3D convex hulls

Computational Geometry [csci 3250]
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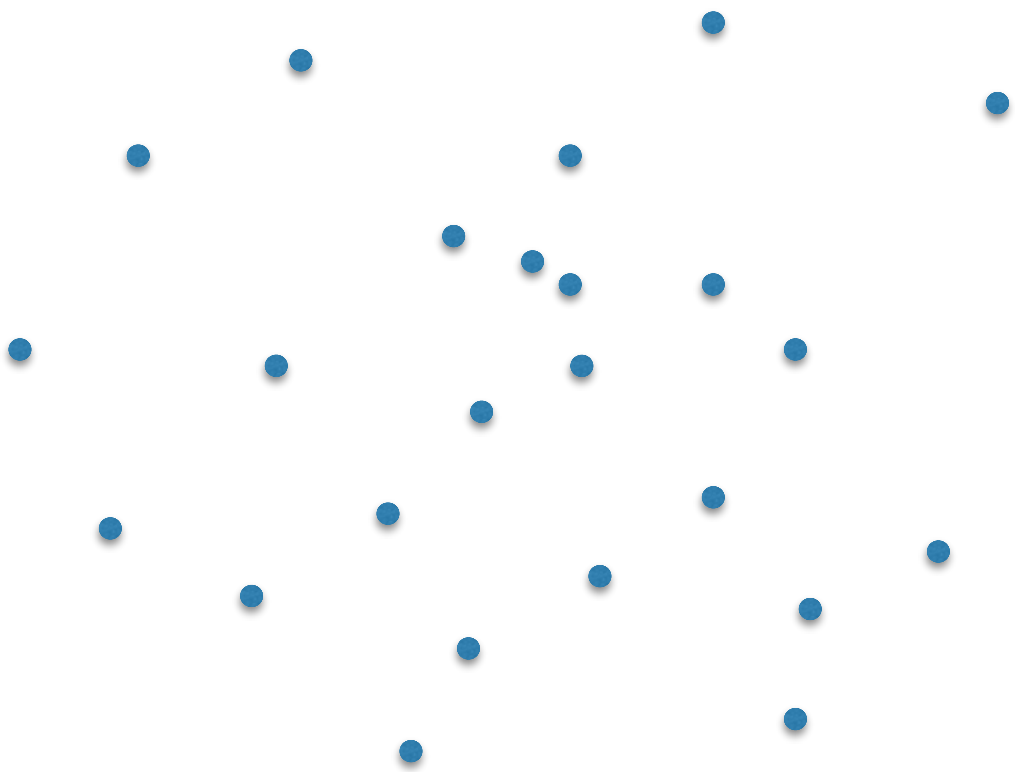
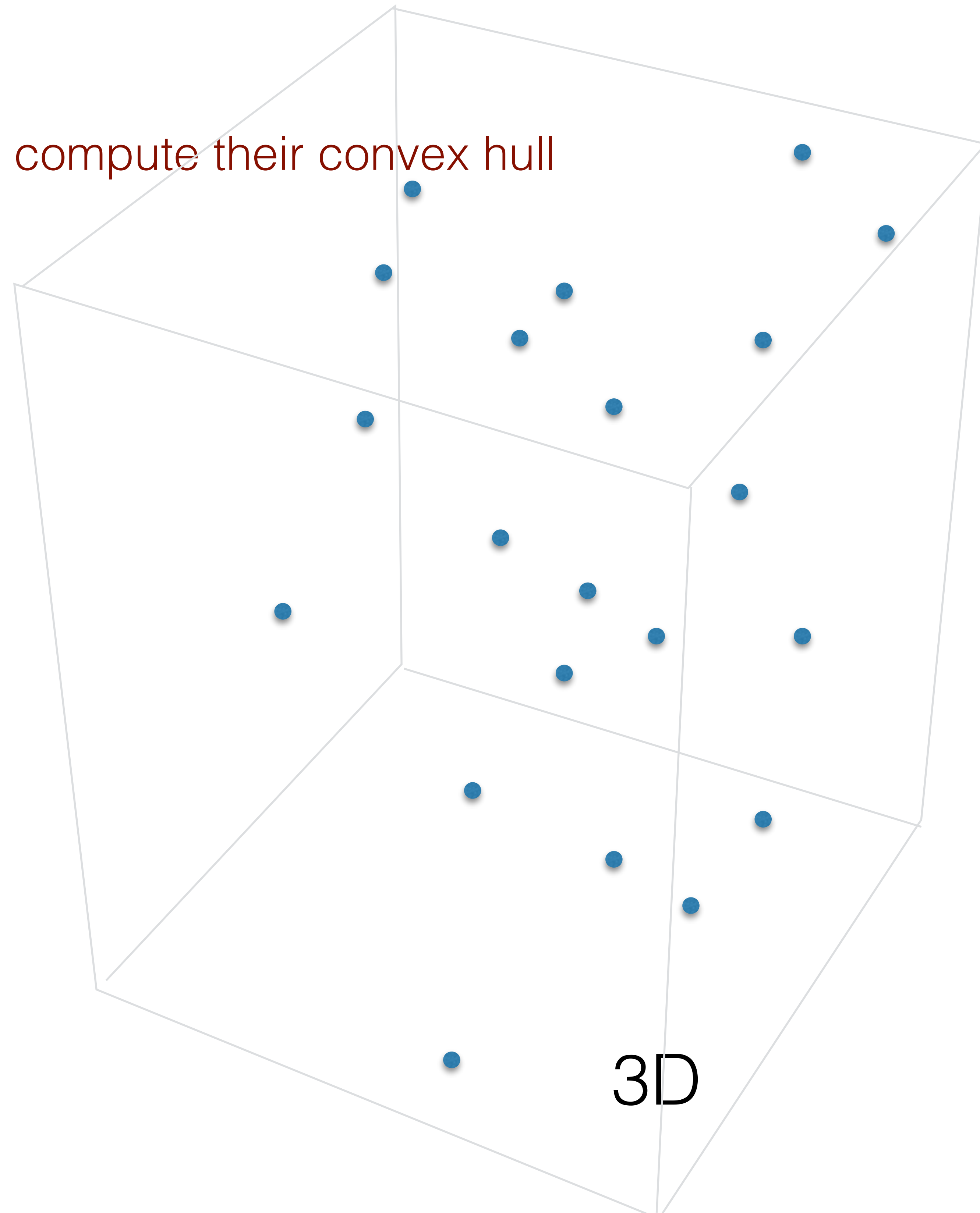
Bowdoin College

Convex Hulls

The problem: Given a set P of points, compute their convex hull

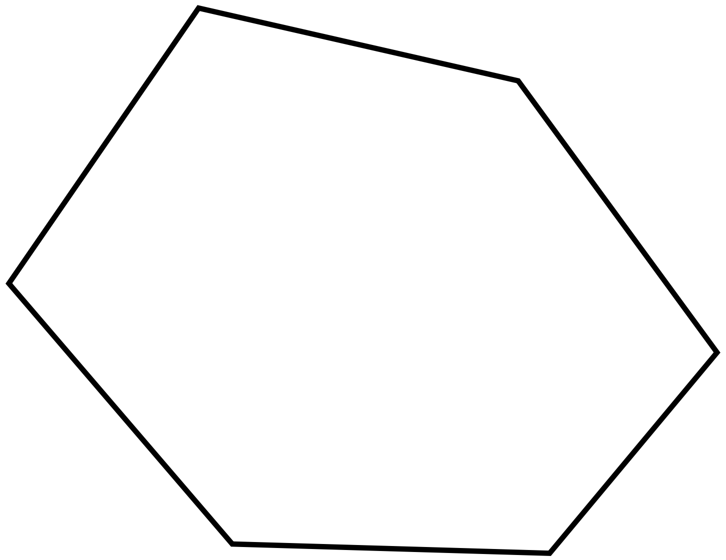


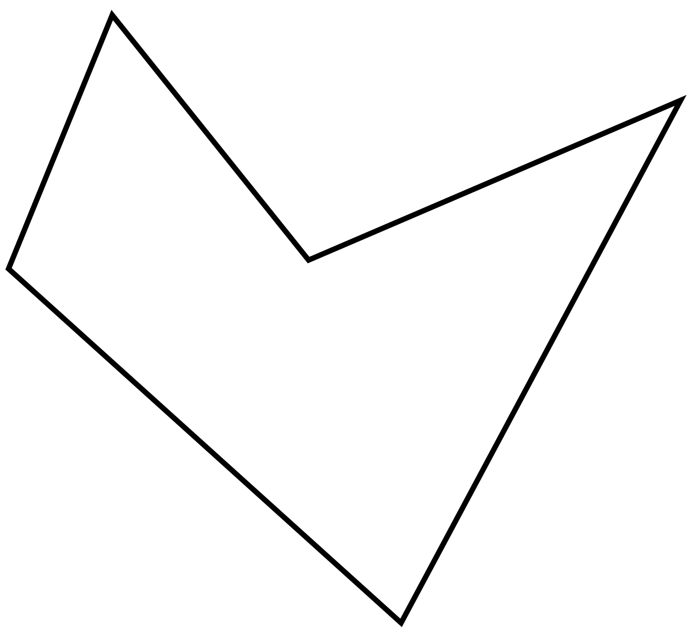
2D



3D

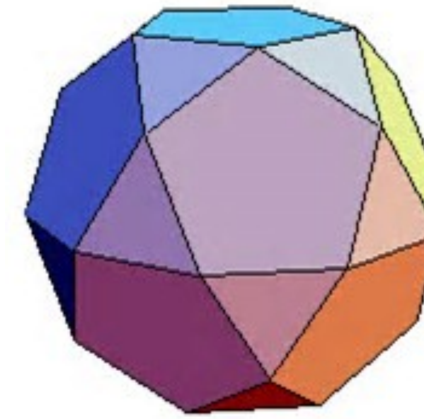
2D

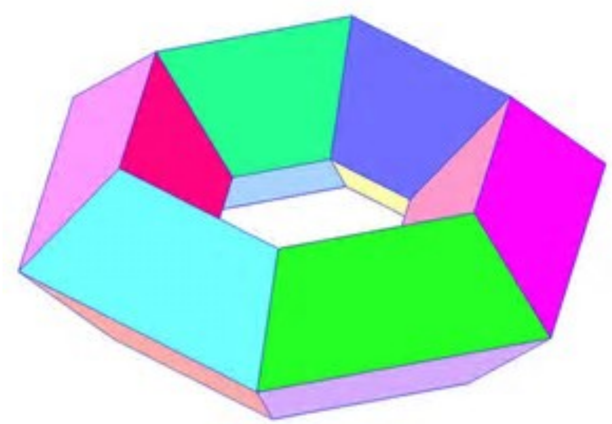




polygon

3D

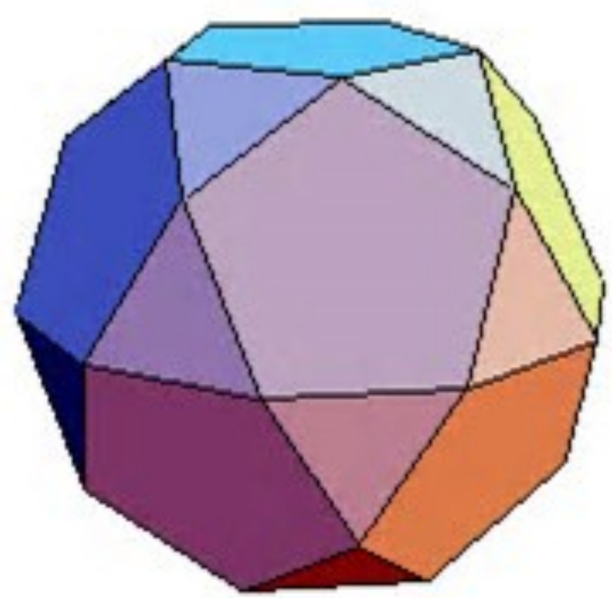


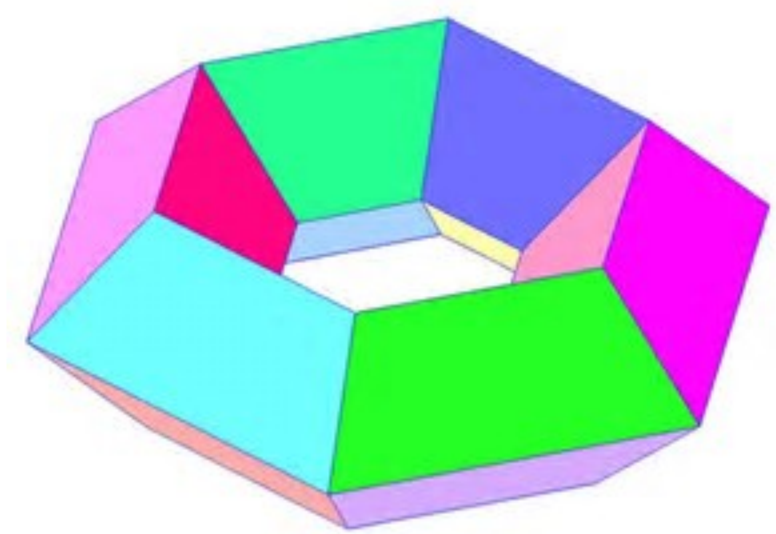


polyhedron

Polyhedron

- region of space whose boundary consists of vertices, edges and faces
- faces intersect properly
- neighborhood of any point on P is homeomorphic to a disk
- surface of P is connected





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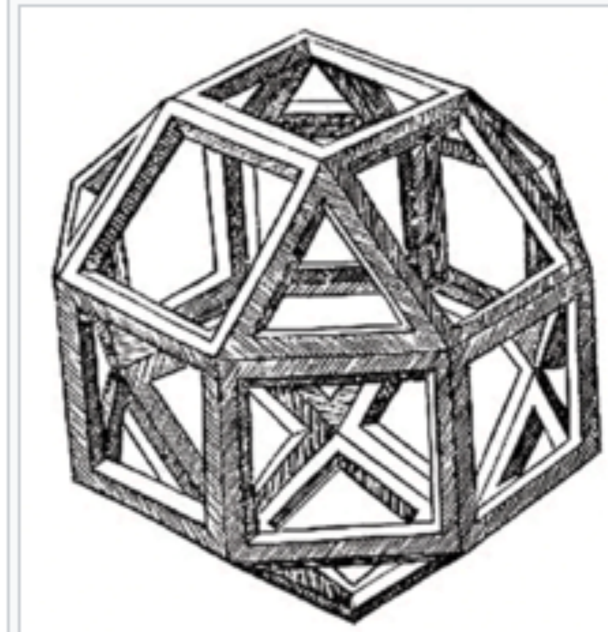
Definition [edit]

Convex polyhedra are well-defined, with several equivalent standard definitions. However, the formal mathematical definition of polyhedra that are not required to be convex has been problematic. Many definitions of "polyhedron" have been given within particular contexts,[1] some more rigorous than others, and there is not universal agreement over which of these to choose. Some of these definitions exclude shapes that have often been counted as polyhedra (such as the self-crossing polyhedra) or include shapes that are often not considered as valid polyhedra (such as solids whose boundaries are not manifolds). As Branko Grünbaum observed,

