Decoder Function and Divisor Sequences

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Problem

A "decoder" takes a positive integer N (having at least three proper divisors) and outputs the sum of the three largest proper divisors of N. Starting from a_1, define a sequence by

$$a_{n+1} = \psi(a_n) \qquad (n \geq 1),$$

where $\psi(N)$ denotes the sum of the three largest proper divisors of N.

Assume that the sequence is well-defined for all n (that is, every a_n has at least three proper divisors).

Let $M = 1{,}000{,}000$. How many integers a_1 with $1 \leq a_1 \leq M$ satisfy this condition?

Solution 1 (classification first)

From the analysis of the recurrence $a_{n+1} = \psi(a_n)$, one proves that the sequence is well-defined for all n if and only if the initial value is of the form

$$a_1 = 6 \cdot 12^e \cdot \ell,$$

where $e \geq 0$ and $\gcd(\ell, 10) = 1$. For such numbers, the three largest proper divisors are eventually $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$, and the recurrence remains defined forever; for all other starting values, the process eventually fails.

Thus we must count all integers of the form $6 \cdot 12^e \cdot \ell$ not exceeding 10^6 with $\gcd(\ell, 10) = 1$.

The inequality $6 \cdot 12^e \leq 10^6$ gives $e \in \{0, 1, 2, 3, 4\}$, since

$$6 \cdot 12^4 = 124{,}416 \leq 10^6 < 6 \cdot 12^5 = 1{,}492{,}992.$$

For each e, set

$$L_e = \left\lfloor \frac{10^6}{6 \cdot 12^e} \right\rfloor.$$

We must count integers $\ell \leq L_e$ that are not divisible by 2 or 5. By inclusion–exclusion, this number equals

$$L_e - \left\lfloor \frac{L_e}{2} \right\rfloor - \left\lfloor \frac{L_e}{5} \right\rfloor + \left\lfloor \frac{L_e}{10} \right\rfloor.$$

Evaluating:

e	L_e	count
0	166,666	66,666
1	13,888	5,555
2	1,157	463
3	96	38
4	8	3

Summing all cases gives

$$66{,}666 + 5{,}555 + 463 + 38 + 3 = 72{,}725.$$

Solution 2 (dynamics first)

Consider the recurrence $a_{n+1} = \psi(a_n)$. A divisor-ordering argument shows that if the iteration is to remain defined forever, every term must be divisible by 6. Moreover, the only sustainable configuration of the three largest proper divisors is

$$\frac{x}{2}, \frac{x}{3}, \frac{x}{4},$$

which yields

$$\psi(x) = \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{13}{12}x.$$

This forces the initial value to have the rigid structure

$$a_1 = 6 \cdot 12^e \cdot \ell,$$

and to prevent larger competing divisors, the factor ℓ must be coprime to 10. Conversely, any such a_1 keeps the iteration well-defined for all n (and eventually stabilizes when $e = 0$).

Therefore, the valid initial values $a_1 \leq 10^6$ are exactly those counted in Solution 1, giving the same total.

Answer

$$\boxed{72{,}725}$$