Endowed Assignment Valuations and Gross Substitutability

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Abstract

This paper is a modified version of my senior thesis and investigates the relationship between various notions of the gross substitutes condition. It connects various mathematical settings in which gross substitutes appears and compares the results and assumptions from different settings. It considers a class of endowed assignment valuations and uses matroid algebra to demonstrate that this class is strictly smaller than the class of gross substitutes valuations.

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1 Technical Background

Our intention is to construct an algorithmic model which takes agents' preferences as inputs and outputs a designation of labor. Let's consider an auction setting in which hospitals are bidding for doctors in a labor market. Let H denote the set of hospitals and D denote the set of doctors. In other words, we are solving a two-sided matching problem in which elements of H may be paired with multiple elements of D, while elements of D may only match with one element from H. Hatfield and Milgrom introduced the contract as convenient mathematical object to formalize a matching of doctors and hospitals [5]. Each potential match between a doctor and a hospital is a contract, $x \in H \times D$. If $x = (h_n, d_m)$, we write $x_h = h_n$ and $x_d = d_m$. A set of contracts, $X' \subset H \times D$, is analogous to an assignment of doctors to hospitals in that $x \in X'$ implies doctor x_d is employed by hospital x_h in assignment X'. An assignment is not allowed to match the same doctor with different hospitals. This language is appropriate since a contractual agreement is drawn up whenever a doctor is hired by a hospital. The contract is a concise and convenient way to include the details of a job offer. For example, the contract can be extended to include wage, and thus can provide a much more realistic representation of the labor market.

Definition 1. *A set of contracts X' is an assignment of doctors to hospitals if for all $d^* \in D$, if there do not exist contracts $(x^*_d, x_h), (x^*_d, x'_h) \in X$ such that $h \neq h'$.*

A stable outcome is one in which no hospital-doctor pair can mutually improve by swapping out their assigned match. This paper is concerned with conditions on hospital preferences that guarantee attainment of competitive equilibrium in a competitive auction/labor matching market.

Definition 2. *A set of contracts $X' \subset X$ is a stable allocation at wages $\bar{w}$ if*

- $C^{\bar{w}}_D(X') = C^{\bar{w}}_H(X') = X'$

- *There does not exist hospital h and set of contracts $X'' \neq C_h^{\bar{w}}(X')$ such that*

$$X'' = C_h^{\bar{w}}(X' \cup X'') \subset C_D^{\bar{w}}(X' \cup X'').$$

1.1 Competitive Equilibrium and the Greedy Algorithm

Perhaps a more familiar definition of stability is that of market competitive equilibrium, the desired outcome of the Kelso-Crawford labor matching system [9]. Kelso and Crawford's matching algorithm first motivated the study of gross substitutes valuations by identifying the gross substitutes condition as necessary and sufficient for their labor matching market to achieve competitive equilibrium [9].

Definition 3. *Consider a set of bidders H in a market where a set of indivisible goods, A, is being offered. For each $h \in H$, let u_h denote the utility function of bidder h. This market is said to achieve competitive equilibrium at price vector $\bar{p}$ if there exists a partition of the set of goods such that $A = \bigcup_{h \in H} S_h$ with $S_h \in C_h^{\bar{p}}(A)$.*

Competitive equilibrium in a Walrasian market is achevied when the economy achieves an efficient outcome on its own. In a Walrasian market, agent make mutually beneficial trades and adjust prices until each agent is as satisfied with his bundle of goods as he would be with any other. Gross substitutes valuations play a key role in determining which economies will be able to reach a stable equilibrium through a natural price setting/good allocating mechanism. The Greedy Algorithm simulates the iterative process of buying and selling goods in a free market. Kelso and Crawford introduced the following version of the greedy algorithm to describe a market for indivisible goods which is guaranteed to achieve competitive equilibrium if agents have gross substitutes preferences [9].

To begin, goods are allocated arbitrarily and prices are fixed at 0 dollars. In the following case, all goods are owned by a single agent. In each iteration, the algorithm checks if there exists agents who do not demand their allocated bundle at the given price. Those agents are endowed with a new bundle that they do demand. However, whenever goods are reallocated because they are included in a demanded bundle, their prices are increased by δ. The Greedy Algorithm captures the classic economic intuition behind supply and demand. When prices are low, excess demand drives prices upwards. After repeated iterations, each good will end up in the hands of the agent who likes it most, because that agent will be willing to pay for the good at a higher price than anyone else.

Algorithm 1: Greedy Algorithm

Result: S_i for all $i \in H$

input: $\delta > 0$ $v_i^{\bar{w}}$ for each $i \in H$

initialization: $S_1 = D$, $S_i = \emptyset$ for all $i \in H$, $i \neq 1$, $w_d = 0$ for all $d \in D$;

while $\exists S_i \notin C_i^{\bar{w}}(D)$ **do**

  Find $X_i \in C_i^{\bar{w}}(D)$. Change prices so that $\forall d \in X_i \backslash D_i$, $w_d = w_d + \delta$

  Change allocations so that $D_i = X_i$ and $D_j = D_j \backslash X_i$ for $j \neq i$

end

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2 Intro to Gross Substitutes

A market consisting of gross substitutes valuations is the condition identified by Kelso and Crawford to guarantee that this greedy algorithm will eventually converge to a competitive equilibrium [9]. Let's frame the Greedy Algorithm in the language of hospitals and doctors and define the gross substitutes condition. We describe doctors' wages through a price vector $\overline{w} \in \mathbb{R}^{|D|}$. Then w_d is the price hospitals must pay to hire doctor d. Let wages be fixed by $\overline{w}$. We will impose the gross substitutes condition on hospitals' preference behavior. Equip each hospital with a real-valued utility function on the power set of available contracts. Let $X = D \times H$ and choose $h \in H$. Then $u_h : 2^X \to \mathbb{R}$ is hospital h's utility function. We recover h's chosen sets of contracts at a given price with a demand correspondence. A hospital's chosen set is the set of contracts it demands at a given price.