> "The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others ... at each stage ... the writers failed to define what are the polyhedra".[2]

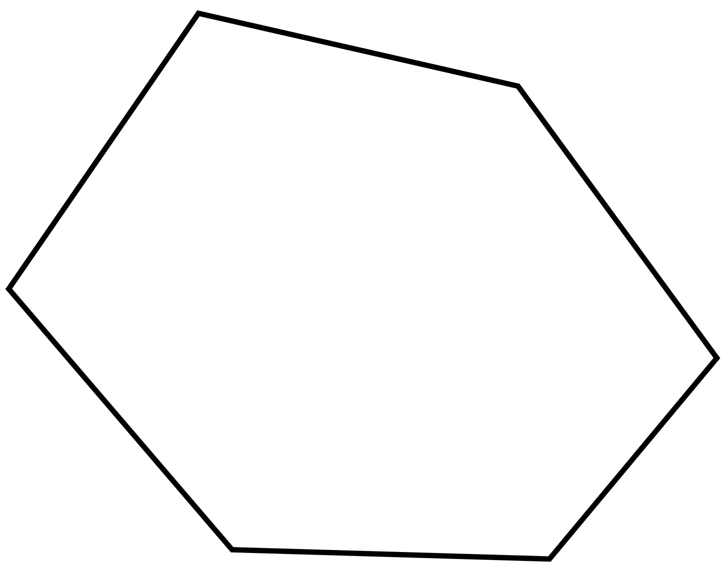


A skeletal polyhedron (specifically, a rhombicuboctahedron) drawn by Leonardo da Vinci to illustrate a book by Luca Pacioli

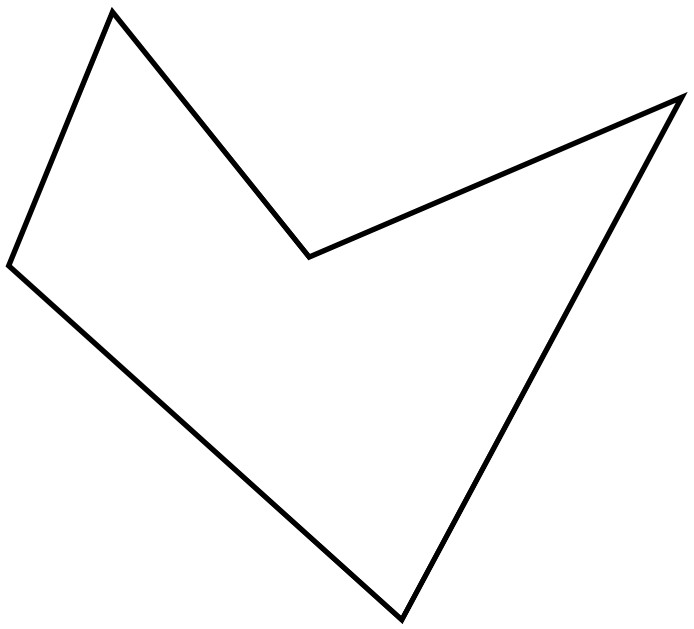
Nevertheless, there is general agreement that a polyhedron is a solid or surface that can be described by its vertices (corner points), edges (line segments connecting certain pairs of vertices), faces (two-dimensional polygons), and sometimes by its three-dimensional interior volume. One can distinguish among these different definitions according to whether they describe the polyhedron as a solid, whether they describe it as a surface, or whether they describe it more abstractly based on its incidence geometry.

Convexity

A polygon P is **convex** if for any p, q in P, the segment pq lies entirely in P.



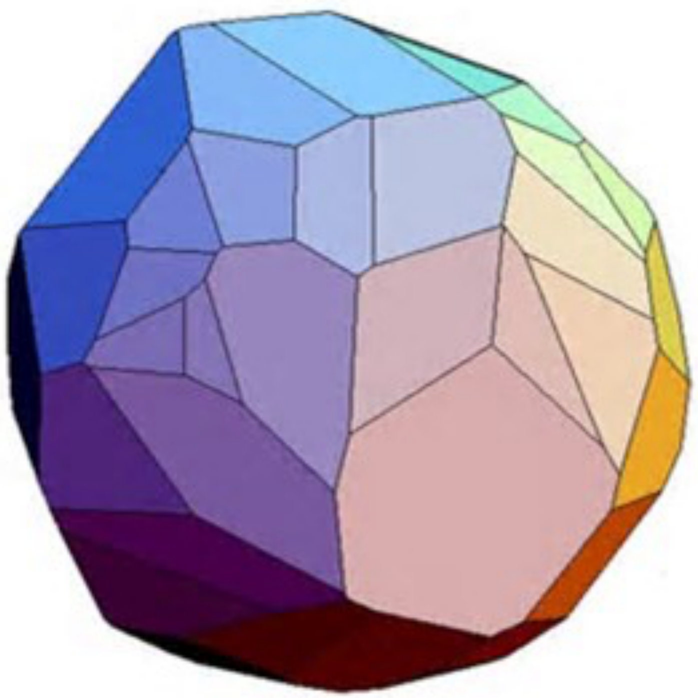
convex



non-convex

Convexity

A polyhedron P is **convex** if for any p, q in P, the segment pq lies entirely in P.



convex

non-convex

convex polyhedron : polytop

digression start

Regular polygons in 2D

- A regular polygon has equal sides and angles

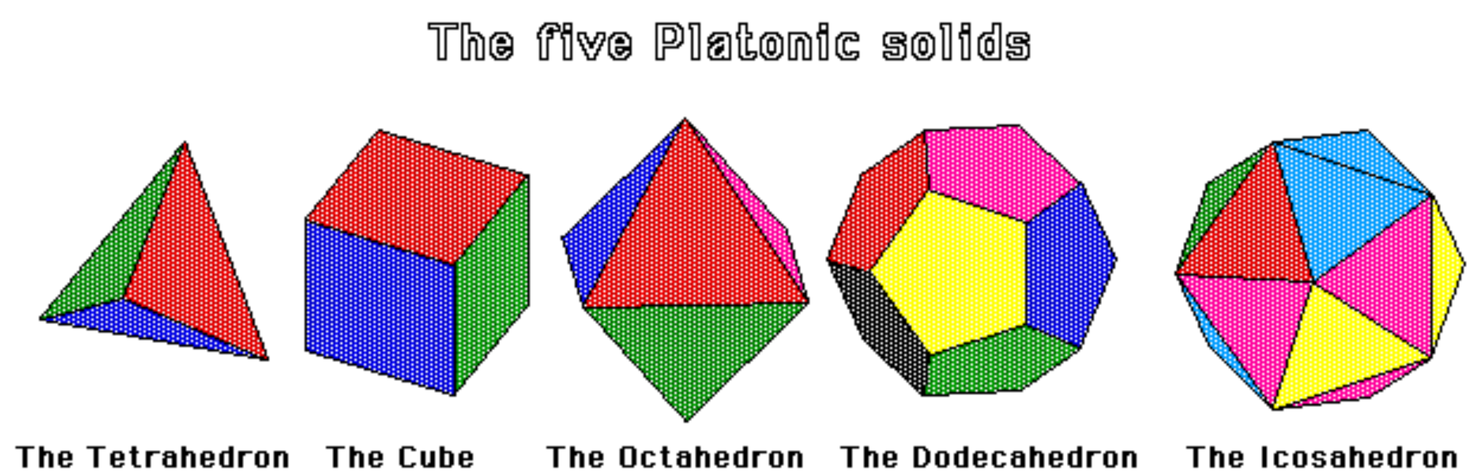
Set of convex regular n-gons

Regular polygons

Regular polytops in 3D

- Regular polytop:
  - faces are congruent regular polygons
  - the number of faces incident to each vertex is the same (and equal angles)

Surprisingly, there exist only 5 regular polytops



The five regular solids discovered by the Ancient Greek mathematicians are:

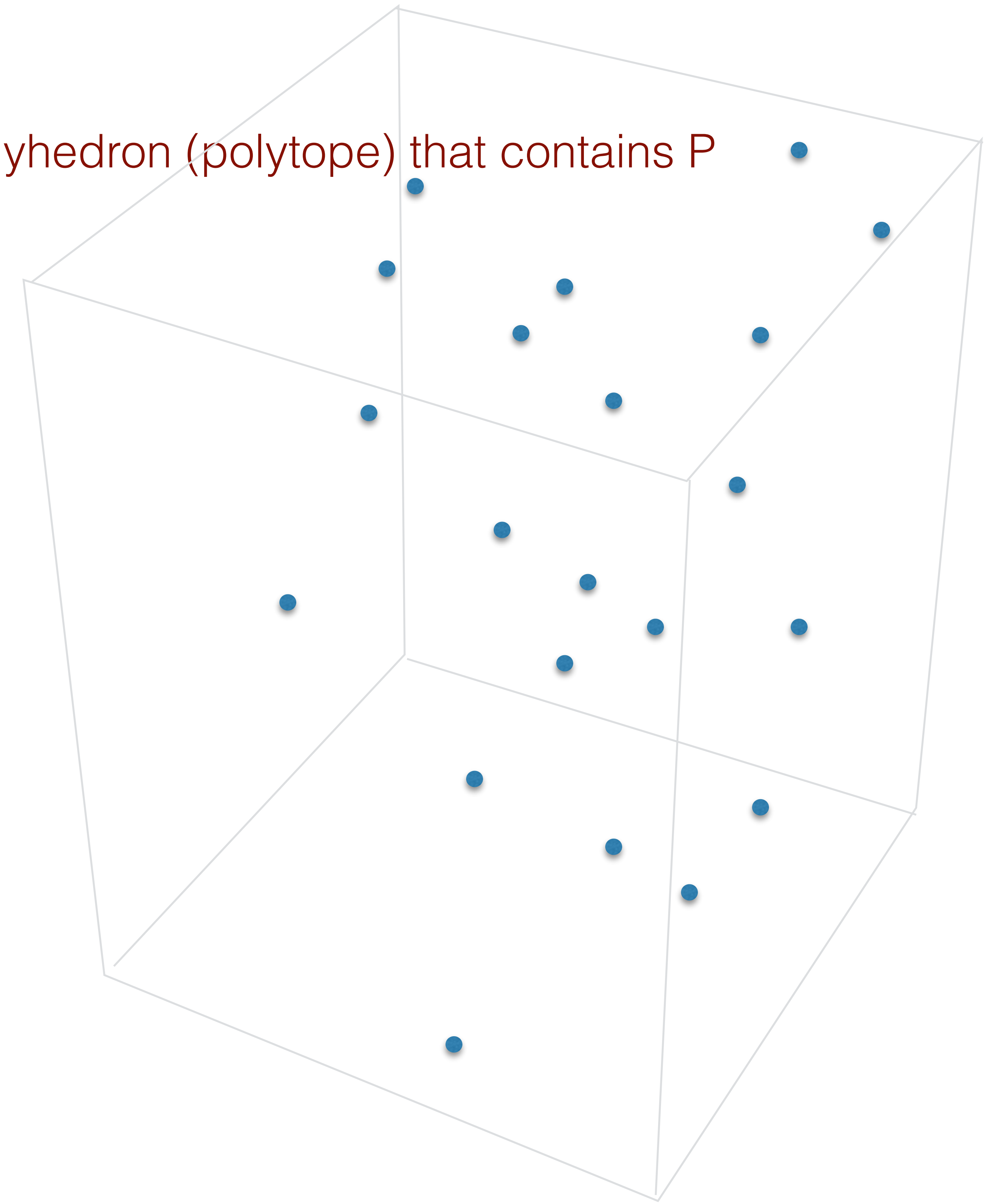
The **Tetrahedron**:	4 vertices	6 edges	4 faces	each with 3 sides
The **Cube**:	8 vertices	12 edges	6 faces	each with 4 sides
The **Octahedron**:	6 vertices	12 edges	8 faces	each with 3 sides
The **Dodecahedron**:	20 vertices	30 edges	12 faces	each with 5 sides
The **Icosahedron**:	12 vertices	30 edges	20 faces	each with 3 sides

The solids are regular because the same number of sides meet at the same angles at each vertex and identical polygons meet at the same angles at each edge.
These five are the only possible regular polyhedra.

