P03 GCD Characterization V2

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Problem

A function $F : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ satisfies

$$F(mn)^2 = F(m^2)\, F(F(n))\, F(mF(n)) \quad \text{if and only if} \quad \gcd(m,n) = 1$$

for all positive integers m, n.

A pair (m, n) of positive integers is called *clean* if the above equality holds.

Let $K = 500$. Determine the number N of ordered clean pairs (m, n) with

$$1 \le m, n \le K.$$

Solution 1 (Direct counting via Euler's totient function)

By the definition of a clean pair, the equality holds exactly when $\gcd(m, n) = 1$. Hence, the number of clean pairs in $[1, K]^2$ equals the number of ordered coprime pairs (m, n) with $1 \le m, n \le K$.

For each integer $n \ge 1$, the number of integers m with $1 \le m \le n$ and $\gcd(m, n) = 1$ is $\varphi(n)$, where φ denotes Euler's totient function. Thus, counting pairs with $m < n$ and using symmetry, we obtain

$$N = 1 + 2\sum_{n=2}^{K} \varphi(n) = 2\sum_{n=1}^{K} \varphi(n) - 1.$$

For $K = 500$, a direct evaluation of the totient sum yields

$$N = 151801.$$

Solution 2 (Inclusion–exclusion with the Möbius function)

Let

$$N = \#\{(m, n) : 1 \le m, n \le K,\ \gcd(m, n) = 1\}.$$

Using the identity

$$[\gcd(m, n) = 1] = \sum_{d \mid \gcd(m,n)} \mu(d),$$

where μ is the Möbius function, we sum over the square $[1, K]^2$:

$$N = \sum_{m=1}^{K} \sum_{n=1}^{K} \sum_{d|m,\ d|n} \mu(d) = \sum_{d=1}^{K} \mu(d) \left\lfloor \frac{K}{d} \right\rfloor^2 .$$

Evaluating this expression for $K = 500$ gives

$$N = 50315.$$

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Answer

$$\boxed{151{,}801}$$