Line Route with All Distances Different

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Problem

Let (a_n) be a real sequence satisfying:

1. *Eventually periodic:* There exist integers $N_0 \geq 1$ and $T \geq 1$ such that

$$a_{n+T} = a_n \quad \text{for all } n \geq N_0.$$

2. *Recurrence:* For all integers $n \geq 7$,

$$a_{n+2} + a_{2n} = a_{n+1} + a_{2n+1}.$$

3. *Square-summable increments:*

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty.$$

Determine the *maximum possible* value of N, the number of distinct real values attained by the sequence (a_n).

Solution 1 (Forcing the tail to be constant)

Define the first differences

$$d_n := a_{n+1} - a_n \qquad (n \geq 1).$$

From the recurrence, for every $n \geq 7$,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

hence

$$d_{n+1} = d_{2n} \qquad (n \geq 7).$$

Equivalently, for every integer $x \geq 8$ (letting $x = n + 1$),

$$d_x = d_{2(x-1)}. \tag{1}$$

Let $h(x) = 2(x - 1)$. Iterating (**??**) gives, for all $k \geq 0$ and $x \geq 2$,

$$d_x = d_{h^k(x)}.$$

We claim that for all $k \geq 0$ and $x \geq 2$,

$$h^k(x) = 2^k(x-2) + 2. \tag{2}$$

This is proved by induction on k. For $k = 0$, $h^0(x) = x = 2^0(x-2) + 2$. Assume true for k. Then

$$h^{k+1}(x) = h(h^k(x)) = 2(h^k(x) - 1) = 2\big(2^k(x-2) + 2 - 1\big) = 2^{k+1}(x-2) + 2,$$

proving (**??**).

Fix any $x \geq 8$. Then by (**??**), the integers $h^k(x) = 2^k(x-2) + 2$ are strictly increasing in k, hence pairwise distinct. Since $d_{h^k(x)} = d_x$ for all $k \geq 0$, if $d_x \neq 0$ then

$$\sum_{n=1}^{\infty} d_n^2 \;\geq\; \sum_{k=0}^{\infty} d_{h^k(x)}^2 = \sum_{k=0}^{\infty} d_x^2 = \infty,$$

contradicting $\sum_{n=1}^{\infty} d_n^2 < \infty$. Therefore,

$$d_n = 0 \quad \text{for all } n \geq 8.$$

Hence

$$a_8 = a_9 = a_{10} = \cdots =: C$$

for some real constant C.

Thus all values of the sequence lie in the set $\{a_1, a_2, \ldots, a_7, C\}$, so the number of distinct values satisfies

$$N \leq 8.$$

Solution 2 (Independent construction and optimality)

We independently determine all sequences satisfying the hypotheses and count the number of distinct values they may attain.

Assume (a_n) satisfies the three given conditions. From the recurrence,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

and the square-summability of $(a_{n+1} - a_n)$, the same iteration argument as above forces

$$a_{n+1} = a_n \quad \text{for all } n \geq 8.$$

Hence there exists a real constant C such that $a_n = C$ for all $n \geq 8$.

Conversely, choose arbitrary real numbers $a_1, a_2, \ldots, a_7, C$ and define

$$a_n := C \quad \text{for all } n \geq 8.$$

Then (a_n) is eventually constant, hence eventually periodic. The increment sequence satisfies

$$(a_{n+1} - a_n)^2 = 0 \quad \text{for all } n \geq 8,$$

so the square-sum condition holds. The recurrence is easily checked directly: for all $n \geq 7$, all involved terms equal C.

Therefore, every solution has all its values in $\{a_1, a_2, \dots, a_7, C\}$, and every such 8-tuple produces a valid solution. Hence

$$N \le 8.$$

To show that 8 is attainable, take $a_1, a_2, \dots, a_7, C$ pairwise distinct, for example

$$a_1 = 14, \qquad a_2 = 12, \qquad a_3 = 10, \qquad a_4 = 8, \qquad a_5 = 6, \qquad a_6 = 4, \qquad a_7 = 2, \qquad C = 0,$$

and define $a_n = C$ for all $n \ge 8$. This sequence satisfies all conditions and attains exactly 8 distinct values.

Answer

The number of distinct values is always at most 8, and this bound is sharp. Hence the maximum possible value of N is

$$\boxed{8}.$$