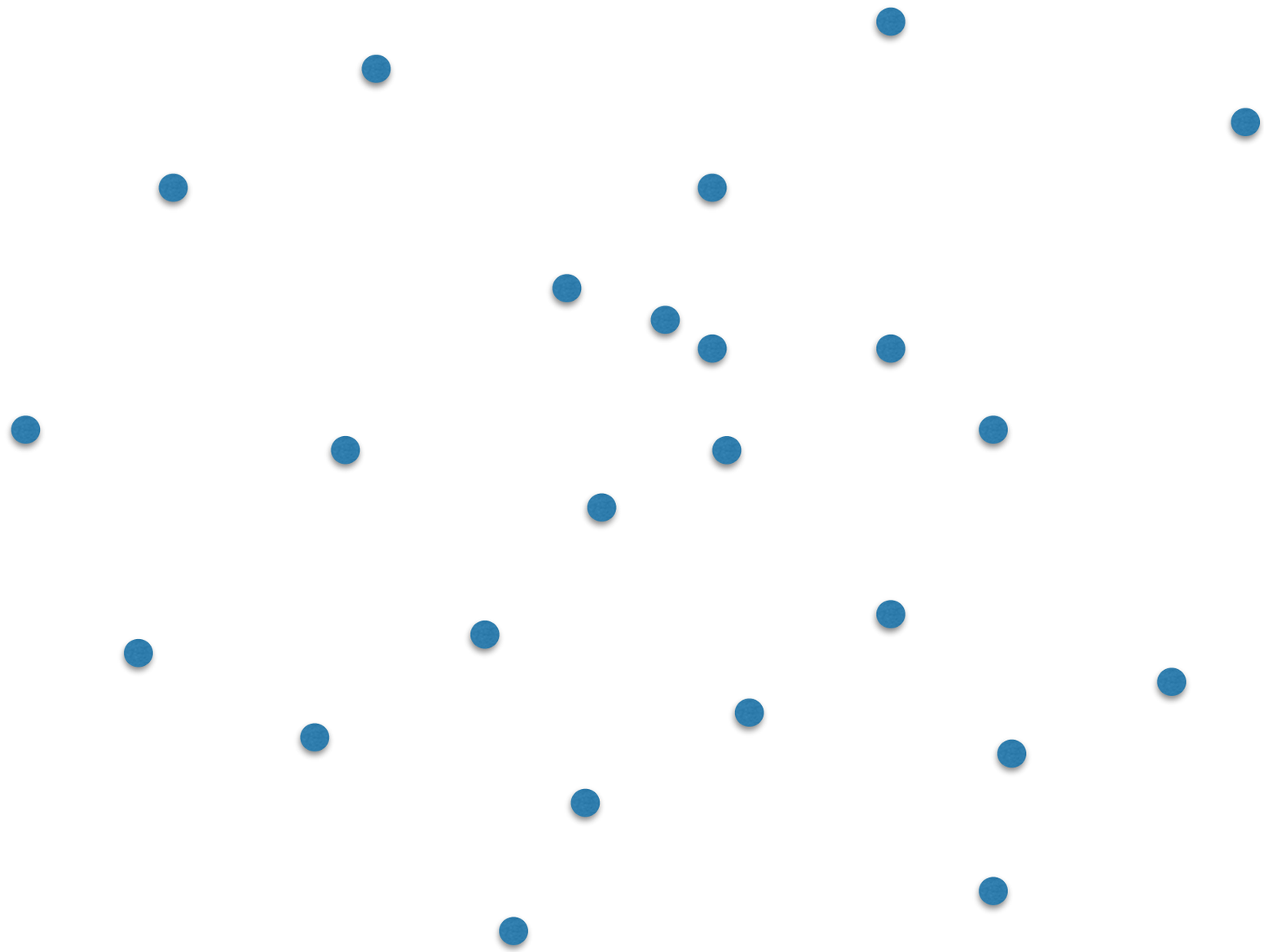
Finding the closest pair

Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

Given an array of points in 2D, find the closest pair.



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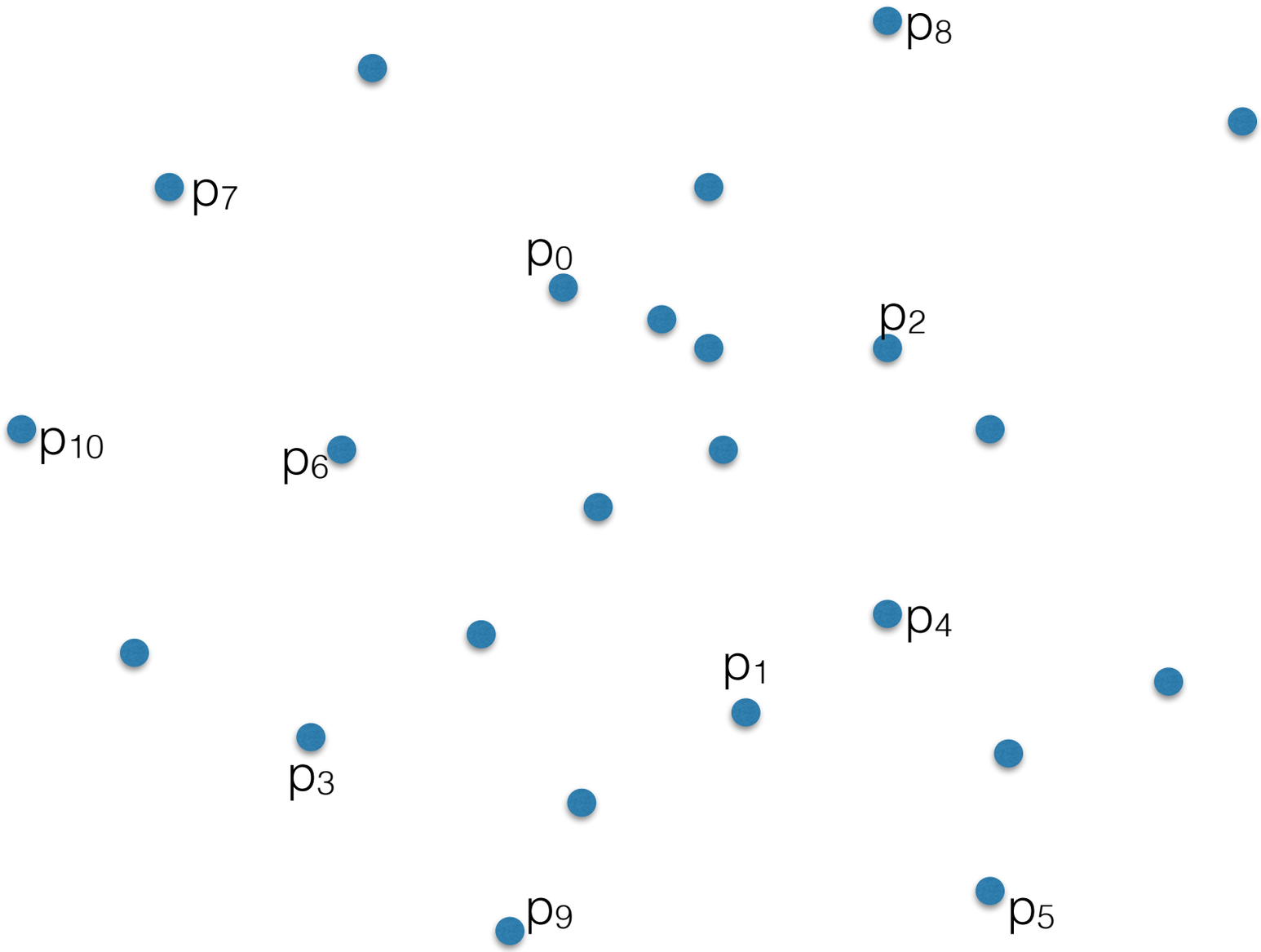

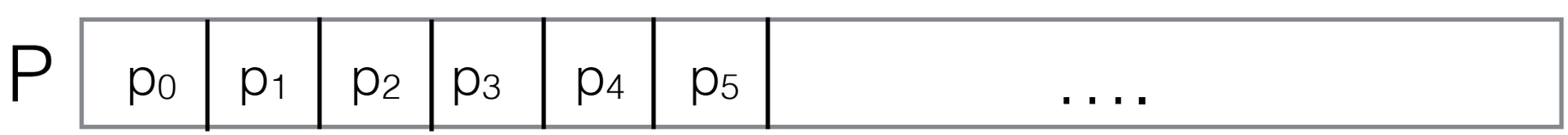

The distance between two points p and q is given by the Euclidian distance given by the formula:

