Line Route with All Distances Different

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Problem

Let $\mathcal{M}$ be the set of all monomials in the commuting variables x_2, x_3, x_5, x_7 with nonnegative integer exponents (including the monomial 1). A function $F : \mathcal{M} \to \mathbb{Z}_{>0}$ satisfies the following property:

For all $A, B \in \mathcal{M}$,

$$F(AB)^2 = F(A^2)\, F(F(B))\, F(A \cdot F(B))$$

holds if and only if A and B have no common variable.

Assume in addition that

$$F(x_2) = 4.$$

Let

$$P = x_2 x_3 x_5 x_7.$$

Compute

$$N = \sum_{D \mid P} F(D),$$

where the sum runs over all monomial divisors D of P.

Solution 1 (Direct approach)

The given identity is invariant under replacing a monomial by its square, since taking $B = 1$ (which is coprime to every monomial) yields

$$F(A)^2 = F(A^2)\, F(1)\, F(A),$$

and hence $F(A) = F(A^2)$ for all A. Thus F depends only on the set of variables dividing the monomial.

Moreover, the identity holds exactly when A and B share no variables. This forces F to be multiplicative on monomials with disjoint variable sets and to fail multiplicativity whenever variables overlap. Consequently, there exists an integer $k \geq 1$ such that

$$F(M) = \mathrm{rad}(M)^k,$$

where $\mathrm{rad}(M)$ is the product of the distinct variables dividing M.

The normalization $F(x_2) = 4$ gives

$$2^k = 4,$$

so $k = 2$. Hence

$$F(M) = \operatorname{rad}(M)^2 \quad \text{for all } M \in \mathcal{M}.$$

Now $P = x_2x_3x_5x_7$ is squarefree, and its monomial divisors correspond to all subsets of $\{x_2, x_3, x_5, x_7\}$. Therefore,

$$\sum_{D|P} F(D) = \sum_{S\subseteq\{2,3,5,7\}} \left(\prod_{p\in S} p\right)^2 = \prod_{p\in\{2,3,5,7\}} (1+p^2).$$

Computing,

$$(1+2^2)(1+3^2)(1+5^2)(1+7^2) = 5 \cdot 10 \cdot 26 \cdot 50 = 65000.$$

Solution 2 (Subset interpretation)

Each monomial corresponds uniquely to the subset of variables dividing it. The condition that A and B share no variable becomes disjointness of subsets. The functional equation then acts as a detector of disjointness.

As in the direct solution, substituting $B = 1$ forces $F(M) = F(M^2)$, so F depends only on the associated subset. Disjointness multiplicativity implies that for some constants $c_p > 1$,

$$F(M) = \prod_{p\in\sigma(M)} c_p,$$

where $\sigma(M)$ is the set of variables dividing M. The "only if" direction of the identity forces all c_p to be equal powers $c_p = p^k$ with a common exponent k. The condition $F(x_2) = 4$ yields $k = 2$, so again $F(M) = \operatorname{rad}(M)^2$.

Thus

$$\sum_{D|P} F(D) = \prod_{p\in\{2,3,5,7\}} (1+p^2) = 65000.$$

Answer

$$\boxed{65000}$$