Line Route with All Distances Different

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Problem

Let (a_n) be a real sequence satisfying:

1. *Eventually periodic:* There exist integers $N_0 \geq 1$ and $T \geq 1$ such that
$$a_{n+T} = a_n \quad \text{for all } n \geq N_0.$$
2. *Recurrence:* For all integers $n \geq 9$,
$$a_{n+2} + a_{2n} = a_{n+1} + a_{2n+1}.$$
3. *Square-summable increments:*
$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty.$$

Determine the *maximum possible* value of N, the number of distinct real values attained by the sequence (a_n).

Solution 1 (Forcing the tail to be constant)

Define the first differences
$$d_n := a_{n+1} - a_n \qquad (n \geq 1).$$
From the recurrence, for every $n \geq 9$,
$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$
hence
$$d_{n+1} = d_{2n} \qquad (n \geq 9).$$
Equivalently, for every integer $x \geq 10$ (letting $x = n + 1$),
$$d_x = d_{2(x-1)}. \tag{1}$$

Let $h(x) = 2(x - 1)$. Iterating (**??**) gives, for all $k \geq 0$ and $x \geq 2$,
$$d_x = d_{h^k(x)}.$$

We claim that for all $k \geq 0$ and $x \geq 2$,

$$h^k(x) = 2^k(x-2) + 2. \tag{2}$$

This is proved by induction on k. For $k = 0$, $h^0(x) = x = 2^0(x-2) + 2$. Assume true for k. Then

$$h^{k+1}(x) = h(h^k(x)) = 2(h^k(x) - 1) = 2\big(2^k(x-2) + 2 - 1\big) = 2^{k+1}(x-2) + 2,$$

proving (**??**).

Fix any $x \geq 10$. Then by (**??**), the integers $h^k(x) = 2^k(x-2)+2$ are strictly increasing in k, hence pairwise distinct. Since $d_{h^k(x)} = d_x$ for all $k \geq 0$, if $d_x \neq 0$ then

$$\sum_{n=1}^{\infty} d_n^2 \;\geq\; \sum_{k=0}^{\infty} d_{h^k(x)}^2 = \sum_{k=0}^{\infty} d_x^2 = \infty,$$

contradicting $\sum_{n=1}^{\infty} d_n^2 < \infty$. Therefore,

$$d_n = 0 \quad \text{for all } n \geq 10.$$

Hence

$$a_{10} = a_{11} = a_{12} = \cdots =: C$$

for some real constant C.

Thus all values of the sequence lie in the set $\{a_1, a_2, \ldots, a_9, C\}$, so the number of distinct values satisfies

$$N \leq 10.$$

Solution 2 (Independent construction and optimality)

We independently determine all sequences satisfying the hypotheses and count the number of distinct values they may attain.

Assume (a_n) satisfies the three given conditions. From the recurrence,

$$a_{n+2} - a_{n+1} = a_{2n+1} - a_{2n},$$

and the square-summability of $(a_{n+1} - a_n)$, the same iteration argument as above forces

$$a_{n+1} = a_n \quad \text{for all } n \geq 10.$$

Hence there exists a real constant C such that $a_n = C$ for all $n \geq 10$.

Conversely, choose arbitrary real numbers $a_1, a_2, \ldots, a_9, C$ and define

$$a_n := C \quad \text{for all } n \geq 10.$$

Then (a_n) is eventually constant, hence eventually periodic. The increment sequence satisfies

$$(a_{n+1} - a_n)^2 = 0 \quad \text{for all } n \geq 10,$$

so the square-sum condition holds. The recurrence is easily checked directly: for all $n \geq 9$, all involved terms equal C.

Therefore, every solution has all its values in $\{a_1, a_2, \ldots, a_9, C\}$, and every such 10-tuple produces a valid solution. Hence

$$N \leq 10.$$

To show that 10 is attainable, take $a_1, a_2, \ldots, a_9, C$ pairwise distinct, for example

$$a_1 = 18, \qquad a_2 = 16, \qquad a_3 = 14, \qquad a_4 = 12, \qquad a_5 = 10, \qquad a_6 = 8, \qquad a_7 = 6, \qquad a_8 = 4, \qquad a_9 = 2,$$

and define $a_n = C$ for all $n \geq 10$. This sequence satisfies all conditions and attains exactly 10 distinct values.

Answer

The number of distinct values is always at most 10, and this bound is sharp. Hence the maximum possible value of N is

$$\boxed{10}.$$