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Langevin approach to plasma kinetics with Coulomb collisions

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The Langevin approach to the kinetics of a collisional plasma is developed. Some collision models are considered, and the corresponding stochastic differential equations are derived. These equations can be regarded as an alternative to the description of a plasma in terms of a distribution function. The method developed here allows one to simulate plasma processes, taking account of both collective kinetics effects and Coulomb collisions. Results of the numerical simulation of the intervention of laser pulses with an overdense plasma are presented. The dependence of the absorption coefficient on the plasma parameters is calculated. The features of the plasma dynamics under the action of intense laser radiation are observed and discussed. The results of numerical tests of the validity of this method are also presented.

1. Introduction

At present, plasma processes are typically investigated in well-developed and distinct approaches (see Krall and Trivelpiece 1973). One approach, described by the Vlasov equation, is the collisionless limit, which is used when collective kinetic effects are dominant. Another approach is the hydrodynamic (or magnetohydrodynamic) limit, when the plasma is considered as a fluid. In addition, in cases where the plasma dynamics is mainly determined by collisional processes in external fields and the self-consistent field can be neglected, the Fokker–Planck approach is also widely used (see e.g. Larroche 1993). At the same time, it is well known that both collective kinetic effects and Coulomb collisions can play an essential role in a great variety of plasma physics problems, such as laser and particle-beam interactions with a plasma, shock waves and plasma expansion, plasma heating and many other phenomena. Besides, as was shown in Anisimov and Ivanov (1975) and Anisimov et al. (1982), taking collisions into account may lead not only to qualitative but also to quantitative changes in the plasma dynamics, even when the collisions frequency ν is much less than the electron plasma frequency ω_{pe} ($\nu/\omega_{pe} \approx 0.005$). The problem is that the Fokker–Planck equation for the collisional plasma dynamics is extremely difficult to deal with, either analytically or via direct numerical investigation.

The Langevin approach to overcoming these difficulties was first proposed by Ivanov and Shvets (1978, 1980). In these papers, the use of stochastic differential equations for the case of particle collisions with the background was developed and applied to the simulation of electrostatic problems (Anisimov and Ivanov 1975;

Anisimov et al. 1982). In recent years, there has been an increasing interest in treating collisional equations via the Langevin approach (Jones et al. 1996; Cadjan and Ivanov 1997; Manheimer et al. 1998). This approach seems to be very attractive and promising because, in principle it allows one to derive the governing equations for particle motion and to introduce collisions into the well-developed particle methods.

At present, particle-in-cell (PIC) methods play a fundamental role in the numerical simulation of collisionless plasma kinetics, and different attempts to generalize these methods to the case of a collisional plasma have been proposed. References and an interesting review of these attempts and other approaches to the simulation of Coulomb collisions can be found in Jones et al. (1996). The main drawbacks of the present methods are that there is some arbitrariness in the construction of numerical algorithms to take account of Coulomb collisions, together with the extremely large number of operations required per particle.

Generally speaking, we believe that if we want to construct an effective method for the simulation of complex nonlinear processes in a plasma, taking account of both collective and collisional effects, then we have to satisfy the following rather obvious but conflicting conditions.

(i) The method should be adequate for the task in hand, i.e. the constructed model should describe collisional effects with the desired accuracy. It should be noted here that for a number of problems the application of simplified (or approximate) models of the collision integral can provide a correct description and ensure good accuracy, and thus there is no need to apply the rather complex Landau collision operator.

(ii) The method should be computationally efficient, i.e. the computational algorithm should not be extremely time-consuming for the computer simulation of the problem under consideration. For example, the number of operations per particle in the binary-collision model (Takizuka and Abe 1977; Ma et al. 1993; Wang et al. 1996) is about N^2, where N is the number of particles in a computational cell. At the same time, the number of operations in collisionless PIC algorithms is of the order of N. Thus it is clear that there are a number of problems for which the application of such collision models can be too time-consuming.

The choice of a particular collision model (or model of the collision integral) is determined by the importance and particular features of the collision processes in a given physical problem. We believe that, in practice, some compromise between accuracy and complexity of the method should be achieved. Otherwise, we restrict ourselves either to a relatively simple setup of the problem, or to a too-rough description of the processes.

We consider the generic kinetic equation for the distribution function f_α of particles of species α (Krall and Trivelpiece 1973):

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} \equiv \frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{x}} + \mathbf{F}_L \frac{\partial f_\alpha}{\partial \mathbf{p}} = I[f_\alpha], \tag{1}$$

where $\mathbf{F}_L = q_\alpha(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B})$ is the Lorentz force and $I[f_\alpha]$ denotes the collision integral. It is well known for Coulomb systems that the change in the particle velocity in an individual collision is infinitesimally small and $I[f_\alpha]$ can be written

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in the form

$$I[f_\alpha] = \frac{\partial}{\partial v_i}\left[-A_i^\alpha(\mathbf{v})f_\alpha + \frac{1}{2}\frac{\partial}{\partial v_k}B_{ik}^\alpha(\mathbf{v})f_\alpha\right], \tag{2}$$

where summation is assumed over repeated indices and $i, k = 1, 2, 3$. For the Landau collision integral the coefficients A_i^α and B_{ik}^α are (Krall and Trivelpiece 1973)

$$A_i^\alpha(\mathbf{v}) = \frac{1}{4\pi}\frac{\partial}{\partial v_i}\sum_\beta L^{\alpha\beta}\frac{m_\alpha + m_\beta}{m_\beta}H^\beta(\mathbf{v}), \tag{3a}$$

$$B_{ik}^\alpha(\mathbf{v}) = \frac{1}{4\pi}\frac{\partial^2}{\partial v_i v_k}\sum_\beta L^{\alpha\beta}G^\beta(\mathbf{v}), \tag{3b}$$

where

$$H^\beta(\mathbf{v}) = \int d\mathbf{v}'\frac{f_\beta(\mathbf{v})}{|\mathbf{v}-\mathbf{v}'|}, \tag{4a}$$

$$G^\beta(\mathbf{v}) = \int d\mathbf{v}' f_\beta(\mathbf{v})|\mathbf{v}-\mathbf{v}'|; \tag{4b}$$

here $L^{\alpha\beta} = \Lambda(4\pi q_\alpha q_\beta/m_\alpha)^2$, Λ is the Coulomb logarithm (Krall and Trivelpiece 1973), and q and m are the charge and mass of a particle respectively. Equations (1)–(4) along with the Maxwell equations make up a complete system for the description of plasma kinetics.

