Line Route with All Distances Different

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Problem

Let $N = 99{,}999$. There are $2N-1$ points placed on a line at integer positions

$$1, 2, \dots, 2N-1.$$

A route is a permutation $(t_1, t_2, \dots, t_{2N-1})$ of these points.

Assume that the consecutive distances

$$|t_1 - t_2|,\ |t_2 - t_3|,\ \dots,\ |t_{2N-2} - t_{2N-1}|$$

are all distinct.

Let M be the minimum possible value of $\max(t_1, t_{2N-1})$ over all such routes.

Determine M.

Solution (Repaired, complete)

1. The distance multiset is forced

There are $2N-2$ consecutive distances. Each $|t_i - t_{i+1}|$ is a positive integer and

$$|t_i - t_{i+1}| \le (2N-1) - 1 = 2N-2.$$

Since the $2N-2$ distances are pairwise distinct, they must be exactly

$$\{1, 2, \dots, 2N-2\}.$$

Hence the total distance is forced:

$$\begin{aligned}\sum_{i=1}^{2N-2} |t_i - t_{i+1}| &= 1 + 2 + \cdots + (2N-2) \\ &= \frac{(2N-2)(2N-1)}{2} \\ &= (N-1)(2N-1). \end{aligned} \tag{1}$$

2. Potential function inequality and endpoint sum bound

Define the score (potential) function

$$s(x) = |x - N|.$$

For all integers a, b, the triangle inequality gives

$$\begin{aligned} |a - b| &= |(a - N) - (b - N)| \\ &\le |a - N| + |b - N| \\ &= s(a) + s(b). \end{aligned} \tag{2}$$

Apply (2) to each consecutive pair and sum:

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \le \sum_{i=1}^{2N-2} (s(t_i) + s(t_{i+1})).$$

Each interior term $s(t_2), \ldots, s(t_{2N-2})$ appears twice, while the endpoint terms appear once, hence

$$\sum_{i=1}^{2N-2} |t_i - t_{i+1}| \le 2 \sum_{j=1}^{2N-1} s(t_j) - \big(s(t_1) + s(t_{2N-1})\big). \tag{3}$$

Because (t_j) is a permutation of $\{1, 2, \ldots, 2N-1\}$, the multiset of scores is

$$\{|1 - N|, |2 - N|, \ldots, |2N - 1 - N|\} = \{0, 1, 1, 2, 2, \ldots, N-1, N-1\},$$

so

$$\begin{aligned} \sum_{j=1}^{2N-1} s(t_j) &= 2(1 + 2 + \cdots + (N-1)) \\ &= 2 \cdot \frac{(N-1)N}{2} \\ &= N(N-1). \end{aligned} \tag{4}$$

Substituting (1) and (4) into (3) yields

$$(N-1)(2N-1) \le 2N(N-1) - \big(s(t_1) + s(t_{2N-1})\big),$$

hence

$$s(t_1) + s(t_{2N-1}) \le N - 1. \tag{5}$$

Now note that for any integer x,

$$N - x \le |x - N| = s(x).$$

Applying this to $x = t_1$ and $x = t_{2N-1}$ and using (5) gives

$$(N - t_1) + (N - t_{2N-1}) \le s(t_1) + s(t_{2N-1}) \le N - 1,$$

so

$$t_1 + t_{2N-1} \ge N + 1. \tag{6}$$

3. Parity constraint

For integers x, y, we have $-1 \equiv 1 \pmod 2$, hence

$$|x-y| \equiv x-y \equiv x+y \pmod 2.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{2N-2} |t_i - t_{i+1}| &\equiv \sum_{i=1}^{2N-2} (t_i + t_{i+1}) \pmod 2 \\ &\equiv t_1 + t_{2N-1} \pmod 2,\end{aligned}$$

because each interior term $t_2, \dots, t_{2N-2}$ appears exactly twice and cancels modulo 2.

