

# Line Route with All Distances Different

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## Problem

Let  $n \geq 3$ . In a depot, there are docking nodes indexed by integer pairs  $(a, b)$  satisfying

$$a > 0, \quad b > 0, \quad a + b \leq n + 1.$$

A *rail* is any straight line in the plane. A rail is called *standard* if it is parallel to one of the three directions

$$x = \text{const}, \quad y = \text{const}, \quad x + y = \text{const}.$$

A rail that is not standard is called *express*.

Exactly  $n$  distinct rails are constructed so that every docking node lies on at least one rail. Let  $k$  be the number of express rails.

### Tasks.

1. Determine all attainable values of  $k$ .
2. Let  $\mathcal{K}$  be the set of all attainable values of  $k$ . Using your set  $\mathcal{K}$ , compute

$$N = 10000 \cdot |\mathcal{K}| + \sum_{k \in \mathcal{K}} k.$$

## Solution 1 (Reduction Using the Long-Rail Lemma)

Assume the following lemma, whose proof can be given using the Alon–Füredi polynomial method:

*Long-rail lemma.* For every  $n \geq 4$ , any collection of  $n$  distinct rails covering all points of

$$T_n = \{(a, b) \in \mathbb{Z}^2 : a > 0, b > 0, a + b \leq n + 1\}$$

must contain at least one of the three boundary rails

$$x = 1, \quad y = 1, \quad x + y = n + 1.$$

Each of these boundary rails is standard. Removing one such rail and translating the remaining configuration reduces the problem from  $n$  to  $n - 1$  while preserving the number of express rails. Hence the set  $\mathcal{K}_n$  of attainable values of  $k$  is the same for all  $n \geq 3$ .

It therefore suffices to determine  $\mathcal{K}$  in the base case  $n = 3$ .

For  $n = 3$ , the set  $T_3$  consists of the six points

$$(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1).$$

One checks directly that:

- $k = 0$  is attainable using the three standard rails  $x = 1$ ,  $y = 1$ , and  $x + y = 4$ ;
- $k = 1$  is attainable using two standard rails and one express rail;
- $k = 2$  is impossible, since any line through two of  $(2, 1), (2, 2), (3, 1)$  is necessarily standard;
- $k = 3$  is attainable using three suitably chosen express rails.

Thus

$$\mathcal{K} = \{0, 1, 3\}.$$

By definition,

$$N = 10000 \cdot |\mathcal{K}| + \sum_{k \in \mathcal{K}} k = 10000 \cdot 3 + (0 + 1 + 3) = 30004.$$

## Solution 2 (Direct Evaluation of the Final Expression)

From the complete analysis of the problem (using the reduction argument and the corrected base case), the set of attainable values of  $k$  is

$$\mathcal{K} = \{0, 1, 3\}.$$

Substituting this into the definition of  $N$ , we obtain

$$N = 10000 \cdot |\mathcal{K}| + \sum_{k \in \mathcal{K}} k = 10000 \cdot 3 + (0 + 1 + 3) = 30004.$$

**Answer**

30004