$$d(p,q) = \sqrt{(x_p-x_q)^2 + (y_p-y_q)^2}$$

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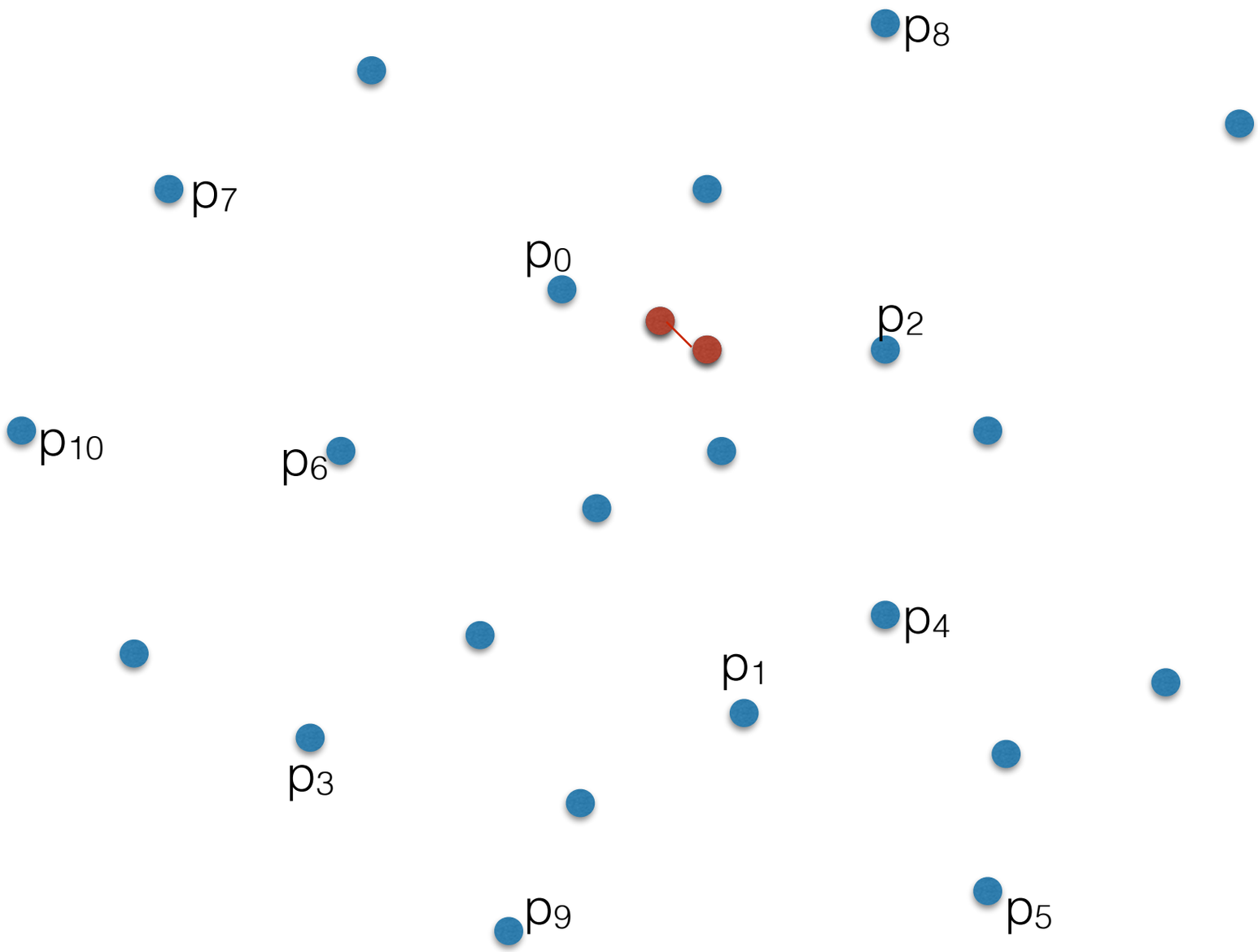

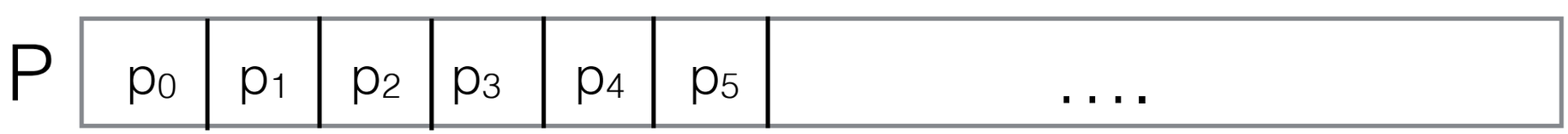

Given an array of points in 2D, find the closest pair.

Brute force:

- mindist = VERY_LARGE_VALUE
- for all distinct pairs of points p_i, p_j
- d = distance (p_i, p_j)
- if (d< mindist): mindist=d

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Hint: use divide-and-conquer

Divide-and-conquer refresher

Divide-and-conquer

mergesort(array A)

- if A has 1 element, there's nothing to sort, so just return it
- else

  //divide input A into two halves, A1 and A2

  - A1 = first half of A
  - A2 = second half of A

  //sort recursively each half

  - sorted_A1 = **mergesort**(array A1)
  - sorted_A2 = **mergesort**(array A2)

  //merge

  - result = merge_sorted_arrays(sorted_A1, sorted_A2)
  - return result

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Analysis: $T(n) = 2T(n/2) + O(n) \Rightarrow O(n \lg n)$

