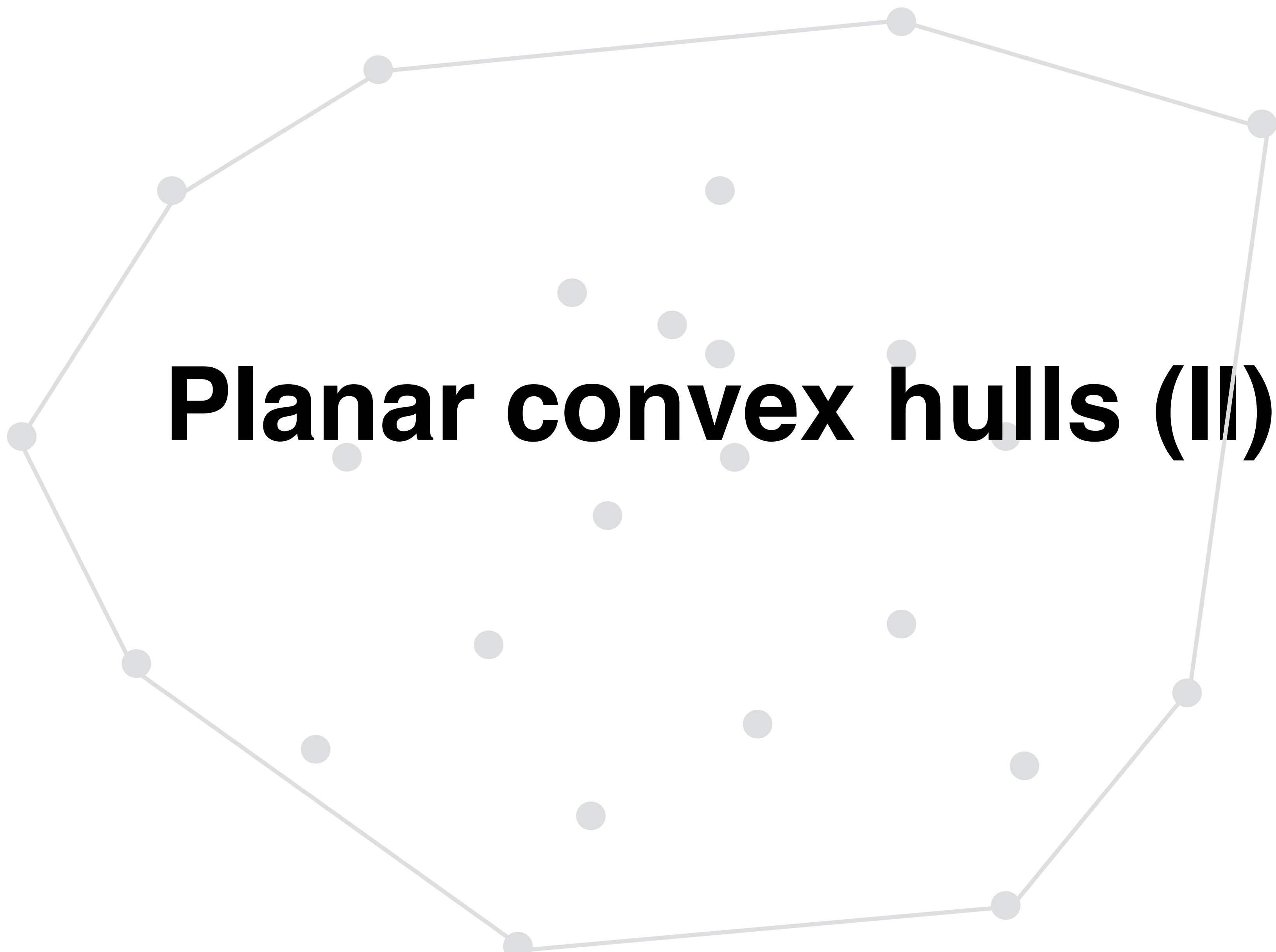
Planar convex hulls (II)

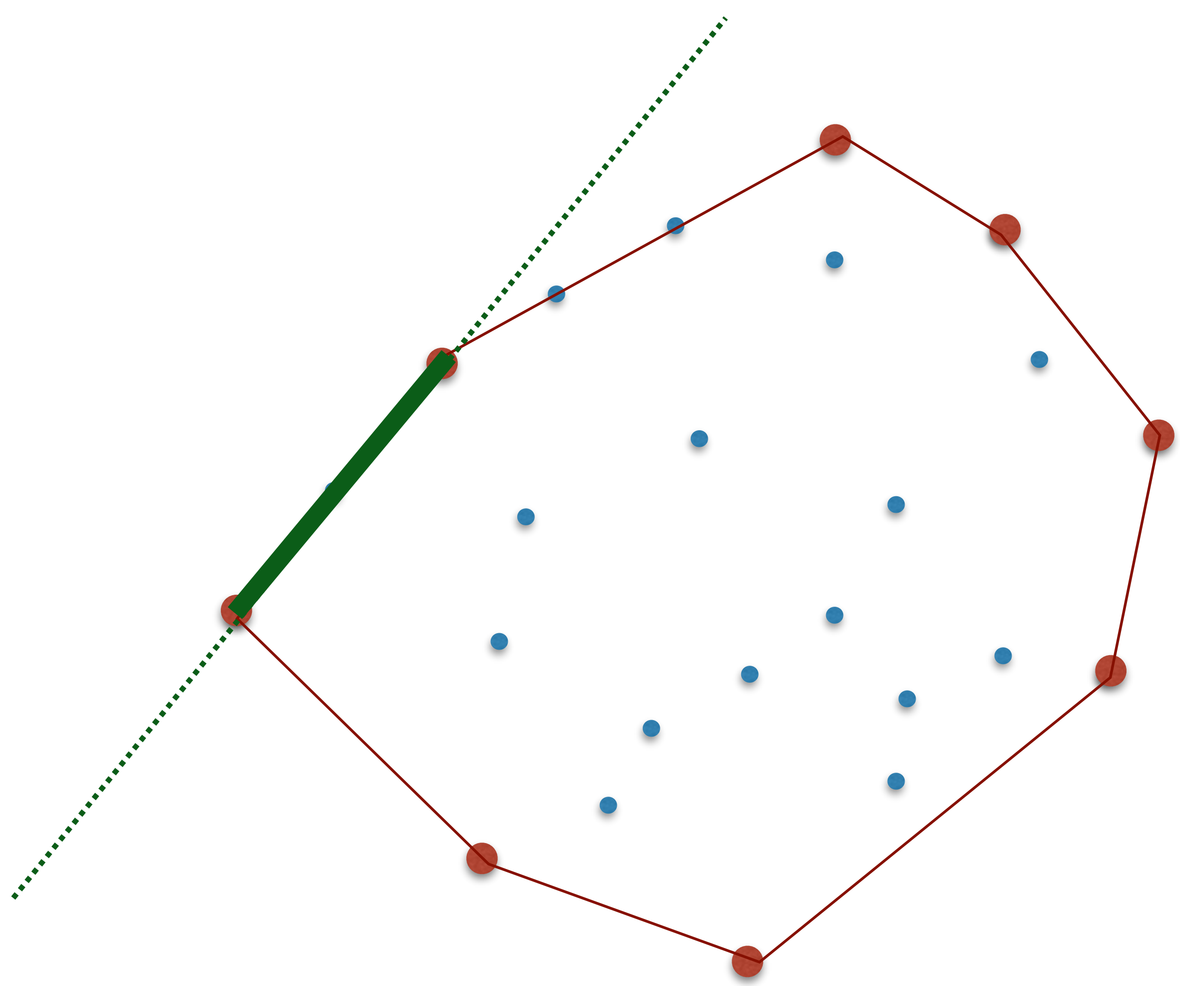


Computational Geometry [csci 3250]

Laura Toma
Bowdoin College

Properties of CH

- All edges of CH are extreme and all extreme edges of P are on the CH
- All points of CH are extreme and all extreme points of P are on the CH
- All internal angles are < 180
- Walking counterclockwise—> left turns
- Points on CH are sorted in radial order wrt a point inside



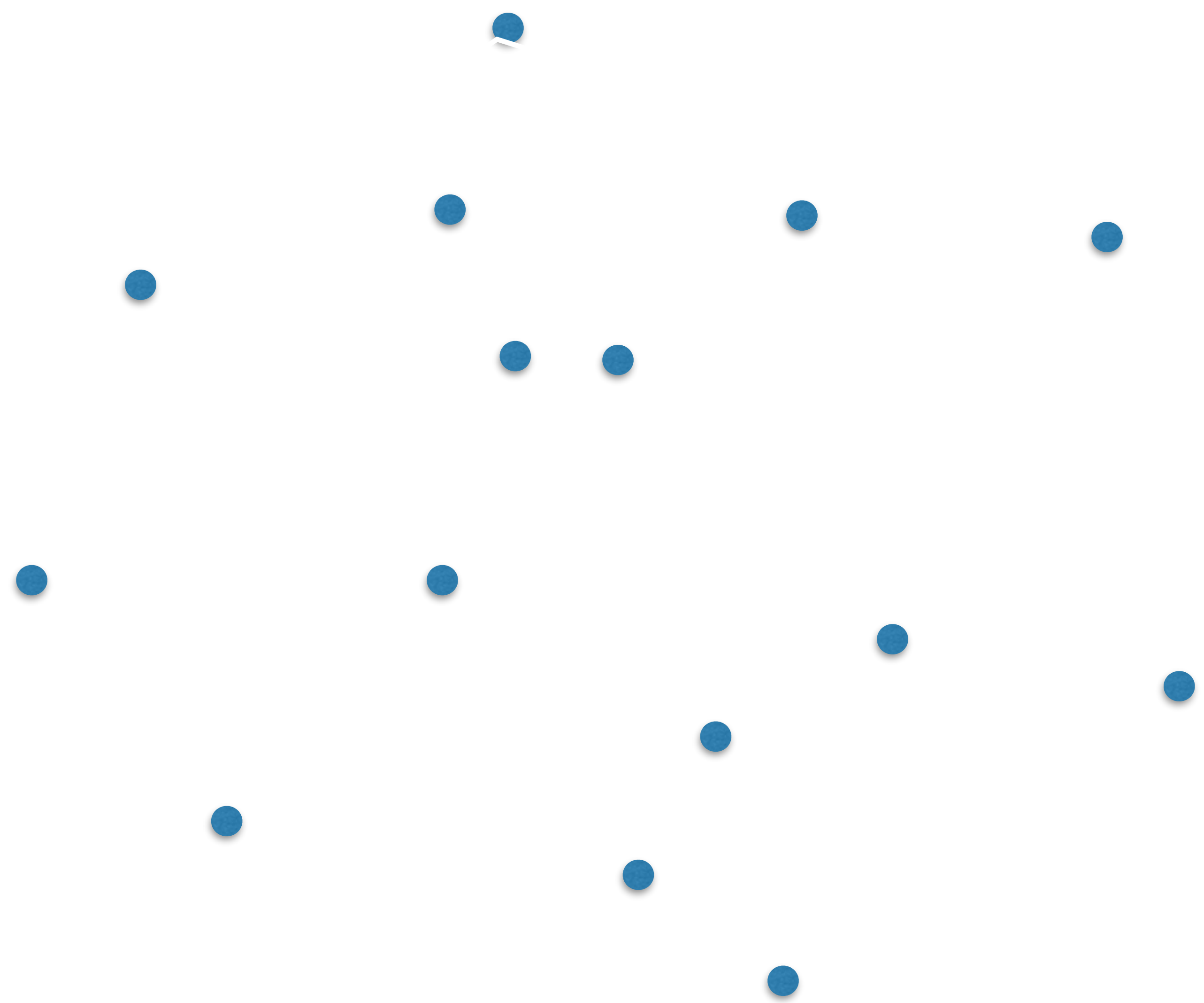
Outline

- Last time:
    - Brute force
    - Gift wrapping
    - Quickhull
    - Graham scan

- Next
  - Andrew’s monotone chain algorithm
  - Exercises
  - Lower bound
  - More algorithms
    - Incremental CH
    - Divide-and-conquer CH

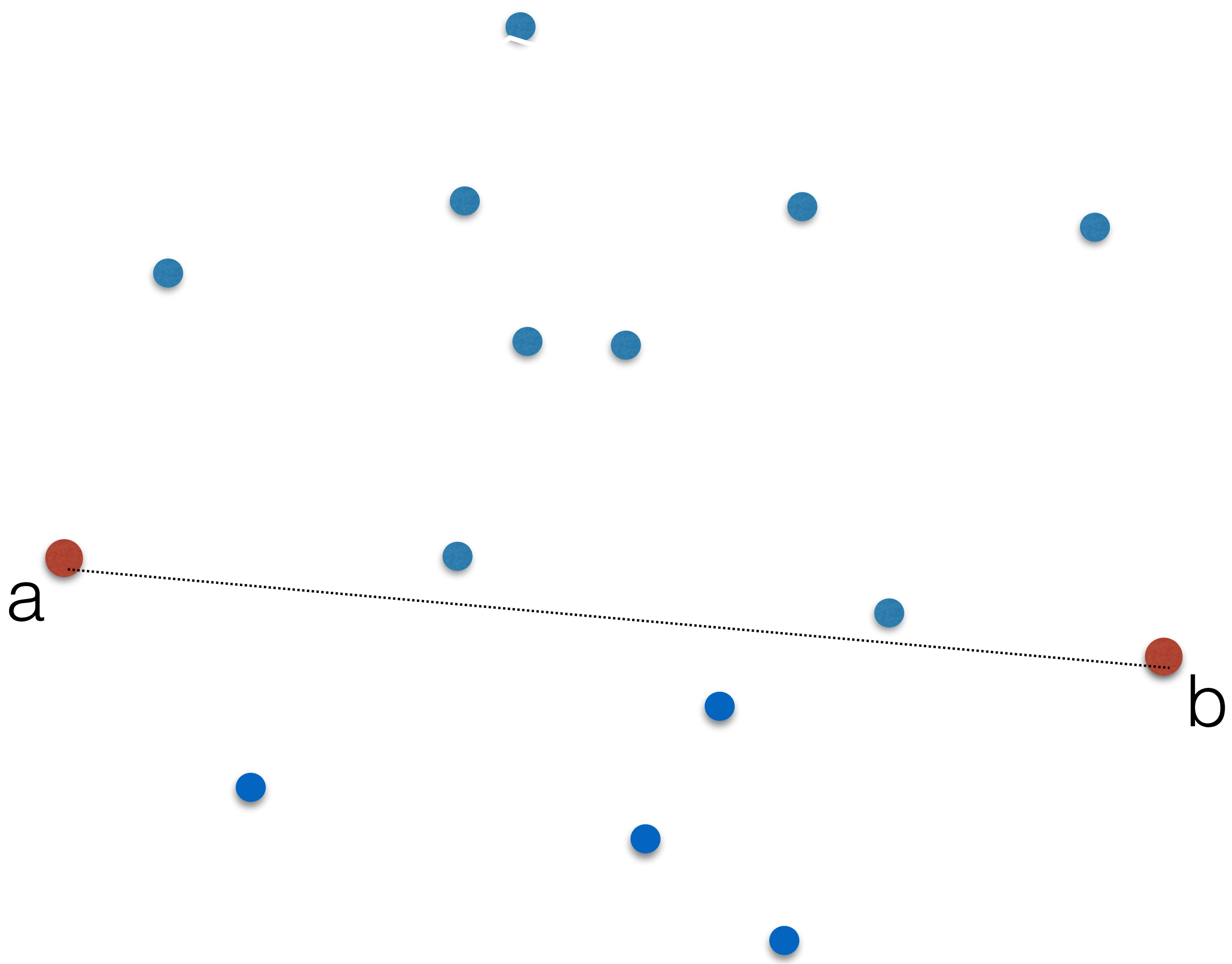
Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Find upper hull and lower hulls separately



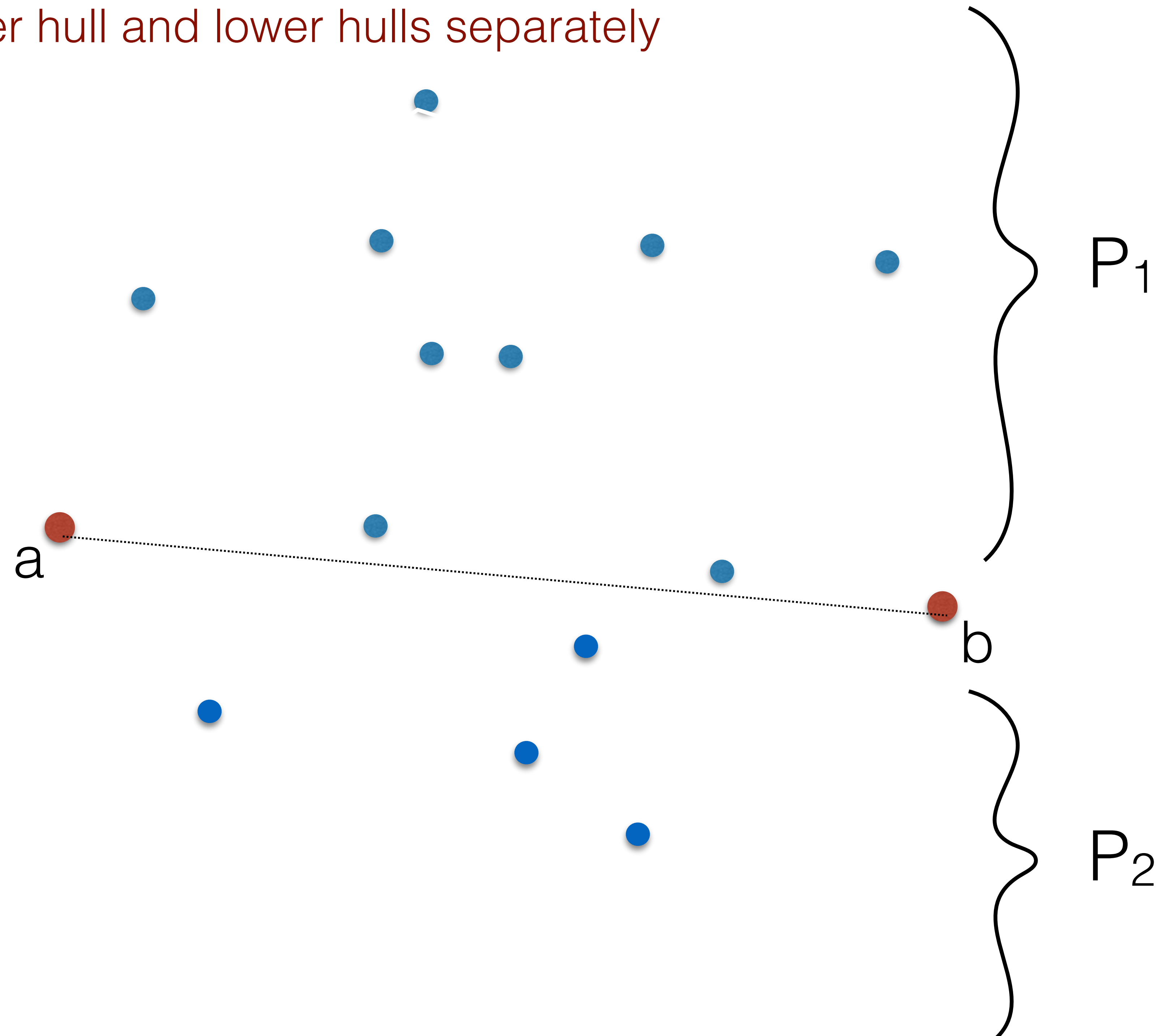
Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Find upper hull and lower hulls separately



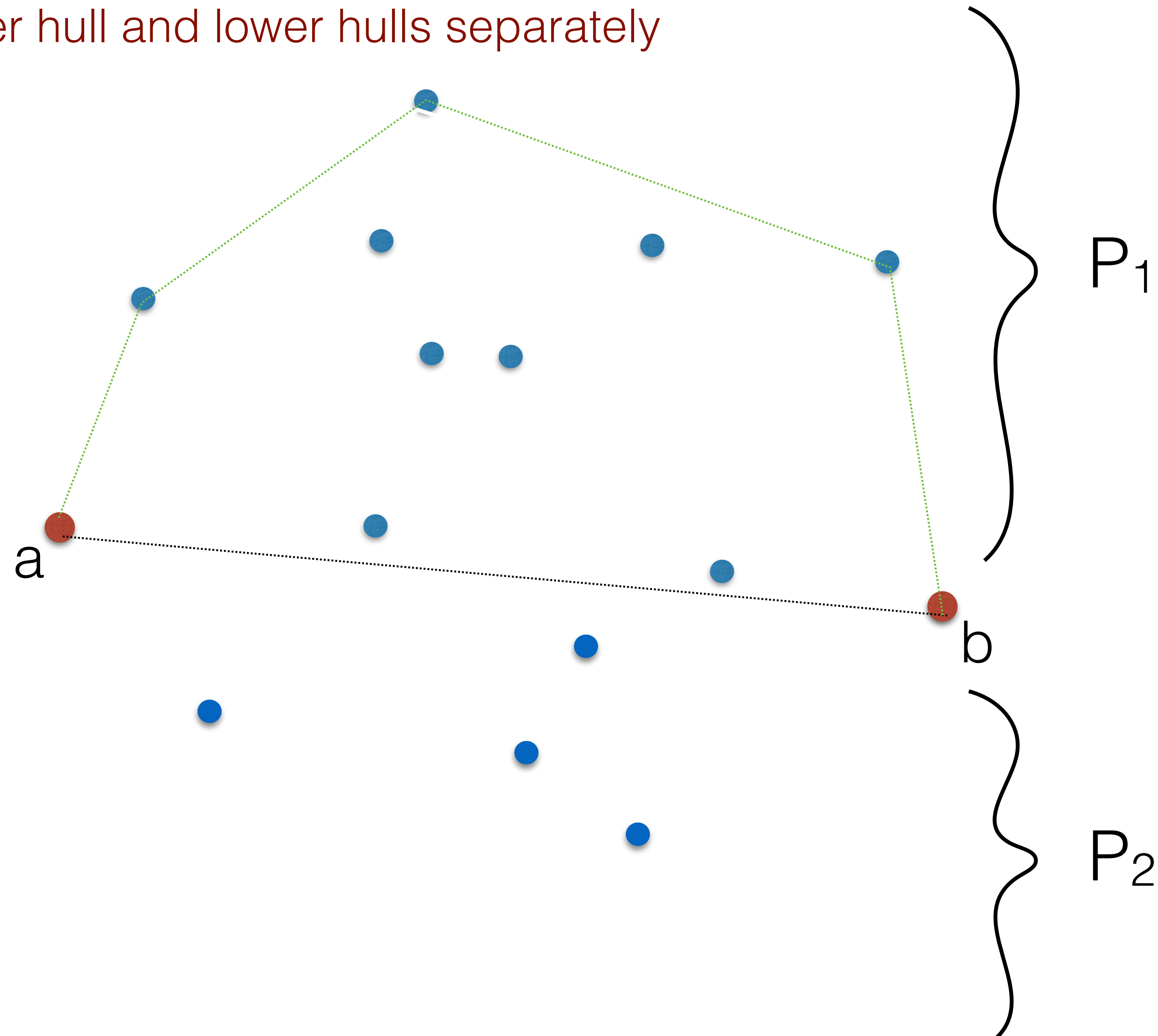
Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Find upper hull and lower hulls separately



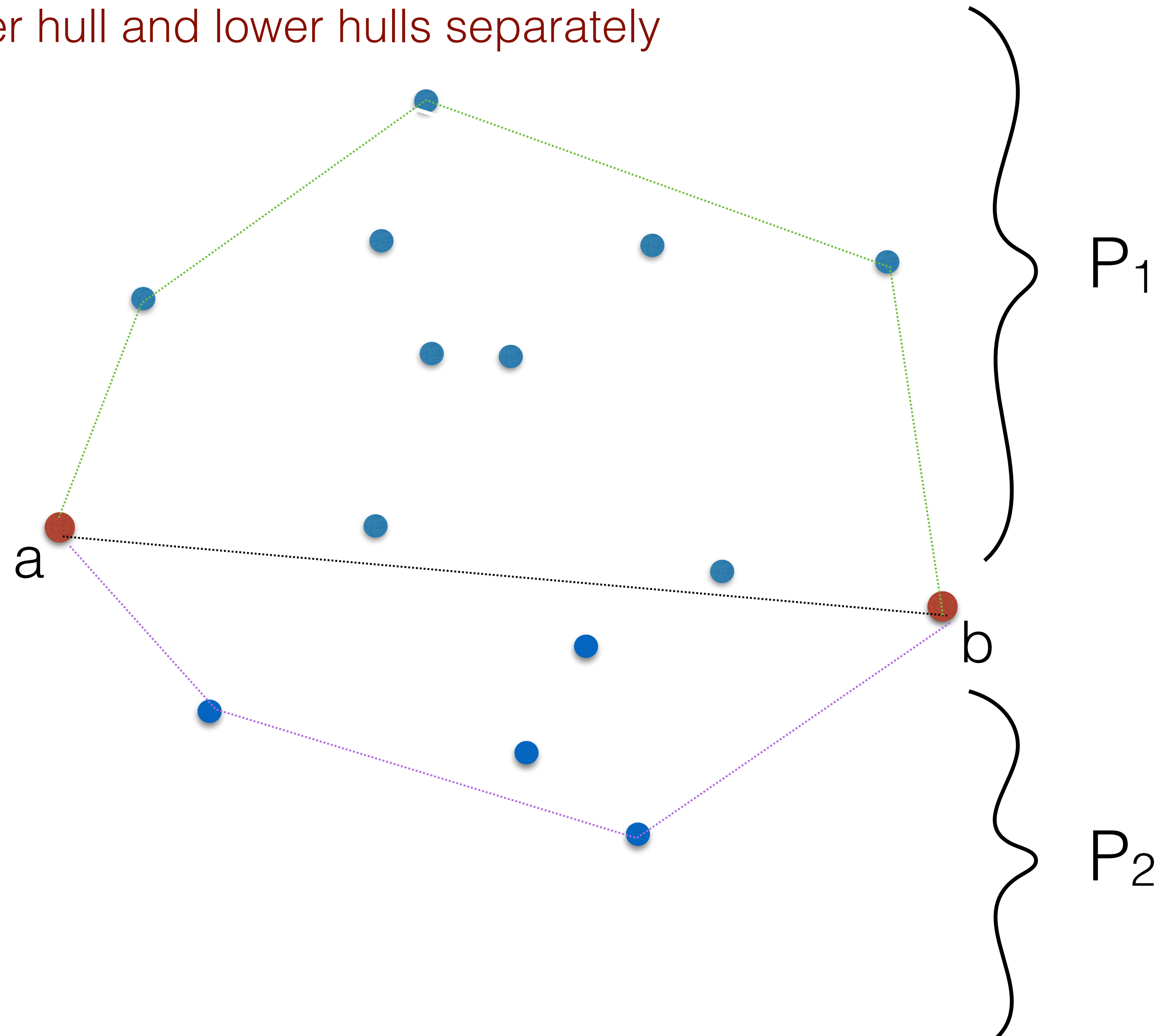
Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Find upper hull and lower hulls separately



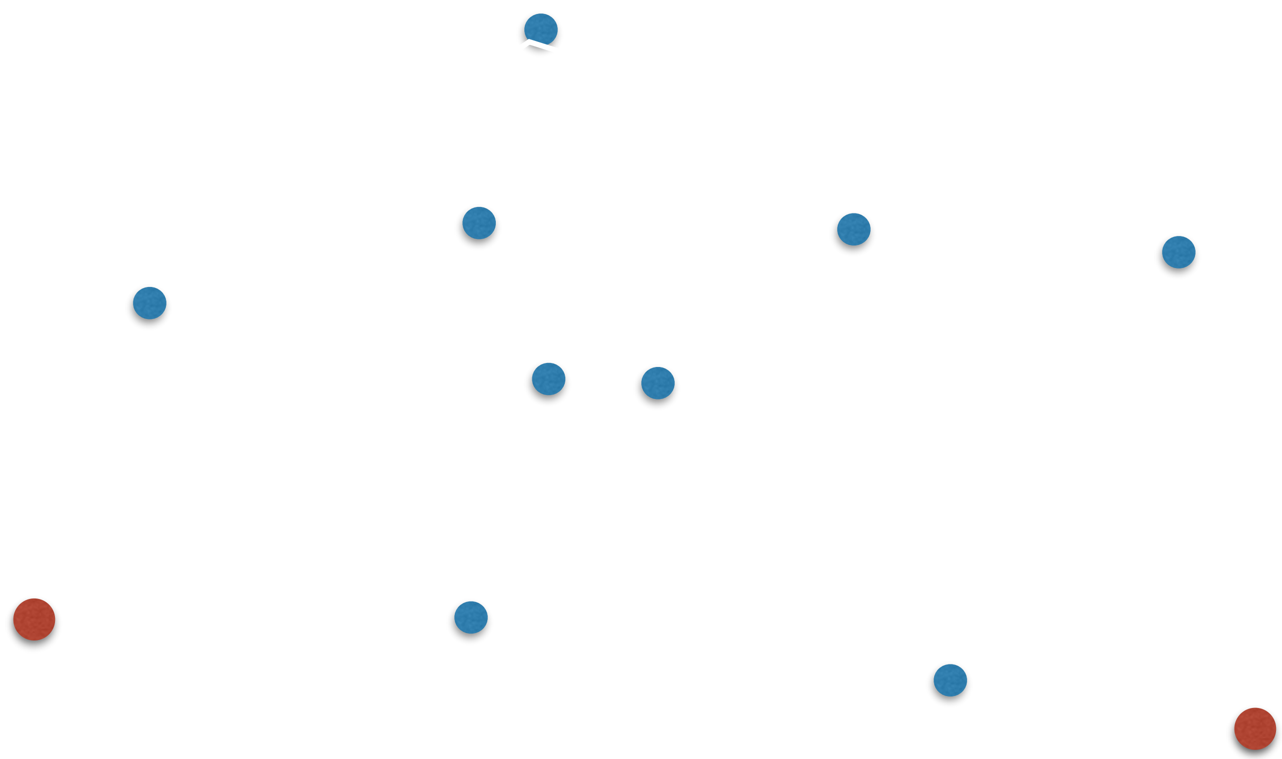
Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Find upper hull and lower hulls separately



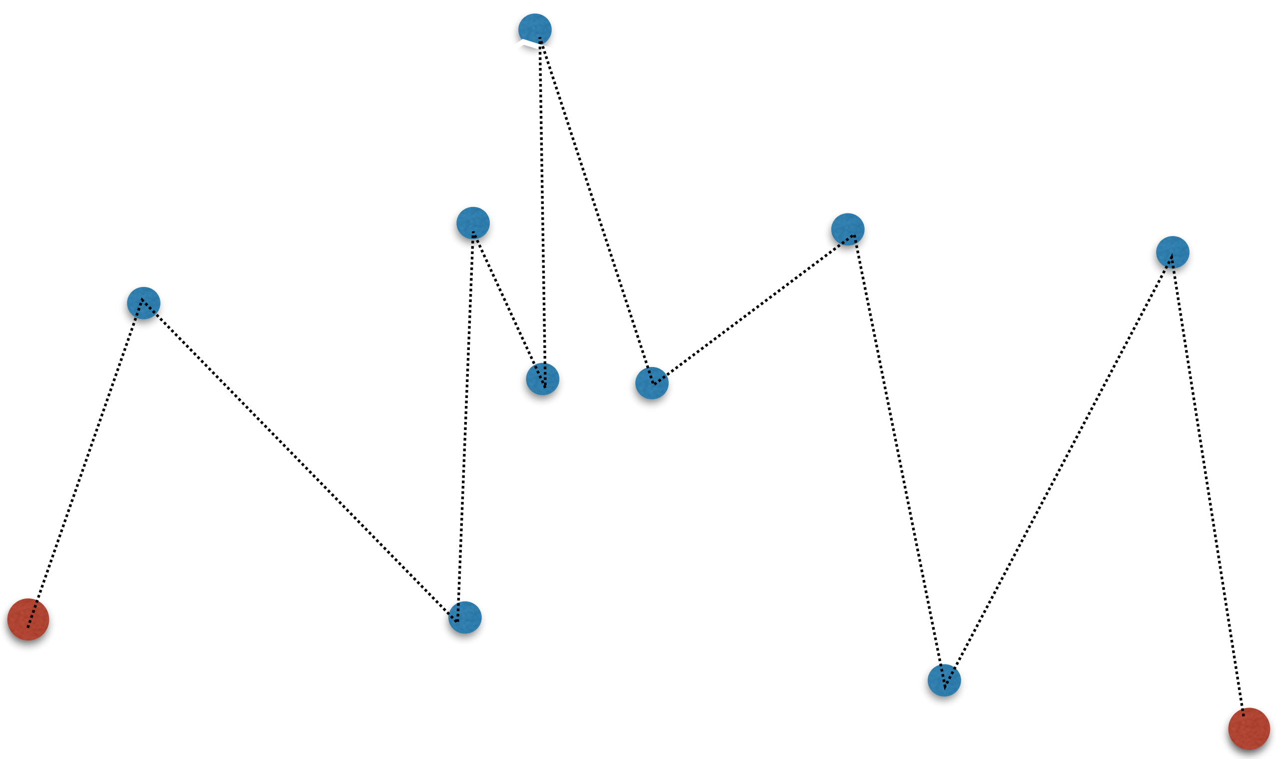
Andrew’s Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



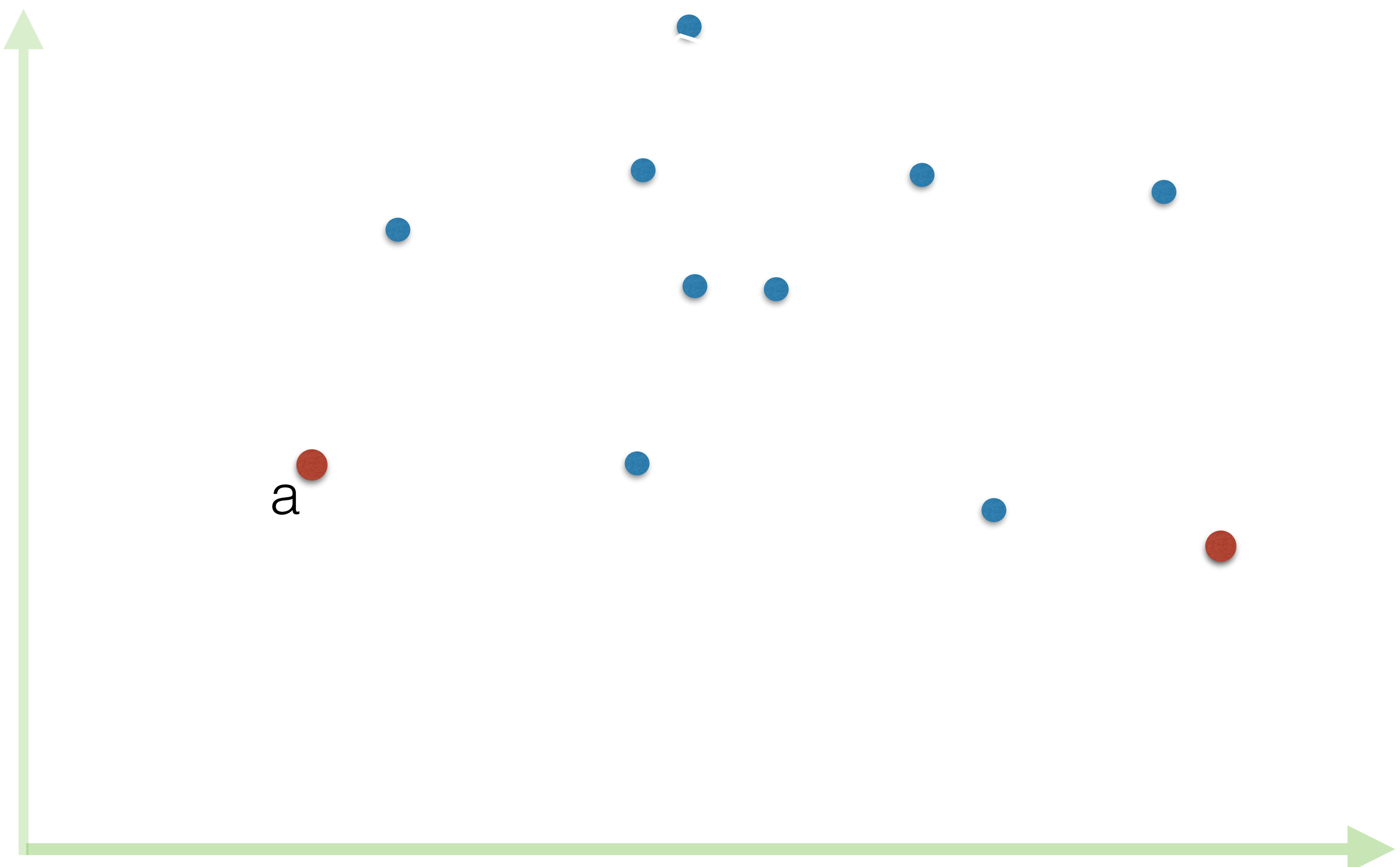
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