Definition 4. *Let $X' \subset X$ and $\bar{w} \in \mathbb{R}^{|X|}$. Let $u_{h^*} : 2^X \to \mathbb{R}$ be hospital h^*'s utility function.*

Then

$$C^{\bar{w}}_{h^*}(X') = max_{S \subset D'}\{u_{h^*}(S) - \sum_{d \in S} w_d\}$$

is h^*'s chosen set of contracts from X' subject to $\bar{w}$ and is such that $\forall x \in C^{\bar{w}}_{h^*}(X')$, $x = (d, h^*)$. Also

$$C^{\bar{w}}_{H}(X') = \bigcup_{h \in H} C_h(X')$$

is the aggregate chosen set of hospitals in H from the set of available contracts X'.

Definition 5. *Let $\bar{w} \in \mathbb{R}^{|D|}$ be a vector of wages for doctors. Then $\bar{w}^{+d^*} \in \mathbb{R}^{|D|}$ is such that $w_d = w^{+d^*}_d$ for all $d \neq d^*$ and $w^{+d^*}_{d*} > w_{d^*}$.*

A hospital's preferences satisfy gross substitutes if demand for each doctor is non-decreasing in the wages of other doctors. In other words, a hospital's preferences satisfy gross substitutes if increasing the wage of any one doctor does not decrease the demand for any other doctor.

Definition 6. *Hospital h has preferences that satisfy the gross substitutes condition if there does not exist a doctor d^*, wage vector $\bar{w}$, and set of contracts X' such that $x = (d, h) \in C^{\bar{w}}_h(X')$, $d \neq d^*$, and $x = (d, h) \notin C^{\bar{w}^{+d^*}}_h(X')$.*

Theorem 1. *If $v^{\bar{w}}_i$ satisfies the gross substitutes property for all $i \in H$, then the Greedy Algorithm will terminate in a finite number of steps to a market that achieves competitive equilibrium at some price $\bar{w}$.*

2.1 Matroids and M-Concavity

A hospital's choice function is a discrete optimization problem. We can glean valuable intuition and results from a variety of mathematical settings which are concerned with such problems. Discrete convex analysis is the name given to the study of convex analysis and matroid theory combined [8]. This field is concerned with a class of real valued set functions that happens to be equivalent to the class of gross substitutes valuations. First, let's get acquainted with the language of matroids, algebraic objects that appear in graph theory, combinatorial optimization, network theory, and many more fields in computer science and mathematics. Matroids abstract the properties of dependence from vector spaces. We will translate our labor market of hospitals and doctors to the language of matroids in order to better understand the gross substitutes condition in our later discussion.

Definition 7. *A matroid M is a pair $(E, \mathcal{I})$ where E is called the ground set and $\mathcal{I}$ is a collection of subsets of E called the independent sets of M and the following are true:*

$$(i)\ \emptyset \in \mathcal{I}$$

$$(ii)\ \forall A' \subset A \subset E, \text{ if } A \in \mathcal{I} \text{ then } A' \in \mathcal{I}$$

$$(iii) \text{ if } A, B \in \mathcal{I} \text{ and } |A| > |B|, \text{ then there exists } x \in A \backslash B \text{ such that } B \cup \{x\} \in \mathcal{I}.$$

2.1.1 Translating to Demand Theory

Let's translate hospital preferences to the language of matroids. Consider hospital h_1. Let M be the matroid that represents hospital h_1's demand decision. Let D denote the set of doctors in the market. Recall that a contract $x = (x_h, x_d)$ is a pairing of hospital h to doctor d. The ground set of M is $X_1 = h_1 \times D$, the set of all possible contracts in the market which involve hospital h_1. The independent sets are $\mathcal{I} = 2^{X_1}$, the set of possible groupings of doctors that hospital h_1 may hire. If hospital h_1 has restrictions on its hiring, perhaps it can only hire a total of two doctors, then $\mathcal{I}$ will instead be equal to the set of all subsets of X_1 that contain a maximum of two contracts.

Property (i) is satisfied as long as our hospital is permitted to choose to hire no doctors. Property (ii) requires that if our hospital can hire a certain group of doctors, it is permitted to hire any subset of that group of doctors. Property (iii) is best illustrated. Suppose h_1 may hire $\{d_1, d_2\}$ and $\{d_1, d_2, d_a, d_b\}$. Then h_1 may also hire $\{d_1, d_2, d_a\}$, $\{d_1, d_2, d_a\}$, and $\{d_1, d_2, d_b\}$.

2.2 M-Concave Functions

An M-Concave function maps from the independent set of a matroid to the real number. In our hospital-doctor matching context, an M-Concave function assigns a utility value to the sets of doctors our hospitals might hire. M-Concave function must satisfy certain properties that are equivalent to the gross substitutes condition on preferences.

Definition 8. *Let N be a set of n items. Let $\mathcal{F}$ be a family of subsets of N. Then a set function $f : \mathcal{F} \to \mathbb{R}$ is M-Concave if for any $X, Y \in \mathcal{F}$ and $i \in X \backslash Y$,*

- *(i)* $(X\backslash\{i\}), (Y \cup \{i\}) \in \mathcal{F}$

  (ii) $f(X) + f(Y) \leq f(X\backslash\{i\}) + f(Y \cup \{i\})$ *or*
- *(i)* $\exists\, j \in Y\backslash X$ *such that* $(X\backslash\{i\} \cup \{j\}), (Y \cup \{i\}\backslash\{j\}) \in \mathcal{F}$

  (ii) $f(X) + f(Y) \leq f(X\backslash\{i\} \cup \{j\}) + f(Y \cup \{i\}\backslash\{j\})$.

First, lets check that the definition of an M-Concave function is compatible with our definition of a matroid. This way, we can easily translate between M-Concave functions and utility functions.

