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A Numerical Method for Solving Incompressible Viscous Flow Problems*

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A numerical method for solving incompressible viscous flow problems is introduced. This method uses the velocities and the pressure as variables and is equally applicable to problems in two and three space dimensions. The principle of the method lies in the introduction of an artificial compressibility δ into the equations of motion, in such a way that the final results do not depend on δ. An application to thermal convection problems is presented. © 1967 Academic Press

INTRODUCTION

The equations of motion of an incompressible viscous fluid are

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho_0}\partial_i p + v\,\Delta u_i + F_i, \quad \Delta \equiv \sum_j \partial_j^2,$$
$$\partial_j u_j = 0,$$

where u_i are the velocity components, p is the pressure, F_i are the components of the external force per unit mass, ρ_0 is the density, v is the kinematic viscosity, t is the time, and the indices i, j refer to the space coordinates x_i, x_j, i, $j = 1, 2, 3$.

Let d be some reference length, and U some reference velocity; we write

$$u_i' = \frac{u_i}{U}, \quad x_i' = \frac{x_i}{d}, \quad p' = \left(\frac{d}{\rho_0 v U}\right) p,$$
$$F_i' = \frac{vU}{d^2} F_i, \quad t' = \left(\frac{v}{d^2}\right) t$$

and drop the primes, obtaining the dimensionless equations

$$\partial_t u_i + R u_j \partial_j u_i = -\partial_i p + \Delta u_i + F_i, \tag{1a}$$
$$\partial_j u_j = 0, \tag{1b}$$

where $R = UD/v$ is the Reynolds number. Our purpose is to present a finite difference method for solving (1a)–(1b) in a domain D in two or three space dimensions, with some appropriate conditions prescribed on the boundary of D.

The numerical solution of these equations presents major difficulties, due in part to the special role of the pressure in the equations and in part to the large amount of computer time which such solution usually requires, making it necessary to devise finite-difference schemes which allow efficient computation. In two-dimensional problems the pressure can be eliminated from the equations using the stream function and the vorticity, thus avoiding one of the difficulties. If, however, a solution in three space dimensions is desired, one is thrown back upon the primary variables, the velocities, and the pressure. In what follows a numerical procedure using these variables is presented; it is equally applicable to two- and three-dimensional problems and is believed to be computationally advantageous even in the two-dimensional case. In the present paper we shall concentrate on the search for steady solutions of the equations; a related method for time-dependent problems will be presented in a forthcoming paper.

Methods using the velocities and the pressure in two-dimensional incompressible flow problems have previously been devised. For example, in [4], Harlow and Welch follow a procedure which appears quite natural—and may indeed in their problem be quite appropriate. It runs as follows: Taking the divergence of Eqs. (1a) one obtains for the pressure an equation of the form

$$\Delta p = Q, \quad \Delta \equiv \sum \partial_j^2, \tag{2}$$

where Q is a quadratic function of the velocities and, eventually, a function also of the external forces. Boundary conditions for (2) can be obtained from (1a) applied at the boundary. There remains, however, the task of ensuring that (1b) is satisfied. This is done by starting the calculation with velocity fields satisfying (1b), making sure that (1b) is always satisfied at the boundary, and solving (2) at every step so that (1b) remains satisfied as time is advanced. An ingenious formulation of the finite difference form of

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Eq. (2) reduces considerably the arithmetic labor necessary to solve it.

In our opinion the main shortcoming of this procedure lies in its treatment of the boundary conditions. In order to satisfy the boundary conditions for (2) derived from (1a) and to satisfy (1b) near the boundary, it is necessary, in the finite-difference formulation, to assign values to the velocity fields at virtual points outside the boundary, and this is a situation where no reflection principle is known to hold.

Were this procedure to be used only for the purpose of obtaining an asymptotic steady solution (which was not the purpose in [4]), it would have additional shortcomings. It would be computationally wasteful to solve (2) at every intermediate step, and moreover, in many problems, to obtain an initial solution satisfying (1b) would be a major problem by itself.

We shall now present a method for solving the system (1a)–(1b), which we believe to be free of these difficulties and computationally more efficient. We shall not use Eq. (2).

