Triangle Incenter and Angle Summation

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Problem

Let ABC be a triangle with incenter I and incircle ω. Points $X, Y \in BC$ are defined as follows:

- the line through X parallel to AC is tangent to ω;
- the line through Y parallel to AB is tangent to ω.

Let AI meet the circumcircle of $\triangle ABC$ again at $P \neq A$.

For an integer $m \geq 2$, let $K_m \in AB$ and $L_m \in AC$ satisfy

$$\frac{AK_m}{AB} = \frac{AL_m}{AC} = \frac{1}{m}.$$

Define the statement $\mathcal{S}(m)$:

For every triangle ABC with $AB < AC < BC$, the identity

$$\angle K_m I L_m + \angle YPX = 180^\circ$$

holds.

Let

$$N = \#\{\, m \in \{2, 3, \dots, 50000\} \mid \mathcal{S}(m) \text{ is true} \,\}.$$

Determine N.

Solution 1 (Direct / Homothety)

Let A_1 be the reflection of A across I. As in the original tangency construction, A_1X and A_1Y are tangents to ω. Hence the quadrilaterals B, P, A_1, X and C, Y, A_1, P are cyclic.

Therefore,

$$\angle APX = \angle A_1BC, \qquad \angle YPA = \angle BCA_1,$$

and

$$\angle YPX = \angle A_1BC + \angle BCA_1.$$

Thus the given identity becomes

$$\angle K_m I L_m = \angle CA_1B. \tag{1}$$

Consider the homothety centered at A with factor m. By construction, it sends K_m to B and L_m to C. It sends I to the point $A + m(I - A)$.

The equality (1) holds for all triangles if and only if this image of I coincides with A_1, which is the reflection of A across I, i.e.

$$A_1 = A + 2(I - A).$$

Hence $m = 2$.

For $m = 2$, the homothety sends (K_2, L_2, I) to (B, C, A_1), giving

$$\angle K_2 I L_2 = \angle C A_1 B,$$

and the angle sum is $180°$. For any $m \neq 2$, equality (1) fails for a generic triangle.

Thus $\mathcal{S}(m)$ holds if and only if $m = 2$.

Solution 2 (Invariance Argument)

From the tangency and cyclicity alone (independent of m), we obtain

$$\angle YPX = 180° - \angle C A_1 B,$$

where A_1 is the reflection of A across I.

Hence

$$\angle K_m I L_m + \angle YPX = 180° \quad \iff \quad \angle K_m I L_m = \angle C A_1 B.$$

The right-hand side depends only on the fixed point A_1, while the left-hand side depends on the positions of K_m, L_m on AB, AC. The only value of m for which the dilation centered at A maps

$$(K_m, L_m, I) \longmapsto (B, C, A_1)$$

is $m = 2$.

Therefore the equality can hold identically for all triangles only when $m = 2$.

Answer

$$\boxed{N = 1}$$