Lemma 1. *Let N be a set of n items and $\mathcal{F}$ a family of subsets of N. Let $f : \mathcal{F} \to \mathbb{R}$ be M-Concave. Then the pair $(N, \mathcal{F})$ is a matroid.*

Proof. We take property (i) for granted as it is obvious.

We check property (ii). Let $X' \subset X \subset N$ with $X \in \mathcal{F}$. We know that X is discrete and so $X = X' \cup \{x_1, ..., x_n\}$ with each $x_i \notin X'$. M-concavity requires that $\forall X \in \mathcal{I}$

and all x_i for $i = 1, ...n$ that $X \backslash x_i \in \mathcal{F}$. Starting with $X \backslash x_1 = X_1 \in \mathcal{F}$ inductively remove one element at a time from X until we achieve $X' = X_n \in \mathcal{F}$.

To see that (iii) holds, note that $|X| > |Y| \implies \exists\, x \in X \backslash Y$. M-concavity requires that $\forall\, X, Y$ and $x \in X \backslash Y$ that $Y \cup \{x\} \in \mathcal{F}$. □

The relationship between gross substitutes valuations, M-Concave valuations, and matroids is investigated by Renato Paes Leme in his algorithmic survey where the following theorem is proven [7].

Theorem 2. *The gross substitutes condition is equivalent to M-concavity on utility functions.*

3 Endowed Assignment Valuations

Hatfield and Milgrom introduce the Endowed Assignment Valuations as a class of grossly substitutable utility functions that can realistically describe hospital preferences. In what follows, we will use matroid theory to understand clearly why the class of EAVs is strictly smaller than the class of gross substitutes valuations. The Endowed Assignment Valuation can be understood from a graph theoretic perspective. Let's begin by introducing the EAV in the vein of Hatfield and Milgrom, in terms of hospital hiring behavior [5].

EAV's are a class of functions that assign utility to bundles of goods. We will proceed with our hospital-doctor notation. Imagine that a hospital is looking to hire n doctors, and each doctor will be hired for a unique position within the hospital according to the doctor's specialties. Hatfield and Milgrom build the class of EAV's up from the set of singleton valuations [5]. Each hospital has a number of positions to fill with doctors, and each position within the hospital will contribute a certain amount of overall utility to the hospital. Singleton valuations capture the amount of utility the hospital gets from a single position within it. By building EAV's from singleton valuations, a doctor will contribute different amounts of utility depending on what position it is employed in.

Definition 9. *Valuation V is a singleton valuation if it is of the form*

$$V(S) = max_{d \in S} \alpha_d$$

such that $\alpha_d \in \mathbb{R}$ is the value of doctor d according to this valuation.

A hospital will have multiple singleton valuations, one for each position, and their EAV will decide which doctors to hire and how to arrange those doctors within their organization in order to maximize the aggregate of their singleton valuations. Hatfield and Milgrom introduce the aggregation operation on singleton valuations so that valuations can be built up to contain many positions [5].

Definition 10. *Let V_1 and V_2 be singleton valuations. The aggregation operation, denoted $\wedge$, behaves in the following way:*

$$(V_1 \wedge V_2)(S) = max_{R \subset S} V_1(R) + V_2(S \backslash R).$$

Definition 11. *Valuation v over set D of doctors is an Assignment Valuation if there exists a set of positions J, and a matrix α of dimension $|D| \times |J|$ such that for any set $X \subset D$,*

$$v(X) = max_z \sum_{d \in X, j \in J} \alpha_{dj} z_{dj},$$

where z varies over all possible assignments of elements in X to elements in J.

Consider the form of an Assignment Valuation. Suppose J is the hospital's set of available positions and S is a set of doctors it might hire. For each doctor $d \in S$ and each position $j \in J$, d provides α_{dj} utility to the hospital if it is assigned to position j. Then the hospital chooses to hire the set of doctors $D \subset S$ that maximizes $v(D') = \sum_{d \in D', j \in J} \alpha_{dj} z_{dj}$

where $z_{dj} \in \{0,1\}$ for all $d \in D'$ and $j \in J$. That is, the $|D'| \times |J|$ identity matrix with entries z_{dj} varies over all possible ways doctors in D' could be assigned to positions in J. To capture the fact that each doctor can only be assigned to one position and each position can only employ one hospital, we require that $\forall j \in J$, $\sum_{d \in D'} z_{dj} \leq 1$ and $\forall d \in D'$, $\sum_{j \in J} z_{dj} \leq 1$.

Given an available set of doctors, assignment valuations describe the maximum utility the hospital can obtain by hiring some of the doctors and arranging them in a particular way. However, hospitals will rarely be starting from scratch in the sense that, when it is choosing to hire, it has no positions already filled. Instead, a hospital will have an endowment of doctors, the doctors it currently employs. Then, the value of hiring a new set of doctors is the difference in the utility it is already getting from its current doctors and the utility it gets from having the new set of doctors made available to it. The Endowment Operation takes an Assignment Valuation and equips it with the tools necessary to capture this choice behavior.

Definition 12. *Let W be an Assignment Valuation on the set of doctors D. The Endowment Operation on W outputs the Endowed Assignment Valuation, V, such that for all $J, S \subset D$,*

$$V(S|J) = W(J \cup S) - W(J).$$

Hatfield and Milgrom's class of Endowed Assignment Valuations is the smallest class of valuations containing singleton valuations and which is closed under the Aggregation and Endowment Operations [5].

3.1 EAV Conjecture

Endowed Assignment Valuations are of particular interest to us because they are a large class of gross substitutes valuations and they capture the nuances of large firm/"hospital" behavior well. The following theorem is attributed to Hatfield and Milgrom who synthesized previous results that assignment valuations satisfy gross substitutes and that a valuation with a submodular indirect profit function is substitutable [5]. Following publication of *Matching with Contracts* where EAVs were first introduced, it remained an open question whether the class of Endowed Assignment Valuations encompasses the class of Gross Substitutes Valuations. The idea seemed plausible because of certain properties of EAVs that are related to M-Concavity.