It is well known that, even in the collisionless limit ($I[f] \equiv 0$), the kinetic equation is still too difficult for direct numerical simulation, and the particle methods are the most widely used algorithms (Birdsall and Langdon 1985). In these methods (PIC methods), instead of direct numerical solution of the kinetic equation, a system of ordinary differential equations (which are the characteristics of the Vlasov equation) for every 'macroparticle' is solved:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F}_L,$$

where $\mathbf{F}_L$ is the long-range electromagnetic force on the particle (Lorentz force). In the case of a collisional plasma, the position of a 'macroparticle' satisfies the usual equation of the collisionless case, but the momentum equation must be modified owing to the short-range Coulomb collisions. Coulomb collisions among ionized particles are described by the Fokker–Planck operator (2), which introduces a 'friction' (A_i) and 'diffusion' (B_{ik}) in velocity space. Thus our purpose is to find the effective 'collisional' force $\mathbf{F}_{\rm coll}$ that acts on the particles, i.e. we are looking for the equation of particle motion in the following form:

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}_L + \mathbf{F}_{\rm coll}.$$

This 'collisional' force can be introduced phenomenologically (Jones et al. 1996), but from our point of view a more mathematically correct approach can be constructed using the stochastic equivalence the Fokker–Planck and Langevin equations. These equations of particle motion (or the expression for $\mathbf{F}_{\rm coll}$) can easily be derived for the description of interspecies collisions, but there are some difficulties in the case of collisions between particles of the same species. The main difficulty in this case is the calculation of the coefficients A_i, and B_{ik} and the required reconstruction of the density function f. To the best of our knowledge, there is no satisfactory

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way to compute these values in a fast and accurate way, and therefore we suppose that in this case the construction of an approximate (or simplified) collision integral would be useful and fruitful for practical applications. In this paper, we describe the general features and development of this method, and make some applications to plasma simulation.

This paper is organized as follows. In Sec. 2 we describe the method, discuss the different approaches simplifying the problem, and derive the equation of motion. Some simulation results on laser–plasma interaction are presented in Sec. 3. In the Appendix, solutions of test problems are presented.

2. Stochastic approach to the Fokker–Planck equation

2.1. Stochastic equivalence of Fokker–Planck and Langevin methods

The starting point of our work is the stochastic equivalence of the Fokker–Planck and Langevin methods. It is known (see e.g. Tikhonov and Mironov 1977) that the Fokker–Planck equation,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i}\left[-A_i(\mathbf{v})f + \frac{1}{2}\frac{\partial}{\partial v_k}B_{ik}(\mathbf{v})f\right] \tag{5}$$

and the Langevin equation

$$\frac{dv_i}{dt} = F_i(\mathbf{v}) + D_{ik}(\mathbf{v})\,\xi_k \tag{6}$$

are alternative methods for the description of a Markovian diffusion process $\mathbf{v}(t)$. Here F_i and D_{ik} are deterministic functions and $\boldsymbol{\xi}(t)$ is random white noise with the following characteristics:

$$\langle \boldsymbol{\xi}(t)\rangle = 0, \qquad \langle \xi_i(t)\,\xi_k(t+\tau)\rangle = \delta_{ik}\,\delta(\tau).$$

These equations are stochastically equivalent (i.e. (5) and (6) define the same process $\mathbf{v}(t)$) if the deterministic functions F_i and D_{ik} are related to A_i and B_{ik} by the expressions (Tikhonov and Mironov 1977)

$$D = B^{1/2}, \qquad F_i = A_i - \chi D_{jk}\,\frac{\partial D_{ik}}{\partial v_j}, \tag{7}$$

where $B = \|B_{ik}\|$ and $D = \|D_{ik}\|$ are symmetric non-negative definite matrices and the parameter χ ($0 \leqslant \chi \leqslant 1$) depends on the definition of the stochastic integral in the integration of (6). So the cases $\chi = 0$ and $\chi = \frac{1}{2}$ correspond to the definitions of a stochastic integral by Ito and by Stratonovich (symmetric integral) respectively (van Kampen 1984).

Thus, to derive the governing equations, we should calculate the values of D_{ik} and F_i using the terms A_i and B_{ik}. The problem is that for the Landau collision integral the coefficients A_i and B_{ik} have an integral dependence on the distribution function. The latter complicates the direct application of this approach in the general case (especially in the case of collisions between particles of the same species). Therefore the approximate calculation of A_i and B_{ik} is useful for the construction of an effective method. This will be one of our main concerns in the present section.

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2.2. Diffusion model

The problem can be simplified for the well-known diffusion model of the collision integral (Artsimovich and Sagdeev 1979)

$$I[f] = \frac{\partial}{\partial v_i}\,\nu_\beta \left(\tilde{v}_i\, f_\alpha + \frac{T_\beta}{m_\beta}\,\frac{\tilde{v}_i \tilde{v}_k}{\tilde{v}^2}\,\frac{\partial}{\partial v_k}\, f_\alpha \right), \tag{8}$$

where $\tilde{v}_i = v_i - u_\beta$, $T_\beta = T_\beta(\mathbf{x}, t)$ and $\mathbf{u}_\beta(\mathbf{x}, t)$ are the temperature and average velocity of species β and ν_β is the collision frequency. This phenomenological model qualitatively correctly describes the basic features of Coulomb collisions such as 'friction' and 'diffusion' in velocity space. Certainly, this model does not reflect all of the properties of scattering in a Coulomb potential, but for many problems it can be regarded as a good approximation (for example, it has been widely used previously in analytical (Zakharov and Karpman 1962) and numerical (Rathman and Denavit 1975) investigations. Using this model and the expressions (7) one can easily derive the corresponding stochastic equation:

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}_L - \nu_\beta \tilde{\mathbf{v}} + \left(\nu_\beta \frac{T_\beta}{m_\beta} \right)^{1/2} \tilde{\mathbf{v}}\,\frac{\tilde{\mathbf{v}} \cdot \boldsymbol{\xi}}{\tilde{v}^2}. \tag{9}$$

This equation has a simple form in which the second term describes the friction and the third term is responsible for the random scattering (diffusion in velocity space). Moreover, for the description of one-dimensional processes in a plasma, a further simplification may be assumed if a rougher account of collisions is acceptable. Naturally, Coulomb collision is a three-dimensional process in velocity space, but, for a rough account, a simpler one-dimensional diffusion model can be applied:

$$I[f] = \frac{\partial}{\partial v}\,\nu_\beta \left(\tilde{v} f_\alpha + \frac{T_\beta}{m_\beta}\,\frac{\partial f_\alpha}{\partial v} \right). \tag{10}$$

This expression leads to an extremely simple equation:

$$\frac{dv}{dt} = F_L - \nu_\beta \tilde{v} + \left(\nu_\beta \frac{T_\beta}{m_\beta} \right)^{1/2} \xi, \tag{11}$$

which resembles the original Langevin equation for Brownian motion. The expressions above presented illustrate how one can obtain the governing equation for particle motion in the simplest case. One of the main advantages of the diffusion model is the simplicity of the corresponding Langevin equation, which ensure a high efficiency of numerical implementation.