digression end

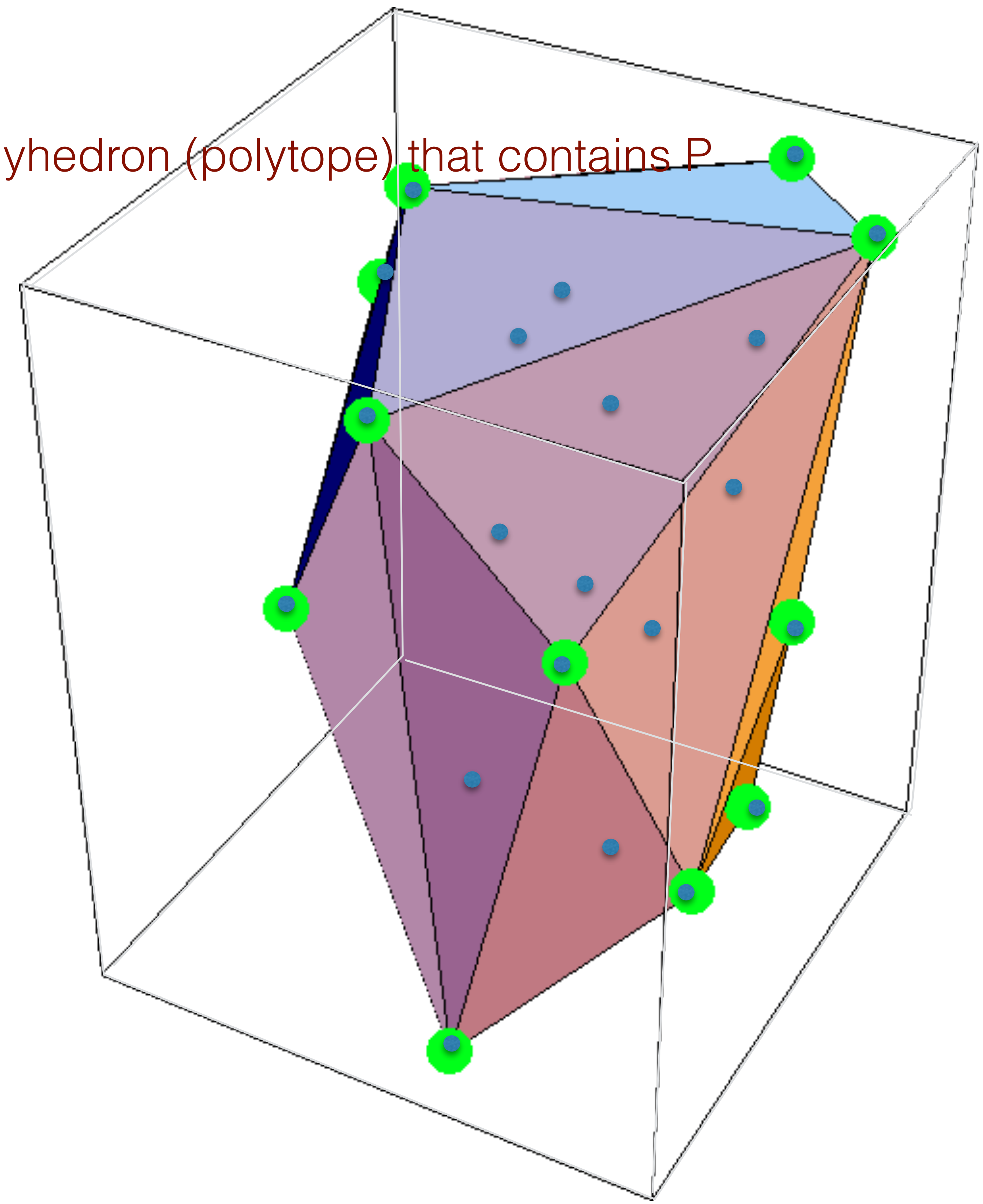
Convex Hulls in 3D

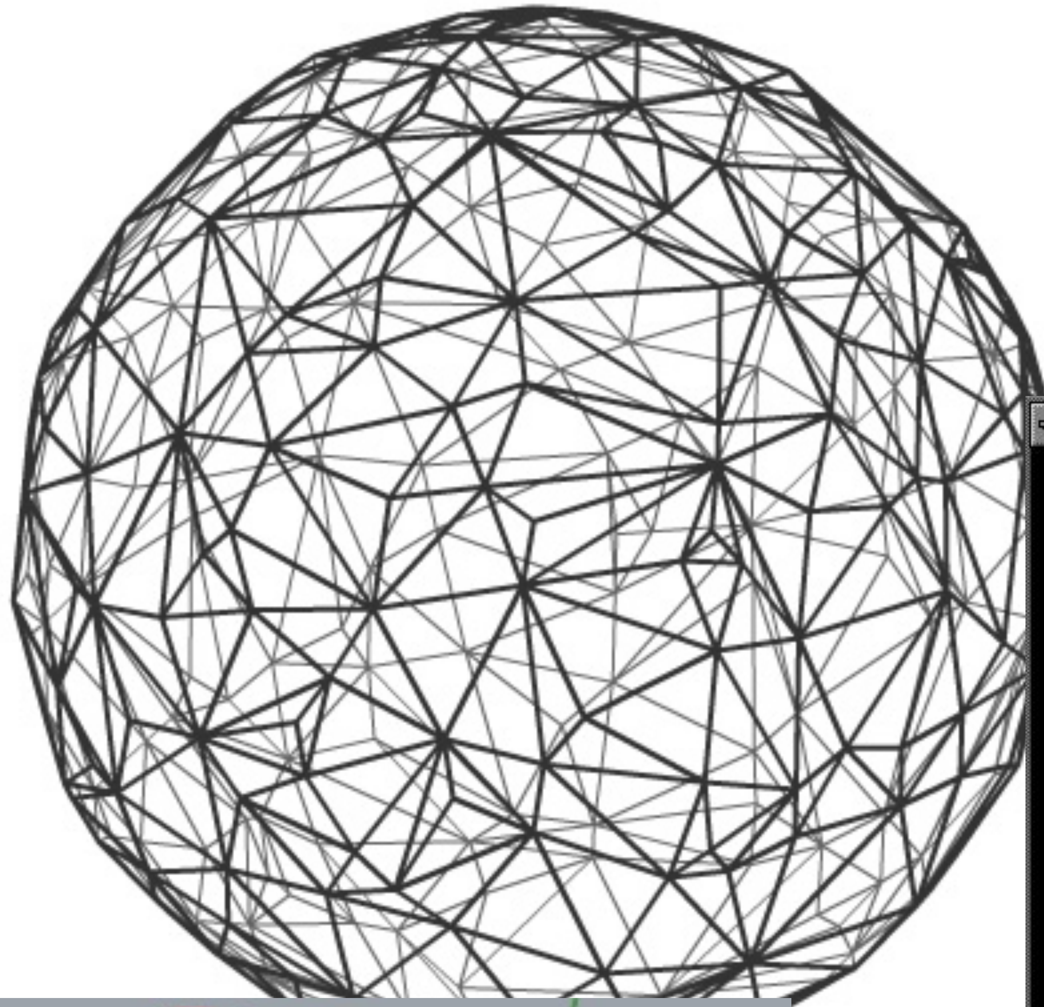
3D convex hull = smallest convex polyhedron (polytope) that contains P