Theorem 3. *Every Endowed Assignment Valuation satisfies the Gross Substitutes property.*

Conjecture 1. *The class of gross substitutes valuations is equal to the class of endowed assignment valuations.*

Renato and Ostrovsky show that the class of Gross Substitutes valuations is strictly larger than the class of Endowed Assignment valuations [10]. Their method of proof is to identify a property that is true of all Endowed Assignment Valuations and then discover a Gross Substitutes Valuation that does not have that property. The property in question is called Strong Exchangeability. Renato and Ostrovsky provide the following definition:

Definition 13. *Valuation function V is strongly exchangeable if for any $\bar{p} \in \mathbb{R}^{|S|}$ and any two sets X and Y that are demanded at $\bar{p}$, there exists a bijective function σ from $X \backslash Y$ to $Y \backslash X$ such that $\forall i \in X \backslash Y$, the bundles $X \cup \{\sigma(i)\} \backslash \{i\}$ and $Y \cup \{i\} \backslash \{\sigma(i)\}$ are also demanded.*

Strong exchangeability requires that if X and Y are both demanded under our valuation function at a given price, then items of each can be swapped out for one another and the resulting sets will still be demanded. This property may look familiar. Compare it to M-Concavity. It is easy to recognize that strongly exchangeable valuations satisfy gross substitutes. The first step in Renato's and Ostrovsky's argument is to show that endowed assignment valuations are always strongly exchangeable. The following is presented as a lemma in their paper [10].

Theorem 4. *Every endowed assignment valuation is strongly exchangeable.*

The fine details of their proof can be found in their original work [10]. We include a sketch of their argument. Consider an endowed assignment valuation

$$V(S) = \max_{d \in S, j \in J} \sum_{d \in S \cup T, j \in J} \alpha_{d,j} z_{dj} - W(T).$$

In this case, T is the endowment of doctors. The valuation function yields an optimal bundle an an optimal matrix z that assigns doctors in $S \cup T$ to positions in J. We suppose that both X and Y are demanded according to this valuation function at some fixed price. This means that $V(X) = V(Y)$ since they are both maximal. Moreover, let z^x be the matrix assigning agents to positions under the assignment that chooses X and let z^y be the matrix assigning agents under the assignment that chooses Y. Now, construct a bipartite graph where nodes are elements of $S \cup J$. For each pair d, j such that $z^x_{dj} = 1$, draw a blue edge connecting the node representing d and the node representing j. Essentially, we are drawing blue edges between doctors and positions to reflect their assignment. Do the same for z^y, but draw these edges in red. Let each edge of the graph have a weight that is equal to α_{dj}, the utility that is contributed by having doctor d in position j under the various assignments. Then the sum of the weights of the blue edges is equal to $V(X)$ and the sum of the weights of the red edges is equal to $V(Y)$. And since $V(X) = V(Y)$, these two sums must also be equal. Since each node in our graph has a degree less than 2, the graph can be decomposed into disjoint cycles and paths. Roughly, this can be seen by picking a node and constructing a path by following along edges until you either get stuck (and have a path) or reach a node that you have already encountered. In this case you stop and have a cycle. Pick $d \in X \backslash Y$. Since d is not a member of Y, it has degree one, and so is the beginning of some path in our graph. Let m be the other end of this path. Suppose $m \notin Y \backslash X$. In this case, our path contains no doctors in $Y \backslash X$, for if it did contain $q \in Y \backslash X$, we arrive at a contradiction, since q has degree 1. This means that each of the doctors that are included in this path are in $X \cup T$. Consider what happens if we swap the colors of the edges along this path. Now the graph represents the bundle $X \backslash \{d\} \cup \{t\}$ for $t \in T$. Swapping the edges will not change their sum, and so the value of this new bundle is equal to the value of X. This is not allowed since the only difference between the new bundle and X is that the new bundle is missing d since t was already contributing utility as part of the endowment. Now we know that m, the other end of the path, is a member of $Y \backslash X$. Switching the colors along this path constructs a representation of the bundles $X \cup \{m\} \backslash \{d\}$ and $Y \cup \{d\} \backslash \{m\}$. But switching the colors leaves the sum of the edge weights unchanged. Therefore, $V(X \cup \{m\} \backslash \{d\}) = V(X) = V(Y) = V(Y \cup \{d\} \backslash \{m\})$. These new sets are also demanded and we have satisfied strong exchangeability.

Theorem 5. *The EAV conjecture is false.*

Renato and Ostrovsky formulate their counter-example with a graph theoretic interpretation of a valuation. They represent objects as the edges of the complete graph with four nodes. Let S represent the set of goods in this configuration. This configuration is relatively simple and can easily be drawn since $|S| = 6$. The substitutes valuation that is not strongly exchangeable is called r and it is defined so that for $X \subset S$, $r(X)$ is the "size of the largest subset of X containing no cycles"[10]. Then, Renato and Ostrovsky pick two bundles of goods and check against the definition for strong exchangeability. First, they determine two sets of edges that maximize the value of r. Any set that contains more than 3 of the 6 edges will necessarily contain a cycle. So, any subsets that have a value of 3 will be maximizers of r and thus demanded. Let X contain three of the four edges that are not intersected by another edge. Let Y contain the two edges that cross each other and the last of the four edges that is not intersected. Consider edge $i \in X$ where i is an edge connected to a node that is connected to an edge in Y. For strong exchangeability to be satisfied, we must be able to identify an edge of Y that we can trade for i and maintain the same value for r. However, each of the three edges in Y shares a node with an edge of X. So swapping i for any $j \in Y$ will result in a cycle, lowering the value of $r(X) = 3$ to $r(X \backslash \{i\} \cup \{j\}) = 2$. What is left to show is that the function r does satisfy gross substitutes. Renato and Ostrovsky call upon the results that their function r has a special property which is equivalent to being M-concave and M-concavity is equivalent to the gross substitutes condition. Earlier, we discussed the relationship between M-concavity and matroidal structures. Specifically, we said that an M-concave set function on items in N, $f : \mathcal{F} \to \mathbb{R}$ induces a matroid with ground set N and independent sets $\mathcal{F}$. Now, let's try to work in the other direction. A matroid rank function is a function on the independent sets of a matroid that outputs the rank of an independent set.

