Quasilinear Utility and the Substitutes Condition

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Abstract

This paper investigates the relationship between various notions of the gross substitutes condition on utility and explores a generalization of gross substitutes that does not require quasilinearity. It connects various mathematical settings in which gross substitutes appears and compares the results and assumptions from different settings. Of specific interest is a generalized version of the Gale-Shapley matching algorithm and the role the substitutes condition on preferences plays in the algorithm's convergence to a stable outcome. It considers a class of endowed assignment valuations and uses matroid algebra to demonstrate that this class is strictly smaller than the class of gross substitutes valuations.

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1 Economics Background

Labor-matching programs offer job hunters and employers an alternative to the standard application process. A labor-matching program is a formal system through which job-hunters are matched with potential employers. At the heart of these systems is a two-sided matching algorithm which intends to pair employees with employers in an optimal way. The National Resident Matching Program serves as a model for how a labor market might be allocated algorithmically. The program has undergone changes since its introduction in 1951, but the principle of it has stayed the same; medical students who are on the hunt for residency at a hospital may voluntarily agree to allow the NRMP to match them with a hospital according to a deferred matching algorithm. Doctors submit their preferences of hospitals while hospitals submit a ranking of doctors and their employment quotas NRMP. Then, employment decisions are made simultaneously by the program, reducing the extreme inefficiency of the market when it was navigated by hospitals and doctors independently.

The market for medical interns has motivated theoretical scrutiny of labor-matching algorithms in both combinatorics and labor economics because it is so well suited for a matching program. Before the implementation of the NRMP, fierce competition between hospitals to recruit top candidates fractured the application process. Hospitals tried to out-bid each other by making job offers to medical students early in their education, so that high-performing students received offers well before completion of their studies. Moreover, students were asked to commit to a program before they could find out whether they would be considered for other positions, leading many doctors to feel disappointed when they had accepted a post early on in the application process only to miss out on a better offer that came about later.

Consequently, hospitals' hiring efforts were frustrated by a lack of commitment from doctors who would accept a position only to renege when presented with a better offer and doctors were unable to make informed employment decisions when they were presented with offers at different times. The NRMP was able to heal this market by ensuring that offers were made simultaneously. The college admissions process presents analogous inefficiencies when considering early-admissions. Parallels can be drawn between the roles of hospitals and universities, and doctors and students. General firm-worker relationships can be studied in the same context, however most job markets are successful in freely allocating labor and do not warrant the use of a formal program.

The algorithm that underpins a labor-matching program uses agents' personal preferences as the basis for how to allocate the market. This means that in a model of a labor-matching program, agents' personal preferences must be formalized so that they can be used as an input of the algorithm. Preferences can be thought of broadly as characterizations of agents' economic behavior. A standard way this is accomplished is by assigning a utility function to each agent in the market so that the algorithm's goal is to create matches between agents that maximize utility. In this setting, the theoretical consideration is that of a two-sided matching process taking place in a market where agents are utility maximizers. Study in this field is concerned with the conditions necessary and sufficient for the market to converge to a desirable outcome. The gross-substitutes condition on utility functions has been identified in a variety of matching/auction settings as sufficient to guarantee convergence of the market to a stable equilibrium and in some cases to an outcome that is optimal for the agents involved. Gross substitutes are of particular interest because if these valuations actually characterize economic behavior, auction settings like labor-matching programs can be successfully employed in the real world. Moreover, we are interested in the assumptions and restrictions we must place on labor-market models in which algorithmic mechanisms allocate the market.

Of interest is whether the class of gross substitutes valuations encompasses the entire range of economic behavior that can be used as an input for a labor-matching algorithm. A variety of equivalent characterizations of this broad class of gross-substitutes valuations has been proposed in an array of mathematical settings. However, each of these characterizations of gross-substitutes imposes the assumption that agents in the market are operating according to some utility function. We examine a setting that has been proposed to generalize the study of labor-matching processes so that the agents in the market do not need to be treated as utility maximizers. Study of Hatfield's and Milgrom's Generalized Gale-Shapley algorithm motivates questions about what ways this model is indeed less restrictive on the type of economic behavior that can be used in a model for labor-matching and how models of gross substitutes valuations are encompassed. We attempt to unify the treatment of labor-matching algorithms from different mathematical settings so as to extend results from different contexts to enrich and complete our understanding of labor-matching algorithms and the characterizations of economic behavior they may be based upon.

1.1 Stable Matching, Optimality, and Why We Care

1.1.1 Stability

Success of a labor matching algorithm requires that participants will respect the match's assignment for them. Hospitals and Doctors are freely acting economic actors and have complete agency in determining their own outcomes. An algorithm is only viable if its outcome determines the actual distribution of labor in the market. A matching in which hospitals and doctors do not feel it is in their best interests to obey is not viable. In other words, we demand the property that an algorithm is effective in assigning a permanent labor allocation in the market. Consider the following example of a matching that is not effective:

Hospital A, Hospital B, Doctor 1, and Doctor 2 are participants in the matching algorithm. The algorithm pairs Doctor 1 with Hospital B and Doctor 2 with Hospital A. However, Doctor 1 and Hospital A each prefer each other to their respective matches. In the absence of a contractual obligation to stick with the algorithm's match, Doctor 1 will leave Hospital B to work for Hospital A, and Hospital A will replace Doctor 2 with Doctor 1 on its roster.

We call the mutual action taken by Hospital A and Doctor 1 a *pareto-improvement*. Then, a match is feasible if no pareto-improvements are possible in that match. Doctor d will only choose to leave his assigned hospital if he can get a job at a hospital he likes better than the one he is matched to. But if no pareto-improvements are possible, all hospitals that doctor d would consider switching to prefer each doctor in its current assignment to doctor d. We call a match in which no pareto-improvements are possible a *stable* match.

Definition 1. *A stable match is a matching of hospitals to doctors such that there does not exist doctors d and d' and hospitals h and h' such that d is matched to h, d' is matched to h' but d prefers h' to h and h' prefers d to d'.*

In order for our algorithm to be successful in assigning labor in such a way that participants will respect, it is a necessary and sufficient condition that the algorithm always produces a stable match. Consider what happens when hospitals and doctors are assigned to a stable match. Each doctor will remain matched to his assigned hospital unless he has the opportunity to work for a hospital he likes better. But, in a stable match no doctor will have the opportunity to work for a preferred hospital since each hospital a doctor prefers likes its own candidates better. Under a stable match, the market is in equilibrium and the market outcome is dictated exactly by the algorithm.

Now, we consider why stability is necessary for an algorithm's outcome. If hospitals and doctors are assigned to an unstable match and are allowed to move freely within the market, they will act upon pareto-improvements and disregard the algorithm's match. However, it is possible to deny hospitals and doctors the opportunity to rearrange themselves, thus forcing even an unstable match to be obeyed. In fact, the participants in the NRMP are contractually obliged to respect the program's decision. But this raises the question of why doctors and hospitals would even consider participating in a program that stripped them of their own agency. The NRMP boasted participation rates of up to 95% in the early years of its implementation. However, the NRMP's match is always stable, and the program would not have been able to attract high rates of participation without this guarantee [3]. To illustrate, suppose the NRMP could not make a stability guarantee. In this case, participants in the NRMP would understand that there is a possibility that contractually obligating themselves to the match might restrict them from being able to achieve a better outcome for themselves and they would not enter into the program in the first place. We have identified "always produces a stable match" as a requirement for an algorithm to be effective at assigning labor. However, producing a stable outcome is not sufficient on its own for an algorithm to be worthy of implementation.

1.1.2 Optimality

Letting the market for medical interns play out freely is always an alternative to using an algorithm to distribute labor. For an algorithm to be justifiably implemented, the algorithm must provide an outcome that is better than the free-market alternative. Moreover, we cannot enforce participation in a labor-matching program. In order to entice doctors and hospitals to take part in the program, the program's algorithm must promise them at least as good of an outcome as they expect to obtain in the free market. With this in mind, we require that an algorithm produces an optimal outcome so as to be compatible with the interests of utility-maximizing participants and to serve the market as a whole.

Optimality may take on a variety of meanings. Usually we think of optimality in terms of maximizing market surplus or making agents as well off as possible. However, it is an economic principle that increasing the surplus of the producer necessarily decreases the surplus of the consumer and vice-versa. This means that in defining a version of optimality, we must make a determination about how to weigh hospital and doctor well-being against each other. In a general labor market, this decision might not be so clear. There are arguments to be made that firms' surplus should be emphasized so as to engender economic activity and productivity. Alternatively, workers have the right to fair wages and working conditions, and their happiness will improve the quality of their work.

In this specific configuration where we are matching student-doctors to residency programs at hospitals or in the analogous setting of the college application process, the market should be designed to prioritize the well-being of the doctor/student. After all, the goal of a residency program is to train residents to become great doctors. Similarly, the goal of a school/university is to produce great minds through a quality education. Moreover, it is a greater challenge to convince individual doctors and students to sign away their free will in deciding where to work and study. The decision to hire/accept is much less personal for hospitals and universities than it is for individuals who are deciding where they will live and work. In addition, hospitals, universities, and most firms are large institutions looking to fill a large number of vacancies; they are already incentivized to participate in a matching program because it saves them administrative hassle.

With this in mind, we propose the following definition of an optimal algorithm for hospital-doctor matching.

Definition 2. *A matching algorithm is doctor-optimal if*

i) it is guaranteed to produce a stable match

ii) the stable match it produces makes each doctor at least as well-off as he would be under any other stable assignment.

In other words, each doctor is as well off as possible. Any hospital he prefers to his match will never employ him, because employment only takes place when the market is in equilibrium. If an algorithm's match is doctor-optimal, it can guarantee to make doctors at least as well off as they would be in a perfectly competitive market setting. If doctors believe the free market operations to be inefficient, then they believe the matching algorithm will make them strictly better off than they would be otherwise.