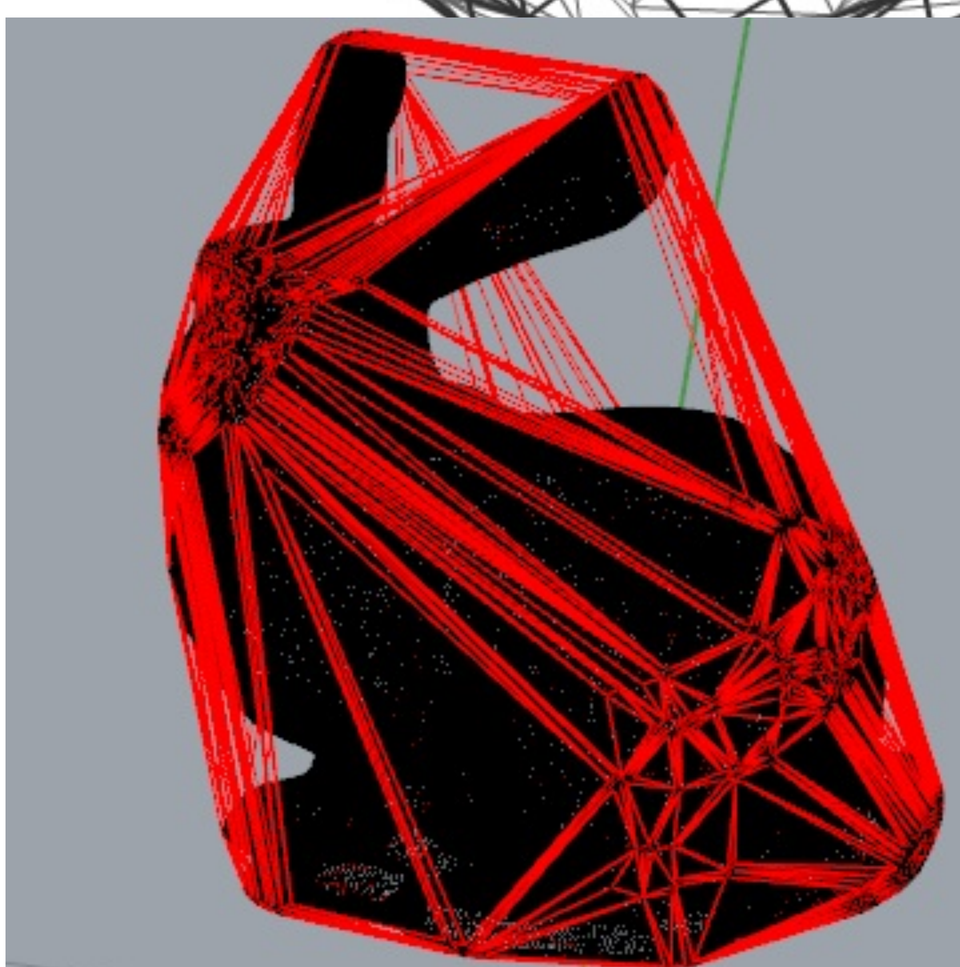


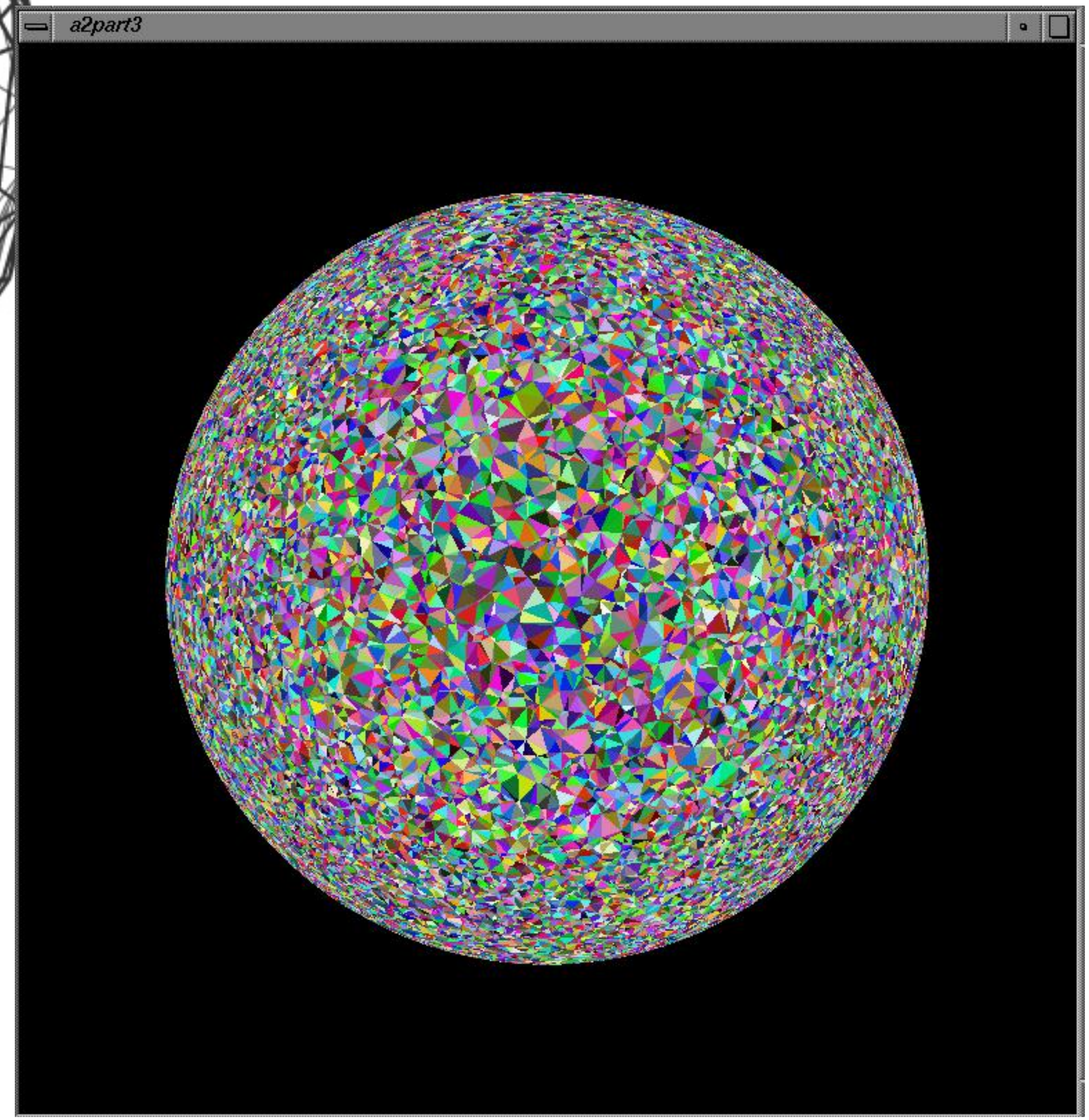
Convex Hulls in 3D

3D convex hull = smallest convex polyhedron (polytope) that contains P



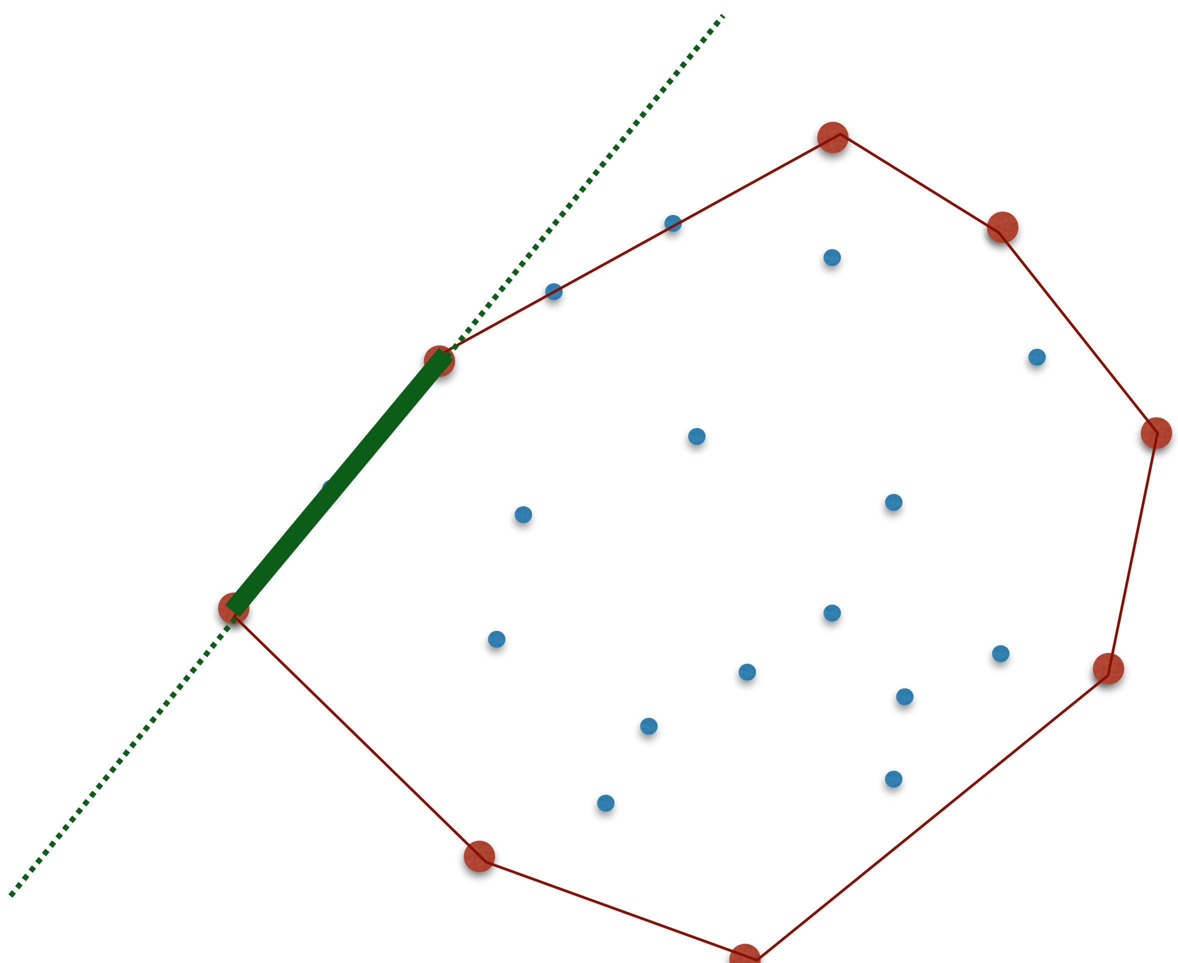





a2part3

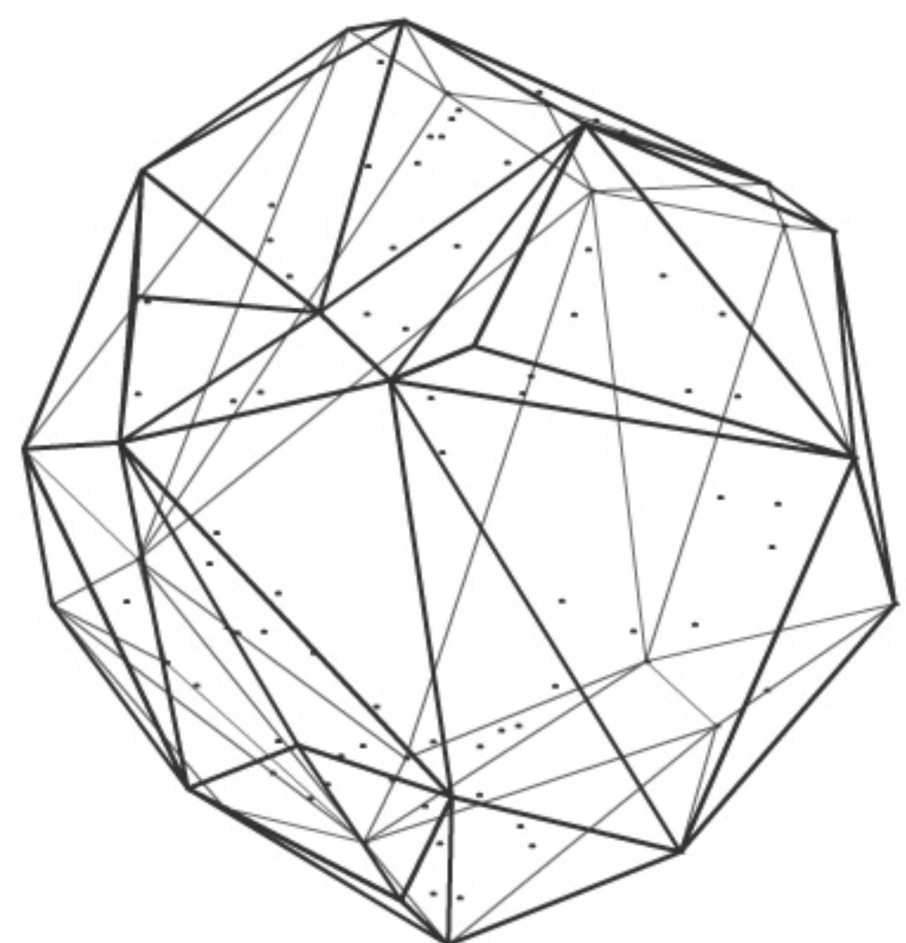
Properties of **2d** hull

- 2d hull consists of all extreme edges and vertices
- All internal angles are < 180
- Walking counterclockwise—> left turns
- Points on hull are sorted in radial order wrt a point inside

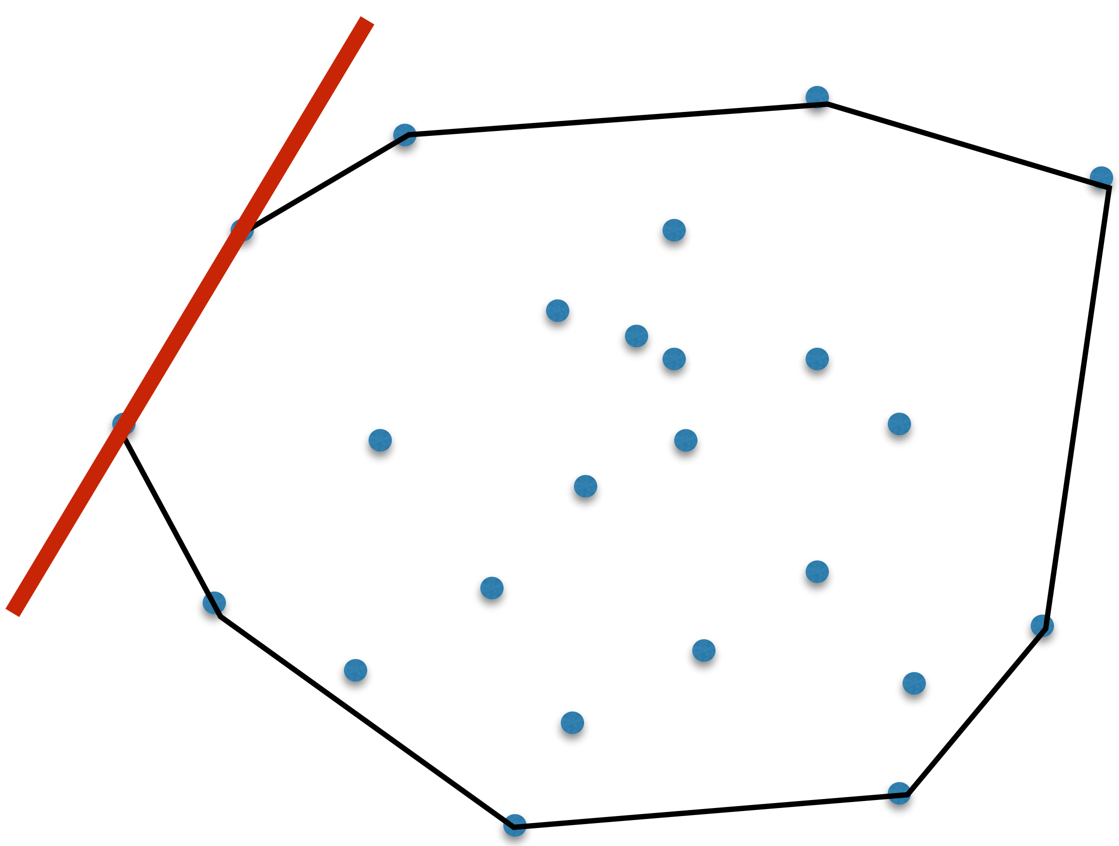


Properties of **3d** hull

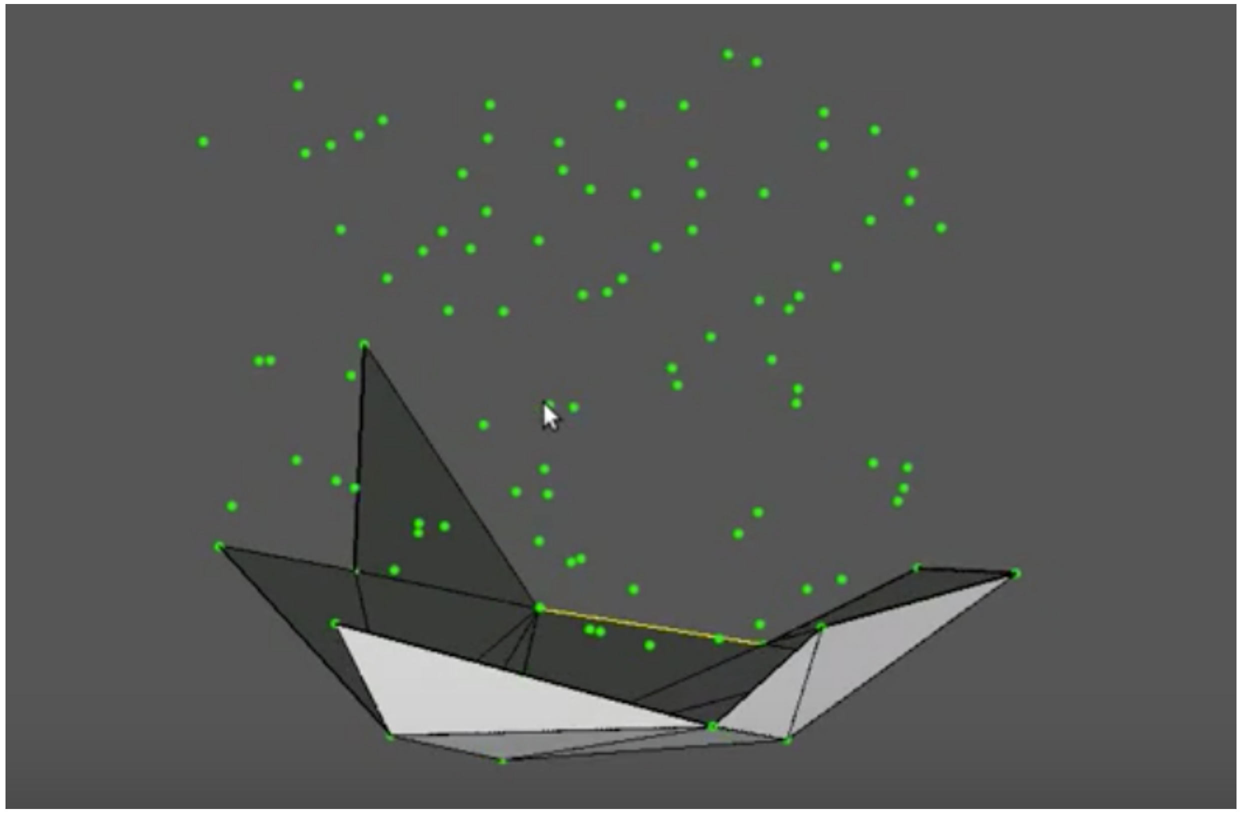
- 3d hull consists of all faces, edges and vertices
- All internal angles between faces are < 180
- ~~Walking counterclockwise—> left turns~~
- ~~Points on CH are sorted in radial order wrt a point inside~~



Faces, edges, vertices on the hull are **extreme**.



2D



3D

Computing the Hull

2D		3D
Naive	$O(n^3)$	
Gift wrapping	$O(nh)$	
Graham scan	$O(n \lg n)$	does not extend..
Quickhull	$O(n \lg n)$, $O(n^2)$	
Incremental	$O(n \lg n)$	
Divide-and-conquer	$O(n \lg n)$	

Naive 3d hull

3d hull: Naive algorithm

Algorithm idea:

- For every triplet of points (pi,pj,pk):
  - check if plane defined by it is extreme
  - if it is, add it to the list of CH faces

- Sketch how to determine if a triplet is extreme and analyze it

  is_extreme(point3d a, point3d b, point3d c, vector<point3d> P)

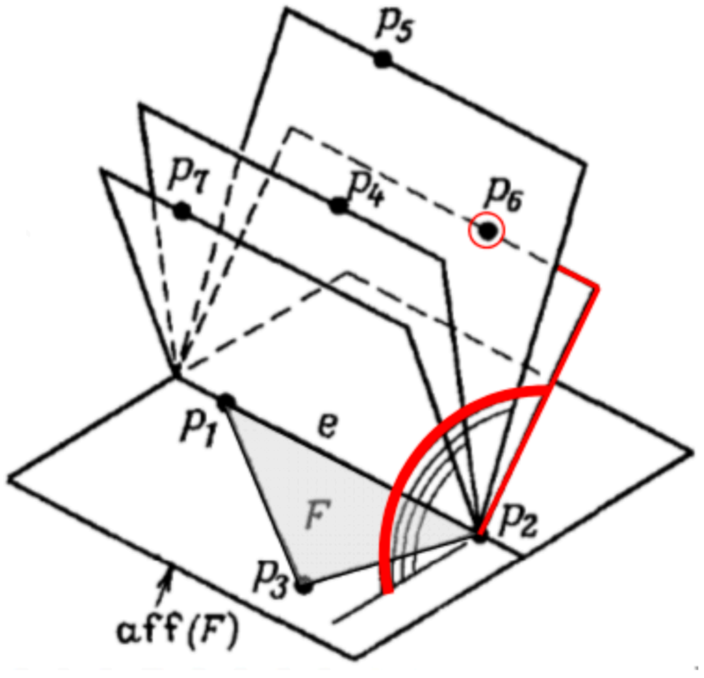
Gift wrapping

3d hull: Gift wrapping

Algorithm

- find a face guaranteed to be on the CH
- REPEAT
    - find an edge e of a face f that's on the CH, and such that the face on the other side of e has not been found.
    - for all remaining points pi, find the angle of (e,pi) with f
    - find point pi with the minimal angle; add face (e,pi) to CH

- Analysis: O(n x F), where F is the number of faces on CH



Mathematical background [edit]

When the two intersecting planes are described in terms of Cartesian coordinates by the two equations

$$a_1 x + b_1 y + c_1 z + d_1 = 0$$
$$a_2 x + b_2 y + c_2 z + d_2 = 0$$

the dihedral angle, φ between them is given by:

$$\cos\varphi = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

An alternative method is to calculate the angle between the vectors, $\mathbf{n}_A$ and $\mathbf{n}_B$, which are normal to the planes.

$$\cos\varphi = \frac{|\mathbf{n}_A \cdot \mathbf{n}_B|}{|\mathbf{n}_A||\mathbf{n}_B|}$$

where $\mathbf{n}_A \cdot \mathbf{n}_B$ is the dot product of the vectors and $|\mathbf{n}_A|\,|\mathbf{n}_B|$ is the product of their lengths. [1]

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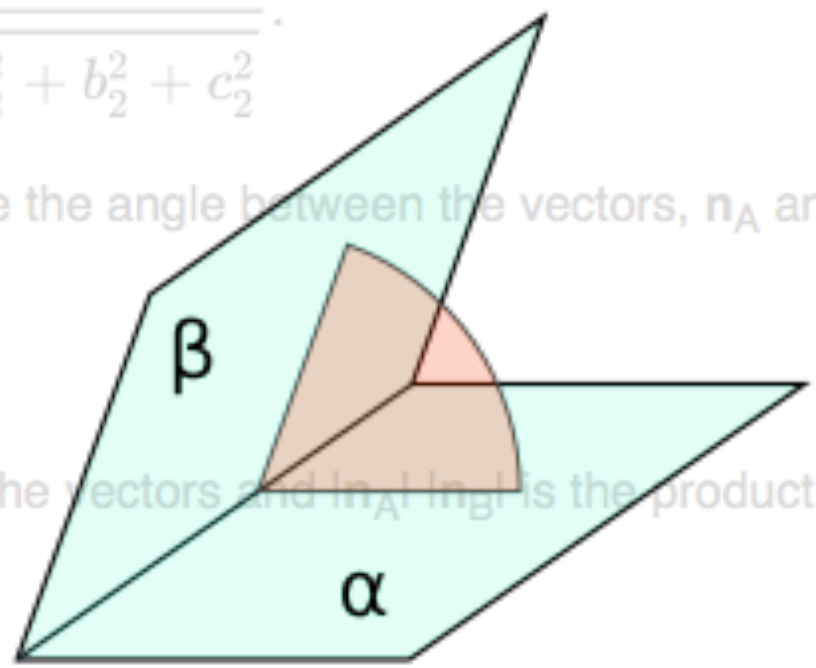
A **dihedral angle** is the angle between two intersecting planes.