Definition 14. *Let N be the ground set of a matroid with independent sets in $\mathcal{F}$. The function $f : \mathcal{F} \to \mathbb{R}$ is a matroid rank function if the value of $f(X)$ for $X \in \mathcal{F}$ is equal to the maximum size of an independent subset of X.*

Lemma 2. *Matroid rank functions are M-concave.*

Now, let's consider our function r. We take the ground set $N = \{1, 2, 3, 4, 5, 6\}$ to be the set of edges of the complete graph with four nodes. We take the independent sets to be the subsets of N for which there is no cycle. Then, a matroid rank function f for $(N, \mathcal{F})$, is such that for $X \subset N$, $f(X)$ is equal to the size of the largest independent set in X. We defined independent sets as the subsets of N which contain no cycles. And so, r is consistent with the definition of a matroid rank function. We conclude that r is a M-concave and thus satisfies gross substitutes. Theorem 10 follows directly since we have identified a gross substitutes valuation that is not an Endowed Assignment Valuation.

3.2 Transversal Matroids

The graph theoretic interpretation of Endowed Assignment Valuations used by Renato and Ostrovsky illuminates the matroidal properties of Endowed Assignment Valuations. Endowed Assignmnet Valuations must have a matroidal structure because they are gross substitutes valuations. In fact, EAV's are closely related to the class of transversal matroids. Recall that the class of Endowed Assignment Valuations are those valuations which can be built up from singleton valuations through the aggregation operation and the endowment operation. We will deconstruct EAV's to recover their graphic representation and then relate this structure to the class of transversal matroids.

3.2.1 The Structure of the EAV

Suppose that hospital h has positions in J and is choosing over the a set of doctors D. For each $j \in J$ there is an associated singleton valuation $v_j : D \to \mathbb{R}$ such that $v_j(d) = \alpha_d^j$ is the utility doctor d produces in position j.

Definition 15. *Let $v_j : D \to \mathbb{R}$ be a singleton valuation taking values $v_j(d) = \alpha_d^j$, The graph that is dual to the singleton valuation v_j is the weighted bipartite graph $G_j = (D, \{j\}, E_j, v_j)$ where $E_j = e_{dj} = \{d, j\}$ exactly when doctor d is available to work at position j.*

Remark 1. *We are slightly changing our definition of singleton valuations here with the graph theoretic interpretation. Before, we considered that each doctor from the available set was able to work at each position. Now, we allow for some doctors to be "prohibited" from working at some positions. Hatfield and Milgrom mentioned that their formulation of EAVs was able to take into consideration hospitals' affirmative action programs [5]. This is precisely why they were able to make that claim. Additionally, this change opens the door for our valuations to consider the qualifications of applicants. Some positions within the hospital may be reserved for certain types of candidates. Candidates will not have an edge connecting them to positions they are ineligible for.*

Remark 2. *For the time being, we are going to forget that we are working with weighted matroids and graphs. This is to make is easier to discuss the structure of the mathematical objects. When the weights again become relevent, we will add them back in to our notation.*

The aggregation operation merged singleton valuations with assignment valuations in order to construct valuations for hospitals with multiple positions to fill. The analogous operation on our graphs is to merge them.

Definition 16. *Let $v_i : D \to \mathbb{R}$ and $v_k : D \to \mathbb{R}$ be EAVs with dual graphs $G_i = (D, \{i\}, E_i)$ and $G_k = (D, \{k\}, E_k)$. Let V be the Assignment Valuation attained by aggregating v_i and v_k. Let α^V be the $|D| \times |J|$ matrix with entries α_{dj} denoting doctor d's utility in position j. The bipartite graph $G_V = (D, \{i, k\}, E_i \cup E_k, V)$ is the graph that is dual to valuation V.*

From here on, for an Assignment Valuation V, we will call the graph that is dual to the valuation $G_V = (D, \{i, k\}, E_V, V)$. In constructing G_V, we have constructed a set system. There is a set in the set system for each position $j \in J$ and that set contains all of the doctors in D who are allowed to fill position j.

Definition 17. *Let $V : 2^D \to \mathbb{R}$ be an Assignment Valuation with dual graph G_V. Define $\mathcal{S} = \{S_j : j \in J\}$ where $S_j = \{d \in D : \exists e_{dj} \in E_V\}$.*

Let's take a look at a simple example. Earlier, we considered hospital h_1 and its Assignment Valuation v_1 generated by the matrix α^1.

$$\alpha^1 = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 10 & 5 \end{pmatrix}$$

Let $J = \{A, B\}$ denote the positions for doctors within h_1. Let $D = \{a, b, c\}$ denote the set of doctors that are available to h_1. Let $G = (D, J, E_{v_1}, v_1)$ be the bipartite graph with vertices in D and J and edges connecting each $d \in D$ to each $j \in J$. Dual to our graph is a set system $\mathcal{S} = \{S_A, S_B, S_C\}$ where $S_A = S_B = S_C = D$. Here, the doctors in S_A are exactly those doctors who are available to fill position A. In our simple example, all doctors are available to fill all positions. Sets in $\mathcal{S}$ are dual to the edges that are connected to the positions in J. So, S_A contains elements which are dual to each edge in G that is connected to vertex A. Each edge is weighted according to the utility of doctors in each position. For example the edge $\{a, A\}$ has weight 4 because doctor a provides a utility of 4 when working in position A.

