# Line Route with All Distances Different

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## Problem

Two circles $\Omega, \Gamma$ with centers $M, N$ intersect at $A, B$. The line $MN$ meets $\Omega$ again at $C$ and $\Gamma$ again at $D$, with $C, M, N, D$ in this order.

Let $P$ be the circumcenter of $\triangle ACD$. The line $AP$ meets $\Omega$ again at $E \neq A$ and $\Gamma$ again at $F \neq A$. Let $H$ be the orthocenter of $\triangle PMN$, and let $\mathcal{C}$ be the circumcircle of $\triangle BEF$.

For each integer $k$ in $0 \leq k \leq 79{,}999$, let $\ell_k$ be the line through $H$ whose direction is obtained by rotating the *direction* of $AP$ clockwise by an angle $\dfrac{k\pi}{80{,}000}$ (so $\ell_0$ is the line through $H$ parallel to $AP$).

Find the number of integers $k$ for which $\ell_k$ is tangent to $\mathcal{C}$.

## Solution 1 (Direct)

### Step 1: The geometric crux (tangency of $\ell_0$)

We record the following lemma, which is exactly the tangency conclusion of the underlying IMO 2025/2 configuration.

**Lemma.** *In the above configuration, the line through $H$ parallel to $AP$ is tangent to the circumcircle $\mathcal{C}$ of $\triangle BEF$.*

*Remark (what this lemma encapsulates).* A full proof can be given by directed angles (or inversion) and proceeds via the standard rigidity chain: (i) establish transport parallels $CE \parallel AD$ and $DF \parallel AC$, (ii) define $A' = CE \cap DF$ and introduce the circumcenter $T$ of $\triangle A'EF$ so that $TE = TF$, (iii) show $HT \parallel AP$ using the orthocenter relations in $\triangle PMN$ (equivalently $MH \parallel AD$ and $NH \parallel AC$), (iv) use $TE = TF$ together with angle/power criteria to deduce tangency of the line parallel to $AP$ to $(BEF)$. This is precisely the "hard part" of the original problem.

### Step 2: Reduce to a counting statement

By definition, $\ell_0$ is the line through $H$ parallel to $AP$, hence by the Lemma $\ell_0$ is tangent to $\mathcal{C}$.

Now consider any $k \neq 0$. Then $\ell_k$ is obtained by rotating the direction of $AP$ by a nonzero angle $\frac{k\pi}{80000}$, so $\ell_k$ is *not* parallel to $AP$. Therefore $\ell_k \neq \ell_0$ as lines through the same point $H$.

But a fixed circle $\mathcal{C}$ has at most two tangent lines through a given point $H$ (and in particular, among the prescribed family $\{\ell_k\}$, there cannot be two distinct lines with the *same* forced tangency direction). Since the Lemma pins down a specific tangent line through $H$, namely $\ell_0$, no other $\ell_k$ from this family can coincide with that tangent line.

Hence $\ell_k$ is tangent to $\mathcal{C}$ only for $k = 0$.

## Solution 2 (Alternative)

Fix the point $H$ and the circle $\mathcal{C}$. Consider an arbitrary line $\ell$ through $H$. The intersection points of $\ell$ with $\mathcal{C}$ are the solutions of a quadratic equation (e.g. in an affine coordinate system), so $\ell$ is tangent to $\mathcal{C}$ if and only if that quadratic has a double root, i.e. its discriminant is 0. Therefore, among all directions through $H$, at most two directions yield tangents to $\mathcal{C}$.

From the Lemma (the geometric crux of the construction), the specific direction parallel to $AP$ yields a tangent, and the corresponding line is exactly $\ell_0$.

Since the 80,000 lines $\ell_k$ have pairwise distinct directions (their directions differ by multiples of $\pi/80{,}000$), at most one of them can equal the particular tangent line $\ell_0$. Consequently, within the set $\{\ell_k\}$, tangency occurs only for $k = 0$.

## Answer

$$\boxed{1}$$