

# P09 Increasing Sequence Gaps V2

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## Problem

Fix  $C = 4021$ . Consider a strictly increasing sequence of positive integers

$$a_0 < a_1 < \cdots < a_m.$$

Assume that for every  $0 \leq n \leq m - 2$ ,

$$a_{n+2} - a_n \leq 4021.$$

Define the gaps  $d_n = a_{n+1} - a_n$  for  $0 \leq n \leq m - 1$ , and suppose that the gaps

$$d_0, d_1, \dots, d_{m-1}$$

are pairwise distinct.

Find the maximum possible value of  $m + 1$ .

## Solution 1 (Gap analysis)

Define  $d_n = a_{n+1} - a_n$  for  $0 \leq n \leq m - 1$ . Since the sequence  $(a_n)$  is strictly increasing, each  $d_n$  is a positive integer.

From the given condition,

$$a_{n+2} - a_n = (a_{n+2} - a_{n+1}) + (a_{n+1} - a_n) = d_{n+1} + d_n \leq 4021$$

for all  $0 \leq n \leq m - 2$ .

Because  $d_{n+1} \geq 1$ , it follows that

$$d_n \leq 4020 \quad \text{for all } n.$$

Hence all gaps belong to the set  $\{1, 2, \dots, 4020\}$ .

Since the gaps are pairwise distinct, we must have

$$m \leq 4020,$$

and therefore

$$m + 1 \leq 4021.$$

We now show that this bound is attainable. Arrange the numbers  $1, 2, \dots, 4020$  in the order

$$4020, 1, 4019, 2, 4018, 3, \dots$$

alternating between the largest unused and smallest unused value. For each adjacent pair in this sequence, the sum is at most 4021.

Let  $d_0, \dots, d_{4019}$  be this ordering, and define  $a_0 = 1$  and

$$a_{n+1} = a_n + d_n.$$

Then  $(a_n)$  is strictly increasing, satisfies  $a_{n+2} - a_n \leq 4021$ , and has 4021 terms.

Thus the maximum possible value of  $m + 1$  is 4021.

## Solution 2 (Graph-theoretic interpretation)

Consider the graph whose vertices are the integers  $1, 2, \dots, 4020$ , with an edge between distinct vertices  $x$  and  $y$  if and only if

$$x + y \leq 4021.$$

A sequence of pairwise distinct gaps  $d_0, \dots, d_{m-1}$  satisfying  $d_n + d_{n+1} \leq 4021$  corresponds exactly to a simple path of length  $m - 1$  in this graph. Hence  $m$  cannot exceed the number of vertices, which is 4020, so  $m + 1 \leq 4021$ .

The alternating ordering

$$4020, 1, 4019, 2, \dots$$

defines a Hamiltonian path in this graph, since every adjacent pair sums to at most 4021. Therefore a path using all 4020 vertices exists, giving  $m = 4020$  and  $m + 1 = 4021$ .

## Answer

4021