We have constructed a weighted graph and a set system to describe the setup of the Assignment Valuation. We did something similar when we established that gross substitutes valuations can be formulated as weighted matroids. We established that for an M-concave function on subsets of N, $f : \mathcal{F} \to \mathbb{R}$, the matroid induced by f is the pair $(N, \mathcal{F})$. Assignment Valuations are gross substitutes valuations. It follows that Assignment Valuations are M-concave and have a matroidal form. We have begun to construct the matroid induced by our Assignment Valuation in our set system $\mathcal{S}$. For an M-concave valuation, the ground set of the induced matroid is the set of doctors to be chosen from and the independent sets are the subsets of doctors that could be chosen. With an Assignment Valuation, the induced matroid has a bit more structure. The Assignment Valuation chooses subsets of doctors and organizes them so that each doctor chosen is matched to a position. Assignment Valuations induce what are called transversal matroids. Transversal matroids are exactly the class of matroids induced by bipartite graphs [11]. The Assignment Valuation will choose a subset of doctors to hire and determine how to allocate those doctors throughout the hospital by identifying a subgraph of G in which edges connect chosen doctors with the position they will work in.

Definition 18. *Let $G = (D, J, E)$ be a bipartite graph with vertices in D and J and edges in E. The transversal matroid induced by G is $M = (D, \mathcal{F})$ with $S \in \mathcal{F}$ if and only if there exists a subgraph $G_S = (S, J_S, E_S) \subset G$ with the following properties:*

(i) $J_S \subset J$

(ii) There exists a bijection $\psi : S \to J_S$ such that $E_S = \{e_{s'j'} : s' \in S, \psi(s') = j'\}$

for all $s \in S$

The independent sets of a transversal matroid are exactly the sets of doctors for which it is feasible for the hospital to hire. For a set of doctors to be independent, there must be a way to match each doctor from that set to a unique position within the hospital. The collection of edges in E_H forms a bijection between elements of S and some subset of positions in J. An edge between a doctor and a position represents a match between the doctor and position in that permutation of possible assignments.

Let's make our example a bit more complicated to illustrate the significance of independent sets in the associated transversal matroid. Suppose that position i is reserved for surgeons. Suppose that doctor a is the only doctor in D who is a surgeon. This means that doctors b and c are ineligible to work in position a. In the graph G_V induced by this valuation, there will be no edges connecting doctors b and c to position i. Let $M_V = (D, \mathcal{F})$ be the transversal matroid induced by this valuation. In this case $\mathcal{F} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $\{b, c\}$ is not independent. There is no subgraph $H_V = (\{b, c\}, \{i, j\}, E_V)$ of G_V with edges forming a bijection between elements of $\{b, c\}$ and $J' \subset \{i, j\}$. The set $\{a, b, c\}$ is also not independent, because three doctors cannot fill only two positions. For the independent sets $\{a, b\}$ and $\{a, c\}$, the subgraphs of G_V are $H_V = (\{a, b\}, \{i, j\}, \{e_{ai}, e_{bj})$ and $H_V = (\{a, c\}, \{i, j\}, \{e_{ai}, e_{cj}\})$. For the singleton independent sets, the subgraphs contain the single edge connecting the doctor to the position it is allowed to fill.

Lemma 3. *Let V be an Assignment Valuation. Let $G = (D, J, E)$ and $M = (D, \mathcal{F})$ be the bipartite graph and transversal matroid induced by V respectively. Then M satisfies the definition of a matroid.*

This lemma is shown by Stallman in *A Gentle Introduction to Matroid Algorithmics* [11]. The only non-obvious property of the transversal matroid is the rule that if $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there exists $x \in X \backslash Y$ such that $Y \cup \{x\} \in \mathcal{F}$. Stallman operates on the graph of an independent set with what he calls an augmenting path to show that for $X, Y \in \mathcal{F}$ with $|X| = |Y| + 1$, there exists $x \in X \backslash Y$ such that $Y \cup \{x\} \in \mathcal{F}$. The property then follows by induction.

It is time to remember that the transversal matroid representing an Endowed Assignment Valuation is weighted. The weights of the independent sets of the matroid will be equal to the total utility derived by the hospital from the assignment of doctors to positions represented by that set. Recall from the definition of the transversal matroid $M_V = (D, \mathcal{F})$ induced by V, that for each independent set $S \in \mathcal{F}$ there exists a bijection $\psi_S : S \to J_S \subset J$. Each independent set of M_V is a possible assignment of doctors to positions in J. For $S \in \mathcal{F}$ and $s \in S$, $\psi(s) = j$ indicates that under the assignment represented by S, doctor s is selected to work in position j.

Definition 19. *Let V be an EAV on a set of doctors D and set of positions J. Let α^V be the $|D| \times |J|$ matrix with entries $\alpha^V_{dj} = v_j(d)$, the utility of doctor d when working in position j. Let $G_V = (D, J, E_V, w)$ be the weighted bipartite graph induced by valuation V. Let $M_V = (D, \mathcal{F}, W)$ be the weighted transversal matroid induced by valuation V.*

Define $w : E_V \to \mathbb{R}$ such that for each edge $e_{dj} = \{d, j\} \in E_V$, $w(e_{dj}) = \alpha^V_{dj}$.

Define $W : \mathcal{F} \to \mathbb{R}$ such that for each $S \in \mathcal{F}$, $W(S) = \sum_{s \in S} w(e_{s\psi_S})$.

So for each subset of doctors $S \in \mathcal{F}$, $W(S)$ is the utility from selecting the set S of doctors to hire. The assignment valuation maximizes $W(S)$ over the independent sets of the matroid.

Theorem 6. *The matroid induced by an Assignment Valuation is a transversal matroid.*

So far, we have dealt with Assignment Valuations and ignored all those valuations which take into consideration the hospital's endowment of doctors. Closing the set of Assignment Valuations under the endowment operation creates the set of Endowed Assignment Valuations. Just like adding a position vertex to the bipartite graph was equivalent to the aggregation operation, there is a corresponding matroid transformation that will capture the endowment operation. Stallman provides the following definiton of a matroid contraction [11].