1.1.3 Introduction to Substitutes

It is clear that any algorithm that is used to assign labor must be such that their output is a stable and optimal match. And so in constructing any model of the labor market, we are interested in what restrictions we must place on the preferences of the economic actors which serve as the algorithm's input. In a variety of algorithmic settings, the gross substitutes condition on agents' utility function has been lauded as the most general restriction which yields the desired result. We will study a more generalized substitutes condition in which the profile of preferences for which the algorithm succeeds is broader than the set of preferences which can be modeled by a utility function.

Before we begin to technically define gross substitutes and substitutes, we should develop the economic intuition behind the conditions so as to determine whether imposing the substitutes condition on economic actors in the model is appropriate. The term *substitutes* refers to the way in which hospitals prefer doctors to each other. From consumer theory, goods are substitutes whenever they essentially serve the same purpose; we are somewhat indifferent between which particular good we have.

Consider the substitutes condition in the market for medical interns. We should expect that hospitals treat doctors as substitutes to one another. Doctors generally serve the same purpose to hospitals as one another. They are units of labor with similar training and abilities and so should be substitutes. If a hospital were to violate the substitutes condition, there would have to be a doctors that a hospital only likes when they are hired together. Any such situation is unlikely since demand for doctors is driven by doctors' individual ability and skills. The hospital may be interested in candidates who work well with others, but teamwork abilities will be assessed in general, not just on how well one works with a single "friend". Now that we have affirmed that the substitutes condition is important and appropriate for our model, we can begin to discover the mathematical properties of substitutes.

2 Technical Background

2.1 Intro to Gross Substitutes

We are unifying concepts from a variety of settings. This means that we require notation that encompasses them all. Let's consider an auction setting in which hospitals are bidding for doctors. Let H denote the set of hospitals and D denote the set of doctors. We describe doctors' wages through a price vector $\overline{w} \in \mathbb{R}^{|D|}$. Then w_d is the price hospitals must pay to hire doctor d. Let wages be fixed by $\overline{w}$. The gross substitutes condition is imposed on hospitals' preference behavior. We must construct some mathematical framework to well-define the preference behavior. Conventionally, this is achieved by equipping each hospital with a real-valued utility function on the power set of available contracts. Let $X = D \times H$ and choose $h \in H$. Then $u_h : 2^X \to \mathbb{R}$ is hospital h's utility function. We recover h's chosen sets of contracts at a given price with a demand correspondence. A hospital's chosen set is the set of contracts it demands at a given price.

Definition 3. *Let $X' \subset X$ and $\bar{w} \in \mathbb{R}^{|X|}$. Let $u_{h^*} : 2^X \to \mathbb{R}$ be hospital h^*'s utility function. Then*

$$C_{h^*}^{\bar{w}}(X') = max_{S \subset D'}\{u_{h^*}(S) - \sum_{d \in S} w_d\}$$

is h^*'s chosen set of contracts from X' subject to $\bar{w}$ and is such that $\forall x \in C_{h^*}^{\bar{w}}(X')$, $x = (d, h^*)$. Also

$$C_H^{\bar{w}}(X') = \bigcup_{h \in H} C_h(X')$$

is the aggregate chosen set of hospitals in H from the set of available contracts X'.

Definition 4. *Let $\bar{w} \in \mathbb{R}^{|D|}$ be a vector of wages for doctors. Then $\bar{w}^{+d^*} \in \mathbb{R}^{|D|}$ is such that $w_d = w_d^{+d^*}$ for all $d \neq d^*$ and $w_{d*}^{+d^*} > w_{d^*}$.*

Now we are ready to define the gross substitutes condition. A hospital's preferences satisfy gross substitutes if demand for each doctor is non-decreasing in the wages of other doctors. In other words, a hospital's preferences satisfy gross substitutes if increasing the wage of any one doctor does not decrease the demand for any other doctor.

Definition 5. *Hospital h has preferences that satisfy the gross substitutes condition if there does not exist a doctor d^*, wage vector $\bar{w}$, and set of contracts X' such that $x = (d, h) \in C_h^{\bar{w}}(X')$, $d \neq d^*$, and $x = (d, h) \notin C_h^{\bar{w}^{+d^*}}(X')$.*

2.2 Stable Matching and Competitive Equilibrium

2.2.1 Stability Condition

Our intention is to construct an algorithmic model which takes agents' preferences as inputs and outputs a designation of labor. In other words, we are solving a two-sided matching problem in which elements of H may be paired with multiple elements of D, while elements of D may only match with one element from H. Hatfield and Milgrom introduced the contract as convenient mathematical object to formalize a matching of doctors and hospitals [5]. Each potential match between a doctor and a hospital is a contract, $x \in H \times D$. If $x = (h_n, d_m)$, we write $x_h = h_n$ and $x_d = d_m$. A set of contracts, $X' \subset H \times D$, is analogous to an assignment of doctors to hospitals in that $x \in X'$ implies doctor x_d is employed by hospital x_h in assignment X'. An assignment is not allowed to match the same doctor with different hospitals.

Definition 6. *A set of contracts X' is an assignment of doctors to hospitals if for all $d^* \in D$, if there do not exist contracts $(x^*_d, x_h), (x^*_d, x'_h) \in X$ such that $h \neq h'$.*

This language is appropriate since a contractual agreement is drawn up whenever a doctor is hired by a hospital. The contract is a concise and convenient way to include the details of a job offer. For example, the contract can be extended to include wage, and thus can provide a much more realistic representation of the labor market. Doctors are able to make decisions based on their preference of hospital and the salary offered. Definition 6 establishes that our solution will be a n to 1 two-sided match.

Definition 7. *A set of contracts $X' \subset X$ is a stable allocation at wages $\bar{w}$ if*

- $C_D^{\bar{w}}(X') = C_H^{\bar{w}}(X') = X'$
- *There does not exist hospital h and set of contracts $X'' \neq C_h^{\bar{w}}(X')$ such that*

$$X'' = C_h^{\bar{w}}(X' \cup X'') \subset C_D^{\bar{w}}(X' \cup X'').$$

Definition 7 defines the stability condition in the context of Hatfield's and Milgrom's matching model. Recall that a stable outcome is one in which no hospital-doctor pair can mutually improve by swapping out their assigned match. Hatfield and Milgrom present the stability condition in terms of sets of contracts [5]. Notice the similarities between Hatfield and Milgrom's stable assignment and a market that has achieved competitive equilibrium. We see below how pareto-improvements are not allowed in Hatfield and Milgrom's definition of stability.

Lemma 1. *Suppose $X' \subset X$ is such that $C_D^{\bar{w}}(X') = C_H^{\bar{w}}(X') = X'$. Then no hospital or doctor would rather reject a contract in X' he is involved in.*

Proof. Let $x = (d, h) \in X'$. Then x is in the chosen set for some hospital. Moreover, each contract can only be in the chosen set of a single hospital because hospitals do not choose contracts they are not involved in. The same is true for doctors. □

Lemma 2. *Suppose that there exists h and set of contracts $X'' \neq C_h(X')$ with $X'' = C_h^{\bar{w}}(X' \cup X'') \subset C_D^{\bar{w}}(X' \cup X'')$. Then it is true that h strictly prefers X'' to X' and each doctor weakly prefers X'' to X'.*

Proof. Since X'' is chosen from a set of contracts containing X', hospital h strictly prefers X'' to X'.

To demonstrate weak preference for X'' by the doctors, we must show that no doctor prefers to reject a contract in X'' whenever both X'' and X' are available. For all $x \in X''$, we have $x \in C_d^{\bar{w}}(X' \cup X'')$ for some $d \in D$. Since hospitals do not accept or reject contracts that do not involve them, it is true that x is not rejected by any doctor. □

2.2.2 Competitive Equilibrium

Perhaps a more familiar definition of stability is that of market competitive equilibrium. Competitive equilibrium is the desired outcome of the Kelso-Crawford labor matching system. Kelso and Crawford's system first motivated the study of gross substitutes valuations by identifying the gross substitutes condition as being necessary and sufficient for their labor matching market to achieve competitive equilibrium [9]. Hatfield and Milgrom cite the Kelso-Crawford labor matching algorithm as the motivation for their study of the more general version of the gross substitutes condition [5].

Definition 8. *Consider a set of bidders H in a market where a set of indivisible goods, A, is being offered. For each $h \in H$, let u_h denote the utility function of bidder h. This market is said to achieve competitive equilibrium at price vector $\bar{p}$ if there exists a partition of the set of goods such that $A = \bigcup_{h \in H} S_h$ with $S_h \in C_h^{\bar{p}}(A)$.*

2.2.3 Compare/Contrast Definitions

It will be useful for us to go between the alternate definitions of stability/equilibrium when we compare results in Hatfield and Milgrom's general setting with properties of gross substitutes. At first, it may seem like the definitions are not compatible since competitive equilibrium captures a one-sided market. A stable assignment is defined in terms of both hospitals and doctors being satisfied with their match. Imagine that doctors are no longer active participants in the market. The market for contracts is now one-sided and stability boils down to an assignment having the properties that $C_H^{\bar{w}}(X') = X'$ and that there does not exist X'' with $X'' = C_h^{\bar{w}}(X' \cup X'')$. Since $C_H^{\bar{w}}$ is the aggregation of the hospital's chosen sets, we recover most of the definition for competitive equilibrium. We lack that $C_H^{\bar{w}}(X')$ is a partition of X'. But recall that each contract can only be chosen by the hospital it is associated with. Our chosen sets will never overlap. The same argument works for doctors if we would like to imagine that they are the bidders. Hatfield's and Milgrom's stability condition on sets of contracts is an intuitive extension of competitive equilibrium to a two sided market. It prevents agents from making pareto improvements and ensures that each agent is allocated a set of contracts it is willing to choose.