2.3. Electron–ion collisions

It is known (Lifshitz and Pitaevskii 1981) that the collision integral can be simplified for the case of electron–ion collisions. Indeed, the existence of a small parameter $m_e/M_{\text{ion}} \ll 1$ enables us to consider the ion distribution function as 'cold', $f_\beta \propto \delta(\mathbf{v} - \mathbf{u}_{\text{ion}})$ in the expressions (3) and (4). Thus the electron–ion collision integral can be written in the form

$$I[f] = \frac{a}{2}\,\frac{\partial}{\partial v_i}\,\frac{\tilde{v}^2 \delta_{ik} - \tilde{v}_i \tilde{v}_k}{\tilde{v}^3}\,\frac{\partial f}{\partial v_k} + \ldots, \tag{12}$$

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where

$$a = \frac{Z\Lambda\omega_{pe}^4}{4\pi n_e} = \frac{3\sqrt{2\pi}}{2}\,\nu_{ei}\,V_{Te}^3,$$
$$\nu_{ei} = \frac{4\sqrt{2\pi}\,Z\Lambda e^4 n_e}{3\,m_e^{1/2}\,T_e^{3/2}};$$

here $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{u}_{\rm ion}$, $V_{Te} = (T_e/m_e)^{1/2}$, ν_{ei} is the electron–ion collision frequency, n_e and T_e are the electron concentration and temperature respectively, and Ze is the ion charge. Here the dots (...) denote higher terms in the perturbation expansion with respect to m_e/M_i. The collision integral (12) leads to the following expressions for the coefficients A_i and B_{ik}:

$$A_i = F_{Li} + a\,\frac{\tilde{v}_i}{\tilde{v}^3} + \ldots, \tag{13a}$$

$$B_{ik} = a\,\frac{\tilde{v}^2\delta_{ik} - \tilde{v}_i\tilde{v}_k}{\tilde{v}^3} + \ldots \quad . \tag{13b}$$

Using the symmetric stochastic integral ($\chi = \frac{1}{2}$) and the relationships (7), one can obtain the corresponding deterministic functions $F_i(\mathbf{v})$ and $D_{ik}(\mathbf{v})$:

$$F_i = F_{Li} + \ldots,$$
$$D_{ik} = \left(\frac{a}{|\tilde{v}|}\right)^{1/2}\left(\delta_{ik} - \frac{\tilde{v}_i\tilde{v}_k}{\tilde{v}^2}\right) + \ldots \quad .$$

Finally, we obtain the governing equation of particle motion in the form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}_L - \boldsymbol{\Omega}(\tilde{v})\tilde{\mathbf{v}} \times (\tilde{\mathbf{v}} \times \boldsymbol{\xi}) + \ldots, \tag{14}$$

where

$$\boldsymbol{\Omega}(\tilde{v}) = \left(\frac{a}{|\tilde{v}|^5}\right)^{1/2}.$$

It is worth noting that if $\mathbf{E} \equiv 0$ and $m_e/M_i = 0$ then (14) describes the rotation of the velocity vector (in the frame of fixed ions), and conserves the energy of a particle, as expected for the case of collisions described by (12) when $m_e/M_i \to 0$. The higher terms in m_e/M_i can be obtained in a similar way. We hope that, owing to the simplicity of (14), it can be used not only for the numerical simulation but also for analytical work.

2.4. General approach

As mentioned above, the basic problems arise in the application of the stochastic approach to the description of collisions between particles of the same species. In general, we propose two relatively distinct ways to resolve this problem.

The first approach consists in the construction and application of an appropriate model of the collision integral, which should be adequate for the physical process under consideration. It is obvious that the model to be constructed must satisfy the conservation laws (for energy and momentum). It is also natural to require that a 'good' model coincide with the Landau collision integral in asymptotic cases. The advantage of this approach is its high efficiency for simulation (about N operations per cell), while the possible cost for the efficiency is the approximate calculation of the coefficients A_i and B_{ik}.

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The second way to overcome the difficulties mentioned above is formally more general but actually more complicated. It requires an effective technique for the fast and accurate numerical calculation of A_i and B_{ik}. In this approach, the kinetic equation should be partially linearized on every time step $[t^n,\ t^{n+1}]$, and the coefficients A_i and B_{ik} should be calculated at $t = t^n$ using $f^n \equiv f(t = t^n)$. This method requires the numerical reconstruction of the distribution function, and therefore it would be rather time-consuming. (about N^2 operations per computational cell, where N is the number of particles in a cell). The advantage of this approach is the possibility of calculating the coefficients A_i and B_{ik} with a desired accuracy, but the drawback is a relatively high computational cost. In the rest of this section, we discuss how this approach can be realized.

Let us consider how the construction of a 'good' model of the collision integral can be performed. The problem consists in finding an approximate expression for the distribution function f, and further calculation of the coefficients in the Fokker–Planck equation. If some information about the dependence of $f(\mathbf{v})$ on the velocity is known a priori then one can write an approximate expression for the distribution function:

$$f(\mathbf{v}) \approx \tilde{f}(\mathbf{v}, \mathbf{a}) = \sum_\gamma \phi_\gamma(\mathbf{v}, a_1, a_2, ...),$$

where ϕ_γ are predetermined functions of $\mathbf{v}$ and the adjustable parameters $a_\gamma = a_\gamma(\mathbf{x}, t)$. The functions ϕ_γ should be chosen in such a way that the coefficients A_i^n and B_{ik}^n can be calculated analytically. In principle, the distribution function obtained from the simulation of the problem in the collisionless regime may be a good hint for finding the approximation $\tilde{f}$. To clarify these rather general speculations, let us construct the 'quasi-Maxwellian' (or multifluid) model. Namely, we represent ϕ in the form

$$\phi_\gamma(\mathbf{v}) = \frac{n_\gamma \exp(-\xi_\gamma^2)}{(V_{T_\gamma}\sqrt{\pi})^3},$$

where

$$V_{T_\gamma}(\mathbf{x}, t) = \left(\frac{2T_\gamma}{m_\gamma}\right)^{1/2}, \qquad \xi_\gamma = \frac{|\mathbf{v} - \mathbf{u}_\gamma(\mathbf{x}, t)|}{V_{T_\gamma}}, \qquad \tilde{\mathbf{v}}_\gamma = \mathbf{v} - \mathbf{u}_\gamma.$$

Substituting this function into the expressions (4), we obtain

$$H = \sum_\gamma H_\gamma(\xi_\gamma), \qquad G = \sum_\gamma G_\gamma(\xi_\gamma),$$

where

$$H_\gamma = \frac{n_\gamma L^{\alpha\gamma}}{4\pi}\frac{\Phi_\gamma - \Psi_\gamma}{\xi_\gamma}, \qquad G_\gamma = 2\frac{n_\gamma L^{\alpha\gamma}}{4\pi}\frac{\Psi_\gamma}{\xi_\gamma}; \tag{15}$$

here $\Phi(\xi) = \int_0^\xi dx\, e^{-x^2}/\sqrt{\pi}$ is the error function, and $\Psi(\xi) = -\frac{1}{2}d(\Phi/\xi)/d\xi$ is the Chandrasekhar function. Substituting these expressions into the expressions (3), after some algebra, we calculate

$$A_i^\alpha = -\sum_\gamma \frac{m_\gamma + m}{m_\gamma V_{T_\gamma}^2} G_\gamma \tilde{v}_{\gamma i}, \tag{16}$$

$$B_{ik}^\alpha = \sum_\gamma \left[H_\gamma \delta_{ik} - (H_\gamma - G_\gamma)\frac{\tilde{v}_{\gamma i}\tilde{v}_{\gamma k}}{\tilde{v}_\gamma^2} \right]. \tag{17}$$