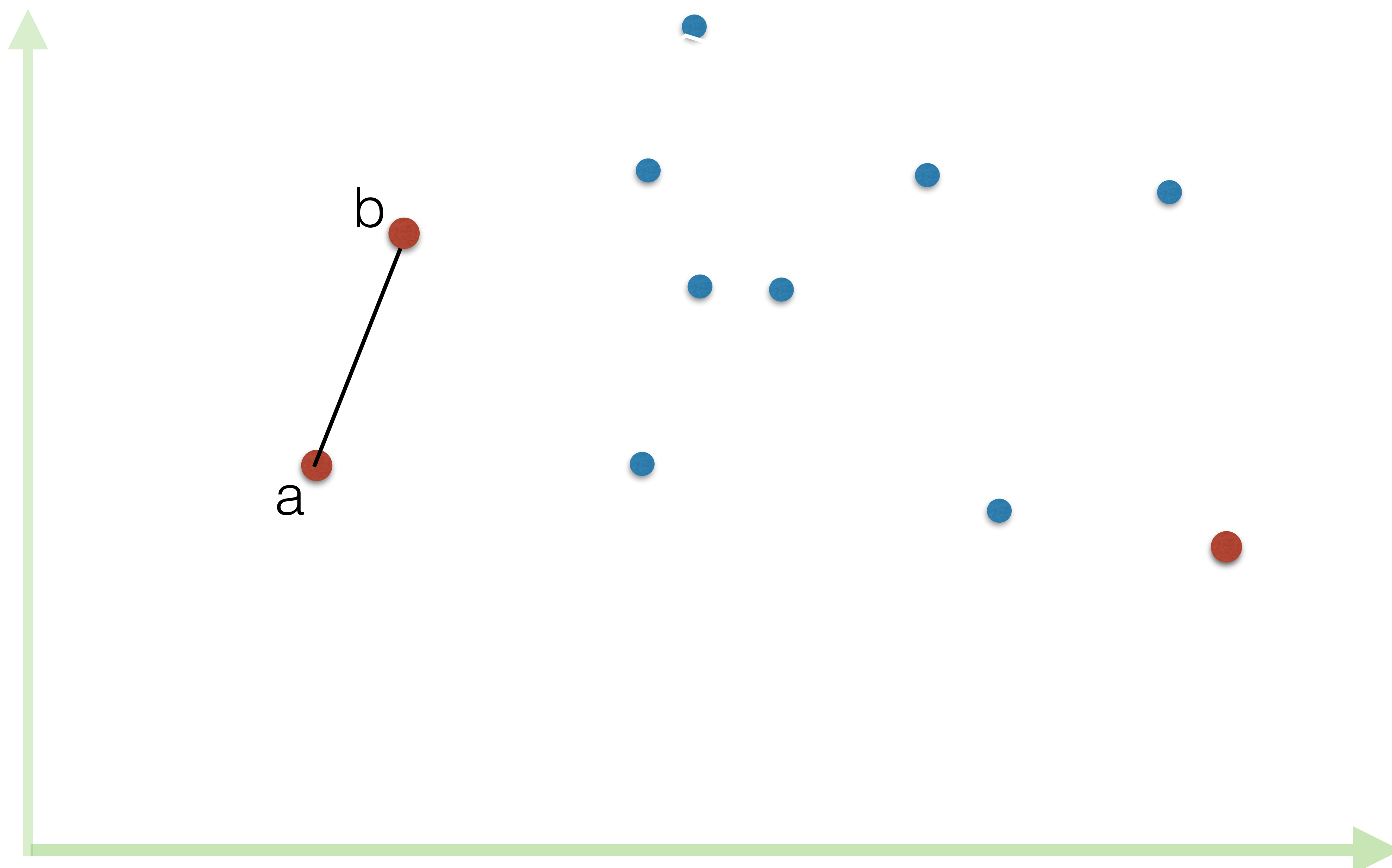
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



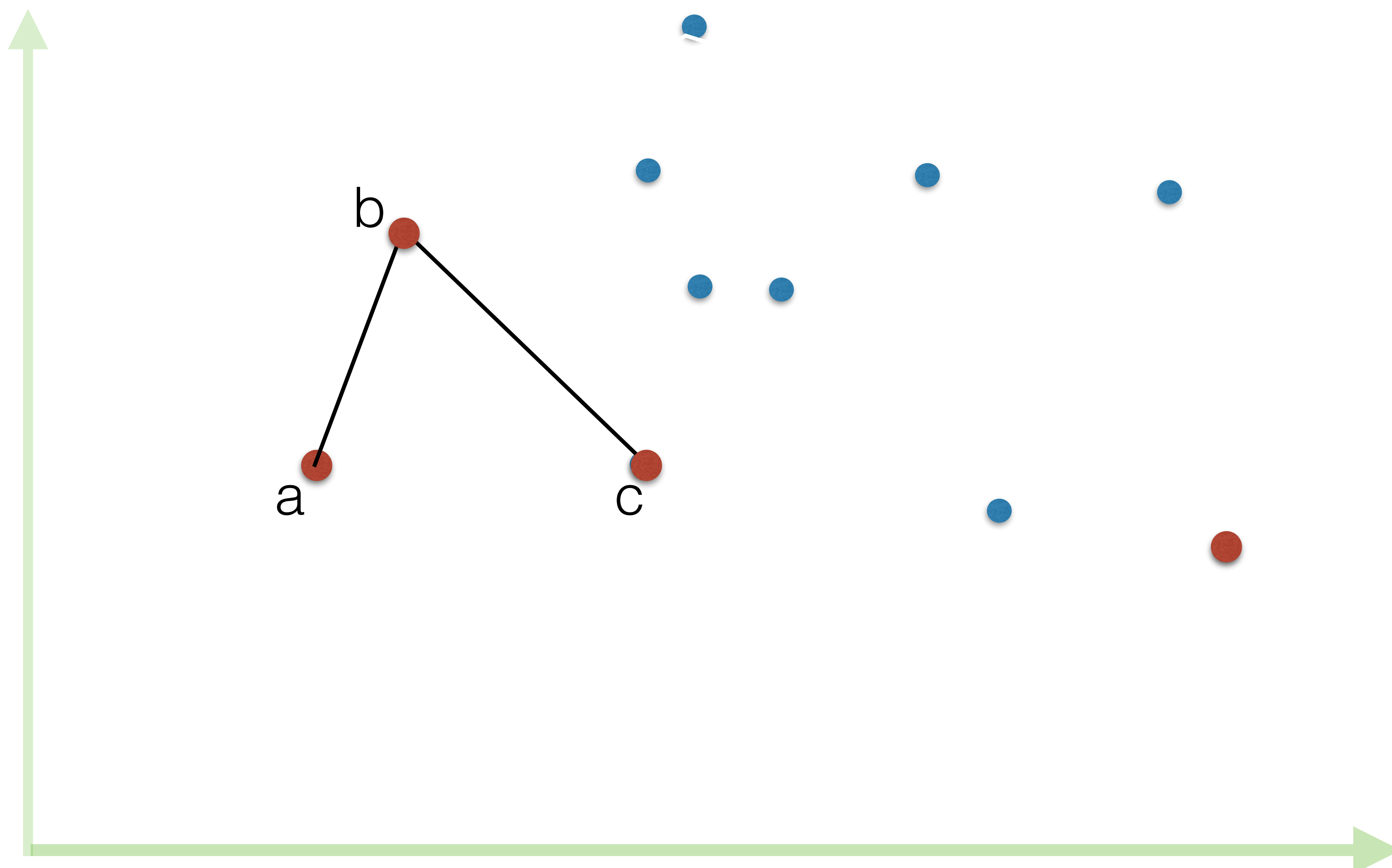
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



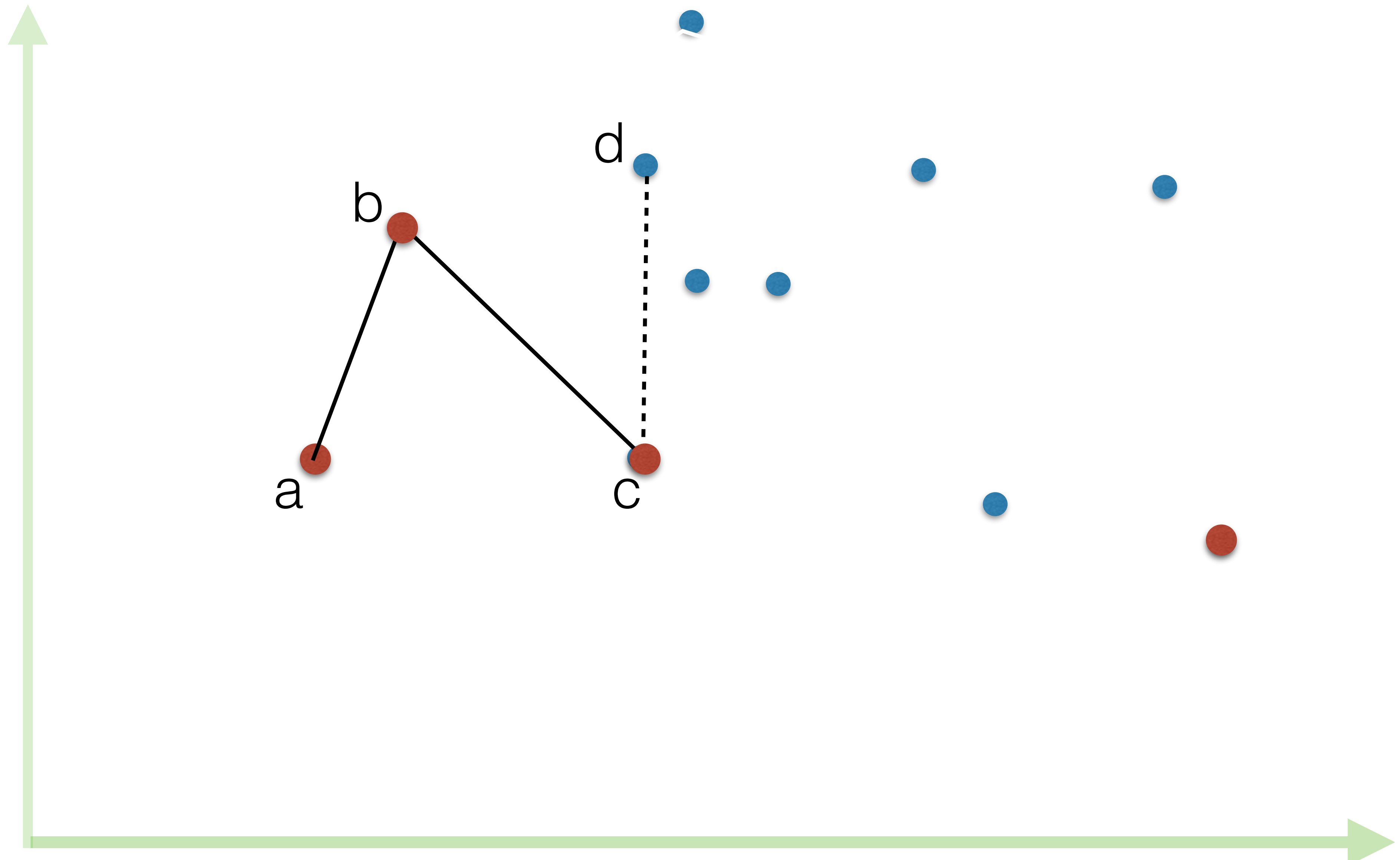
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



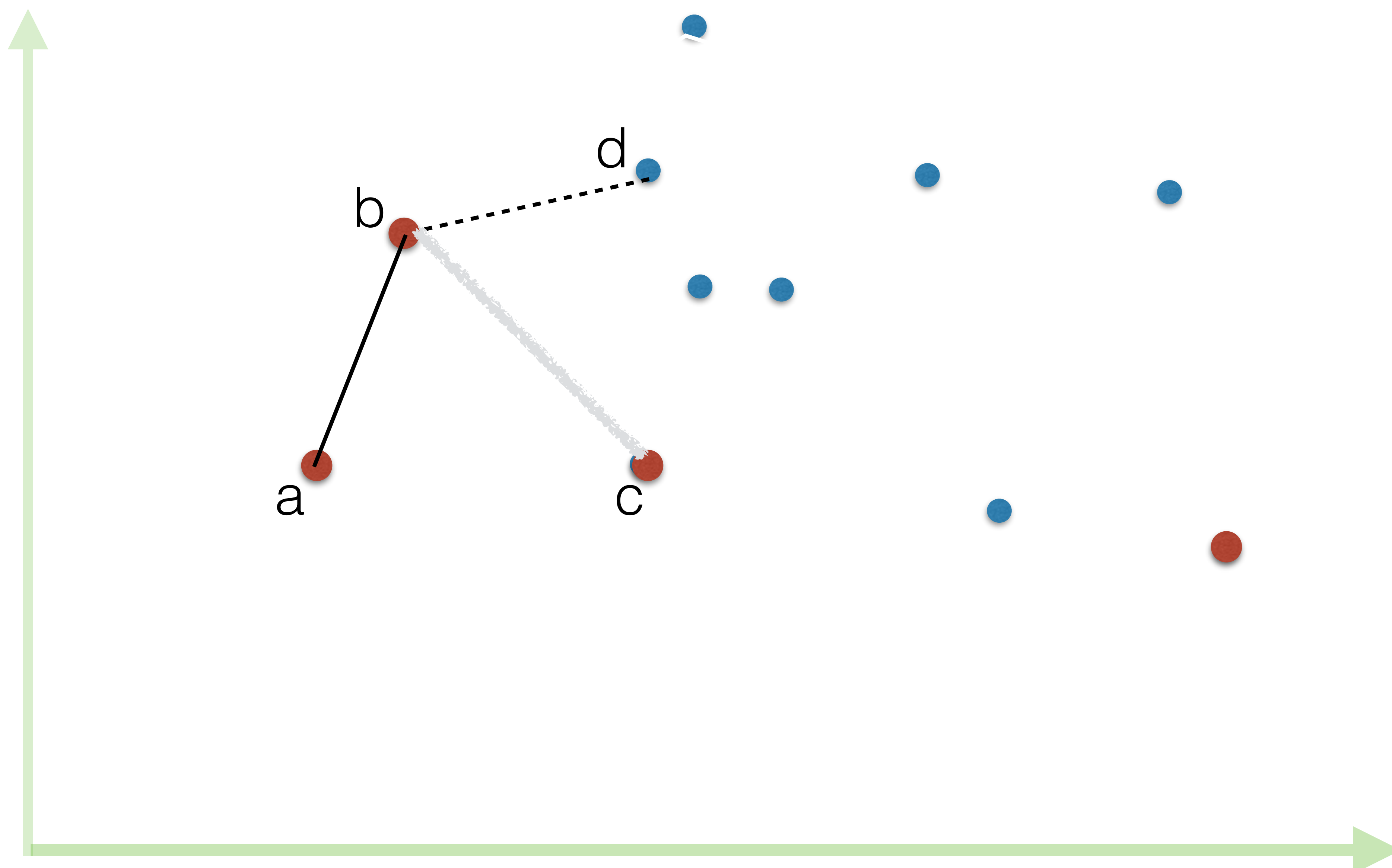
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



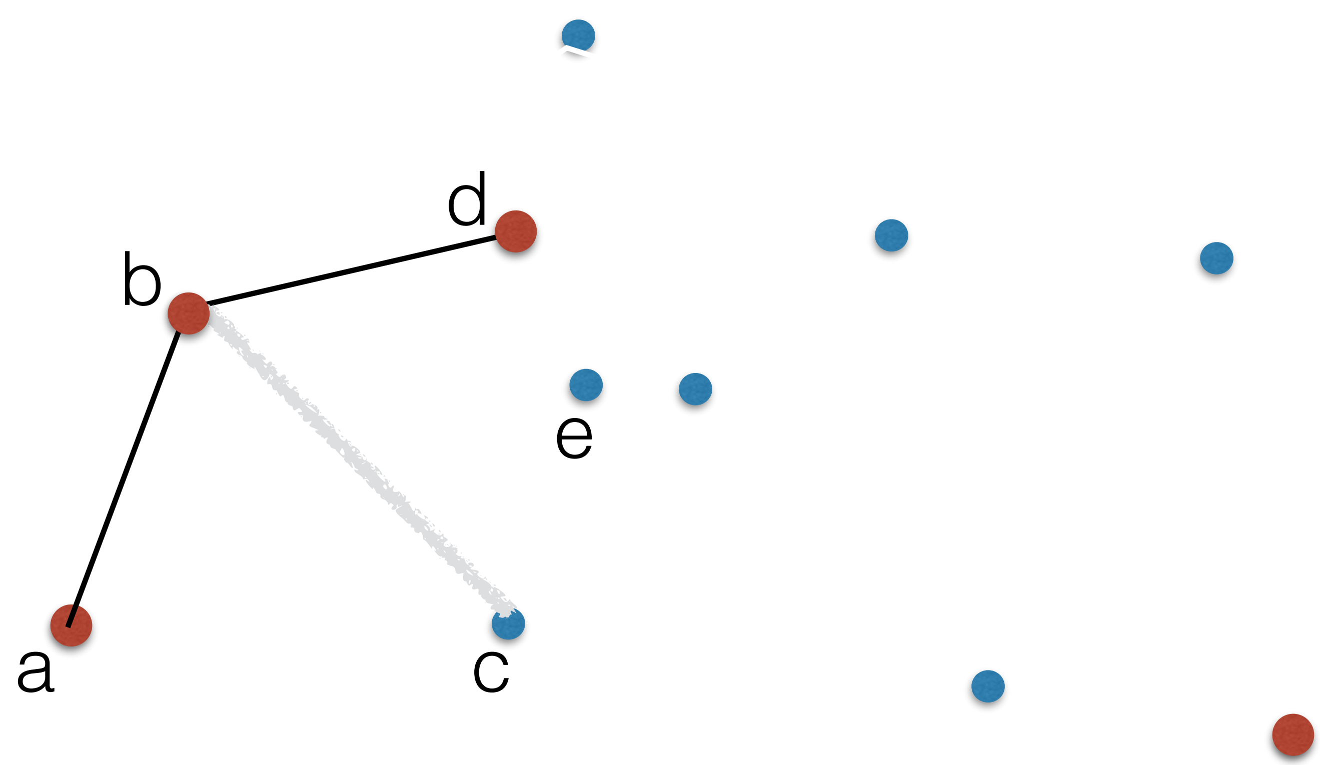
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



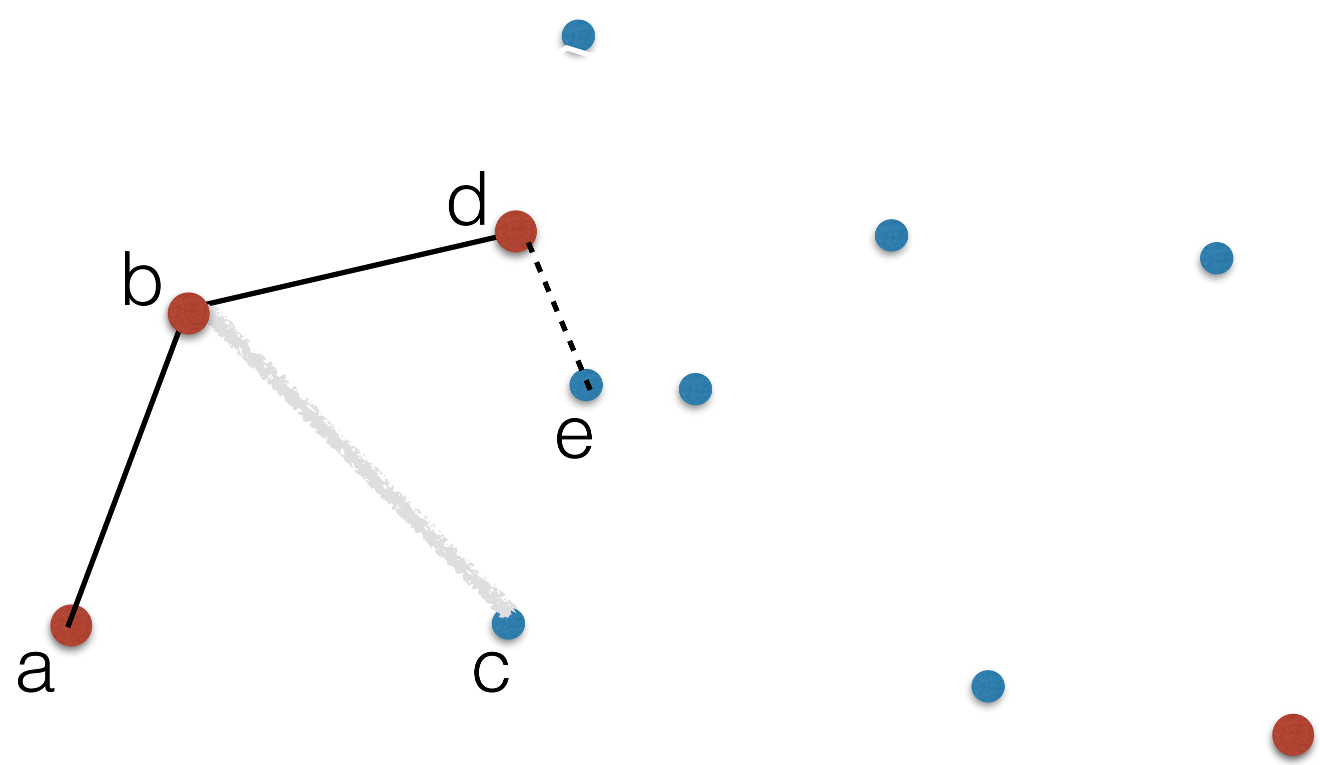
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



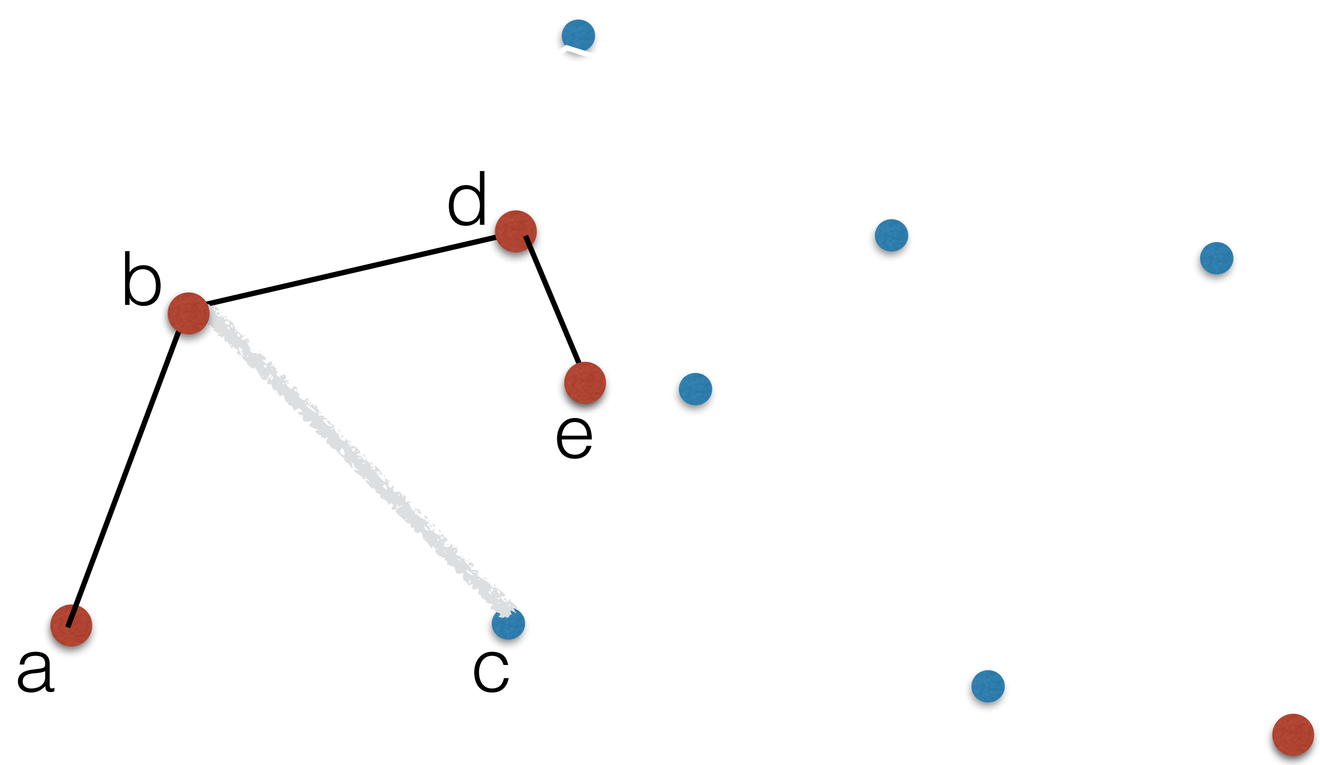
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



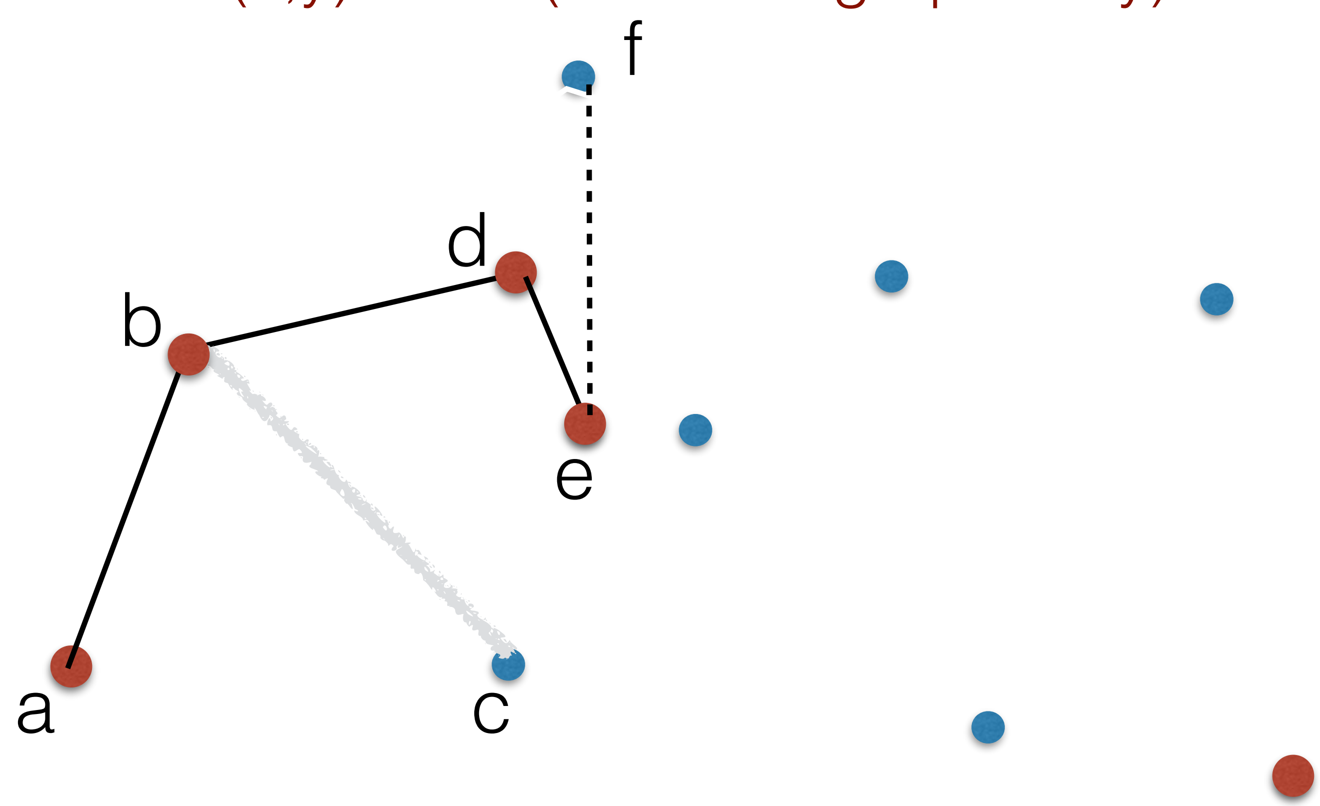
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



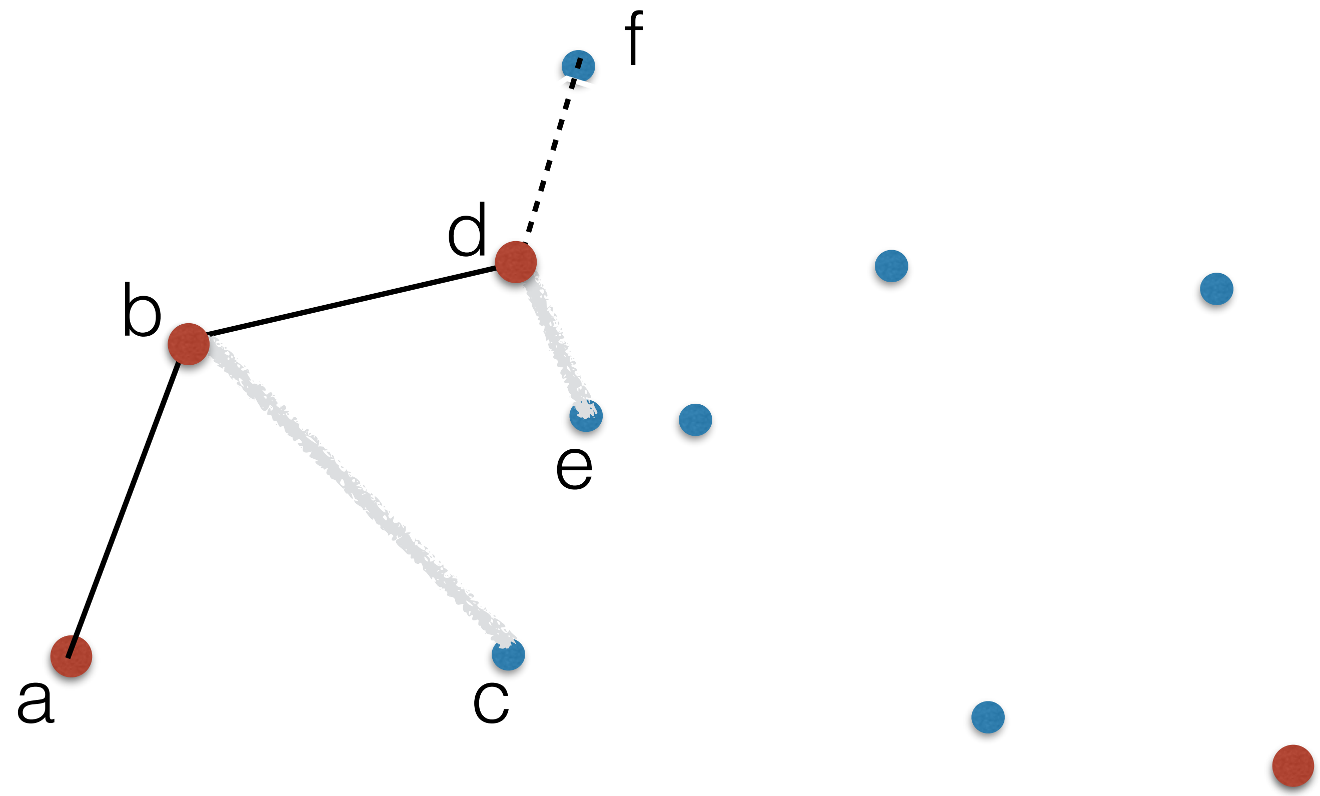
Andrew’s Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



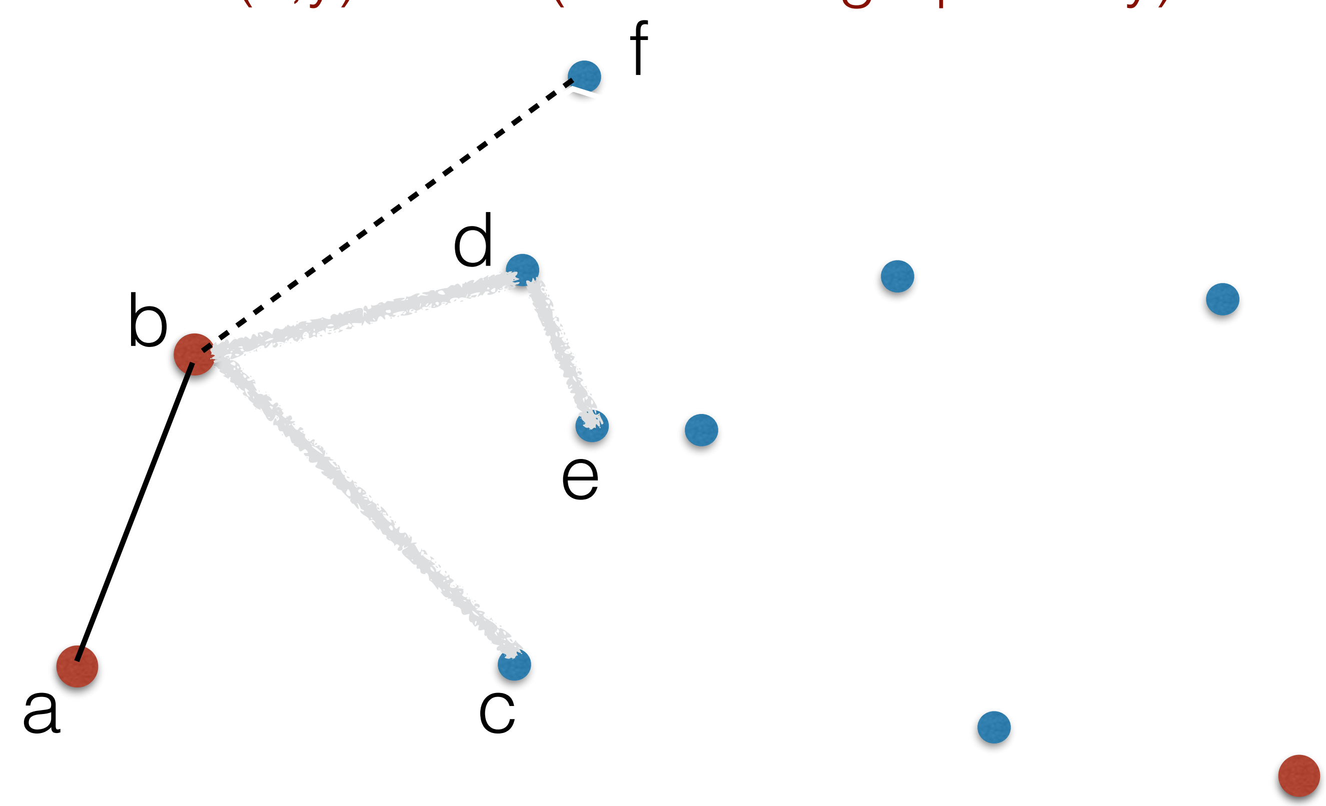
Andrew’s Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



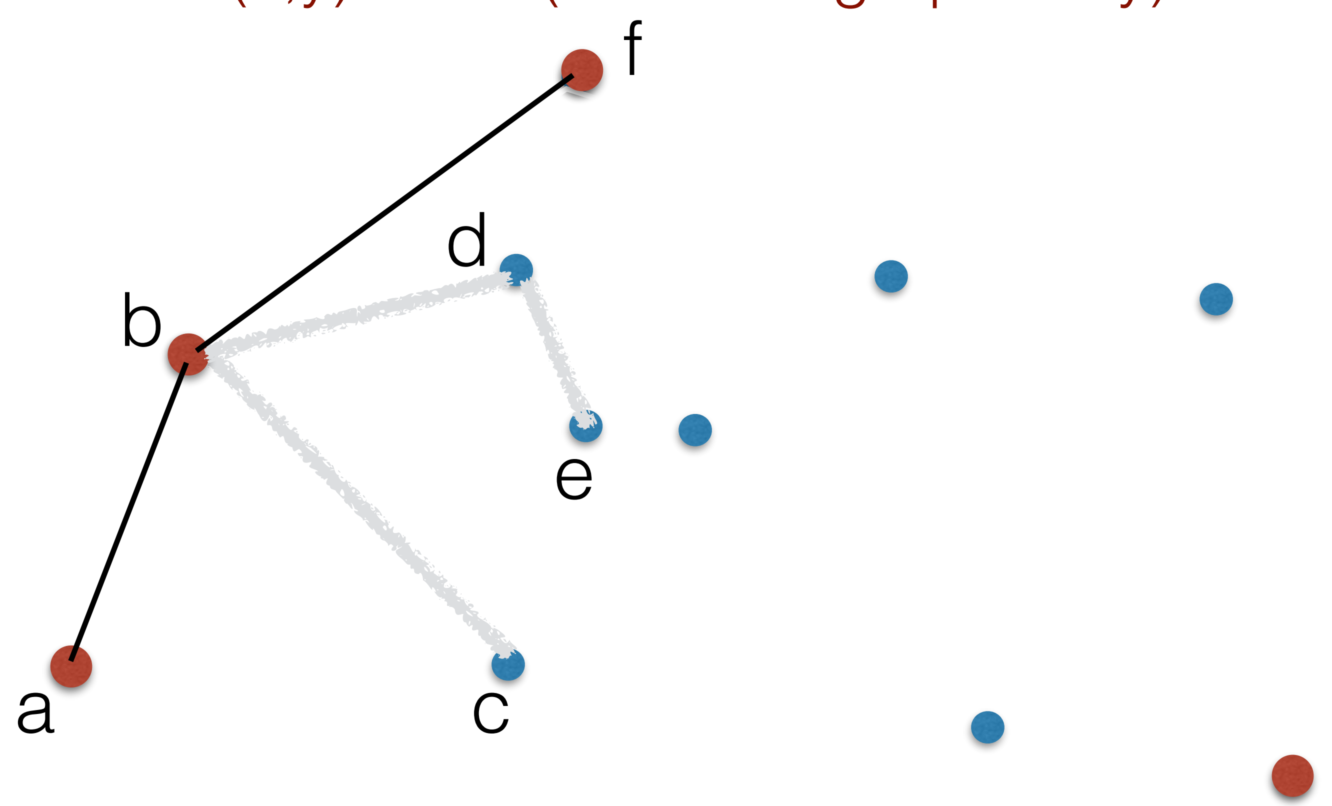
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