D&C, in general

```
DC(input P)
   if P is small, solve and return
   else
      //divide
      divide input P into two halves, P1 and P2
      //recurse
      result1 = DC(P1)
      result2 = DC(P2)
      //merge
      do_something_to_figure_out_result_for_P
      return result
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Analysis: T(n) = 2T(n/2) + O(merge phase)

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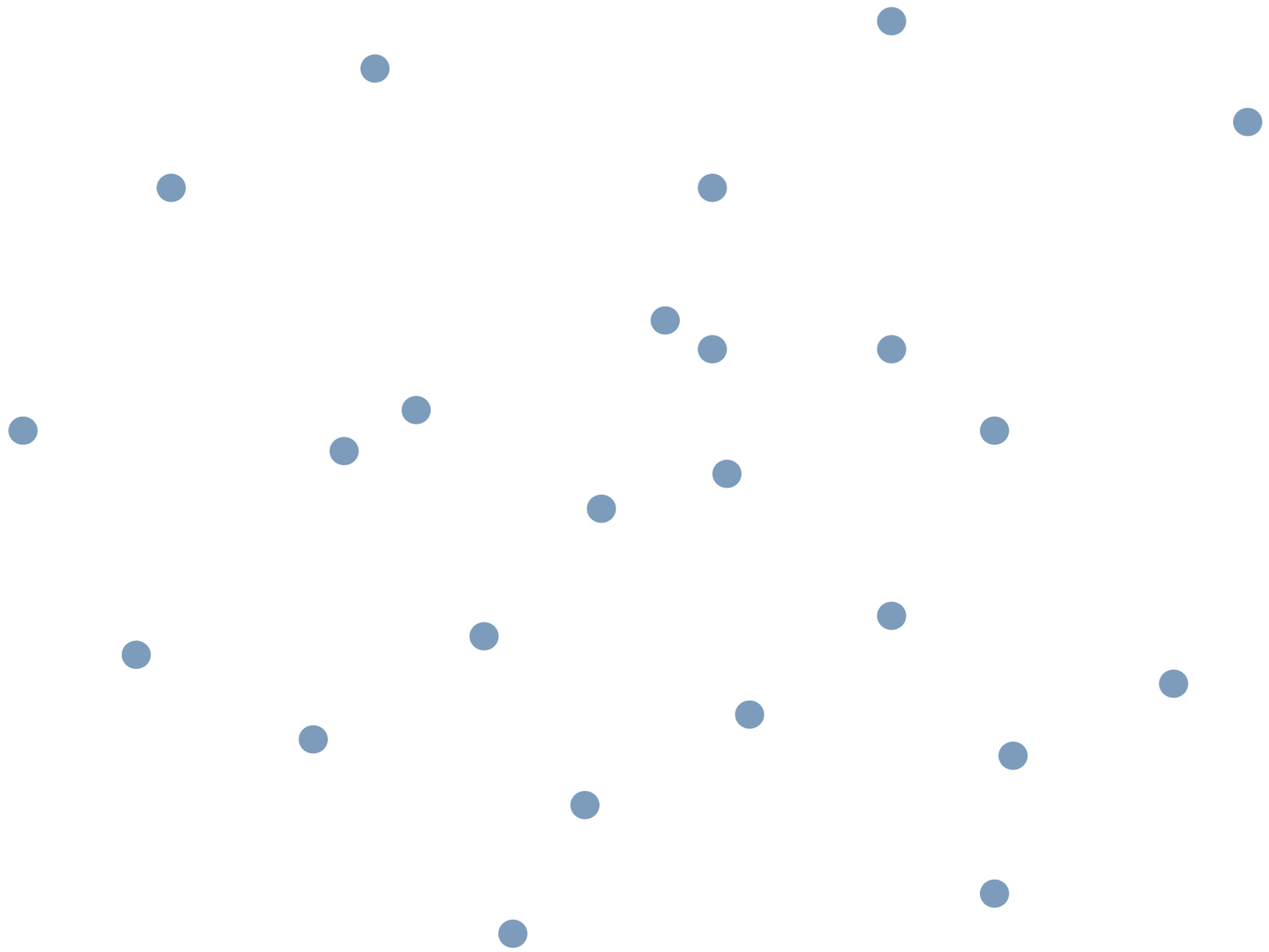
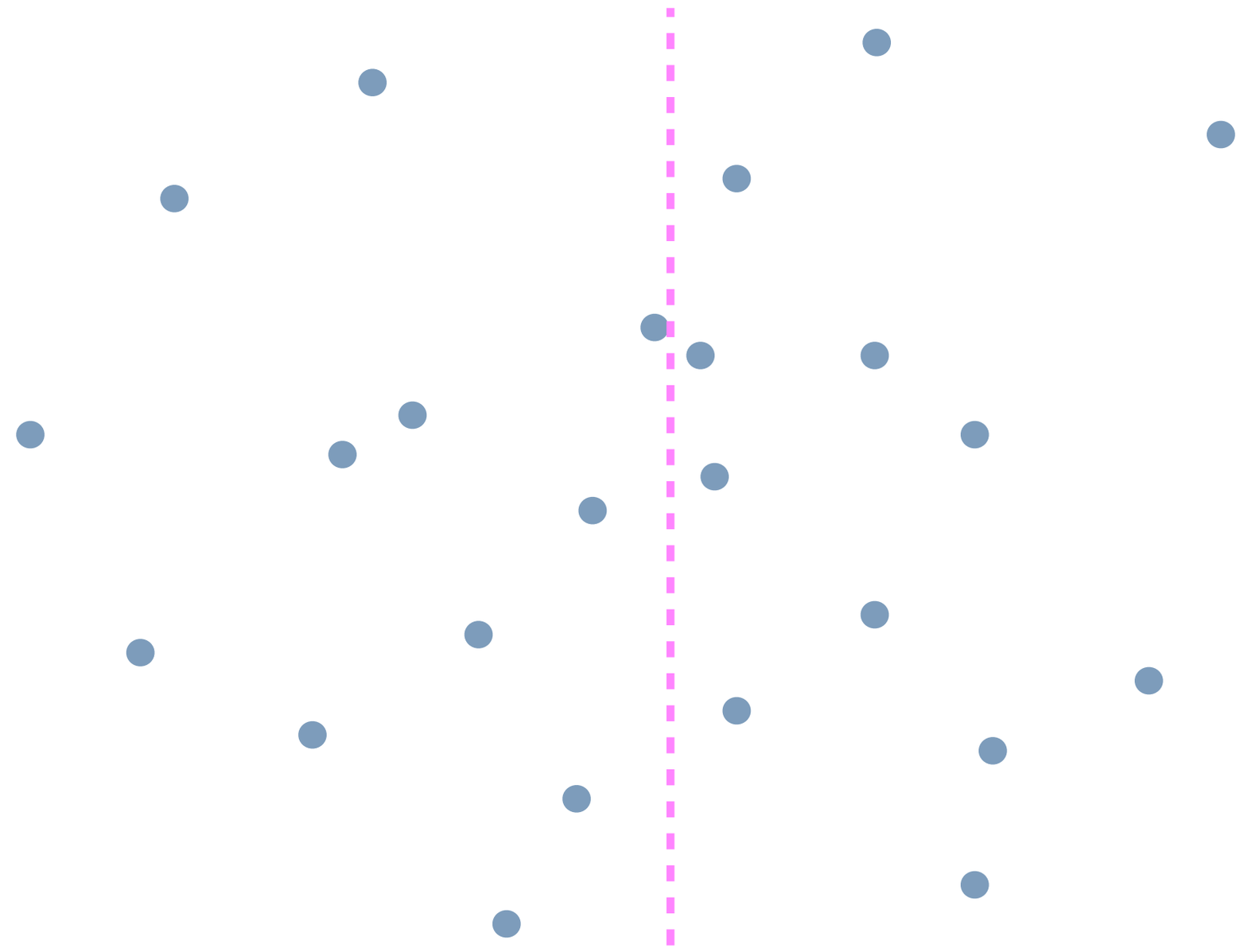
- if merge phase is **O(n)**: T(n) = 2T(n/2) + **O(n)** => O(n lg n)
- if merge phase is **O(n lg n)**: T(n) = 2T(n/2) + **O(n lg n)** => O(n $\lg^2 n$)

Closest pair, divide-and-conquer



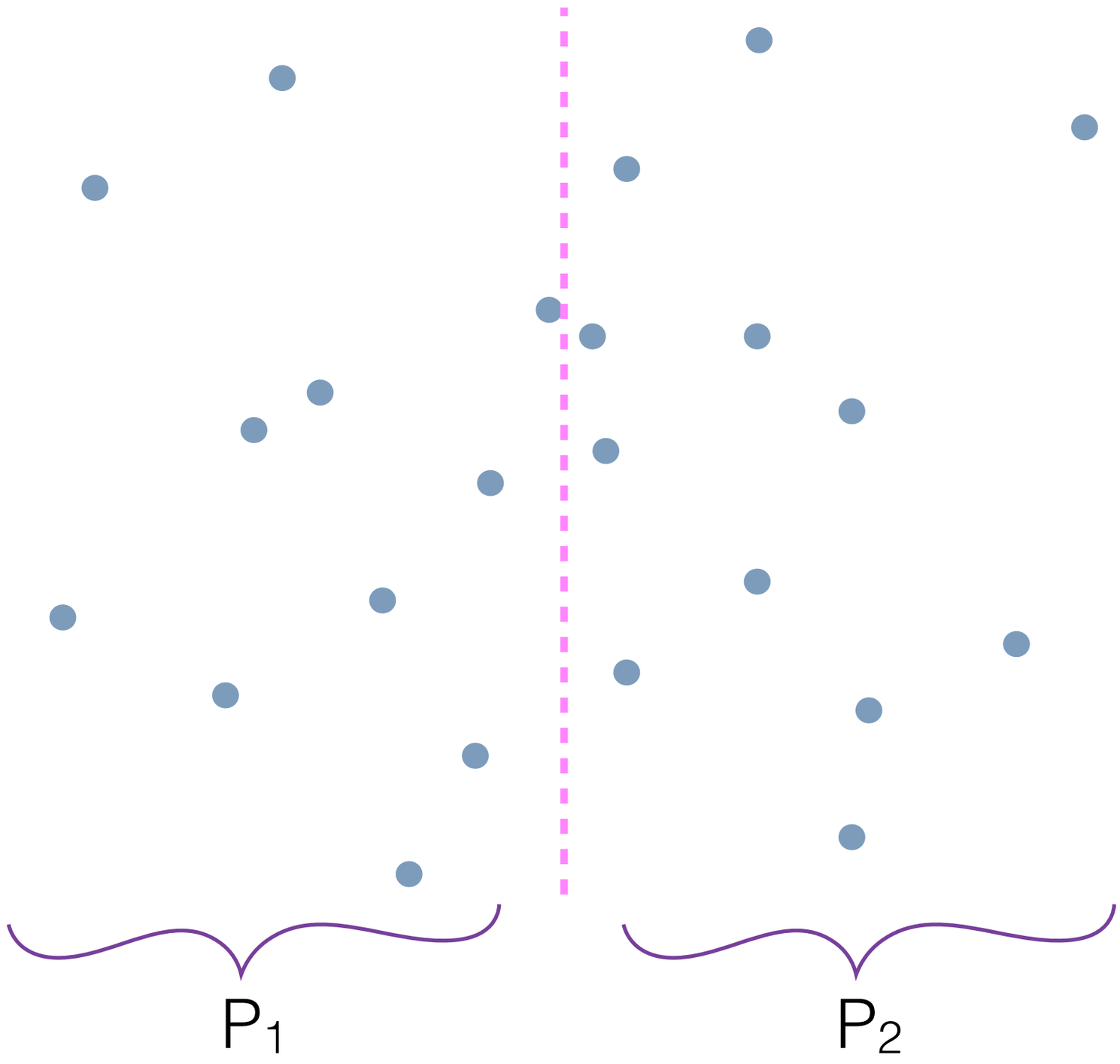
Closest pair, divide-and-conquer

- find vertical line that splits P in half



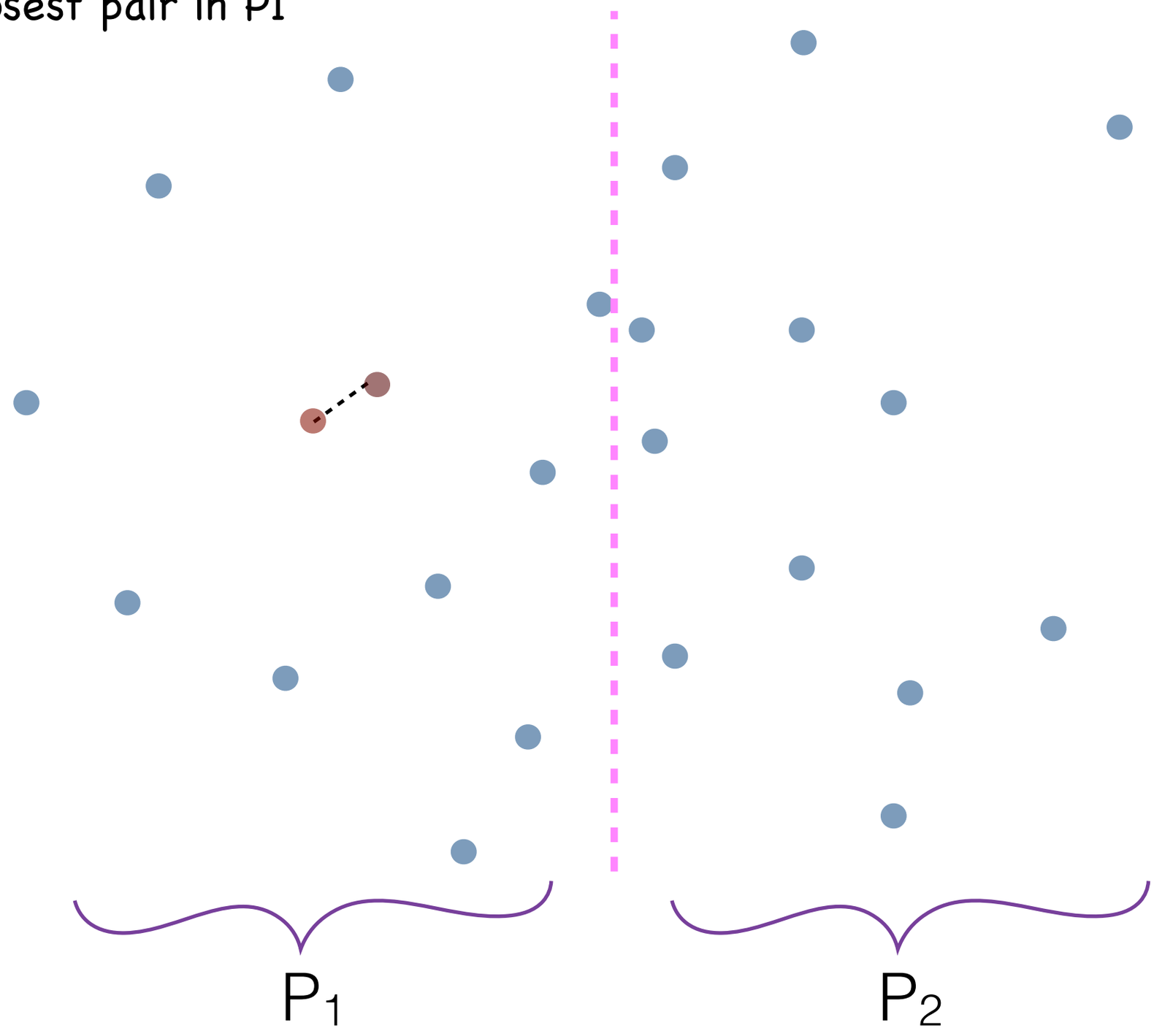
Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line



Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1



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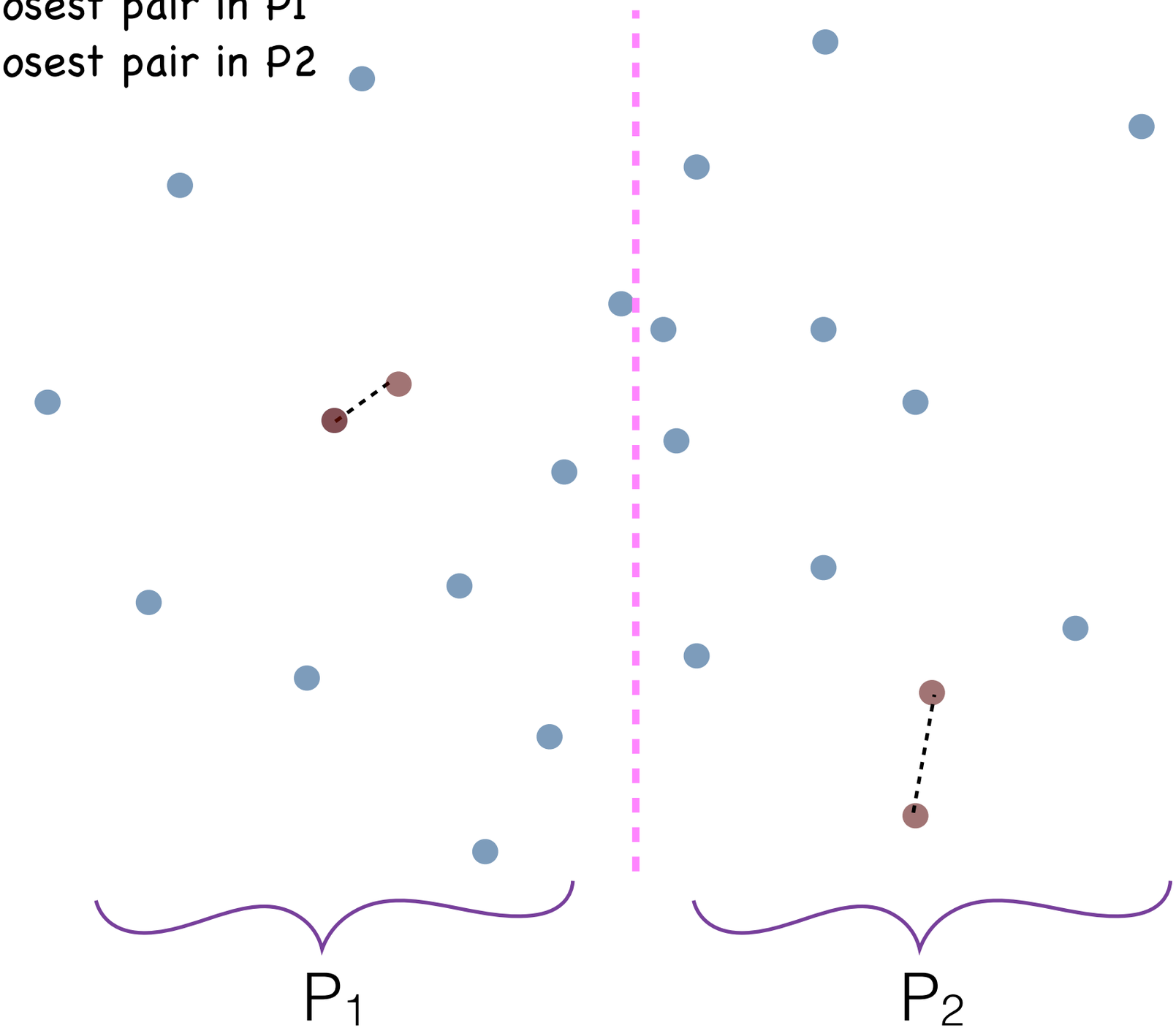

Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
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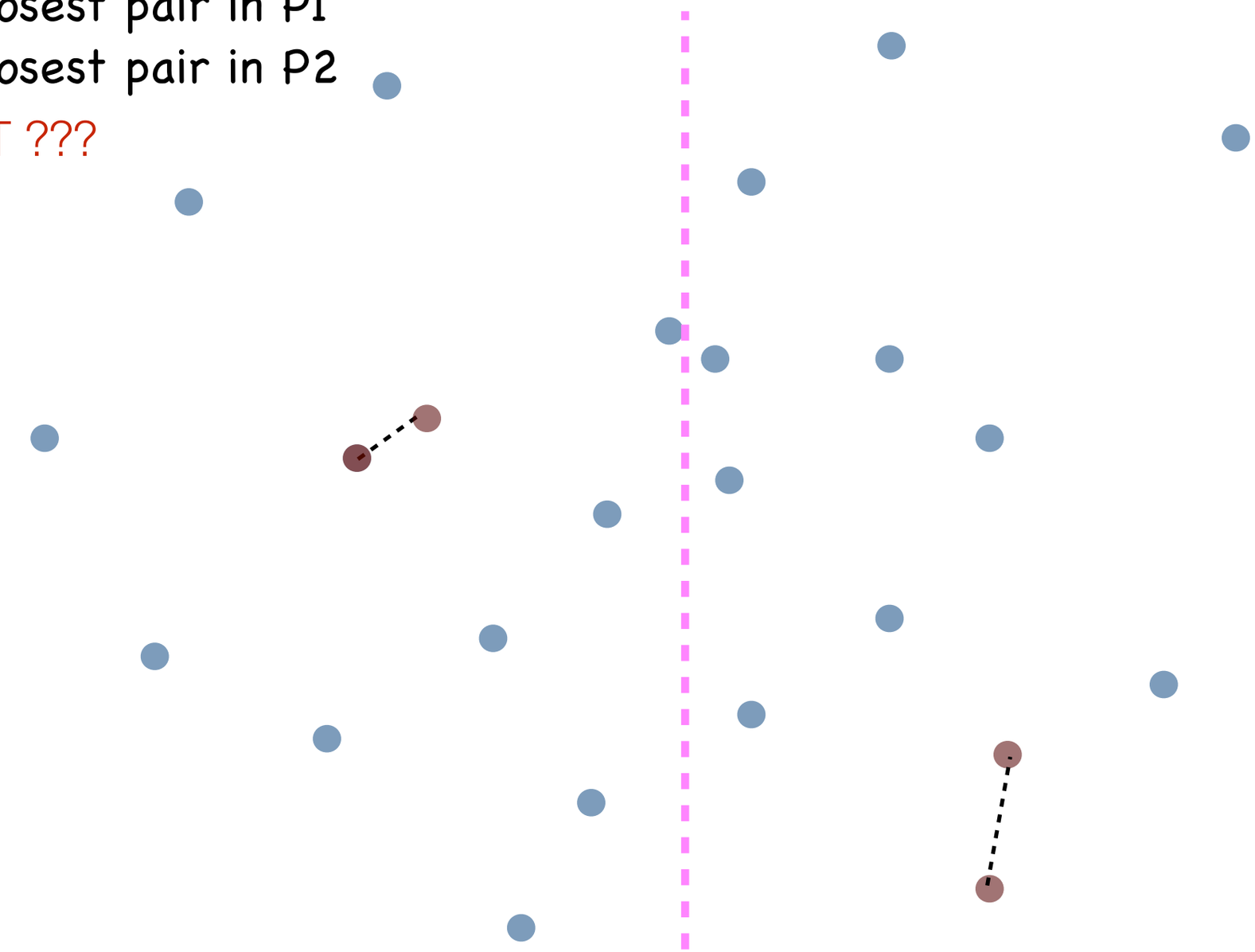
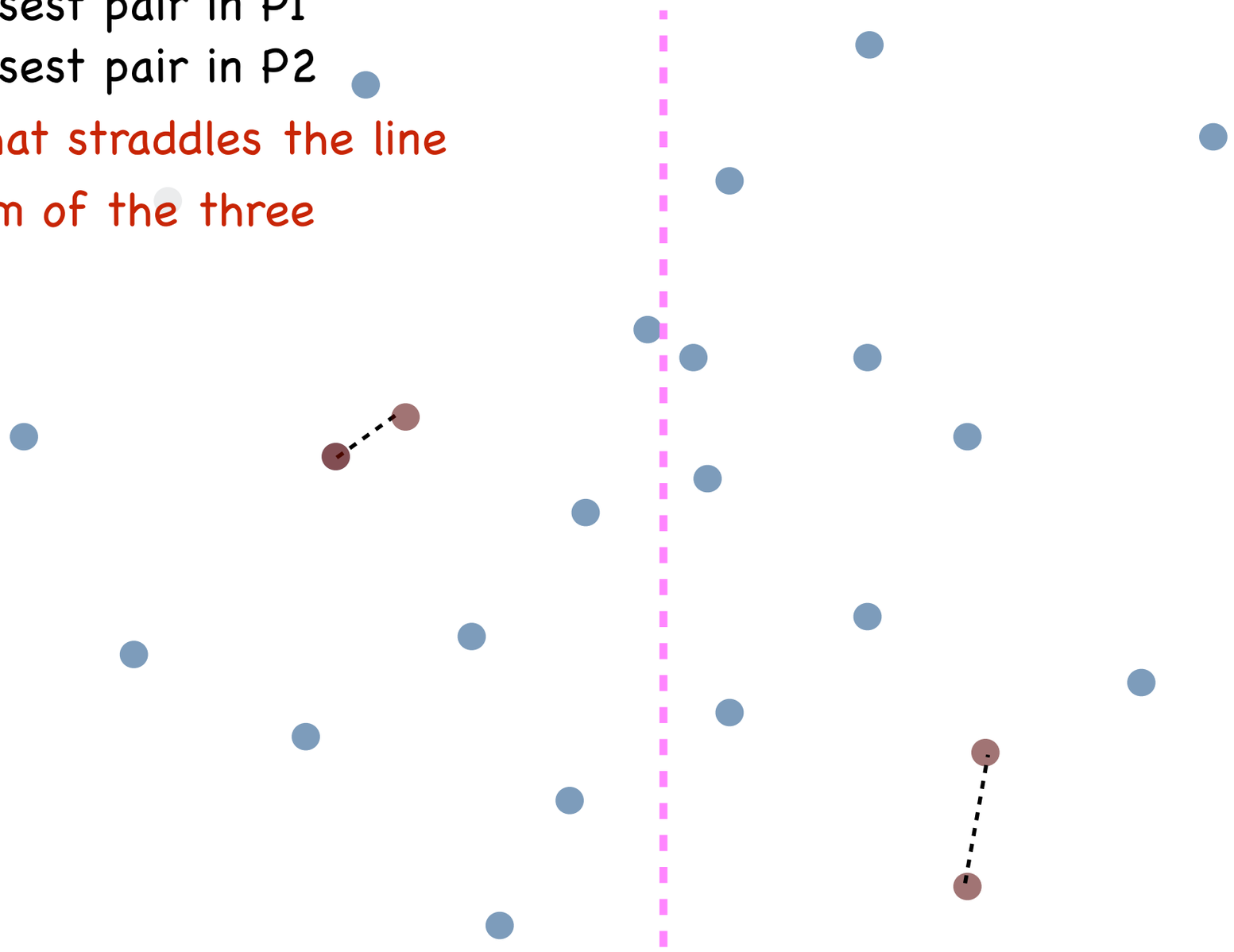
- //…… NOW WHAT ???



