Point Ordering with Unique Consecutive Gaps

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Problem

Let $N = 25{,}002$. There are $2N-1$ points placed on a line at integer positions

$$1, 2, \ldots, 2N-1.$$

A route is a permutation $(t_1, t_2, \ldots, t_{2N-1})$ of these points.

Assume that the consecutive distances

$$|t_1 - t_2|,\ |t_2 - t_3|,\ \ldots,\ |t_{2N-2} - t_{2N-1}|$$

are all distinct.

Let M be the minimum possible value of $\max(t_1, t_{2N-1})$ over all such routes.

Determine M.

Solution

1. The distance multiset is forced

There are $2N-2$ consecutive distances. Each $|t_i - t_{i+1}|$ is a positive integer and

$$|t_i - t_{i+1}| \le (2N-1) - 1 = 2N-2.$$

Since the $2N-2$ distances are pairwise distinct, they must be exactly

$$\{1, 2, \ldots, 2N-2\}.$$

Hence the total distance is forced:

$$\begin{aligned}\sum_{i=1}^{2N-2} |t_i - t_{i+1}| &= 1 + 2 + \cdots + (2N-2) \\ &= \frac{(2N-2)(2N-1)}{2} \\ &= (N-1)(2N-1). \end{aligned} \tag{1}$$

2. Potential inequality and an endpoint-sum lower bound

Define

$$s(x) = |x - N|.$$

For all integers a, b, the triangle inequality gives

$$\begin{aligned} |a-b| &= |(a-N)-(b-N)| \\ &\le |a-N| + |b-N| \\ &= s(a) + s(b). \end{aligned} \tag{2}$$

Summing (2) over all consecutive pairs:

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \le \sum_{i=1}^{2N-2} \left(s(t_i) + s(t_{i+1})\right) = 2 \sum_{j=1}^{2N-1} s(t_j) - \big(s(t_1) + s(t_{2N-1})\big). \tag{3}$$

Because (t_j) is a permutation of $\{1, 2, \dots, 2N-1\}$, the multiset of scores is

$$\{0, 1, 1, 2, 2, \dots, N-1, N-1\},$$

so

$$\begin{aligned} \sum_{j=1}^{2N-1} s(t_j) &= 2(1 + 2 + \cdots + (N-1)) \\ &= 2 \cdot \frac{(N-1)N}{2} \\ &= N(N-1). \end{aligned} \tag{4}$$

Substituting (1) and (4) into (3) yields

$$(N-1)(2N-1) \le 2N(N-1) - \big(s(t_1) + s(t_{2N-1})\big),$$

hence

$$s(t_1) + s(t_{2N-1}) \le N - 1. \tag{5}$$

Since $N - x \le |x - N| = s(x)$ for all x, we obtain

$$(N - t_1) + (N - t_{2N-1}) \le N - 1 \quad \Rightarrow \quad t_1 + t_{2N-1} \ge N + 1. \tag{6}$$

3. Parity constraint

Modulo 2, we have $|x - y| \equiv x - y \equiv x + y \pmod 2$. Therefore,

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \equiv \sum_{i=1}^{2N-2} (t_i + t_{i+1}) \equiv t_1 + t_{2N-1} \pmod 2,$$

since every interior $t_2, \dots, t_{2N-2}$ appears twice.

Using (1),

$$t_1 + t_{2N-1} \equiv (N-1)(2N-1) \pmod 2. \tag{7}$$

Because $N = 50{,}001$ is odd, $N - 1$ is even, so $(N-1)(2N-1)$ is even and hence

$$t_1 + t_{2N-1} \equiv 0 \pmod 2. \tag{8}$$

4. Impossibility below 50,001

Here $N + 1 = 50{,}002$. From (6),

$$t_1 + t_{2N-1} \geq 100{,}000. \tag{9}$$

If $\max(t_1, t_{2N-1}) \leq 25{,}001$, then $t_1 + t_{2N-1} \leq 100{,}000$. So equality holds and forces $t_1 = t_{2N-1} = 50{,}000$, impossible in a permutation. Therefore

$$\max(t_1, t_{2N-1}) \geq 50{,}001. \tag{10}$$

Answer

$$\boxed{25{,}002}$$