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Using the expressions (16) and (17), together with (7), one can calculate the terms F_i and D_{ik}, and obtain the corresponding Langevin equation. Nevertheless, this is a relatively expensive way from a computational point of view, because the calculation of $D = B^{1/2}$ requires a relatively large number of operations. It should be noted here that the change in the total kinetic energy due to collisions depends on the sum of the diagonal terms of the matrix B_{ik}:

$$\left(\frac{\partial E_{\rm kin}}{\partial t}\right)_{\rm coll} = \partial_t \sum_{k=1}^{3} \langle \tfrac{1}{2} m v_k^2 \rangle = \sum_{k=1}^{3} \int (A_k v_k + \tfrac{1}{2} B_{kk}) f \, d^3 v,$$

and the change in momentum due to collisions

$$\left(\frac{\partial \langle m v_k \rangle}{\partial t}\right)_{\rm coll} = \int m A_k f \, d^3 v$$

does not depend on the term B_{ik}. Thus the terms $\tilde{v}_{\gamma i}\tilde{v}_{\gamma k}/|\tilde{v}_\gamma|^2$ $(i \neq k)$ characterize the angular relaxation of kinetic energy in the plasma, and do not influence the evolution of the total kinetic energy due to collisions. To construct a simpler (but more efficient in practice) model, we suggest the following substitution in the expression (17):

$$\frac{\tilde{v}_{\gamma i}\tilde{v}_{\gamma k}}{|\tilde{v}_\gamma|^2} \to \frac{\tilde{v}_i \tilde{v}_k}{|\tilde{v}|^2},$$

where $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{u}$, and $\mathbf{u}$ is the hydrodynamic plasma velocity. Thus the 'diffusion' coefficient can be written in the following form:

$$B_{ik} = H\delta_{ik} - (H - G)\frac{\tilde{v}_i \tilde{v}_k}{\tilde{v}^2}.$$

The latter allows us to obtain the expression for D_{ik} easily:

$$D_{ik} = H^{1/2}\delta_{ik} - (H^{1/2} - G^{1/2})\frac{\tilde{v}_i \tilde{v}_k}{\tilde{v}^2}. \tag{18}$$

After some calculation, we obtain the following expression:

$$\Delta F_i \equiv D_{jk}\frac{\partial D_{ik}}{\partial v_j} = \frac{1}{2}\left\{\frac{\partial H}{\partial v_i} + \frac{\tilde{v}_i \tilde{v}_j}{\tilde{v}^2}\frac{\partial (G - H)}{\partial v_j}\right\} - 2[H - (HG)^{1/2}]\frac{\tilde{v}_i}{\tilde{v}^2}.$$

Using (15), the expression in braces can be rewritten in the form

$$\left\{ \dots \right\} = \sum_\gamma \frac{1}{\tilde{v}_\gamma}\left[H'_\gamma \tilde{\mathbf{v}}_\gamma + (G'_\gamma - H'_\gamma)\frac{\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}_\gamma}{\tilde{v}^2}\,\tilde{\mathbf{v}}\right],$$

where

$$H'_\gamma = \frac{n_\gamma L^{\alpha\gamma}}{4\pi}\,\frac{3\Psi_\gamma - \Phi_\gamma}{V_{T_\gamma}\,\xi_\gamma^2},$$

$$G'_\gamma = 2\frac{n_\gamma L^{\alpha\gamma}}{4\pi}\,\frac{\Phi_\gamma - (2\xi_\gamma^2 + 3)\Psi_\gamma}{V_{T_\gamma}\,\xi_\gamma^2}.$$

Thus we have

$$\Delta\mathbf{F} = [(HG)^{1/2} - H]\frac{\tilde{\mathbf{v}}}{\tilde{v}^2} + \frac{1}{2}\sum_\gamma \frac{1}{\tilde{v}_\gamma}\left[G'_\gamma \mathbf{v}_\gamma + (G'_\gamma - H'_\gamma)\frac{\tilde{\mathbf{v}} \times (\tilde{\mathbf{v}} \times \tilde{\mathbf{v}}_\gamma)}{\tilde{v}^2}\right]. \tag{19}$$

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Substituting the expressions (16) and (19) into the equation (7), we obtain the 'friction' force:

$$\mathbf{F}_{\rm fr} = -\sum_{\gamma} \frac{m_\gamma + m}{m_\gamma V_{T_\gamma}^2} G_\gamma \mathbf{v}_\gamma + \chi \Delta \mathbf{F}. \tag{20}$$

Here, as mentioned above, the parameter χ depends on the definition of the stochastic integral. In this way, the Langevin equation for particle motion can be written in the form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}_L + \mathbf{F}_{\rm fr} + G^{1/2}\,\boldsymbol{\xi} - (H^{1/2} - G^{1/2}) \frac{\tilde{\mathbf{v}} \times (\tilde{\mathbf{v}} \times \boldsymbol{\xi})}{\tilde{v}^2}. \tag{21}$$

This equation is probably satisfactory for many problems of interest. Indeed, it becomes 'exact' (i.e. equivalent stochastically to the description of collisions by the Landau operator) in the following cases:

(i) if $\xi \to \infty$ (e.g. if the temperature is high enough); then the 'collisional' force also tends to zero, and we obtain the collisionless limit;

(ii) in the hydrodynamic limit, when the deviation of the distribution function from local Maxwellian is weak;

(iii) in the case of interaction with a background plasma, which can be regarded as a fluid;

(iv) for $\xi \gg 1$, the terms $\mathbf{F}_{\rm fr}$ and G are small in comparison with the last term in (21), and the 'collisional' force describes rotation in velocity space, as expected for the case of collisions described by the Landau collision operator in this limit; analogously, this equation is applicable for the description of electron–ion collisions, because $m_e/M_i \ll 1$.

Thus the derived equation has an 'exact' asymptotic, and in general it can be regarded as an approximate wide-range model. For the cases when this model is strongly unsatisfactory, the construction of more adequate models can be performed using the technique described above.

Finally, let us discuss how to calculate the coefficients in the Fokker–Planck equation numerically in the general case. In PIC codes, it is impractical (in terms of number of particles, computational time and statistical fluctuations) to actually compute the expressions (4) as multiple integrals and then perform numerical differentiation. It is well known that in PIC methods, the distribution function is expressed in terms of disordered points – 'macroparticles'. For the numerical calculation of A_i and B_{ik} we propose to use the integral interpolant of the distribution function:

$$f_\beta(\mathbf{v}) \approx \sum_{k}^{N} W(\mathbf{v} - \mathbf{v}_k, h),$$

where the summation index k denotes a particle label, and the summation is over all particles. Here W is an interpolating kernel, with

$$W(\mathbf{v} - \mathbf{v}', h) \to \delta(\mathbf{v} - \mathbf{v}') \quad \text{as} \quad h \to 0,$$

$$\int W(\mathbf{v} - \mathbf{v}', h)\, d\mathbf{v}' = \frac{1}{N}.$$

In addition, we should choose the kernel in such a way that the integrals (4) can

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be calculated analytically. This allows us to calculate the expressions (3) for every particle, and to perform the summation over all particles to obtain the coefficients A_i and B_{ik} in a simple way. For example, a Gaussian function may be chosen as a kernel, and the final equation will have a form analogous to (21). It should be noted here that a somewhat similar approach is applied in smoothed particle hydrodynamics (SPH), which has been developed within the last two decades (see e.g. Monaghan 1992).