2.2.4 The Important Result

The idea behind competitive equilibrium in a free market is that the economy will sort through goods and resources and be able to achieve an efficient outcome on its own. The way this happens is by agents making mutually beneficial trades and prices adjusting until each agent is as satisfied with his bundle as he would be with any other. Gross substitutes valuations play a key role in determining which economies will be able to reach a stable equilibrium through a natural price setting/good allocating mechanism.

Algorithm 1: Greedy Algorithm

Result: S_i for all $i \in H$

input: $\delta > 0$ $v_i^{\bar{w}}$ for each $i \in H$

initialization: $S_1 = D$, $S_i = \emptyset$ for all $i \in H$, $i \neq 1$, $w_d = 0$ for all $d \in D$;

while $\exists S_i \notin C_i^{\bar{w}}(D)$ **do**

  Find $X_i \in C_i^{\bar{w}}(D)$. Change prices so that $\forall d \in X_i \backslash D_i$, $w_d = w_d + \delta$

  Change allocations so that $D_i = X_i$ and $D_j = D_j \backslash X_i$ for $j \neq i$

end

The Greedy Algorithm is meant to simulate what happens naturally in an economy. Kelso and Crawford introduced this version of the greedy algorithm to describe a market for indivisible goods [9]. To begin, goods are allocated arbitrarily and prices are fixed at 0 dollars. In this case, all of the goods are owned by one agent. As long as there is an agent who does not demand the bundle of goods allocated to them, the market gets updated. For the unhappy agent, we find a bundle that he does demand and give it to him. The price of each good in that demanded bundle is increased by δ. What should happen is that each good will end up with the agent that likes it most, because that agent will be willing to pay for the good at a higher price than anyone else. A market consisting of gross substitutes valuations is the condition identified by Kelso and Crawford to guarantee that this greedy algorithm will eventually converge to a competitive equilibrium [9]. The greedy algorithm captures the classic economic intuition behind supply and demand. When prices are low, excess demand drives prices upwards. This phenomenon is captured by updating the prices (wages of doctors) $w_d = w_d + \delta$ at each iteration.

Theorem 1. *If $v_i^{\bar{w}}$ satisfies the gross substitutes property for all $i \in H$, then the Greedy Algorithm will terminate in a finite number of steps to a market that achieves competitive equilibrium at some price $\bar{w}$.*

A proof of theorem 1 is elegantly presented in Renato Leme's algorithmic survey of the gross substitutes condition [7].

2.3 M-Concavity

So far, we have committed to the assumption that hospital preferences are modeled by a utility function. However, notice that our definition of gross substitutes does not explicitly mention a utility function. It merely places restrictions on the chosen sets of goods which are utility maximizers for the agent. If we are interested in studying gross substitutes valuations, we would to define them in terms of the properties of the valuation itself. In essence, the hospital's choice function is a discrete optimization problem. Murota in *Discrete Convex Analysis* presents the following definition of an M-concave function [8]. That M-concave functions and utility functions in the class of gross substitutes preferences are equivalent will allow us to extend results from convex analysis to our setting and help uncover algebraic properties of gross substitutes preferences. Discrete convex analysis is the name given to the study of convex analysis and matroid theory combined [8]. Discrete convex analysis is concerned with a class of real valued set functions that is equivalent to the class of gross substitutes valuations.

Definition 9. *Let N be a set of n items. Let $\mathcal{F}$ be a family of subsets of N. Then a set function $f : \mathcal{F} \to \mathbb{R}$ is M-Concave if for any $X, Y \in \mathcal{F}$ and $i \in X \backslash Y$,*

- *(i)* $(X\backslash\{i\}), (Y \cup \{i\}) \in \mathcal{F}$

  (ii) $f(X) + f(Y) \leq f(X\backslash\{i\}) + f(Y \cup \{i\})$ *or*

- *(i)* $\exists\ j \in Y\backslash X$ *such that* $(X\backslash\{i\} \cup \{j\}), (Y \cup \{i\}\backslash\{j\}) \in \mathcal{F}$

(ii) $f(X) + f(Y) \leq f(X\backslash\{i\} \cup \{j\}) + f(Y \cup \{i\}\backslash\{j\})$.

2.3.1 Matroids

Definition 10. *A matroid M is a pair $(E, \mathcal{I})$ where E is called the ground set and $\mathcal{I}$ is a collection of subsets of E called the independent sets and the following are true:*

$$(i)\ \emptyset \in \mathcal{I}$$

$$(ii)\ \forall A' \subset A \subset E, \text{ if } A \in \mathcal{I} \text{ then } A' \in \mathcal{I}$$

$$(iii)\ \text{if } A, B \in \mathcal{I} \text{ and } |A| > |B|, \text{ then there exists } x \in A\backslash B \text{ such that } B \cup \{x\} \in \mathcal{I}.$$

M-concave functions have a matroidal structure. They are called matroid based valuations.Let's check that our M-concave structure is actually a matroid.

Lemma 3. *Let N be a set of n items and $\mathcal{F}$ a family of subsets of N. Let $f : \mathcal{F} \to \mathbb{R}$ be M-Concave. Then the pair $(N, \mathcal{F})$ is a matroid.*

Proof. We take property (i) for granted.

We check property (ii). Let $X' \subset X \subset N$ with $X \in \mathcal{F}$. We know that X is discrete and so $X = X' \cup \{x_1, ..., x_n\}$ with each $x_i \notin X'$. M-concavity requires that $\forall X \in \mathcal{I}$ and all x_i for $i = 1, ...n$ that $X\backslash x_i \in \mathcal{F}$. Starting with $X\backslash x_1 = X_1 \in \mathcal{F}$ inductively remove one element at a time from X until we achieve $X' = X_n \in \mathcal{F}$.

To see that (iii) holds, note that $|X| > |Y| \implies \exists\, x \in X\backslash Y$. M-concavity requires that $\forall\ X, Y$ and $x \in X\backslash Y$ that $Y \cup \{x\} \in \mathcal{F}$. $\square$

2.3.2 Translating to Demand Theory

We must ensure that it makes sense to talk about hospital preferences in terms of an M-concave function and matroids. Take X, the set of contracts, as the ground set for our matroid. Let $X' \subset X$ be a set of contracts available to hospital h. Then the family of independent sets, $\mathcal{F}$, is a set of possible assignments. The utility function $u_h^{\bar{w}}$ sends bundles of contracts to $\mathbb{R}$. Using the above translation, we can apply the equivalence of gross substitutes and M concavity to hospital preferences. The relationship between gross substitutes valuations, M concave valuations, and matroids is investigated by Renato Paes Leme in his algorithmic survey where the following theorem is proven [7].

Theorem 2. *The gross substitutes condition is equivalent to M-concavity on utility functions.*

2.4 Quasilinearity

So far, we have adhered to the utility-maximizing theory of demand which contends that purchasing behavior is governed by a utility function that is maximized by picking an optimal bundle of goods. Preferences that are determined by a convex utility function are called quasilinear. They are the preferences that are most commonly studied in economic theory. However, there are profiles of demand behavior that are impossible to describe with a utility function. We will investigate how key results in matching theory differ when preferences are not assumed to be quasilinear. In this sense, we will identify quasilinearity as somewhat of a necessary condition for certain useful results about matching algorithms that are applied broadly throughout the literature.

We say that a bidder's preferences are dictated by utility function u if preferences are defined such that $u(a) > u(b) \iff$ good a is preferred to good b. To understand a bidder with quasilinear preferences, we need to update our understanding of utility functions. The bidder will derive overall utility from both the goods he has purchased and the amount of money he has. This means that we can include the price of a good in the agent's utility function. To formalize, let $\bar{a} = \{a_1, ..., a_2\}$ be a bundle of goods and let $\bar{p}$ be a price vector. Let $u(\bar{a}^{\bar{p}})$ be representative of amount of utility the bidder gets from owning good a. This utility function takes into account how much money the bidder has by subtracting the price of the good he purchases from his utility. The bidder's preferences are quasilinear if the bidder's utility function can be decomposed into two separate components. One component calculates the utility from the goods owned. The other component subtracts off the price that must be paid for the goods.

Definition 11. *An agent's preferences are quasilinear if they can be represented by a utility function that is linear in price. That is to say, agent's preferences are described by a utility function such that for all bundles of goods, $\bar{a}^{\bar{p}} = \{a_1, ..., a_n, p(\bar{a})\}$,*

$$u(\bar{a}^{\bar{p}}) = v(\bar{a}) - p(\bar{a})$$

for a convex function $v : A \to \mathbb{R}$.

In economics, the quasilinear property is captured by requiring that indifference curves are convex and parallel. Quasilinearity requires that the way that bundles of goods are valued is independent of the price of the bundles. The convexity requirement encompasses the economic notion of diminishing marginal returns. So far in our configuration of hospital demand we have been dealing with quasilinear preferences since choice behavior is such that $C_{h^*}^{\bar{w}}(X') = \max_{S \subset D'}\{u_{h^*}(S) - \sum_{d \in S} w_d\}$.

2.4.1 Preferences Without Quasilinearity

It is rare to find economics literature dealing with preferences that are not assumed to be quasilinear. The concept of prices is that they are a unit of measurement by which to compare the worth of different things. That preferences are linear in price means that the way goods are valued does not change with respect to how much they cost. For example, my car gives me the same amount of utility that it would if I had paid more for it. However, in the hospital-doctor matching setting Hatfield and Milgrom constructed a functional model of the market for medical interns that in which hospital preferences are not required to be quasilinear.