THE METHOD OF ARTIFICIAL COMPRESSIBILITY

We introduce the auxiliary system of equations

$$\begin{gathered}\partial_t u_i + R\partial_j(u_i u_j) = -\partial_i p + \Delta u_i + F_i,\\ \partial_t \rho + \partial_j u_j = 0, \quad p = \rho/\delta.\end{gathered} \tag{3}$$

An alternative form for the first of these equations is

$$\partial_t u_i + R u_j \partial_j u_i = -\partial_i p + \Delta u_i + F_i. \tag{3'}$$

We shall call ρ the artificial density, δ the artificial compressibility, and $p = \rho/\delta$ the artificial equation of state. t is an auxiliary variable whose role is analogous to that of time in a compressible flow problem.

If, as the calculation progresses, the solution of (3) converges to a steady solution, i.e. one which does not depend on t, this solution is a steady solution of (1) and does not depend on δ; δ appears as a disposable parameter, analogous to a relaxation parameter. The system (3) is not a purely artificial construction, as can be seen by comparing it with the equations of motion of a compressible fluid with a small Mach number.

Equations (3) contain an artificial sound speed

$$c = 1/\delta^{1/2}$$

and relative to that speed the artificial Mach number M is

$$M = \frac{R}{c}\max_D \left(\sum u_i^2\right)^{1/2}.$$

It is clearly necessary that $M < 1$.

It now remains to replace the system (3) or (3′) by a finite difference system, and

(a) show that the finite difference approximation to (3) is stable,

(b) demonstrate that the solution of the difference system does indeed tend to a steady limit,

(c) find a value of δ and of any other parameter in the finite difference system such that the steady limit is reached as fast as possible, and show that the resulting procedure is indeed efficient.

(d) show that the steady limit of the difference system does tend to a steady solution of (1) as the mesh width tends to zero.

The author has not been able to carry out this program analytically, forcing heavy reliance on the numerical evidence.

It is not indispensable that the solution of the differential system (3) tend to a steady limit, as long as the solution of the difference system does. It is, however, believed that the solution of (3) does tend to a steady limit, at least in the absence of external forces, under quite general conditions. This can be proved in the limiting case $R = 0$, for problems in which the velocities are prescribed at the boundary. By linearity it is sufficient to consider the case of zero velocities at the boundary. From (3) the following equality can be obtained:

$$\frac{1}{2}\partial_t \int_D \left(\frac{1}{2}u_i u_i + \frac{\rho^2}{\delta}\right) dV = -\int_D \sum_{i,j} (\partial_i u_j)^2 \, dV.$$

The integrands on both sides are positive; hence the u_i tend to the limit $u_i = 0$, and p to a limit independent of t. From (3) one sees that this limit is independent of the x_i and therefore is a constant.

THE FINITE-DIFFERENCE APPROXIMATION

The system (3) can be used with various difference schemes. In the one adopted here, after some experimentation, the inertia and pressure terms are differenced according to the leap-frog scheme, i.e., both time and space derivatives are replaced by central differences, and the viscous dissipation terms are differenced according to the Dufort–Frankel pattern, in which a second derivative such as

$$\partial_1^2 u$$

is replaced by

$$\frac{1}{\Delta x_1^2}\left(u_{i+1}^n + u_{i-1}^n - u_i^{n+1} - u_i^{n-1}\right), \quad u_i^n \equiv u(i\,\Delta x_1, n\,\Delta t).$$

Δt, Δx_1 are, respectively, the "time"- and space-variable increments.

Equations (3) then become, in the two-dimensional case, and in the absence of external forces,

