P03 Monomial Disjointness V2

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Problem

Consider the set $\mathcal{M}$ of all monomials in the commuting variables x_2, x_3, x_7, x_13 with non-negative integer exponents (including the monomial 1). A function $F : \mathcal{M} \to \mathbb{Z}_{>0}$ satisfies the following property:

For all $A, B \in \mathcal{M}$,

$$F(AB)^2 = F(A^2)\,F(F(B))\,F(A \cdot F(B))$$

holds if and only if A and B have no common variable.

Assume in addition that

$$F(x_2) = 16.$$

Let

$$P = x_2x_3x_7x_13.$$

Find

$$N = \sum_{D|P} F(D),$$

where the sum runs over all monomial divisors D of P.

Solution 1 (Direct approach)

The given identity is invariant under replacing a monomial by its square, since taking $B = 1$ (which is coprime to every monomial) yields

$$F(A)^2 = F(A^2)\,F(1)\,F(A),$$

and hence $F(A) = F(A^2)$ for all A. Thus F depends only on the set of variables dividing the monomial.

Moreover, the identity holds exactly when A and B share no variables. This forces F to be multiplicative on monomials with disjoint variable sets and to fail multiplicativity whenever variables overlap. Consequently, there exists an integer $k \geq 1$ such that

$$F(M) = \mathrm{rad}(M)^k,$$

where $\mathrm{rad}(M)$ is the product of the distinct variables dividing M.

The normalization $F(x_2) = 16$ gives

$$2^k = 16,$$

so $k = 4$. Hence

$$F(M) = \operatorname{rad}(M)^4 \quad \text{for all } M \in \mathcal{M}.$$

Now $P = x_2 x_3 x_7 x_1 3$ is squarefree, and its monomial divisors correspond to all subsets of $\{x_2, x_3, x_7, x_1 3\}$. Therefore,

$$\sum_{D|P} F(D) = \sum_{S \subseteq \{2,3,7,13\}} \left(\prod_{p \in S} p \right)^4 = \prod_{p \in \{2,3,7,13\}} (1 + p^4).$$

Computing,

$$(1 + 2^4)(1 + 3^4)(1 + 7^4)(1 + 13^4) = 17 \cdot 82 \cdot 2402 \cdot 28562 = 95636658056.$$

Solution 2 (Subset interpretation)

Each monomial corresponds uniquely to the subset of variables dividing it. The condition that A and B share no variable becomes disjointness of subsets. The functional equation then acts as a detector of disjointness.

As in the direct solution, substituting $B = 1$ forces $F(M) = F(M^2)$, so F depends only on the associated subset. Disjointness multiplicativity implies that for some constants $c_p > 1$,

$$F(M) = \prod_{p \in \sigma(M)} c_p,$$

where $\sigma(M)$ is the set of variables dividing M. The "only if" direction of the identity forces all c_p to be equal powers $c_p = p^k$ with a common exponent k. The condition $F(x_2) = 16$ yields $k = 4$, so again $F(M) = \operatorname{rad}(M)^4$.

Thus

$$\sum_{D|P} F(D) = \prod_{p \in \{2,3,7,13\}} (1 + p^4) = 95636658056.$$

Answer

$$\boxed{95636658056}$$