Preferences that are not quasilinear cannot be defined by a utility function. Instead, non-quasilinear preferences are defined in terms of an economic agent's choice behavior. For example, let h be a hospital with preferences that are not quasilinear. Hospital h's preferences are defined by the law of revealed preference. From a set of contracts, the hospital chooses its favorite subset of those contracts to sign. The contracts that h chooses to sign are preferred to the contracts that are not chosen. The choice function defines h's preference. Unlike in the quasilinear case, there is no explicit rule for determining which contracts will be chosen. Hatfield and Milgrom use a very general definition for these types of preferences [5]. Each hospital, h, has choice behavior that is essentially unrestricted, except we require that h only chooses contracts for which it is named in. In other words, hospital h is only allowed to choose contracts $x = (x_d, x_h)$ such that $x_h = h$. It doesn't make sense for a hospital to choose a contract that names a different hospital. Hatfield and Milgrom conveniently defined choice behavior as a mechanism to reveal hospital preferences without referring to price. The hospital choice function will still apply when price is reintroduced, but for now we have a simple and general way to model preferences that need not be quasilinear.

Definition 12. *Let X be a set of contracts. Let X' be a set of contracts available to hospital h^*. Hospital h^*'s chosen set given X' is $C_{h^*}(X') \subset X'$ with the restriction that $\forall x \in C_{h^*}(X')$, $x = (d, h^*)$.*

3 Hospital-Doctor Matching and the Generalized Gale-Shapley Algorithm

3.1 The Deferred Matching Algorithm

D. Gale and L.S. Shapley introduced their "Deferred Matching Algorithm" in 1962 as a way of solving a two-sided matching problem in which agents of each group has a preference ranking on the other group [1]. The Deferred Matching Algorithm was motivated by the college admissions process. The algorithm matches students to colleges in such a way that is desirable to each college and each student. Gale and Shapley introduced the Deferred Matching Algorithm as a generalization of the stable marriage algorithm which creates a 1-1 matching between a set of "men" and "women" according to agents' tastes in a marital partner. The "Deferred Matching Algorithm" is an extension of the stable marriage problem in that it facilitates a pairing of elements from two disjoint sets, however it allows elements of one set to be matched with multiple elements from the other. In this way, Gale and Shapley presented a model of a standard admissions process in which applicants may only attend one college or work for one hospital whereas colleges or hospitals are able to admit/hire many applicants. The National Resident Matching Program has an equivalent task of matching doctors with hospitals, and the"Deferred Matching Algorithm" is identical to the process used by the NRMP to assign hospital-doctor pairs [4].

The formal algorithm takes a set of doctors D, a set of hospitals H, and each agent's preferences as inputs in order to facilitate a fair and efficient allocation of the market for medical interns. In the original configuration of the Deferred Matching Algorithm, preferences are defined by a ranking. Each hospital has a ranking on the set of doctors and each doctor has a ranking on the set of hospitals. Each hospital has a maximum number of doctors it wishes to hire. The preferences of hospital h are denoted by $>_h$ and q_h is the quota of hospital h. In order to keep track of the matching of doctors to hospitals, each hospital is given a waitlist W_h containing the doctors it chooses to hire. At the algorithm's initialization, each waitlist is empty. As the algorithm runs, wait lists get filled until a stable match is obtained.

3.1.1 The Market for Medical Interns

A breakdown of the Deferred Matching Algorithm demonstrates that the algorithm's process mimics a real-world application cycle. Gale and Shapley devised a configuration in which hospital and doctors preferences were submitted as a ranking of individual hospitals and doctors:

(i) Each doctor applies to its"favorite" hospital.

(ii) Each hospital receives a number of applications. Each hospital selects its favorites and adds those doctors to its waitlist. All other doctors are rejected. If a hospital receives fewer applications than its quota, it rejects no doctors and has open positions.

(iii) Doctors who are currently on a waitlist do nothing. Each doctor who was rejected by its favorite hospital applies to its second favorite hospital.

(iv) Again, each hospital adds its favorites to its waitlist. This time, the hospital must choose between all of the applicants from this round and from the first round. This means that a doctor who is on a waitlist may later be removed from that waitlist if someone better comes along. How disappointing!

The process continues until all doctors are either on a waitlist or have applied to all of the hospitals he is willing to work for. The final waitlists become the assignment of doctors to hospitals produced by the Deferred Matching Algorithm. We can describe the Gale Shapley algorithm in more generality through the language of chosen sets. Let H denote the set of hospitals and D denote the set of doctors. Without loss of generality, suppose that each doctor would rather work for all $h \in H$ than have no work and that each hospital would rather hire each $d \in D$ than have a vacancy. For each $H' \subset H$ let $C_d(H') = h$ such that h is doctor d's most preferred hospital from H'. For each $h \in H$ let q_h be h's quota, the number of doctors it wants to hire. For each $D' \subset D$, let $C_h(D')$ be hospital h's favorite q_h doctors in D', supposing that D' contains at least q_h doctors. If the number of doctors in D' is less than q_h, $C_h(D')$ consists of all doctors in D' and $|C_h(D')| < q_h$.

Let A_h^i be the set of doctors that applied to hospital h in iteration i. Let W_h^i be hospital h's waitlist at iteration i. Let R_h^i be the set of doctors that hospital h has rejected applications from through iteration i. Let H_d^i be the set of hospitals that have not yet rejected doctor d through iteration i.

Algorithm 2: Gale Shapley Algorithm

Result: W_h^i

input: H, D, $C_d(H')$ $\forall d \in D$ and $H' \subset H$, $C_h(D')$ $\forall h \in H$ and $D' \subset D$

initialization: $R_h^0 = \emptyset$ $\forall h \in H$, $A_h^0 = \emptyset$ $\forall h \in H$, $W_h^0 = \emptyset$ $\forall h \in H$, $H_d^1 = H$ $\forall d \in D$;

while $\exists d \in D$ *such that* $d \notin W_h^i$ $\forall h \in H$ *and* $d \notin R_h^i \forall h \in H$ **do**

  $\forall h \in H$, $A_h^i = \{d : C_d(H_d^i) = h\}$,

  $W_h^i = C_h(A_h^i)$,

  $R_h^i = R_h^{i-1} \cup (A_h^i \backslash W_h^i)$.

  $\forall d \in D$, $H_d^{i+1} = H_d^i \backslash \{h : d \in R_h^i\}$

end

The algorithm terminates at iteration k. Let $W_h^k = W_h$. Each W_h is the set of doctors that h hires.

3.1.2 Optimality of Deferred Matching Algorithm

The NRMP's policy states that "the ranking of applicants by a program director and the ranking of programs by an applicant establishes a binding commitment to offer or to accept a position if a match results" [3]. It is surprising that the NRMP program maintains high participation rates when participation in the program means giving up the right to autonomous choice in what, for doctors, is one of the most monumental decisions of their careers. Almost full participation in the NRMP reveals doctors' beliefs that the NRMP's match will be preferable to any match they could achieve on their own terms through a traditional application process. The 95% of eligible doctors who opted into this centralized process of course weren't forced to do so [4]. In some sense, the NRMP's match is the "best" option for hospitals and doctors. Participating agents in the NRMP program must believe that they aren't missing out on a better opportunity by signing away their right to choose their own match. If hospitals and doctors thought that they could potentially improve their situation after the algorithm makes its assignments, they would have no reason to sign the match agreement contract.

Define doctor preferences over the set of hospitals in accordance with each doctor's submitted ranking of the hospitals from favorite to least favorite. Doctors are not allowed to be indifferent between hospitals. Doctors exclude those hospitals that they would never accept a job offer from. Doctors' rankings should also be reasonable in that

(i) $h_1, h_2 \in H$ either $h_1 >_d h_2$ or $h_2 >_d h_1$ and

(ii) If $h_1 >_d h_2$ and $h_2 >_d h_3$, then $h_1 >_d h_3$.

Mathematically, each doctor has a complete and total ordering over the hospitals it would consider working for. Since doctors are matched to at most one hospital in any assignment, we can conveniently extend their preferences over hospitals to describe a total ordering of possible assignments X.

Similarly define each hospital's preferences over the set of doctors according to their ranking submissions. Since hospitals hire sets of doctors rather than individual doctors, this model is lacking in realism in terms of modeling hospital preferences.

Hospital preferences for doctors are described by a total ordering on D. This restricts our model from accounting for more nuanced expression of hospital preferences over the *sets* of doctors they choose between.

Recall that the mutual action taken by Hospital A and Doctor 1 from earlier is called a *pareto-improvement.* We would like our algorithm to provide an assignment in which no pareto-improvements are be possible. We call such an assignment *stable.* We have previously defined stability, but now we make a definition that is specific to the case where preferences are modeled by rankings.

Definition 13. *Let X be an assignment of doctors to hospitals. X is called stable if there do not exist doctors d and d' and hospitals h and h' such that $(d, h), (d', h') \in X$ and $h' >_d h$ and $d >_{h'}> d'$.*

The following theorem provides an explanation for why the NRMP program was able to convince participants to sign the match agreement contract. If doctors and hospitals believe that there will be no feasible way for them to improve their outcome after the algorithm has created their pairings, they will have no desire to break contract.

