Periodic Sequence with Recurrence Constraint

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Problem

Let (a_n) be a real sequence satisfying:

1. *Eventually periodic:* There exist integers $N_0 \geq 1$ and $T \geq 1$ such that
$$a_{n+T} = a_n \quad \text{for all } n \geq N_0.$$
2. *Recurrence:* For all integers $n \geq 1$,
$$a_{n+2} + a_{2n} = a_{n+1} + a_{2n+1}.$$
3. *Square-summable increments:*
$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty.$$

Determine the *maximum possible* value of N, the number of distinct real values attained by the sequence (a_n).

Solution 1 (Forcing the tail to be constant)

Define the first differences
$$d_n := a_{n+1} - a_n \qquad (n \geq 1).$$
From the recurrence, for every $n \geq 1$,
$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$
hence
$$d_{n+1} = d_{2n} \qquad (n \geq 1).$$
Equivalently, for every integer $x \geq 2$ (letting $x = n + 1$),
$$d_x = d_{2(x-1)}. \tag{1}$$

Let $h(x) = 2(x - 1)$. Iterating (**??**) gives, for all $k \geq 0$ and $x \geq 2$,
$$d_x = d_{h^k(x)}.$$

We claim that for all $k \geq 0$ and $x \geq 2$,

$$h^k(x) = 2^k(x-2) + 2. \tag{2}$$

This is proved by induction on k. For $k = 0$, $h^0(x) = x = 2^0(x-2) + 2$. Assume true for k. Then

$$h^{k+1}(x) = h(h^k(x)) = 2(h^k(x) - 1) = 2\big(2^k(x-2) + 2 - 1\big) = 2^{k+1}(x-2) + 2,$$

proving (**??**).

Fix any $x \geq 3$. Then by (**??**), the integers $h^k(x) = 2^k(x-2) + 2$ are strictly increasing in k, hence pairwise distinct. Since $d_{h^k(x)} = d_x$ for all $k \geq 0$, if $d_x \neq 0$ then

$$\sum_{n=1}^{\infty} d_n^2 \;\geq\; \sum_{k=0}^{\infty} d_{h^k(x)}^2 = \sum_{k=0}^{\infty} d_x^2 = \infty,$$

contradicting $\sum_{n=1}^{\infty} d_n^2 < \infty$. Therefore,

$$d_n = 0 \quad \text{for all } n \geq 3.$$

Hence

$$a_3 = a_4 = a_5 = \cdots =: C$$

for some real constant C.

Thus all values of the sequence lie in the set $\{a_1, a_2, C\}$, so the number of distinct values satisfies

$$N \leq 3.$$

Solution 2 (Independent construction and optimality)

We independently determine all sequences satisfying the hypotheses and count the number of distinct values they may attain.

Assume (a_n) satisfies the three given conditions. From the recurrence,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

and the square-summability of $(a_{n+1} - a_n)$, the same iteration argument as above forces

$$a_{n+1} = a_n \quad \text{for all } n \geq 3.$$

Hence there exists a real constant C such that $a_n = C$ for all $n \geq 3$.

Conversely, choose arbitrary real numbers a_1, a_2, C and define

$$a_n := C \quad \text{for all } n \geq 3.$$

Then (a_n) is eventually constant, hence eventually periodic. The increment sequence satisfies

$$(a_{n+1} - a_n)^2 = 0 \quad \text{for all } n \geq 3,$$

so the square-sum condition holds. The recurrence is easily checked directly: it holds for $n = 1, 2$ by substitution, and for $n \geq 3$ all involved terms equal C.

Therefore, every solution has all its values in $\{a_1, a_2, C\}$, and every such triple produces a valid solution. Hence

$$N \leq 3.$$

To show that 3 is attainable, take a_1, a_2, C pairwise distinct, for example

$$a_1 = 10, \qquad a_2 = 5, \qquad C = 0,$$

and define $a_n = C$ for all $n \geq 3$. This sequence satisfies all conditions and attains exactly three distinct values.

Answer

The number of distinct values is always at most 3, and this bound is sharp. Hence the maximum possible value of N is

$$\boxed{3}.$$