P07 Delayed Recurrence V3

Laureano Arcanio   (February 2026)

Problem

Let (a_n) be a real sequence satisfying:

1. *Eventually periodic:* There exist integers $N_0 \geq 1$ and $T \geq 1$ such that

$$a_{n+T} = a_n \quad \text{for all } n \geq N_0.$$

2. *Recurrence:* For all integers $n \geq 9$,

$$a_{n+2} + a_{2n} = a_{n+1} + a_{2n+1}.$$

3. *Square-summable increments:*

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty.$$

Determine the *maximum possible* value of N, the number of distinct real values attained by the sequence (a_n).

Solution 1 (Forcing the tail to be constant)

Define the first differences
$$d_n := a_{n+1} - a_n \qquad (n \geq 1).$$
From the recurrence, for every $n \geq 9$,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

hence
$$d_{n+1} = d_{2n} \qquad (n \geq 9).$$
Equivalently, for every integer $x \geq 10$ (letting $x = n + 1$),

$$d_x = d_{2(x-1)}. \tag{1}$$

Let $h(x) = 2(x - 1)$. Iterating (**??**) gives, for all $k \geq 0$ and $x \geq 2$,

$$d_x = d_{h^k(x)}.$$

1

We claim that for all $k \geq 0$ and $x \geq 2$,

$$h^k(x) = 2^k(x - 2) + 2. \tag{2}$$

This is proved by induction on k. For $k = 0$, $h^0(x) = x = 2^0(x - 2) + 2$. Assume true for k. Then

$$h^{k+1}(x) = h(h^k(x)) = 2(h^k(x) - 1) = 2\left(2^k(x - 2) + 2 - 1\right) = 2^{k+1}(x - 2) + 2,$$

proving (**??**).

Fix any $x \geq 10$. Then by (**??**), the integers $h^k(x) = 2^k(x - 2) + 2$ are strictly increasing in k, hence pairwise distinct. Since $d_{h^k(x)} = d_x$ for all $k \geq 0$, if $d_x \neq 0$ then

$$\sum_{n=1}^{\infty} d_n^2 \geq \sum_{k=0}^{\infty} d_{h^k(x)}^2 = \sum_{k=0}^{\infty} d_x^2 = \infty,$$

contradicting $\sum_{n=1}^{\infty} d_n^2 < \infty$. Therefore,

$$d_n = 0 \quad \text{for all } n \geq 10.$$

Hence

$$a_{10} = a_{11} = a_{12} = \cdots =: C$$

for some real constant C.

Thus all values of the sequence lie in the set $\{a_1, a_2, \ldots, a_9, C\}$, so the number of distinct values satisfies

$$N \leq 10.$$

Solution 2 (Independent construction and optimality)

We independently determine all sequences satisfying the hypotheses and count the number of distinct values they may attain.

Assume (a_n) satisfies the three given conditions. From the recurrence,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

and the square-summability of $(a_{n+1} - a_n)$, the same iteration argument as above forces

$$a_{n+1} = a_n \quad \text{for all } n \geq 10.$$

Hence there exists a real constant C such that $a_n = C$ for all $n \geq 10$.

Conversely, choose arbitrary real numbers $a_1, a_2, \ldots, a_9, C$ and define

$$a_n := C \quad \text{for all } n \geq 10.$$

Then (a_n) is eventually constant, hence eventually periodic. The increment sequence satisfies

$$(a_{n+1} - a_n)^2 = 0 \quad \text{for all } n \geq 10,$$

so the square-sum condition holds. The recurrence is easily checked directly: for all $n \geq 9$, all involved terms equal C.

2

Therefore, every solution has all its values in $\{a_1, a_2, \ldots, a_9, C\}$, and every such 10-tuple produces a valid solution. Hence

$$N \le 10.$$

To show that 10 is attainable, take $a_1, a_2, \ldots, a_9, C$ pairwise distinct, for example

$$a_1 = 18, \quad a_2 = 16, \quad a_3 = 14, \quad a_4 = 12, \quad a_5 = 10, \quad a_6 = 8, \quad a_7 = 6, \quad a_8 = 4, \quad a_9 = 2,$$

and define $a_n = C$ for all $n \ge 10$. This sequence satisfies all conditions and attains exactly 10 distinct values.

Answer

The number of distinct values is always at most 10, and this bound is sharp. Hence the maximum possible value of N is

$$\boxed{10}.$$