of 0.556 Calories per kilogram. It takes five trees to burn for four hours [23]. Each tree produces 8,600 btu/lb, which is roughly 4781 Cal/g [2]. If we assume each tree is about 1,000 kg and using the charcoal to start flames, this results in roughly $4.781 * 10^9$ Calories for one tree, and $2.3905 * 10^{10}$ Calories to keep the fire burning for about four hours [23]. Generalizing this to the dragons, they will expend over $1.4343 * 10^{11}$ Calories per day (i.e. 24 hours) to keep their internal fires burning.

Summing it all together, each dragon will need to consume over $1.434302264 * 10^{11}$ Calories per day to account for Calorie expenditure and maintain their weight and internal fire. Utilizing knowledge from above that an average sheep is about 173,264 Calories, this results in one dragon needing to consume roughly 827,813 sheep per day to sustain their energy and growth.

2.3. Predator-Prey Model

To introduce the dragons into different environments, we added another variable to our Lotka-Volterra equations. Utilizing the Heaviside function, we were able to introduce the dragons at a specific time. This was useful in our studies of how the other species interacted before the dragons, and then what happens when the dragons appear. With the same variables as before, we can model these new predator-prey eqations as such:

$$\frac{dx}{dt} = \alpha x - \beta xy - \omega x * Heaviside(t - t_0) \tag{3}$$

$$\frac{dy}{dt} = \delta xy - \gamma y - \omega y * Heaviside(t - t_0) \tag{4}$$

where ω is a constant parameter and t_0 is the time in which the dragons are introduced. In order to be as accurate as possible, we used the dragons' metabolic rate as the value of ω. This equation can be approximated as:

$$MetabolicRate = B_0 M^{\alpha} e^{-E/kT} \tag{5}$$

where B_0 is a constant of proportionality, M is body mass of a dragon, α is a scaling coefficient (approximated as 3/4), E is activation energy (approximated as 0.65 eV), κ is Boltzmann's constant (8.6173324×10^{-5} ev/K), and T is body temperature of a dragon (measured in Kelvins) [22].

For our model of the dragons, we set B_0 to 2,483,439 because using the information in *Section 2.2*, this is the amount of sheep the three dragons will eat per day. We modeled the dragons at four years old, thus making their body mass equivalent to 2,560 kg. Because we assumed the fire in the dragons measured about $1{,}000°C$, we can calculate the dragons' internal body temperature to be about 1,273 Kelvins.

The sheep population in modern-day UK is assumed to be approximately 22 million [20]. A common predator of sheep are known as "canids", carnivorous animals that include domestic dogs, wolves, coyotes, foxes, and other dog-like animals [7]. Using numbers from fox and wolf populations in the UK, the canid population can be approximated to roughly 300,000 [10][21]. Due to the high nature of these population numbers, we will scale down the population sizes by a factor of 100,000 for our model. Therefore, in MATLAB the prey population will start at 220 and the predator population will be 3. B_0 will also be scaled by 100,000 and will result in $B_0 = 24.83439$, since that accounts for sheep populations, as well.

