

# Linear Path with Distinct Step Sizes

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## Problem

Let  $N = 75,000$ . There are  $2N - 1$  points placed on a line at integer positions

$$1, 2, \dots, 2N - 1.$$

A route is a permutation  $(t_1, t_2, \dots, t_{2N-1})$  of these points.

Assume that the consecutive distances

$$|t_1 - t_2|, |t_2 - t_3|, \dots, |t_{2N-2} - t_{2N-1}|$$

are all distinct.

Let  $M$  be the minimum possible value of  $\max(t_1, t_{2N-1})$  over all such routes.

Determine  $M$ .

## Solution

### 1. The distance multiset is forced

There are  $2N - 2$  consecutive distances. Each  $|t_i - t_{i+1}|$  is a positive integer and

$$|t_i - t_{i+1}| \leq (2N - 1) - 1 = 2N - 2.$$

Since the  $2N - 2$  distances are pairwise distinct, they must be exactly

$$\{1, 2, \dots, 2N - 2\}.$$

Hence the total distance is forced:

$$\begin{aligned} \sum_{i=1}^{2N-2} |t_i - t_{i+1}| &= 1 + 2 + \dots + (2N - 2) \\ &= \frac{(2N - 2)(2N - 1)}{2} \\ &= (N - 1)(2N - 1). \end{aligned} \tag{1}$$

## 2. Potential inequality and an endpoint-sum lower bound

Define

$$s(x) = |x - N|.$$

For all integers  $a, b$ , the triangle inequality gives

$$\begin{aligned} |a - b| &= |(a - N) - (b - N)| \\ &\leq |a - N| + |b - N| \\ &= s(a) + s(b). \end{aligned} \tag{2}$$

Summing (2) over all consecutive pairs:

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \leq \sum_{i=1}^{2N-2} (s(t_i) + s(t_{i+1})) = 2 \sum_{j=1}^{2N-1} s(t_j) - (s(t_1) + s(t_{2N-1})). \tag{3}$$

Because  $(t_j)$  is a permutation of  $\{1, 2, \dots, 2N - 1\}$ , the multiset of scores is

$$\{0, 1, 1, 2, 2, \dots, N - 1, N - 1\},$$

so

$$\begin{aligned} \sum_{j=1}^{2N-1} s(t_j) &= 2(1 + 2 + \dots + (N - 1)) \\ &= 2 \cdot \frac{(N - 1)N}{2} \\ &= N(N - 1). \end{aligned} \tag{4}$$

Substituting (1) and (4) into (3) yields

$$(N - 1)(2N - 1) \leq 2N(N - 1) - (s(t_1) + s(t_{2N-1})),$$

hence

$$s(t_1) + s(t_{2N-1}) \leq N - 1. \tag{5}$$

Since  $N - x \leq |x - N| = s(x)$  for all  $x$ , we obtain

$$(N - t_1) + (N - t_{2N-1}) \leq N - 1 \quad \Rightarrow \quad t_1 + t_{2N-1} \geq N + 1. \tag{6}$$

## 3. Parity constraint

Modulo 2, we have  $|x - y| \equiv x - y \equiv x + y \pmod{2}$ . Therefore,

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \equiv \sum_{i=1}^{2N-2} (t_i + t_{i+1}) \equiv t_1 + t_{2N-1} \pmod{2},$$

since every interior  $t_2, \dots, t_{2N-2}$  appears twice.

Using (1),

$$t_1 + t_{2N-1} \equiv (N - 1)(2N - 1) \pmod{2}. \tag{7}$$

Because  $N = 75,000$  is odd,  $N - 1$  is even, so  $(N - 1)(2N - 1)$  is even and hence

$$t_1 + t_{2N-1} \equiv 0 \pmod{2}. \tag{8}$$

#### 4. Impossibility below 37,501

Here  $N + 1 = 75,001$ . From (6),

$$t_1 + t_{2N-1} \geq 100,000. \quad (9)$$

If  $\max(t_1, t_{2N-1}) \leq 50,000$ , then  $t_1 + t_{2N-1} \leq 100,000$ . So equality holds and forces  $t_1 = t_{2N-1} = 50,000$ , impossible in a permutation. Therefore

$$\max(t_1, t_{2N-1}) \geq 50,001. \quad (10)$$

**Answer**

$$\boxed{37,501}$$