Definition 20. *Let $M = (D, \mathcal{F})$ be a transversal matroid. The contraction of the matroid M with respect to $d \in D$ is the matroid $M' = (D \backslash \{d\}, \mathcal{F}')$ such that $X' \in \mathcal{F}'$ if and only if $X' \cup \{d\} \in \mathcal{F}$.*

Essentially, a contraction on a transversal matroid looks like removing one of the vertices in D from the graph. For an Endowed Assignment Valuation, it corresponds to removing a doctor from consideration in the valuation, perhaps because the position doctor is already employed by the hospital. For Endowed Assignment Valuation $V(S|T)$ where T is the set of doctors already employed by S, the contraction operation removes from S any doctors that belong to T. The contraction operation is closed under the class of matroid valuations since each independent set in $\mathcal{F}'$ is a subset of an independent set in $\mathcal{F}$.

Representing the EAV as a weighted cube is a convenient way to grasp exactly how the contraction operation corresponds to the endowment operation on a transversal matroid. Let $C = \{i_1, ...i_{|D|}\}$ such that each $i_j \in \{0, 1\}$ be the cube of dimension $|D|$ with points of the cube representing subsets of doctors. For example, in the three doctor case, the point $\{1, 1, 0\}$ indicates the bundle of doctors that includes d_1 and d_2 and does not include d_3. Each point on the cube is also given a weight that indicates the utility derived from hiring that particular subset of doctors. The independent sets of the matroid as a cube are the points of the cube. The contraction operation removes a doctor d from the ground set of the matroid. On the cube C, this is equivalent to contracting the cube to the $|D| - 1$ cube, where the dimension lost is the one that represents doctor d. Under the endowment operation, we wish to assign a value to set of doctors in the scenario where doctor d is already hired. Then, the contraction of cube C is the set of points on C such that $i_d = 1$. If C' is the contraction of cube C, then $C' = \{i_1, ...i_{d-1}, 1, i_{d+1}, ...i_{|D|}\}$. Now, the endowed assignment valuation on C' evaluates the utility of subsets of doctors under the condition that doctor d is already chosen. The class of transversal matroids is not closed under the contraction operation. So, it is true that the entire class of Assignment Valuations can be described as valuations on a transversal matroid. The class of Endowed

Assignment Valuations is slightly larger, containing all transversal matroidal valuations and those matroidal valuations on a matroid that are the contraction of a that are on the contraction of a transversal.

3.3 An Alternate and More Direct Proof the EAV Conjecture Is False

That EAVs are so closely related to transversal matroids illuminates why the EAV conjecture is false. Recall that the EAV conjecture was that the class of Endowed Assignment Valuations is equal to the class of Gross Substitutes valuations. We understand that the Gross Substitutes property is equivalent to M concavity and the class M concave functions is the class of finite matroid based valuations. Then the relationship between the size of the class of EAVs and the size of the class of Gross Substitutes valuations is similar to the relationship between the size of the set of transversal matroids relative to the set of finite based matroids. Transversal matroids make up only a small subset of the set of all matroids. Therefore, we should not expect EAVs to come close to the size of class of Gross Substitutes valuations.

We move into another field of study to develop a proof of that the EAV conjecture is false that does not require the intermediate step of comparing definitions to an intermediate property (strong exchangeability in the case of Ostrovsky and Leme). The study of combinatorial auctions is interested in describing bidding behavior in an auction for differentiated goods. A bid is another word for valuation. Types of bids are characterized mathematically, often starting with unit demand functions and building more complicated valuations through an aggregation operation. The construction of different types of bids is analogous to the construction of Assignment Valuations and Endowed Assignment Valuations. Of particular interest is the OR and XOR bid. We are interested in the class of OR of XOR bids of singleton valuations. We will call this class of bid the OXS bid. XOR and OR are operations that are performed on valuations in order to construct a more complicated class of valuations. Bidding language differs slightly from the demand language we are used to.

Definition 21. *Let D be a set of items. A singleton bid is a pair (S, p) where $S \subset D$ and $p \in \mathbb{R}$ where p represents the utility from having set S.*

An XOR bid on a set of singleton bids chooses the singleton bid that provides the highest utility and chooses that particular set of items to purchase. The XOR bid is equivalent to the valuation that determines the utility produced by a single position in an EAV. The XOR bid on singletons is a singleton bid itself, outputting a single item and its corresponding utility.

Definition 22. *Let D be a set of items. An XOR bid on a set of bids $(S_1, p_1), ..., (S_n, p_n)$ is (S_i, p_i) such that $p_i \geq p_j$ for all $j \leq n$. The value of the bid is p_i and the set of items purchased is S_i.*

The OR operation is equivalent to the aggregation operation on singleton valuations which gave us the class of assignment valuations. We have a collection of XOR bids, and apply the OR operation so that the value of the aggregated bid is equal to the sum of the XOR singleton bids.

Definition 23. *Let D be a set of items. Let (S_1, p_1) and (S_2, p_2) be bids. Then The OR operation on (S_1, p_1) and (S_2, p_2) creates the bid $(S_1 \cup S_2, p_1 + p_2)$.*

The OR of XOR operation on singleton valuation takes a collection of singleton XOR bids, maximizes their value, and then applies the OR operation to aggregate the total value of the bid as the sum of value of the maximized XOR bids. It is for this reason that the class of OR of XOR bids, which we call OXS bids, is equivalent to the class of Assignment Valuations and is the class of valuations based on transversal matroids. The XOR and OR operations on bids construct different classes of bids/valuations. For example, the XOR of OR operation on singleton valuations reverses the role of OR and XOR in our above construction so that a collection of bids is first aggregated and then maximized over. Lehmann, Lehmann, and Nisan were interested in the relationship between different classes of bids [6]. In particular, they compared the size of the OXS class to the class of Gross Substitutes valuations. We have seen Lemme's example of a Gross Substitutes valuation that is not an EAV. We also have some insight as to why OXS is contained by the class of Endowed Assignmnet Valuations. Indeed, Lehmann, Lehmann, and Nisan demonstrate that the class of OXS valuations is contained by the the class of Gross substitutes valuations through a clever example of a Gross Substitutes valuation that is not in OXS [6]. As we have seen with our work on transversals, the relationship between OXS and EAVs is close. Here, we adapt Lehmann's, Lehmann's, and Nisan's counterexample to the language of Endowed Assignment Valuations to provide an alternate proof that the EAV conjecture is false.

