P06 Rails on Triangular Nodes V2

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Problem

Let $n \geq 3$. On a triangular grid, there are nodes indexed by integer pairs (a, b) satisfying

$$a > 0, \qquad b > 0, \qquad a + b \leq n + 1.$$

A *rail* is any straight line in the plane. A rail is called *standard* if it is parallel to one of the three directions

$$x = \text{const}, \qquad y = \text{const}, \qquad x + y = \text{const}.$$

A rail that is not standard is called *express*.

Exactly n distinct rails are constructed so that every docking node lies on at least one rail. Let k be the number of express rails.

Tasks.

1. Determine all attainable values of k.
2. Let $\mathcal{K}$ be the set of all attainable values of k. Using your set $\mathcal{K}$, compute

$$N \;=\; 12000 \cdot |\mathcal{K}| \;+\; \sum_{k \in \mathcal{K}} k.$$

Solution 1 (Reduction Using the Long-Rail Lemma)

Assume the following lemma, whose proof can be given using the Alon–Füredi polynomial method:

Long-rail lemma. For every $n \geq 4$, any collection of n distinct rails covering all points of

$$T_n = \{(a, b) \in \mathbb{Z}^2 : \; a > 0, \; b > 0, \; a + b \leq n + 1\}$$

must contain at least one of the three boundary rails

$$x = 1, \qquad y = 1, \qquad x + y = n + 1.$$

Each of these boundary rails is standard. Removing one such rail and translating the remaining configuration reduces the problem from n to $n - 1$ while preserving the number of express rails. Hence the set $\mathcal{K}_n$ of attainable values of k is the same for all $n \geq 3$.

It therefore suffices to determine $\mathcal{K}$ in the base case $n = 3$.

For $n = 3$, the set T_3 consists of the six points

$$(1,1),(1,2),(1,3),(2,1),(2,2),(3,1).$$

One checks directly that:

- $k = 0$ is attainable using the three standard rails $x = 1$, $y = 1$, and $x + y = 4$;
- $k = 1$ is attainable using two standard rails and one express rail;
- $k = 2$ is impossible, since any line through two of $(2,1),(2,2),(3,1)$ is necessarily standard;
- $k = 3$ is attainable using three suitably chosen express rails.

Thus

$$\mathcal{K} = \{0,1,3\}.$$

By definition,

$$N = 12000 \cdot |\mathcal{K}| + \sum_{k \in \mathcal{K}} k = 12000 \cdot 3 + (0 + 1 + 3) = 36004.$$

Solution 2 (Direct Evaluation of the Final Expression)

From the complete analysis of the problem (using the reduction argument and the corrected base case), the set of attainable values of k is

$$\mathcal{K} = \{0,1,3\}.$$

Substituting this into the definition of N, we obtain

$$N = 12000 \cdot |\mathcal{K}| + \sum_{k \in \mathcal{K}} k = 12000 \cdot 3 + (0 + 1 + 3) = 36004.$$

Answer

$$\boxed{36004}$$