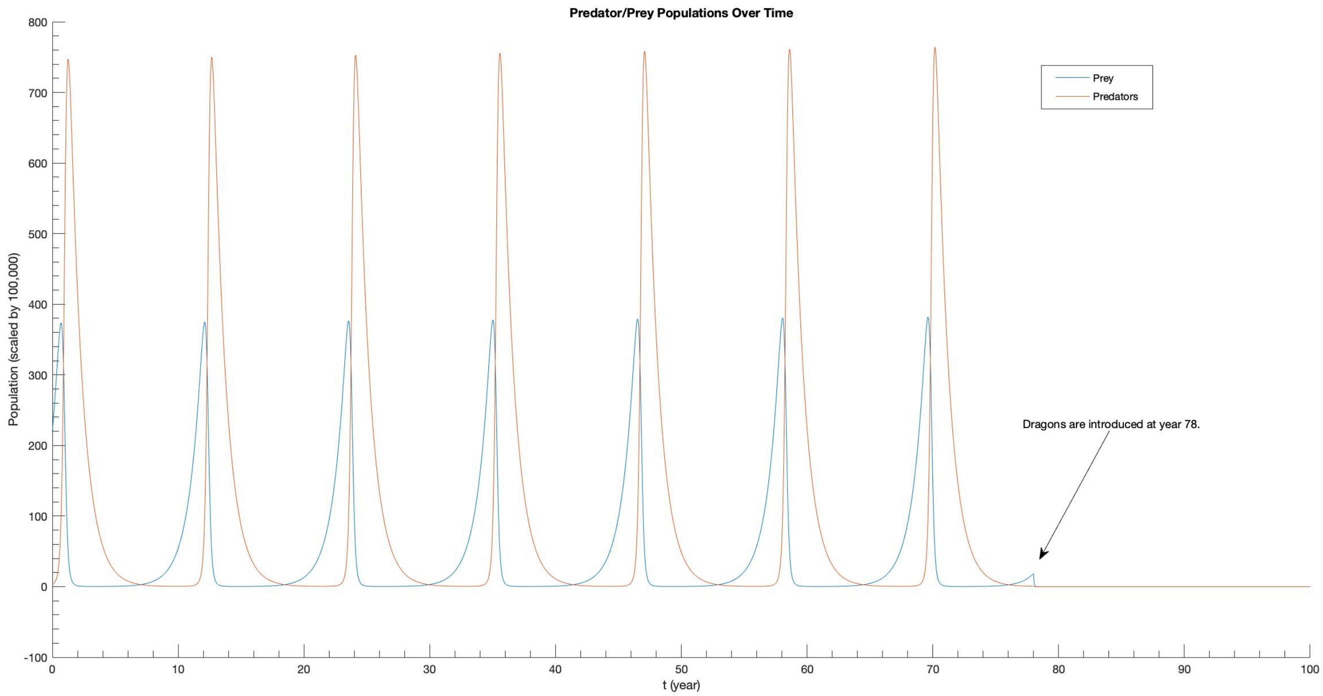

Figure 2: Predator-Prey Model with dragons introduced at t=78 years.

2.4. Analysis of Results

Species will die off. Since dragons are considered apex predators, introducing them to differing environments will result in other species dying off. As seen in Figure 2, the predator-prey model will begin the natural course, but will immediately decline to zero once the dragons are introduced. Because of the amount of food needed per dragon, they will wipe out entire populations in order to sustain themselves. This is modeled using sheep populations. We can see that since the three dragons need to consume 2,483,439 sheep per day, they will completely eliminate the sheep populations in short amounts of time.

Dragons have limited time. Seeing how quickly the three dragons wipe out animal populations, it is evident that the dragons will not be left with enough to eat. Smaller animal populations will not be able to reproduce, thus leaving the dragons without enough food to sustain themselves. Because they only eat meat, the amount of Calories required to sustain the dragons will actually be their downfall. They will eat quicker than the other populations can reproduce.

3. Future Work

Future studies include further calculations for other animals. For the sake of our project, we generalized the data found from sheep to other animal populations. However, due to the difference in weight and Calories per animal, there could be some discrepancies in our overall numbers. Researching other animals and populations would provide a better idea about how quickly the dragons would wipe out major population sizes. Along with this idea, we would of like to further research where large game animals are most concentrated. This plays into the climate element of the report because the dragons would survive the longest where the most food is widely available. We say large game because dragons are going to focus on first hunting the larger animals. Sheep was a great starting point for this idea.

More research about different climates and native animals would also be beneficial. For this project, we assumed that the dragons would eat animals at the same rate, regardless of the climate the dragons were in. It was also assumed their internal fire burned with the same amount of Calories expended. However, these two assumptions pose a problem regarding energy expenditure. The dragons' energy expenditure is surely going to differ in different climates. For example, they will need to expend more energy and Calories to heat their internal flames in the colder, Arctic temperatures compared to the warmer, temperate climates. By conducting more research about different energy expenditure in other climates, better conclusions can be drawn about how long the dragons will survive.

Further steps can also be taken in regard to the MATLAB model. We used the dragons' metabolic rate as the constant for the added Heaviside function. As seen in our conclusions, the dragons immediately kill off animal species in each of the climates. This poses as a problem because we cannot see exactly how long they can survive in each of the different areas. By researching different constants, we can gain a better idea about the dragons' survival time in each climate. This would hopefully create a more detailed model, outlining how long it takes the dragons to wipe out entire animal populations.