Theorem 3. *The Gale-Shapley algorithm produces a stable assignment of doctors to hospitals.*

Proof. Let X be the assignment of doctors in D to hospitals in H that the Gale-Shapley algorithm produces. Let $(d, h), (d', h') \in X$. Suppose $h' >_d h$. However, doctor d is matched with hospital h. This means that doctor d applied to hospital h. Doctor d will have applied to hospital h' and been rejected by hospital h' at earlier rounds of the application process since it only applies to its less preferred hospitals after being rejected by those it more prefers. However, hospital h' did not reject doctor d' at any point in the process. Hospital h' must prefer doctor d' to doctor d, otherwise it would not have rejected d in favor of keeping d' on its waiting list. □

In earlier discussion of the NRMP, we determined that stability is only necessary, not sufficient for a matching algorithm to be practical. The other property identified was *doctor optimality*. We said that an assignment X is *doctor-optimal* if it is stable and all doctors weakly prefer assignment X to any alternative stable assignment. Doctor-optimality guarantees that all doctors are as happy with their match as they would be in any feasible outcome. Provided that doctors are convinced that the following theorem applies in the real world, they guarantee themselves their best employment opportunity by participating in the NRMP.

Theorem 4. *The Gale-Shapley algorithm produces a doctor-optimal assignment of doctors to hospitals.*

Proof. Let X^* be the assignment produced by Algorithm 1. Suppose that there exists an alternative stable assignment X that doctor d^* prefers to X^*. Then $(d^*, h^*) \in X^*$, $(d^*, h) \in X$, and $h^* <_{d^*} h$. In the Gale-Shapley Algorithm, doctors apply to hospitals in order of their preferences. Doctor d^* prefers h to h^*, so it must have applied to h before it applied to h^*. Doctor d^* must have been rejected by h, otherwise it would have remained on h's waitlist and never applied to h^*. So, d^* must have applied to and been rejected by a hospital for which there exists an alternative stable allocation in which d^* is paired to that hospital. The rest of the proof shows that at any iteration of Algorithm 1, no doctor is rejected by a hospital for which there exists a stable assignment in which that doctor is paired to that hospital, proving that X^* is the most preferred stable assignment for each doctor.

Inductively assume that up to iteration n, no doctor d has been rejected by a hospital h for which there exists a stable assignment X with $(d, h) \in X$. At iteration n, suppose d^* applies to and is rejected by hospital h. Then $d^* \notin W_h^n = \{d_i : i = 1, ..., q_h\}$ and $d^* <_h d_i \; \forall d_i \in W_h^n$.

Consider an assignment X in which $(d^*, h) \in X$. Suppose, for contradiction, that X is stable. There must exist $d_i \in W_h^n$ with $(d_i, h) \notin X$, because h hires at most q_h hospitals. We know, however, that $d^* <_h< d_i$. Let $(d_i, h') \in X$. Since X is stable, we know that $h' <_{d_i}< h$ by the inductive assumption and the logic that d_i applied to all hospitals it prefers to h before applying to h. But then X is unstable.

$\square$

3.1.3 Limitations of Deferred Matching Algorithm

Gale and Shapley admit that their interest in the Deferred Matching Algorithm belonged in the "world of mathematical make-believe" [1]. Of course, any model of human behavior relies on unrealistic assumptions, but real-life algorithms like the NRMP are a testament to the efficacy of this type of research. Of course, if we intend to use these models to design matching mechanisms in the real world, we should strive to make them as realistic as possible, especially if we would like to extend them to free-market settings in which decision-making behavior is unrestricted.

The Deferred Matching Algorithm is founded on a simple understanding of bidder behavior. It supposes that the entire question of what makes a bidder best-off can be summed up by a ranking or favorite to least favorite. This may be suitable to explain doctors' choice behavior since they are solely motivated to gain employment from their preferred hospital. Still, when it comes to modeling hospital preferences and consequent hiring choices, the Deferred Matching configuration is not satisfactory. Hospitals hire sets of doctors and may prefer to hire a certain collection of doctors to hiring just a few of their favorites. Moreover, doctors will perform differently in different positions. Consider that hospitals have various departments and not all doctors will be suitable to work in all departments. The NRMP matching algorithm can be improved by allowing for more general hospital choice behavior. We will investigate a generalization of the Deferred Matching Algorithm. The new setting allows for our actors to operate by general choice behavior defined by chosen and rejected sets. Hatfield and Milgrom extend the Deferred Matching Algorithm to apply to a much broader class of preferences than the Deferred Matching Algorithm and provide a more realistic model of the hospital-doctor matching process [5].

3.2 Generalized Gale-Shapley

3.2.1 Introduction

John Hatfield and Paul Milgrom presented their "generalized Gale-Shapley" algorithm with the intention of garnering a more nuanced understanding of the market for medical interns and analogous auction settings. The hospital-doctor setting can easily be replaced by a worker-firm relationship and there are strong connections to auctions of indivisible goods. We will explore Hatfield's and Milgrom's generalized algorithm and connect the key results they discovered to a range of mathematical settings. Of particular interest is the role of quasilinear preferences in their model and how it is possible for a greedy algorithm to take as inputs preferences that are not necessarily quasilinear. We should keep in mind that the hospital-doctor matching setting is identical to the worker-firm labor matching setting developed by Kelso and Crawford that is used ubiquitously throughout the literature on auctions and models for labor markets [9]. Not only will our analysis glean insights into the famous Gale-Shapley algorithm and its extensions, but it will help to understand how the literature on auctions for indivisible goods can be expanded to further generality.

3.2.2 Original Setup

Hatfield and Milgrom built upon notation from the Deferred Matching Algorithm to construct a less assumptive model of the market for medical interns. We have adopted some of their notation in order to restate their "Generalized Gale-Shapley Algorithm".

This model consists of the set D of doctors, the set H of hospitals, and the set X of contracts. Hatfield and Milgrom introduced the contract as convenient mathematical object to formalize what we have called an assignment and the hospital-doctor pairs that make up an assignment. Each potential match between a doctor and a hospital is a contract, $x \in H \times D$. If $x = (h_n, d_m)$, we write $x_h = h_n$ and $x_d = d_m$. A set of contracts, $X \in H \times D$, is analogous to an assignment in that $x \in X$ implies doctor x_d is employed by hospital x_h in assignment X. This language is appropriate since a contractual agreement is drawn up whenever a doctor is hired by a hospital. The contract is a concise and convenient way to include the details of a job offer. For example, it can be extended to include wage, and thus allows for a much more realistic representation of the labor market in which doctors make decisions based on their preference of hospital and the salary offered and in which hospitals are price-driven.

For all its convenience, the language of contracts is potentially confusing. In the following algorithm, doctors and hospitals will be presented with sets of contracts and be asked to choose contracts they like and reject contracts they dislike. In formalizing this choice-behavior, we might say "contract x is doctor d's favorite contract from the set of contracts X". We will say this despite the fact that there are contracts in X that do not involve doctor d and for which doctor d will not consider when making its choice decisions. The Deferred Matching Algorithm involves multiple iterations in which doctors apply to hospitals and hospitals decide whether to reject or accept applicants. We generalize this process by writing "doctor d offers contract $x = (d, h)$ to hospital h" when doctor d applies to hospital h. Similarly, hospital h rejects or accepts contract $x = (d, h)$ if it denies or accepts d's application. The instance of the the generalized algorithm that is equivalent to the Deferred Matching Algorithm is referred to as the "Doctor-Offering Algorithm" by Hatfield and Milgrom [5].

Hatfield and Milgrom formalize decision-making behavior through "chosen sets" and "rejected sets". In the Deferred Matching setting, a doctor's chosen set from a set of contracts X is its most preferred contract in X. At each iteration, hospitals reveal their chosen sets from the set of applications to it by adding some doctors to their waitlist and rejecting others. In the general setting, we make as few assumptions about chosen and rejected sets as possible. Naturally, we require that a doctor's chosen set contains at most one contract. A doctor's chosen set can be empty if it would prefer to accept no contracts to the contracts it is offered. We say nothing about hospitals' chosen sets at this time but remark that the decision-making behavior of hospitals is central to the study of equilibrium in this model.

The orignal Gale-Shapley algorithm and the NIMP algorithm dealt with doctor choice in terms of their "favorite" hospital. We can equivalently describe these preferences with a utility-maximizing function to describe doctor choice. We equip each doctor with a utility function $u_d : X \to \mathbb{R}$ over the set of contracts so that $x_1 <_d< x_2 \iff u_d(x_1) < u_d(x_2)$.

Definition 14. *Let X be a set of contracts. Let X' be a set of contracts available to doctor d. Doctor d's chosen set given X' is*

$$C_d^{\bar{w}}(X') = arg\ max_{x \in X'}\{u_d(x) : x \in X'\}.$$

We will use the definition of hospitals' chosen sets that does not require quasilinearity. Choice behavior is defined by the function $C_h(X')$ for sets of contracts in X. We require that hospitals sign at most one contract with each doctor. It is important to notice that hospital preferences need not be quasilinear in the algorithm; the generalized Gale Shapley algorithm is a version of a greedy algorithm that does not require quasilinearity. The definition we use for hospital preferences does not refer to wage. It is also convenient to describe the set of contracts that are not chosen by a hospital or doctor.

