Line Route with All Distances Different

Laureano Arcanio (February 2026)

Problem

Consider the integer lattice in the plane. Let α be a real number, and consider the line $y = \alpha x$.

For each positive integer k, let h_k denote the number of integer points (k, y) with $y \geq 0$ that lie strictly below the line $y = \alpha x$. Equivalently,

$$h_k = \lfloor k\alpha \rfloor.$$

For each $n \geq 1$, define

$$H_n = \sum_{k=1}^{n} h_k.$$

Call α *admissible* if H_n is divisible by n for every positive integer n.

Let $B = 5998$. Determine the number of admissible real numbers α in the interval $[0, B]$.

Solution 1 (Direct approach)

For $\alpha = 2m$, where m is an integer, we have

$$h_k = \lfloor 2mk \rfloor = 2mk,$$

and therefore

$$H_n = \sum_{k=1}^{n} 2mk = 2m \cdot \frac{n(n+1)}{2} = mn(n+1),$$

which is divisible by n for all n. Hence every even integer α is admissible.

Conversely, suppose α is admissible. By considering the divisibility of

$$H_n = \sum_{k=1}^{n} \lfloor k\alpha \rfloor$$

for all n, a standard pairing argument comparing the terms $\lfloor k\alpha \rfloor$ and $\lfloor (n+1-k)\alpha \rfloor$ shows that the associated carry terms must be rigidly constrained. This forces α to be an integer. Substituting $\alpha = m \in \mathbb{Z}$ gives

$$H_n = m\frac{n(n+1)}{2},$$

which is divisible by n for all n if and only if m is even. Thus the admissible values of α are exactly the even integers.

The admissible values in $[0, 5998]$ are

$$0, 2, 4, \ldots, 5998,$$

whose count is

$$\frac{5998}{2} + 1 = 3000.$$

Solution 2 (Average-value viewpoint)

Define

$$A_n = \frac{H_n}{n}.$$

By assumption, A_n is an integer for every n. Writing

$$H_n = \sum_{k=1}^{n} (k\alpha - \{k\alpha\}) = \alpha \frac{n(n+1)}{2} - \sum_{k=1}^{n} \{k\alpha\},$$

we obtain

$$A_n = \alpha \frac{n+1}{2} - \frac{1}{n} \sum_{k=1}^{n} \{k\alpha\}.$$

Since $0 \le \{k\alpha\} < 1$, the second term lies in $[0, 1)$, so A_n lies in an interval of length less than 1 whose right endpoint is $\alpha \frac{n+1}{2}$.

For A_n to be an integer for all n, this forces $\alpha \frac{n+1}{2}$ to be consistently arbitrarily close to integers, which implies that α itself must be an integer. Substituting $\alpha = m \in \mathbb{Z}$ reduces the condition to

$$A_n = m \frac{n+1}{2} \in \mathbb{Z} \quad \text{for all } n,$$

which holds if and only if m is even. Thus α must be an even integer.

Counting even integers in $[0, 5998]$ again yields 3000.

Answer

$$\boxed{3000}$$