$$
\begin{aligned}
u^{n+1}_{1(i,j)} - u^{n-1}_{1(i,j)} = {} & -R\frac{\Delta t}{\Delta x_1}\left((u^n_{1(i+1,j)})^2 - (u^n_{1(i-1,j)})^2\right) \\
& - R\frac{\Delta t}{\Delta x_2}\left(u^n_{1(i,j+1)}u^n_{2(i,j+1)} - u^n_{1(i,j-1)}u^n_{2(i,j-1)}\right) \\
& + \frac{2\,\Delta t}{\Delta x_1^2}\left(u^n_{1(i+1,j)} + u^n_{1(i-1,j)} - u^{n+1}_{1(i,j)} - u^{n-1}_{1(i,j)}\right) \\
& + \frac{2\,\Delta t}{\Delta x_2^2}\left(u^n_{1(i,j+1)} + u^n_{1(i,j-1)} - u^{n+1}_{1(i,j)} - u^{n-1}_{1(i,j)}\right) \\
& - \frac{\Delta t}{\Delta x_1}\frac{1}{\delta}\left(\rho^n_{i,j+1} - \rho^n_{i,j-1}\right), \\
u^{n+1}_{2(i,j)} - u^{n-1}_{2(i,j)} = {} & -R\frac{\Delta t}{\Delta x_1}\left(u^n_{1(i+1,j)}u^n_{2(i+1,j)} - u^n_{1(i-1,j)}u^n_{2(i-1,j)}\right) \qquad (4) \\
& - R\frac{\Delta t}{\Delta x_2}\left((u^n_{2(i,j+1)})^2 - (u^n_{2(i,j-1)})^2\right) \\
& + \frac{2\,\Delta t}{\Delta x_1^2}\left(u^n_{2(i+1,j)} + u^n_{2(i-1,j)} - u^{n+1}_{2(i,j)} - u^{n-1}_{2(i,j)}\right) \\
& + \frac{2\,\Delta t}{\Delta x_2^2}\left(u^n_{2(i,j+1)} + u^n_{2(i,j-1)} - u^{n+1}_{2(i,j)} - u^{n-1}_{2(i,j)}\right) \\
& - \frac{\Delta t}{\Delta x_2}\frac{1}{\delta}\left(\rho^n_{i,j+1} - \rho^n_{i,j-1}\right), \\
\rho^{n+1}_{i,j} - \rho^{n-1}_{i,j} = {} & -\frac{\Delta t}{\Delta x_1}\left(u_{1(i+1,j)} - u_{1(i-1,j)}\right) \\
& - \frac{\Delta t}{\Delta x_2}\left(u_{2(i,j+1)} - u_{2(i,j-1)}\right),
\end{aligned}
$$

with

$$\rho^n_{i,j} \equiv \rho(i\,\Delta x_1, j\,\Delta x_2, n\,\Delta t), \quad u^n_{m(i,j)} \equiv u_m(i\,\Delta x_1, j\,\Delta x_2, n\,\Delta t).$$

Similar expressions are used in the three-dimensional case.

It is also necessary to approximate the equation

$$\partial_t \rho = -\partial_j u_j$$

at the boundary. Suppose the boundary is the line $x_2 = 0$, represented by $j = 1$ (see Fig. 1). A reasonable approximation is

$$
\begin{aligned}
\rho^{n+1}_{i,1} - \rho^{n-1}_{i,1} = {} & -2\frac{\Delta t}{\Delta x_2}\left(u^n_{2(i,2)} - u^n_{2(i,1)}\right) \\
& - \frac{\Delta t}{\Delta x_1}\left(u^n_{1(i+1,1)} - u^n_{1(i-1,1)}\right).
\end{aligned}
\qquad (5)
$$

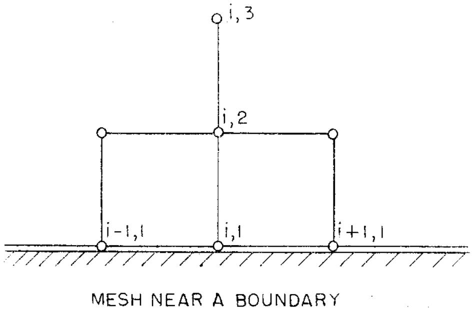

FIG. 1. Mesh near a boundary.

When $\rho^{n+1} \to \rho^{n-1}$, this expression tends to

$$2\frac{1}{\Delta x_2}\left(u^n_{2(i,2)} - u^n_{2(i,1)}\right) + \frac{1}{\Delta x_1}\left(u^n_{1(i+1,1)} - u^n_{1(i-1,1)}\right) = 0,$$

which approximates $\partial_j u_j = 0$ on the boundary to order Δx_2. A possible second-order approximation is

$$
\begin{aligned}
\rho^{n+1}_{i,1} - \rho^{n-1}_{i,1} = {} & -4\frac{\Delta t}{\Delta x_2}\left(u^n_{2(i,2)} - u^{n+1}_{2(i,1)}\right) + \frac{\Delta t}{\Delta x_2}\left(u^n_{2(i,2)} - u^{n-1}_{2(i,1)}\right) \\
& - \frac{\Delta t}{\Delta x_1}\left(u^n_{1(i+1,1)} - u^n_{1(i-1,1)}\right).
\end{aligned}
$$

One notices that these formulas contain three levels in appearance only, for, since $u^{n+1}_{i,j}$ does not depend on $u^n_{i,j}$, the calculation splits into two unrelated calculations on two intertwined meshes, one of which can be omitted. If this is done, the nth and $(n + 1)$st "time" levels can be considered as one level.

