

P10 Rotated Direction Count V2

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Problem

Two circles Ω, Γ with centers M, N intersect at A, B . The line MN meets Ω again at C and Γ again at D , with C, M, N, D in this order.

Let P be the circumcenter of $\triangle ACD$. The line AP meets Ω again at $E \neq A$ and Γ again at $F \neq A$. Let H be the orthocenter of $\triangle PMN$, and let \mathcal{C} be the circumcircle of $\triangle BEF$.

For each integer k in $1 \leq k \leq 89,999$, let ℓ_k be the line through H whose direction is obtained by rotating the *direction* of AP clockwise by an angle $\frac{k\pi}{90,000}$ (so ℓ_0 is the line through H parallel to AP).

Find the number of integers k for which ℓ_k is tangent to \mathcal{C} .

Solution 1 (Direct)

Step 1: The geometric crux (tangency of ℓ_0)

We record the following lemma, which is exactly the tangency conclusion of the underlying IMO 2025/2 configuration.

Lemma. *In the above configuration, the line through H parallel to AP is tangent to the circumcircle \mathcal{C} of $\triangle BEF$.*

Remark (what this lemma encapsulates). A full proof can be given by directed angles (or inversion) and proceeds via the standard rigidity chain: (i) establish transport parallels $CE \parallel AD$ and $DF \parallel AC$, (ii) define $A' = CE \cap DF$ and introduce the circumcenter T of $\triangle A'EF$ so that $TE = TF$, (iii) show $HT \parallel AP$ using the orthocenter relations in $\triangle PMN$ (equivalently $MH \parallel AD$ and $NH \parallel AC$), (iv) use $TE = TF$ together with angle/power criteria to deduce tangency of the line parallel to AP to (BEF) . This is precisely the “hard part” of the original problem.

Step 2: Reduce to a counting statement

By definition, ℓ_0 is the line through H parallel to AP , hence by the Lemma ℓ_0 is tangent to \mathcal{C} .

Now consider any $k \neq 0$. Then ℓ_k is obtained by rotating the direction of AP by a nonzero angle $\frac{k\pi}{90000}$, so ℓ_k is *not* parallel to AP . Therefore $\ell_k \neq \ell_0$ as lines through the same point H .

But a fixed circle \mathcal{C} has at most two tangent lines through a given point H (and in particular, among the prescribed family $\{\ell_k\}$, there cannot be two distinct lines with the *same* forced tangency direction). Since the Lemma pins down a specific tangent line through H , namely ℓ_0 , no other ℓ_k from this family can coincide with that tangent line.

Since $k = 0$ is not in the allowed range, none of the ℓ_k can coincide with ℓ_0 . Therefore no k yields tangency.

Solution 2 (Alternative)

Fix the point H and the circle \mathcal{C} . Consider an arbitrary line ℓ through H . The intersection points of ℓ with \mathcal{C} are the solutions of a quadratic equation (e.g. in an affine coordinate system), so ℓ is tangent to \mathcal{C} if and only if that quadratic has a double root, i.e. its discriminant is 0. Therefore, among all directions through H , at most two directions yield tangents to \mathcal{C} .

From the Lemma (the geometric crux of the construction), the specific direction parallel to AP yields a tangent, and the corresponding line is exactly ℓ_0 .

Since the 89,999 lines ℓ_k have pairwise distinct directions (their directions differ by multiples of $\pi/90,000$), none of them equals ℓ_0 because $k = 0$ is excluded. Consequently, within the set $\{\ell_k\}$, no k yields tangency.

Answer