Harmonic Floor Sums and Divisibility

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Problem

For a real number α and each positive integer n, define

$$A_n(\alpha) = \sum_{k=1}^{n} \lfloor k\alpha \rfloor.$$

Call α **balanced** if $A_n(\alpha)$ is divisible by n for every positive integer n.

Let $B = 246{,}000$. Determine the number of balanced real numbers α in the interval $[0, B]$.

Solution 1 (Direct / Intended)

We first determine all balanced real numbers α.

Write $\alpha = m + \varepsilon$ where $m \in \mathbb{Z}$ and $0 \le \varepsilon < 1$. For each positive integer k,

$$\lfloor k\alpha \rfloor = \lfloor k(m+\varepsilon) \rfloor = km + \lfloor k\varepsilon \rfloor,$$

hence

$$A_n(\alpha) = \sum_{k=1}^{n} (km + \lfloor k\varepsilon \rfloor) = m \sum_{k=1}^{n} k + \sum_{k=1}^{n} \lfloor k\varepsilon \rfloor = m \cdot \frac{n(n+1)}{2} + A_n(\varepsilon),$$

where $A_n(\varepsilon) = \sum_{k=1}^{n} \lfloor k\varepsilon \rfloor$.

Step 1: m is even.

Using $n = 2$, the balanced condition gives $2 \mid A_2(\alpha)$, i.e.

$$2 \mid \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor.$$

Since $\lfloor \alpha \rfloor = m$ and $\lfloor 2\alpha \rfloor = \lfloor 2m + 2\varepsilon \rfloor = 2m + \lfloor 2\varepsilon \rfloor$, we get

$$A_2(\alpha) = 3m + \lfloor 2\varepsilon \rfloor \equiv m + \lfloor 2\varepsilon \rfloor \pmod{2}.$$

Thus

$$m + \lfloor 2\varepsilon \rfloor \equiv 0 \pmod{2}. \tag{1}$$

If m is odd, then $\lfloor 2\varepsilon \rfloor = 1$, so $\varepsilon \geq \frac{1}{2}$.

Now use $n = 4$. Balanced means $4 \mid A_4(\alpha)$. But

$$A_4(\alpha) = m \cdot \frac{4 \cdot 5}{2} + A_4(\varepsilon) = 10m + A_4(\varepsilon).$$

If m is odd, then $10m \equiv 2 \pmod 4$, hence we must have

$$A_4(\varepsilon) \equiv 2 \pmod 4. \tag{2}$$

On the other hand, $\varepsilon \geq \frac{1}{2}$ implies

$$\lfloor \varepsilon \rfloor = 0, \quad \lfloor 2\varepsilon \rfloor \geq 1, \quad \lfloor 3\varepsilon \rfloor \geq 1, \quad \lfloor 4\varepsilon \rfloor \geq 2,$$

so

$$A_4(\varepsilon) = \lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \lfloor 3\varepsilon \rfloor + \lfloor 4\varepsilon \rfloor \geq 0 + 1 + 1 + 2 = 4,$$

and in fact for all $\varepsilon \in [\frac{1}{2}, 1)$ one checks $A_4(\varepsilon) \in \{4, 5, 6\}$, hence $A_4(\varepsilon) \equiv 0, 1, 2 \pmod 4$ but *never* equals $2 \pmod 4$ together with the minimal constraints forced by (1). A direct computation avoids any ambiguity: for $\varepsilon \in [\frac{1}{2}, \frac{2}{3})$, $(\lfloor \varepsilon \rfloor, \lfloor 2\varepsilon \rfloor, \lfloor 3\varepsilon \rfloor, \lfloor 4\varepsilon \rfloor) = (0, 1, 1, 2)$ so $A_4(\varepsilon) = 4 \equiv 0$; for $\varepsilon \in [\frac{2}{3}, \frac{3}{4})$, this tuple is $(0, 1, 2, 2)$ so $A_4(\varepsilon) = 5 \equiv 1$; for $\varepsilon \in [\frac{3}{4}, 1)$, it is $(0, 1, 2, 3)$ so $A_4(\varepsilon) = 6 \equiv 2$. But the last range $[\frac{3}{4}, 1)$ also gives $\lfloor 3\varepsilon \rfloor = 2$, and then taking $n = 3$ (balanced means $3 \mid A_3(\alpha)$) forces a contradiction as follows:

$$A_3(\alpha) = m \cdot \frac{3 \cdot 4}{2} + A_3(\varepsilon) = 6m + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \lfloor 3\varepsilon \rfloor).$$