Definition 15. *Let X be a set of contracts. Let X' be a set of contracts available to hospital h^*. Hospital h^*'s chosen set given X' is $C_{h^*}(X') \subset X'$ with the restriction that $\forall x \in C_{h^*}(X')$, $x = (d, h^*)$.*

Definition 16. *Let X be a set of contracts. Let X' be a set of contracts available to agent a. Agent a's rejected set given X' is $R_a(X') = X' - C_a(X')$.*

Remark 1. *The set of contracts chosen by any doctor from $X' \subset X$ is $C_D(X') = \bigcup_{d \in D} C_d(X')$. Similarly define $C_H(X')$, $R_D(X')$, and $R_H(X')$.*

We now have all the equipment necessary to describe the Generalized Gale-Shapley algorithm, as presented by Hatfield and Milgrom. In a broad sense, the Generalized Gale Shapley Algorithm is an iterated application of an isotone operator $F : X \times X \to X \times X$ defined by

$$F_1(X') = X - R_H(X')$$

$$F_2(X') = X - R_D(X')$$

and

$$F(X_a, X_b) = F_1(X_b), F_2(F_1(X_b)).$$

The Gale-Shapley algorithm is encompassed by the iterative application of F in the following way:

Algorithm 3: Hospital-Doctor Matching Algorithm

Result: $X_D(t) \cap X_H(t)$

inputs: H, D, $X \subset H \times D$, $C_d(X')\ \forall d \in D$ and $X] \subset X$, $C_h(X')\ \forall h \in H$ and $X' \subset X$.

initialization: $X_D(0) = \emptyset$, $X_H(0) = X$

while $(X_H(t), X_D(t)) \neq (X_D(t-1), X_H(t-1))$ **do**

  $X_D(t) = X - R_H(t-1)$;

  $X_H(t) = X - R_D(X_D(t-1))$;

end

Algorithm 3 operates as an instance of the operator F and ties together the operator F and the market for medical interns. Let X consist of all possible contracts between doctors in D and hospitals in H. At iteration t, let $X_D(t)$ denote the set of contracts that have not yet been rejected by hospitals. $X_D(t)$ consists of contracts that have not yet been offered and contracts that have been offered and not rejected through iteration t. Let $X_H(t)$ denote the set of contracts that have been offered by doctors through iteration t. This set includes all contracts that have been accepted or rejected by hospitals thus far in the application process. The algorithm progresses recursively according to the following operations.

$$X_D(t) = X - R_H(t-1)$$

$$X_H(t) = X - R_D(X_D(t-1))$$

The algorithm initializes at $(X_D(0), X_H(0)) = (X, \emptyset)$ so that the process begins with no contracts having been rejected or offered. Then $F(X_D(t), X_H(t)) = (X - R_H(X_H), X - R_D(X - R_H(X_H)))$ defines the recursive function that Hatfield and Milgrom rigorously study. At first, it might not be explicitly clear that the Deferred Matching Algorithm is encompassed by this iterative process. Suppose that each doctor has a total ordering, $>_d$ on the set of hospitals and its chosen set given X' is the contract in X' that matches it with its favorite hospital. That is,

$$C_d(X') = \begin{cases} \emptyset & \{x \in X' | x_D = d, x >_d \emptyset\} = \emptyset \\ \max_{>_d}\{x \in X' | x_D = d\} & \text{otherwise} \end{cases}$$

Suppose that each hospital has a total ordering, $>_h$ on the set of doctors so that

$$C_h(X') = \{x_1, ..., x_{q_h} : \forall i \leq q_h \text{ and } k > q_h,\, x_i >_h x_k\}.$$

If hospital h is named in less contracts than its quota in X', its chosen set consists of all contracts in X' that name it. The chosen and rejected sets for doctors and hospitals describe the application criteria used in the Deferred Matching Algorithm from before. Each doctor's chosen set is its favorite hospital and each hospital's chosen set is its favorite set of doctors that fills its quota. In the first iteration, doctors apply to their favorite hospital. Each doctor offers the contract in $C_d(X)$. No contracts have been rejected yet, so $X_D(1) = X_D(0) = X$. Thus, the cumulative set of contracts offered in the first iteration is $X_H(1) = C_D(X) = X - R_D(X)$ where $X = X_D(1)$. We recover $X_H(1) = X - R_D(X_D(1))$. Next, hospitals choose to accept or reject the applications they received. They are presented with the set of applications $C_D(X) = X_H(1)$. The set of contracts hospitals accept (add to their wait-list) are those in $C_H(X_H(1))$. The set of contracts in X that have yet to be rejected are $X_D(2) = X - R_H(X_H(1))$. The application process continues, and the set of not-yet-rejected contracts monotonically shrinks while the set of previously offered contracts monotonically grows. That is, $X_D(t+1) \subset X_D(t)$ and $X_H(t) \subset X_H(t+1)$.

The algorithm terminates when a fixed point is reached. Recall that the Deferred Matching Algorithm terminated when each doctor was either on a hospital's wait-list or had been rejected by all hospitals in H. When this happens, $X_H(t) = X_H(t+1)$. Each doctor that is on a waitlist will offer no new contract, because any contract it prefers to its current contract must have been rejected by the respective hospital in a previous iteration. Each doctor that has been rejected by all hospitals has no new contracts to offer. Then $X_D(t) = X - R_H(X_H(t-1)) = X - R_H(X_H(t)) = X_D(t+1)$. A fixed point has been reached. Let (X_D, X_H) denote the fixed point that the generalized algorithm terminates at. $X_D \cap X_H$ is precisely the set of contracts in X that were offered by some doctor and that were never rejected by the respective hospital. At any iteration $X_D(t) \cap X_H(t)$ is set of contracts that are currently on a wait-list. Then $X_D \cap X_H$ is the final assignment of doctors to hospitals made by the generalized algorithm. This outcome is the same that is reached by the Deferred Matching Algorithm.

3.3 Generality of Hatfield's and Milgrom's configuration

The generalized Gale Shapley algorithm describes the market for medical interns in a realistic and informative way. What makes this algorithm special is that it operates without the assumption that hospital preferences are quasilinear. We noted that the class of gross substitutes valuations is intriguing because when preferences in a market model are gross substitutes, a greedy algorithm can successfully allocate the market to reach competitive equilibrium. However, the procedure for finding the stable market outcome relied on adjusting prices and defining demand in terms of prices of goods. The Greedy Algorithm would make no sense if we allowed for valuations that are not quasilinear to be used as inputs. Nevertheless, Hatfield and Milgrom determine a property of hospital preferences that guarantees their hospital-doctor matching algorithm converges to a stable outcome. Their model is able to represent a broader class of preferences than the set of gross substitutes valuations in a version of a greedy algorithm for constructing a market allocation.

Hatfield and Milgrom present their substitutes condition as the necessary restriction on hospital preferences to guarantee convergence to a stable assignment. Recall that the practical Deferred Matching Algorithm always converges to a stable outcome. This is because a representation of hospital preferences as a linear ordering of doctors always satisfies Hatfield and Milgrom's substitutes condition. However, there are profiles of hospital preferences that are less restrictive than a general ordering that we would like to include in our model. Hatfield and Milgrom show that as long as these preferences satisfy the following substitutes condition, they may be used as inputs for the Hospital-Doctor matching algorithm and the algorithm will always converge to a stable and doctor optimal allocation.

Definition 17. *Elements of X are substitutes for hospital h if for all subsets $X' \subset X'' \subset X$, we have $R_h(X') \subset R_h(X'')$*

To see that the total-ordering preferences from the Deferred Matching algorithm satisfies the substitutes condition, consider a hospital h with quota q_h and sets of contracts $X' \subset X$. Let $x \in R_h(X')$. Let $C_h(X') = \{x_1, ..., x_{q_h}\}$. We know that $x <_h x_i \; \forall i = 1, ..., q_h$. We also know that $C_h(X') \subset X$. Then contract x is preferred to at least q_h contracts in X, and must be rejected by h from the set X as well.

3.3.1 Equivalent Results

The central theorem from *Matching with Contracts* provides the generalized result that the greedy algorithm yields a stable allocation, provided that hospital preferences satisfy the substitutes condition [5].

Theorem 5. *If hospital preferences satisfy the substitutes condition, then the Hospital-Doctor matching algorithm will converge to a stable outcome.*

Our definition of stability implicitly assumes quasilinearity because the choice sets involved depend on wage. However, removing wage from the choice sets changes nothing about the definition's meaning. We discussed how to describe preferences without quasilinearity through chosen sets. Now, we rework the definition of stability to apply outside of the quasilinearity restriction in that same way.

The Hospital-Doctor matching algorithm outputs a set of contracts $X_D \cap X_H$ that is meant to describe the final assignment of doctors to hospitals. Let's check that, provided the hospital preferences satisfy substitutes, $X_H \cap X_D$ is a stable set of contracts. Before we check against the properties of a stable outcome, we must determine that the Hospital-Matching algorithm produces an outcome at all. There is no guarantee that the algorithm will ever terminate. Notice that when hospital and doctor preferences satisfy the substitutes condition, $F : X \times X \to X \times X$ is an isotone mapping from a finite lattice to itself. Hatfield and Milgrom apply Tarski's Fixed Point theorem to their lattice of contracts in order to prove that their algorithm converges to a stable outcome.

Theorem 6. *Let F be an isotone operator on a finite lattice $X \times X$ ordered by $<$. Let X_0 be such that $\forall X' \in X \times X$, $X_0 \geq X'$. Let $F_{X \times X}$ denote the set of fixed points in $X \times X$. Inductively define $X_i = F(X_{i-1})$. The sequence $\{X_i : i = 1, 2, ...\} \to X^*$ where $X^* \geq X'$ for all $X' \in F_{X \times X}$.*

As long as F is isotone, the iterated application of F will reach a fixed point within a finite number of steps. The substitutes condition is key here, because it guarantees that F is in fact isotone. Recall that contracts are substitutes for hospitals whenever $X' \subset X'' \implies R_h(X') \subset R_h(X'')$. Clearly, R_H is isotone. Since doctor preferences are singletons, the law of revealed preference requires that R_d is isotone as well. This is the method of proof used by Hatfield and Milgrom to prove their main result [5].

