Positive Integer Sequence Properties

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Problem

Let $C = 2025$. Consider a strictly increasing sequence of positive integers

$$a_0 < a_1 < \cdots < a_m.$$

Assume that for every $0 \le n \le m-2$,

$$a_{n+2} - a_n \le 2025.$$

Define the gaps $d_n = a_{n+1} - a_n$ for $0 \le n \le m-1$, and suppose that the gaps

$$d_0, d_1, \ldots, d_{m-1}$$

are pairwise distinct.

Determine the maximum possible value of $m+1$.

Solution 1 (Gap analysis)

Define $d_n = a_{n+1} - a_n$ for $0 \le n \le m-1$. Since the sequence (a_n) is strictly increasing, each d_n is a positive integer.

From the given condition,

$$a_{n+2} - a_n = (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n) = d_{n+1} + d_n \le 2025$$

for all $0 \le n \le m-2$.

Because $d_{n+1} \ge 1$, it follows that

$$d_n \le 2024 \quad \text{for all } n.$$

Hence all gaps belong to the set $\{1, 2, \ldots, 2024\}$.

Since the gaps are pairwise distinct, we must have

$$m \le 2024,$$

and therefore

$$m + 1 \le 2025.$$

We now show that this bound is attainable. Arrange the numbers $1, 2, \ldots, 2024$ in the order

$$2024,\ 1,\ 2023,\ 2,\ 2022,\ 3,\ \ldots$$

alternating between the largest unused and smallest unused value. For each adjacent pair in this sequence, the sum is at most 2025.

Let $d_0, \ldots, d_{2023}$ be this ordering, and define $a_0 = 1$ and

$$a_{n+1} = a_n + d_n.$$

Then (a_n) is strictly increasing, satisfies $a_{n+2} - a_n \leq 2025$, and has 2025 terms.

Thus the maximum possible value of $m + 1$ is 2025.

Solution 2 (Graph-theoretic interpretation)

Consider the graph whose vertices are the integers $1, 2, \ldots, 2024$, with an edge between distinct vertices x and y if and only if

$$x + y \leq 2025.$$

A sequence of pairwise distinct gaps $d_0, \ldots, d_{m-1}$ satisfying $d_n + d_{n+1} \leq 2025$ corresponds exactly to a simple path of length $m - 1$ in this graph. Hence m cannot exceed the number of vertices, which is 2024, so $m + 1 \leq 2025$.

The alternating ordering

$$2024,\ 1,\ 2023,\ 2,\ \ldots$$

defines a Hamiltonian path in this graph, since every adjacent pair sums to at most 2025. Therefore a path using all 2024 vertices exists, giving $m = 2024$ and $m + 1 = 2025$.

Answer

$$\boxed{2025}$$