

## KINEMATICAL DIFFERENTIAL EQUATIONS IN TERMS OF ORIENTATION ANGLES

In this appendix, the kinematical differential equations associated with each of twenty-four sets of angles describing the orientation of a rigid body  $B$  in a reference frame  $A$  are tabulated. To use the tables, proceed as follows: let  $a_1, a_2, a_3$  be a dextral set of mutually perpendicular unit vectors fixed in the reference frame  $A$ , and let  $b_1, b_2, b_3$  be a similar such set fixed in the body  $B$ . Regard  $b_i$  as initially aligned with  $a_i$  ( $i = 1, 2, 3$ ); select the type of rotation sequence of interest (i.e., body-three, body-two, space-three, or space-two); letting  $\theta_1, \theta_2, \theta_3$  denote the amounts (in radians) of the first, the second, and the third rotation, respectively, pick the rotation sequence of interest [for example, 3-1-2 (corresponding to a  $\theta_1 b_3, \theta_2 b_1, \theta_3 b_2$  body-three sequence or a  $\theta_1 a_3, \theta_2 a_1, \theta_3 a_2$  space-three sequence)]; finally, locate the table corresponding to the rotation sequence chosen. The table contains the relationships between  $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$  and  $\omega_1, \omega_2, \omega_3$ , where  $\omega_i \triangleq {}^A\omega^B \cdot b_i$  ( $i = 1, 2, 3$ ),  ${}^A\omega^B$  being the angular velocity of  $B$  in  $A$ .

### Body-three: 1-2-3

$\omega_1 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_1 = (\omega_1 c_3 - \omega_2 s_3)/c_2$
$\omega_2 = -\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_2 = \omega_1 s_3 + \omega_2 c_3$
$\omega_3 = \dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_3 = (-\omega_1 c_3 + \omega_2 s_3)s_2/c_2 + \omega_3$

**Body-three: 2-3-1**

$\omega_1 = \dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_1 = (\omega_2 c_3 - \omega_3 s_3)/c_2$
$\omega_2 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_2 = \omega_2 s_3 + \omega_3 c_3$
$\omega_3 = -\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_3 = \omega_1 + (-\omega_2 c_3 + \omega_3 s_3)s_2/c_2$

**Body-three: 3-1-2**

$\omega_1 = -\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_1 = (-\omega_1 s_3 + \omega_3 c_3)/c_2$
$\omega_2 = \dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_2 = \omega_1 c_3 + \omega_3 s_3$
$\omega_3 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_3 = (\omega_1 s_3 - \omega_3 c_3)s_2/c_2 + \omega_2$

**Body-three: 1-3-2**

$\omega_1 = \dot{\theta}_1 c_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_1 = (\omega_1 c_3 + \omega_3 s_3)/c_2$
$\omega_2 = -\dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_2 = -\omega_1 s_3 + \omega_3 c_3$
$\omega_3 = \dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_3 = (\omega_1 c_3 + \omega_3 s_3)s_2/c_2 + \omega_2$

**Body-three: 2-1-3**

$\omega_1 = \dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_1 = (\omega_1 s_3 + \omega_2 c_3)/c_2$
$\omega_2 = \dot{\theta}_1 c_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_2 = \omega_1 c_3 - \omega_2 s_3$
$\omega_3 = -\dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_3 = (\omega_1 s_3 + \omega_2 c_3)s_2/c_2 + \omega_3$

**Body-three: 3-2-1**

$\omega_1 = -\dot{\theta}_1 s_2 + \dot{\theta}_3$	$\dot{\theta}_1 = (\omega_2 s_3 + \omega_3 c_3)/c_2$
$\omega_2 = \dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_2 = \omega_2 c_3 - \omega_3 s_3$
$\omega_3 = \dot{\theta}_1 c_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_3 = \omega_1 + (\omega_2 s_3 + \omega_3 c_3)s_2/c_2$

