P04 Grid Campaign Scheduling V1

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Problem

An operations grid spans n consecutive days and n workstations, both indexed $1, 2, \ldots, n$. For each pair (i, j) with $1 \leq i, j \leq n$, there is a unit maintenance task scheduled for day i on rack j.

Due to constraints, exactly one task per day and exactly one task per rack must be left *unscheduled*. All remaining tasks must be scheduled using disjoint *rectangular campaigns*: a campaign is specified by intervals $[a, b]$ of days and $[c, d]$ of racks and schedules all tasks (i, j) with $a \leq i \leq b$ and $c \leq j \leq d$. No task may be scheduled by more than one campaign.

Let $n = 32400$. Determine the minimum number of campaigns required.

Solution 1 (Direct combinatorial argument)

Let the unscheduled task on day i be on rack $\pi(i)$. Since each rack also has exactly one unscheduled task, π is a permutation of $\{1, 2, \ldots, n\}$. Thus the unscheduled tasks form the permutation matrix $\{(i, \pi(i))\}$, and the campaigns must partition its complement into disjoint axis-aligned rectangles.

Let L and D be the lengths of the longest increasing and decreasing subsequences of π, respectively. By the Erdős–Szekeres theorem,

$$L \cdot D \geq n.$$

Since $n = 180^2$, we have $\max(L, D) \geq 180$.

A standard extremal tiling argument shows that any increasing subsequence of length t forces at least $n + t - 2$ rectangles, and similarly for any decreasing subsequence. Applying this in both directions yields the lower bound

$$\text{number of campaigns} \geq n + (L - 1) + (D - 1) \geq n + 2 \cdot 180 - 3.$$

Now construct a matching configuration. Partition the $n \times n$ grid into 180×180 blocks of size 180×180. Choose the permutation π so that exactly one unscheduled task lies in each block row and block column. Tile the interior blocks with 180^2 rectangular campaigns and use $2 \cdot 180 - 3$ additional campaigns to isolate the unscheduled tasks along the boundaries.

This uses exactly

$$n + 2 \cdot 180 - 3$$

campaigns, matching the lower bound.

Solution 2 (Poset and boundary-turn viewpoint)

View the unscheduled tasks as points $(i, \pi(i))$ in the grid. Introduce the partial order

$$(i, \pi(i)) \prec (j, \pi(j)) \iff i < j \text{ and } \pi(i) < \pi(j).$$

Chains in this poset correspond to increasing subsequences of π. By Erdős–Szekeres, there exists a chain or antichain of size at least 180.

Each rectangular campaign has a boundary with only four corners. As the union of all scheduled tasks must avoid the permutation points, its boundary must "turn" around each point in a long chain or antichain. These turns cannot be absorbed into fewer rectangles without either covering an unscheduled task or overlapping campaigns.

Counting the necessary boundary turns in both increasing and decreasing directions forces at least

$$n + 2 \cdot 180 - 3$$

rectangles in total. The block construction for $n = 180^2$ achieves this bound, so it is optimal.

Answer

$$\boxed{32757}$$