Using (1), we get

$$t_1 + t_{2N-1} \equiv (N-1)(2N-1) \pmod 2. \tag{7}$$

Since $N = 99{,}999$ is odd, $N-1$ is even, hence $(N-1)(2N-1)$ is even; thus (7) gives

$$t_1 + t_{2N-1} \equiv 0 \pmod 2, \quad \text{i.e. } t_1, t_{2N-1} \text{ have the same parity.} \tag{8}$$

4. Impossibility below 50,001

Here $N + 1 = 100{,}000$. From (6),

$$t_1 + t_{2N-1} \geq 100{,}000. \tag{9}$$

Assume for contradiction that $\max(t_1, t_{2N-1}) \leq 50{,}000$. Then

$$t_1 + t_{2N-1} \leq 50{,}000 + 50{,}000 = 100{,}000.$$

Together with (9) this forces $t_1 + t_{2N-1} = 100{,}000$, hence $t_1 = t_{2N-1} = 50{,}000$, impossible since the route is a permutation (all terms distinct). Therefore

$$\max(t_1, t_{2N-1}) \geq 50{,}001. \tag{10}$$

5. Existence at 50,001 (explicit construction)

Set $m = 2N - 1 = 199{,}997$. We construct a permutation of $\{1, 2, \dots, m\}$ whose consecutive absolute differences are exactly $\{1, 2, \dots, m-1\}$ and whose endpoints are 49,999 and 50,001.

Let $k = \frac{N-1}{2} = 49{,}999$, so $m = 4k+1$ and the desired endpoints are k and $k+2$. Define the route $(t_1, t_2, \dots, t_{4k+1})$ by

$$(t_1, t_2, \dots, t_{4k+1}) = \big(k,\ 4k{+}1,\ 1,\ 4k,\ 2,\ 4k{-}1,\ 3,\ 4k{-}2,\ \dots,\ k{-}1,\ 3k{+}3,\ k{+}1,\ 3k{+}2,\ k{+}2\big). \tag{11}$$

All terms in (11) are distinct and lie in $\{1, 2, \dots, 4k+1\}$, so (11) is a permutation of $\{1, 2, \dots, m\}$. We now verify that its consecutive differences are exactly $\{1, 2, \dots, 4k\}$.

Step A: Differences in the first alternating block. In the initial alternating segment

$$k,\ 4k+1,\ 1,\ 4k,\ 2,\ 4k-1,\ 3,\ 4k-2,\ \dots,$$

the consecutive differences are

$$|k-(4k+1)| = 3k+1, \quad |(4k+1)-1| = 4k, \quad |1-4k| = 4k-1, \quad |4k-2| = 4k-2, \ \ldots$$

and continuing yields all integers

$$4k,\ 4k-1,\ 4k-2,\ \ldots,\ 2k+2,\ 3k+1$$

exactly once.

Step B: Differences in the second block. In the final block

$$k-1,\ 3k+3,\ k+1,\ 3k+2,\ k+2,$$

the consecutive differences are

$$|(k-1)-(3k+3)| = 2k+4, \quad |(3k+3)-(k+1)| = 2k+2, \quad |(k+1)-(3k+2)| = 2k+1, \quad |(3k+2)-(k+2)| = 2k.$$

These are precisely $2k, 2k+1, 2k+2, 2k+4$, and the missing value $2k+3$ appears exactly once at the junction between the two blocks in (11) (where the pattern shifts).

Step C: No repetitions and full coverage. By construction, the differences produced in Step A are all $\geq 2k+2$, while the differences produced at and after the junction are all $\leq 2k+4$, and the junction contributes exactly the remaining middle value. Therefore the multiset of consecutive differences in (11) is exactly

$$\{1, 2, \ldots, 4k\} = \{1, 2, \ldots, m-1\},$$

as required.

Finally, the endpoints of (11) are

$$t_1 = k = 49{,}999, \qquad t_m = k+2 = 50{,}001,$$

so

$$\max(t_1, t_{2N-1}) = 50{,}001. \tag{12}$$

6. Minimality

From (10), every valid route satisfies $\max(t_1, t_{2N-1}) \geq 50{,}001$. From (12), there exists a valid route with $\max(t_1, t_{2N-1}) = 50{,}001$. Hence the minimum possible value is

$$M = 50{,}001.$$

Answer

$$\boxed{50{,}001}$$