3. Simulation of laser–plasma interaction

With the use of the approach developed above, let us consider the problem of the interaction of short laser pulses with an overdense plasma, which has attracted increasing interest in recent years (Yang et al. 1995). The expansion of the heated target can be neglected for sufficiently short laser pulses, and the ions can be regarded as immobile with a step-like density profile. The problem under consideration is then formulated as follows. The immobile ions of the plasma fill the half-space $x \geqslant X_i$, i.e. the density of the ion component is equal to n for $x \geqslant X_i$ and is zero for $x < X_i$. The laser light propagates along the x axis and has main frequency ω_0, where $\omega_0 < \omega_{pe}$. The simulation parameters are chosen close to those described in Yang et al. (1995).

We first consider the absorption of laser light of low intensity (i.e. the linear regime), which was simulated in the collisionless limit by Yang et al. (1995). They studied the influence of the sheath on absorption, and showed that the value of the absorption coefficient from the simulation using a the realistic reflection condition (when the particles are reflected by the longitudinal field of the double layer at the plasma boundary) can be greater than the prediction of linear theory by up to 50%. This discrepancy was explained (Yang et al. 1995) by the fact that the linear theory assumes instant reflection of electrons at the laser–plasma interface (and the fact that the unperturbed distribution of the electron density coincides with that of the ions), while the finite transit time in the sheath region may play an important role. So, let us consider the absorption of laser light for a wide range of plasma parameters including both the collisional and collisionless regimes. For the step-like density profile and the condition of instant reflection of electrons, the theoretical value of the absorption coefficient η_{ab} can be calculated as follows:

$$\eta_{ab} = 1 - \left| \frac{ik_0 E_+ - E'_+}{ik_0 E_+ + E'_+} \right|^2 . \tag{22}$$

Here E_+ and $E'_+ = dE/dx$ are the values of the electric field and its derivative at the plasma boundary (at $x = X_i + 0$), $E_y \propto e^{i(k_0 x - \omega_0 t)}$ is the electric field of the incident wave in vacuum, ($k_0 = \omega_0/c$). The values E_+ and E'_+ are related by the expression (Lifshitz and Pitaevskii 1981)

$$\frac{E(+0)}{E'(+0)} = -\frac{2}{\pi} \int_0^\infty \frac{dq}{q^2 - \epsilon^t (\omega_0, qV_{Te}/c)}. \tag{23}$$

Following the usual perturbation technique (Lifshitz and Pitaevskii 1981), the dielectric permittivity is calculated by linearizing the kinetic equation with the col-

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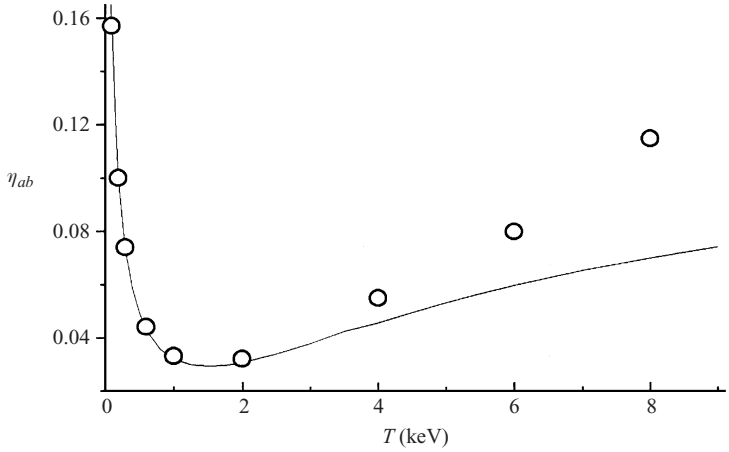

Figure 1. Temperature dependence of the absorption coefficient η_{ab} ($Z = 3$, $n_e = 25n_{\rm cr}$, $\lambda_0 = 0.35\,\mu$m): solid curves, linear theory (instant reflection); circles, simulation results (sheath reflection).

lision integral (12) around perturbations of the form $e^{i(kx-\omega_0 t)}$:

$$\epsilon^t(\omega_0,k) = 1 - \frac{\omega_{pe}^2}{\omega_0}\int_{-\infty}^{\infty}\frac{\exp(-v^2/2V_{Te}^2)}{\omega_0 - kv_x + ia/v^3}\left(\frac{v_y}{V_{Te}}\right)^2\frac{d^3v}{(V_{Te}\sqrt{2\pi})^3}$$
$$= 1 - \left(\frac{\omega_{pe}}{\omega_0}\right)^2\pi\int_0^{\infty}\left[2F - (1-F^2)\ln\left(\frac{F-1}{F+1}\right)\right]\frac{x^3\exp(-\frac{1}{2}x^2)}{kV_{Te}/\omega_0}\frac{dx}{(\sqrt{2\pi})^3}, \tag{24}$$

where

$$F = \frac{x^3 + ia/\omega_0}{xkV_{Te}/\omega_0}.$$

The values of the dielectric permittivity ϵ^t and the absorption coefficient η_{ab} can be obtained analytically only in a number of limits (Lifshitz and Pitaevskii 1981), such as those of the normal low- and high-frequency skin effects, the anomalous skin effect, and so-called 'sheath inverse bremsstrahlung'. For arbitrary values of the plasma parameters, we calculated the values of ϵ^t and η_{ab} numerically (the dependence of the Coulomb logarithm on temperature was also taken into account).

The dependence of the absorption coefficient on temperature is shown in Fig. 1. The laser-light wavelength $\lambda = 0.35\,\mu$m, the ion charge $Z = 3$ and the electron concentration $n_e = 25n_{\rm cr}$, where $n_{\rm cr}$ is the critical density for the incident wave. At relatively low temperature (the decreasing part of the curve in Fig. 1) the collisional regime of absorption is dominant while the collisionless mechanism plays the basic role at high temperature (the increasing part of the curve). The good agreement of the simulation results (circles) and the theoretical values of η_{ab} at low temperatures confirms that the peculiar features of electron reflections at the laser–plasma interface are not significant when the spatial dispersion is small enough. As can be seen in Fig. 1, with increasing plasma temperature, the absorption coefficient from the simulation using realistic reflection becomes greater than the prediction of linear theory. This result is in a good agreement with the data from Yang et al. (1995) when the temperature is high enough. The result of simulation of the absorp-

