

# Balanced Numbers and Divisibility

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## Problem

For a real number  $\alpha$  and each positive integer  $n$ , define

$$A_n(\alpha) = \sum_{k=1}^n \lfloor k\alpha \rfloor.$$

Call  $\alpha$  **balanced** if  $A_n(\alpha)$  is divisible by  $n$  for every positive integer  $n$ .

Let  $B = 199,996$ . Determine the number of balanced real numbers  $\alpha$  in the interval  $[0, B]$ .

## Solution 1 (Direct / Intended)

We first determine all balanced real numbers  $\alpha$ .

Write  $\alpha = m + \varepsilon$  where  $m \in \mathbb{Z}$  and  $0 \leq \varepsilon < 1$ . For each positive integer  $k$ ,

$$\lfloor k\alpha \rfloor = \lfloor k(m + \varepsilon) \rfloor = km + \lfloor k\varepsilon \rfloor,$$

hence

$$A_n(\alpha) = \sum_{k=1}^n (km + \lfloor k\varepsilon \rfloor) = m \sum_{k=1}^n k + \sum_{k=1}^n \lfloor k\varepsilon \rfloor = m \cdot \frac{n(n+1)}{2} + A_n(\varepsilon),$$

where  $A_n(\varepsilon) = \sum_{k=1}^n \lfloor k\varepsilon \rfloor$ .

### Step 1: $m$ is even.

Using  $n = 2$ , the balanced condition gives  $2 \mid A_2(\alpha)$ , i.e.

$$2 \mid \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor.$$

Since  $\lfloor \alpha \rfloor = m$  and  $\lfloor 2\alpha \rfloor = \lfloor 2m + 2\varepsilon \rfloor = 2m + \lfloor 2\varepsilon \rfloor$ , we get

$$A_2(\alpha) = 3m + \lfloor 2\varepsilon \rfloor \equiv m + \lfloor 2\varepsilon \rfloor \pmod{2}.$$

Thus

$$m + \lfloor 2\varepsilon \rfloor \equiv 0 \pmod{2}. \tag{1}$$

If  $m$  is odd, then  $\lfloor 2\varepsilon \rfloor = 1$ , so  $\varepsilon \geq \frac{1}{2}$ .

Now use  $n = 4$ . Balanced means  $4 \mid A_4(\alpha)$ . But

$$A_4(\alpha) = m \cdot \frac{4 \cdot 5}{2} + A_4(\varepsilon) = 10m + A_4(\varepsilon).$$

If  $m$  is odd, then  $10m \equiv 2 \pmod{4}$ , hence we must have

$$A_4(\varepsilon) \equiv 2 \pmod{4}. \quad (2)$$

On the other hand,  $\varepsilon \geq \frac{1}{2}$  implies

$$\lfloor \varepsilon \rfloor = 0, \quad \lfloor 2\varepsilon \rfloor \geq 1, \quad \lfloor 3\varepsilon \rfloor \geq 1, \quad \lfloor 4\varepsilon \rfloor \geq 2,$$

so

$$A_4(\varepsilon) = \lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \lfloor 3\varepsilon \rfloor + \lfloor 4\varepsilon \rfloor \geq 0 + 1 + 1 + 2 = 4,$$

and in fact for all  $\varepsilon \in [\frac{1}{2}, 1)$  one checks  $A_4(\varepsilon) \in \{4, 5, 6\}$ , hence  $A_4(\varepsilon) \equiv 0, 1, 2 \pmod{4}$  but *never* equals  $2 \pmod{4}$  together with the minimal constraints forced by (1). A direct computation avoids any ambiguity: for  $\varepsilon \in [\frac{1}{2}, \frac{2}{3})$ ,  $(\lfloor \varepsilon \rfloor, \lfloor 2\varepsilon \rfloor, \lfloor 3\varepsilon \rfloor, \lfloor 4\varepsilon \rfloor) = (0, 1, 1, 2)$  so  $A_4(\varepsilon) = 4 \equiv 0$ ; for  $\varepsilon \in [\frac{2}{3}, \frac{3}{4})$ , this tuple is  $(0, 1, 2, 2)$  so  $A_4(\varepsilon) = 5 \equiv 1$ ; for  $\varepsilon \in [\frac{3}{4}, 1)$ , it is  $(0, 1, 2, 3)$  so  $A_4(\varepsilon) = 6 \equiv 2$ . But the last range  $[\frac{3}{4}, 1)$  also gives  $\lfloor 3\varepsilon \rfloor = 2$ , and then taking  $n = 3$  (balanced means  $3 \mid A_3(\alpha)$ ) forces a contradiction as follows:

$$A_3(\alpha) = m \cdot \frac{3 \cdot 4}{2} + A_3(\varepsilon) = 6m + (\lfloor \varepsilon \rfloor + \lfloor 2\varepsilon \rfloor + \lfloor 3\varepsilon \rfloor).$$