Angle between two planes (α, β, green) in a third plane (pink) which cuts the line of intersection at right angles

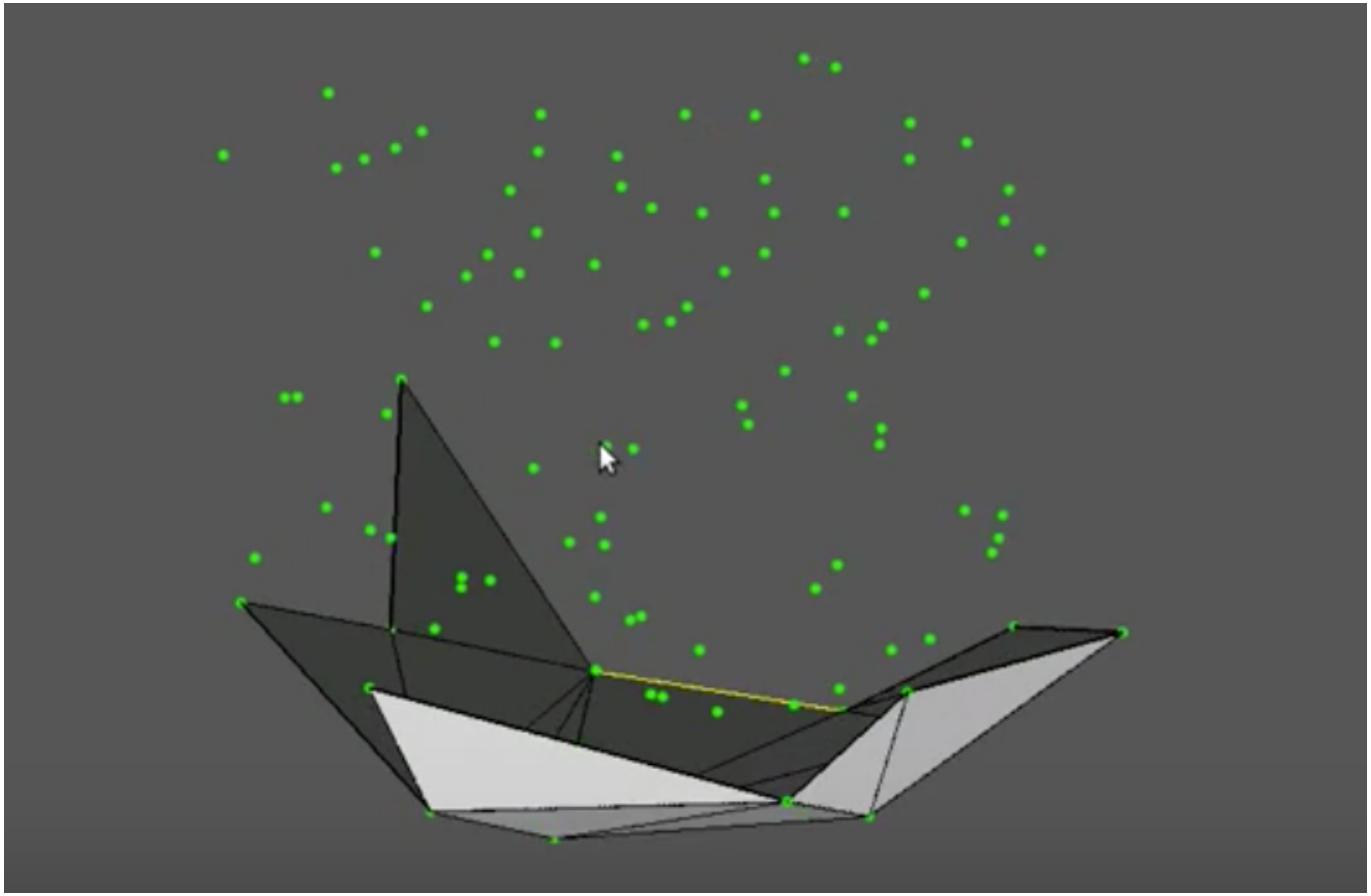
3d hull: Gift wrapping

Algorithm

- find a face guaranteed to be on the CH
- REPEAT
    - find an edge e of a face f that's on the CH, and such that the face on the other side of e has not been found.
    - for all remaining points pi, find the angle of (e,pi) with f
    - find point pi with the minimal angle; add face (e,pi) to CH

- To think
    - finding first face?
    - How to keep track of the hull? we'll need to store the connectivity (what faces are adjacent, for an edge which faces its adjacent to, etc)
    - How to keep track of the boundary of the hull (the edges that have only one face discovered)?

Gift wrapping in 3D



- YouTube
    - Video of CH in 3D (by Lucas Benevides)

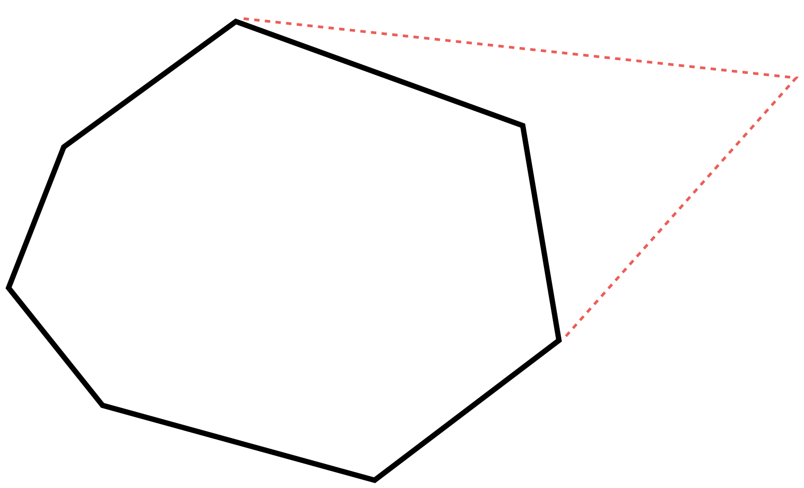
From 2D to 3D

2D		3D
Naive	$O(n^3)$	$O(n^4)$
Gift wrapping	$O(nh)$	$O(n \times F)$
Graham scan	$O(n \lg n)$	does not extend to 3D
Quickhull	$O(n \lg n)$, $O(n^2)$	
Incremental	$O(n \lg n)$	
Divide-and-conquer	$O(n \lg n)$	

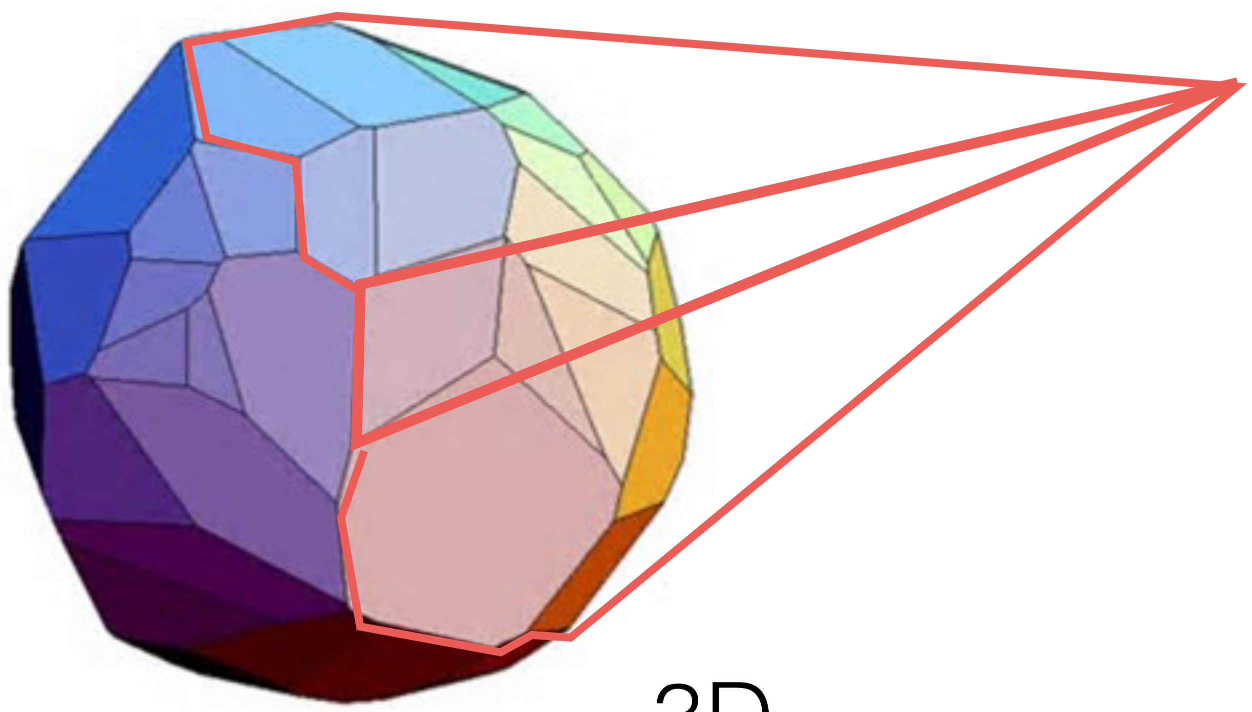
Incremental 3D hull

Incremental 3d hull

- sort points lexicographically
- initialize hull H = {p1,p2,p3}
- for i= 4 to n
  - //invariant: H represents the CH of $p_1..p_{i-1}$
  - add p_i to H and update H to represent the CH of $p_1..p_i$