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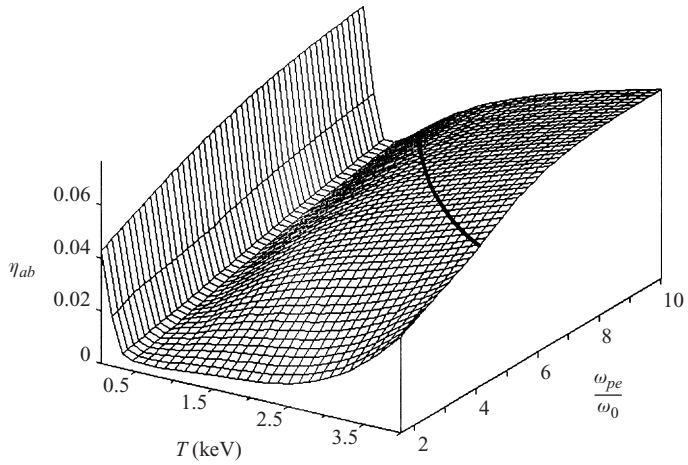

Figure 2. Absorption coefficient η_{ab} as a function of plasma parameters. Here T is the plasma temperature, and ω_{pe} and ω_0 are the plasma frequency and the frequency of the laser light.

tion coefficient for a wide range of plasma parameters (temperature and plasma frequency) is shown in Fig. 2. The bold curve separates the region of plasma parameters (large temperatures and densities) where the spatial dispersion is large enough and the value of η_{ab} is greater than the prediction of linear theory. Thus the technique developed here allows one to calculate correctly the absorption coefficient not only in the collisional and collisionless limits but also for arbitrary values of the plasma parameters.

Let us now consider the most interesting effects observed during the computer simulation of the strongly nonlinear regime of laser–plasma interaction (Cadjan and Ivanov 1997). The intensity of external radiation is $10^{16}\,\mathrm{W\,cm^{-2}}$ and the initial temperature of the plasma is $T = 80\,\mathrm{eV}$. The initial frequency of electron–ion collisions is $\nu_{ei}/\omega_{pe} = 0.1$ and the plasma density is $n = 4n_{\mathrm{cr}}$. It should be noted that in this case the unperturbed plasma frequency $\omega_{pe} = 2\omega_0$ is equal to the frequency of oscillation of the ponderomotive force. The setup for the simulation is similar to that considered above.

It turns out that, with similar parameters, the longitudinal electric field of the double layer at the laser–plasma interface can play a very important role. At $x < X_i$, the quasistatic electric field at the boundary can be roughly estimated as follows $E_x \approx ne\lambda_d$ ($eE_x/m_e\omega_0 c \approx \omega_{pe}V_{Te}/\omega_0 c$), where λ_d is the Debye length. During the energy absorption and plasma heating, this field grows, and its magnitude can be comparable to or greater than that of the electric field of the incident wave. At $x > X_i$, the ponderomotive force pushes plasma electrons from the interface into the exterior of the plasma, and the maximum electron density occurs at $x > X_i$. The latter causes additional changes in the structure of the field of the double layer. The high-frequency component of the force ($\propto \mathbf{j} \times \mathbf{B}$) acting upon the plasma boundary leads to the creation of an oscillating component in the longitudinal electric field. Thus a strong electric field E_x with steady-state and oscillating components can be formed at the laser–plasma interface.

In Fig. 3 the electron density profile and the distribution of the longitudinal electric field E_x are shown. One can see that the maximum value of E_x (the sum of the steady-state and the oscillating components) exceeds the amplitude of electric

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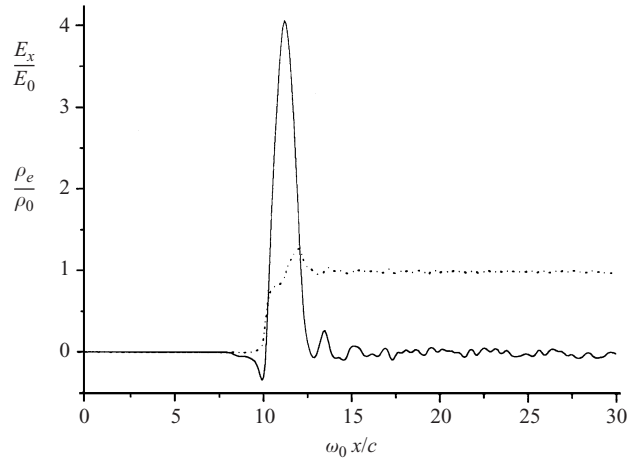

Figure 3. Profiles of the electron concentration ρ_e (dotted curve) and the longitudinal electric field E_x (solid curve). The parameters are $n_e = 4n_{\rm cr}$, $\nu_{ei}/\omega_{pe} = 0.1$, and $t = 325\omega_0^{-1}$. ρ_0 is the initial concentration of electrons and E_0 is the amplitude of the electric field of the incident wave.

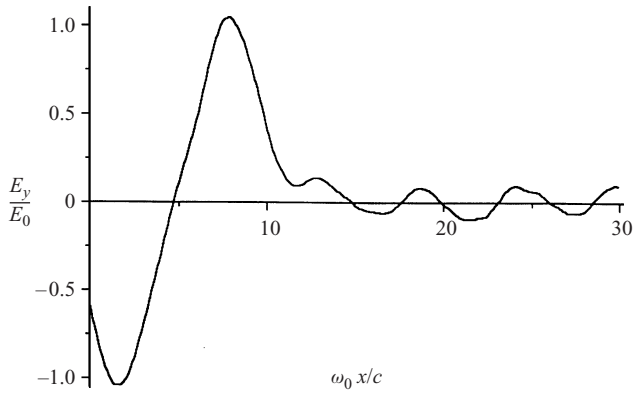

Figure 4. Spatial distribution of the transverse electric field E_y for $n_e = 4n_{\rm cr}$, $\nu_{ei}/\omega_{pe} = 0.1$ and $t = 325\omega_0^{-1}$.

field of the incident wave several times. The strong nonlinearity of the processes occurring in the region of the double layer leads to the generation of high-frequency radiation. It is clear in Fig. 4 that this radiation propagates deep into the plasma. One more peculiarity of the laser–plasma interaction observed in the simulation is the formation of high-energy electrons. In the region of the laser–plasma interface, the electrons that are accelerated in the non-uniform oscillating field E_x are pushed deep into the plasma, where the amplitude of E_x is weaker. In addition, the high-frequency transverse electromagnetic field propagating into the plasma can cause the formation of a longitudinal plasma wave due to the development of decay instability. This plasma wave can lead to additional acceleration of electrons. As a result, a non-Maxwellian distribution function can be formed even far from the laser–plasma interface. From a formal point of view, the collisions must lead to a decrease in the heated-electron fraction, but, taking into account that the effective force (14) acting on an electron is a decreasing function of particle velocity, one can

