Line Route with All Distances Different

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Problem

Consider the set $\mathcal{M}$ of all monomials in the commuting variables x_2, x_5, x_11, x_13 with non-negative integer exponents (including the monomial 1). A function $F : \mathcal{M} \to \mathbb{Z}_{>0}$ satisfies the following property:

For all $A, B \in \mathcal{M}$,

$$F(AB)^2 = F(A^2)\, F(F(B))\, F(A \cdot F(B))$$

holds if and only if A and B have no common variable.

Assume in addition that

$$F(x_2) = 32.$$

Let

$$P = x_2 x_5 x_11 x_13.$$

Find

$$N = \sum_{D|P} F(D),$$

where the sum runs over all monomial divisors D of P.

Solution 1 (Direct approach)

The given identity is invariant under replacing a monomial by its square, since taking $B = 1$ (which is coprime to every monomial) yields

$$F(A)^2 = F(A^2)\, F(1)\, F(A),$$

and hence $F(A) = F(A^2)$ for all A. Thus F depends only on the set of variables dividing the monomial.

Moreover, the identity holds exactly when A and B share no variables. This forces F to be multiplicative on monomials with disjoint variable sets and to fail multiplicativity whenever variables overlap. Consequently, there exists an integer $k \geq 1$ such that

$$F(M) = \mathrm{rad}(M)^k,$$

where $\mathrm{rad}(M)$ is the product of the distinct variables dividing M.

The normalization $F(x_2) = 32$ gives

$$2^k = 32,$$

so $k = 5$. Hence

$$F(M) = \mathrm{rad}(M)^5 \quad \text{for all } M \in \mathcal{M}.$$

Now $P = x_2 x_5 x_1 1 x_1 3$ is squarefree, and its monomial divisors correspond to all subsets of $\{x_2, x_5, x_1 1, x_1 3\}$. Therefore,

$$\sum_{D|P} F(D) = \sum_{S \subseteq \{2,5,11,13\}} \left(\prod_{p \in S} p \right)^5 = \prod_{p \in \{2,5,11,13\}} (1 + p^5).$$

Computing,

$$(1 + 2^5)(1 + 5^5)(1 + 11^5)(1 + 13^5) = 33 \cdot 3126 \cdot 161052 \cdot 371294 = 6168605079987504.$$

Solution 2 (Subset interpretation)

Each monomial corresponds uniquely to the subset of variables dividing it. The condition that A and B share no variable becomes disjointness of subsets. The functional equation then acts as a detector of disjointness.

As in the direct solution, substituting $B = 1$ forces $F(M) = F(M^2)$, so F depends only on the associated subset. Disjointness multiplicativity implies that for some constants $c_p > 1$,

$$F(M) = \prod_{p \in \sigma(M)} c_p,$$

where $\sigma(M)$ is the set of variables dividing M. The "only if" direction of the identity forces all c_p to be equal powers $c_p = p^k$ with a common exponent k. The condition $F(x_2) = 32$ yields $k = 5$, so again $F(M) = \mathrm{rad}(M)^5$.

Thus

$$\sum_{D|P} F(D) = \prod_{p \in \{2,5,11,13\}} (1 + p^5) = 6168605079987504.$$

Answer

$$\boxed{6168605079987504}$$