

# Line Route with All Distances Different

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## Problem

Consider the set  $\mathcal{M}$  of all monomials in the commuting variables  $x_2, x_3, x_5, x_1$  with non-negative integer exponents (including the monomial 1). A function  $F : \mathcal{M} \rightarrow \mathbb{Z}_{>0}$  satisfies the following property:

For all  $A, B \in \mathcal{M}$ ,

$$F(AB)^2 = F(A^2) F(F(B)) F(A \cdot F(B))$$

holds if and only if  $A$  and  $B$  have no common variable.

Assume in addition that

$$F(x_2) = 8.$$

Let

$$P = x_2 x_3 x_5 x_1.$$

Find

$$N = \sum_{D|P} F(D),$$

where the sum runs over all monomial divisors  $D$  of  $P$ .

## Solution 1 (Direct approach)

The given identity is invariant under replacing a monomial by its square, since taking  $B = 1$  (which is coprime to every monomial) yields

$$F(A)^2 = F(A^2) F(1) F(A),$$

and hence  $F(A) = F(A^2)$  for all  $A$ . Thus  $F$  depends only on the set of variables dividing the monomial.

Moreover, the identity holds exactly when  $A$  and  $B$  share no variables. This forces  $F$  to be multiplicative on monomials with disjoint variable sets and to fail multiplicativity whenever variables overlap. Consequently, there exists an integer  $k \geq 1$  such that

$$F(M) = \text{rad}(M)^k,$$

where  $\text{rad}(M)$  is the product of the distinct variables dividing  $M$ .

The normalization  $F(x_2) = 8$  gives

$$2^k = 8,$$

so  $k = 3$ . Hence

$$F(M) = \text{rad}(M)^3 \quad \text{for all } M \in \mathcal{M}.$$

Now  $P = x_2x_3x_5x_11$  is squarefree, and its monomial divisors correspond to all subsets of  $\{x_2, x_3, x_5, x_11\}$ . Therefore,

$$\sum_{D|P} F(D) = \sum_{S \subseteq \{2,3,5,11\}} \left( \prod_{p \in S} p \right)^3 = \prod_{p \in \{2,3,5,11\}} (1 + p^3).$$

Computing,

$$(1 + 2^3)(1 + 3^3)(1 + 5^3)(1 + 11^3) = 9 \cdot 28 \cdot 126 \cdot 1332 = 42293664.$$

## Solution 2 (Subset interpretation)

Each monomial corresponds uniquely to the subset of variables dividing it. The condition that  $A$  and  $B$  share no variable becomes disjointness of subsets. The functional equation then acts as a detector of disjointness.

As in the direct solution, substituting  $B = 1$  forces  $F(M) = F(M^2)$ , so  $F$  depends only on the associated subset. Disjointness multiplicativity implies that for some constants  $c_p > 1$ ,

$$F(M) = \prod_{p \in \sigma(M)} c_p,$$

where  $\sigma(M)$  is the set of variables dividing  $M$ . The “only if” direction of the identity forces all  $c_p$  to be equal powers  $c_p = p^k$  with a common exponent  $k$ . The condition  $F(x_2) = 8$  yields  $k = 3$ , so again  $F(M) = \text{rad}(M)^3$ .

Thus

$$\sum_{D|P} F(D) = \prod_{p \in \{2,3,5,11\}} (1 + p^3) = 42293664.$$

**Answer**

42293664