I. EFFECTS IN THE MAGNETOSPHERE

The equation for the EM modes in a plasma coupled to a dilaton field is

$$\frac{\left(n^2-1\right)\left(\omega_p^2\left(n^2\cos^2(\theta)-1\right)-\left(n^2-1\right)\omega^2\gamma_c^3(n\,v\cos(\theta)-1)^2\right)}{\omega^2\gamma_c^3(n\,v\cos(\theta)-1)^2}-g^2B^2\frac{\left(n^2-1\right)\left(n^2\cos(2\theta)+n^2-2\right)}{2\left(m^2+\left(n^2-1\right)\omega^2\right)}=0 \tag{1}$$

The interior of the magnetosphere, at $r \ll r_A$ has

$$A_p=\frac{\omega_p^2\gamma_c}{\omega^2}\gg 1\,, \tag{2}$$

while the exterior, at $r \gg r_A$ has

$$A_p \ll 1\,. \tag{3}$$

Among the modes, solutions of Eq. (1), one must chose the ones that propagate as transverse at larger angles. The trivial mode that is always transverse and always a solution is

$$n_1=1\,. \tag{4}$$

Since we already know this solution, we can divide the Eq. (1) by (n^2-1) and look for the other ones. We work in the small angles approximation $\theta \ll 1$. Moreover, the velocity of the plasma is close to $c=1$, $\gamma_c \gg 1$ and $v \approx 1-\frac{1}{2\gamma_c}$.

A. Magnetosphere interior

In the magnetosphere exterior, where $A_p \gg 1$ we can take $v=1$. We pick the mode that becomes transverse when $\theta \gg \theta_* = (\frac{\omega_p^2\gamma_c^-3}{\omega^2})^{1/4}$ [LS: there's a typo in the definition of this quantity in the draft] ,

$$n_2=1+\frac{\theta^2}{4}-\sqrt{\theta^4+\theta_*^4}+F(g,\omega,\theta,\theta_*,m). \tag{5}$$

In the expression above, the function F is zero in the absence of the coupling g, and was introduced to find the coupled solution perturbatively. At first order in F and g^2, we obtain [LS: different from the draft]

$$F(g,\omega,\theta,\theta_*,m)=\frac{B^2g^2\left(G\left(\theta,\theta_*\right)+\theta^2\right)^2\left(\theta^2G\left(\theta,\theta_*\right)+\theta^4+8\theta_*^4\right)}{2\left(\left(G\left(\theta,\theta_*\right)+\theta^2\right)^2+16\theta_*^4\right)\left(\theta_*^4\omega^2\left(G\left(\theta,\theta_*\right)-\theta^2-8\right)+m^2\left(G\left(\theta,\theta_*\right)+\theta^2\right)\right)} \tag{6}$$

B. Magnetosphere exterior

In the magnetosphere exterior, where $A_p \ll 1$ and $|1-n| \ll \gamma^{-2}$, the transverse modes at zero coupling $g=0$ are $n_1=1$ and [LS: different from the draft]

$$n_2=1-2A_p\theta^2+O(\theta^4,A_p^2)\ . \tag{7}$$

With the same approximations, we arrive, for $g\neq 0$, at

$$n_2=1+\frac{B^2g^2}{2\left(B^2g^2+m^2\right)}\theta^2++O(\theta^4,A_p) \tag{8}$$

The new mode is bigger than the $g=0$ contribution, which was therefore discarded [LS: Are we so sure about this?] .