This scheme is stable for Δt small enough and is entirely explicit. The presence of the dissipation terms suppresses the instabilities to which the nondissipative leap-frog scheme is susceptible. The known inaccuracy of the Dufort–Frankel scheme is of no relevance if only the asymptotic steady solution is sought. In fact, if we consider the Dufort–Frankel scheme,

$$
\begin{aligned}
u^{n+1}_{i,j} - u^{n-1}_{i,j} = {} & 2\frac{\Delta t}{\Delta x^2}\left(u^n_{i+1,j} + u^n_{i-1,j} + u^n_{i,j+1}\right. \\
& \left. + u^n_{i,j-1} - 2u^{n+1}_{i,j} - 2u^{n-1}_{i,j}\right) + 2\,\Delta t f,
\end{aligned}
\qquad (6)
$$

which, for $\Delta t = o(\Delta x)$ approximates the equation

$$\partial_t u = \Delta u + f, \quad \Delta \equiv \partial_1^2 + \partial_2^2,$$

then, if we write

$$\omega = 8\frac{\Delta t}{\Delta x^2}\left(1 + 4\frac{\Delta t}{\Delta x^2}\right)^{-1}, \tag{7}$$

we see that (6) is nothing but the usual relaxation method for the solution of the 5-point Laplace difference equation, with relaxation parameter ω. Returning to the general system (4), we see that, since that system is stable only for Δt small enough, ω [defined by (7)] can take values only in an interval $0 \le \omega \le \omega_c < 2$. This is a familiar situation (see, e.g., [1]).

δ plays a role similar to that of a relaxation coefficient. Suppose the u_i are such that, at some point the finite-difference analog of (1b) is not satisfied; for example,

$$\frac{1}{\Delta x_1}\left(u^n_{1(i+1,j)} - u^n_{1(i-1,j)}\right) + \frac{1}{\Delta x_2}\left(u_{2(i,j+1)} - u_{2(i,j-1)}\right) < 0$$

so that $\rho^{n+1}_{i,j} > \rho^{n-1}_{i,j}$. Then a "density" gradient is formed which, through the terms $-\partial_i \rho/\delta$ in the momentum equations, will at the next step increase the velocity components pointing away from the point (i, j), thus increasing $\rho_{i,j}$ and bringing the equation of continuity closer to being satisfied. This sequence resembles a relaxation step.

A stability analysis of (4) shows that if the boundary conditions consist of prescribed velocities, the system is stable when

$$\max_D \max_{\phi_i} \left\{\frac{1}{2}|\Sigma| + \frac{1}{2}[\Sigma^2 + 4c^2(\psi_1^2 + \psi_2^2 + \psi_3^2)]^{1/2}\right\} \le 1,$$

where

$$\psi_i = \frac{\Delta t}{\Delta x_i}\sin\phi_i \quad 0 \le \phi_i \le 2\pi, \quad i = 1, 2, 3,$$

$$\Sigma = u_1\psi_1 + u_2\psi_2 + u_3\psi_3, \quad c = \delta^{-1/2}.$$

If one ensures that the flow is subsonic with respect to the artificial sound speed, the above condition is satisfied when

$$\Delta t \le \frac{2}{n^{1/2}(1 + 5^{1/2})}(\min_i \Delta x_i)\delta^{1/2},$$

where n is the number of space dimensions.

If other types of boundary conditions are imposed, e.g., if the derivatives of the velocities are prescribed at the boundary, one has to ensure that no instabilities arise due to boundary effects. For details, see [3].

In fact, we have at our disposal two parameters, Δt and δ, to be assigned values which make convergence to the steady solution as rapid as possible. The stability condition restricts the range of permissible values of these parameters.

Finally, the accuracy of the finite-difference scheme can be improved in two-dimensional problems with the use of staggered nets. This was not done here because our programs were written with three-dimensional problems in view. Slight modifications of the scheme were found necessary in problems involving singular points on the boundary.