Closest pair, divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- find closest pair that straddles the line
- return the minimum of the three



Closest pair, divide-and-conquer

FindClosestPair(P)

//basecase

- if P has 1 point, return infinity
- if P has 2 points, return their distance
- else
  - find vertical line that splits P in half
  - let P1, P2 = set of points to the left/right of line
  - d_1 = FindClosestPair(P1)
  - d_2 = FindClosestPair(P2)

//compute closest pair across

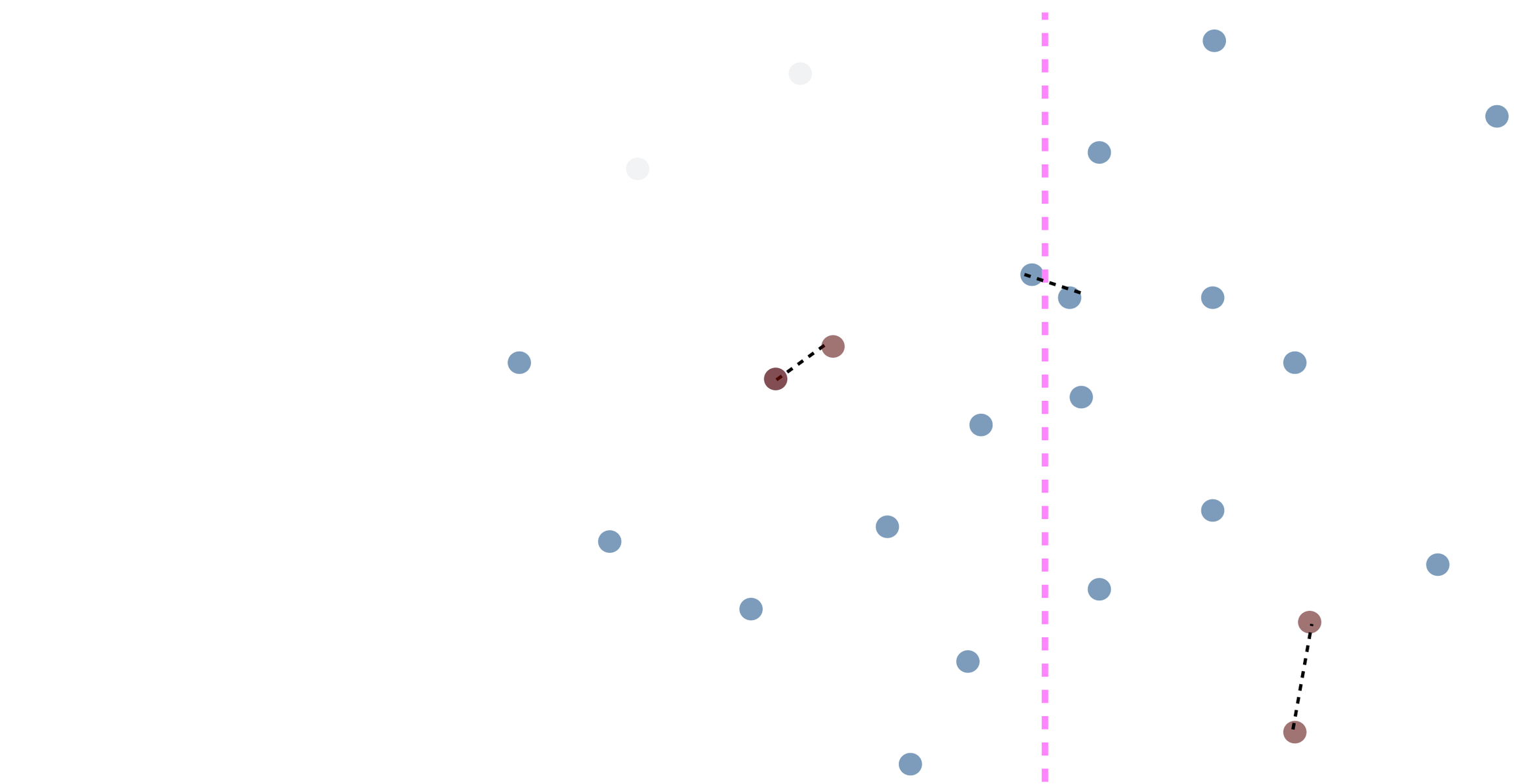
- mindist=infinity
- for each p in P_1, for each q in P_2
  - compute distance d(p,q)
  - mindist = min{d_1, d_2, d(p,q)}

//return smallest of the three

- return min {d_1, d_2, mindist}

Is this correct?

Closest pair, divide-and-conquer



Closest pair, divide-and-conquer

Correct, because the closest pair in P must be one of:

- **Both points are in P1:** then it is found by the recursive call on P1
- **Both points are in P2:** then it is found by the recursive call on P2
- **One point is in P1 and one in P2:** then it is found in the merge phase, because the merge phase considers all such pairs

Closest pair, divide-and-conquer

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Running time?

Closest pair, divide-and-conquer