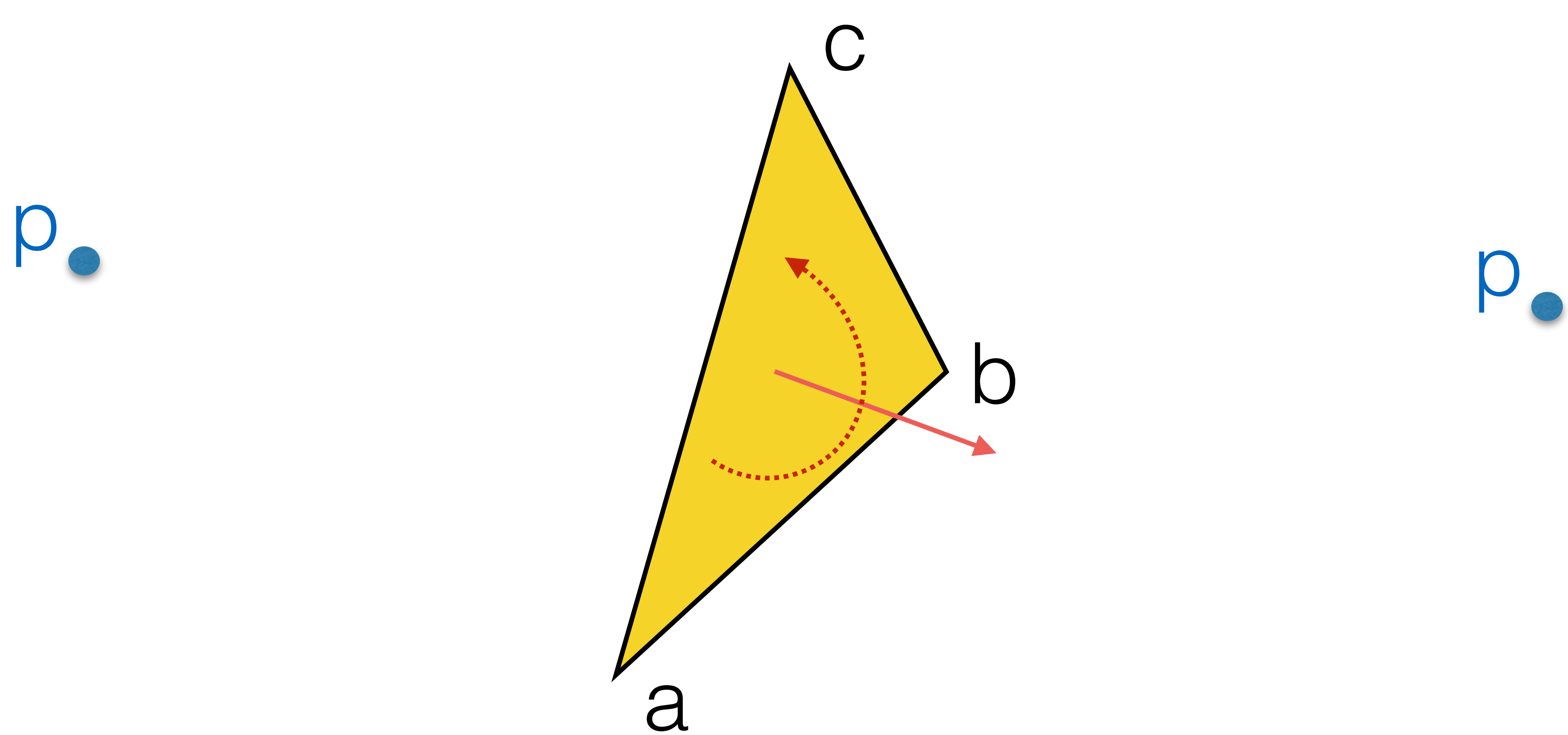


2D



3D

Terminology: Point in front/behind face



ps is left of (behind) abc

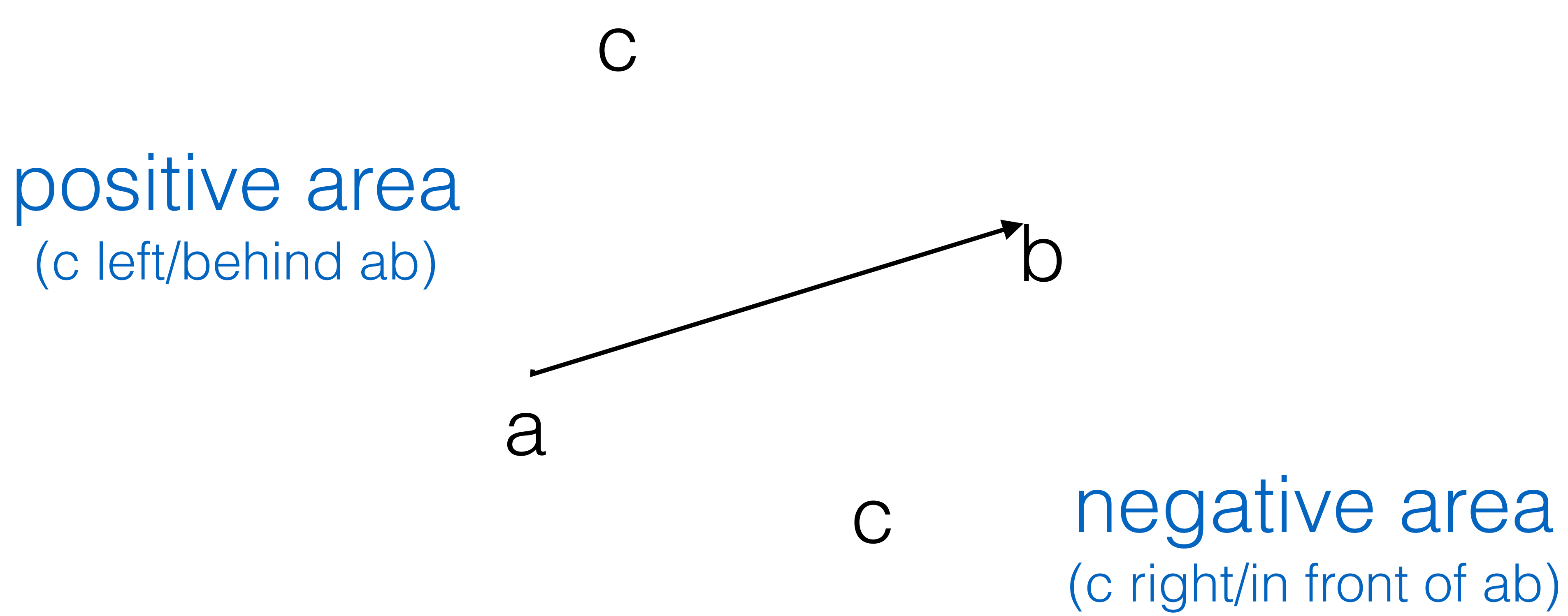
abc not visible from p

p is right of (in front) abc

abc visible from p

2D

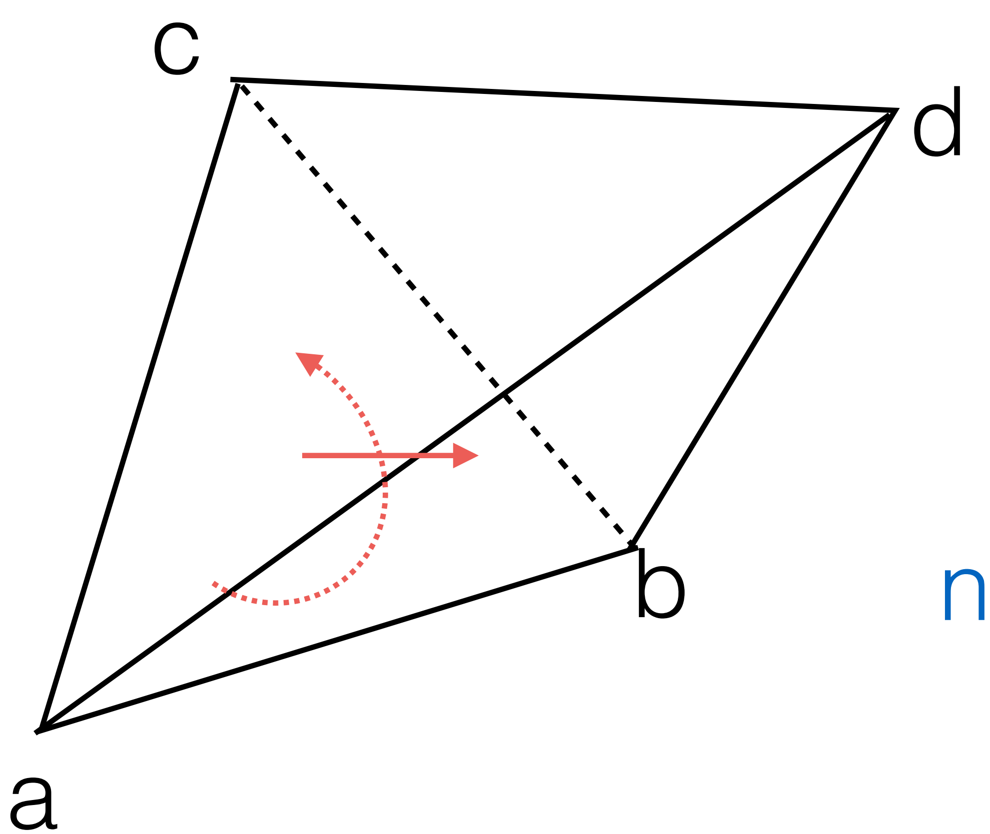
$$2\ \text{signedArea}(a,b,c) = \det \begin{vmatrix} a.x & a.y & 1 \\ b.x & b.y & 1 \\ c.x & c.y & 1 \end{vmatrix}$$



3D

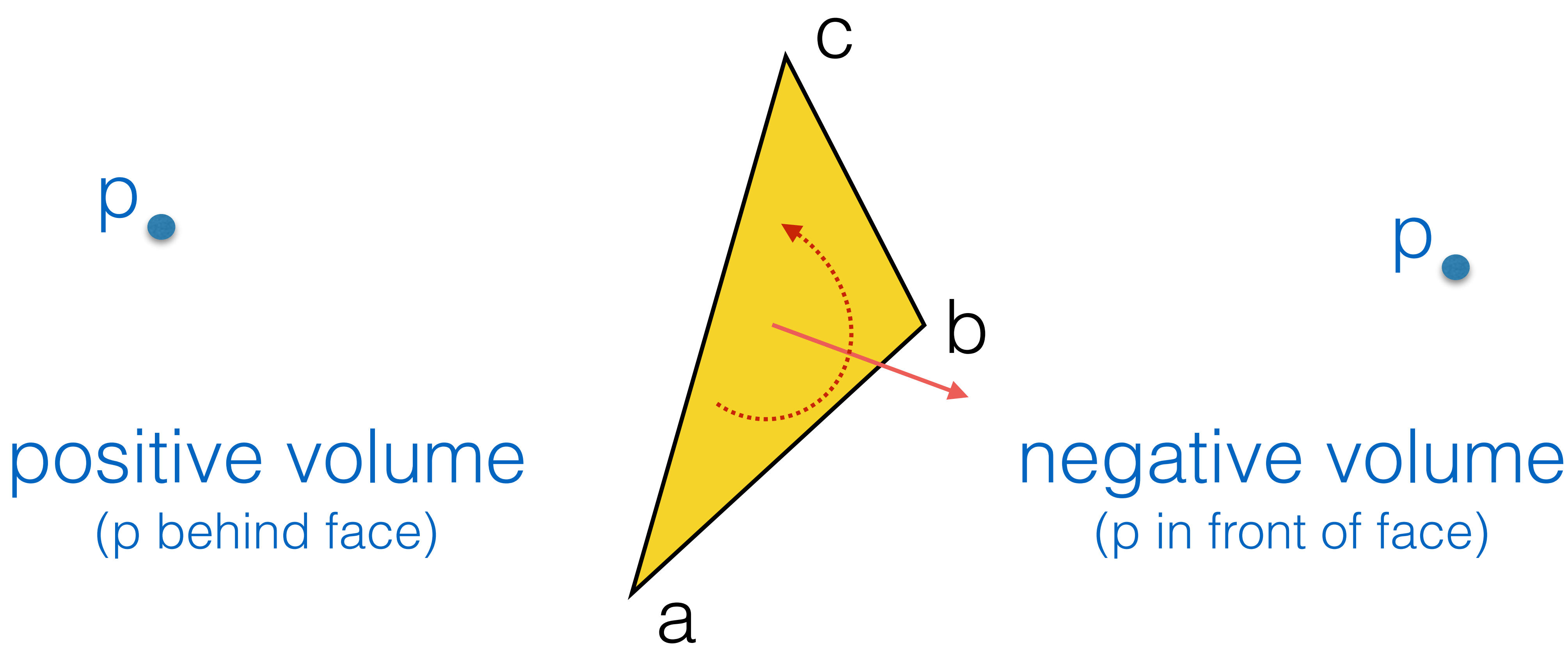
$$6 \text{ signedVolume}(a,b,c,d) = \det \begin{vmatrix} a.x & a.y & a.z & 1 \\ b.x & b.y & b.z & 1 \\ c.x & c.y & c.z & 1 \\ d.x & d.y & d.z & 1 \end{vmatrix}$$

positive volume
(p behind face)



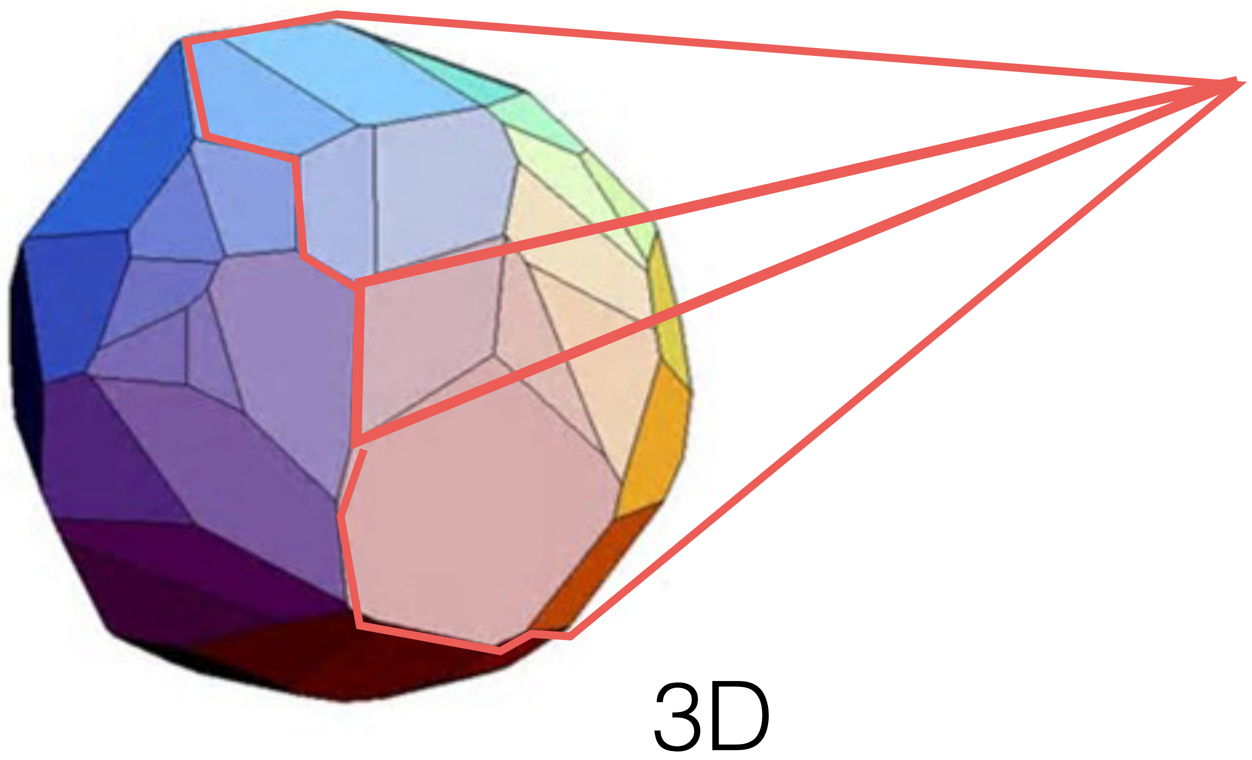
negative volume
(d in front of face)

- Assume all faces oriented counterclockwise so that their normals determined by the right-hand rule point towards the **outside** of P.



is_visible(a,b,c,p):   return signedVolume(a,b,c,p) < 0

Incremental 3d hull



The visible faces are precisely those that need to be discarded

The edges on the boundary of the visible region are the basis of the cone

Incremental 3d hull

- sort points lexicographically
- initialize H for p1, p2, p3, p4
- for i = 5 to n do:
    - for each face f of H do: check if f is visible from pi
    - if no faces are visible
        - discard pi (pi must be inside H)
    - else
        - find border edge of all visible faces
        - for each border edge e construct a face (e,pi) and add to H
        - for each visible face f: delete f from H

Incremental 3d hull

- Analysis:
  - We can start at previous vertex pi-1, find its neighboring faces, determine if they are visible, and continue from there. For each face that we determine to be visible, that face will be deleted.
  - 2D: a vertex on the hull is connected to 2 edges. Those edges may be deleted later, and they can be "charged" to the vertex
  - 3D: All faces (e, pi) added at step i are now connected to vertex p_i. The number of faces incident to a vertex pi is not constant. Some or all of these faces may be deleted later.
    - Overall in 3D running time adds up to $O(n^2)$

Divide & conquer 3D hull

3d hull: divide & conquer

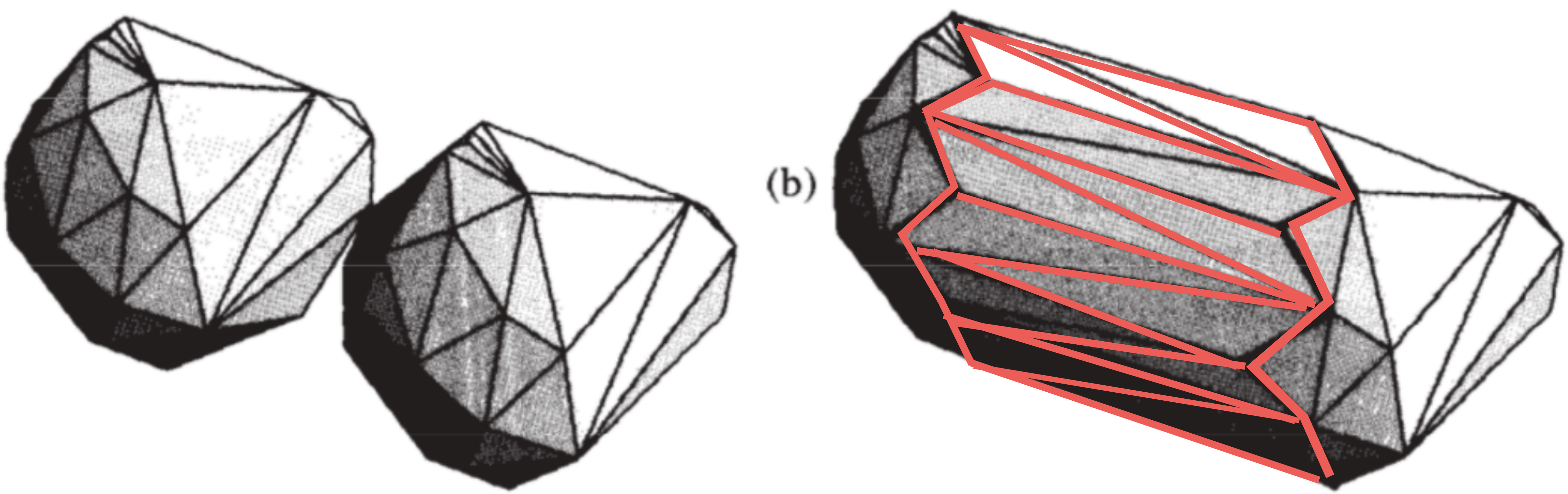
The same idea as 2D algorithm

- divide points in two halves P1 and P2
- recursively find CH(P1) and CH(P2)
- merge