TABLE I

Errors in Test Problem

N	$E(u_1)$	$E(u_2)$	$E(p)$
0	1.	0.	8.
100	0.1053	2.0×10^{-2}	7.04
200	1.03×10^{-2}	1.5×10^{-3}	0.61
300	7.7×10^{-4}	1.0×10^{-4}	0.22
400	7.1×10^{-5}	1.7×10^{-5}	1.6×10^{-2}
500	6.5×10^{-6}	3.8×10^{-6}	8.6×10^{-3}
600	1.2×10^{-6}	3.9×10^{-7}	7.8×10^{-3}
700	4.1×10^{-7}	1.2×10^{-7}	5.2×10^{-4}
800	7.2×10^{-8}	1.7×10^{-8}	1.3×10^{-4}
900	2.3×10^{-8}	6.5×10^{-9}	3.6×10^{-5}
1000	5.4×10^{-9}	1.5×10^{-9}	9.5×10^{-6}
1100	1.5×10^{-9}	4.2×10^{-10}	2.5×10^{-6}
1200	3.9×10^{-10}	1.1×10^{-10}	6.6×10^{-7}
1300	1.0×10^{-10}	$3. \times 10^{-11}$	1.7×10^{-7}
1400	$3. \times 10^{-11}$	$1. \times 10^{-11}$	4.6×10^{-8}
1500	$1. \times 10^{-11}$	less than 5×10^{-12}	1.2×10^{-8}

A SIMPLE TEST PROBLEM

The system (1) with $F_i = 0$ will now be solved in a square domain D: $0 \le x_1 \le 1$, $0 \le x_2 \le 1$ with the boundary conditions

$$u_1 = 4x_2(1 - x_2), \quad u_2 = 0 \quad \text{on the lines } x_1 = 0, x_1 = 1,$$

$$u_1 = u_2 = 0 \quad \text{on the lines } x_2 = 0, x_2 = 1.$$

This is a simple problem, designed to test our method. The domain D represents a segment of a channel. The reference velocity in the Reynolds number is the maximum velocity in the channel, and the reference length d is the width of the channel. The steady solution is known analytically; it is

$$u_1 = 4x_2(1 - x_2), \quad u_2 = 0, \quad p = C - x_1 \quad \text{in } D,$$

where C is an arbitrary constant.

The equation of continuity is represented at the boundary by the formula (5). The higher-accuracy formula was also tried, and the results are very similar.

In Table I results of a sample computation are presented. The initial values for Eqs. (3)—or, if one prefers, the initial guess at the steady solution—are very unfavorable: $u_1 = u_2 = 0$ everywhere except at the boundary, $p = 0$ everywhere. These initial values are very unfavorable because u_1 is discontinuous, and therefore, in the first steps, $\partial_t \rho$ becomes very large. Convergence is much faster when the initial values are smooth, or when they incorporate some advance knowledge regarding the final solution. These initial values were chosen to demonstrate the convergence of the procedure even under unfavorable conditions. In Table I the Reynolds number R is 1, $\delta = 0.00032$; 19 mesh points were used in each space direction. N is the number of steps, $E(u_1)$, $E(u_2)$, $E(p)$ are the errors, i.e., the maxima of the differences between the computed solution and the analytic solution given above. The constant C in the computed pressure is determined from the values of p on the line $x_1 = 0$.

It should be kept in mind that every step is very simple, being entirely explicit.

The optimal value of δ, δ_{opt}, has to be determined from a preliminary test computation; it is independent of Δx. Δt is determined from the relation

$$\Delta t = 0.6 \cdot \Delta x \cdot \delta^{1/2}$$

so that the stability requirement is met. δ_{opt} is not sharply defined; for $R = 0$, all values of δ between 0.006 and 0.05 lead to approximately the same rate of convergence.

The channel flow problem was solved for values of R varying between 0 and 1000. The method converged for all these values, although convergence was very slow for the higher values of R. δ_{opt} decreases as R increases.

The problem in this section is particularly simple; the analytic steady solution is known, and it satisfies the finite-difference equations exactly. The method was of course applied to less trivial problems, one of which will now be described.

THERMAL CONVECTION IN A FLUID LAYER HEATED FROM BELOW. THE TWO-DIMENSIONAL CASE

Suppose a plane layer of fluid, of thickness d and infinite lateral extent, in the field of gravity, is heated from below. The lower boundary $x_3 = 0$ is maintained at a temperature T_0, the upper boundary $x_3 = d$ at a temperature T_1, with $T_0 - T_1$ positive. (x_3 is the vertical coordinate.) The warmer fluid at the bottom of the layer expands and tends to move upwards; this tendency is inhibited by the viscous stresses.

The equations governing the fluid motions are, in the Boussinesq approximation (see [2, 3]),

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho_0} \partial_i p + v \, \Delta u_i - g(1 - \alpha(T - T_0))\varepsilon_i,$$
$$\partial_t T + u_j \partial_j T = k \, \Delta T, \quad \partial_j u_j = 0,$$

where k is the coefficient of thermal conductivity, g is the force of gravitation, T the temperature, α the coefficient of thermal expansion of the fluid, and ε_i are the components of the unit vector pointing upwards.

