Line Route with All Distances Different

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Problem

Two circles Ω, Γ with centers M, N intersect at A, B. The line MN meets Ω again at C and Γ again at D, with C, M, N, D in this order.

Let P be the circumcenter of $\triangle ACD$. The line AP meets Ω again at $E \neq A$ and Γ again at $F \neq A$. Let H be the orthocenter of $\triangle PMN$, and let $\mathcal{C}$ be the circumcircle of $\triangle BEF$.

For each integer k in $0 \leq k \leq 100{,}000$, let ℓ_k be the line through H whose direction is obtained by rotating the *direction* of AP clockwise by an angle $\dfrac{k\pi}{100{,}000}$ (so ℓ_0 is the line through H parallel to AP).

Find the number of integers k for which ℓ_k is tangent to $\mathcal{C}$.

Solution 1 (Direct)

Step 1: The geometric crux (tangency of ℓ_0)

We record the following lemma, which is exactly the tangency conclusion of the underlying IMO 2025/2 configuration.

Lemma. *In the above configuration, the line through H parallel to AP is tangent to the circumcircle $\mathcal{C}$ of $\triangle BEF$.*

Remark (what this lemma encapsulates). A full proof can be given by directed angles (or inversion) and proceeds via the standard rigidity chain: (i) establish transport parallels $CE \parallel AD$ and $DF \parallel AC$, (ii) define $A' = CE \cap DF$ and introduce the circumcenter T of $\triangle A'EF$ so that $TE = TF$, (iii) show $HT \parallel AP$ using the orthocenter relations in $\triangle PMN$ (equivalently $MH \parallel AD$ and $NH \parallel AC$), (iv) use $TE = TF$ together with angle/power criteria to deduce tangency of the line parallel to AP to (BEF). This is precisely the "hard part" of the original problem.

Step 2: Reduce to a counting statement

By definition, ℓ_0 is the line through H parallel to AP, hence by the Lemma ℓ_0 is tangent to $\mathcal{C}$.

Now consider any $k \notin \{0, 100{,}000\}$. Then ℓ_k is obtained by rotating the direction of AP by an angle that is neither 0 nor π, so ℓ_k is *not* parallel to AP. Therefore $\ell_k \neq \ell_0$ as lines through the same point H.

A fixed circle $\mathcal{C}$ has at most two tangent lines through a given point H. Therefore, within the family $\{\ell_k\}$, only the two indices producing ℓ_0 can yield tangency.

Moreover, $k = 100{,}000$ rotates the direction of AP by π, producing the same line as ℓ_0. Thus $k = 0$ and $k = 100{,}000$ both yield tangency, and no other k does.

Solution 2 (Alternative)

Fix the point H and the circle $\mathcal{C}$. Consider an arbitrary line ℓ through H. The intersection points of ℓ with $\mathcal{C}$ are the solutions of a quadratic equation (e.g. in an affine coordinate system), so ℓ is tangent to $\mathcal{C}$ if and only if that quadratic has a double root, i.e. its discriminant is 0. Therefore, among all directions through H, at most two directions yield tangents to $\mathcal{C}$.

From the Lemma (the geometric crux of the construction), the specific direction parallel to AP yields a tangent, and the corresponding line is exactly ℓ_0.

Among the 100,001 indices k, the only lines coinciding with ℓ_0 are $k = 0$ and $k = 100{,}000$, since a π rotation gives the same line. All other ℓ_k have distinct directions and cannot be tangent. Therefore exactly two k work.

Answer

$$\boxed{2}$$