- merge in O(n) time ==> O(n lg n) algorithm

Merge

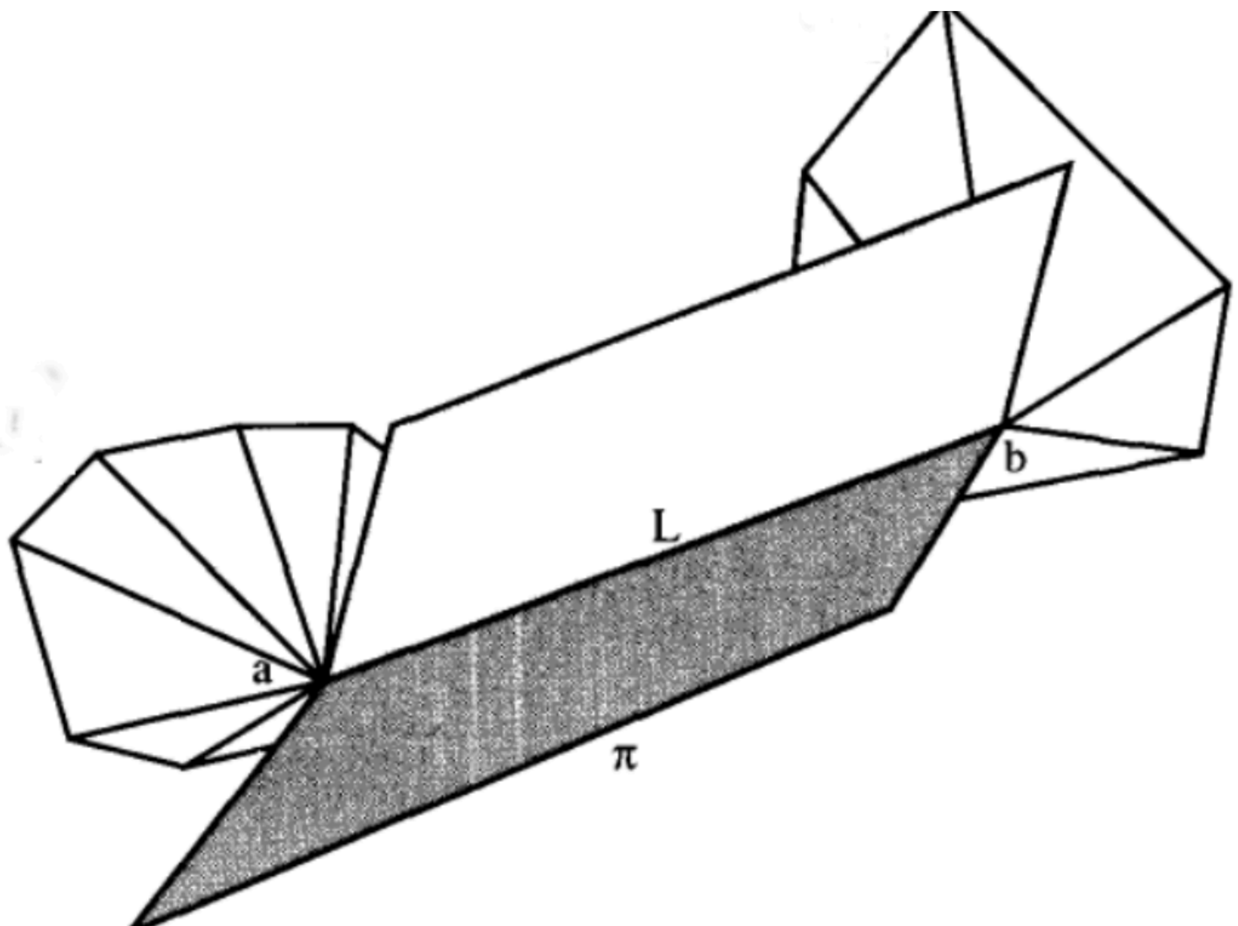
- Idea: Start with the lower tangent, wrap around, find one face at a time.

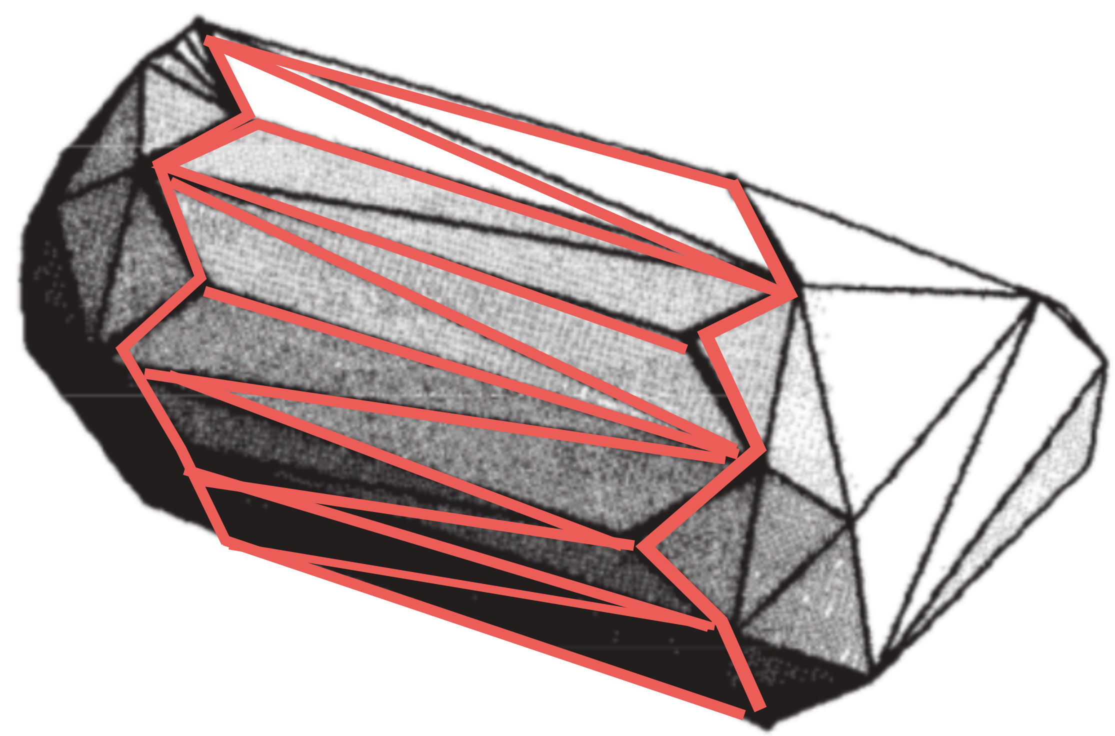


Merged hull: cylinder without end caps

Merge

- Idea: Start with the lower tangent, wrap around, find one face at a time.





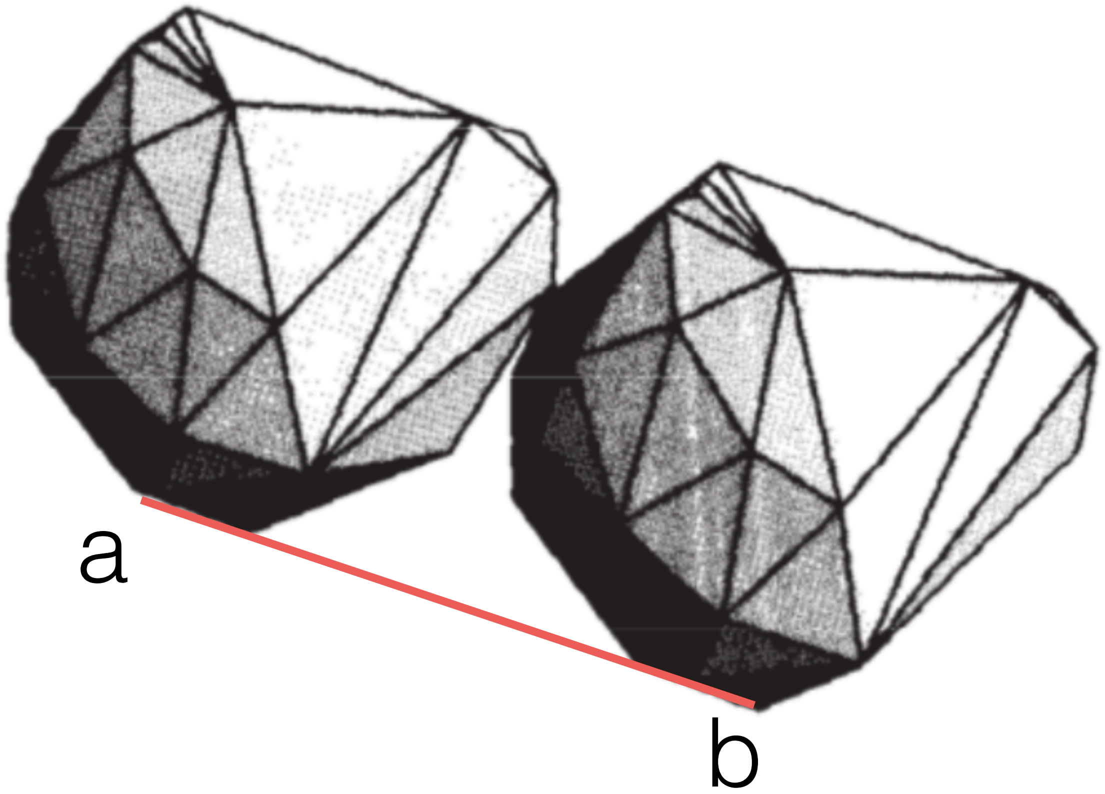
- Let PI be a plane that supports the hull from below

Claim:

- When we rotate PI around ab, the first vertex hit is a vertex adjacent to a or b
- Vertex c has the smallest angle among all neighbors of a,b

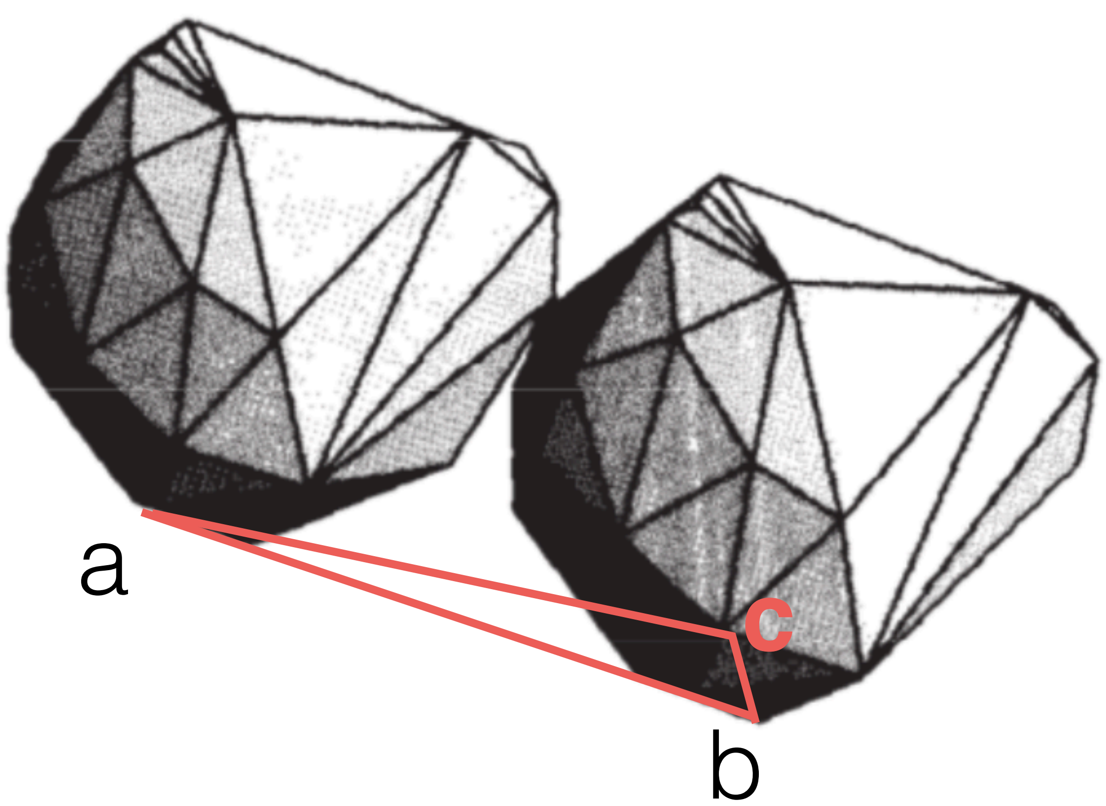
Merge

1. Find a common tangent ab



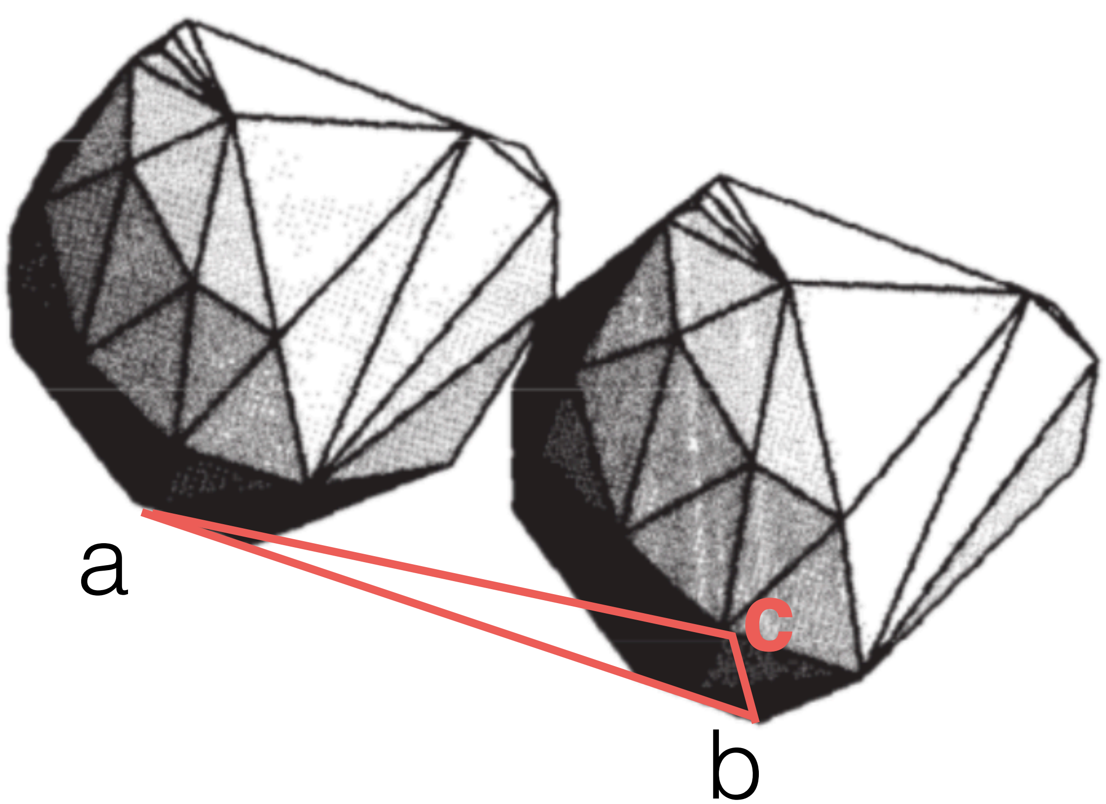
Merge

1. Find a common tangent ab
2. Consider all neighbor vertices of a,b and find the vertex with smallest angle (wrt the plane through ab).



Merge

1. Find a common tangent ab
2. Consider all neighbor vertices of a,b and find the vertex with smallest angle (wrt the plane through ab).
3. Repeat from edge ac.