We write

$$u_i' = \frac{d}{v} u_i, \quad T' = \frac{T - T_1}{T_0 - T_1}, \quad t' = \frac{v^2}{d} t,$$
$$x_i' = \frac{x_i}{d}, \quad p' = \frac{1}{\rho_0} \left(\frac{d}{v} \right)^2 p$$

and drop the primes. The equations become

$$\begin{aligned} &\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \Delta u_i - \frac{R^*}{\sigma q} (1 - q(T - 1))\varepsilon_i, \\ &\partial_t T + u_j \partial_j T = \frac{1}{\sigma} \Delta T, \quad \partial_j u_j = 0, \end{aligned} \tag{8}$$

where $R^* = [\alpha \beta g d^3 (T_0 - T_1)](kv)^{-1}$ is the Rayleigh number, $\sigma = v/k$ the Prandtl number, and $q = \alpha(T_0 - T_1)$. It is assumed that the upper and lower boundaries are rigid, i.e., $u_i = 0$, $i = 1, 2, 3$ on $x_3 = 0$ and $x_3 = 1$.

It is known from the linearized stability theory that, for $R^* < R_c$, the state of rest is stable with respect to infinitesimal perturbations, where $R_c = 1707.62$ is the critical Rayleigh number (see [2]). This is taken to mean that for $R^* < R_c$ no convective motion can be maintained in the layer. When $R^* = R_c$ steady infinitesimal convection can first appear; the various field quantities are given by

$$\begin{aligned} &u_3 = W(x_3)\phi, \\ &u_1 = \frac{1}{a^2} W(x_3) \partial_1 \phi, \quad u_2 = \frac{1}{a^2} W(x_3) \partial_2 \phi, \\ &T = T(x_3)\phi, \end{aligned} \tag{9}$$

where $\phi = \phi(x_1, x_2)$ determines the horizontal planform of the motion and satisfies

$$(\partial_1^2 + \partial_2^2)\phi = -a^2 \phi.$$

$W(x_3)$, $T(x_3)$ are certain fully determined functions, $a = 3.117$, and the amplitude is of course undetermined.

In two-dimensional convection $u_1 = 0$ and the motion is independent of x_1. ϕ has then the form

$$\phi = \cos a x_2$$

when $R^* = R_c$. The motion is periodic in x_2 with period $2\pi/a$. In this section we shall confine ourselves to two-dimensional problems.

When $R^* > R_c$ it is known from experiment that steady

TABLE II

Nu as a Function of R^*/R_c

R^*/R_c	$M = 30, N = 26$	$M = 30, N = 28$
2	1.754	1.759
3	2.093	2.099
4	2.309	2.317
5	2.478	2.482
6	2.608	2.620
7	2.728	2.735
8	2.833	2.841
9	2.927	2.936
10	3.008	3.021
11	3.086	3.098
12	3.161	3.172
13	3.232	3.241

convection sets in, at least when R^* is not too large. We shall assume that the motion remains periodic, with a period equal to the period of the first unstable mode (9) of the linearized theory. There is no difficulty in trying other periods. The periodicity assumption is physically very reasonable. We are interested in determining the amplitude of the motions, and more specifically, the magnitude of the heat transfer, measured by the dimensionless Nusselt number *Nu*. *Nu* is the ratio of the total heat transfer to the heat transfer which would have occurred if no convective motion were present; in our dimensionless variables it is simply

$$Nu = \frac{a}{2\pi} \int_0^{2\pi/a} (\sigma u_3 T - \partial_3 T)\, dx_2$$

and does not depend on x_3 when the convection is steady. For $R^* \leq R_c$, $Nu = 1$. It can be seen from (8) that the only physical parameters in the problem are R^* and σ; the solution does not depend on q, except inasmuch as R^* depends on q. Changing q in (8) simply implies a change in the definition of the pressure. We shall study the dependence of *Nu* on R^* and σ.