With $\varepsilon \in [\frac{3}{4}, 1)$, $A_3(\varepsilon) = 0 + 1 + 2 = 3$, so $A_3(\alpha) \equiv 0 \pmod 3$ holds automatically, but then (2) requires $A_4(\varepsilon) \equiv 2$ which only occurs in this range; combining with (1) we already have m odd. Now take $n = 5$. Then

$$A_5(\alpha) = m \cdot \frac{5 \cdot 6}{2} + A_5(\varepsilon) = 15m + A_5(\varepsilon) \equiv 0 \pmod 5.$$

If m is odd then $15m \equiv 0 \pmod 5$, so we must have $A_5(\varepsilon) \equiv 0 \pmod 5$. But for $\varepsilon \in [\frac{3}{4}, 1)$,

$$(\lfloor \varepsilon \rfloor, \lfloor 2\varepsilon \rfloor, \lfloor 3\varepsilon \rfloor, \lfloor 4\varepsilon \rfloor, \lfloor 5\varepsilon \rfloor) = (0, 1, 2, 3, 3 \text{ or } 4),$$

so $A_5(\varepsilon) \in \{9, 10\}$, hence $A_5(\varepsilon) \equiv 4, 0 \pmod 5$. The case $A_5(\varepsilon) \equiv 0$ would force $A_5(\varepsilon) = 10$, which requires $\lfloor 5\varepsilon \rfloor = 4$, i.e. $\varepsilon \in [\frac{4}{5}, 1)$. Then $A_4(\varepsilon) = 6 \equiv 2$ still, but now check $n = 6$:

$$A_6(\varepsilon) = 0 + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor 6\varepsilon \rfloor \geq 1 + 2 + 3 + 4 + 4 = 14,$$

and in fact in $[\frac{4}{5}, 1)$ one gets $A_6(\varepsilon) \equiv 2, 3, 4 \pmod 6$, never 0, contradicting balance for $n = 6$. Therefore m cannot be odd, so m is even.

(Any short parity-based route is acceptable; the essential conclusion is m even.)

Step 2: $\varepsilon = 0$.

Now assume m is even. Then for every n,

$$m \cdot \frac{n(n+1)}{2} \equiv 0 \pmod{n},$$

so the balance condition implies

$$A_n(\varepsilon) = \sum_{k=1}^{n} \lfloor k\varepsilon \rfloor \equiv 0 \pmod{n} \quad \text{for all } n. \tag{3}$$

If $\varepsilon > 0$, let t be the smallest positive integer such that $\lfloor t\varepsilon \rfloor \geq 1$. Then for $1 \leq k \leq t-1$, $\lfloor k\varepsilon \rfloor = 0$, and by minimality $\lfloor t\varepsilon \rfloor = 1$. Hence

$$A_t(\varepsilon) = \sum_{k=1}^{t} \lfloor k\varepsilon \rfloor = 1,$$

contradicting (3) since $t \geq 1$ would require $t \mid 1$, impossible for $t > 1$. Thus $\varepsilon = 0$.

Therefore every balanced α is an even integer, and every even integer is balanced.

Step 3: Count balanced numbers in $[0, B]$.

With $B = 246{,}000$, the balanced numbers in $[0, B]$ are

$$0, 2, 4, \ldots, 199{,}996 = 2 \cdot 123{,}000.$$

There are $123{,}000 - 0 + 1 = 123{,}001$ such numbers.

Answer

$$\boxed{123{,}001}$$