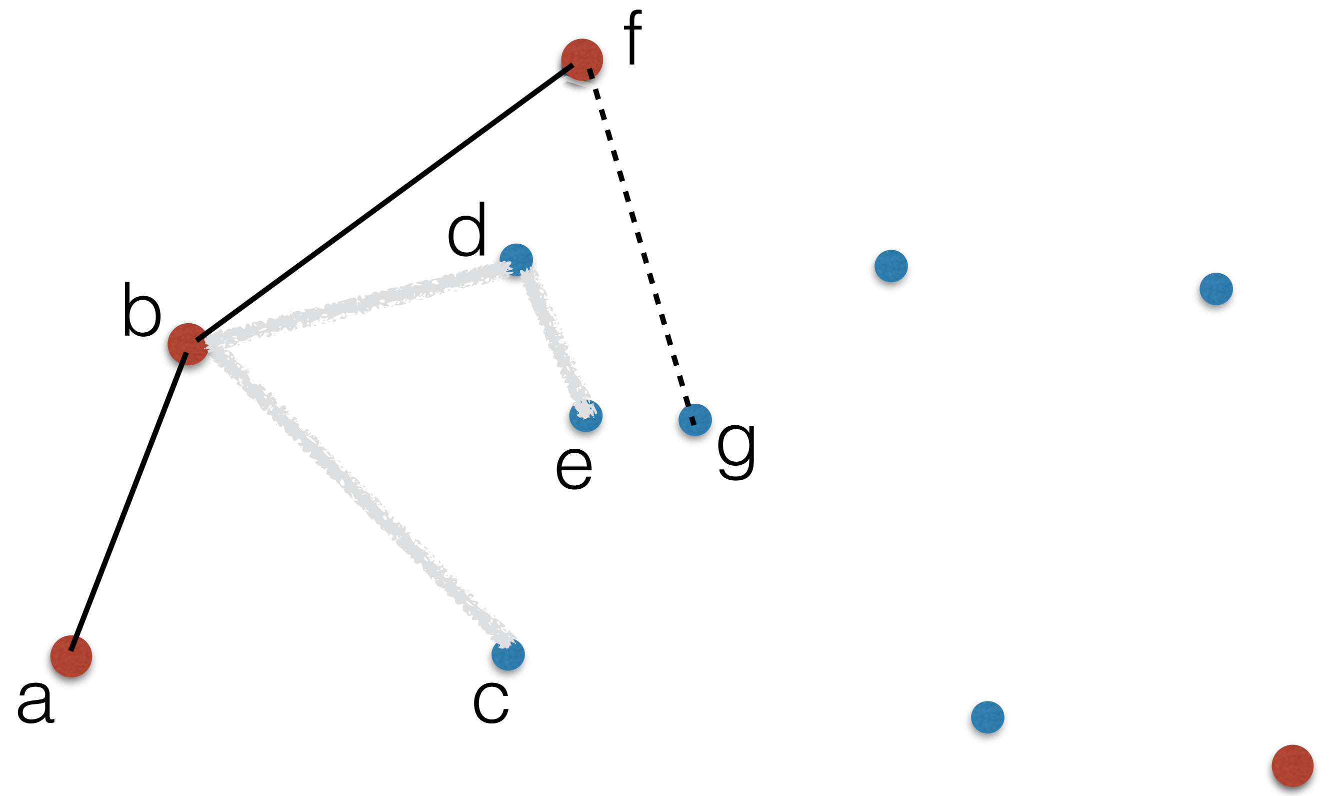
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



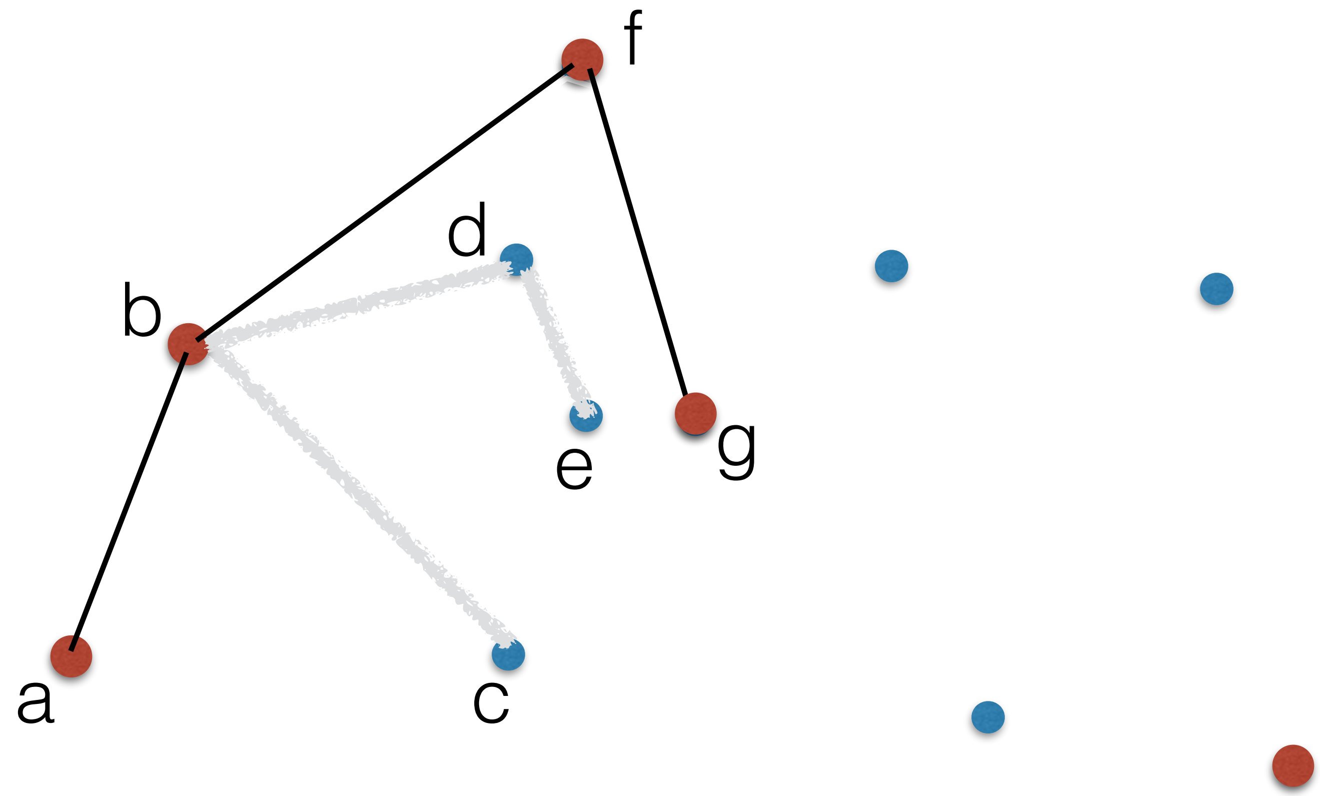
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



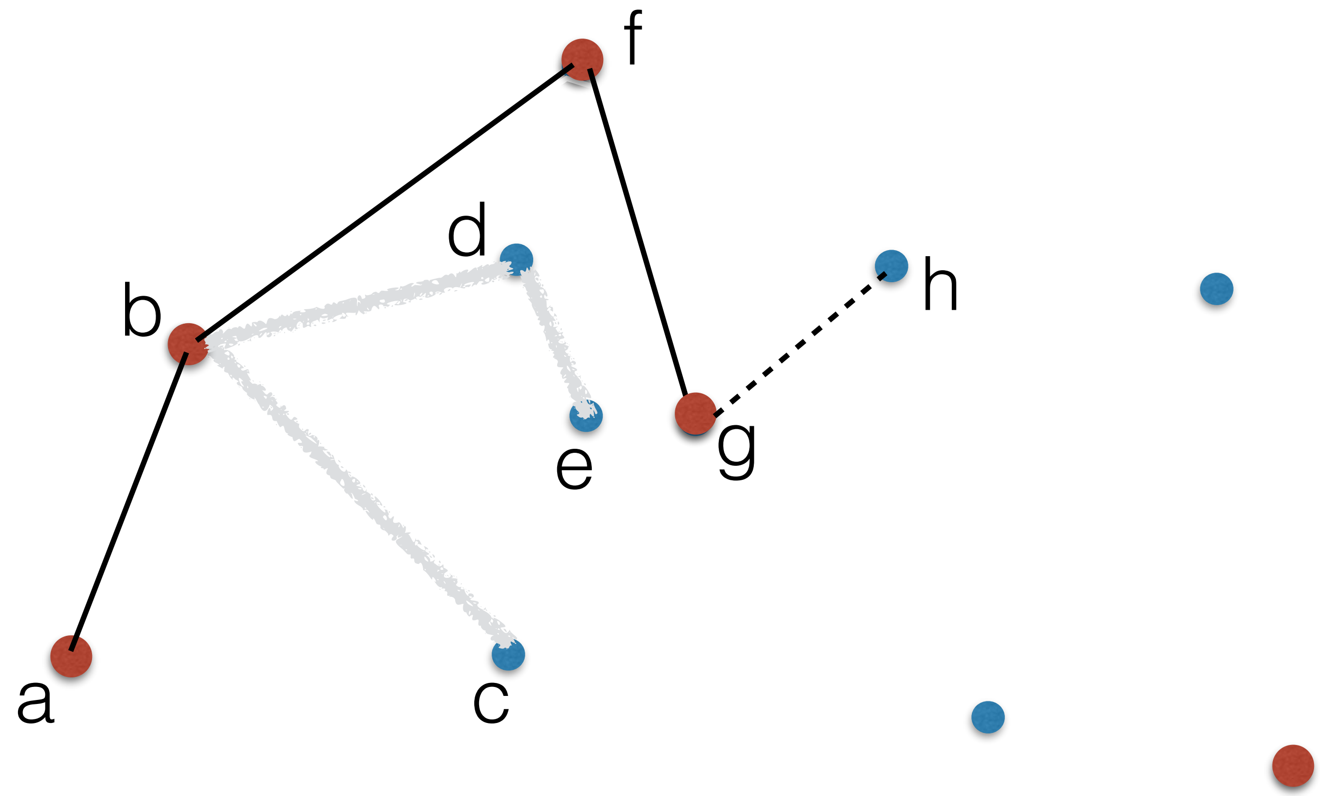
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



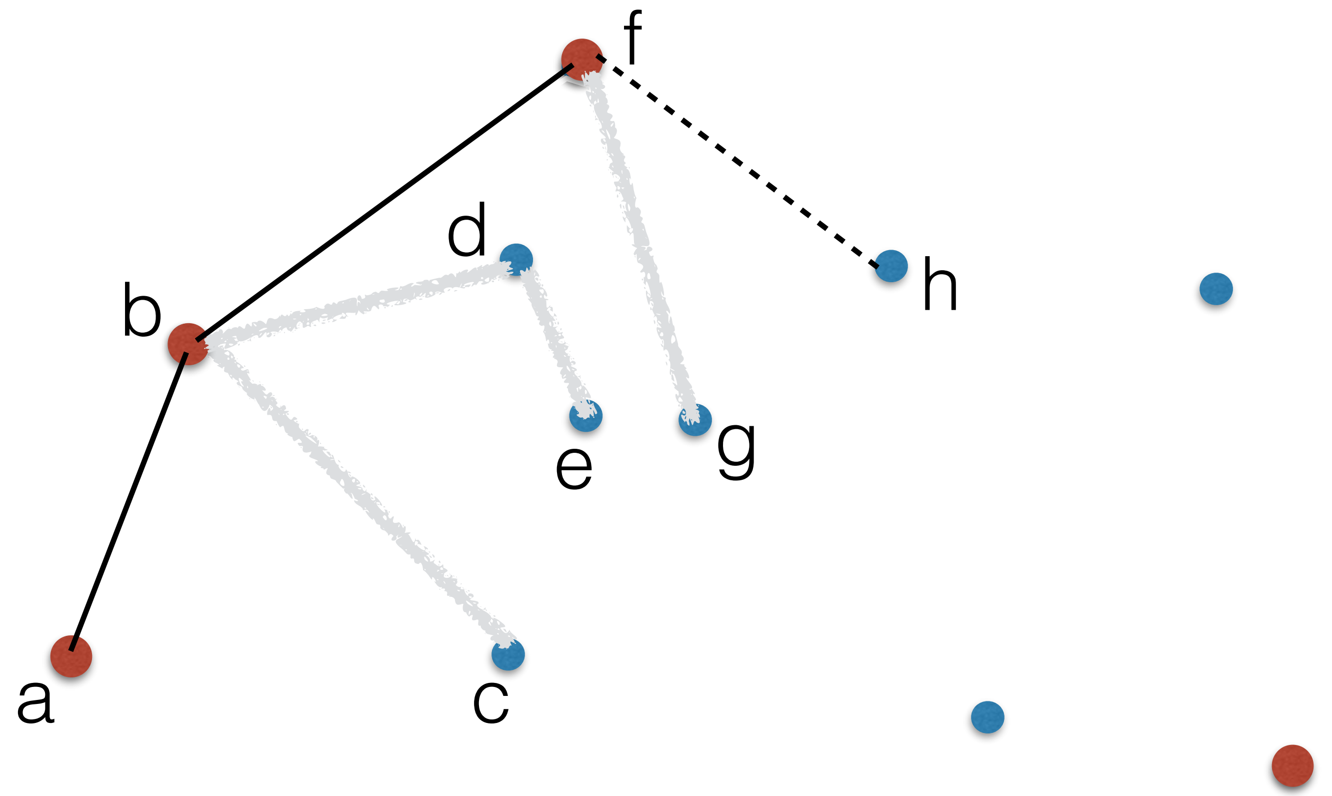
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



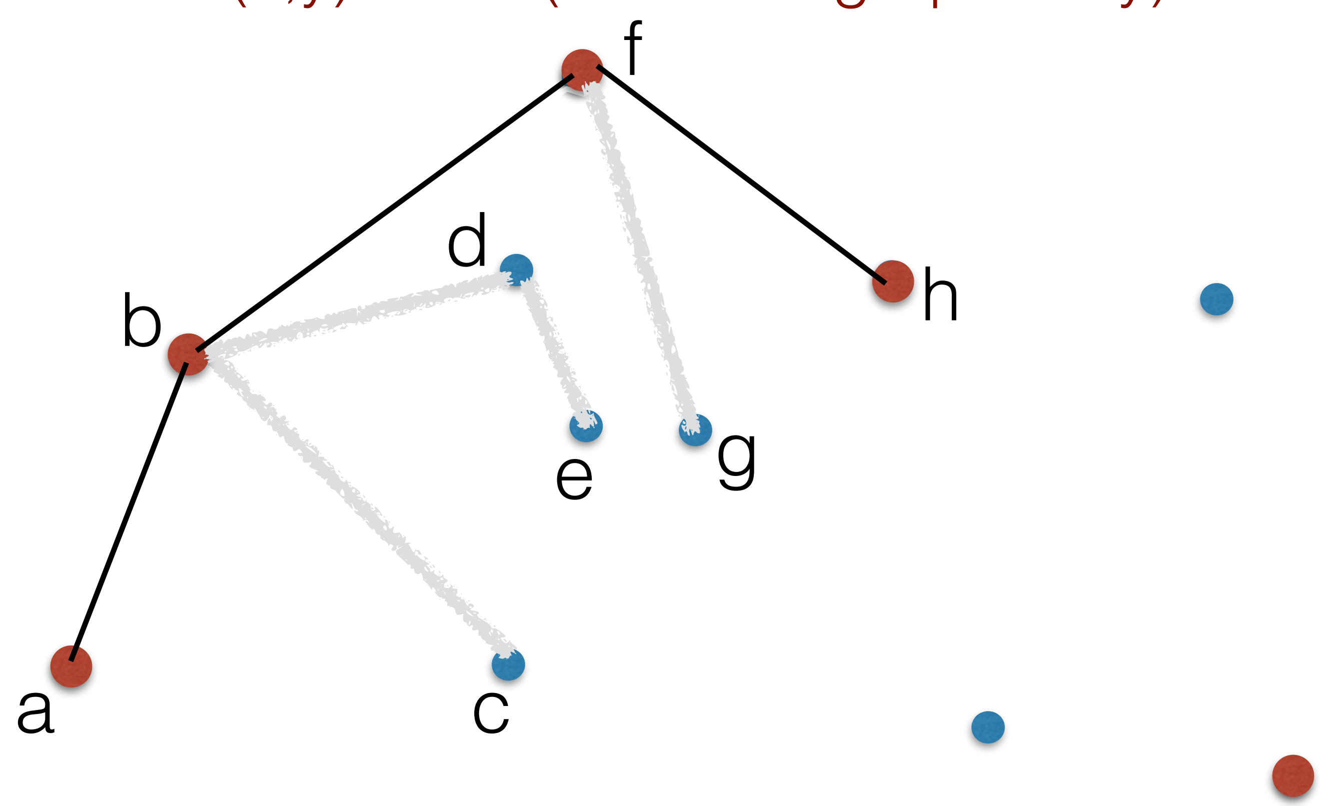
Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



Andrew's Monotone Chain Algorithm (1979)

- Goal: find the CH of P1
- Idea: Traverse points in (x,y) order (i.e. lexicographically)



and so on..

Andrew's Monotone Chain Algorithm (1979)

- Alternative to Graham's scan
- Idea: Traverse points in (x,y) lexicographic order (instead of radial order)
- Runs in sort + scan
- Sorting lexicographically is faster than sorting radially

Convex hull: summary

Naive	$O(n^3)$	
Gift wrapping	$O(nh)$	1973
Graham scan	$O(n \lg n)$	1972
Andrew monotone	$O(n \lg n)$	1979
Quickhull	$O(n^2)$	1977

Can we do better?

Lower bound

What is a lower bound?

- Given an algorithm A, its **worst-case running time** is the **largest** running time on any input of size n

$T_A(n) = \max_{|P|=n} \{ T(n) \mid T(n) \text{ is the running time of algorithm A on input P} \}$

What is a lower bound?

- Given an algorithm A, its **worst-case running time** is the **largest** running time on any input of size n

$$T_A(n) = \max_{|P|=n} \{ T(n) \mid T(n) \text{ is the running time of algorithm A on input P} \}$$

- A lower bound for CH: What is the **worst-case running time** of the **best possible** CH algorithm?

$$\min_A \{ T_A(n) \}$$

Consider **all** possible CH algorithms A

Take the overall smallest worst-case running time

What is a lower bound?

- Lower bounds depend on the machine model.
- The standard model is the decision tree (comparison) model.
- Example: Sorting lower bound in the comparison model is $\Omega(n \lg n)$

How do we prove lower bounds?

- Prove directly
    - Theorem: Any sorting algorithm that uses only comparisons must take at least $\Omega(n \lg n)$ in the worst case.
    - Proof: We saw this in Algorithms..
- Or via **reduction** from a problem known to have a lower bound
    - We'll use this to show that any algorithm for ConvexHull must have worst-case complexity $\Omega(n \lg n)$

Lower bounds by reduction

- We know that

  sorting >= $\Omega(n \lg n)$

- If we could show that ConvexHull is at least as hard as sorting

  ConvexHull $>=$ **sorting**

- This would imply that

  ConvexHull $>=$ $\Omega(n \lg n)$

How do we show that Convex Hull is at least as hard as sorting?

- We'll show that we can use ConvexHull to Sort
  - for any P, there exists some instance of the CH problem that sorts P, and we can build this instance in O(n) time

sorting

sort (array P)

- Input: array P
- Output: Sorted P

O(n)

reduces to

convex hull

convexHull(P)

- Input: array P
- Output: convex hull of P

sort (array P)

- create a set P' of points from P
- find CH(P')
- use the convex hull to infer sorted order of P

Sorting via CH

- ANALYSIS:
  - run CH(P)
  - O(n) to create P' and infer sorted order of P

sort (array P)

- create a set P' of points from P
- find CH(P')
- use the convex hull to infer sorted order of P

- Therefore this gives us an O(CH(n)) + O(n) algorithm for sorting

- But we know that we cannot sort faster than $\Omega(n \lg n)$ in the worst case
- Therefore we cannot solve CH faster than $\Omega(n \lg n)$ in the worst case

How do we show that Convex Hull is at least as hard as sorting?

We can use ConvexHull to Sort

$$\boxed{\text{sorting}} \xrightarrow[\text{reduces to}]{O(n)} \boxed{\text{convex hull}}$$

sorting is $\Omega(n \lg n)$

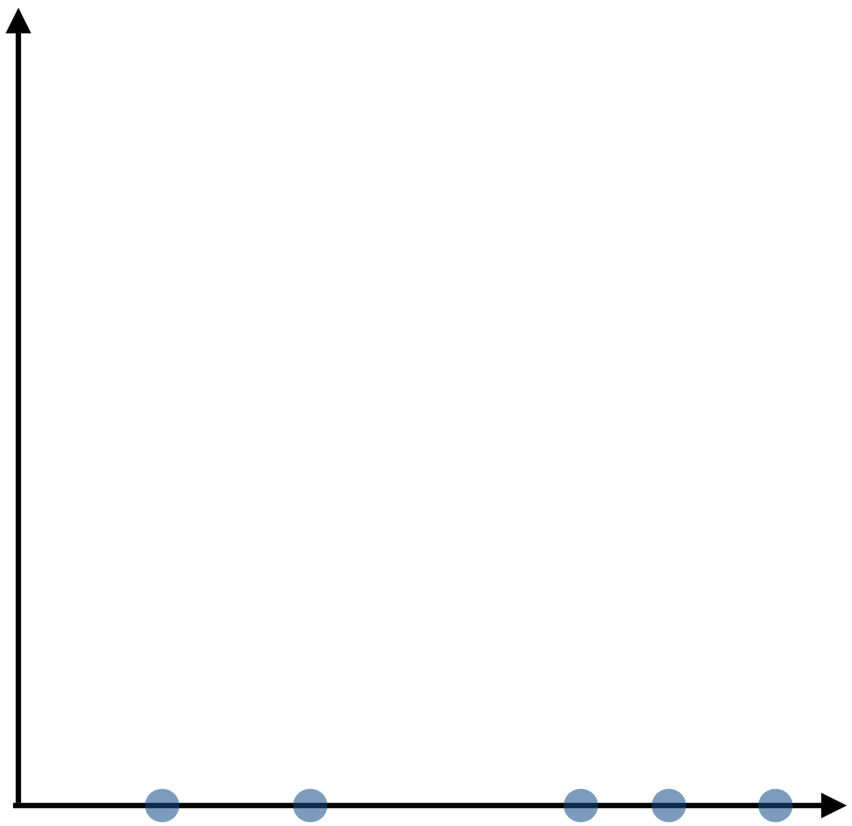
CH must be $\Omega(n \lg n)$

Sort = O(n) + O(Convex Hull)

Sorting “reduces” in O(n) to Convex Hull

Sorting reduces to CH

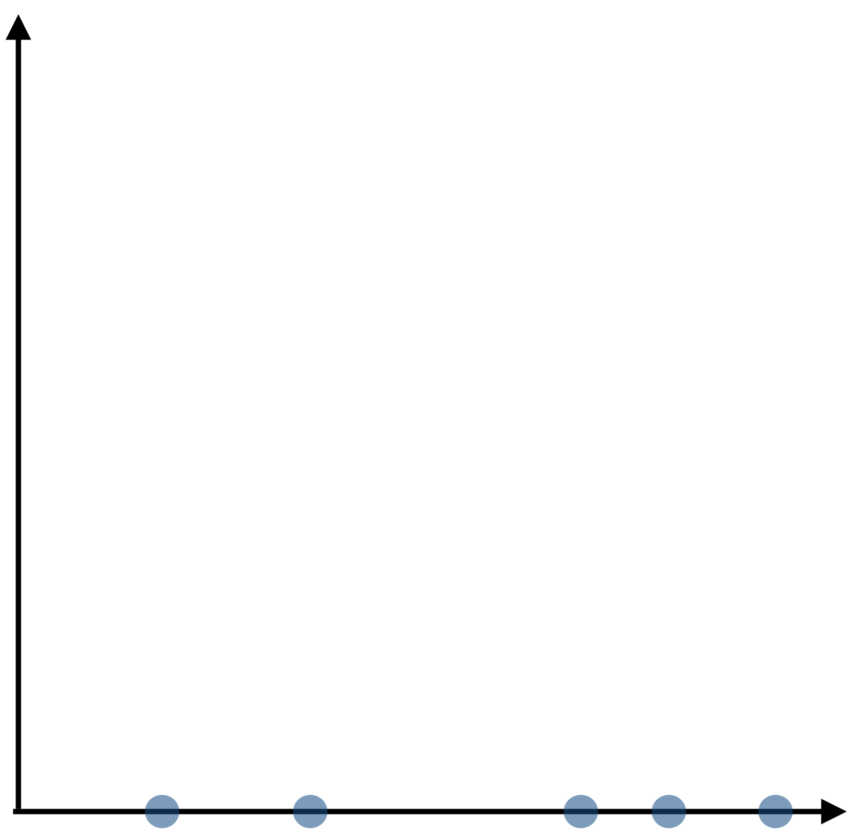
- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.

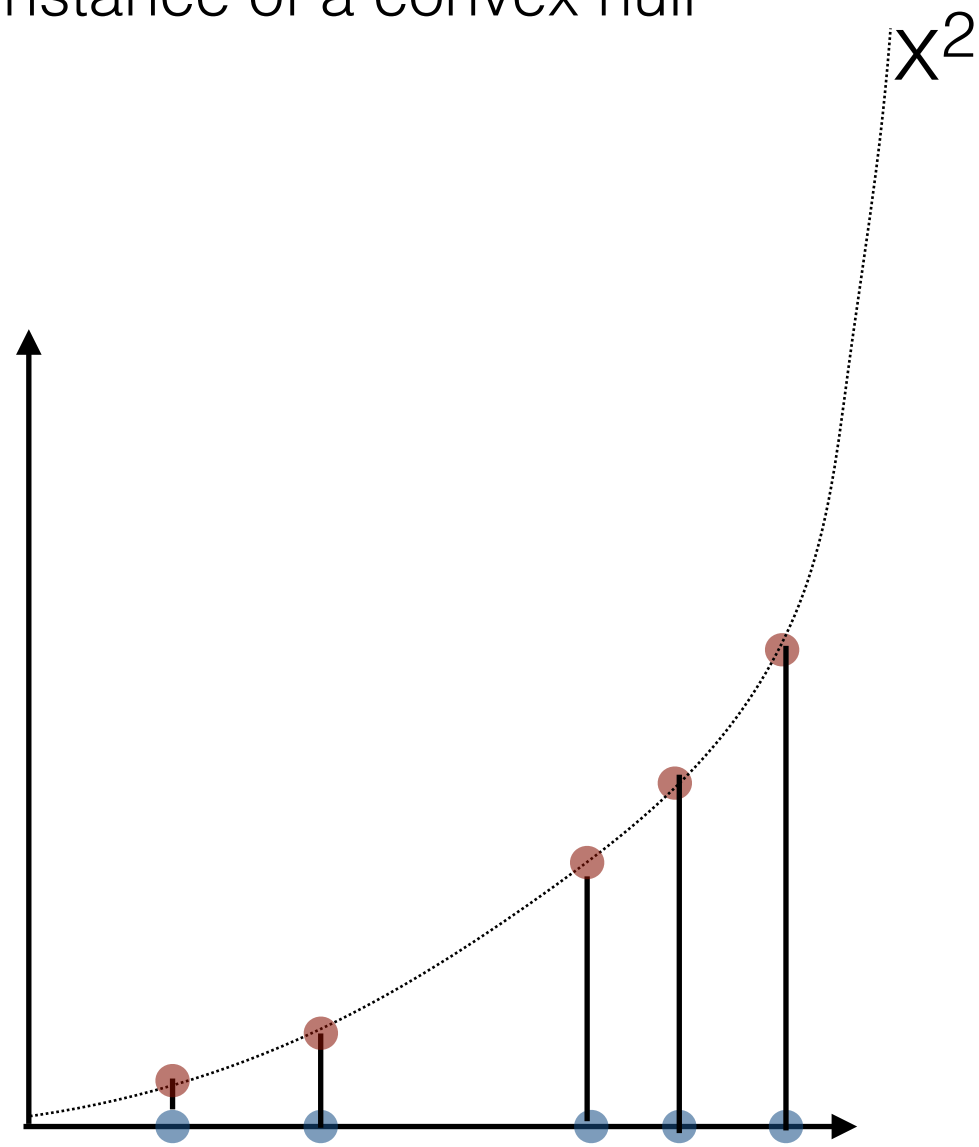


- Let P be the set of points x_1, x_2, …$x_{n.}$ to sort

Sorting reduces to CH

- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.

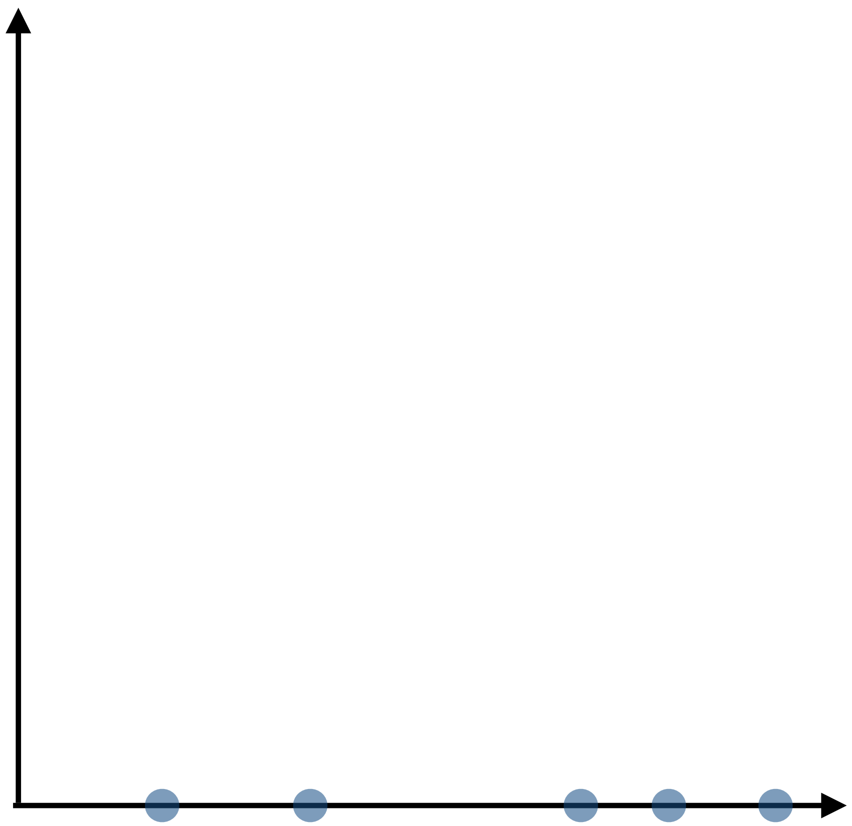


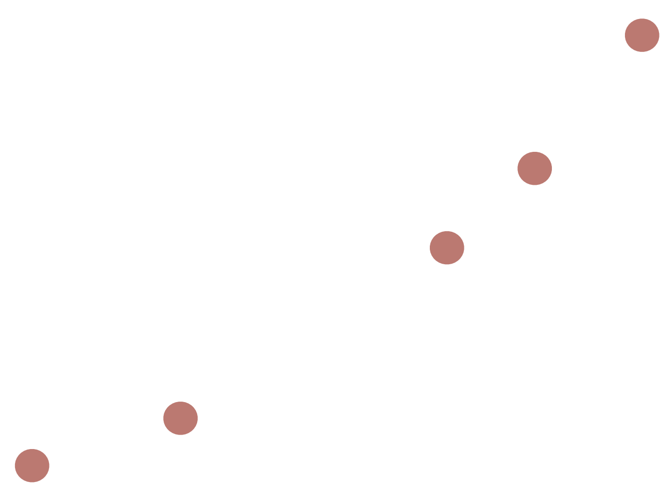


- Let P be the set of points x_1, x_2, …x_n. to sort
- Let P’: set of 2D points (x_i, x_i^2).

Sorting reduces to CH

- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.

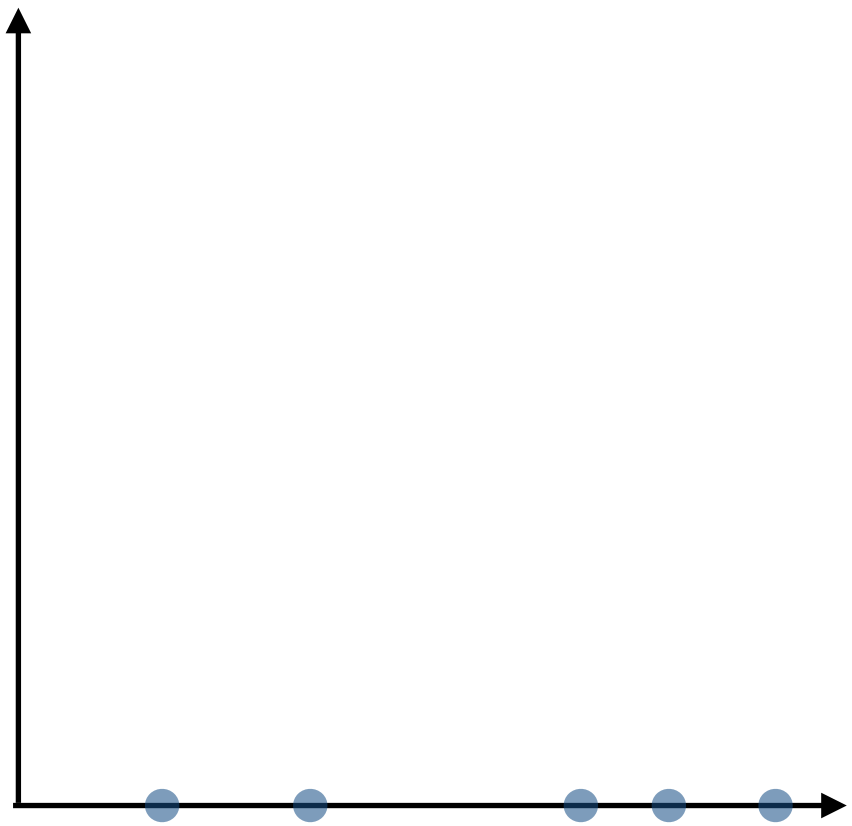




- Let P be the set of points x_1, x_2, …x_n. to sort
- Let P': set of 2D points (x_i, x_i^2).

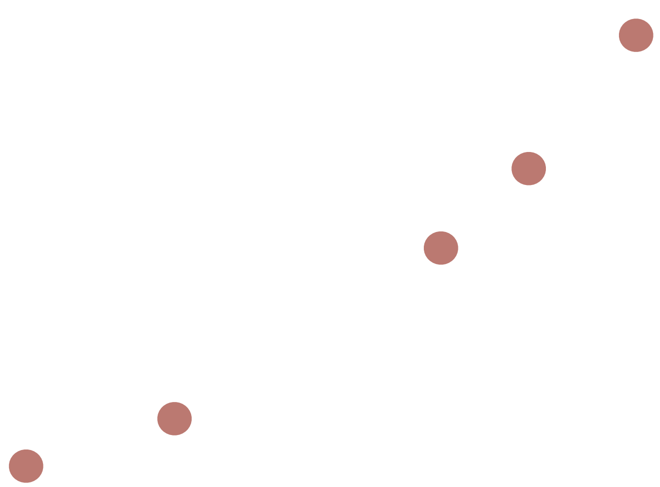
Sorting reduces to CH

- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.



- Let P be the set of points x_1, x_2, …x_n. to sort

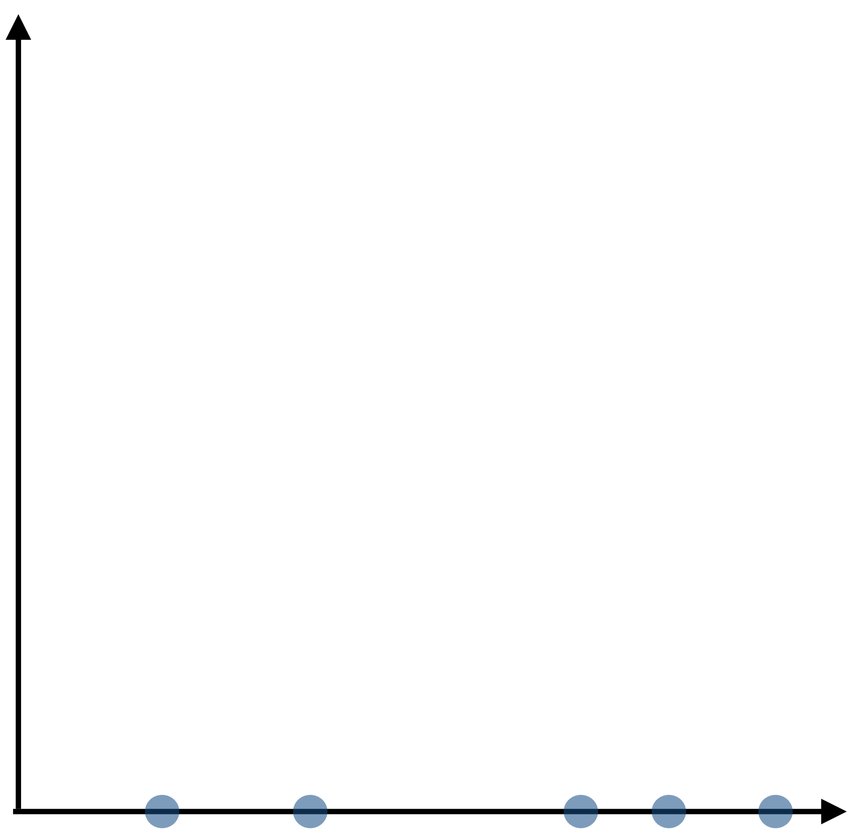
- Run CH(P') to find their convex hull



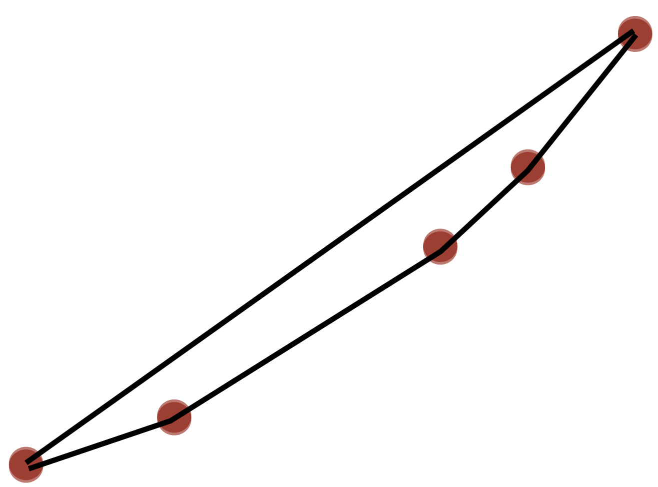
- Let P': set of 2D points (x_i, x_i^2).

Sorting reduces to CH

- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.