The auxiliary system used for finding steady solutions of (8) is

$$\begin{aligned} &\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \Delta u_i - \frac{R^*}{\sigma q}(1 - q(T-1))\varepsilon_i, \\ &\partial_t T + u_j \partial_j T = \frac{1}{\sigma}\Delta T, \quad \partial_t \rho = -\partial_j u_j \end{aligned} \qquad (10)$$

with the artificial equation of state, either

$$p = \frac{R^*}{\sigma q \delta}(\rho - 1 - q(T-1))$$

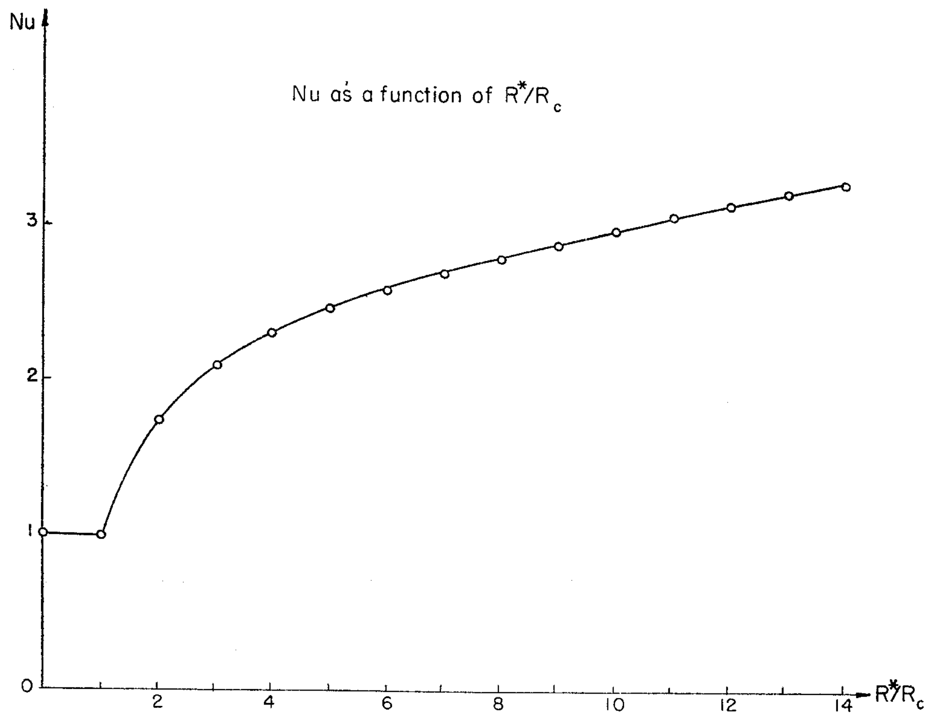

FIG. 2. *Nu* as a function of R^*/R_c.

TABLE III

Nu as a Function of σ

σ	$M = 30, N = 26$	$M = 30, N = 28$
20.0	2.67	2.68
6.8	2.68	2.69
1.0	2.73	2.73
0.2	2.68	2.68

or

$$p = (R^*/\sigma q \delta)\rho.$$

The artificial sound speed c is in both cases

$$c = (R/\sigma q \delta)^{1/2}$$

δ is the artificial compressibility. The results are not affected by which equation of state is used. It should be noted that t in (10) does not represent real time.

The finite-difference scheme is a straightforward extension of the scheme presented before, i.e., a combined leapfrog and Dufort–Frankel scheme. It was found that the steady state is reached with less computing effort when the nonlinear terms are differenced in a nonconservative form, as in (3′). It was also observed that the computation proceeds with greatest efficiency when Δt is as large as possible, and hence, in view of the stability condition, when c is as small as possible. Since the artificial Mach number M has to be smaller than 1, q and δ were chosen in practice so as to have $M \sim 0.5$–0.8, thus allowing for possible velocity overshoots. A rough trial computation was usually made for every class of problems to determine the order of magnitude of M.

For every value of R^* and σ it is necessary to determine how many mesh points are needed to produce an accurate value of *Nu*. Serious errors may ensue when too few points are used. Every series of calculations was therefore performed at least twice, and the results accepted only if they had been approximately reproduced by two different calculations with differing meshes. As is to be expected, the number of points required increases with the Rayleigh number.

The initial data for the various problems consist of a zero-order solution on which a perturbation is imposed. The zero-order solution is

$$u_2 = u_3 = 0, \quad T = 1 - x_3,$$

with ρ and p obtained by solving numerically the finite-difference equations in the absence of motion. The perturbation which produces the fastest convergence to the steady solution was found to be one in which the temperature alone is perturbed, by adding to it a multiple of the temperature field of the first unstable mode of the linearized theory.

The steady state is assumed to have been reached when two conditions are satisfied: (a) The Nusselt number evaluated at the lower boundary has varied by less than 0.2% over 100 steps, and (b) the Nusselt number evaluated at the lower boundary and the Nusselt number evaluated at midlayer differ by less than 0.2%.