**Body-two: 1-2-1**

$\omega_1 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_1 = (\omega_2 s_3 + \omega_3 c_3)/s_2$
$\omega_2 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_2 = \omega_2 c_3 - \omega_3 s_3$
$\omega_3 = \dot{\theta}_1 s_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_3 = \omega_1 - (\omega_2 s_3 + \omega_3 c_3)c_2/s_2$

**Body-two: 1-3-1**

$\omega_1 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_1 = (-\omega_2 c_3 + \omega_3 s_3)/s_2$
$\omega_2 = -\dot{\theta}_1 s_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_2 = \omega_2 s_3 + \omega_3 c_3$
$\omega_3 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_3 = \omega_1 + (\omega_2 c_3 - \omega_3 s_3)c_2/s_2$

**Body-two: 2-1-2**

$\omega_1 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_1 = (\omega_1 s_3 - \omega_3 c_3)/s_2$
$\omega_2 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_2 = \omega_1 c_3 + \omega_3 s_3$
$\omega_3 = -\dot{\theta}_1 s_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_3 = (-\omega_1 s_3 + \omega_3 c_3)c_2/s_2 + \omega_2$

**Body-two: 2-3-2**

$\omega_1 = \dot{\theta}_1 s_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_1 = (\omega_1 c_3 + \omega_3 s_3)/s_2$
$\omega_2 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_2 = -\omega_1 s_3 + \omega_3 c_3$
$\omega_3 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_3 = -(\omega_1 c_3 + \omega_3 s_3)c_2/s_2 + \omega_2$

**Body-two: 3-1-3**

$\omega_1 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_1 = (\omega_1 s_3 + \omega_2 c_3)/s_2$
$\omega_2 = \dot{\theta}_1 s_2 c_3 - \dot{\theta}_2 s_3$	$\dot{\theta}_2 = \omega_1 c_3 - \omega_2 s_3$
$\omega_3 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_3 = -(\omega_1 s_3 + \omega_2 c_3)c_2/s_2 + \omega_3$

**Body-two: 3-2-3**

$\omega_1 = -\dot{\theta}_1 s_2 c_3 + \dot{\theta}_2 s_3$	$\dot{\theta}_1 = (-\omega_1 c_3 + \omega_2 s_3)/s_2$
$\omega_2 = \dot{\theta}_1 s_2 s_3 + \dot{\theta}_2 c_3$	$\dot{\theta}_2 = \omega_1 s_3 + \omega_2 c_3$
$\omega_3 = \dot{\theta}_1 c_2 + \dot{\theta}_3$	$\dot{\theta}_3 = (\omega_1 c_3 - \omega_2 s_3)c_2/s_2 + \omega_3$

**Space-three: 1-2-3**

$\omega_1 = \dot{\theta}_1 - \dot{\theta}_3 s_2$	$\dot{\theta}_1 = \omega_1 + (\omega_2 s_1 + \omega_3 c_1)s_2/c_2$
$\omega_2 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_2 = \omega_2 c_1 - \omega_3 s_1$
$\omega_3 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_3 = (\omega_2 s_1 + \omega_3 c_1)/c_2$

**Space-three: 2-3-1**

$\omega_1 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_1 = (\omega_1 c_1 + \omega_3 s_1)s_2/c_2 + \omega_2$
$\omega_2 = \dot{\theta}_1 - \dot{\theta}_3 s_2$	$\dot{\theta}_2 = -\omega_1 s_1 + \omega_3 c_1$
$\omega_3 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_3 = (\omega_1 c_1 + \omega_3 s_1)/c_2$

**Space-three: 3-1-2**

$\omega_1 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_1 = (\omega_1 s_1 + \omega_2 c_1)s_2/c_2 + \omega_3$
$\omega_2 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_2 = \omega_1 c_1 - \omega_2 s_1$
$\omega_3 = \dot{\theta}_1 - \dot{\theta}_3 s_2$	$\dot{\theta}_3 = (\omega_1 s_1 + \omega_2 c_1)/c_2$