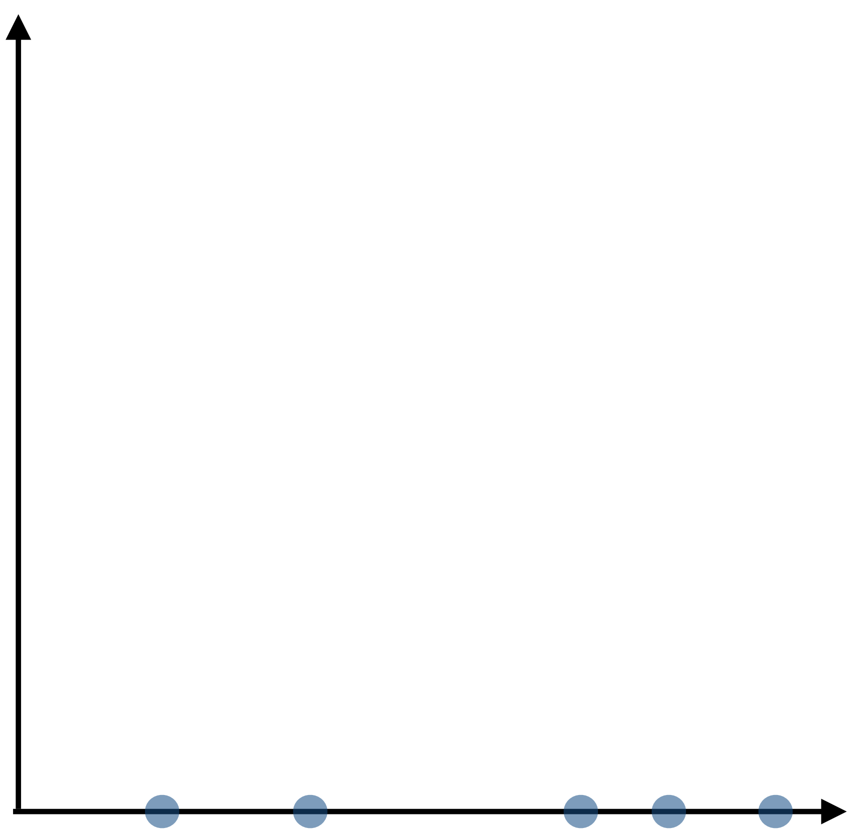
- Run CH(P') to find their convex hull



- Let P be the set of points x_1, x_2, …$x_{n.}$ to sort
- Let P': set of 2D points (x_i, x_i^2).

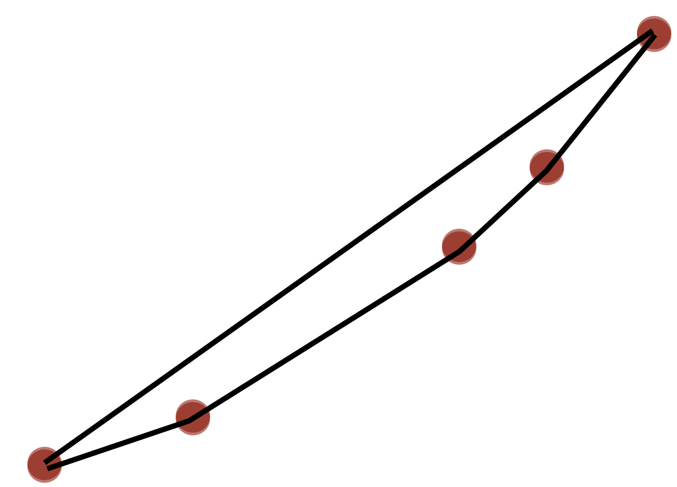
Sorting reduces to CH

- Assume we are given a set of of numbers x_1, x_2, …x_n to sort.
- Our goal is to argue that there exists some instance of a convex hull problem that sorts our numbers.



- Let P be the set of points x_1, x_2, …$x_{n.}$ to sort

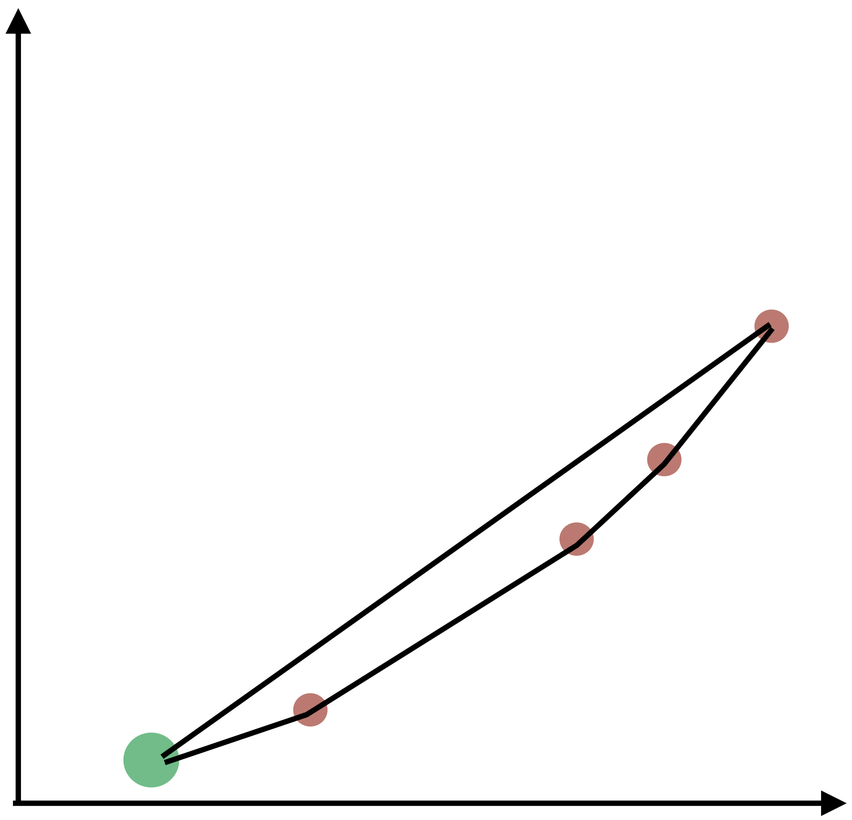
- Run CH(P') to find their convex hull
- They fall on a parabola, so every point is on the hull



- Let P': set of 2D points (x_i, x_i^2).

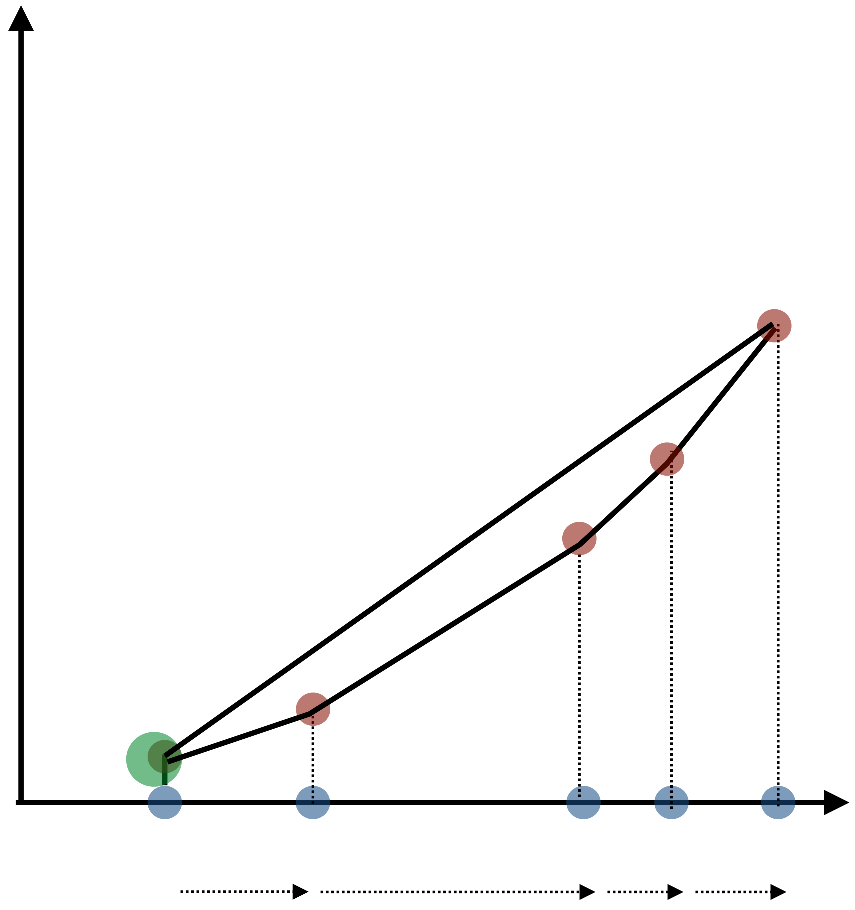
Sorting reduces to CH

- Find the lowest point on the hull, and walk from in ccw order.



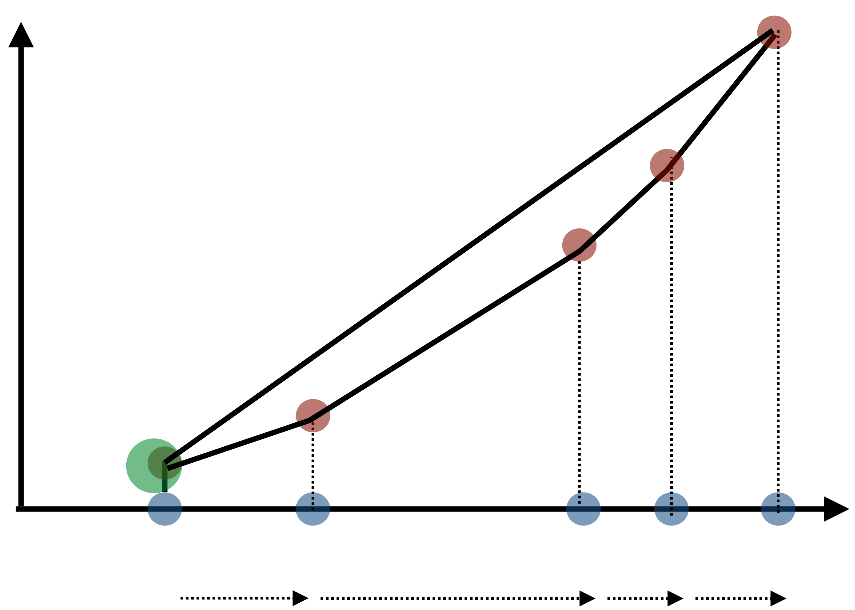
Sorting reduces to CH

- Find the lowest point on the hull, and walk from in ccw order.
- This is sorted order!



Sorting reduces to CH

- Input: set of points x_1, x_2, …x_n
  - Form a set of 2D points (x_i, x_i^2).
  - Run the CH algorithm to construct their convex hull.
  - Find the lowest point on the hull, and walk from in ccw order.   This is sorted order!



- If we could find the CH faster than n lg n, then we could sort faster than n lg n! Impossible!

Sorting reduces to CH

- What we actually proved is that
  - Any CH algorithm **that produces the boundary in order** must take Omega (n lg n) in the worst case.
- If we did not want the boundary in order, can the CH be constructed faster?
  - It was an open problem for a while
  - Finally, it was established quite recently that a convex hull algorithm, even if it does not produce the boundary in order, still needs $\Omega(n \lg n)$ in the worst case

- Yes, Graham scan is the ultimate CH algorithm but…
  - not output sensitive
  - does not extend to 3D

- The (re)search continues

An incremental algorithm for CH

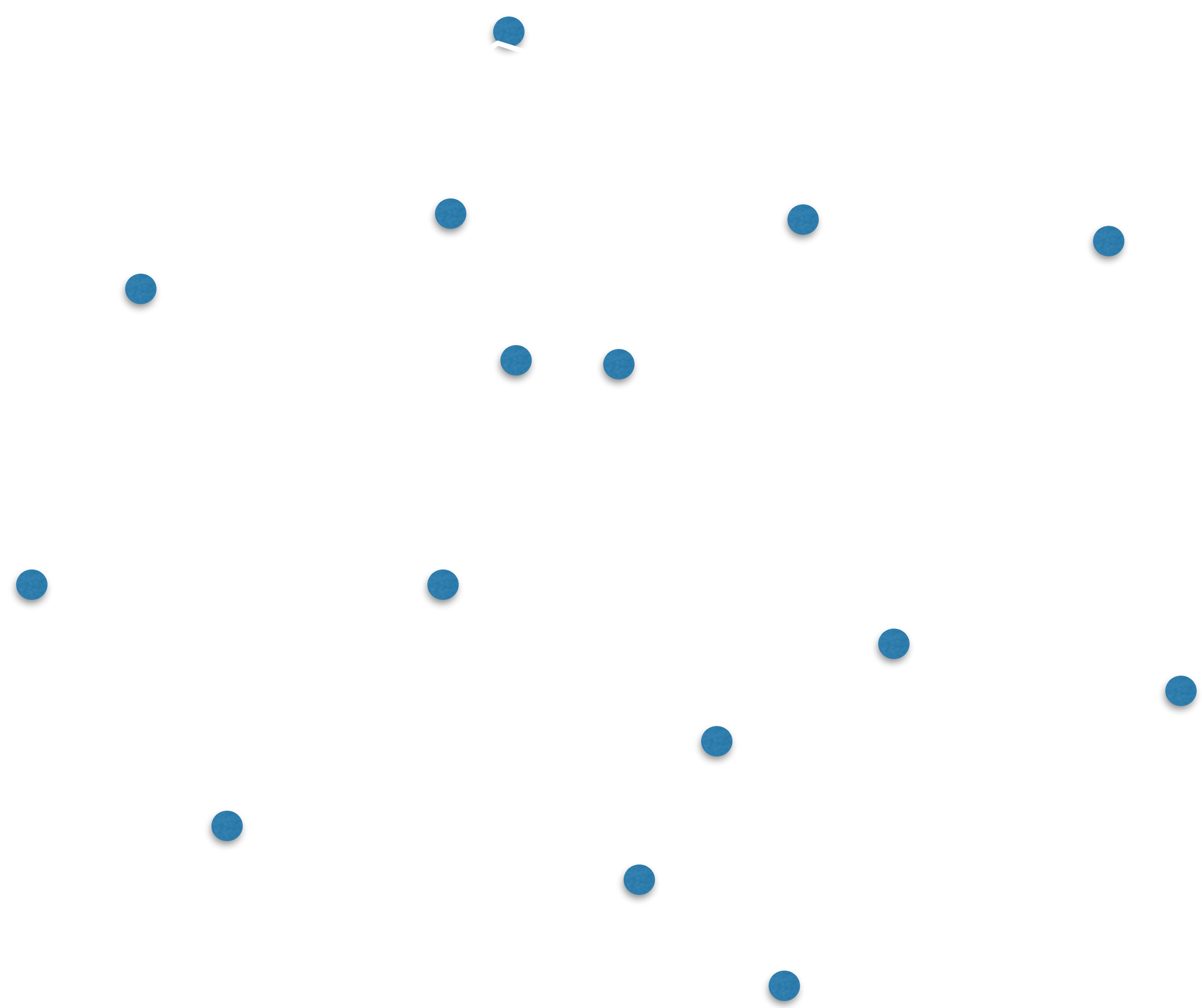
Incremental algorithms

- Goal: solve problem P
- Idea: traverse points one at a time and solve the problem for points seen so far

- Incremental Algorithm
  - initialize solution S = initial solution
  - for i=1 to n
    - //S represents solution of $p_1 \ldots\ldots p_{i-1}$
    - update S to represent solution of $p_1 \ldots . p_{i-1}\ p_i$

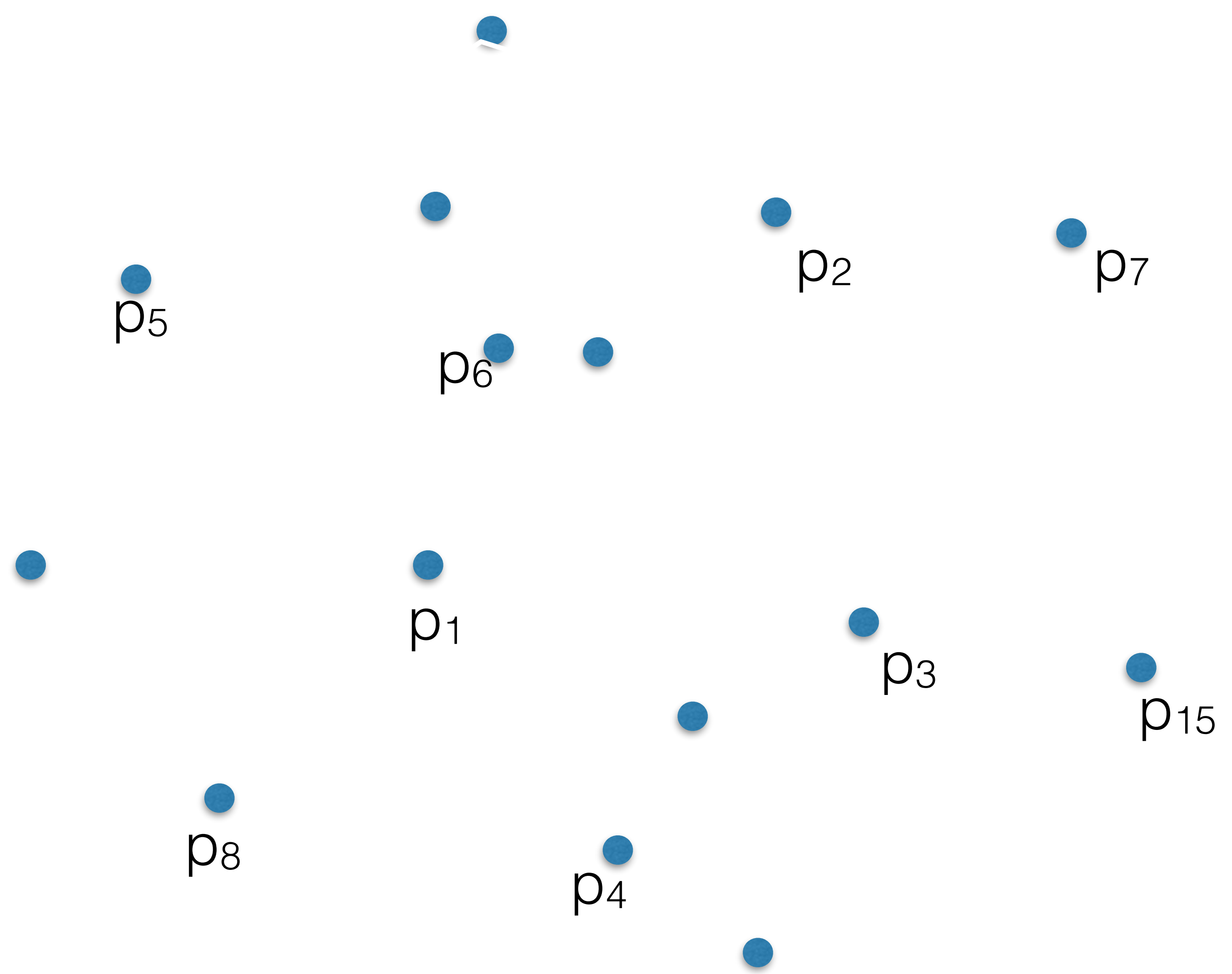
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



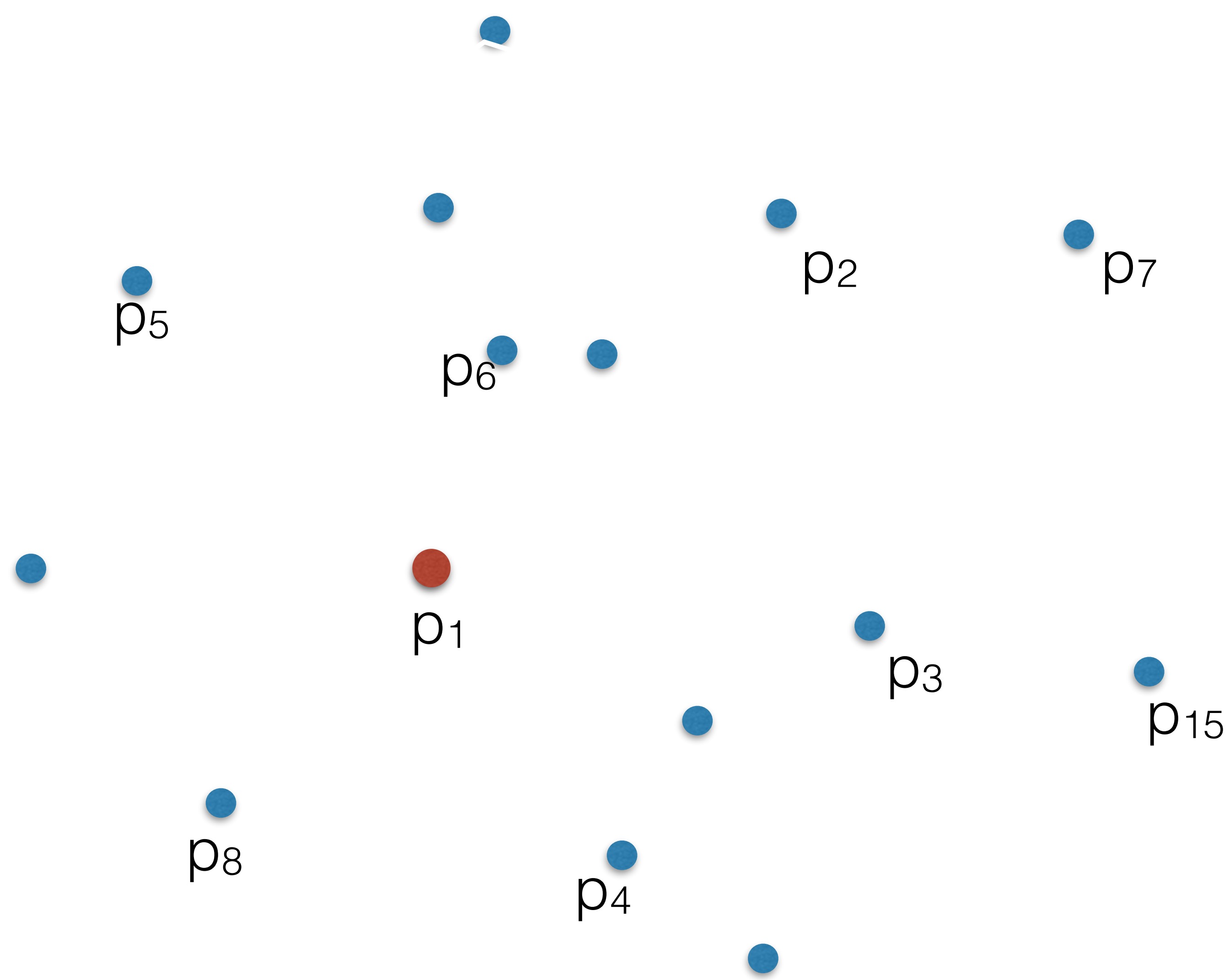
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



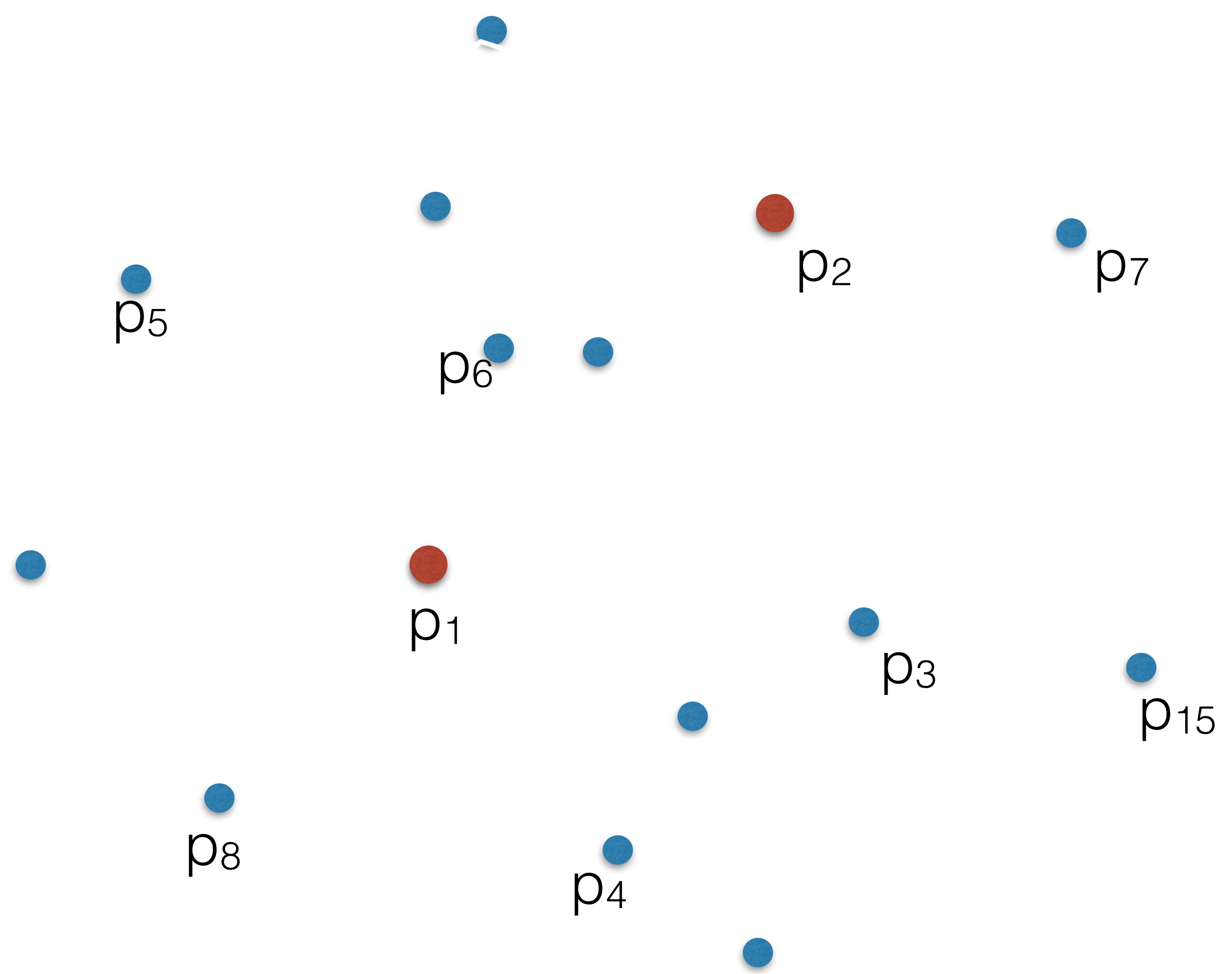
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



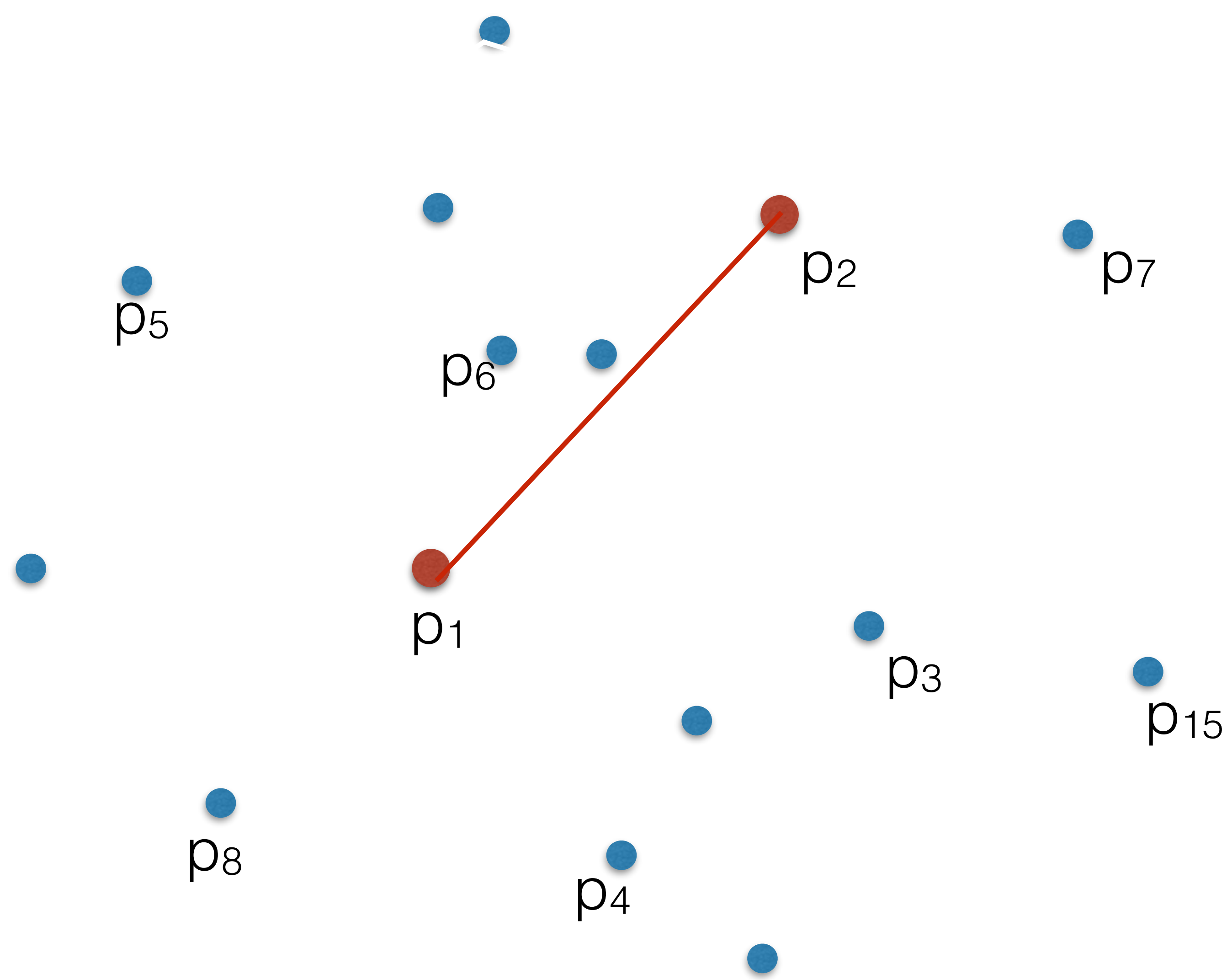
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



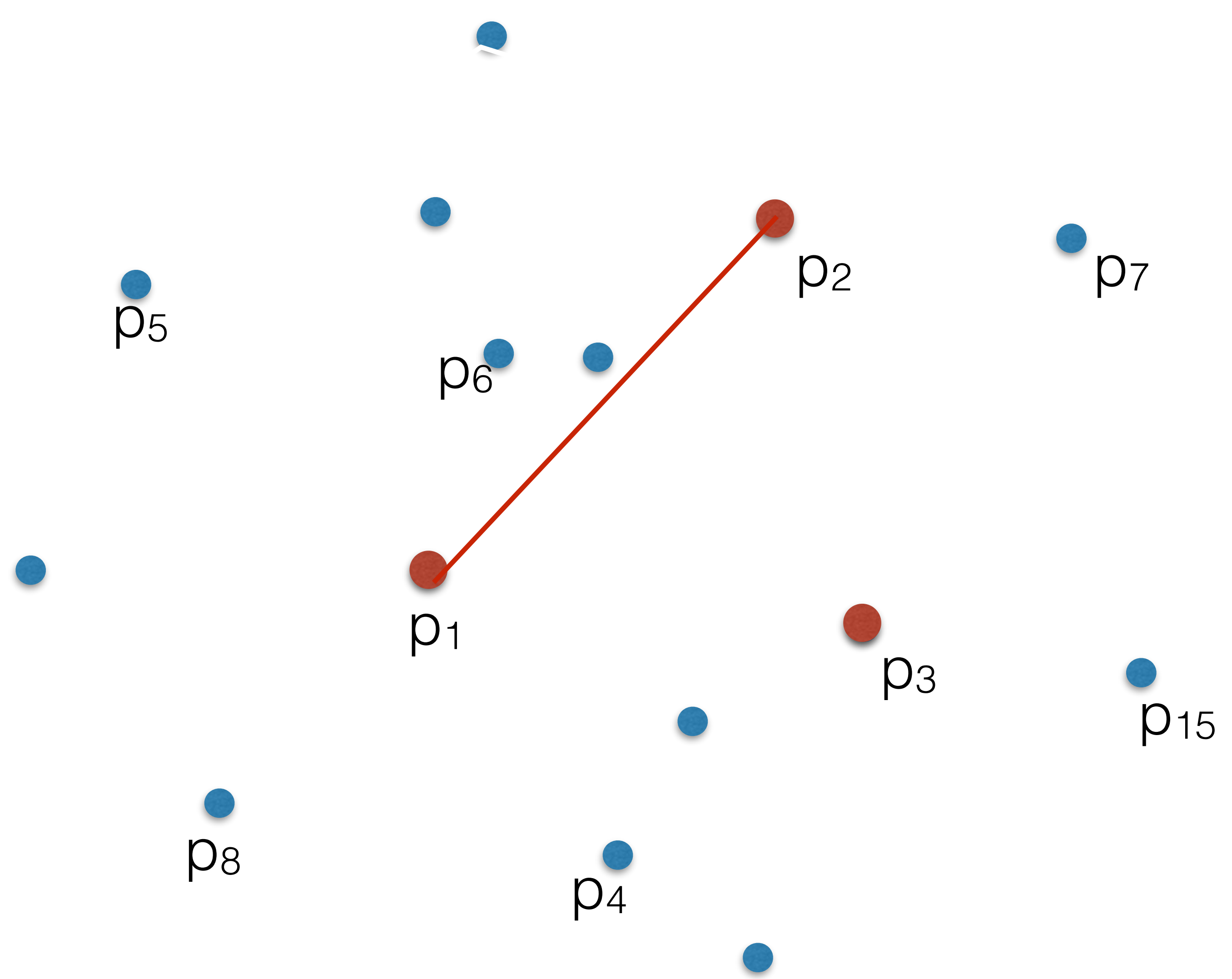
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



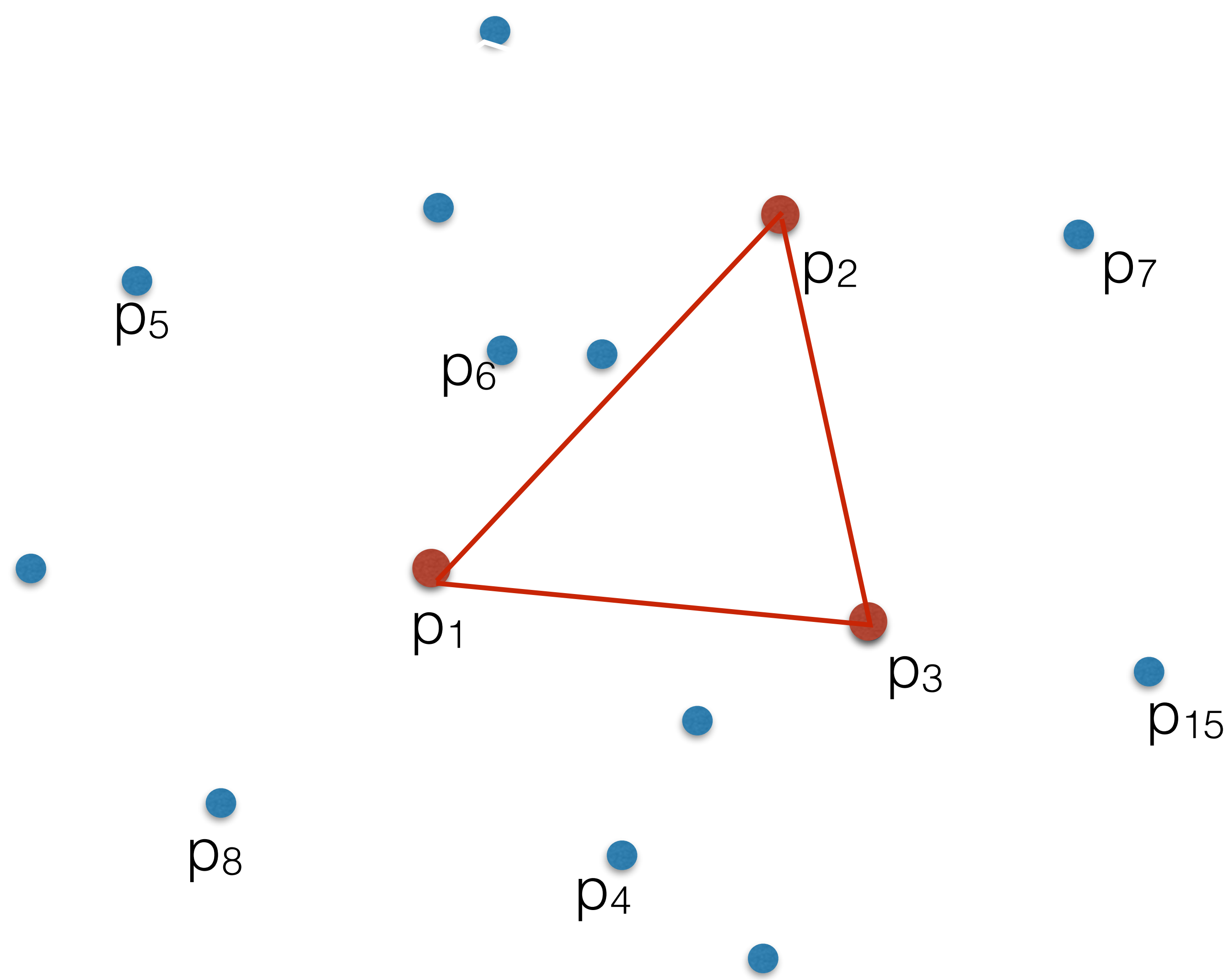
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