4. Conclusions

Our group reached several conclusions about the three dragons after creating a MATLAB simulation and making assumptions based off our model. The first question the MCM prompt stated involved the ecological impact and requirements of the dragons. The ecological impact inflected by the dragons is depleting animal species. Mostly large game animals, such as deer, elk, sheep, goats, etc., will be driven to extinction due to how much food dragons need to survive and continue growing. This is shown in Figure 2, which represents the Predator-Prey Model. You can see the sharp decline of both predators and prey not long after the dragons are introduced at year 78.

The next large area that was questioned by the MCM prompt was the energy expenditures, caloric intake, and what it takes to sustain these large creatures. Our group found that energy expenditures are quite high for dragons, assuming they have an internal fire burning within them that they must maintain. The fire the dragons breathe is around 1,000$°C$, which takes roughly $1.4343 * 10^{11}$ calories per day to maintain that hot, internal fire. This equates to the dragons having to eat about 2,483,439 sheep per day to sustain their energy and growth. We have already touched on it a little bit, but we also computed calculations on the dragons' metabolism rates. Since basal metabolic rate is measured in energy over time, we found that a dragon would need about a little over one sheep per day, without accounting for internal fire.

The last major question posed by the MCM prompt was related to how much area is needed to support and sustain dragons, and the effects of different climates. Our group realized early on that the area needed to support the dragons was not of major concern. The dragons could survive and sustain themselves as long as other animal populations continue to reproduce at a fast enough rate. Then we looked into the three major climates; temperate, arid (tropical), and arctic. We are assuming the dragons were born into temperate climates, which is most ideal. If dragons were introduced to our Earth, we analyzed that dragons will thrive and survive the longest in temperate regions, but once food sources are depleted, they will move onto tropical and then to arctic. They would eventually die off in arctic climates, as it would require the most energy to survive and keep their internal body temperatures up; however, there would not be enough food. Overall, it was clear to our group that dragons would not be sustainable for very long if introduced to our world, or in the *Game of Thrones* realm.

5. Distribution of Work

Rachel: Research; wrote “Introduction/Motivation”, “Climates”, “Similar Animal Research”, “Conclusions” sections; helped write ”Future Work” section; contributed to presentation slides

Chad: Research; ODE models in MatLab and figuring out the right constants

Ben: Research; wrote “Metabolism” section; worked on presentation slides

Laura: Research; ODE models in MatLab; helped write “Metabolism” section; wrote “Assumptions”, “Lotka-Volterra Equations”, “Energy Usage”, “Predator-Prey Model”, “Analysis of Results”, “Future Work” sections; worked on presentation slides

Appendix

Matlab Code

```
%contstants for equations - parameters describing species interactions
a = 1;
b = .01;  %interaction
c = .02;  %interaction
d = 1;

%constant for Heaviside function for introducing dragons - dragon's metabolic rate
B0 = 24.83439;  %choose it = constant of proportionality will be # of sheep the dragons eat, scaled by 100,000
Malpha = 2560^(0.75);  %M = body mass; alpha = scaling coefficient - bigger animals expend more energy
E = 0.65;  %activation energy
k = 8.6173324*(10.^-5);  %Boltzmann's constant
T = 1273;  %body temperature in Kelvins
eterm = exp(-E./(k*T));
e = B0*Malpha*eterm;

%when dragons introduced, they are now larger predators and everything is prey
%(with the balance of nature still in tact for the most part)

%diff.eq. - instantaneous growth rates of the populations
%x are the prey, y are the predators
%initial populations scaled by 100,000 [prey,predators]
x0 = [220,3];
%model over years
t = 0:0.01:100;
%f=(dx;dy)
f = @(t,x) [a*x(1) - b*x(1)*x(2) - e*heaviside(t-78)*x(1);c*x(1)*x(2) - d*x(2) - e*heaviside(t-78)*x(2)];
%solve ode
[t,y] = ode23(f,t,x0);

plot(t,y);
title('Predator/Prey Populations Over Time')
xlabel('t (year)')
ylabel('Population (scaled by 100,000)')
legend('Prey','Predators','Location','North')
```

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