

P09 Increasing Sequence Gaps V2

Laureano Arcanio (February 2026)

Problem

Fix $C = 4021$. Consider a strictly increasing sequence of positive integers

$$a_0 < a_1 < \dots < a_m.$$

Assume that for every $0 \leq n \leq m - 2$,

$$a_{n+2} - a_n \leq 4021.$$

Define the gaps $d_n = a_{n+1} - a_n$ for $0 \leq n \leq m - 1$, and suppose that the gaps

$$d_0, d_1, \dots, d_{m-1}$$

are pairwise distinct.

Find the maximum possible value of $m + 1$.

Solution 1 (Gap analysis)

Define $d_n = a_{n+1} - a_n$ for $0 \leq n \leq m - 1$. Since the sequence (a_n) is strictly increasing, each d_n is a positive integer.

From the given condition,

$$a_{n+2} - a_n = (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n) = d_{n+1} + d_n \leq 4021$$

for all $0 \leq n \leq m - 2$.

Because $d_{n+1} \geq 1$, it follows that

$$d_n \leq 4020 \quad \text{for all } n.$$

Hence all gaps belong to the set $\{1, 2, \dots, 4020\}$.

Since the gaps are pairwise distinct, we must have

$$m \leq 4020,$$

and therefore

$$m + 1 \leq 4021.$$

We now show that this bound is attainable. Arrange the numbers $1, 2, \dots, 4020$ in the order

$$4020, 1, 4019, 2, 4018, 3, \dots$$

alternating between the largest unused and smallest unused value. For each adjacent pair in this sequence, the sum is at most 4021.

Let d_0, \dots, d_{4019} be this ordering, and define $a_0 = 1$ and

$$a_{n+1} = a_n + d_n.$$

Then (a_n) is strictly increasing, satisfies $a_{n+2} - a_n \leq 4021$, and has 4021 terms.

Thus the maximum possible value of $m + 1$ is 4021.

Solution 2 (Graph-theoretic interpretation)

Consider the graph whose vertices are the integers $1, 2, \dots, 4020$, with an edge between distinct vertices x and y if and only if

$$x + y \leq 4021.$$

A sequence of pairwise distinct gaps d_0, \dots, d_{m-1} satisfying $d_n + d_{n+1} \leq 4021$ corresponds exactly to a simple path of length $m - 1$ in this graph. Hence m cannot exceed the number of vertices, which is 4020, so $m + 1 \leq 4021$.

The alternating ordering

$$4020, 1, 4019, 2, \dots$$

defines a Hamiltonian path in this graph, since every adjacent pair sums to at most 4021. Therefore a path using all 4020 vertices exists, giving $m = 4020$ and $m + 1 = 4021$.

Answer

4021