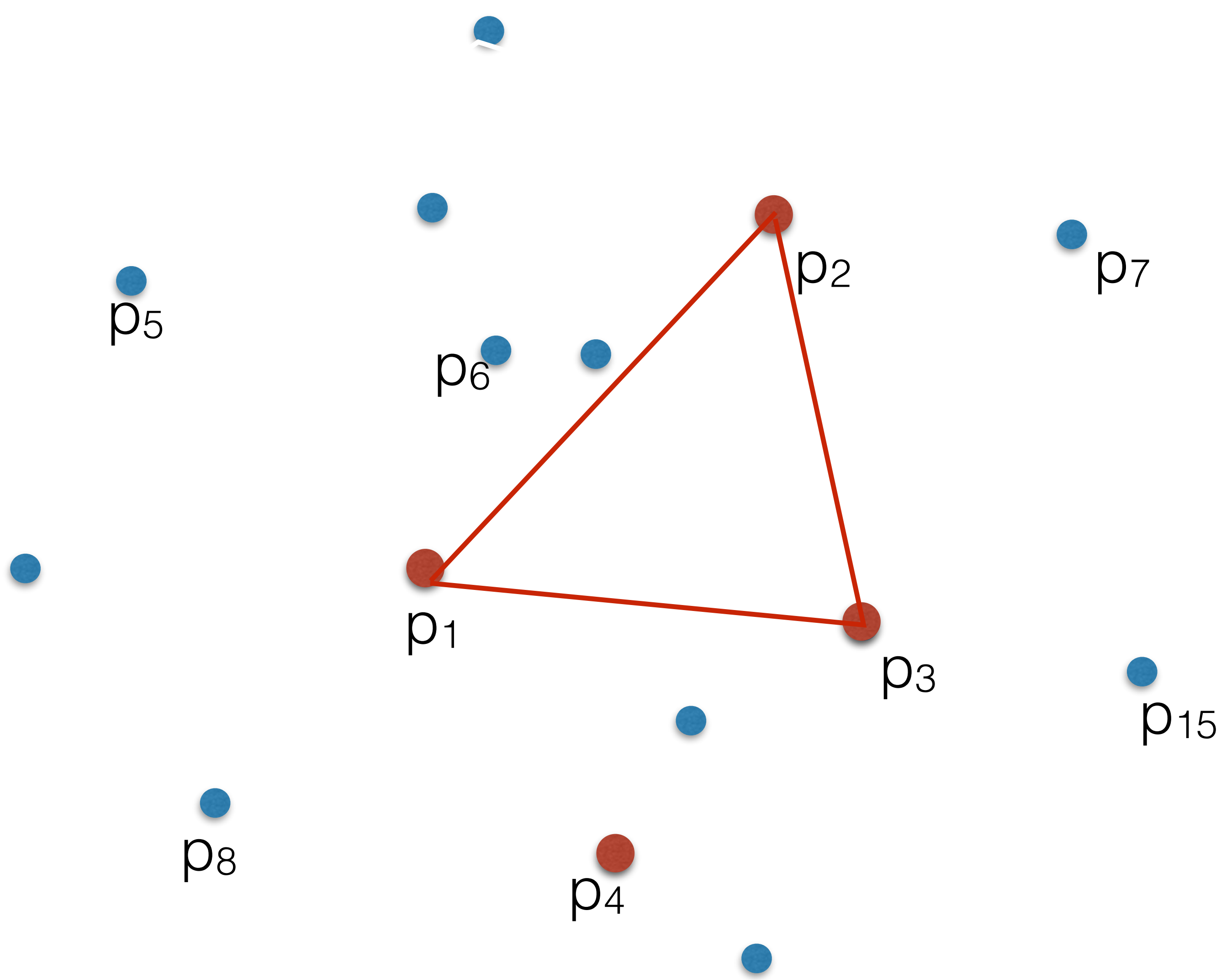
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



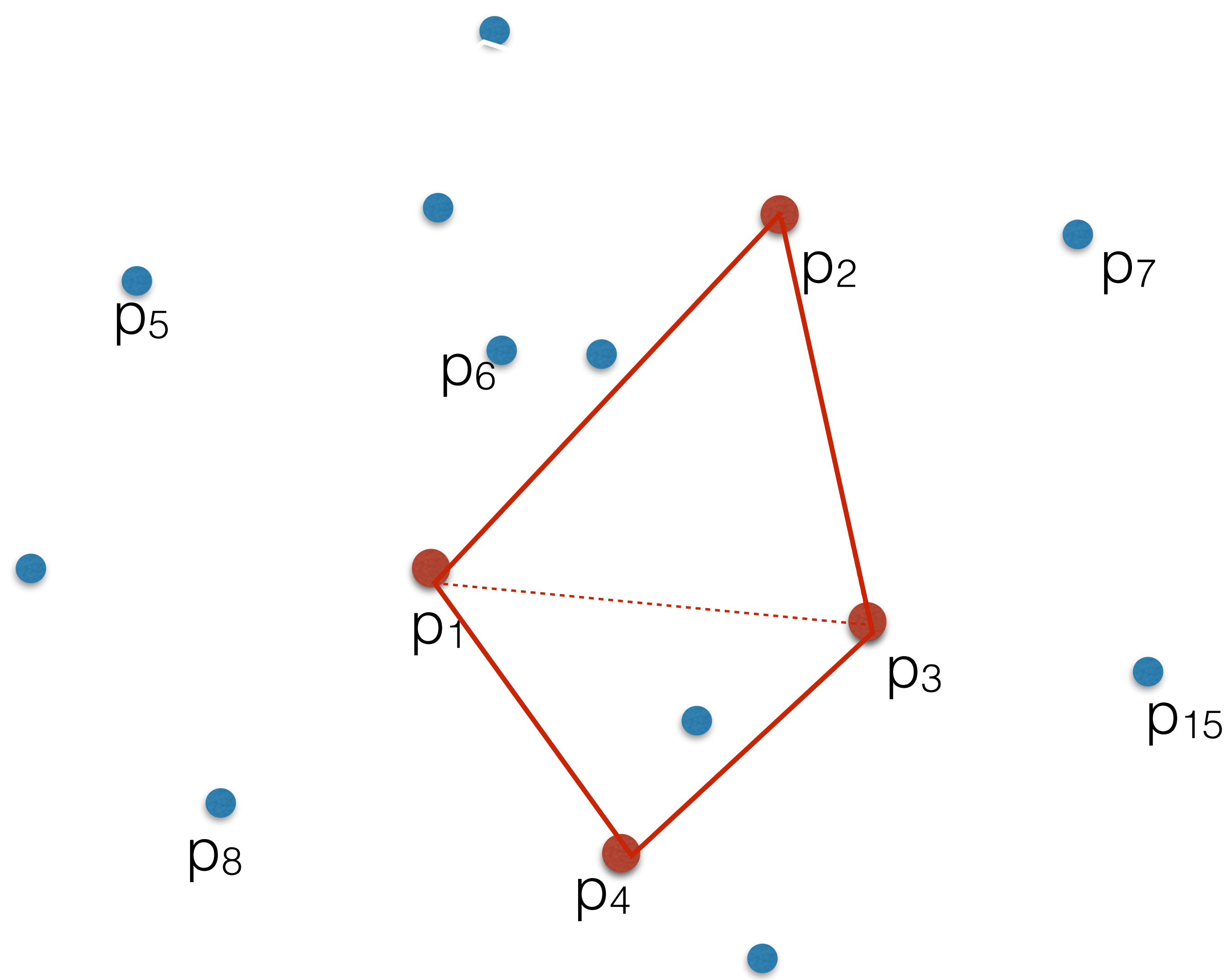
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



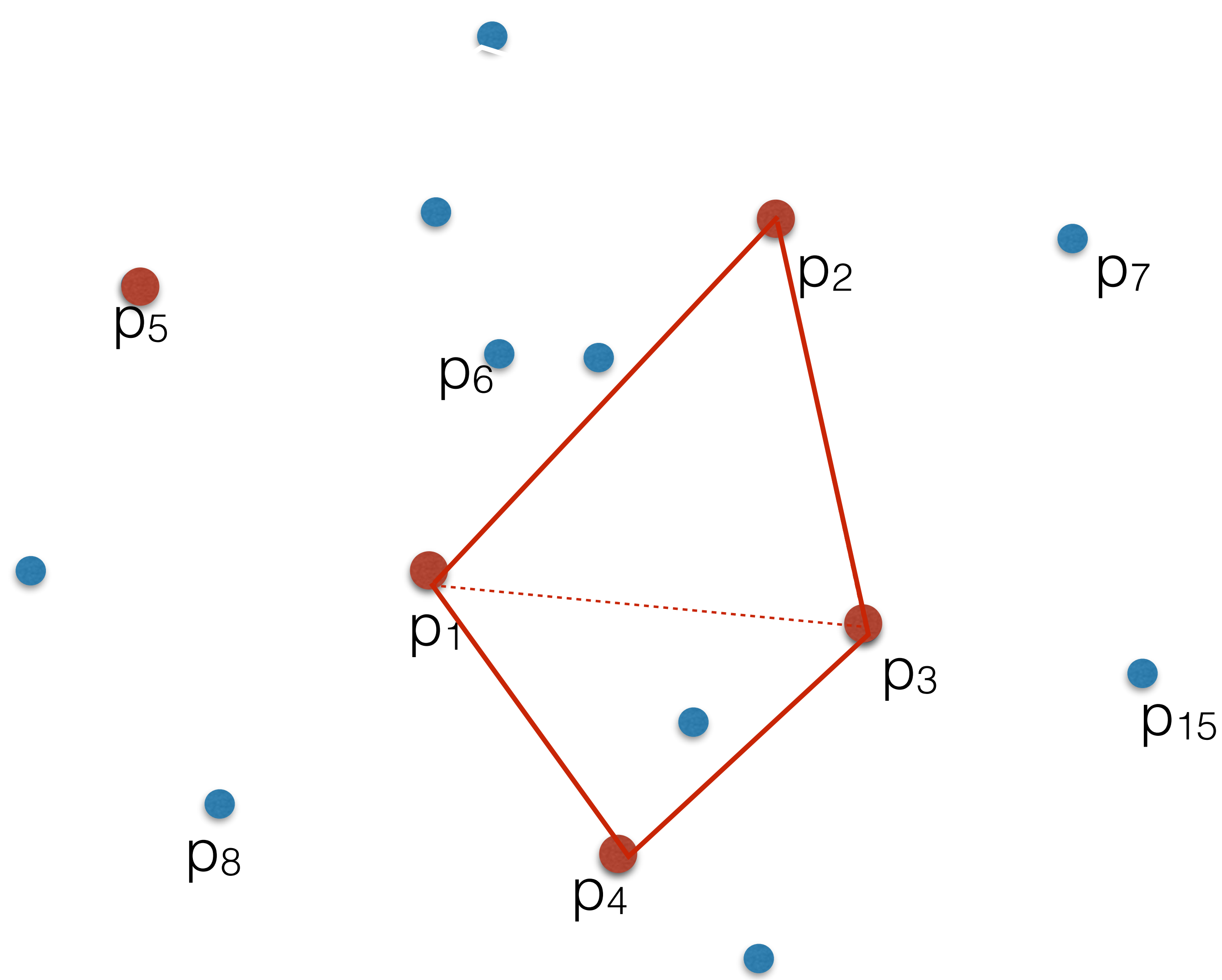
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



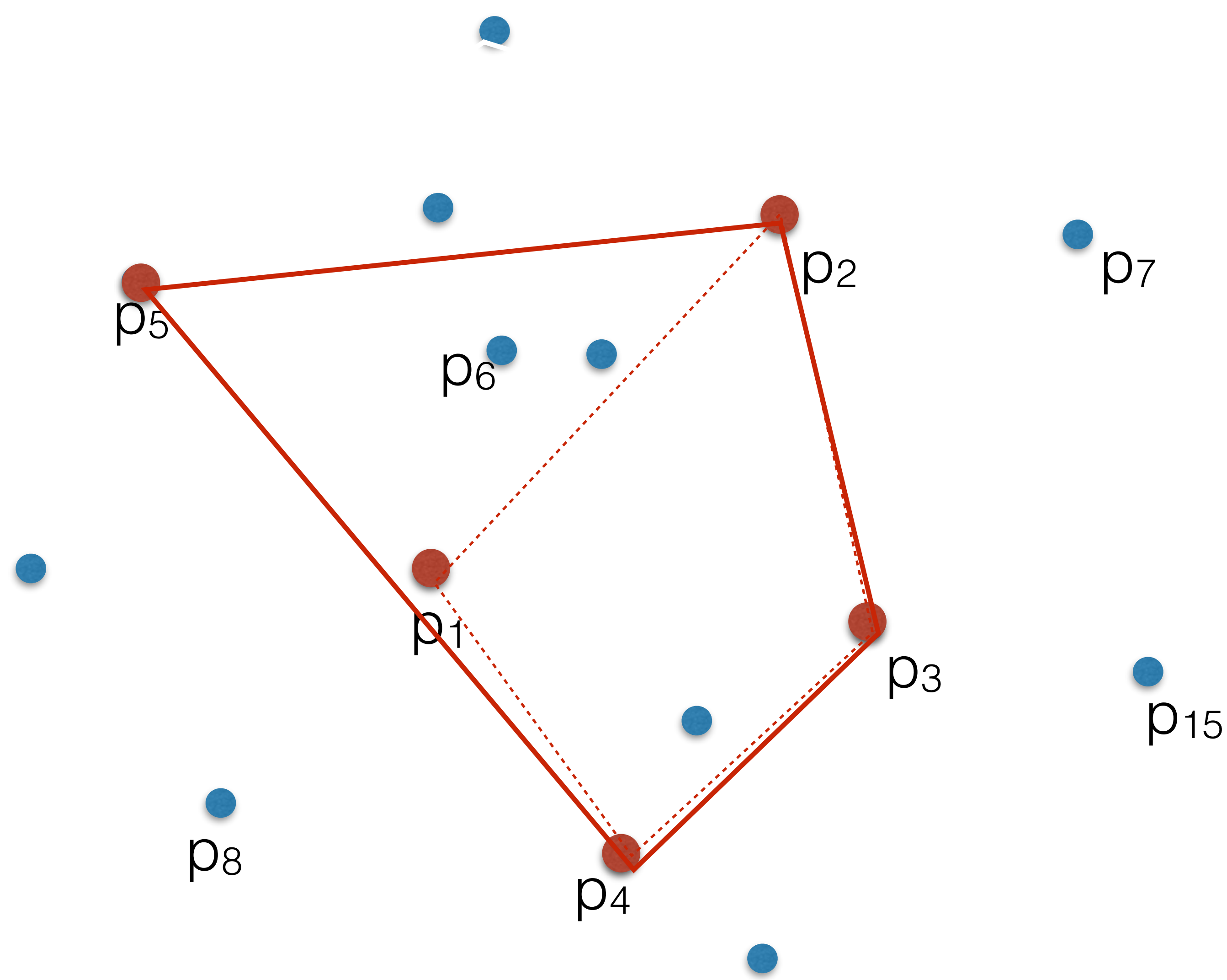
Incremental algo for CH

- CH = {}
- for i=1 to n
  - //CH represents the CH of $p_1..p_{i-1}$
  - update CH to represent the CH of $p_1..p_i$



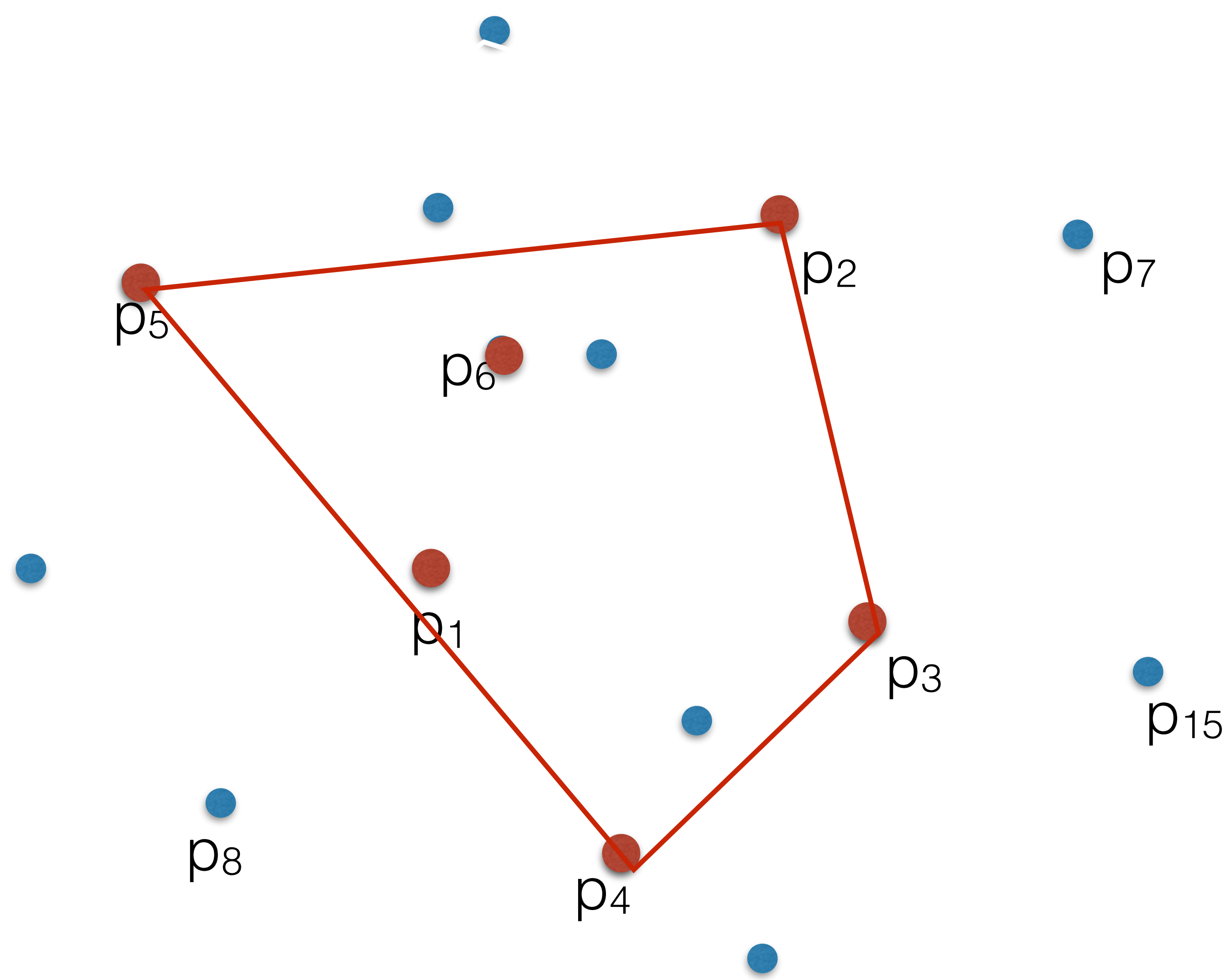
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



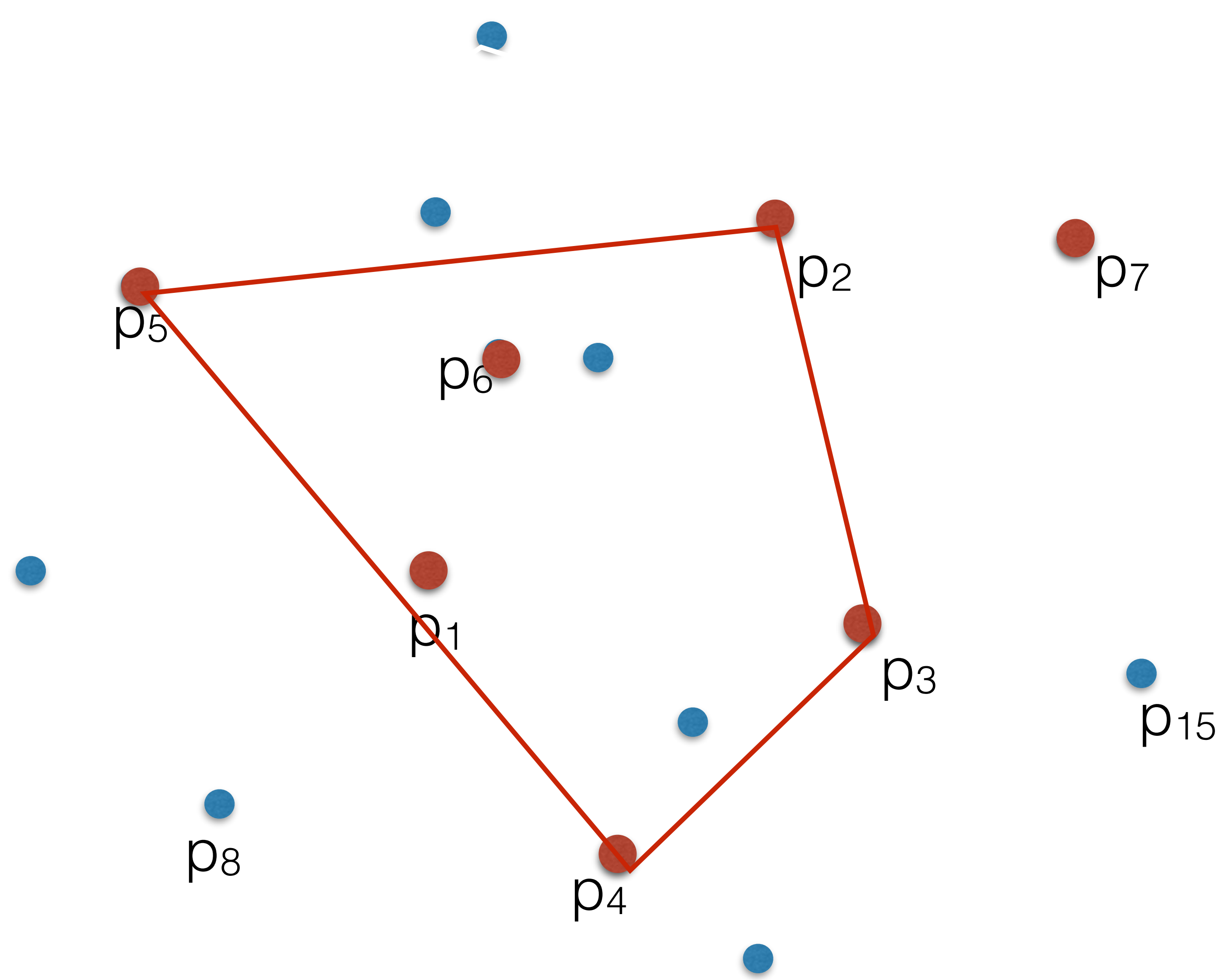
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



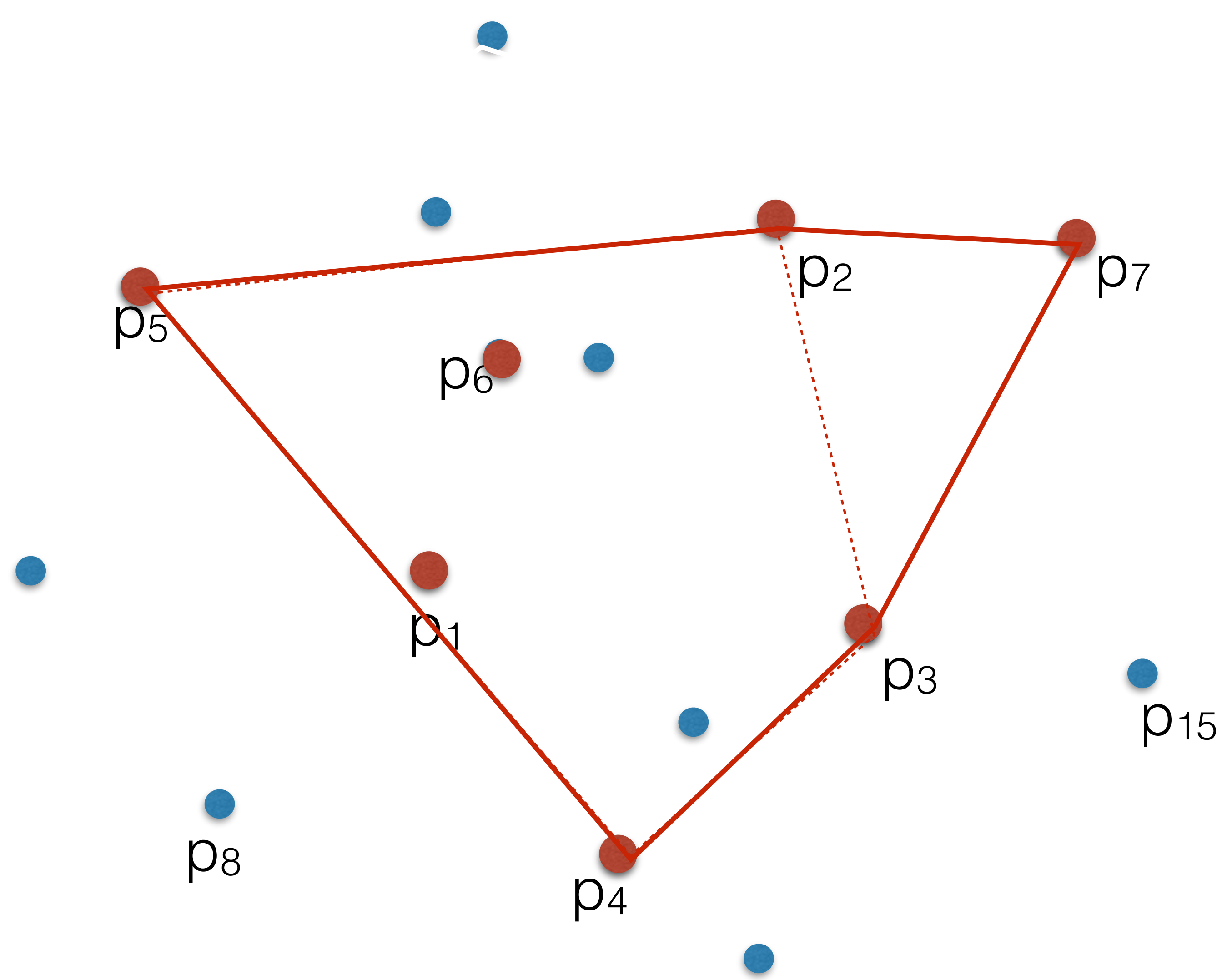
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



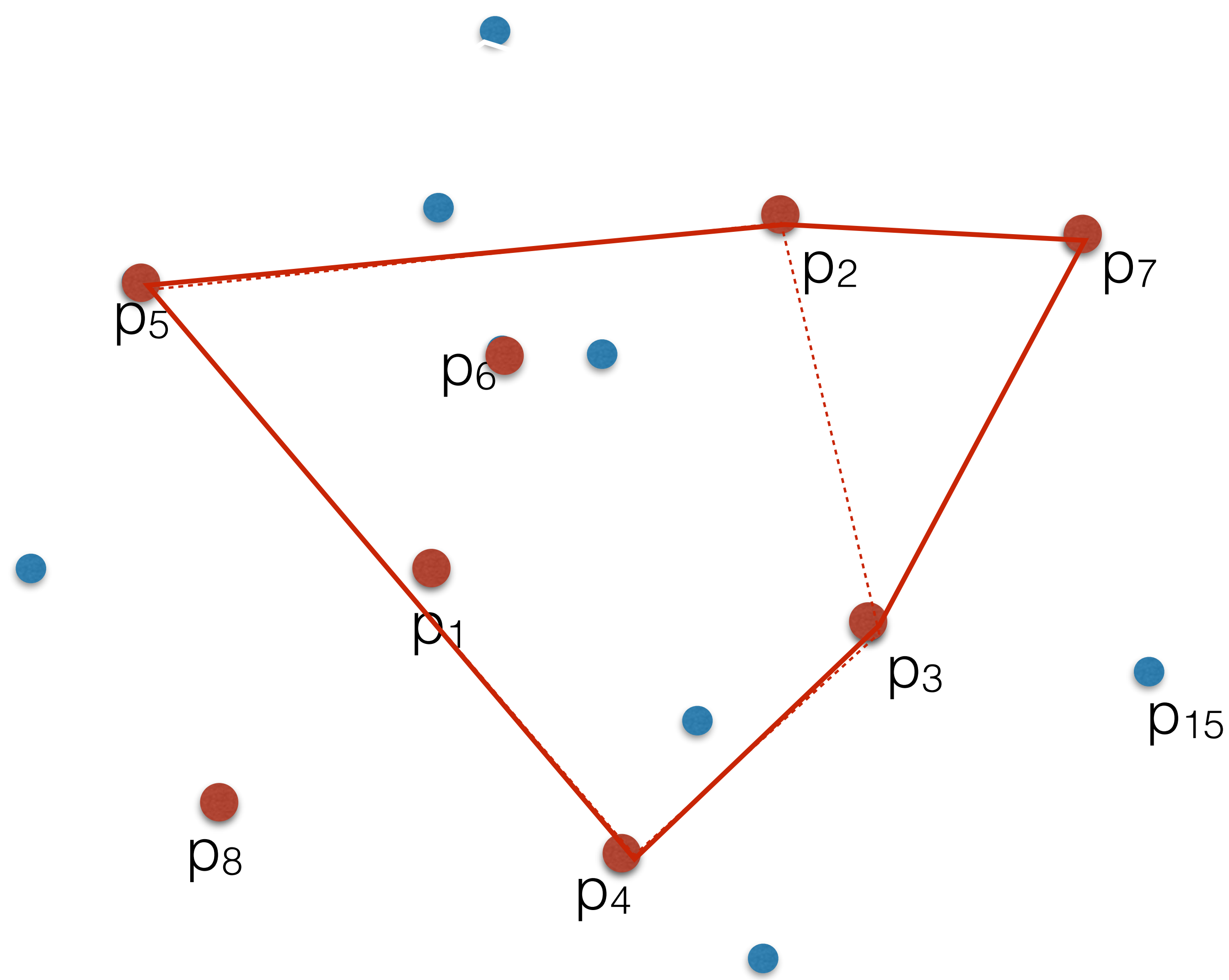
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



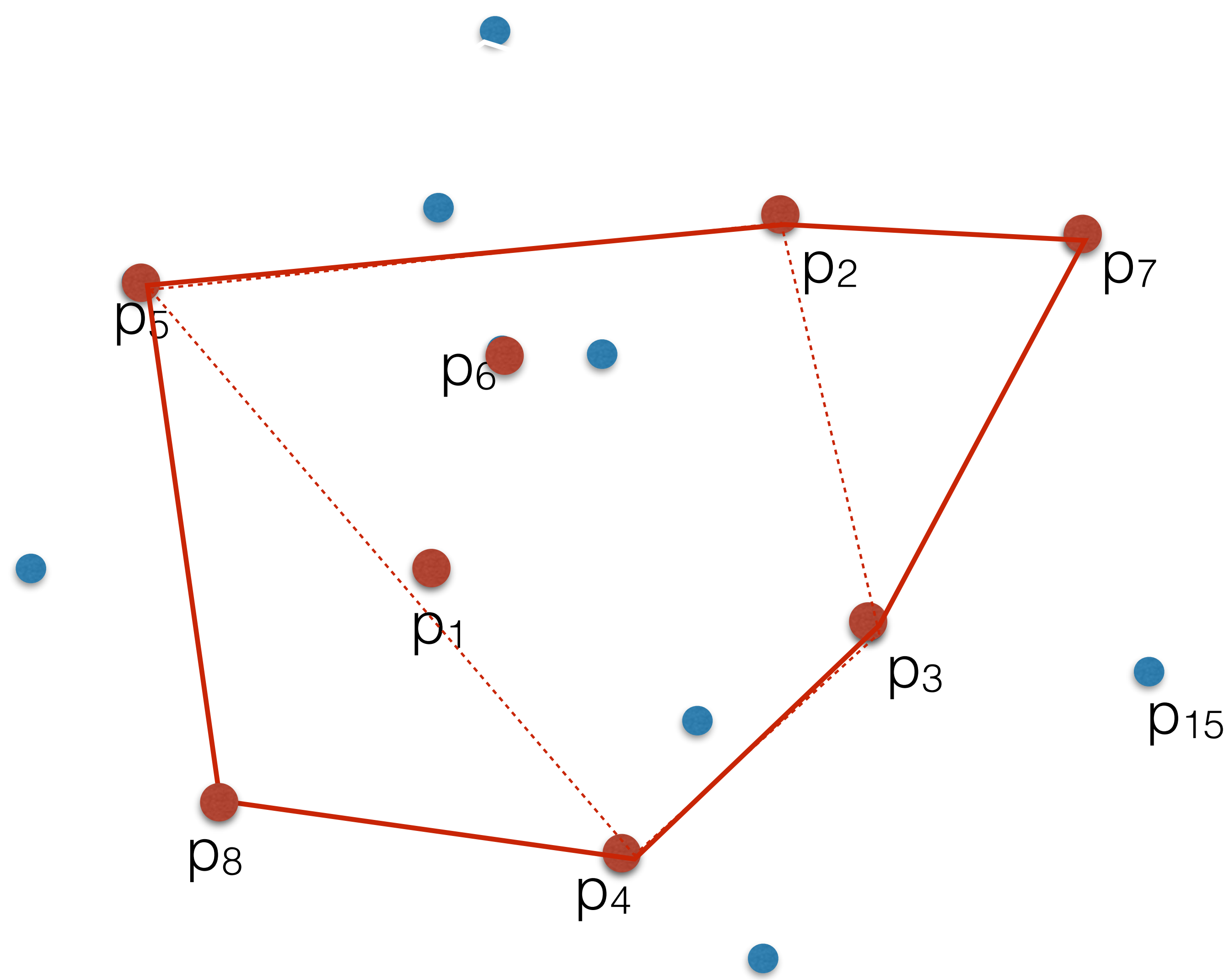
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



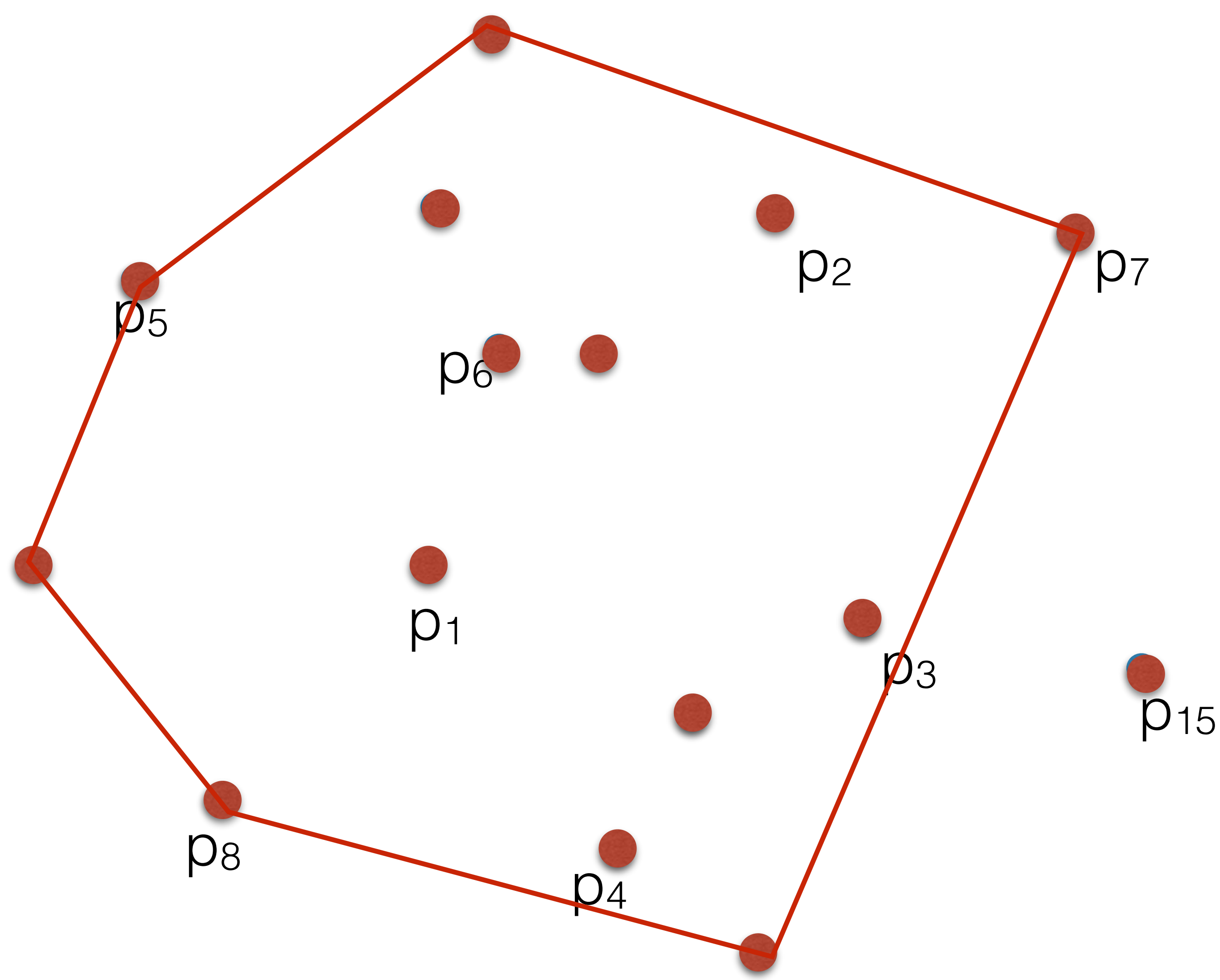
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



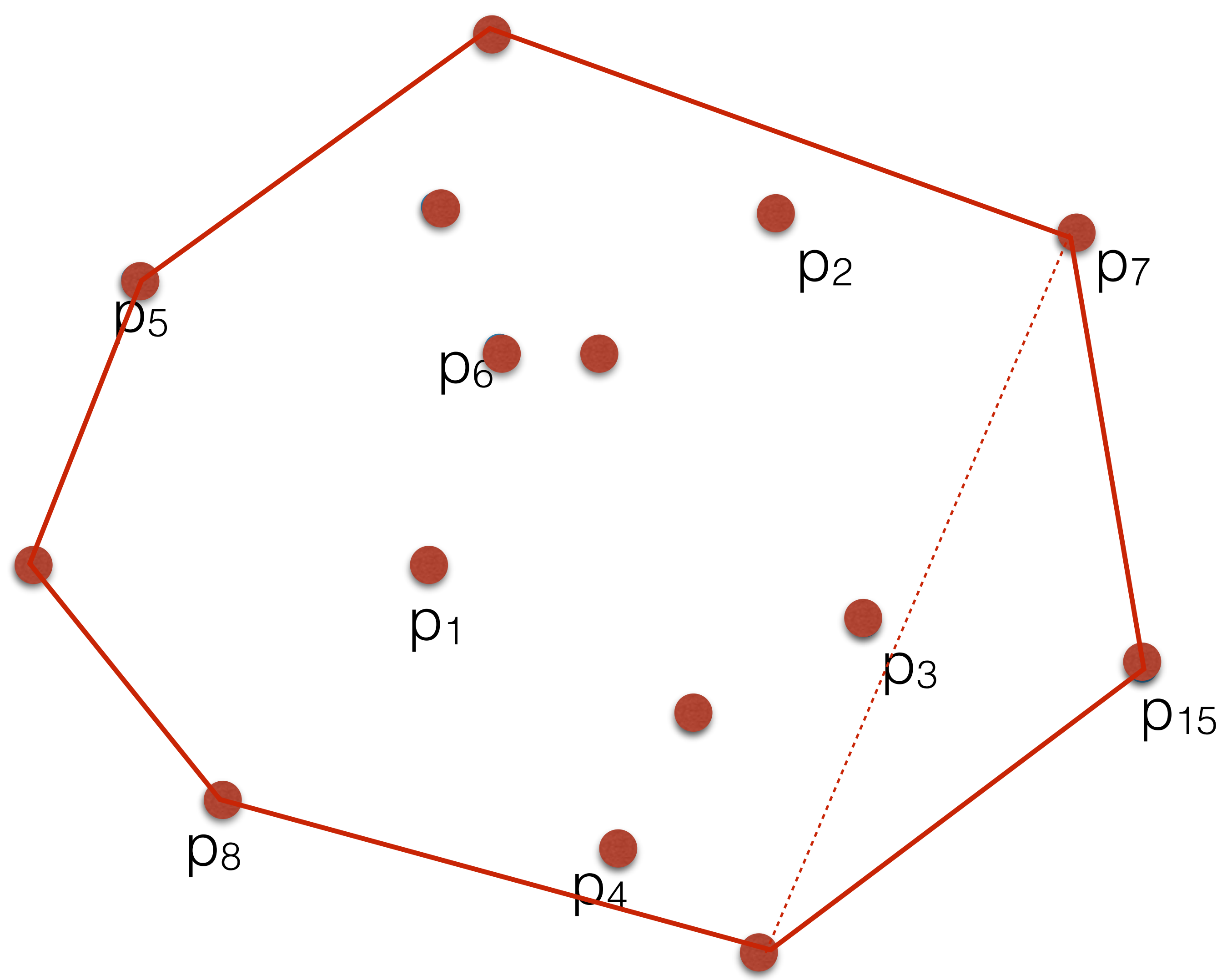
Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$



Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$

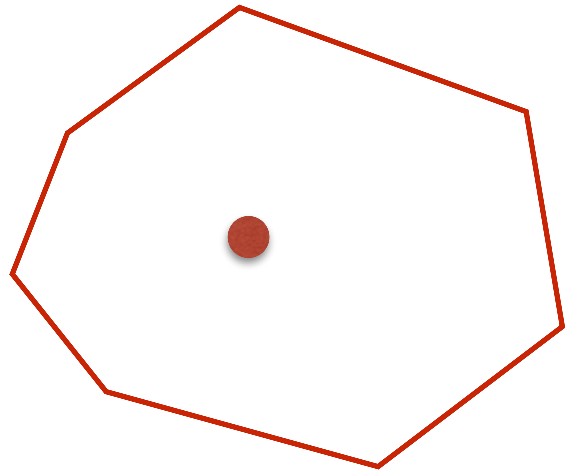
- The basic operation is adding a point to a convex polygon
    - How many cases?
    - How to handle each case?