**Space-three: 1-3-2**

$\omega_1 = \dot{\theta}_1 + \dot{\theta}_3 s_2$	$\dot{\theta}_1 = \omega_1 + (-\omega_2 c_1 + \omega_3 s_1) s_2 / c_2$
$\omega_2 = \dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_2 = \omega_2 s_1 + \omega_3 c_1$
$\omega_3 = \dot{\theta}_2 c_1 - \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_3 = (\omega_2 c_1 - \omega_3 s_1) / c_2$

**Space-three: 2-1-3**

$\omega_1 = \dot{\theta}_2 c_1 - \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_1 = (\omega_1 s_1 - \omega_3 c_1) s_2 / c_2 + \omega_2$
$\omega_2 = \dot{\theta}_1 + \dot{\theta}_3 s_2$	$\dot{\theta}_2 = \omega_1 c_1 + \omega_3 s_1$
$\omega_3 = \dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_3 = (-\omega_1 s_1 + \omega_3 c_1) / c_2$

**Space-three: 3-2-1**

$\omega_1 = \dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 c_2$	$\dot{\theta}_1 = (-\omega_1 c_1 + \omega_2 s_1) s_2 / c_2 + \omega_3$
$\omega_2 = \dot{\theta}_2 c_1 - \dot{\theta}_3 s_1 c_2$	$\dot{\theta}_2 = \omega_1 s_1 + \omega_2 c_1$
$\omega_3 = \dot{\theta}_1 + \dot{\theta}_3 s_2$	$\dot{\theta}_3 = (\omega_1 c_1 - \omega_2 s_1) / c_2$

**Space-two: 1-2-1**

$\omega_1 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_1 = \omega_1 - (\omega_2 s_1 + \omega_3 c_1) c_2 / s_2$
$\omega_2 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_2 = \omega_2 c_1 - \omega_3 s_1$
$\omega_3 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_3 = (\omega_2 s_1 + \omega_3 c_1) / s_2$

**Space-two: 1-3-1**

$\omega_1 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_1 = \omega_1 + (\omega_2 c_1 - \omega_3 s_1) c_2 / s_2$
$\omega_2 = \dot{\theta}_2 s_1 - \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_2 = \omega_2 s_1 + \omega_3 c_1$
$\omega_3 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_3 = (-\omega_2 c_1 + \omega_3 s_1) / s_2$

**Space-two: 2-1-2**

$\omega_1 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_1 = (-\omega_1 s_1 + \omega_3 c_1) c_2 / s_2 + \omega_2$
$\omega_2 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_2 = \omega_1 c_1 + \omega_3 s_1$
$\omega_3 = \dot{\theta}_2 s_1 - \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_3 = (\omega_1 s_1 - \omega_3 c_1) / s_2$

**Space-two: 2-3-2**

$\omega_1 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_1 = -(\omega_1 c_1 + \omega_3 s_1) c_2 / s_2 + \omega_2$
$\omega_2 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_2 = -\omega_1 s_1 + \omega_3 c_1$
$\omega_3 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_3 = (\omega_1 c_1 + \omega_3 s_1) / s_2$

**Space-two: 3-1-3**

$\omega_1 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_1 = -(\omega_1 s_1 + \omega_2 c_1) c_2 / s_2 + \omega_3$
$\omega_2 = -\dot{\theta}_2 s_1 + \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_2 = \omega_1 c_1 - \omega_2 s_1$
$\omega_3 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_3 = (\omega_1 s_1 + \omega_2 c_1) / s_2$

**Space-two: 3-2-3**

$\omega_1 = \dot{\theta}_2 s_1 - \dot{\theta}_3 c_1 s_2$	$\dot{\theta}_1 = (\omega_1 c_1 - \omega_2 s_1) c_2 / s_2 + \omega_3$
$\omega_2 = \dot{\theta}_2 c_1 + \dot{\theta}_3 s_1 s_2$	$\dot{\theta}_2 = \omega_1 s_1 + \omega_2 c_1$
$\omega_3 = \dot{\theta}_1 + \dot{\theta}_3 c_2$	$\dot{\theta}_3 = (-\omega_1 c_1 + \omega_2 s_1) / s_2$