By Tarski's Fixed Point Theorem, the Generalized Gale Shapley Algorithm terminates and $X_D \cap X_H$ exists. For this assignment to be stable, we must have that $C_D(X_D \cap X_H) = C_H(X_D \cap X_H) = X_D \cap X_H$. Let $x \in X_D \cap X_H$. Suppose $x \notin C_H(X_D \cap X_H)$. Then contract x is rejected by some hospital h. But we know that $x \in X_D$, the set of contracts that have not yet been rejected by hospitals. So, $X_D \cap X_H \subset C_H(X_D \cap X_H)$. Suppose $x \notin C_D(X_D \cap X_H)$. This time x is either rejected by a doctor or was never offered up. But since $x \in X_H$, we know that x has been offered. This is a contradiction since we said x was rejected by its doctor. To check the second condition for stability, let $x \notin X_D \cap X_H$ be a contract such that $x \in X''$ with $X'' = C_h((X_D \cap X_H) \cup X'') \subset C_D((X_D \cap X_H) \cup X'')$. Either x was rejected or it was never offered. Otherwise, it would be in the final assignment. If x was never offered, it could never have been in a choice set for any doctor. If x was rejected by a hospital, it cannot be in a choice set for hospital h. Hatfield's and Milgrom's proof that the substitutes condition on preferences guarantees their greedy algorithm converges to a stable outcome requires no mention of wages or utility functions.

3.3.2 Equivalent Definitions

The substitutes condition is similar to the gross substitutes property in that each of these properties is used as a restriction on preferences in order for a greedy algorithm to succeed in producing a stable market allocation. In fact, the conditions are equivalent in the quasilinear case. A valuation has the gross substitutes property if an increase in the price of one good does not decrease the demand for any other. On the other hand, a valuation has the substitutes property if an expansion in a hospital's opportunity set does not shrink the set of contracts it demands. In the quasilinear case, opportunity sets are determined by prices. Goods can either be affordable or unaffordable to our agents. In choosing a demand set, agents determine which goods have a low enough price for purchasing the good to be beneficial to them. We can think about the price increase as a reduction of the hospital's opportunity set. If the hospital demands a doctor from a larger set and the set is reduced but still contains that doctor, then the hospital will still demand that doctor.

To explicitly translate between gross substitutes and substitutes, we need to update notation a bit. Notice that gross substitutes refers to a situation in which the set of goods available remains the same but the prices of those goods is different. Alternatively, the substitutes condition refers to a situation in which the set of available goods (contracts) is changing. I propose the following notation to unite the substitutes and gross substitutes conditions. Let W be a finite set of potential wages. Assume we are working with a single hospital so that the relevant set of contracts is $X = D \times W$. The contract $x = (d, w)$ pairs hospital h with doctor d and specifies that d gets paid w. Let $X_{\bar{w}}$ denote the set of contracts with wages greater than or equal to the wages specified in $\bar{w}$. That is, if $x = (d, w') \in X_{\bar{w}}$, then $w' \geq w_d$ where w_d is the dth component of $\bar{w}$. Since hospitals are profit maximizing, the only contracts that are relevant to them from $X_{\bar{w}}$ are those which specify a low wage. For example if $x = (d^*, w)$ and $x' = (d^*, w + \epsilon)$ are both contracts in $X_{\bar{w}}$, the contract x' will never be chosen because the hospital can get the same doctor for cheaper by choosing x. This way, the chosen set of contracts from $X_{\bar{w}}$ represents the set of contracts that the hospital demands when wages are fixed at $\bar{w}$. Now, we can easily talk about price changes in terms of growing/shrinking sets of contracts.

Theorem 7. *The substitutes condition is equivalent to the gross substitutes condition in the quasilinear case.*

Proof. Let $C^{\bar{w}}$ satisfy the substitutes condition. Let X' be a collection of contracts and fix wages at $\bar{w}$. For gross substitutes to be satisfied, we require that if $d \in C^{\bar{w}}(X')$ and $d = d^*$, then $d \in C^{\bar{w}^{d+}}(X')$. Let $X'_{\bar{w}} \subset X'$ be such that $x = (d', w'_d) \in X'_{\bar{w}} \implies w'_d \geq w_d$. Similarly define $X'_{\bar{w}+d}$. Then $C^{\bar{w}}(X') = C(X'_{\bar{w}})$ and $C^{\bar{w}^{d+}}(X') = X'_{\bar{w}+d}$. By definition, $X'_{\bar{w}^{d+}} \subset X'_{\bar{w}}$. By the substitutes condition, $R(X'_{\bar{w}^{d+}}) \subset R(X'_{\bar{w}})$, which gives us the desired result. $\square$

3.3.3 A True Generalization

Hatfield and Milgrom were unique in their treatment of the substitutes condition in that they did not require the quasilinearity assumption to guarantee convergence to a stable allocation in their algorithm. That substitutes valuations made their operator F isotone was enough to prove their key result. However, much of the existing literature studies quasilinear valuations. M-Concavity has been widely used as the sufficient condition on valuation functions to produce a stable equilibrium in greedy algorithms [7]. In the absence of the quasilinearity condition, Hatfield and Milgrom's substitutes valuations need not satisfy M-Concavity. Consider the following example of substitutes preferences that violates M-Concavity.

Suppose doctor h_1 has preferences over the set $D = \{d_1, d_2, d_3\}$ such that $\{d_3\} > \{d_1, d_2\} > \{d_2\} > \{d_1\} > \emptyset > \{d_1, d_3\} > \{d_2, d_3\}$. We are supposing that this hospital has a quota of two doctors. It is easily verified that these preferences satisfy the substitutes condition. Any proper subset of $\{d_1, d_2, d_3\}$ containing d_1 or d_2 will reject either d_1, d_s, or both and the rejected set of $\{d_1, d_2, d_3\}$ is $\{d_1, d_2\}$. Similar logic applies to the two-good sets. Taking $X = \{d_1, d_2\}$, $Y = \{d_3\}$ and $i = d_1$. For h_1's preferences to be encompassed by an M-Concave utility function, there must exist M-concave $f : D \to \mathbb{R}$ that yields demanded sets in accordance with the substitutes preferences above. The definition of M-concavity requires that for d_3 and sets $\{d_3\}, \{d_1, d_2\}$,

$$\text{(i) } f(\{d_3\}) + f(\{d_1, d_2\}) \leq f(\emptyset) + f(\{d_1, d_2, d_3\})$$

or

$$\text{(ii) There exists } d_i \text{ such that } f(\{d_3\}) + f(\{d_1, d_2\}) \leq f(\{d_i\} + f(\{d_j, d_3\})$$

where $i \neq 3$ and $j \neq i$.

The first property cannot be true because $\{d_1, d_2, d_3\}$ is not a viable set to demand. The hospital only has two positions available. The second property must also be false. $\{d_3\}$ is the most preferred set, so it must give utility larger than $\{d_j, d_3\}$ and the hospital prefers the set $\{d_1, d_2\}$ to $\{d_1\}$ and $\{d_2\}$.

3.3.4 Is Substitutes Valid?

Intuitively, a hospital's choice behavior satisfies the substitutes condition if whenever a contract is chosen by that hospital from a set of contracts, the hospital still chooses that contract from any smaller set of contracts containing it. Notice the similarity between the substitutes condition and the law of revealed preference. The substitutes condition is indeed the requirement that hospitals treat doctors as if they are substitutes for one another, not complementary goods. Notice that under the substitutes condition, there can exist no doctor who, when made available to a hospital, increases demand for another doctor. Increase in availability of one doctor can only decrease or have no effect on demand for the others. To see how complementarity of doctors is banned by the substitutes condition, suppose that doctors d and d' are complements for hospital h. This means that h does not want d' if it cannot have d, but if it can have d, it wants d' as well. We translate to the language of choice sets. There exists a set of contracts X' such that $d' \in X'$, $d \notin X'$ and $d' \in R_h(X')$. When d becomes available to the hospital, it will add d' to its chosen set. That is, $d' \notin R_h(X' \cup \{d\})$. This violates substitutes because $X' \subset X' \cup \{d\}$ and $R_h(X') \not\subset R_h(X \cup \{d\})$.

The substitutes condition is not a restrictive assumption in the market for medical interns. In fact, we should expect the condition from any reasonable hospital. Doctors generally serve the same purpose to hospitals as one another. They are units of labor with similar training and abilities and so should be substitutes. If a hospital were to violate the substitutes condition, there would have to be a doctor that significantly increases the utility of another doctor to that hospital. For example, suppose two doctors are friends and the one friend is only motivated to work hard in the presence of the other. Any such situation is unlikely since demand for doctors is driven by doctors' individual ability and skills. The hospital may be interested in candidates who work well with others, but teamwork abilities will be assessed in general, not just on how well one works with a single "friend". The same is true in the general worker-firm environment.

The necessity of the substitutes condition is another matter. It is unclear whether there are preferences that are realistic to expect of hospitals which satisfy substitutes but not gross substitutes. In losing the assumption that preferences are quasilinear, important practical results about our model are lost. The law of aggregate demand is a crucial factor of most models of the economy. For example, the "downward sloping demand" curve is downward sloping because the supply-demand model obeys the law of aggregate demand. The law of aggregate demand states that as prices rise, demand does not increase. The gross substitutes restriction on preferences implies the law of aggregate demand, but there is no implication when preferences are substitutes.

Definition 18. *Hospital h with preferences defined by the choice function obeys the law of aggregate demand if whenever the price of a single doctor increases, the size of the demanded set of hospital h does not decrease.*

The law of aggregate demand is crucial in economic analysis and intuition. In abandoning the quasilinearity assumption in moving from the class of gross substitutes valuations to the class of substitutes valuations, this essential piece of demand theory is lost. For this reason, it is reasonable to question whether any preferences that are outside of the class of gross substitutes are valid in a model of economic behavior and whether or not Hatfield and Milgrom's generalization realistically expands the class of preferences to be modeled in the two-sided matching setting.

4 Endowed Assignment Valuations

4.1 Building EAV

Hatfield and Milgrom introduce Endowed Assignment Valuations as a class of utility functions with properties that, not only describe the hospital preferences realistically, but that yield rich insights into matching theory from a variety of mathematical perspectives [5]. The Endowed Assignment Valuation can be understood from a graph theoretic perspective. Let's begin by introducing the EAV in the vein of

Hatfield and Milgrom [5]. The EAV was motivated by the need for a gross-substitutes valuation that captures the nature of hospital preferences provided their institutional structure.