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Running time?

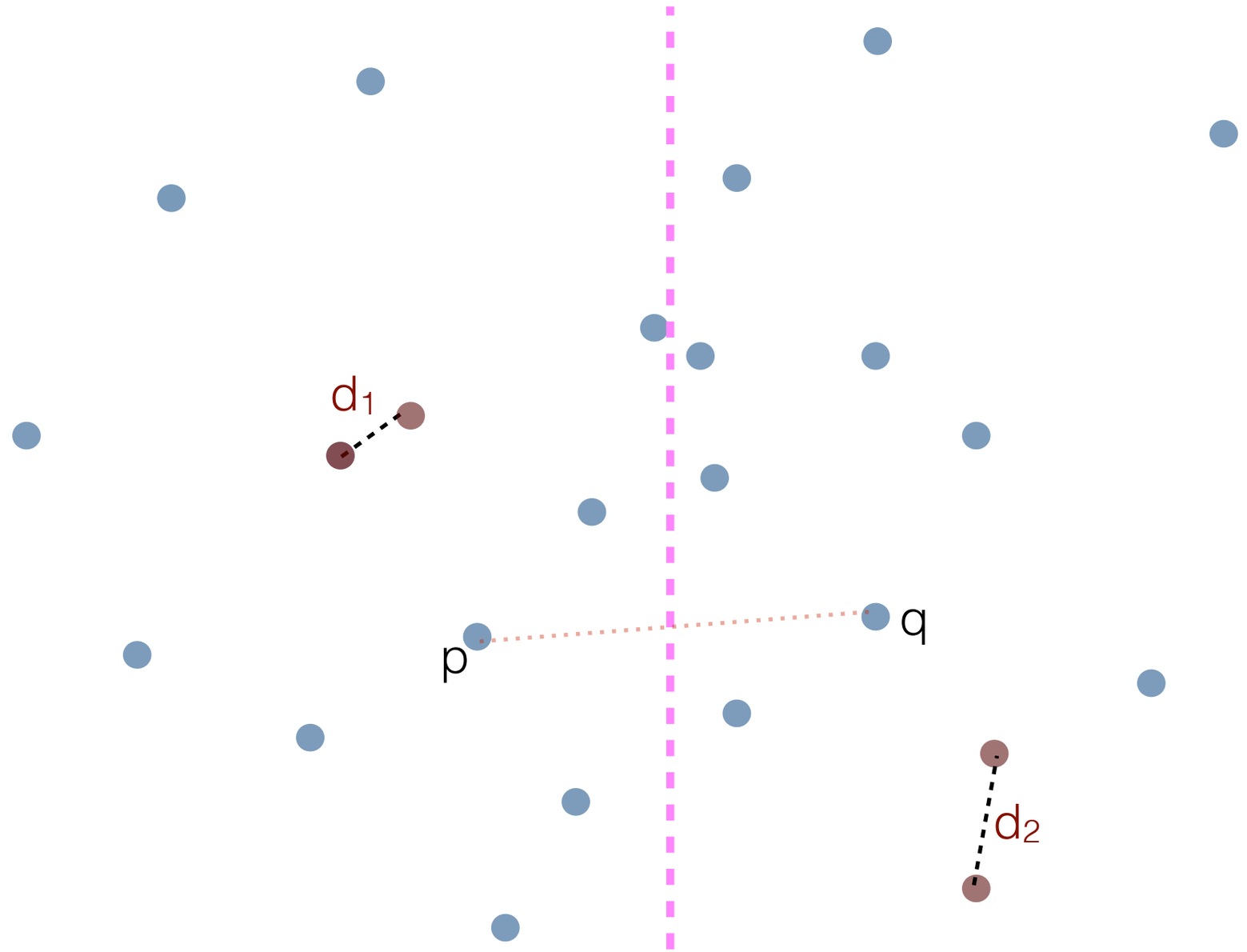
$T(n) = 2T(n/2) + O(n^2)$

solves to $O(n^2)$

Can we do better?

Refining the merge

Do we need to examine all pairs (p,q), with p in P_1, q in P_2?

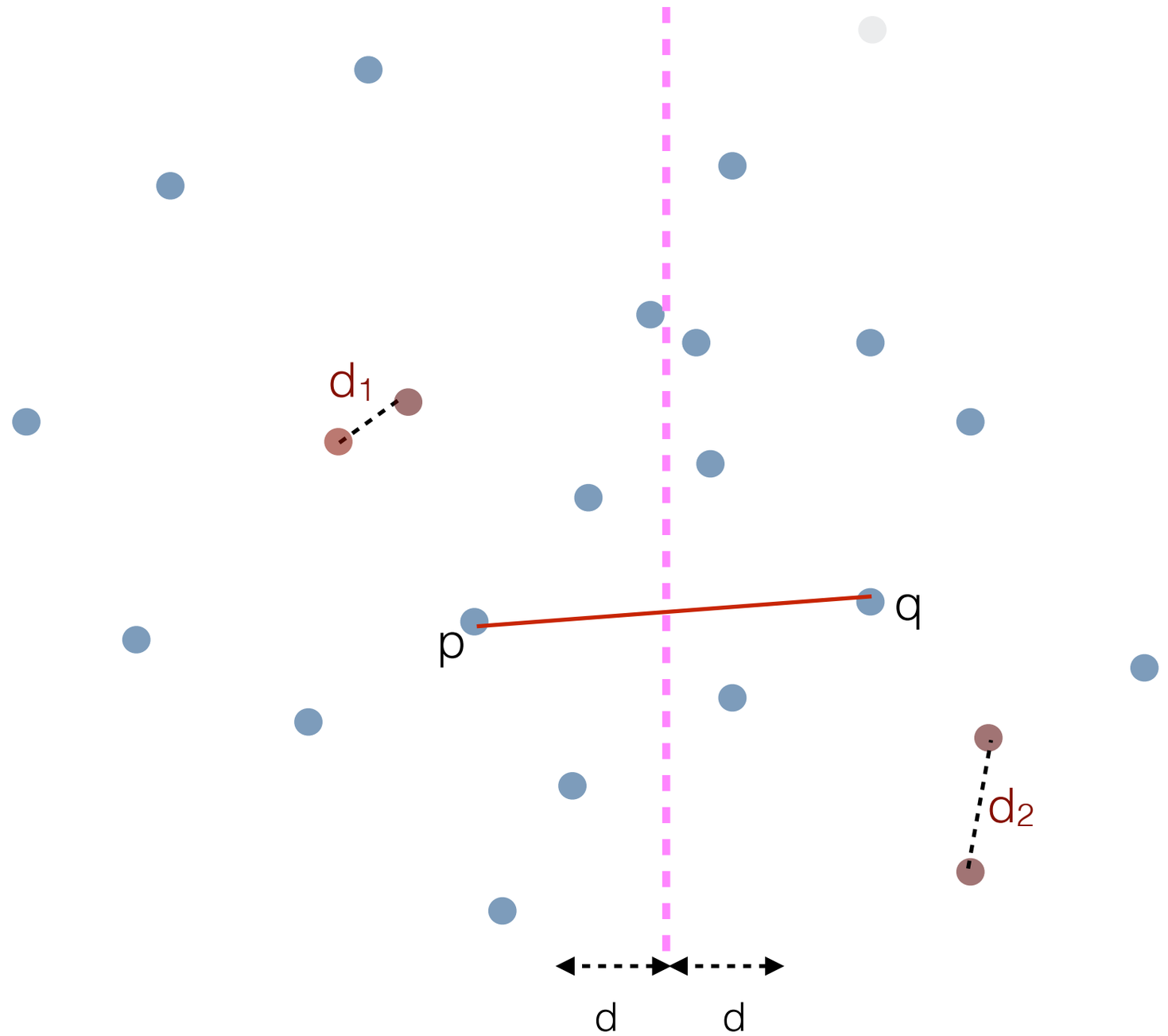


Can (p,q) be the closest pair?

Why not?

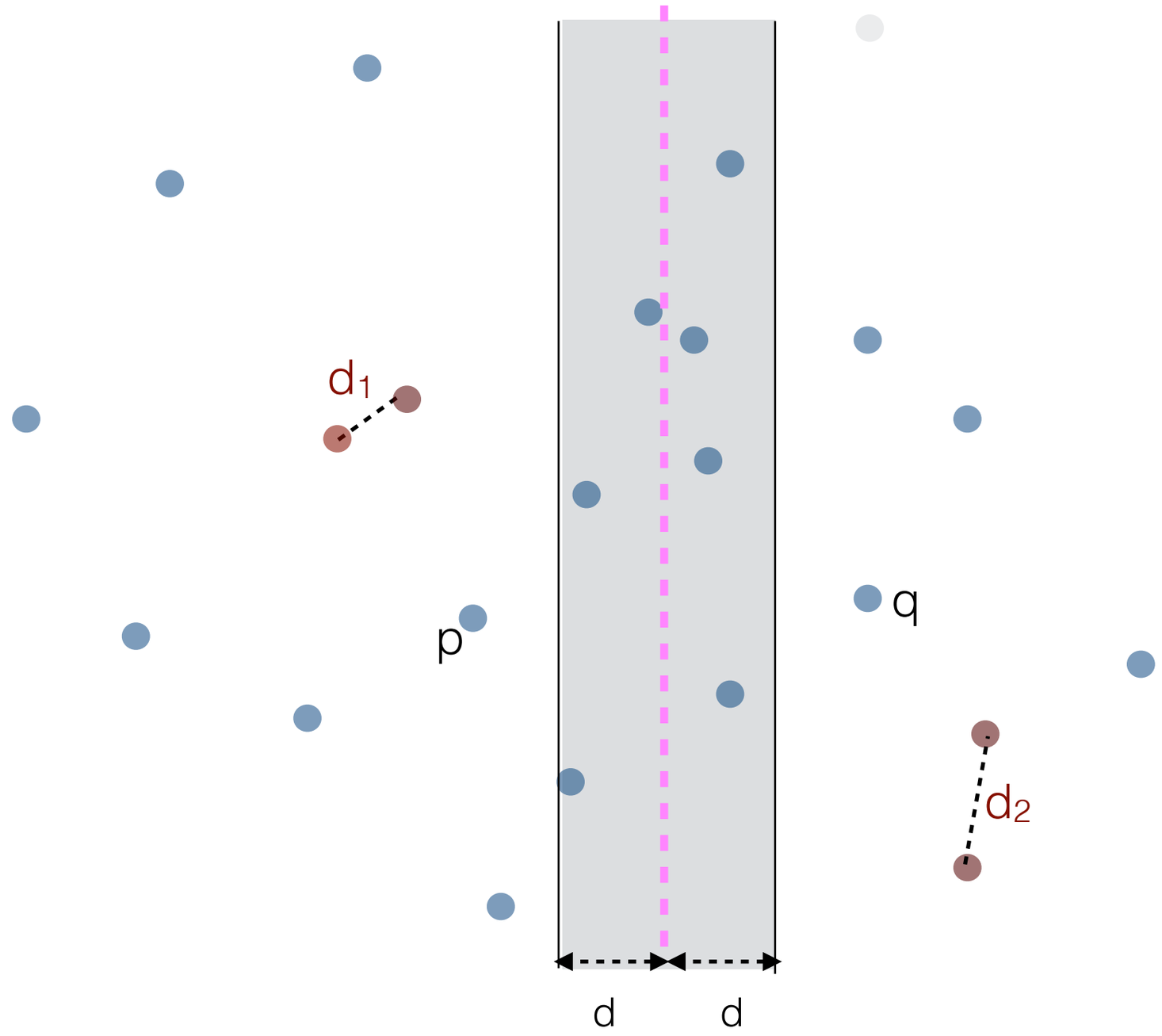
Notation: $d = \min \{d_1, d_2\}$

Claim: In order for dist(p,q) to be smaller than d, it must be that both the horizontal and vertical distance between p and q must be smaller than d.



Notation: $d = \min \{d_1, d_2\}$

==> In order to be candidates for closest pair, points p, q must lie in the d-by-d strip centered at the median.



Refining the merge

FindClosestPair(P)

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  - find vertical line that splits P in half
  - let P1, P2 = set of points to the left/right of line
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  - traverse P_1 and select all points P_1' in the strip
  - traverse P_2 and select all points P_2' in the strip
  - for each p in P_1'
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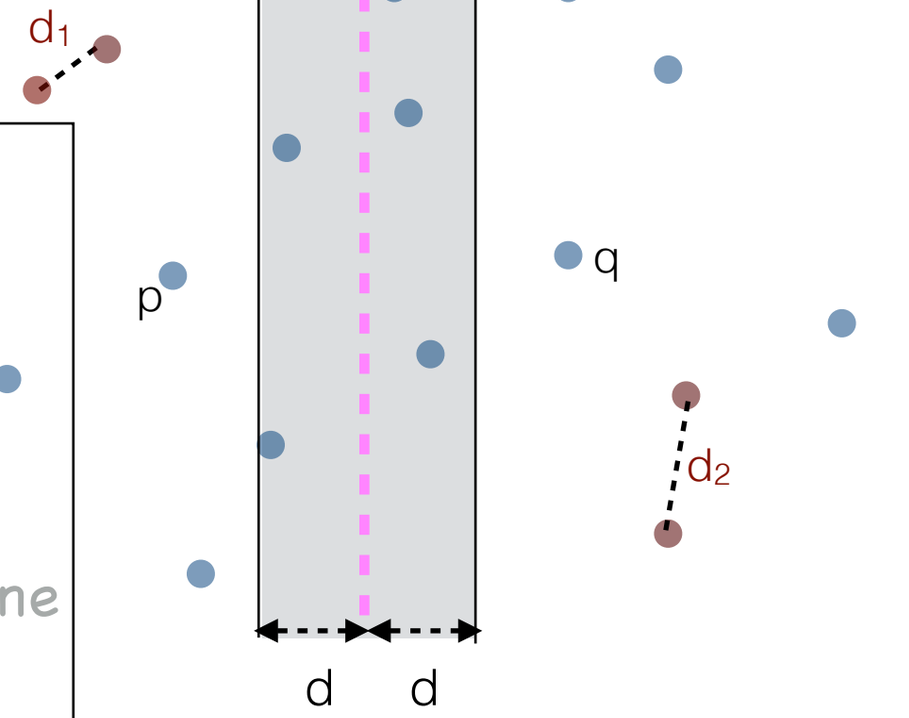

Running time?

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  - for each p in P_1'
    - for each point q in P_2'
