P08 Distinct Distance Route V1

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Problem

Let $N = 75{,}001$. There are $2N - 1$ points placed on a line at integer positions

$$1, 2, \ldots, 2N - 1.$$

A route is any permutation $(t_1, t_2, \ldots, t_{2N-1})$ of these points.
   Assume that the consecutive distances

$$|t_1 - t_2|, \ |t_2 - t_3|, \ \ldots, \ |t_{2N-2} - t_{2N-1}|$$

are all distinct.
   Let M be the minimum possible value of $\max(t_1, t_{2N-1})$ over all such routes.
   Determine M.

Solution

1. The distance multiset is forced

There are $2N - 2$ consecutive distances. Each $|t_i - t_{i+1}|$ is a positive integer and

$$|t_i - t_{i+1}| \le (2N - 1) - 1 = 2N - 2.$$

Since the $2N - 2$ distances are pairwise distinct, they must be exactly

$$\{1, 2, \ldots, 2N - 2\}.$$

Hence the total distance is forced:

$$
\begin{aligned}
\sum_{i=1}^{2N-2} |t_i - t_{i+1}| &= 1 + 2 + \cdots + (2N - 2) \\
&= \frac{(2N - 2)(2N - 1)}{2} \\
&= (N - 1)(2N - 1).
\end{aligned}
\tag{1}
$$

2. Potential inequality and an endpoint-sum lower bound

Define
$$s(x) = |x - N|.$$

For all integers a, b, the triangle inequality gives
$$
\begin{aligned}
|a - b| &= |(a - N) - (b - N)| \\
&\leq |a - N| + |b - N| \\
&= s(a) + s(b).
\end{aligned}
\tag{2}
$$

Summing (2) over all consecutive pairs:
$$
\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \leq \sum_{i=1}^{2N-2} \big(s(t_i) + s(t_{i+1})\big) = 2 \sum_{j=1}^{2N-1} s(t_j) - \big(s(t_1) + s(t_{2N-1})\big).
\tag{3}
$$

Because (t_j) is a permutation of $\{1, 2, \ldots, 2N - 1\}$, the multiset of scores is
$$\{0, 1, 1, 2, 2, \ldots, N - 1, N - 1\},$$

so
$$
\begin{aligned}
\sum_{j=1}^{2N-1} s(t_j) &= 2(1 + 2 + \cdots + (N - 1)) \\
&= 2 \cdot \frac{(N-1)N}{2} \\
&= N(N - 1).
\end{aligned}
\tag{4}
$$

Substituting (1) and (4) into (3) yields
$$(N - 1)(2N - 1) \leq 2N(N - 1) - \big(s(t_1) + s(t_{2N-1})\big),$$

hence
$$s(t_1) + s(t_{2N-1}) \leq N - 1.
\tag{5}$$

Since $N - x \leq |x - N| = s(x)$ for all x, we obtain
$$(N - t_1) + (N - t_{2N-1}) \leq N - 1 \quad \Rightarrow \quad t_1 + t_{2N-1} \geq N + 1.
\tag{6}$$

3. Parity constraint

Modulo 2, we have $|x - y| \equiv x - y \equiv x + y \pmod 2$. Therefore,
$$
\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \equiv \sum_{i=1}^{2N-2} (t_i + t_{i+1}) \equiv t_1 + t_{2N-1} \pmod 2,
$$

since every interior $t_2, \ldots, t_{2N-2}$ appears twice.

Using (1),
$$t_1 + t_{2N-1} \equiv (N - 1)(2N - 1) \pmod 2.
\tag{7}$$

Because $N = 75{,}001$ is odd, $N - 1$ is even, so $(N - 1)(2N - 1)$ is even and hence
$$t_1 + t_{2N-1} \equiv 0 \pmod 2.
\tag{8}$$

4. Impossibility below 37,502

Here $N + 1 = 75{,}002$. From (6),

$$t_1 + t_{2N-1} \geq 75{,}002. \tag{9}$$

If $\max(t_1, t_{2N-1}) \leq 37{,}501$, then $t_1 + t_{2N-1} \leq 75{,}002$. So equality holds and forces $t_1 = t_{2N-1} = 37{,}501$, impossible in a permutation. Therefore

$$\max(t_1, t_{2N-1}) \geq 37{,}502. \tag{10}$$

Answer

$$\boxed{37{,}502}$$