Let $D = \{a, b, c, d\}$. Let $V : 2^D \to \mathbb{R}$ be a valuation such that $V(\{x\}) = 10$ $\forall x \in D$, $V(\{a, c\}) = V(\{b, d\}) = 15$, and $V(S) = 19$ for all other subsets of D. In this instance, we ignore the endowment aspect of the EAV. It is as if we treated each of these values as if the endowment were the empty set. Whether V is an EAV comes down to whether V is an Assignment Valuation. For V to be an Assignment Valuation, there must exist a set of positions J and $|D| \times |J|$ matrix α such that $V(S) = \max_z \sum_{x \in S, j \in J} \alpha_{xj} z_{xj}$.

Suppose matrix α does exist. A few things must be true of α for it to induce valuation V.

1. For each $x \in D$, there must exist $j_x^{10} \in J$ such that $\alpha_{xj} = 10\ \forall x \in D$. This is necessary for the value of singletons to each be equal to 10.

2. For all $x \in D$ and $j \neq j_x^{10}$, $\alpha_{xj} \leq 5$. To see why this is true, without loss of generality, let $x = a$ and suppose $\alpha_{aj} > 5$. Then $V(\{a, c\}) \geq 10 + \alpha_{aj} > 10 + 5$ since it is possible to assign a to position j_a^{10} and c to position j.

However, $V(\{a,b\}) = 19$. This means that there exist $j,k \in J$ such that $\alpha_{aj} + \alpha_{bk} = 19$. According to our two conclusions from before, this cannot be. If $\alpha_{aj} = 10$, then $\alpha_{bk} \leq 5$ and vice versa. If neither α_{aj} nor α_{bk} is equal to 10, then their sum cannot be greater than 10.

Lehmann, Lehmann, and Nisan provide a proof-sketch to illustrate that the valuation V has the gross substitutes property. We will confirm that V satisfies the definition of an M concave function. Recall that for a valuation to be M concave, for all X, Y and $i \in X \backslash Y$, one of the following must be true:

1. $V(X) + V(Y) \leq V(X \backslash \{i\}) + V(Y \cup \{i\})$
2. $V(X) + V(Y) \leq V(X \backslash \{i\} \cup \{j\}) + V(Y \cup \{i\} \backslash \{j\})$.

Whenever X and Y are singletons and whenever $|X| = |Y| = 3$, property 2 is satisfied with equality because all sets of size 1 have the same value and all sets of size 3 have the same value. Whenever $|X| = 3$ pr $|X| = 4$ and $|Y| = 1$, property 2 is satisfied because the left hand side will always be equal to 29 while the right hand side is at least 30. The rest of the cases can be broken down into the following.

- $|X| = 2$ and $|Y| = 1$

  Let $X = \{a,b\}$, $Y = \{a\}$, and $i = b$.

$$V(\{a,b\}) + V(\{a\}) = V(\{a\}) + V(\{a,b\})$$

$$19 + 10 = 10 + 19.$$

  Let $X = \{a,b\}$, $Y = \{c\}$, and $i = a$, $j = c$.

$$V(\{a,b\}) + V(\{c\}) = V(\{b,c\} + V(\{a\})$$

$$19 + 10 = 19 + 10.$$

Let $X = \{a, c\}$, $Y = \{a\}$, and $i = c$.

$$V(\{a, c\}) + V(\{a\}) = V(\{a\}) + V(\{a, c\}).$$

$$15 + 10 = 10 + 15.$$

Let $X = \{a, c\}$, $Y = \{b\}$, and $i = c$.

$$V(\{a, c\}) + V(\{b\}) < V(\{a\}) + V(\{a, b\}).$$

$$15 + 10 < 10 + 19.$$

- $|X| = 2$ and $|Y| = 2$

  Let $X = \{a, b\}$, $Y = \{c, d\}$, and $i = b$, $j = d$.

$$V(\{a, b\}) + V(\{c, d\}) < V(\{a, d\}) + V(\{c, b\}).$$

$$19 + 19 = 19 + 19.$$

  Let $X = \{a, c\}$, $Y = \{b, d\}$, and $i = c$, $j = b$.

$$V(\{a, c\}) + V(\{b, d\}) < V(\{a, b\}) + V(\{b, c\}).$$

$$15 + 15 < 19 + 19.$$

- $|X| = 3$ and $|Y| = 2$

  Let $X = \{a, b, c\}$, $Y = \{a, b\}$, and $i = c$.

$$V(\{a, b, c\}) + V(\{a, b\}) = V(\{a, b\}) + V(\{a, b, c\}).$$

$$19 + 19 = 19 + 19.$$

Let $X = \{a, b, c\}$, $Y = \{a, c\}$, and $i = b$.

$$V(\{a, b, c\}) + V(\{a, c\}) = V(\{a, c\}) + V(\{a, b, c\}).$$

$$19 + 15 = 15 + 19.$$

Let $X = \{a, b, c\}$, $Y = \{d\}$, and $i = a$.

$$V(\{a, b, c\}) + V(\{d\}) < V(\{b, c\}) + V(\{a, d\}).$$

$$19 + 10 < 19 + 19.$$

Let $X = \{a, b, c\}$, $Y = \{d\}$, and $i = b$.

$$V(\{a, b, c\}) + V(\{d\}) < V(\{a, c\}) + V(\{b, d\}).$$

$$19 + 10 < 15 + 15.$$

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