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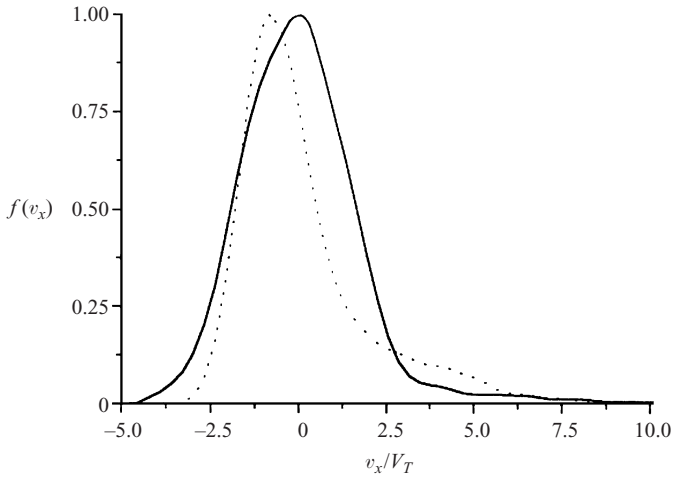

Figure 5. The electron distribution functions $f(v_x)$ for $20c/\omega_0 \geqslant x \geqslant 30c/\omega_0$: dotted curve, $\nu_{ei}/\omega_{pe} = 0$; solid curve, $\nu_{ei}/\omega_{pe} = 0.1$. Here V_T is the initial thermal velocity, $n_e = 4n_{\rm cr}$ and $t = 325\omega_0^{-1}$.

assume that the influence of collisions on the heated electrons will also decrease with increasing particle velocity. So, we can say that collisions described by the expression (12) behave like a filter – they decrease the fraction of heated particles with relatively low energies and practically do not influence the more strongly heated electrons, i.e. the 'tail' of the distribution function is not really sensitive to electron–ion collisions. For the cases of zero and non-zero electron–ion collision frequency, the difference in the distribution functions $f(v_x)$ $(20c/\omega_0 \geqslant x \geqslant 30c/\omega_0)$ is demonstrated in Fig. 5.

Note that an increase in the ratio $n/n_{\rm cr}$ leads to a decrease in the amplitude of the oscillating longitudinal electric field, as well as to a decrease in the generation of higher harmonics at the plasma boundary and to a decrease in the formation of heated electrons. We conclude by mentioning that regimes of laser–plasma interaction similar to that discussed above require a kinetic description with account taken of both collective kinetic effects and collisional processes.

4. Conclusions

We have presented a stochastic method for the description of a collisional plasma. This Langevin approach can be regarded as an alternative to the description of the plasma in terms of a distribution function. It has been proposed to derive the governing equations for plasma-particle motion analytically using an accurate mathematical procedure rather than by introducing the collisions in a phenomenological way. These equations of particle motion are the usual dynamic equations of the collisionless case plus an additional term that describes the collisions. This term is found using the stochastic equivalence of the Fokker–Planck and Langevin approaches. For the case of electron–ion collisions, the governing equations have been derived directly. The more complex case of intraspecies collisions has been discussed, and a general approach has been presented. A 'quasi-Maxwellian' model has been considered, which coincides with the Landau integral of collision for different asymptotic

cases. Using this model, we have derived the corresponding stochastic differential equations, which allow one to describe both interspecies and intraspecies collisions in the same simple and effective manner.

The implementation of the presented method is straightforward and convenient owing to its simplicity. It has the advantages of collisionless PIC algorithms, such as easy application to vectorized and parallel computation. Numerical examples have been presented to show the validity and versatility of the method developed here. It has been shown that this approach and the resulting collisional particle-in-cell code are reliable tools for investigating plasma kinetics in the collisionless and hydrodynamic limits as well as in intermediate regimes. The problem of laser–plasma interaction has been considered, and the absorption coefficient calculated for a wide range of plasma parameters. The influence of electron–ion collisions on the evolution of heated electrons has been analysed.

The development of the stochastic method has been motivated by studies of many interesting and relatively unexplored processes involving collisions and collective effects in plasma physics. The method presented here is applicable to the simulation of these processes, and it is potentially useful for investigations in previously inaccessible regimes of a semicollisional plasma.

Appendix A. Test problem

Using the governing equations for particle motion presented in Sec. 2, we have created a collisonal PIC code for the simulation of plasma kinetics, taking account of Coulomb collisions.

As a first test problem, let us consider the conductivity of a Lorentz plasma in an external electric field $\delta E = E_0 e^{i\omega_0 t}$ of small amplitude (i.e. the linear regime). The theoretical value of the conductivity σ is easily calculated by linearizing the kinetics equation with the collision integral (12) around a perturbation of the form $\delta f \propto e^{i\omega_0 t}$:

$$\begin{aligned}\sigma &= \frac{\omega_{pe}^2}{4\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2/2V_{Te}^2)}{i\omega_0 + a/v^3}\left(\frac{v_x}{V_{Te}}\right)^2\frac{d^3v}{(V_{Te}\sqrt{2\pi})^3}\\ &= \frac{1}{4\pi}\frac{\omega_{pe}^2}{\omega_0}\frac{1}{3\sqrt{\frac{1}{2}\pi}}\int_0^{\infty}\frac{\exp(-\frac{1}{2}r^2)}{ir^3 + 3\sqrt{\frac{1}{2}\pi}\,\nu_{ei}/\omega_0}\,r^7 dr\,. \end{aligned} \tag{A 1}$$

Analytical expressions for the plasma conductivity can be found in the following limits:

$$\sigma = \left(\frac{\omega_{pe}}{\omega_0}\right)^2\frac{\nu_{ei} - i\omega_0}{4\pi} \qquad \text{for } \nu_{ei} \ll \omega_0,$$

$$\sigma = \left(\frac{\omega_{pe}}{\nu_{ei}}\right)^2\frac{32\nu_{ei}/3\pi - i70\omega_0/\pi}{4\pi} \qquad \text{for } \nu_{ei} \gg \omega_0.$$

For arbitrary values of ω_{pe} and ω_0 the integral (A 1) was calculated numerically. The dependences of $\sigma_r = \mathrm{Re}(4\pi\sigma\omega_0/\omega_{pe}^2)$ and $\sigma_i = \mathrm{Im}(4\pi\sigma\omega_0/\omega_{pe}^2)$ on the parameter ν_{ei}/ω_0 is shown in Fig. A.1. Good agreement of numerical and theoretical results was obtained (2000 'macroparticles' were used in the simulation). Note that the computational error decreases with growing number of 'macroparticles'.

