

# Disjointness via Functional Equation

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## Problem

A function  $F : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$  satisfies

$$F(mn)^2 = F(m^2) F(F(n)) F(mF(n)) \quad \text{if and only if} \quad \gcd(m, n) = 1$$

for all positive integers  $m, n$ .

A pair  $(m, n)$  of positive integers is called *clean* if the above equality holds.

Let  $K = 1000$ . Determine the number  $N$  of ordered clean pairs  $(m, n)$  with

$$1 \leq m, n \leq K.$$

## Solution 1 (Direct counting via Euler's totient function)

By the definition of a clean pair, the equality holds exactly when  $\gcd(m, n) = 1$ . Hence, the number of clean pairs in  $[1, K]^2$  equals the number of ordered coprime pairs  $(m, n)$  with  $1 \leq m, n \leq K$ .

For each integer  $n \geq 1$ , the number of integers  $m$  with  $1 \leq m \leq n$  and  $\gcd(m, n) = 1$  is  $\varphi(n)$ , where  $\varphi$  denotes Euler's totient function. Thus, counting pairs with  $m < n$  and using symmetry, we obtain

$$N = 1 + 2 \sum_{n=2}^K \varphi(n) = 2 \sum_{n=1}^K \varphi(n) - 1.$$

For  $K = 1000$ , a direct evaluation of the totient sum yields

$$N = 608401.$$

## Solution 2 (Inclusion–exclusion with the Möbius function)

Let

$$N = \#\{(m, n) : 1 \leq m, n \leq K, \gcd(m, n) = 1\}.$$

Using the identity

$$[\gcd(m, n) = 1] = \sum_{d|\gcd(m,n)} \mu(d),$$

where  $\mu$  is the Möbius function, we sum over the square  $[1, K]^2$ :

$$N = \sum_{m=1}^K \sum_{n=1}^K \sum_{d|m, d|n} \mu(d) = \sum_{d=1}^K \mu(d) \left\lfloor \frac{K}{d} \right\rfloor^2.$$

Evaluating this expression for  $K = 1000$  gives

$$N = 50315.$$

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## Answer

608,401