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      - mindist = min{d_1, d_2, d(p,q)}
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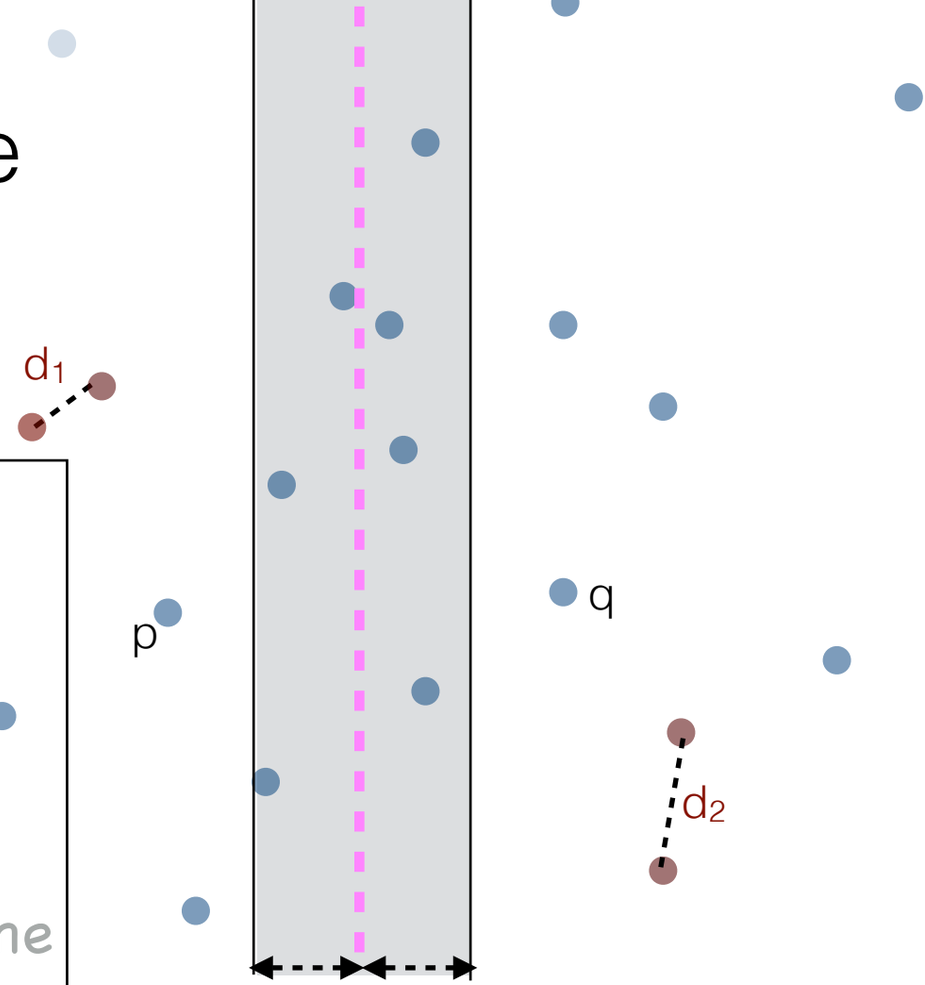
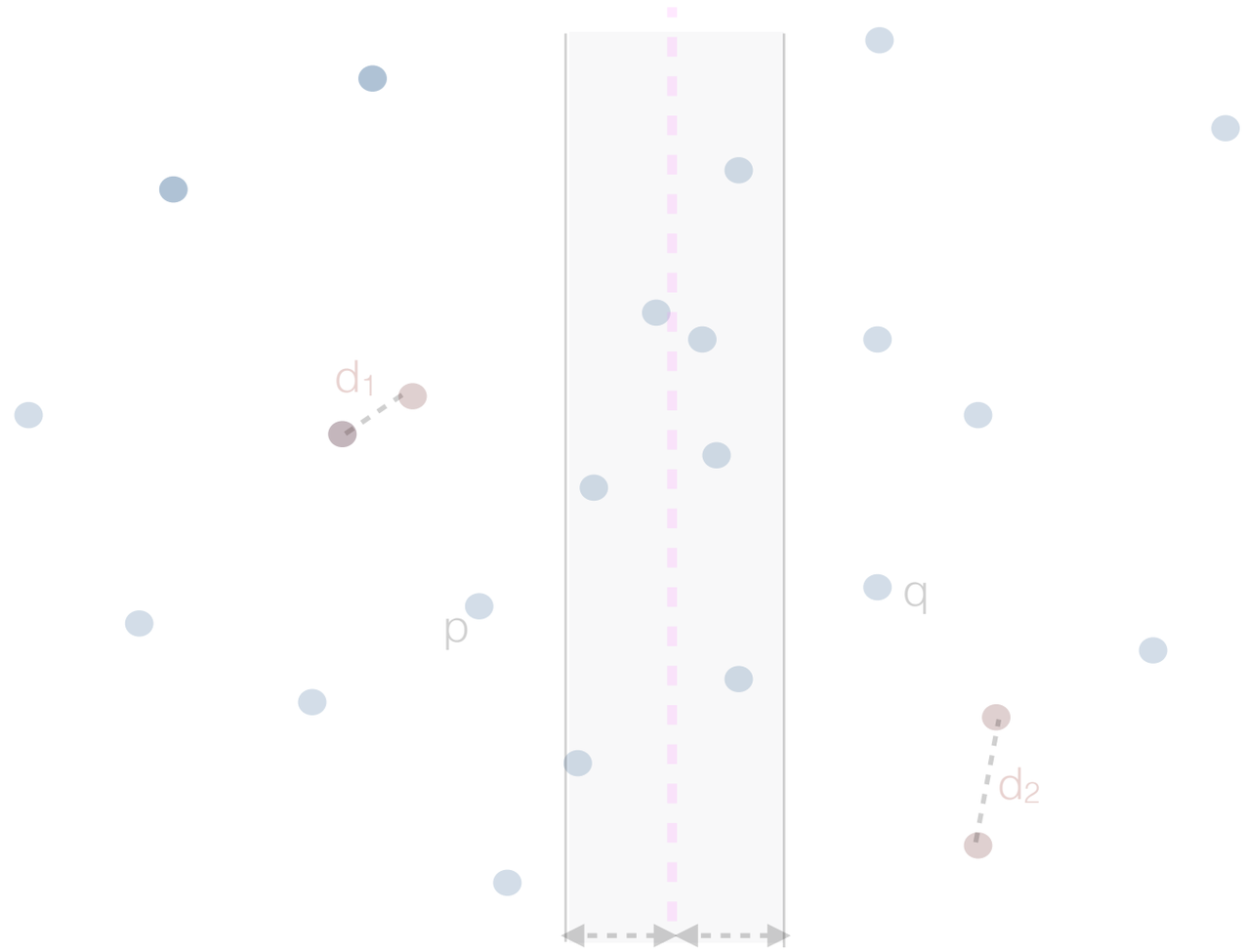


Running time?

It's possible that all n/2 points on either side lie inside the strip

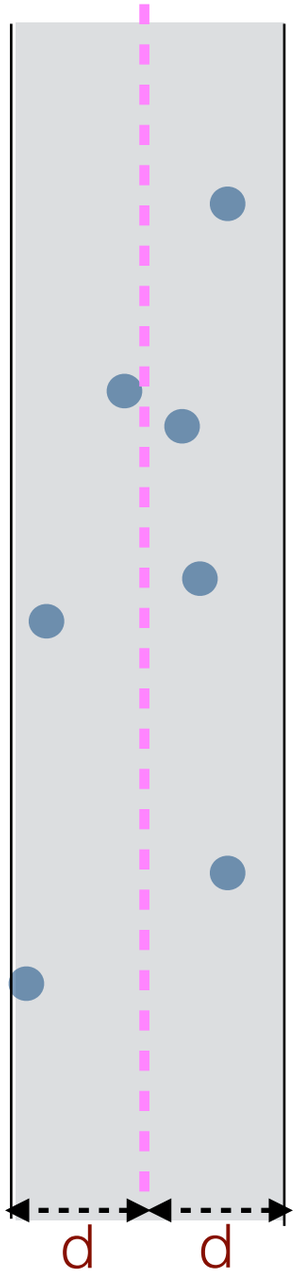
Refining the merge

- Show an example where the strip may contain Omega(n) points.
- What does this imply for the running time?



Refining the merge

- Filtering the points in the strip is not enough..
- But, we can show that the points in the strip have a special structure which will enable us to merge faster

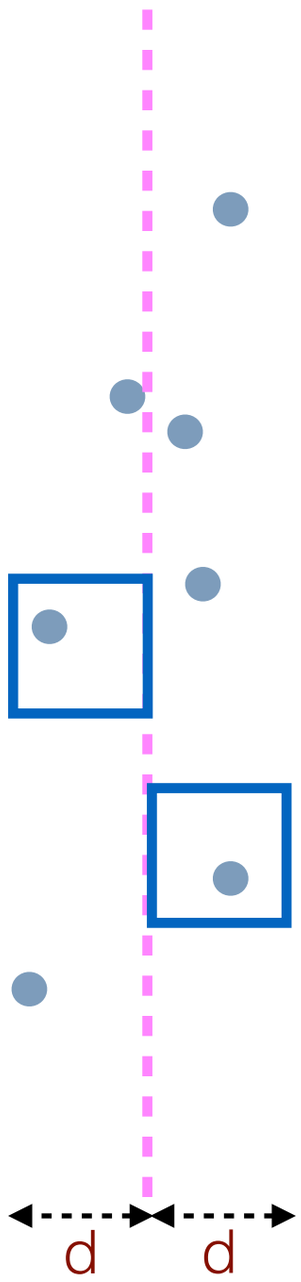


Refining the merge

Points on both sides are “sparse”

Any pair of points in P_1 must be at least d away

Any pair of points in P_2 must be at least d away

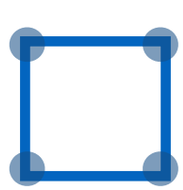


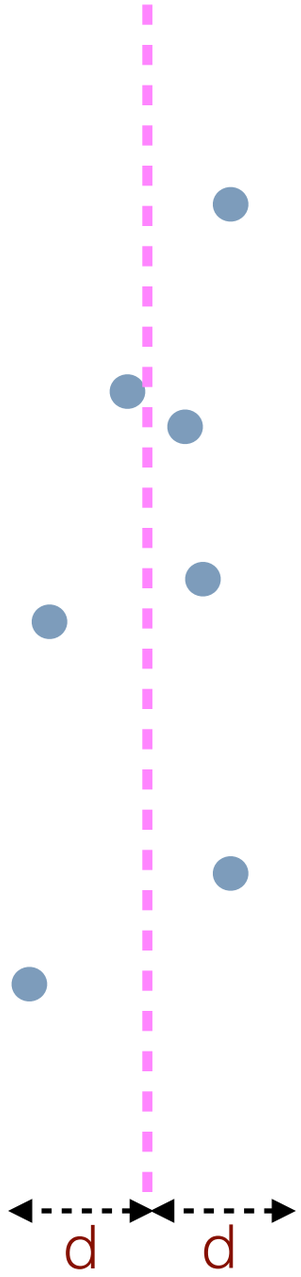
Refining the merge

Points on both sides are “sparse”

Any pair of points in P_1 must be at least d away

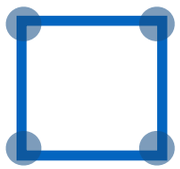
Any square with side d contains at most 4 points of P_1





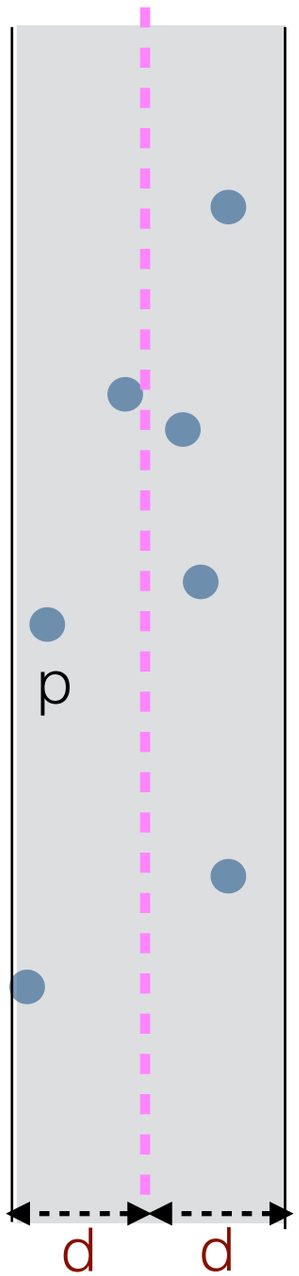
Any pair of points in P_2 must be at least d away

Any square with side d contains at most 4 points of P_1



