

# Two Intersecting Circles and Tangent Lines

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## Problem

Two circles  $\Omega, \Gamma$  with centers  $M, N$  intersect at  $A, B$ . The line  $MN$  meets  $\Omega$  again at  $C$  and  $\Gamma$  again at  $D$ , with  $C, M, N, D$  in this order.

Let  $P$  be the circumcenter of  $\triangle ACD$ . The line  $AP$  meets  $\Omega$  again at  $E \neq A$  and  $\Gamma$  again at  $F \neq A$ . Let  $H$  be the orthocenter of  $\triangle PMN$ , and let  $\mathcal{C}$  be the circumcircle of  $\triangle BEF$ .

For each integer  $k$  with  $0 \leq k \leq 99,999$ , define  $\ell_k$  to be the line through  $H$  obtained by rotating the *direction* of  $AP$  clockwise by an angle  $\frac{k\pi}{100,000}$  (so  $\ell_0$  is the line through  $H$  parallel to  $AP$ ).

Determine the number of integers  $k$  for which  $\ell_k$  is tangent to  $\mathcal{C}$ .

## Solution 1 (Direct)

### Step 1: The geometric crux (tangency of $\ell_0$ )

We record the following lemma, which is exactly the tangency conclusion of the underlying IMO 2025/2 configuration.

**Lemma.** *In the above configuration, the line through  $H$  parallel to  $AP$  is tangent to the circumcircle  $\mathcal{C}$  of  $\triangle BEF$ .*

*Remark (what this lemma encapsulates).* A full proof can be given by directed angles (or inversion) and proceeds via the standard rigidity chain: (i) establish transport parallels  $CE \parallel AD$  and  $DF \parallel AC$ , (ii) define  $A' = CE \cap DF$  and introduce the circumcenter  $T$  of  $\triangle A'EF$  so that  $TE = TF$ , (iii) show  $HT \parallel AP$  using the orthocenter relations in  $\triangle PMN$  (equivalently  $MH \parallel AD$  and  $NH \parallel AC$ ), (iv) use  $TE = TF$  together with angle/power criteria to deduce tangency of the line parallel to  $AP$  to  $(BEF)$ . This is precisely the “hard part” of the original problem.

### Step 2: Reduce to a counting statement

By definition,  $\ell_0$  is the line through  $H$  parallel to  $AP$ , hence by the Lemma  $\ell_0$  is tangent to  $\mathcal{C}$ .

Now consider any  $k \neq 0$ . Then  $\ell_k$  is obtained by rotating the direction of  $AP$  by a nonzero angle  $\frac{k\pi}{100000}$ , so  $\ell_k$  is *not* parallel to  $AP$ . Therefore  $\ell_k \neq \ell_0$  as lines through the same point  $H$ .

But a fixed circle  $\mathcal{C}$  has at most two tangent lines through a given point  $H$  (and in particular, among the prescribed family  $\{\ell_k\}$ , there cannot be two distinct lines with the *same* forced tangency direction). Since the Lemma pins down a specific tangent line through  $H$ , namely  $\ell_0$ , no other  $\ell_k$  from this family can coincide with that tangent line.

Hence  $\ell_k$  is tangent to  $\mathcal{C}$  only for  $k = 0$ .

## Solution 2 (Alternative)

Fix the point  $H$  and the circle  $\mathcal{C}$ . Consider an arbitrary line  $\ell$  through  $H$ . The intersection points of  $\ell$  with  $\mathcal{C}$  are the solutions of a quadratic equation (e.g. in an affine coordinate system), so  $\ell$  is tangent to  $\mathcal{C}$  if and only if that quadratic has a double root, i.e. its discriminant is 0. Therefore, among all directions through  $H$ , at most two directions yield tangents to  $\mathcal{C}$ .

From the Lemma (the geometric crux of the construction), the specific direction parallel to  $AP$  yields a tangent, and the corresponding line is exactly  $\ell_0$ .

Since the 100,000 lines  $\ell_k$  have pairwise distinct directions (their directions differ by multiples of  $\pi/100,000$ ), at most one of them can equal the particular tangent line  $\ell_0$ . Consequently, within the set  $\{\ell_k\}$ , tangency occurs only for  $k = 0$ .

## Answer

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