

# P07 Bounded Increment V2

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## Problem

Let  $(a_n)$  be a real sequence satisfying:

1. *Eventually periodic:* There exist integers  $N_0 \geq 1$  and  $T \geq 1$  such that

$$a_{n+T} = a_n \quad \text{for all } n \geq N_0.$$

2. *Recurrence:* For all integers  $n \geq 1$ ,

$$a_{n+2} + a_{2n} = a_{n+1} + a_{2n+1}.$$

3. *Square-summable increments:*

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty.$$

Determine the *maximum possible* value of  $N$ , the number of distinct real values attained by the sequence  $(a_n)$ .

## Solution 1 (Forcing the tail to be constant)

Define the first differences

$$d_n := a_{n+1} - a_n \quad (n \geq 1).$$

From the recurrence, for every  $n \geq 1$ ,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

hence

$$d_{n+1} = d_{2n} \quad (n \geq 1).$$

Equivalently, for every integer  $x \geq 2$  (letting  $x = n + 1$ ),

$$d_x = d_{2(x-1)}. \tag{1}$$

Let  $h(x) = 2(x - 1)$ . Iterating (??) gives, for all  $k \geq 0$  and  $x \geq 2$ ,

$$d_x = d_{h^k(x)}.$$

We claim that for all  $k \geq 0$  and  $x \geq 2$ ,

$$h^k(x) = 2^k(x - 2) + 2. \quad (2)$$

This is proved by induction on  $k$ . For  $k = 0$ ,  $h^0(x) = x = 2^0(x - 2) + 2$ . Assume true for  $k$ . Then

$$h^{k+1}(x) = h(h^k(x)) = 2(h^k(x) - 1) = 2(2^k(x - 2) + 2 - 1) = 2^{k+1}(x - 2) + 2,$$

proving (??).

Fix any  $x \geq 3$ . Then by (??), the integers  $h^k(x) = 2^k(x - 2) + 2$  are strictly increasing in  $k$ , hence pairwise distinct. Since  $d_{h^k(x)} = d_x$  for all  $k \geq 0$ , if  $d_x \neq 0$  then

$$\sum_{n=1}^{\infty} d_n^2 \geq \sum_{k=0}^{\infty} d_{h^k(x)}^2 = \sum_{k=0}^{\infty} d_x^2 = \infty,$$

contradicting  $\sum_{n=1}^{\infty} d_n^2 < \infty$ . Therefore,

$$d_n = 0 \quad \text{for all } n \geq 3.$$

Hence

$$a_3 = a_4 = a_5 = \dots =: C$$

for some real constant  $C$ .

Thus all values of the sequence lie in the set  $\{a_1, a_2, C\}$ , so the number of distinct values satisfies

$$N \leq 3.$$

## Solution 2 (Independent construction and optimality)

We independently determine all sequences satisfying the hypotheses and count the number of distinct values they may attain.

Assume  $(a_n)$  satisfies the three given conditions. From the recurrence,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

and the square-summability of  $(a_{n+1} - a_n)$ , the same iteration argument as above forces

$$a_{n+1} = a_n \quad \text{for all } n \geq 3.$$

Hence there exists a real constant  $C$  such that  $a_n = C$  for all  $n \geq 3$ .

Conversely, choose arbitrary real numbers  $a_1, a_2, C$  and define

$$a_n := C \quad \text{for all } n \geq 3.$$

Then  $(a_n)$  is eventually constant, hence eventually periodic. The increment sequence satisfies

$$(a_{n+1} - a_n)^2 = 0 \quad \text{for all } n \geq 3,$$

so the square-sum condition holds. The recurrence is easily checked directly: it holds for  $n = 1, 2$  by substitution, and for  $n \geq 3$  all involved terms equal  $C$ .

Therefore, every solution has all its values in  $\{a_1, a_2, C\}$ , and every such triple produces a valid solution. Hence

$$N \leq 3.$$

To show that 3 is attainable, take  $a_1, a_2, C$  pairwise distinct, for example

$$a_1 = 10, \quad a_2 = 5, \quad C = 0,$$

and define  $a_n = C$  for all  $n \geq 3$ . This sequence satisfies all conditions and attains exactly three distinct values.

## Answer

The number of distinct values is always at most 3, and this bound is sharp. Hence the maximum possible value of  $N$  is

$$\boxed{3}.$$