As a second test, we present the solution of the temperature relaxation problem

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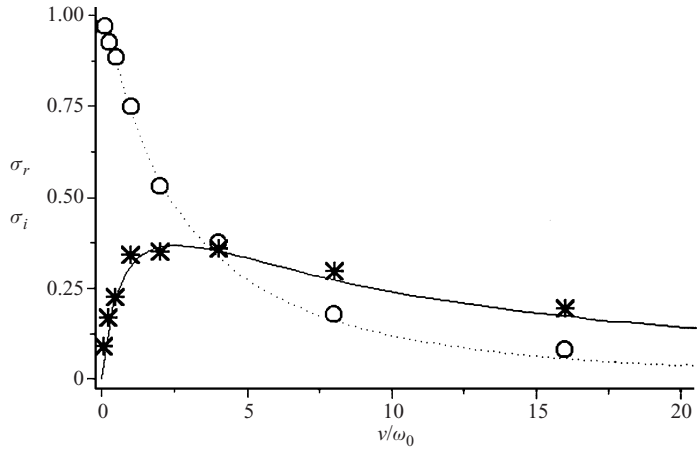

Figure A.1. Dependence of the plasma conductivity on the plasma parameters: solid curve/asterisks, $\sigma_r = \mathrm{Re}(4\pi\sigma\omega_0/\omega_{pe}^2)$; dotted curve/circles, $\sigma_i = \mathrm{Im}(4\pi\sigma\omega_0/\omega_{pe}^2)$ (curves, linear theory; asterisks/circles, simulation results). Here ν is the electron–ion collision frequency, and ω_{pe} and ω_0 are the plasma frequency and the frequency of external electric field.

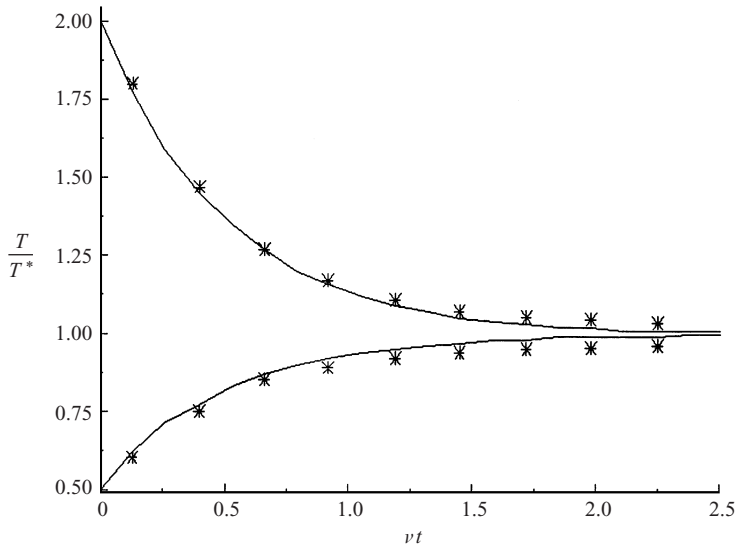

Figure A.2. Test problem showing relaxation of two plasma components with different initial temperatures $T_1 = \frac{1}{2}T_\infty$ and $T_2 = 2T_\infty$: solid curves, analytical solutions; asterisks, simulation results.

in a two-component plasma. We consider the evolution of a plasma consisting of two species α and β with Maxwellian distributions and initial temperatures $T_{\alpha 0}$ and $T_{\beta 0}$ respectively. For the sake of simplicity we assume that the masses of the two species are the same. This allows us to write easily an analytical solution for the temperature relaxation:

$$T_i(t) = T_\infty + (T_{i0} - T_\infty)\exp(-2\nu t),$$

where $i = \alpha, \beta$, $T_\infty = \frac{1}{2}(T_{\alpha 0} + T_{\beta 0})$ and ν is the collision frequency. The initial values

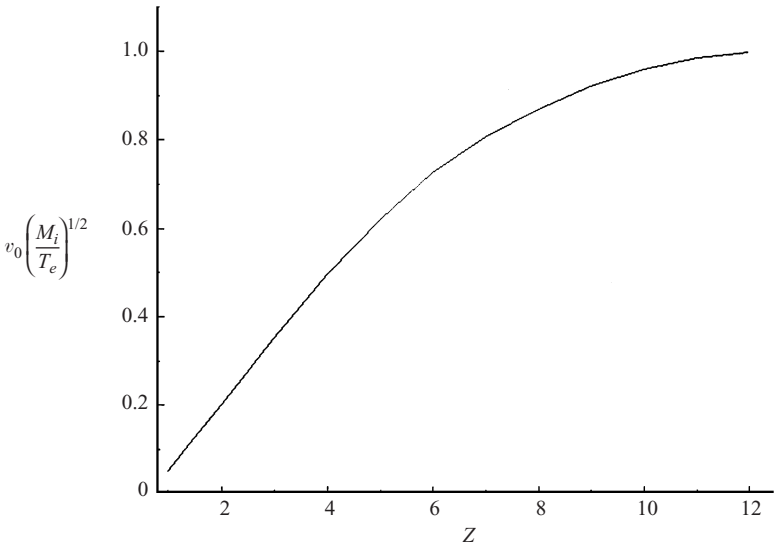

Figure A.3. Dependence of the averaged velocity of admixed ions as a function of their charge Z. Here $\nu_{ii}/\omega_{pi} = 0.01$ and $\omega_{pi}t = 160$.

are as follows: $T_{\alpha 0} = \frac{1}{2}T_\infty$ and $T_{\beta 0} = 2T_\infty$, and the number of 'macroparticles' in the simulation is 3000. Figure A.2 shows a comparison of the analytical solution and numerical results. The convergence of the method with growth in the number of particles is also observed.

Let us now consider the problem of acceleration of admixed ions during plasma expansion. The spectrum of the accelerated ions is important for plasma diagnostics. In the hydrodynamic limit, Mulser (1971) showed that the energy of admixed ions does not depend on their charge Z. In the collisionless limit, a self-similar solution for the plasma expansion was constructed by Gurevich et al. (1965). Using this solution, Gurevich et al. (1971) showed that the kinetic energy of admixed ions depends on their charge as Z^2/M (i.e. a linear growth of the average velocity with increasing Z should be observed). The results of a numerical simulations of this problem were presented by Anisimov et al. (1982). If the concentration of admixed ions is small enough, they do not significantly influence the electric field, and therefore the acceleration of admixed ions can be considered in an external field determined by the dynamics of the electrons and ions of the background plasma. In addition, it can be assumed that the electron density satisfies Boltzman's distribution, because the characteristic time of the development of the process is determined by the ion motion (Anisimov et al. 1982). Because the frequency of collisions between the admixed ions and the background-plasma components grows with increasing Z, one could expect to observe the results predicted by collisionless and hydrodynamic theory at small and large values of Z respectively. Let us consider the acceleration of aluminium ions during the expansion of a hydrogen plasma, and determine the dependence of the average velocity of admixed ions on Z. Initially, the isothermal plasma fills a half-space with a sharp boundary (the details of the problem setup and an analysis of the influence of Coulomb collisions on the acceleration of admixture ions were presented in Anisimov et al. 1982). The dynamics of the background plasma was simulated by the conventional collisionless

PIC method. The dynamics of admixed ions was described by the equations (21) derived in Sec. 2. The average velocity of aluminium ions as a function of their charge Z is shown in Fig. A.3. This figure, clearly shows the linear dependence at small values of Z, which was predicted by the collisionless theory. It can also be seen that for large Z the average velocity of admixed ions has a weak dependence on Z; that is this dependence becomes weaker with growth in Z. This corresponds to the results of the hydrodynamic theory. In this way, we have demonstrated the validity and versatility of the method developed here for the simulation of plasma kinetics in a wide range of parameters, including both the collisionless and hydrodynamic regimes.

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