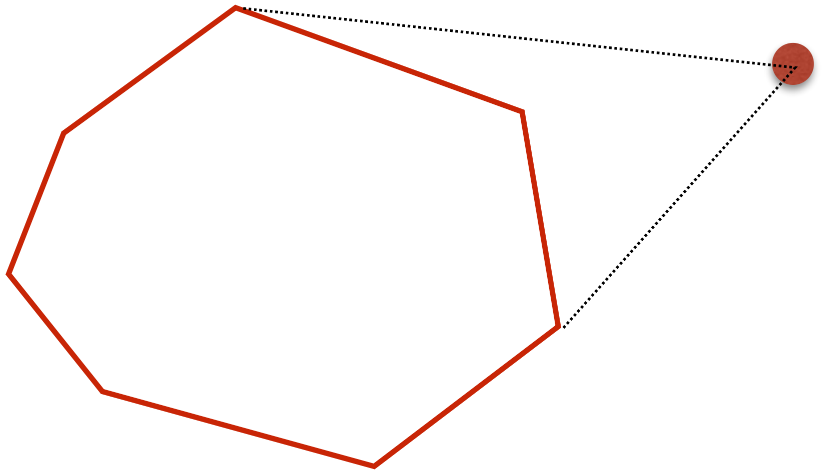
- Class work: Pick a set of point, Simulate the approach and try to answer these questions.

Incremental algo for CH

- CH = {}
- for i=1 to n
    - //CH represents the CH of $p_1..p_{i-1}$
    - update CH to represent the CH of $p_1..p_i$

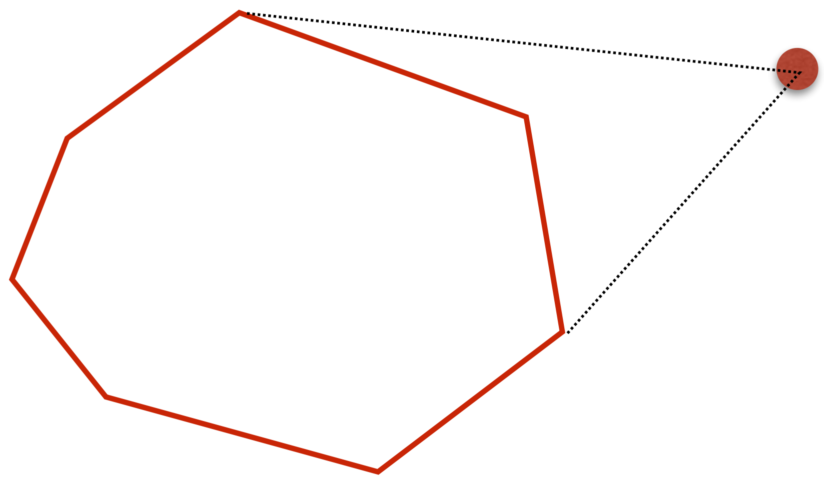
- The basic operation is adding a point to a convex polygon
    - CASE 1: p is in polygon
    - CASE 2: p outside polygon





Incremental algo for CH

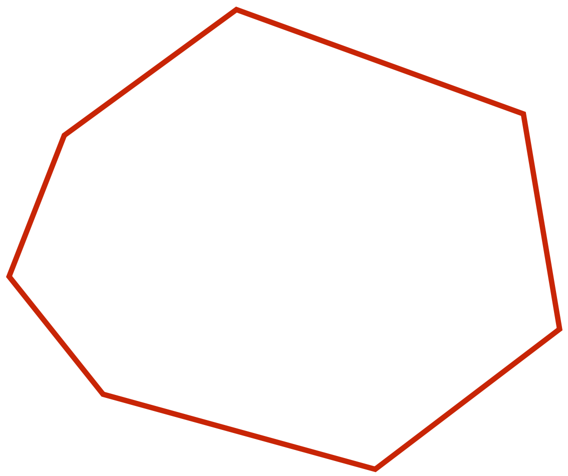
- Issues to solve
  - What's a good representation for a polygon?
  - We need a point-in-polygon test ?
  - How to handle CASE 2 ?



Representing a polygon

A polygon is represented as a list of vertices in boundary order.

(the convention is counter-clockwise order)

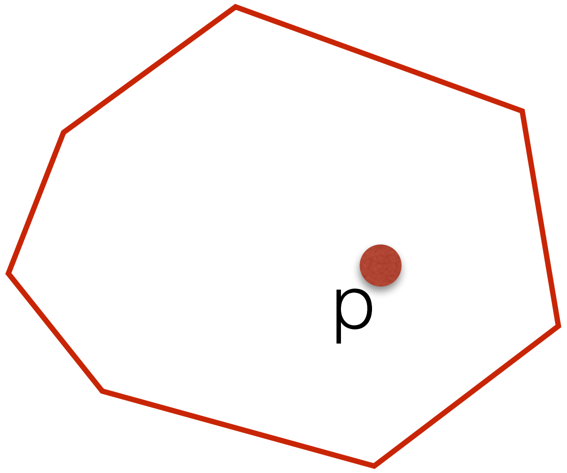


```
typedef struct _polygon{
        int k; //number of vertices
        Point* vertices; //the vertices, ccw in boundary order
  } Polygon;
```

or

```
Vector<Point>        //note: the vertices, ccw in boundary order
```

Point in convex polygon

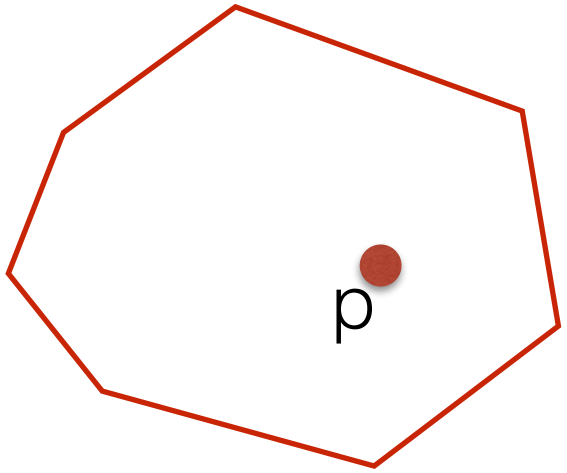


//return TRUE iff p on the boundary or inside H; H is convex a polygon

bool point_in_polygon(point p, polygon H)

What has to be true in order for p to be inside?

Point in convex polygon



//return TRUE iff p on the boundary or inside H; H is convex a polygon

bool point_in_polygon(point p, polygon H)

  //p is inside if and only if it is on or to the left of all edges, oriented ccw

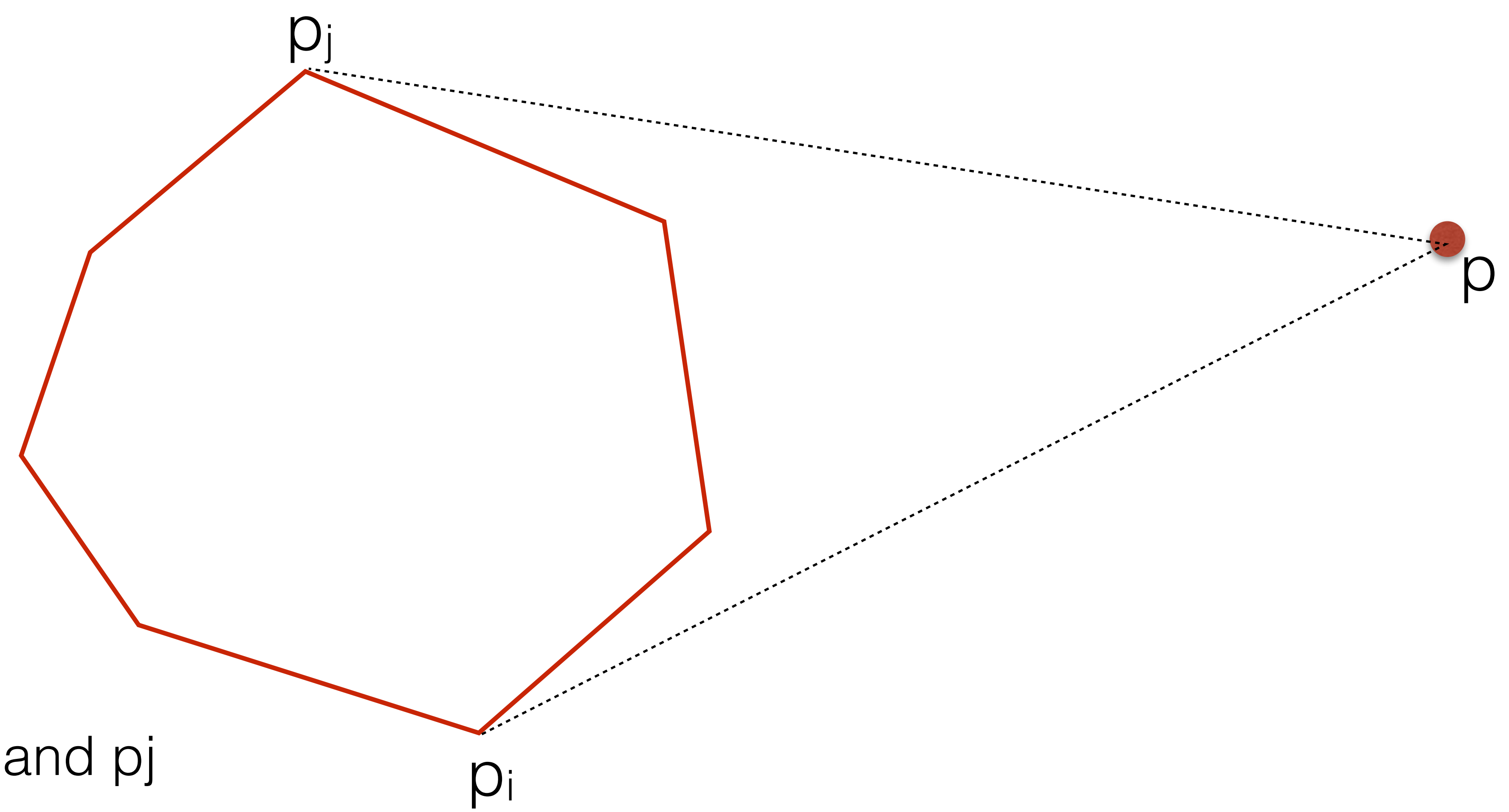
  //note: this is NOT true for a non-convex polygon — can you show a

  //counter-example?

Analysis:

- O(k) where k is the size of the polygon

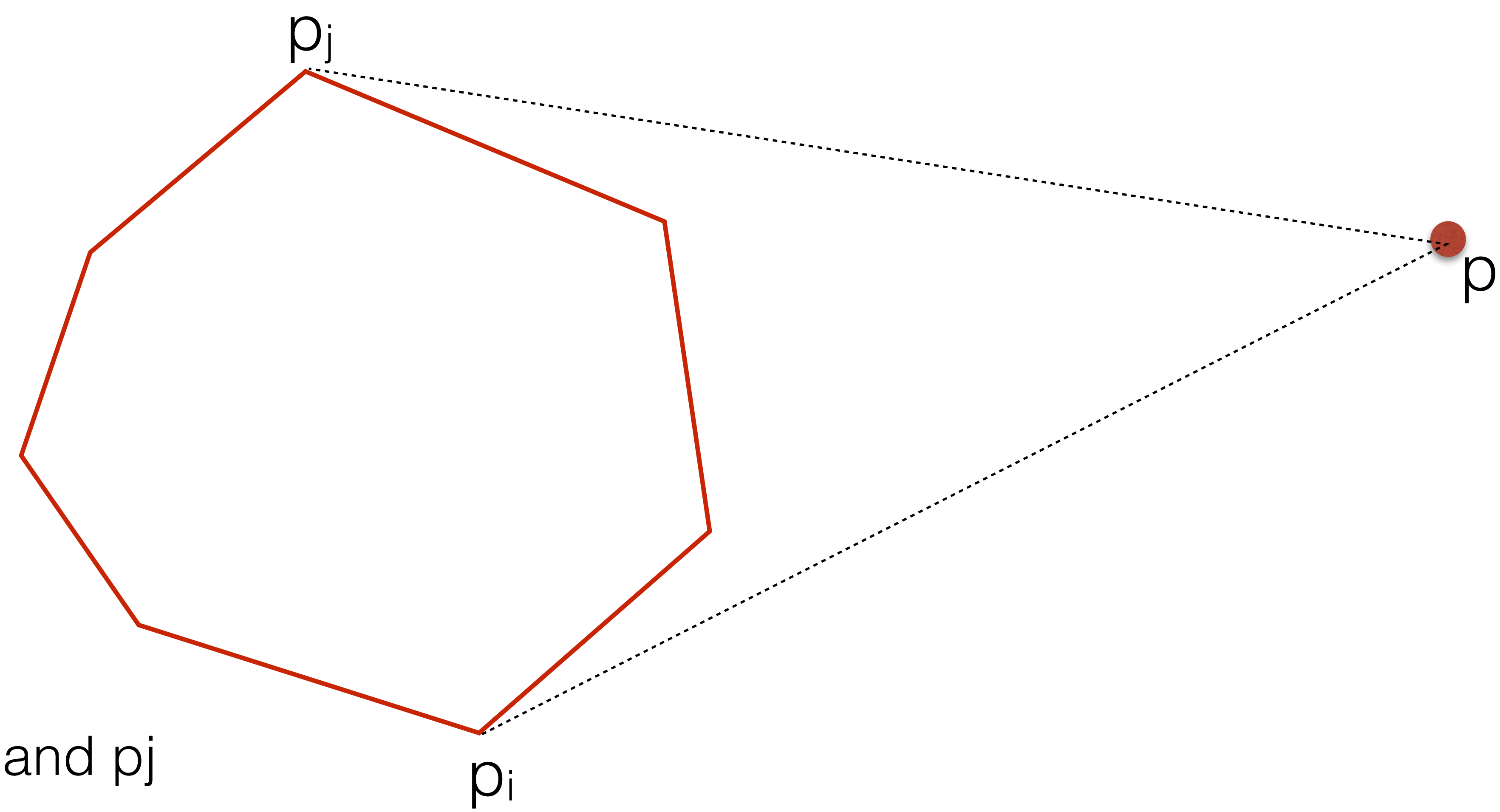
Case 2:



We want to find pi and pj

IDEAS?

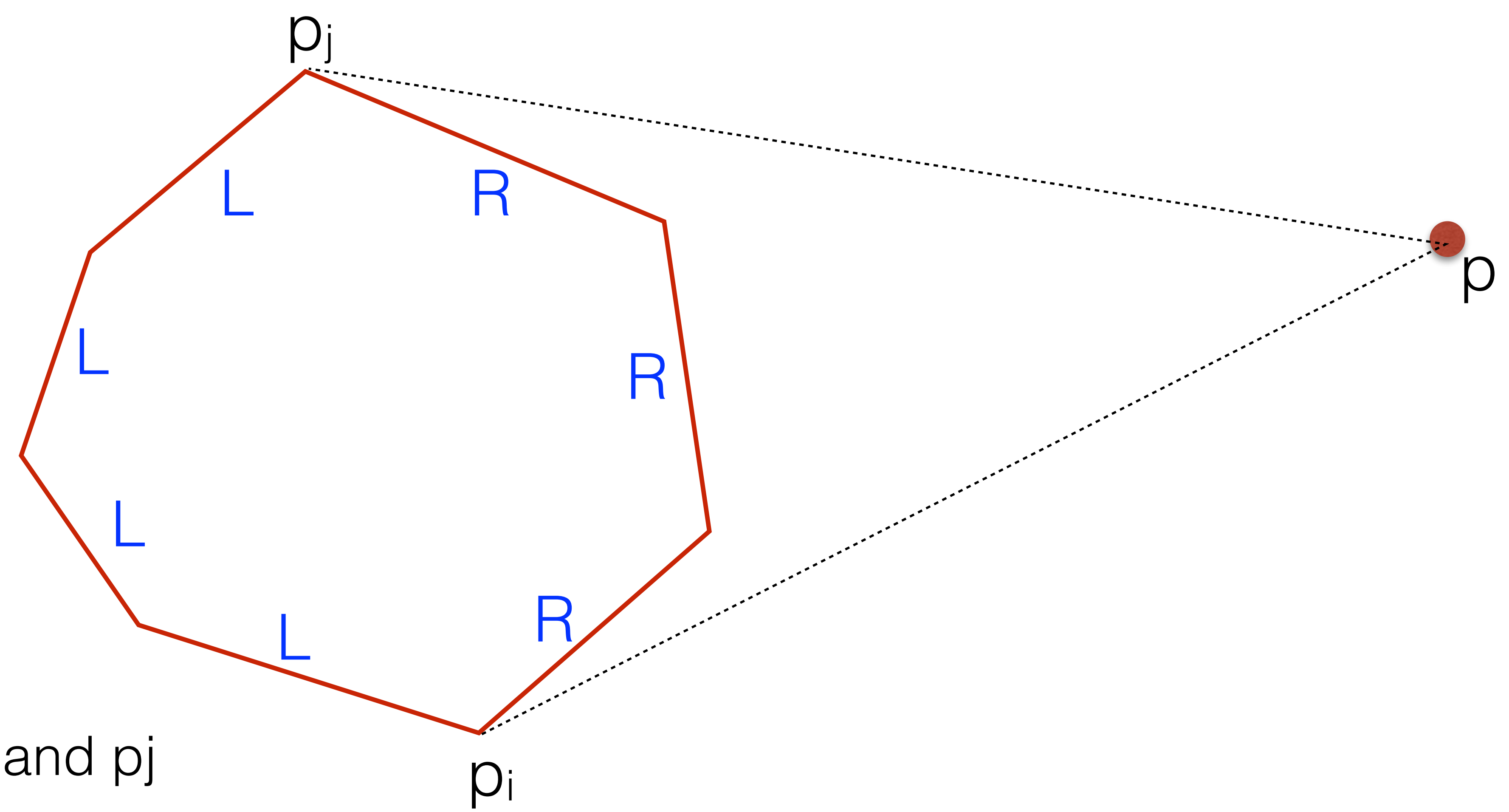
Case 2:



We want to find pi and pj

Hint: Check the orientation of p wrt the edges of the polygon.

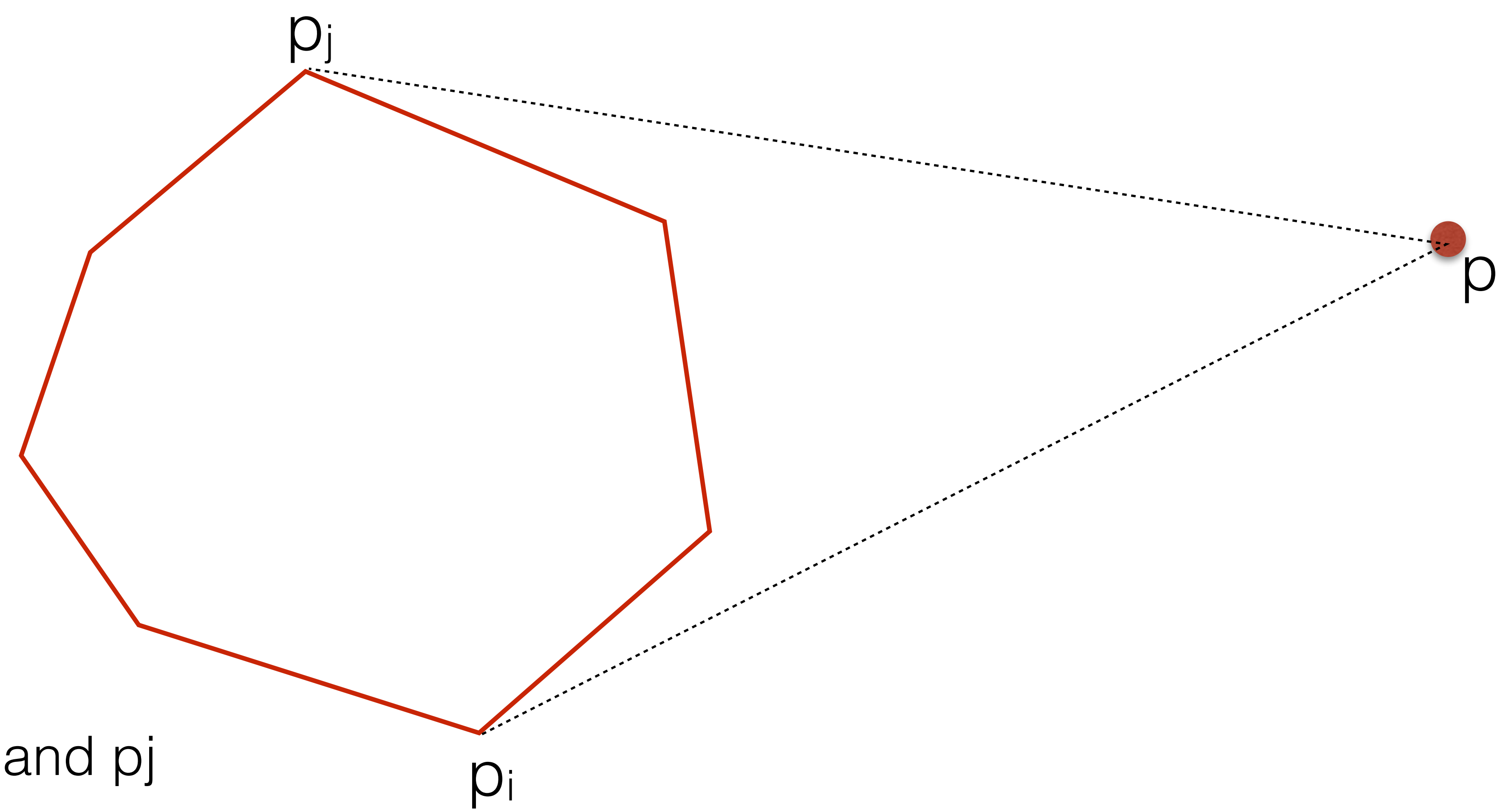
Case 2:



We want to find pi and pj

Hint: Check the orientation of p wrt the edges of the polygon.

Case 2:



We want to find pi and pj

Hint: Check the orientation of p wrt the edges of the polygon.

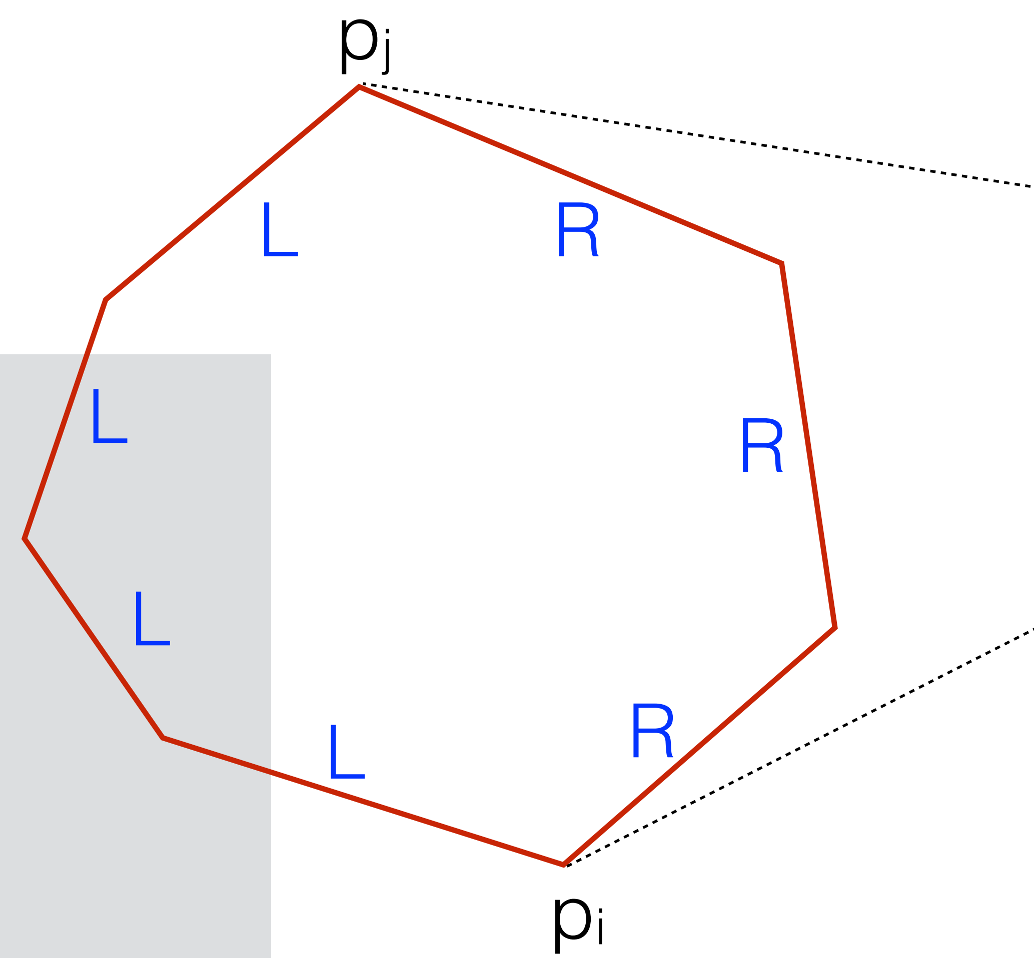
What do you notice? How can we use this to find the tangent points? Sketch an algorithm. How long does it take?

Finding tangent points

Input: point p outside H

polygon H = [p_0, p_1,…, p_{k-1}] convex

- *for i=0 to k-1 do*
  - *prev = ((i == 0)? k-1: i-1);*
  - *next = (i==k-1)? 0; k+1);*
  - *if XOR (p is left-or-on (p_{prev}, p_i), p is left-or-on(p_i, p_{next}))*
  - *then p_i is a tangent point*



After finding p_i and p_j, how would you update H?

Back to an incremental algorithm for CH

Incremental algo for CH

- *H = [p1, p2, p3]*
- *for i=4 to n do*
  - *//add p_i to H*
  - *if point_in_polygon(p, H)*
    - *//do nothing*
  - *else*
    - *find p_i the tangent point where orientation changes from L to R*
    - *find p_j the tangent point where orientation changes from R to L*

      //note: p_i not necessarily before p_j in the vertex array of H
    - *cut out the part from p_i to p_j in H (note: view H as wrapping around) and replace it with vertex p*

Incremental algo for CH

- *H = [p1, p2, p3]*
- *for i=4 to n do*
  - *//add p_i to H*
  - *if point_in_polygon(p, H)*
    - *//do nothing*
  - *else*
    - *find p_i the tangent point where orientation changes from L to R*
    - *find p_j the tangent point where orientation changes from R to L*

    //note: p_i not necessarily before p_j in the vertex array of H

    - *cut out the part from p_i to p_j in H (note: view H as wrapping around) and replace it with vertex p*

Simulate the algorithm on a couple of examples.
Think how p_i could come before p_j in H or the other way around.

Incremental algo for CH

- *H = [p1, p2, p3]*
- *for i=4 to n do*
  - *//add p_i to H*
  - *if point_in_polygon(p, H)*
    - *//do nothing*
  - *else*
    - *find p_i the tangent point where orientation changes from L to R*
    - *find p_j the tangent point where orientation changes from R to L*

      //note: p_i not necessarily before p_j in the vertex array of H
    - *cut out the part from p_i to p_j in H (note: view H as wrapping around) and replace it with vertex p*

Analysis:

Incremental algo for CH

- *H = [p1, p2, p3]*
- *for i=4 to n do*
  - *//add p_i to H*
  - *if point_in_polygon(p, H)* ← O(i)
    - *//do nothing*
  - *else*
    - *find p_i the tangent point where orientation changes from L to R* ← O(i)
    - *find p_j the tangent point where orientation changes from R to L*

    //note: p_i not necessarily before p_j in the vertex array of H
    - *cut out the part from p_i to p_j in H (note: view H as wrapping around) and replace it with vertex p*

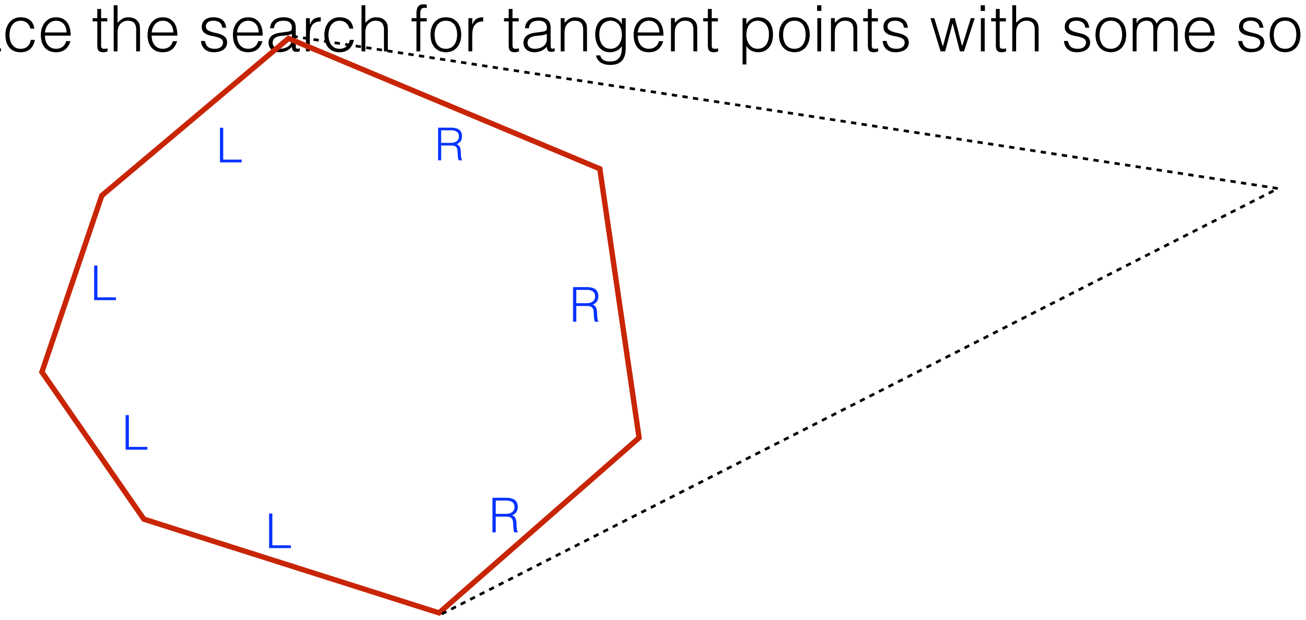
Analysis: $\text{SUM}_i\ O(i) = O(n^2)$

Incremental algo for CH

- The "straightforward" incremental algorithm is $O(n^2)$
- Improvement:
  - pre-sort the points by their x-coordinates and add them in this order

    What does this give us?

