

# P10 Homothety Tangent V3

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## Problem

Let  $ABC$  be a triangle with incenter  $I$  and incircle  $\omega$ . Points  $X, Y \in BC$  are defined as follows:

- the line through  $X$  parallel to  $AC$  is tangent to  $\omega$ ;
- the line through  $Y$  parallel to  $AB$  is tangent to  $\omega$ .

Let  $AI$  meet the circumcircle of  $\triangle ABC$  again at  $P \neq A$ .

For an integer  $m \geq 2$ , let  $K_m \in AB$  and  $L_m \in AC$  satisfy

$$\frac{AK_m}{AB} = \frac{AL_m}{AC} = \frac{1}{m}.$$

Define the statement  $\mathcal{S}(m)$ :

*For every triangle  $ABC$  with  $AB < AC < BC$ , the identity*

$$\angle K_m I L_m + \angle Y P X = 180^\circ$$

*holds.*

Let

$$N = \#\{m \in \{2, 3, \dots, 200000\} \mid \mathcal{S}(m) \text{ is true}\}.$$

Determine  $N$ .

## Solution 1 (Direct / Homothety)

Let  $A_1$  be the reflection of  $A$  across  $I$ . As in the original tangency construction,  $A_1X$  and  $A_1Y$  are tangents to  $\omega$ . Hence the quadrilaterals  $B, P, A_1, X$  and  $C, Y, A_1, P$  are cyclic.

Therefore,

$$\angle APX = \angle A_1BC, \quad \angle YPA = \angle BCA_1,$$

and

$$\angle YPX = \angle A_1BC + \angle BCA_1.$$

Thus the given identity becomes

$$\angle K_m I L_m = \angle CA_1B. \tag{1}$$

Consider the homothety centered at  $A$  with factor  $m$ . By construction, it sends  $K_m$  to  $B$  and  $L_m$  to  $C$ . It sends  $I$  to the point  $A + m(I - A)$ .

The equality (1) holds for all triangles if and only if this image of  $I$  coincides with  $A_1$ , which is the reflection of  $A$  across  $I$ , i.e.

$$A_1 = A + 2(I - A).$$

Hence  $m = 2$ .

For  $m = 2$ , the homothety sends  $(K_2, L_2, I)$  to  $(B, C, A_1)$ , giving

$$\angle K_2 IL_2 = \angle CA_1 B,$$

and the angle sum is  $180^\circ$ . For any  $m \neq 2$ , equality (1) fails for a generic triangle.

Thus  $\mathcal{S}(m)$  holds if and only if  $m = 2$ .

## Solution 2 (Invariance Argument)

From the tangency and cyclicity alone (independent of  $m$ ), we obtain

$$\angle YPX = 180^\circ - \angle CA_1 B,$$

where  $A_1$  is the reflection of  $A$  across  $I$ .

Hence

$$\angle K_m IL_m + \angle YPX = 180^\circ \iff \angle K_m IL_m = \angle CA_1 B.$$

The right-hand side depends only on the fixed point  $A_1$ , while the left-hand side depends on the positions of  $K_m, L_m$  on  $AB, AC$ . The only value of  $m$  for which the dilation centered at  $A$  maps

$$(K_m, L_m, I) \mapsto (B, C, A_1)$$

is  $m = 2$ .

Therefore the equality can hold identically for all triangles only when  $m = 2$ .

## Answer

$N = 1$