How can we use this?

- Consider a point p in P_1'
- We don't need to compute the distances from p to all points in P_2'



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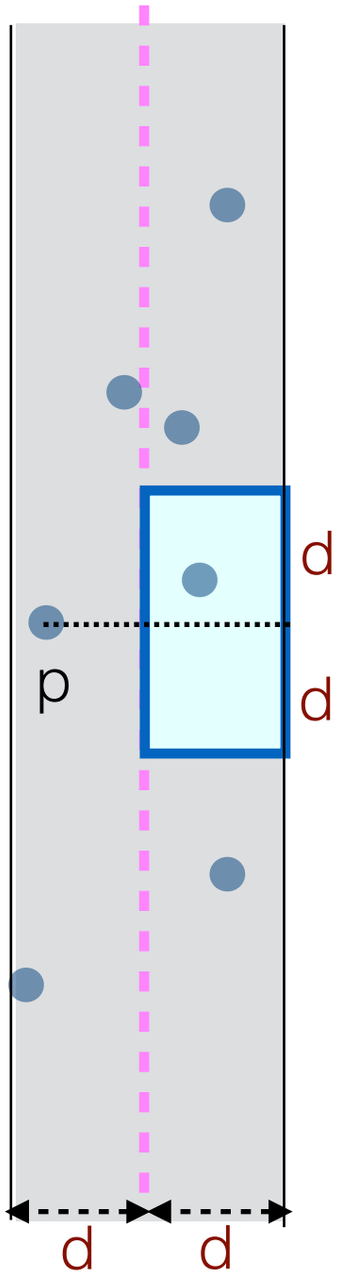

- CLAIM: All points of P_2' within distance d of p are vertically above or below p by at most d
  ==> they must lie in a rectangle d x 2d

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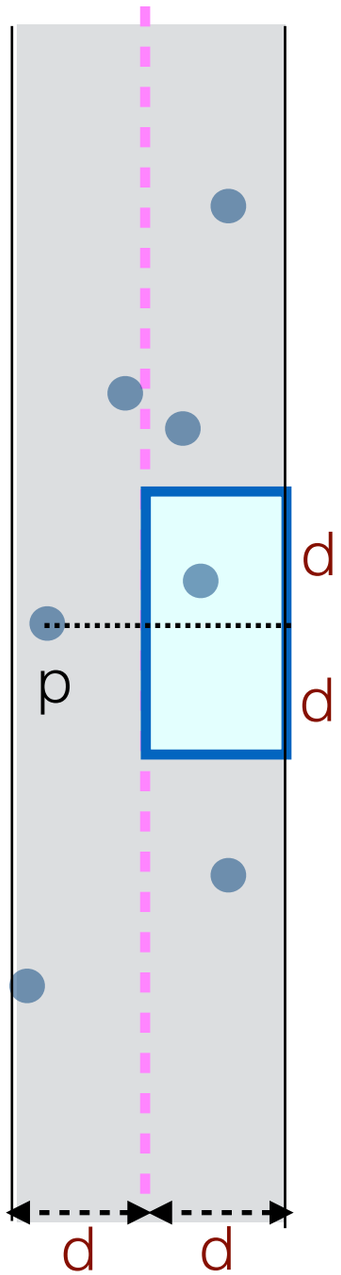

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- How many points q of P_2' can there be in a rectangle of size d x 2d? (knowing that any pair of points in P_2' must be at least d away).

Refining the merge

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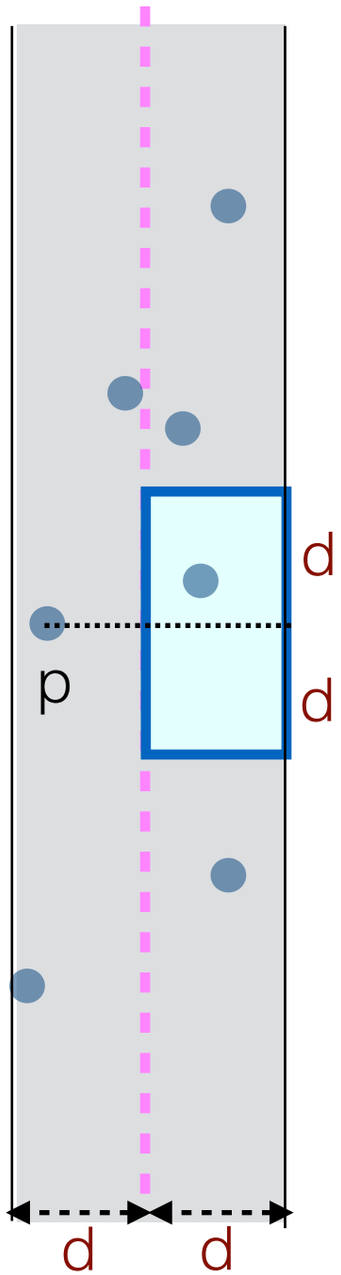

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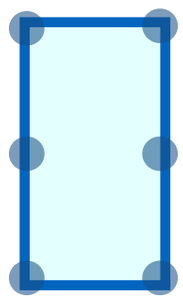
  ==> they must lie in a rectangle d x 2d

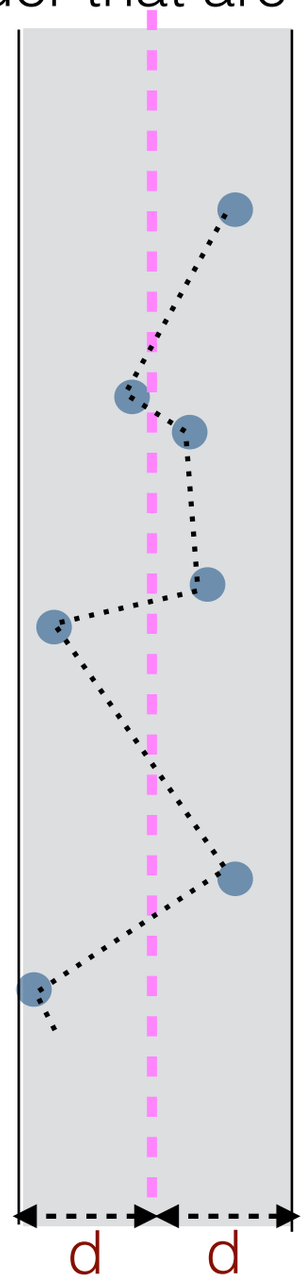
- How many points q of P_2' can there be in a rectangle of size d x 2d? (knowing that any pair of points in P_2' must be at least d away).

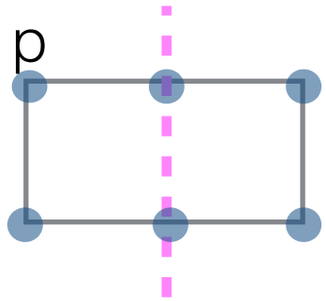


=> So for every p in P_1', we only need to check at most 6 points!

Refining the merge

- An elegant (simple?) way to do this is by traversing the points in P_1' and P_2' in y-order