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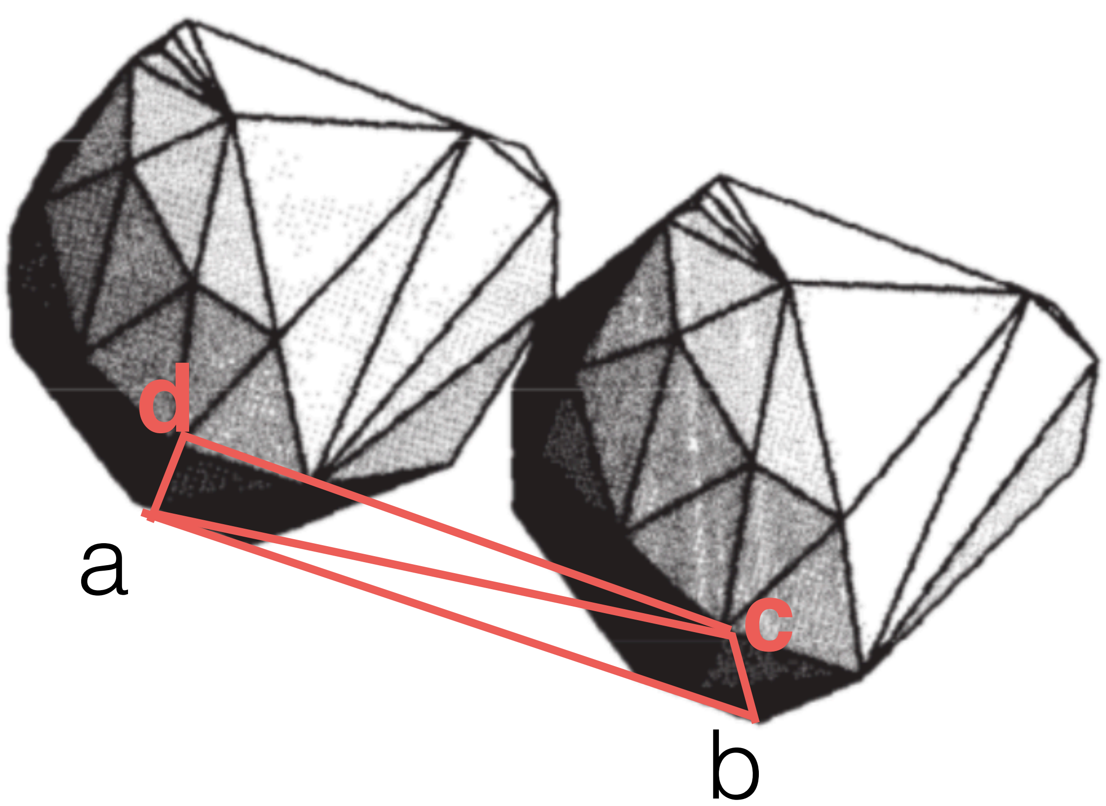

Merge

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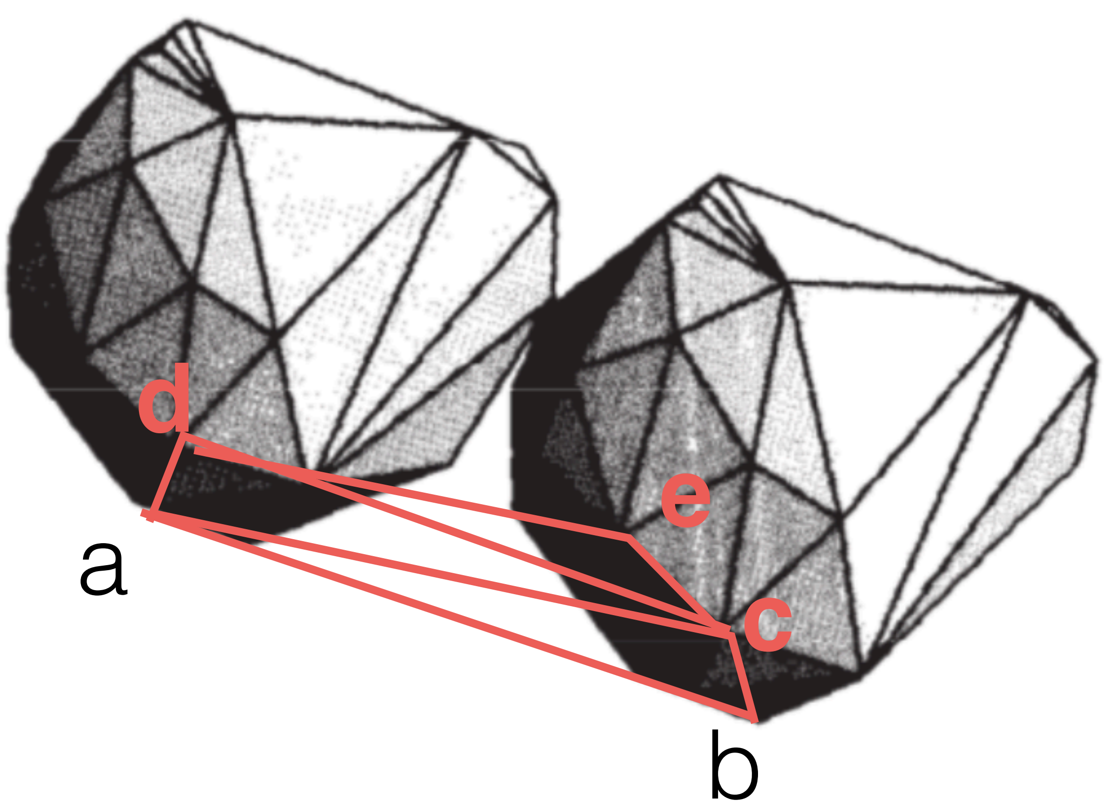

Merge

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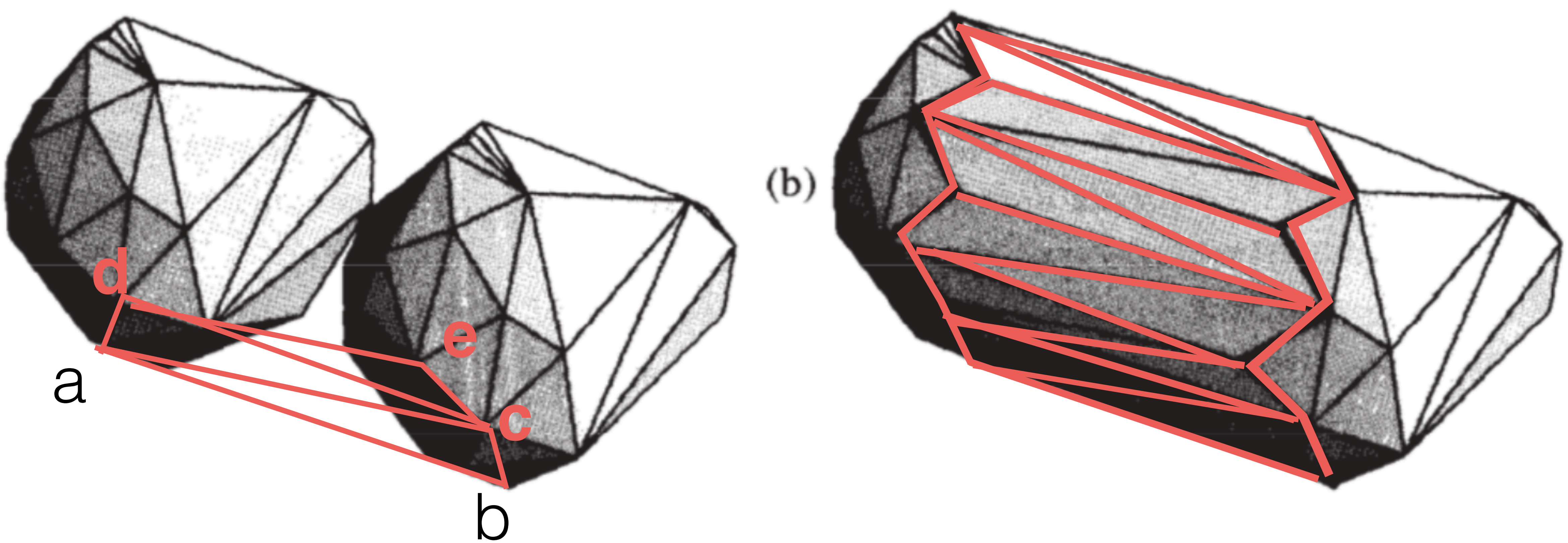
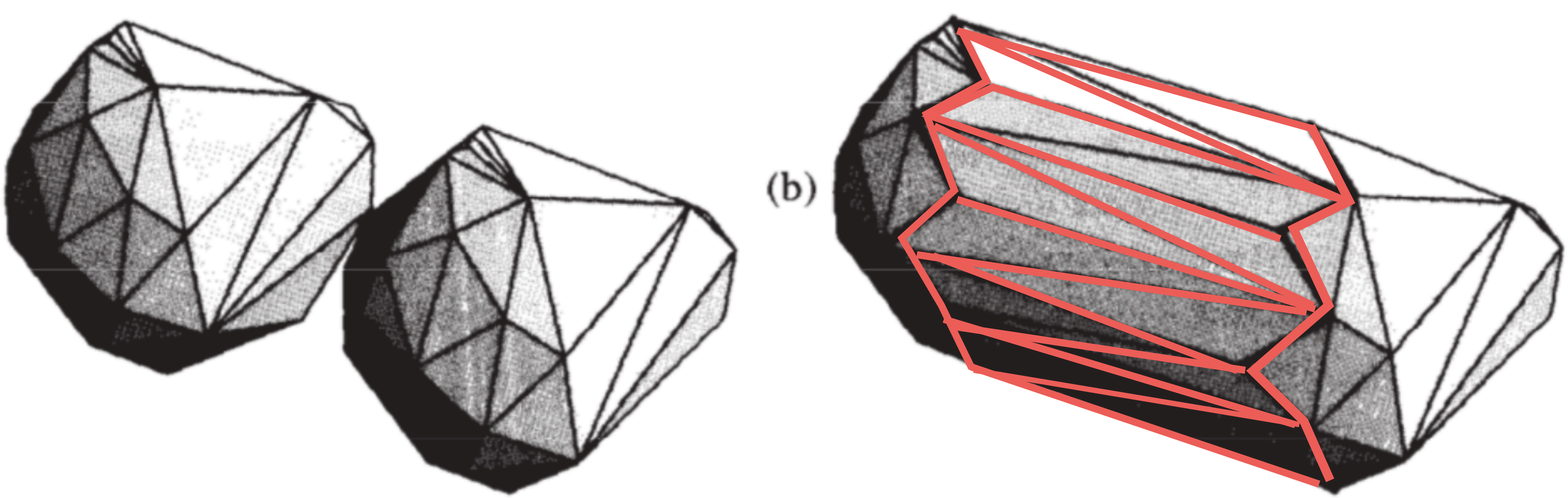
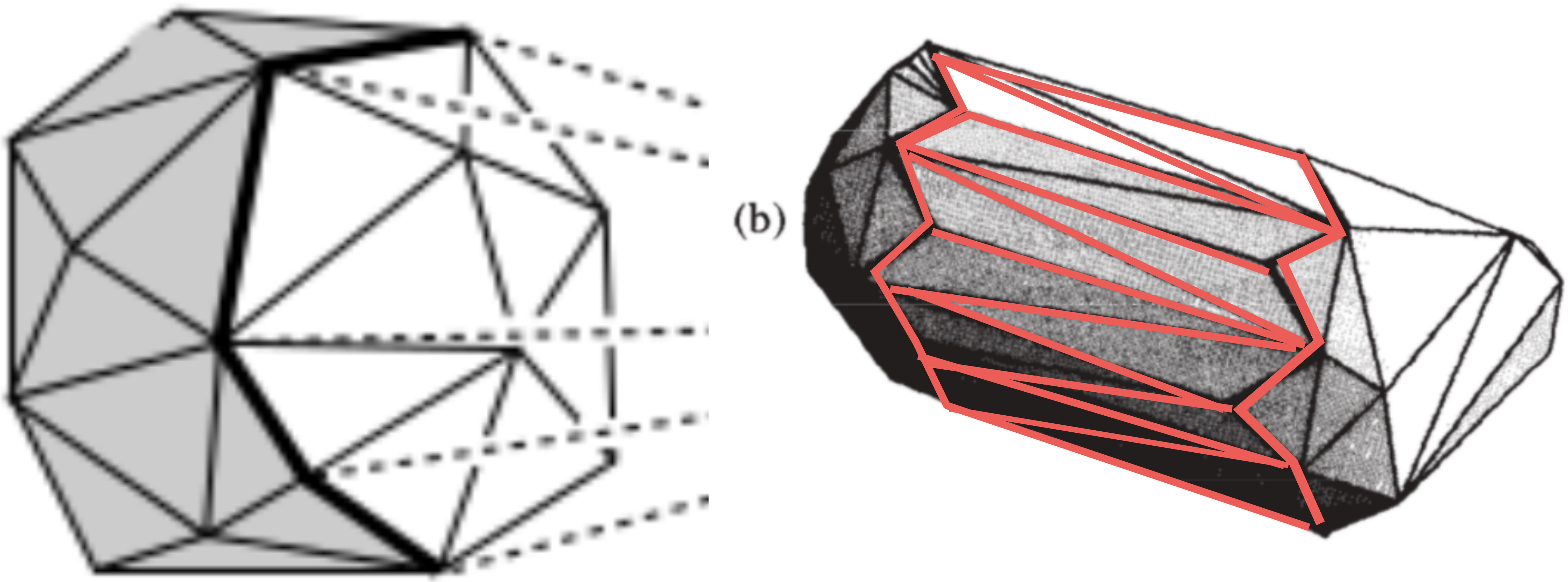


Merge

1. Find a common tangent ab
2. Consider all neighbor vertices of a,b and find the vertex with smallest angle (wrt the plane through ab).
3. Repeat from edge ac.
4. Delete hidden faces



The hidden faces



- Find the edges on the “boundary” of the cylinder
- BFS or DFS faces “towards” the cylinder
- All faces reached are inside

3d hull: Summary

- Theoretically important and elegant
- Of all algorithms that extend to 3D, D&C is the only one that achieves optimal (n lg n)
- Difficult to implement
- The slower algorithms (quickhull, incremental) preferred in practice

3D hull summary

2D		3D
Naive	$O(n^3)$	$O(n^4)$
Gift wrapping	$O(nh)$	$O(n \times F)$
Graham scan	$O(n \lg n)$	does not extend to 3D
Quickhull	$O(n \lg n)$, $O(n^2)$	
Incremental	$O(n \lg n)$	$O(n^2)$
Divide-and-conquer	$O(n \lg n)$	$O(n \lg n)$

Euler's formula

- Euler noticed a remarkable regularity in the number of vertices, edges and faces of a polyhedron (w/o holes).

- **Euler's formula: V - E + F = 2**

- One proof idea:
    - flatten the polygon to a plane
    - prove the formula for a tree
    - prove for any planar graph by induction on E