Table II displays the variation of *Nu* with R^* for $\sigma = 1$ (see also Fig. 2). M is the number of mesh points in the x_2 direction, and N the number of mesh points in the x_3

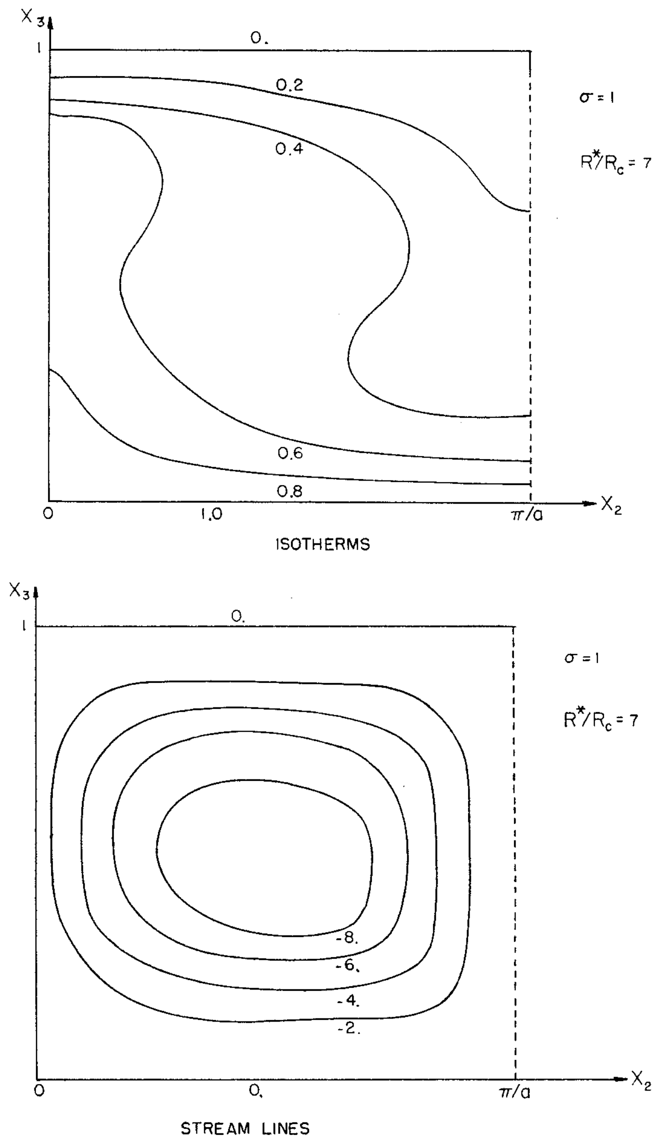

FIG. 3. (a) isotherms; (b) stream lines.

direction. These results are in good agreement with some results obtained by G. Veronis and P. Schneck [5].

Table III gives an indication about the way *Nu* varies with σ, for $R^*/R_c = 7$. It is seen that *Nu* does not vary very much with σ, as already discovered by Veronis [6] with another type of boundary conditions.

For the sake of completeness, typical isotherm and stream line configurations are represented in Figs. 3a and b. They were obtained with $R^*/R_c = 7$, $\sigma = 1$, $M = 30$, $N = 28$. The stream function ψ was obtained from the computed velocities and affords a further check on the results, since the conditions

$$\partial_3\psi = 0, \quad \partial_2\psi = 0$$

are satisfied at the upper and lower walls.

THERMAL CONVENTION IN THREE SPACE DIMENSIONS

In three space dimensions not only the amplitude of the motions is to be determined, but also their spatial configuration. For $R^* = R_c$ the function ϕ in (9) can be any periodic solution of

$$(\partial_1^2 + \partial_2^2 + a^2)\phi = 0, \quad a = 3.117.$$

This corresponds to the fact that a given wave vector can be broken up into two orthogonal components in an infinite number of ways, with arbitrary amplitudes and phases. It is reasonable to assume that the cell patterns are made up of polygons whose union covers the (x_1, x_2)-plane; possible cell shapes are hexagons, rectangles, and rolls (i.e., two-dimensional convection cells). For $R^* > R_c$, the nonlinear terms in the equations determine which cell pattern actually occurs.

The numerical method described in this article is applicable; some computational results were described in [3]. The conclusion to be drawn from them is that the preferred cellular mode is the roll, but that even this preferred mode is subject to instabilities. A search for possible values of R, σ, and a for which such instabilities do not occur will be described elsewhere.

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