- A point p needs to check only the points following it, and there can be at most 5 points following p in y-order that are within d from p.





Note: Assume no duplicate points.

Refining the merge

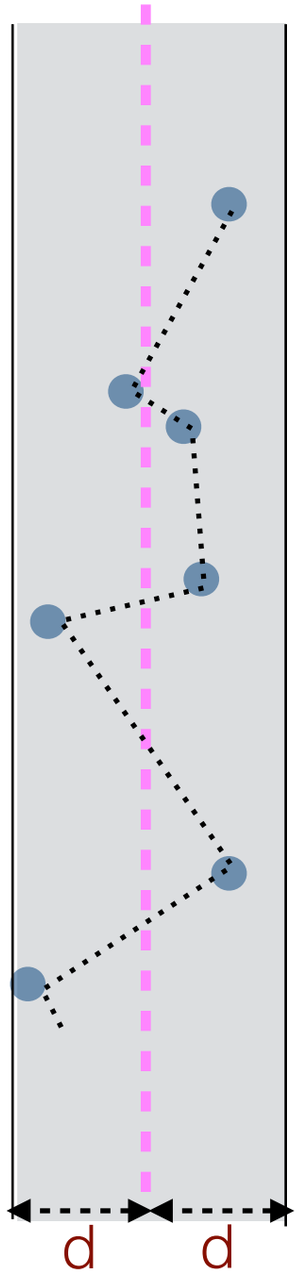
closestPair(P)

//divide

- find vertical line l that splits P in half
- let P_1, P_2 = set of points to the left/right of line
- d_1 = closestPair(P_1)
- d_2 = closestPair(P_2)

//merge

- let d = min{d_1, d_2}
- Strip= empty
- for all p in P_1: if $x_p > x_l - d$: add p to Strip
- for all p in P_2: if $x_p < x_l + d$: add p to Strip
- sort Strip by y-coord
- initialize mindist=d
- for each p in Strip in sorted order
  - compute its distance to the 5 points that come after it in sorted order
  - if any of these is smaller than mindist, update mindist
- return mindist



Refining the merge

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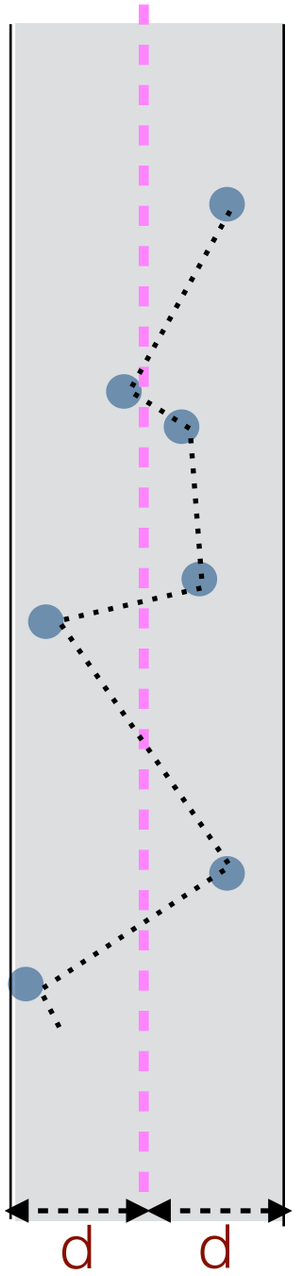

Analysis: $T(n) = 2T(n/2) + O(n \lg n) \Rightarrow O(n \lg^2 n)$

Closest pair, divide-and-conquer

- The sorted list of points in the strip can be obtained from pre-sorting P

closestPair(P_sorted_by_x, P_sorted_by_y)

- Sort P at the beginning and maintain it through recursion
- Final recurrence is T(n) = 2T(n/2) + O(n), which solves to O(n lg n)