- It was shown that $O(n \lg n)$ incremental algorithm is possible.
  - avoid re-computing all orientations every time
  - replace the search for tangent points with some sort of binary search



A divide-and-conquer algorithm for CH

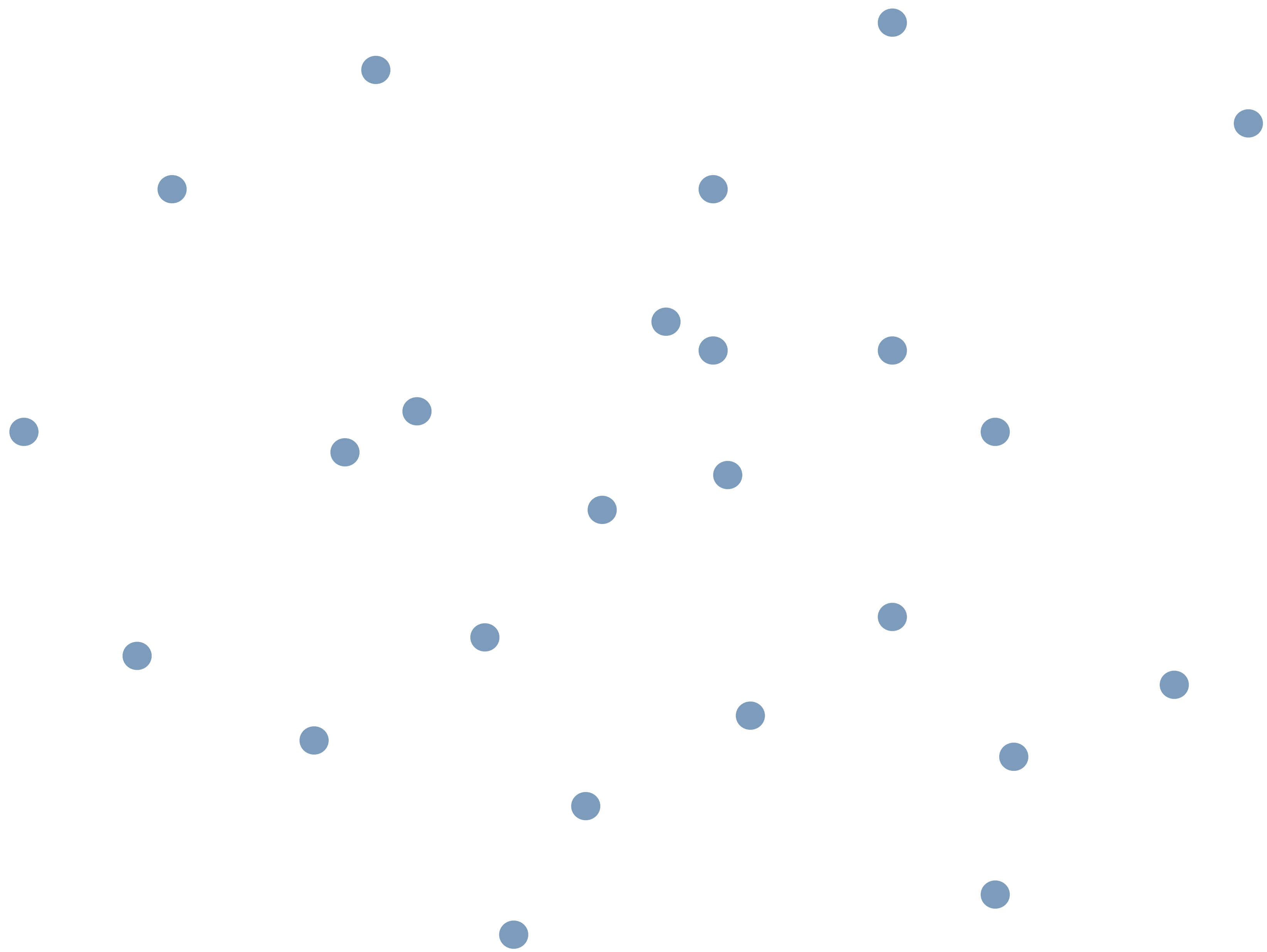
Divide-and-conquer

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
    //recurse
    result1 = DC(P1)
    result2 = DC(P2)
    //merge
    do_something_to_figure_out_result_for_P
    return result
```

Analysis: T(n) = 2T(n/2) + O(merge phase)

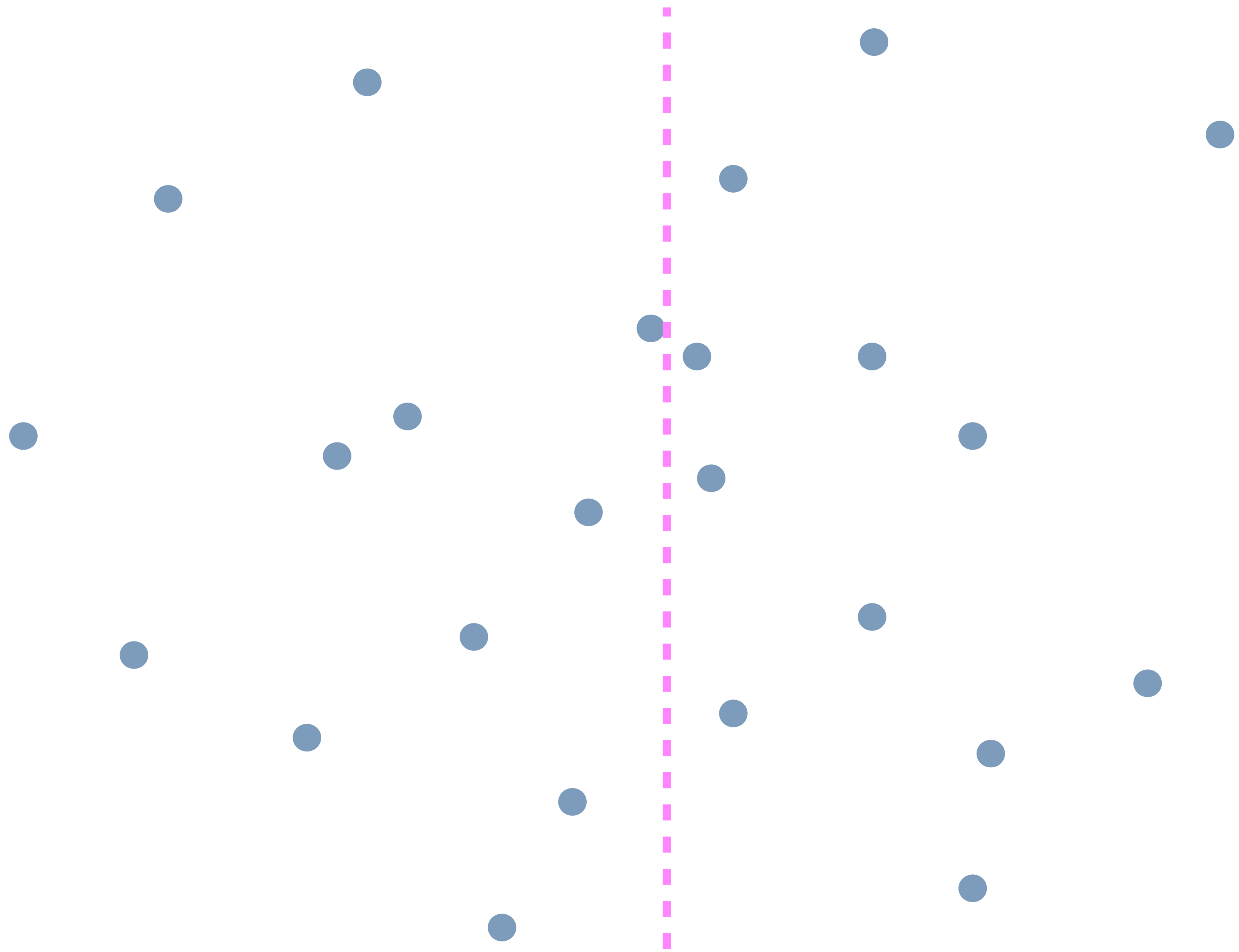
- if merge phase is **O(n)**: T(n) = 2T(n/2) + **O(n)** => O(n lg n)

CH via divide-and-conquer



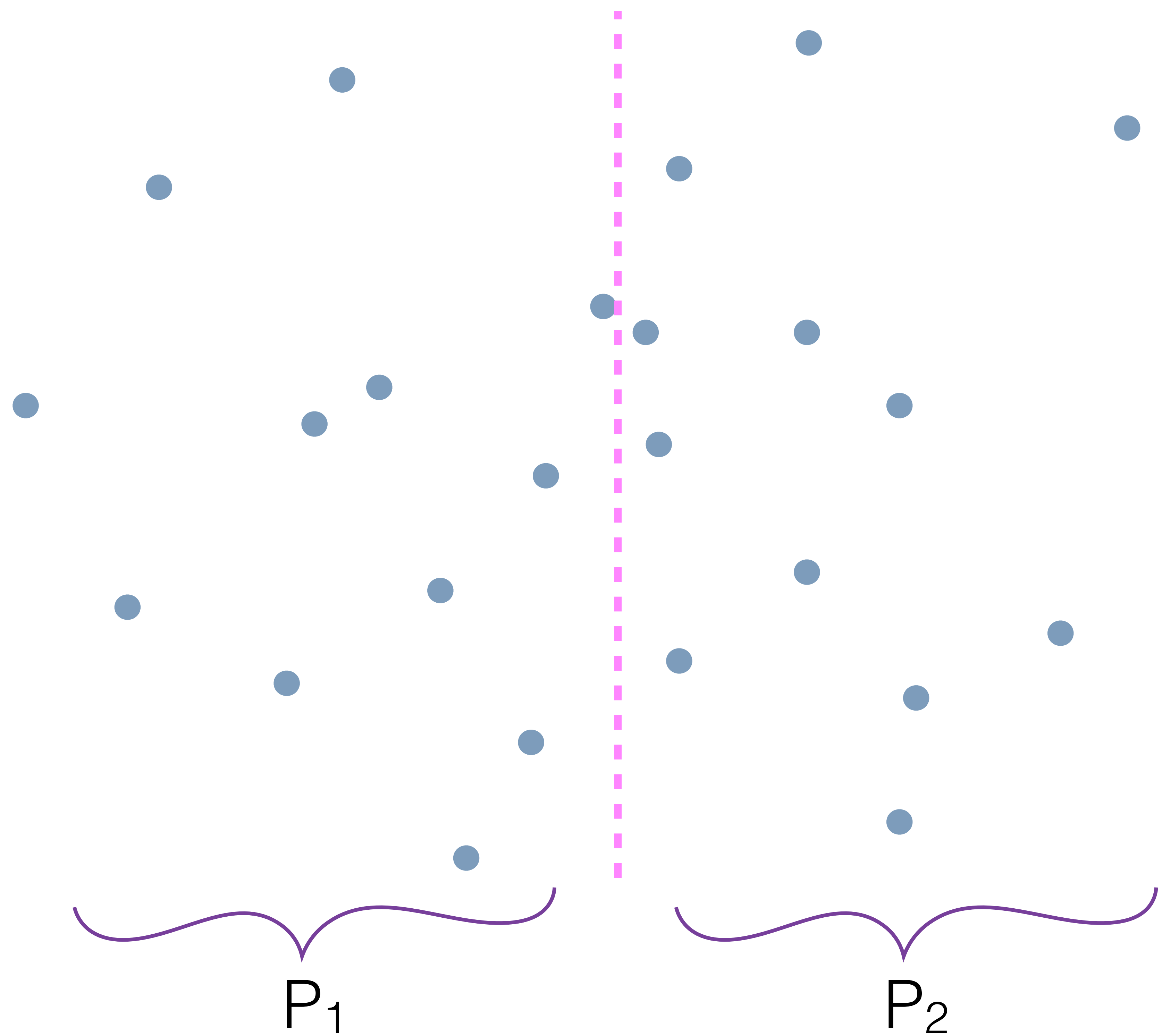
CH via divide-and-conquer

- find vertical line that splits P in half



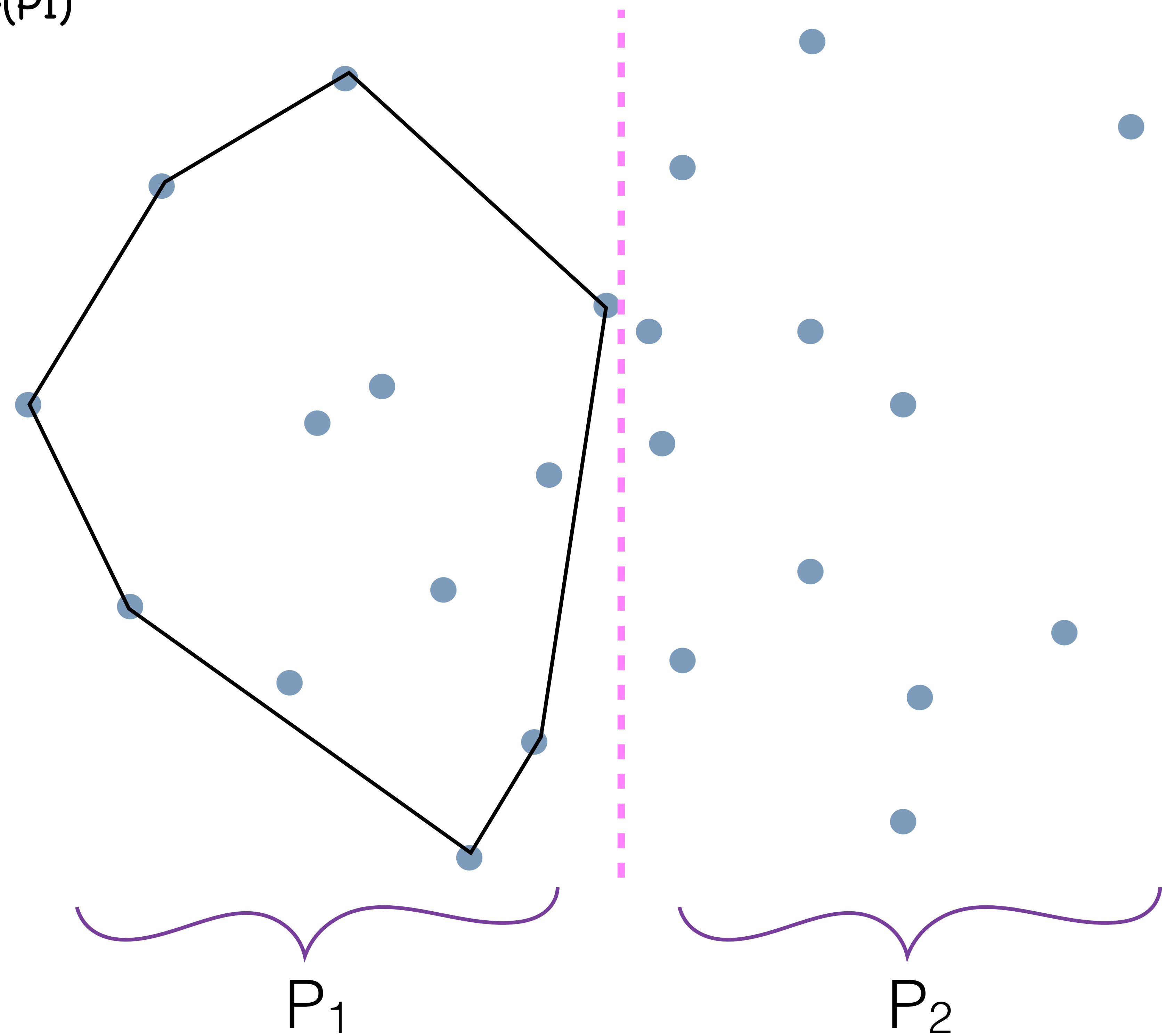
CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line



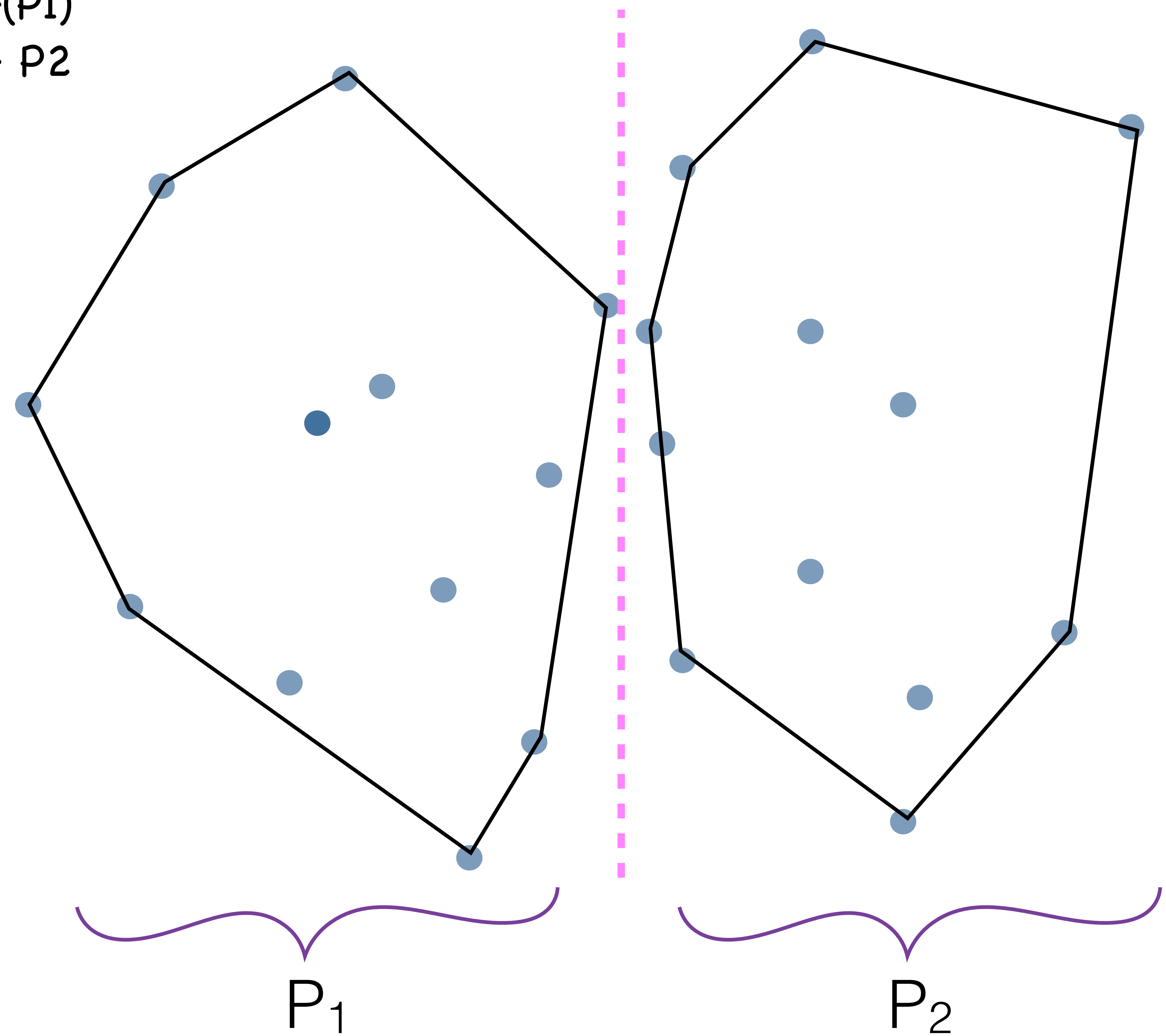
CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)



CH via divide-and-conquer

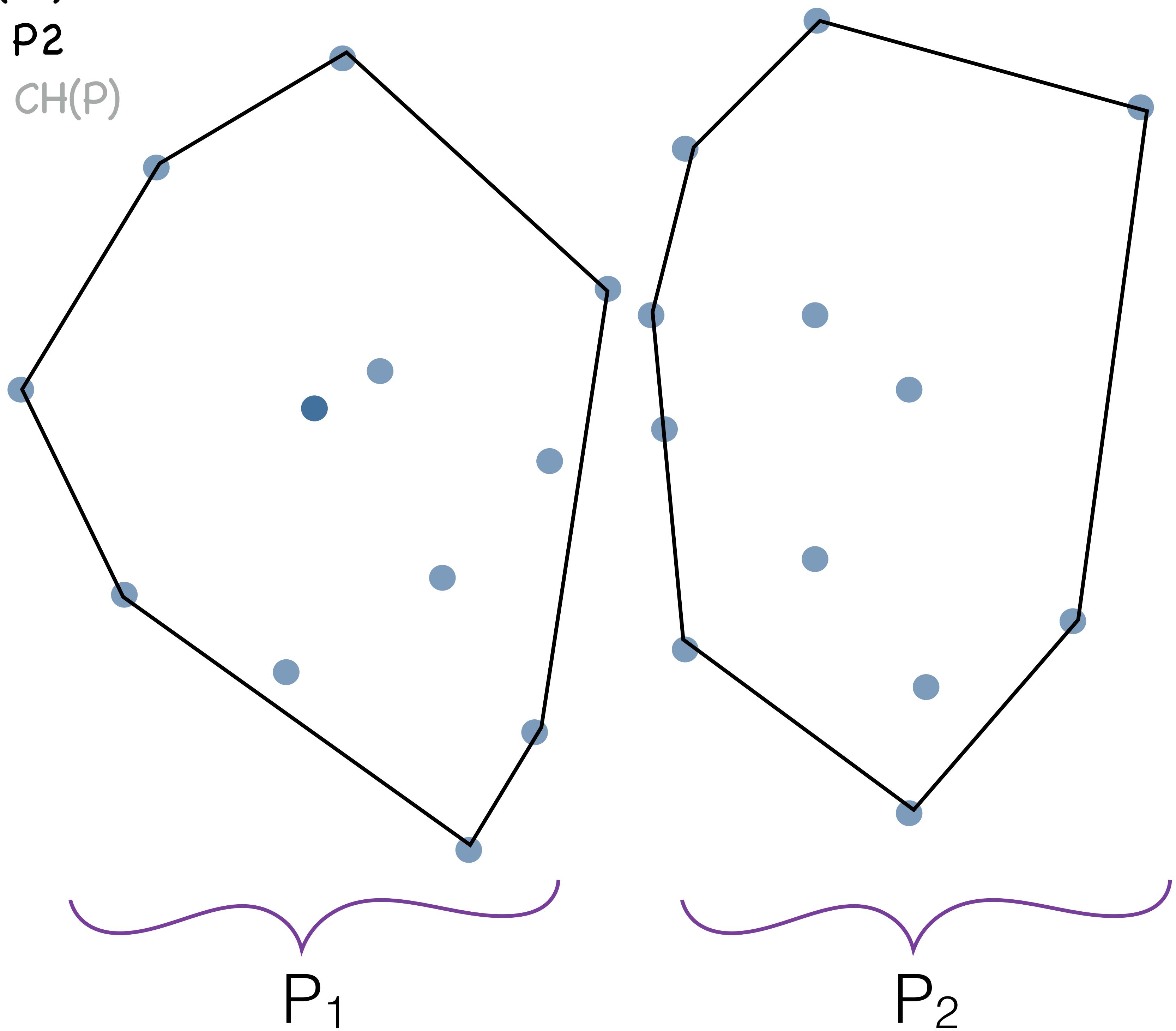
- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)
- recursively find CH P2



CH via divide-and-conquer

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)
- recursively find CH P2

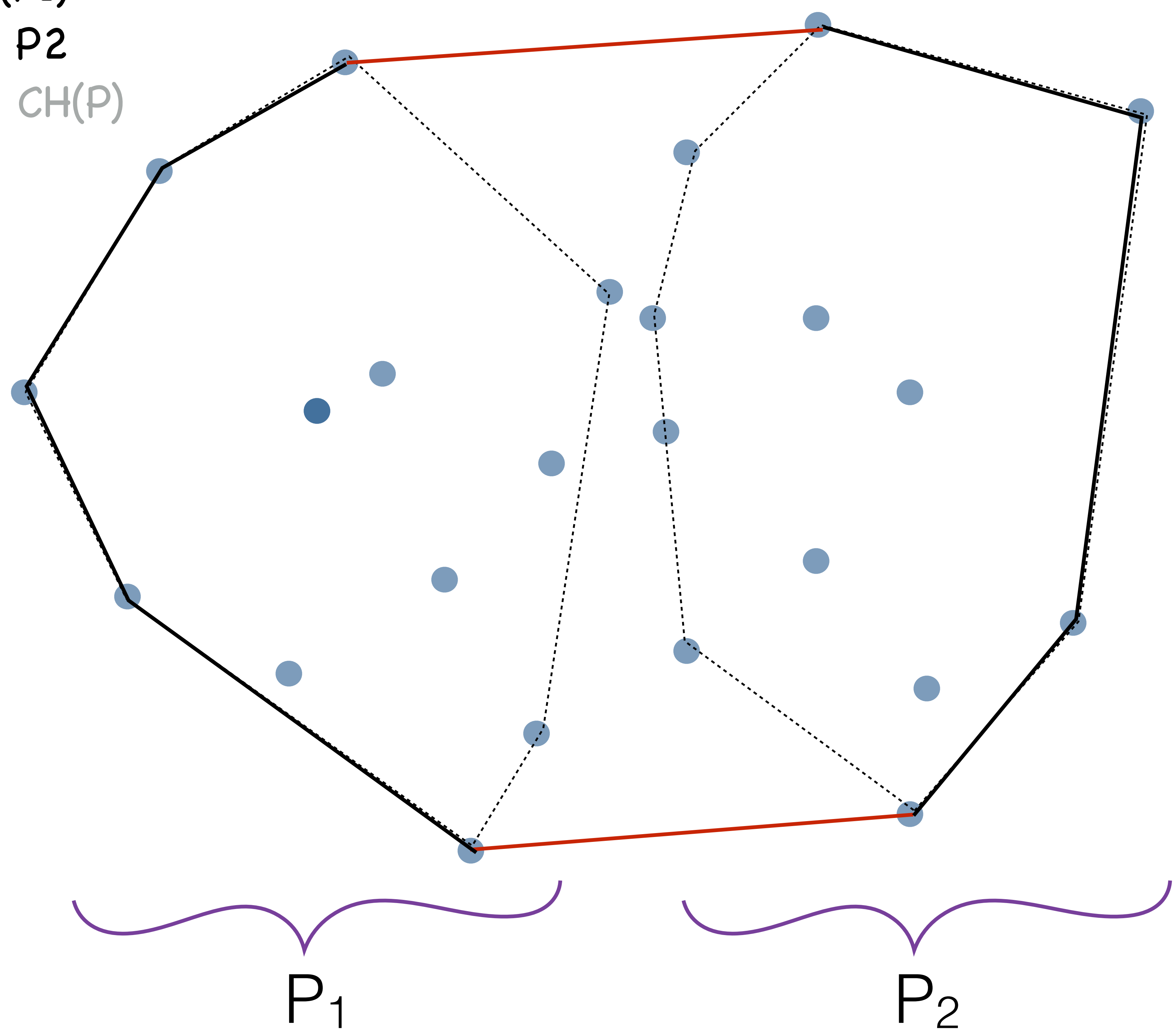
//now get somehow CH(P)



CH via divide-and-conquer

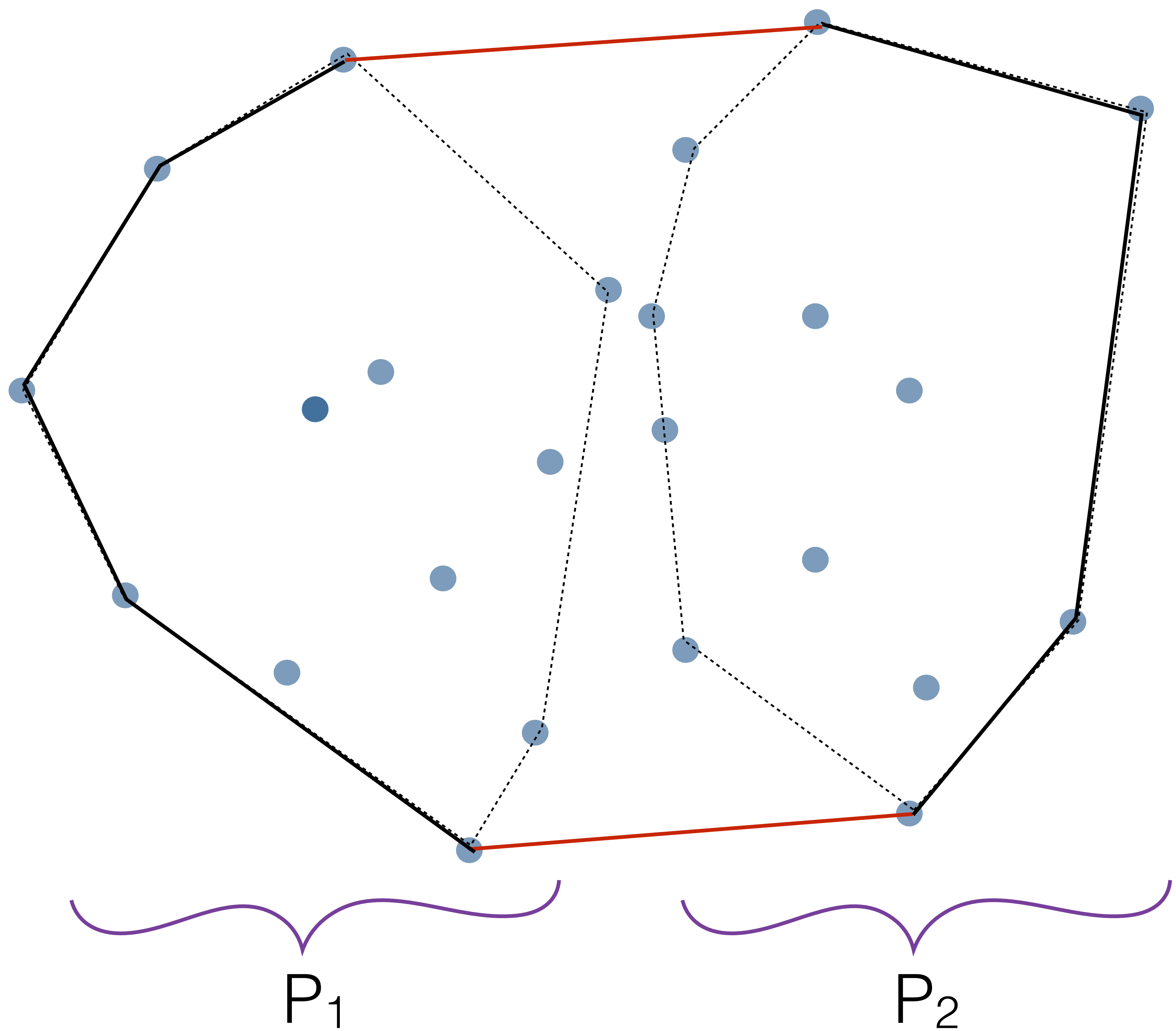
- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find CH(P1)
- recursively find CH P2

//now get somehow CH(P)



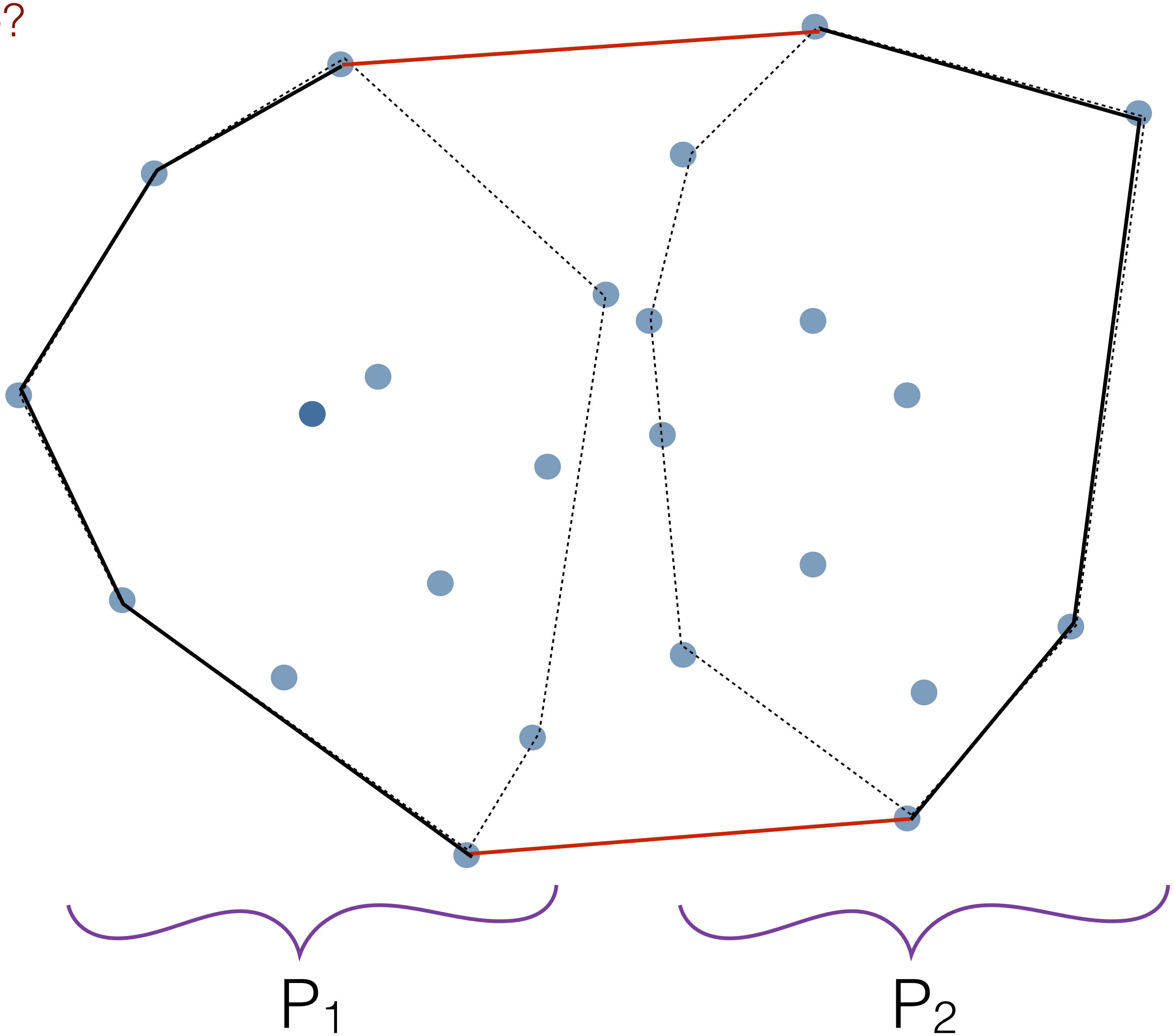
Merging two hulls..in linear time

- Need to find the two tangents

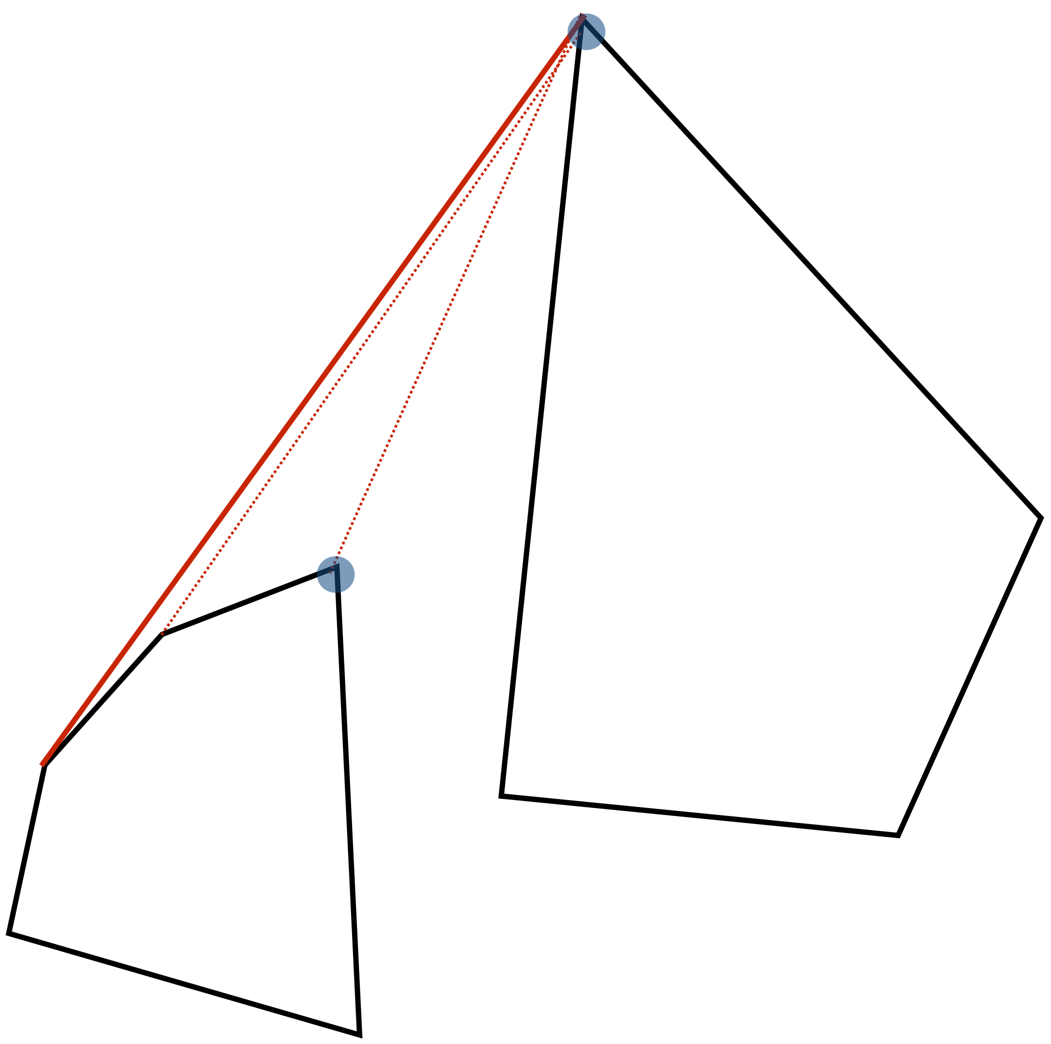


Merging two hulls..in linear time

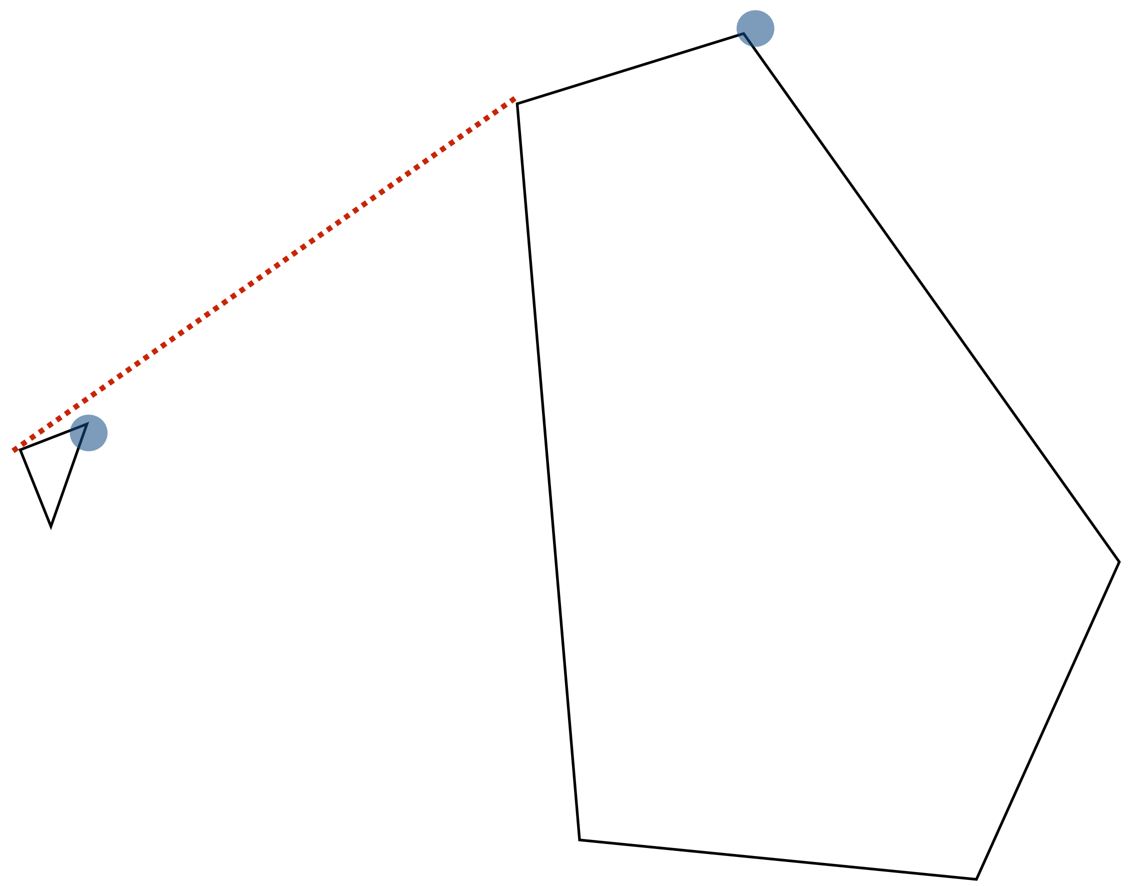
- Here it looks like the upper tangent is between the top points in P_1 and P_2
- Is that always true?



Not necessarily…



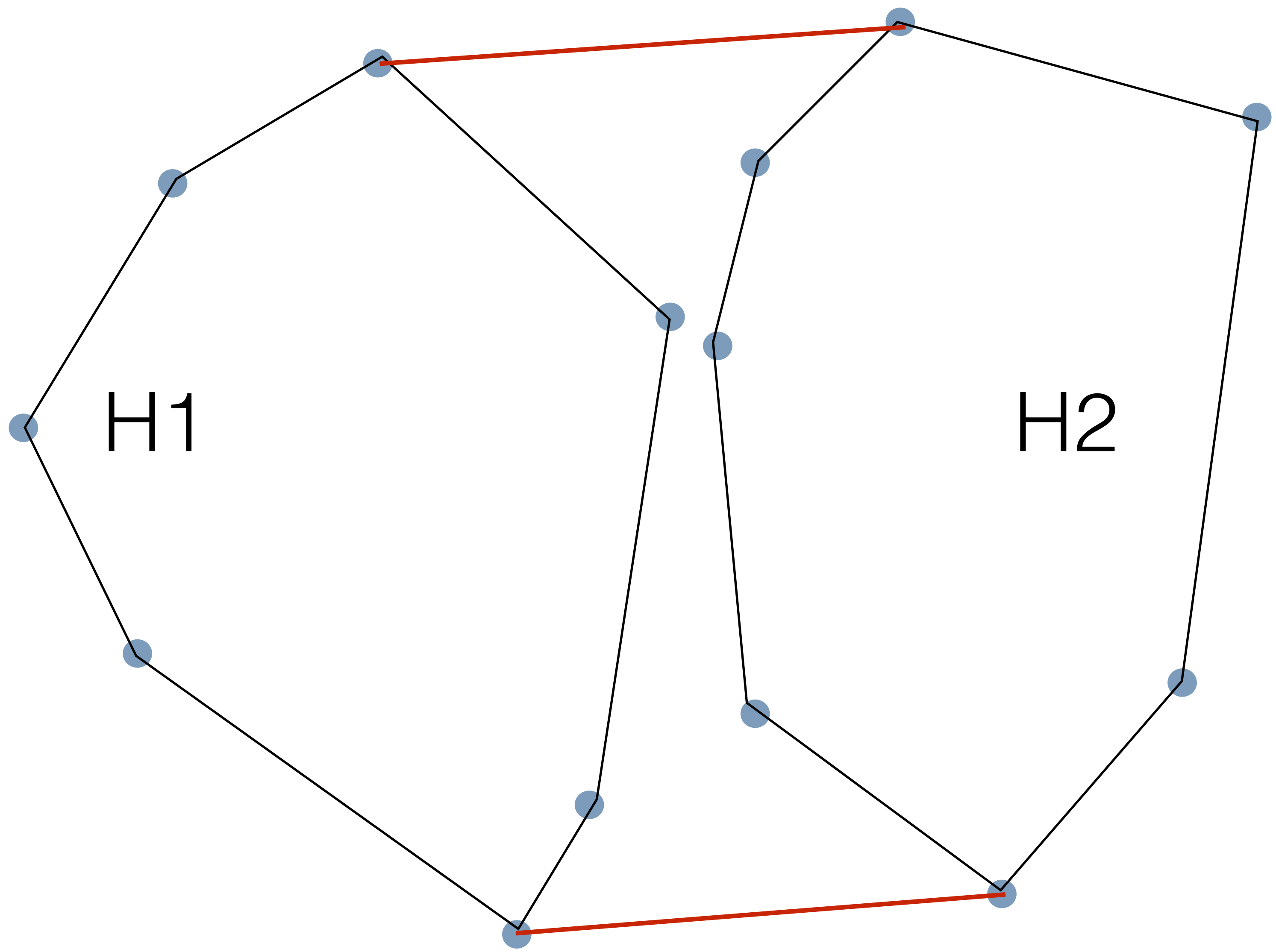
The top-most point overall is on the CH, but not necessarily on the upper tangent



Merging two hulls..in linear time

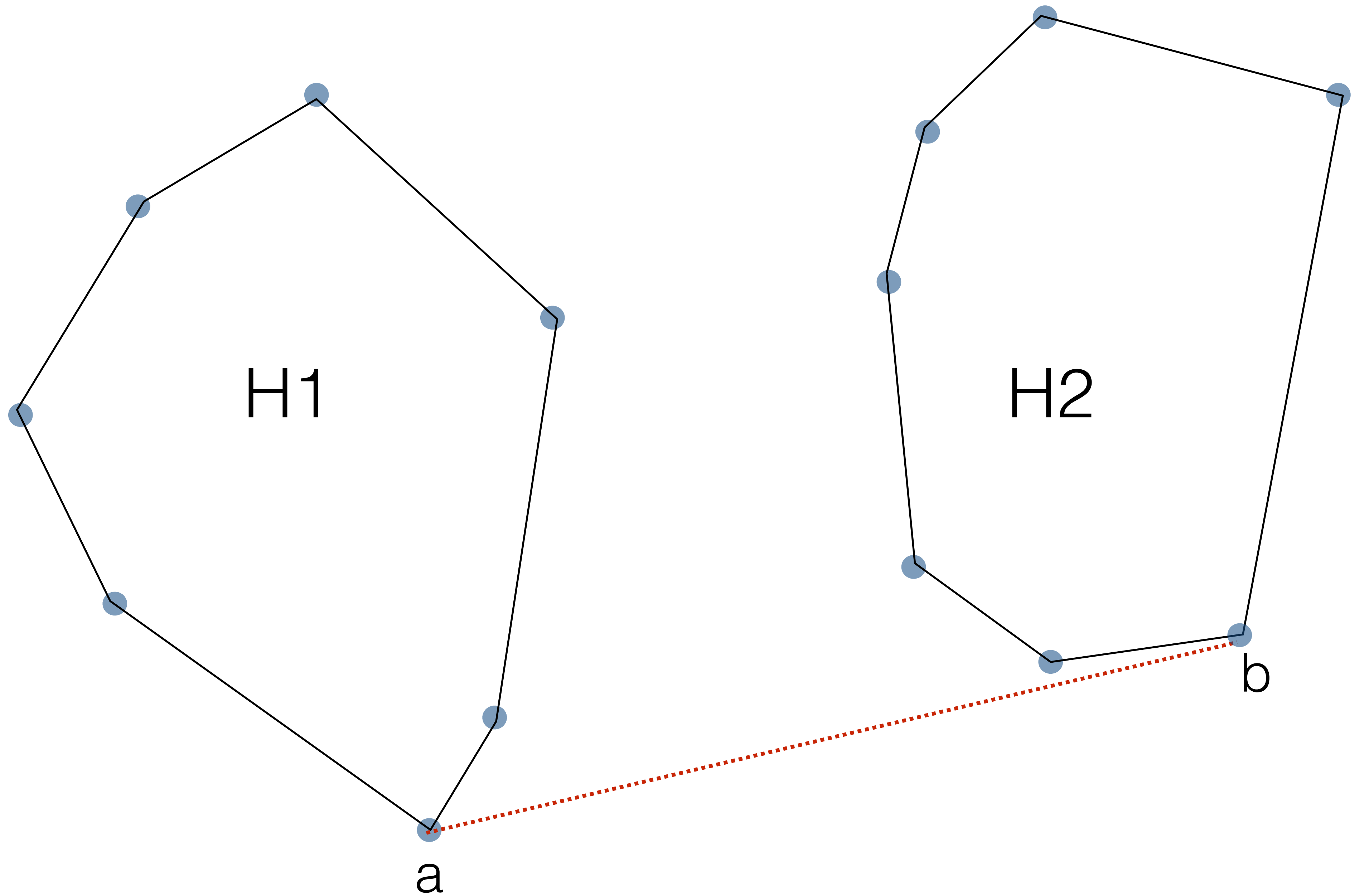
- Naive algorithm: try all segments (a,b) with a in H_1 and b in H_2