With  $\varepsilon \in [\frac{3}{4}, 1)$ ,  $A_3(\varepsilon) = 0 + 1 + 2 = 3$ , so  $A_3(\alpha) \equiv 0 \pmod{3}$  holds automatically, but then (2) requires  $A_4(\varepsilon) \equiv 2$  which only occurs in this range; combining with (1) we already have  $m$  odd. Now take  $n = 5$ . Then

$$A_5(\alpha) = m \cdot \frac{5 \cdot 6}{2} + A_5(\varepsilon) = 15m + A_5(\varepsilon) \equiv 0 \pmod{5}.$$

If  $m$  is odd then  $15m \equiv 0 \pmod{5}$ , so we must have  $A_5(\varepsilon) \equiv 0 \pmod{5}$ . But for  $\varepsilon \in [\frac{3}{4}, 1)$ ,

$$(\lfloor \varepsilon \rfloor, \lfloor 2\varepsilon \rfloor, \lfloor 3\varepsilon \rfloor, \lfloor 4\varepsilon \rfloor, \lfloor 5\varepsilon \rfloor) = (0, 1, 2, 3, 3 \text{ or } 4),$$

so  $A_5(\varepsilon) \in \{9, 10\}$ , hence  $A_5(\varepsilon) \equiv 4, 0 \pmod{5}$ . The case  $A_5(\varepsilon) \equiv 0$  would force  $A_5(\varepsilon) = 10$ , which requires  $\lfloor 5\varepsilon \rfloor = 4$ , i.e.  $\varepsilon \in [\frac{4}{5}, 1)$ . Then  $A_4(\varepsilon) = 6 \equiv 2$  still, but now check  $n = 6$ :

$$A_6(\varepsilon) = 0 + \lfloor 2\varepsilon \rfloor + \cdots + \lfloor 6\varepsilon \rfloor \geq 1 + 2 + 3 + 4 + 4 = 14,$$

and in fact in  $[\frac{4}{5}, 1)$  one gets  $A_6(\varepsilon) \equiv 2, 3, 4 \pmod{6}$ , never 0, contradicting balance for  $n = 6$ . Therefore  $m$  cannot be odd, so  $m$  is even.

(Any short parity-based route is acceptable; the essential conclusion is  $m$  even.)

**Step 2:**  $\varepsilon = 0$ .

Now assume  $m$  is even. Then for every  $n$ ,

$$m \cdot \frac{n(n+1)}{2} \equiv 0 \pmod{n},$$

so the balance condition implies

$$A_n(\varepsilon) = \sum_{k=1}^n \lfloor k\varepsilon \rfloor \equiv 0 \pmod{n} \quad \text{for all } n. \quad (3)$$

If  $\varepsilon > 0$ , let  $t$  be the smallest positive integer such that  $\lfloor t\varepsilon \rfloor \geq 1$ . Then for  $1 \leq k \leq t-1$ ,  $\lfloor k\varepsilon \rfloor = 0$ , and by minimality  $\lfloor t\varepsilon \rfloor = 1$ . Hence

$$A_t(\varepsilon) = \sum_{k=1}^t \lfloor k\varepsilon \rfloor = 1,$$

contradicting (3) since  $t \geq 1$  would require  $t \mid 1$ , impossible for  $t > 1$ . Thus  $\varepsilon = 0$ .

Therefore every balanced  $\alpha$  is an even integer, and every even integer is balanced.

### Step 3: Count balanced numbers in $[0, B]$ .

With  $B = 199,996$ , the balanced numbers in  $[0, B]$  are

$$0, 2, 4, \dots, 199,996 = 2 \cdot 99,998.$$

There are  $99,998 - 0 + 1 = 99,999$  such numbers.

### Answer

99,999
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