  Too slow. => $O(n^2)$ merge, $O(n^2 \lg n)$ CH algorithm



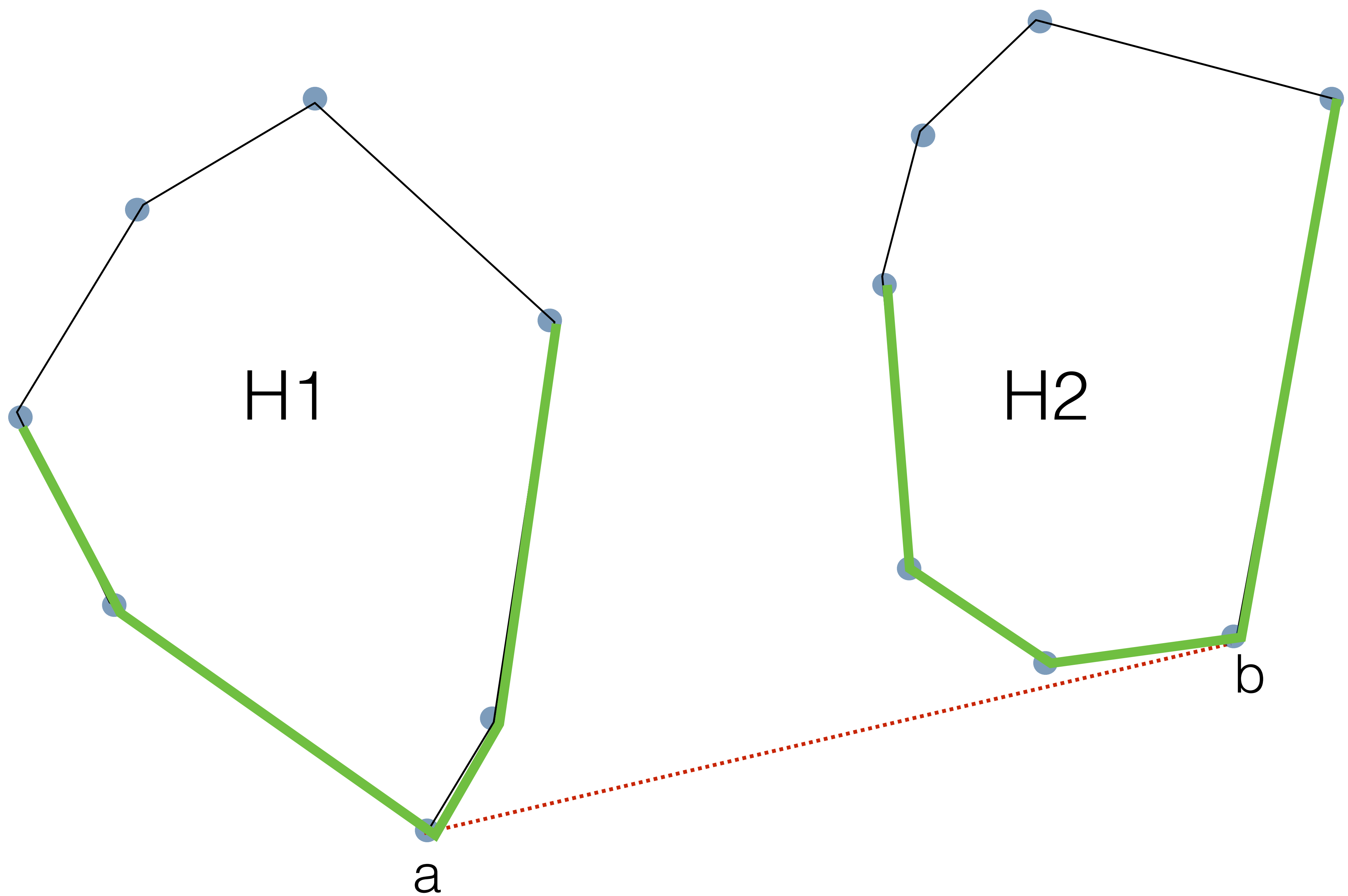
Finding the lower tangent

- Claim: All points in H1 and H2 are to the left of ab



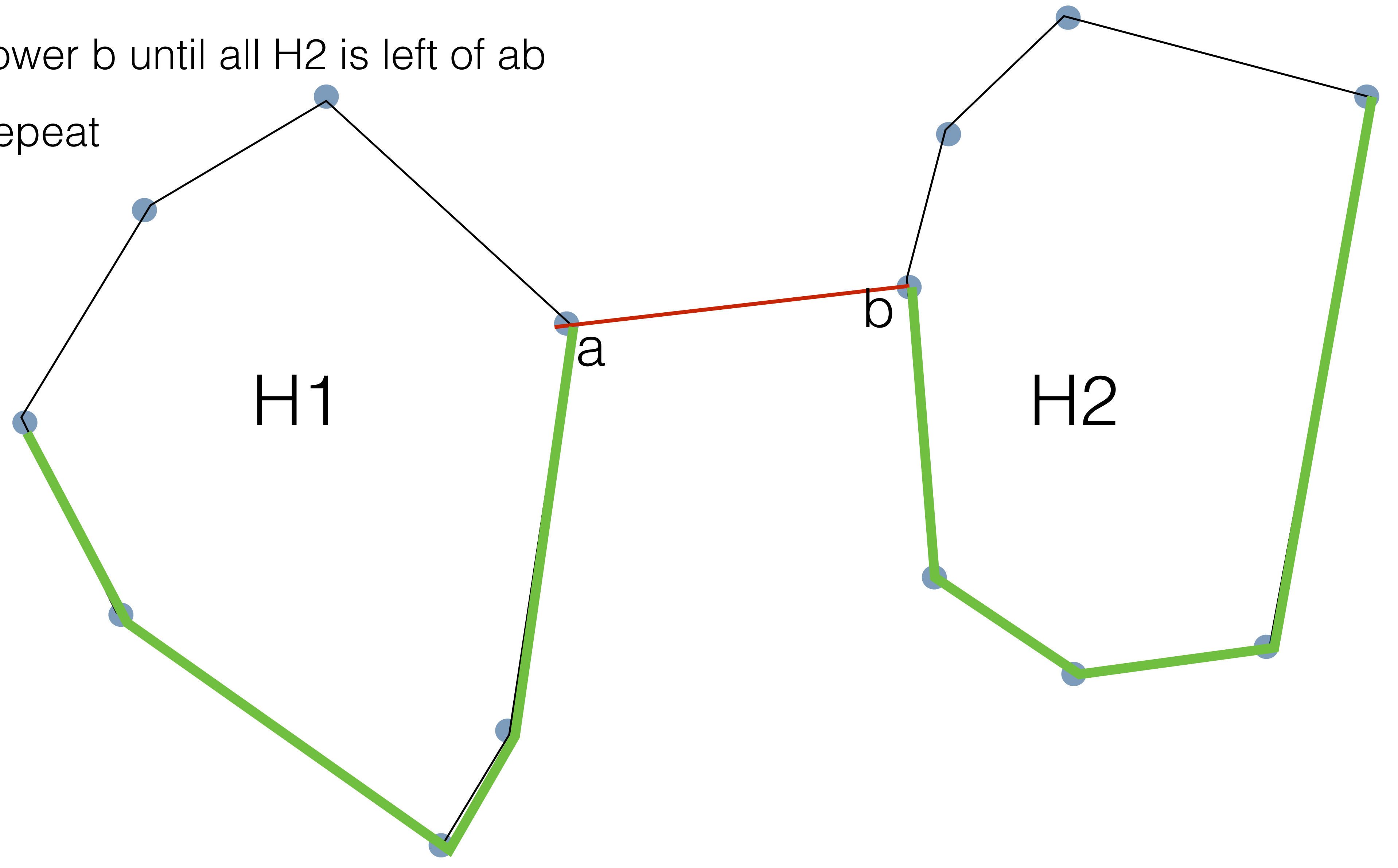
Finding the lower tangent

- Claim: Points a,b are on the lower hulls of H1 and H2, respectively.



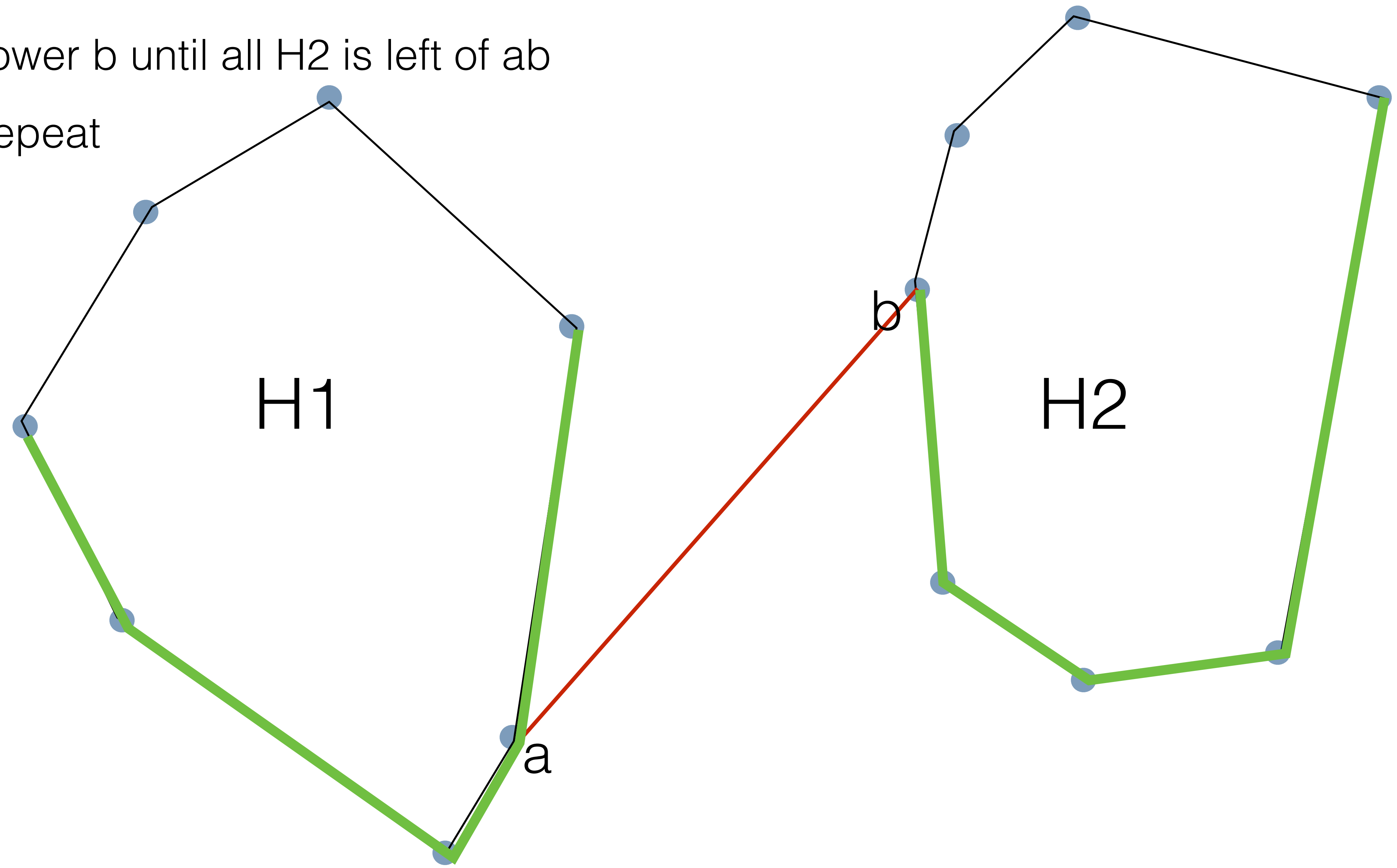
Finding the lower tangent

- Idea:
  - start with a = rightmost point in H1, b = leftmost point in H2
  - lower a until all H1 is left of ab
  - lower b until all H2 is left of ab
  - repeat



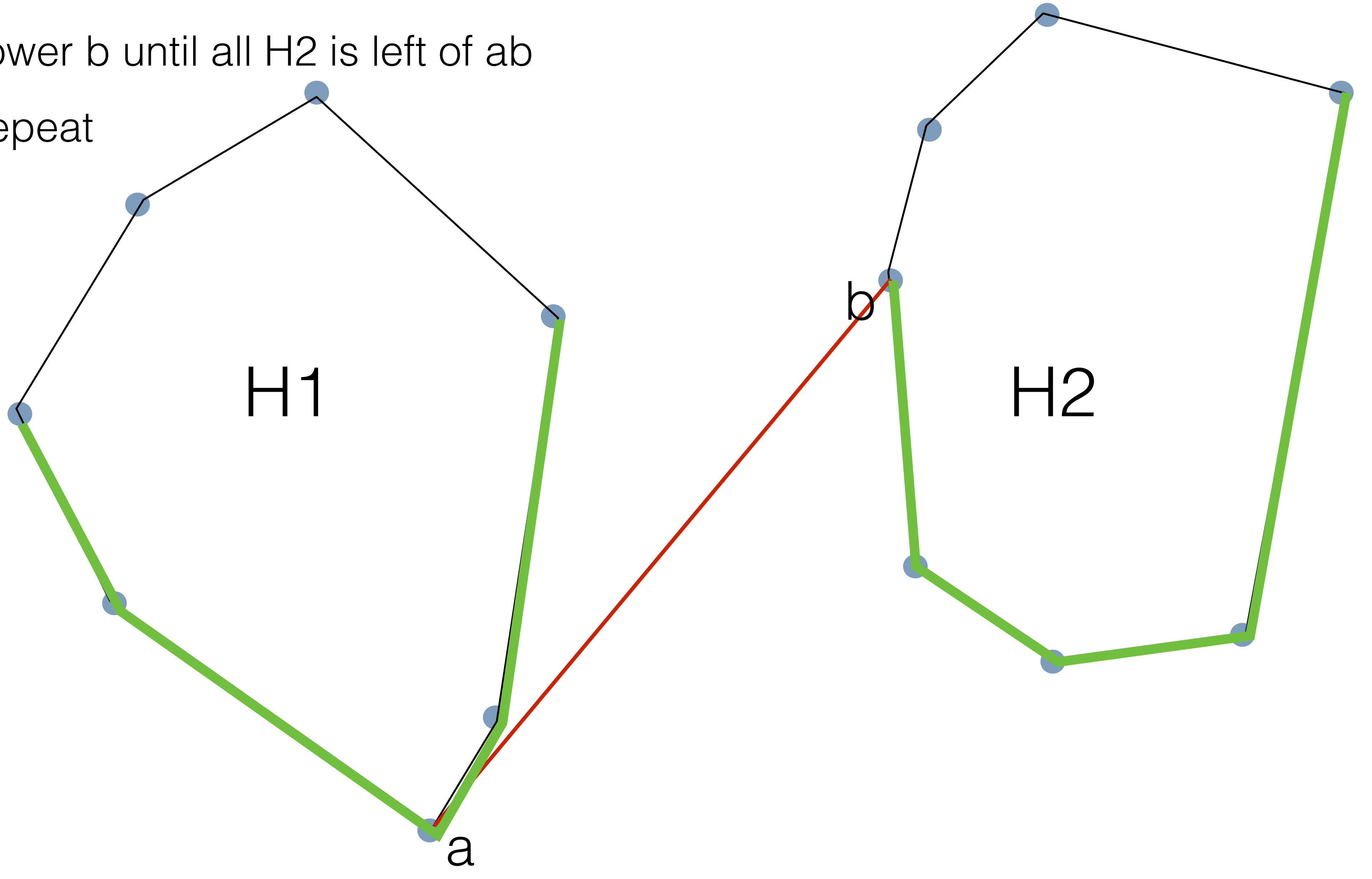
Finding the lower tangent

- Idea:
  - start with a = rightmost point in H1, b = leftmost point in H2
  - lower a until all H1 is left of ab
  - lower b until all H2 is left of ab
  - repeat



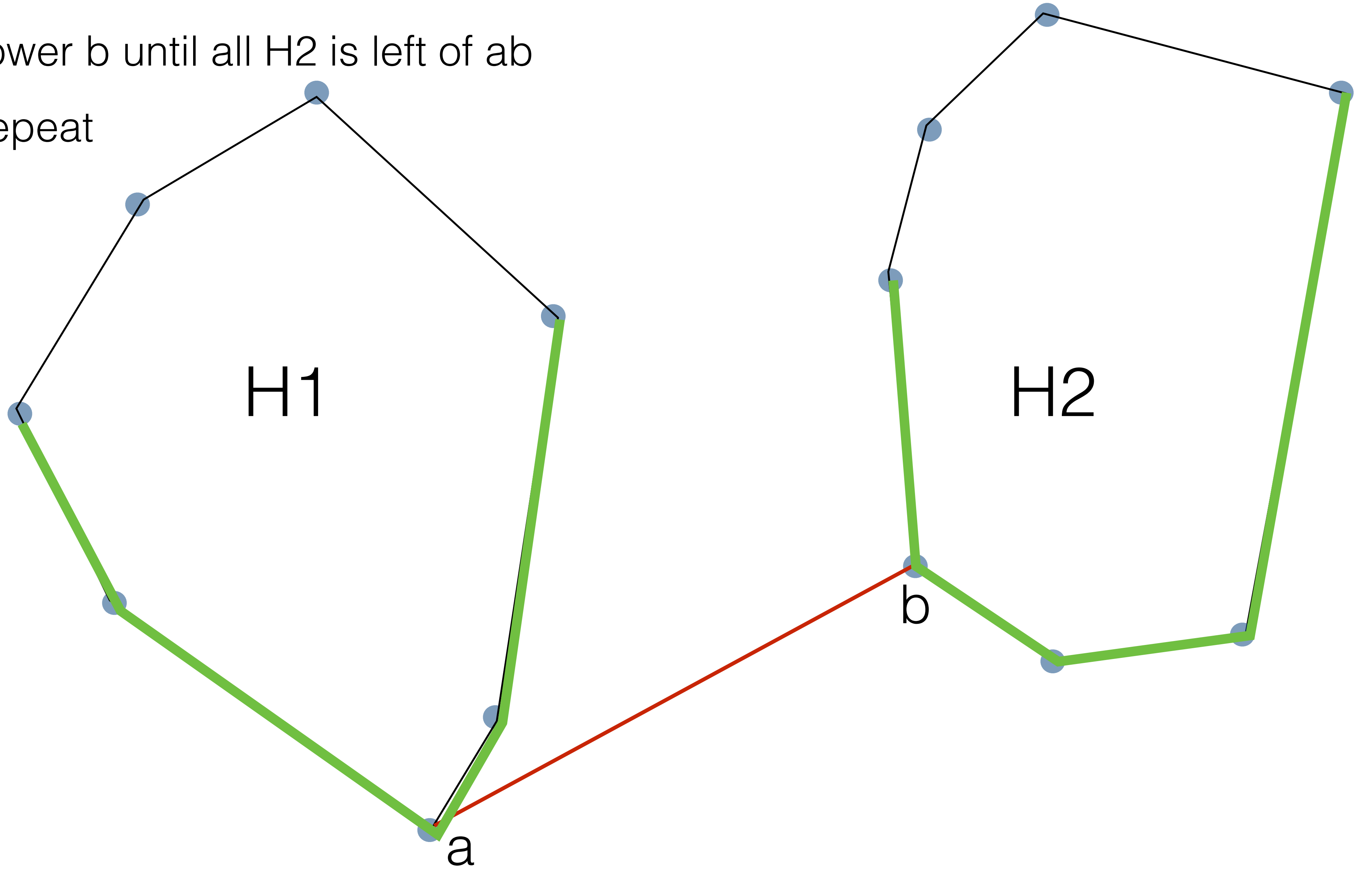
Finding the lower tangent

- Idea:
  - start with a = rightmost point in H1, b = leftmost point in H2
  - lower a until all H1 is left of ab
  - lower b until all H2 is left of ab
  - repeat



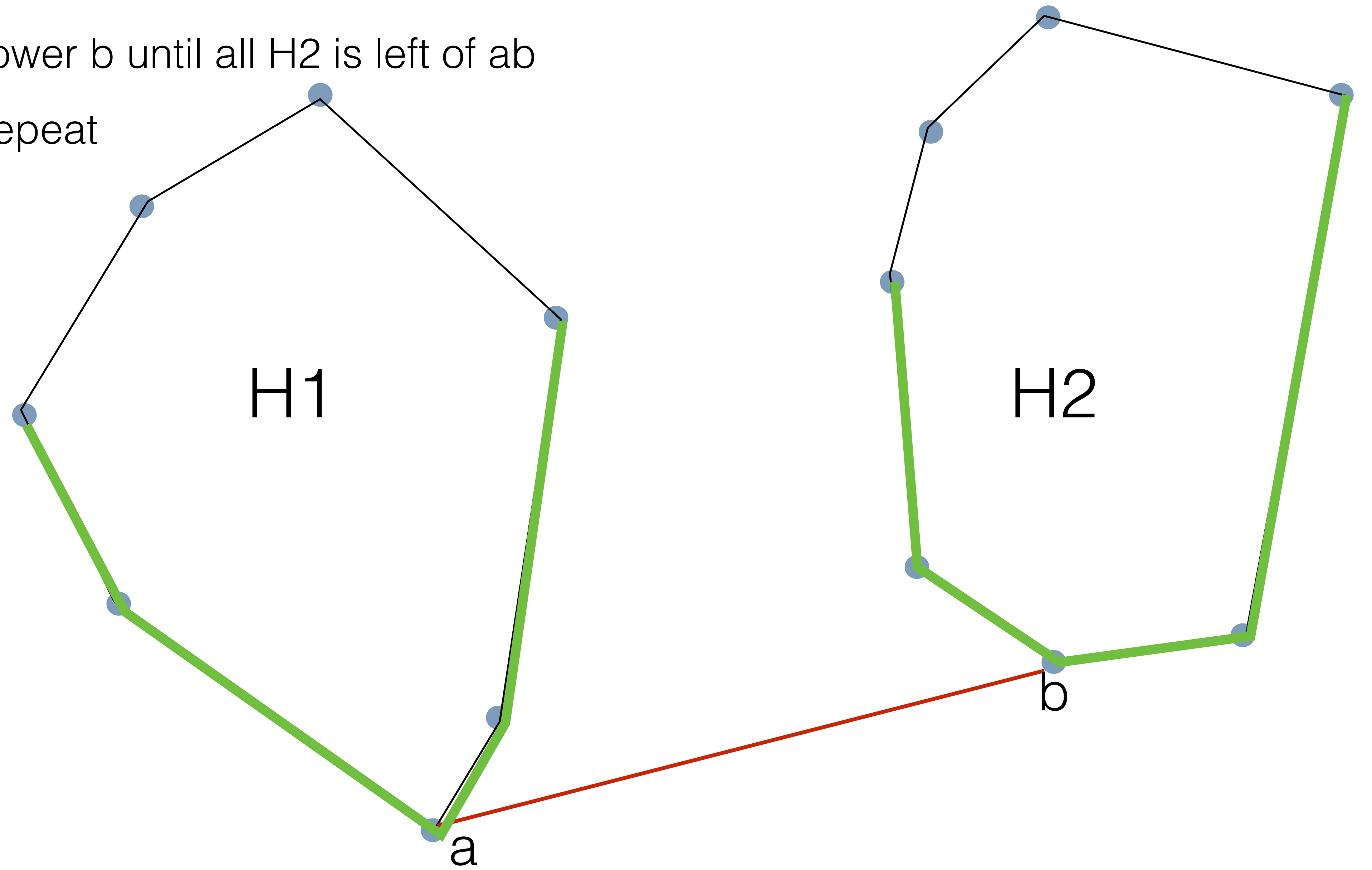
Finding the lower tangent

- Idea:
  - start with a = rightmost point in H1, b = leftmost point in H2
  - lower a until all H1 is left of ab
  - lower b until all H2 is left of ab
  - repeat



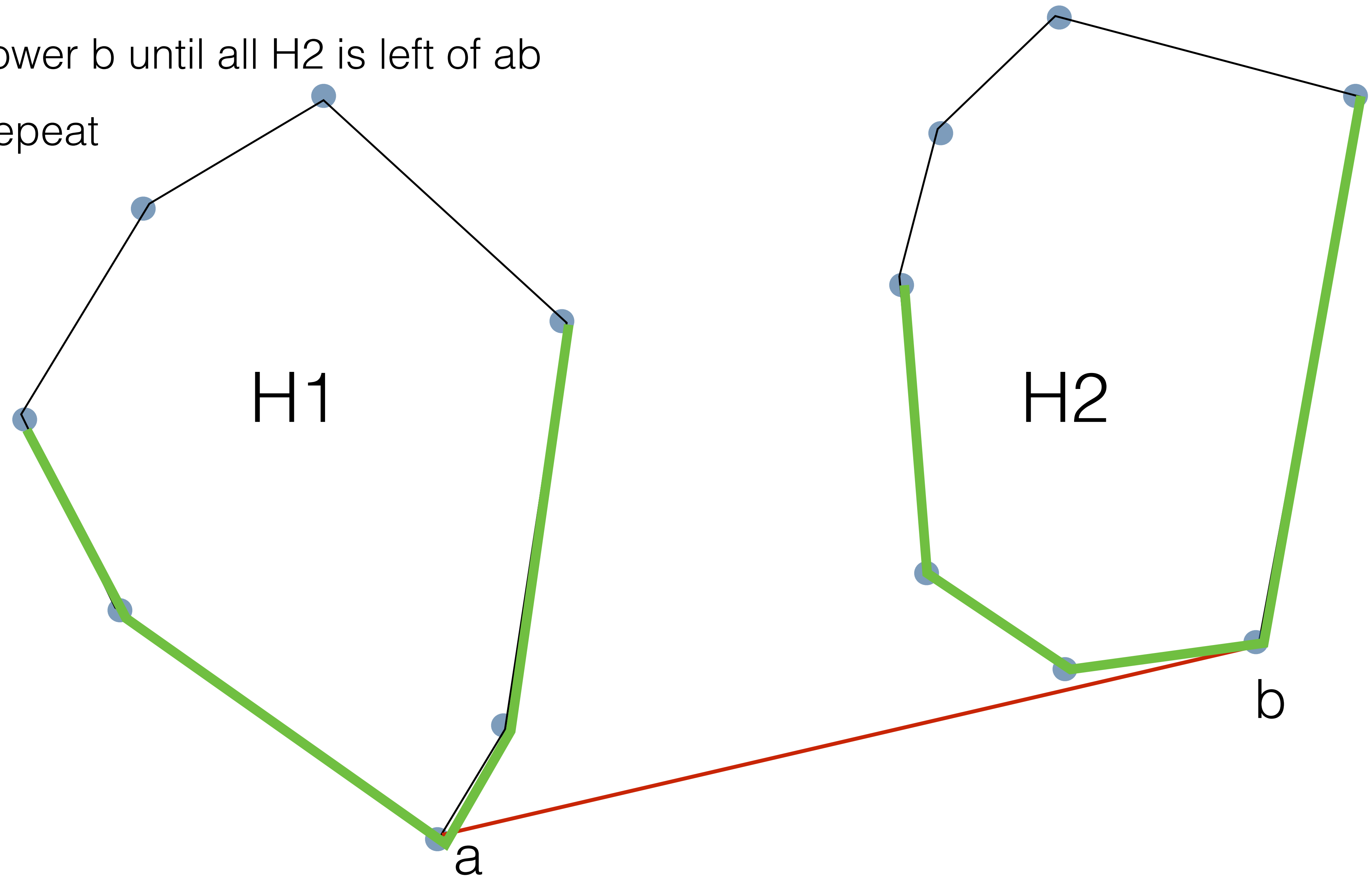
Finding the lower tangent

- Idea:
    - start with a = rightmost point in H1, b = leftmost point in H2
    - lower a until all H1 is left of ab
    - lower b until all H2 is left of ab
    - repeat



Finding the lower tangent

- Idea:
  - start with a = rightmost point in H1, b = leftmost point in H2
  - lower a until all H1 is left of ab
  - lower b until all H2 is left of ab
  - repeat



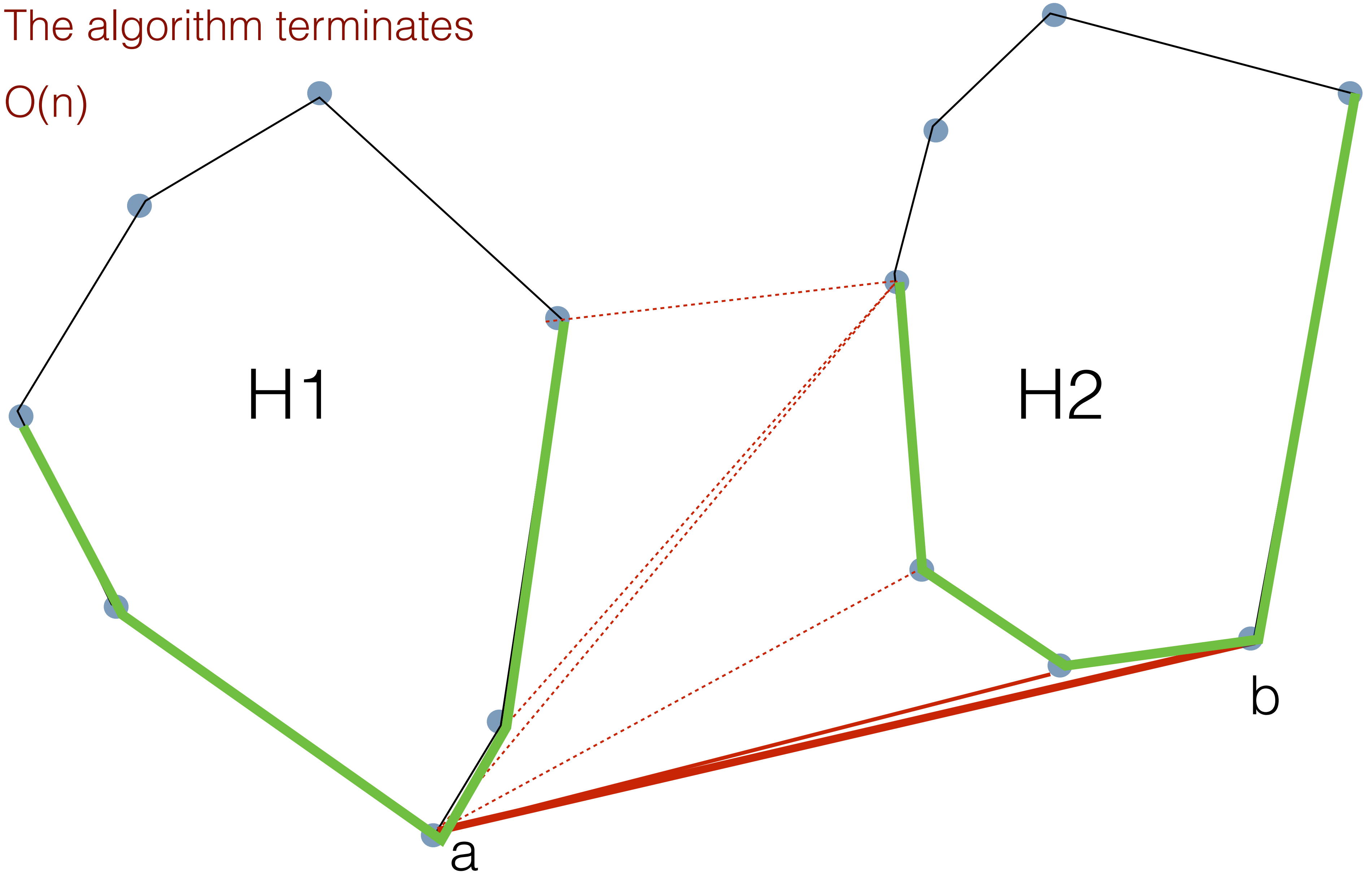
(why) does this work?

Claim: At any point during the algorithm, segment ab cannot intersect the interior of the polygons

==> a cannot move into the upper hull of P1, b cannot move into the upper hull of P2

==> The algorithm terminates

==> O(n)



CH via divide-and-conquer

- Yet another illustration of divide-and-conquer paradigm
- Runs in O(n lg n)
- Extends to O(n lg h)
- Extends nicely to 3D