We will proceed with our hospital-doctor notation. Imagine that a hospital is looking to hire n doctors, and each doctor will be hired for a unique position within the hospital according to the doctor's specialties. Our hospital already has a staff of doctors. The new doctors hired will be allocated across the hospital along with those doctors that are already employed. A real-world hospital places value on a set of new hires in a way that depends on the set of doctors it is currently employing. For example, if a hospital already has many capable doctors working for it, then it will value a new set of doctors less than if it were badly in need of man-power. A hospital that has a world-class surgeon will value a mediocre surgeon much less than it would if it had a shortage of surgeons.

Endowed Assignment Valuations provide a means to capture these nuances of hospital preferences. Generally speaking, EAV's are a class of functions that assign utility to bundles of goods. Hatfield and Milgrom build the class of EAV's up from the set of singleton valuations [5]. Each hospital has a number of positions to fill with doctors, and each position within the hospital will contribute a certain amount of overall utility to the hospital. Singleton valuations capture the amount of utility the hospital gets from a single position within it. By building EAV's from singleton valuations, a doctor will contribute different amounts of utility depending on what position it is employed in.

Definition 19. *Valuation V is a singleton valuation if it is of the form*

$$V(S) = max_{d \in S} \alpha_d$$

such that $\alpha_d \in \mathbb{R}$ is the value of doctor d according to this valuation.

A hospital will have multiple singleton valuations, one for each position, and their EAV will decide both which doctors to hire and how to arrange those doctors in order to maximize the aggregate of the singleton valuations for each position. Hatfield and Milgrom introduce the aggregation operation on singleton valuations so that valuations can be built up to contain many positions [5].

Definition 20. *Let V_1 and V_2 be singleton valuations. The aggregation operation, denoted $\wedge$, behaves in the following way:*

$$(V_1 \wedge V_2)(S) = max_{R \subset S} V_1(R) + V_2(S \backslash R).$$

Remark 2. *The class of Assignment Valuations captures the valuation functions that are constructed by aggregating singleton valuations.*

Definition 21. *The set of assignment valuations is the smallest set that contains the set of singleton valuations and is closed under the assignment operation.*

Consider the form of an Assignment Valuation. Suppose J is the hospital's set of available positions and S is a set of doctors who could be hired. For each doctor $d \in S$ and each position $j \in J$, d provides α_{dj} utility to the hospital if it is assigned to position j. Then the hospital chooses to hire the set of doctors $D \subset S$ that maximizes $v(D') = \sum_{d \in D', j \in J} \alpha_{dj} z_{dj}$ where $z_{dj} \in \{0, 1\}$ for all $d \in D'$ and $j \in J$. That is, the $|D'| \times |J|$ identity matrix with entries z_{dj} varies over all possible ways doctors in D' could be assigned to positions in J. To capture the fact that each doctor can only be assigned to one position and each position can only employ one hospital, we require that $\forall j \in J$, $\sum_{d \in D'} z_{dj} \leq 1$ and $\forall d \in D'$, $\sum_{j \in J} z_{dj} \leq 1$. Renato and Ostrovsky present an alternative definition for Assignment Valuations that captures this form [10].

Definition 22. *Valuation v over set D of doctors is an Assignment Valuation if there exists a set of positions J, and a matrix α of dimension $|D| \times |J|$ such that for any set $X \subset D$,*

$$v(X) = max_z \sum_{d \in X, j \in J} \alpha_{dj} z_{dj},$$

where z varies over all possible assignments of elements in X to elements in J.

Given an available set of doctors, assignment valuations describe the maximum utility the hospital can obtain by hiring some of the doctors and arranging them in a particular way. However, hospitals will rarely be starting from scratch in the sense that, when it is choosing to hire, it has no positions already filled. Instead, a hospital will have an endowment of doctors, the doctors it currently employs. Then, the value of hiring a new set of doctors is the difference in the utility it is already getting from its current doctors and the utility it gets from having the new set of doctors made available to it. The Endowment Operation takes an Assignment Valuation and equips it with the tools necessary to capture this choice behavior.

Definition 23. *Let W be an Assignment Valuation on the set of doctors D. The Endowment Operation on W outputs the Endowed Assignment Valuation, V, such that for all $J, S \subset D$,*

$$V(S|J) = W(J \cup S) - W(J).$$

Hatfield and Milgrom's class of Endowed Assignment Valuations is the smallest class of valuations containing singleton valuations and which is closed under the Aggregation and Endowment Operations [5].

4.1.1 An Example

Let's take a look at at an example of a hospital-doctor market where hospitals model their preferences using Endowed Assignment Valuations. Let $H = \{h_1, h_2\}$ with each hospital having a quota of 2. Let $D = \{d_1, d_2, d_3\}$. We will let h_1 and h_2 have endowed assignment valuations so that $v_1(X) = \max_z \sum_{d_i \in X, j \in J} \alpha^1_{ij} z_{ij}$ and $v_2(X) = \max_z \sum_{d_i \in X, j \in J} \alpha^2_{ij} z_{ij}$.

$$\alpha^1 = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 10 & 5 \end{pmatrix}$$

and

$$\alpha^2 = \begin{pmatrix} 2 & 5 & 6 \\ 4 & 6 & 4 \end{pmatrix}$$

For this example, let

$$p = \begin{pmatrix} 4 & 2 & 5 \end{pmatrix}$$

EAV's satisfy the Gross Substitutes condition and so we expect that the Greedy Algorithm on these preferences will output sets that are demanded by the hospitals.

$$D(v_1, p) = \text{argmax}_{S \subset D} \{v_1(S) - p(S)\}$$

is the demanded set for hospital 1.

We check $v_1(S) - p(S)$ for all possible allocations of doctors to hospital

1. Since hospital 1 has a quota of 2, the possible allocations are $\{d_1\}$, $\{d_2\}$, $\{d_3\}$, $\{d_1, d_2\}$, $\{d_1, d_3\}$, $\{d_2, d_3\}$. We calculate $v(S)$ for each subset by allocating doctors in S to the positions within the hospital that optimizes utility. As such, $v_1(\{d_1\}) = 4$ and we assign d_1 to the first position within hospital 1. Similarly, $v_1(\{d_2\}) = 10$ and $v_1(\{d_3\}) = 5$. We optimize the productivity of $S = \{d_1, d_2\}$ by assigning d_1 to hospital 1's first position and d_2 to hospital 2's second position. So, $v(\{d_1, d_2\}) = 14$. Similarly, $v(\{d_1, d_3\}) = 9$ and $v(\{d_2, d_3\}) = 13$.

Subjecting hospital 1 to prices p, we find that

$$D(v_1, p) = \{\{d_2\}, \{d_1, d_2\}\}.$$

We calculate the demanded set for hospital 2 in a similar fashion to obtain

$$D(v_2, p) = \{\{d_2, d_3\}\}.$$

Let's implement the Greedy Oracle for h_1.

- $X = \emptyset$
- Solve for $d_i \in D$ such that d_i solves $\max_{d \in D} v(\{d_i\}|\emptyset) - p_i = v_1(\{d\}) - p_i$.

$$v_1(\{d_1\}) - p_1 = 0$$

$$v_1(\{d_2\}) - p_2 = 8$$

and

$$v_1(\{d_3\}) - p_3 = 5.$$

Then $d_i = d_2$. Since $v_1(\{d_2\}|\emptyset) - p_2 > 0$, we let $X = \{d_2\}$ and continue to the next iteration.

- Solve for $d \in D\backslash\{d_2\}$ such that d_i solves $\max_{d \in D\backslash\{d_2\}} v_1(\{d_i\}|\{d_2\}) - p_i$.

  We check

$$v_1(\{d_1\}|\{d_2\}) - p_1 = v_1(\{d_1, d_2\}) - v_1(\{d_2\}) - p_1 = 14 - 10 - 4 = 0$$

  and

$$v_1(\{d_3\}|\{d_2\}) - p_3 = v_1(\{d_2, d_3\}) - v_1(\{d_2\}) - p_3 = 13 - 10 - 5 = -2.$$

  So, $d_i = d_1$ but $v_1(\{d_1\}|\{d_2\}) \leq 0$. The algorithm terminates and

$$G(v_1, p) = X = \{d_2\} \in D(v_1, p).$$

Now, let's do the same for h_2.

- $X = \emptyset$

- Solve for $d_i \in D$ such that d_i maximizes $v_2(\{d_i\}|\emptyset) - p_i = v_2(\{d_i\}) - p_i$.

$$v_2(\{d_1\}) - p_1 = 0$$

$$v_2(\{d_2\}) - p_2 = 4$$

  and

$$v_2(\{d_3\}) - p_3 = 1.$$

So, $d_i = d_2$ and since $v_2(\{d_2\}|\emptyset) - p_2 > 0$, we let $X = \{d_2\}$ and move on to the next iteration.

- Solve for d_i such that d_i solves $\max_{d \in D \setminus \{d_2\}} v_2(\{d\}|\{d_2\}) - p_2$.

  We check for

$$v_2(\{d_1\}|\{d_2\}) - p_1 = v_2(\{d_1, d_2\}) - v_2(\{d_2\}) - p_1 = -1$$

  and

$$v_2(\{d_3\}|\{d_2\}) - p_3 = v_2(\{d_2, d_3\}) - v(\{d_2\}) - p_3 = 1.$$

  So, $d_i = d_3$ and since $v_2(\{d_3\}|\{d_2\})$, we add d_3 to X so that $X = \{d_2, d_3\}$.

  Now hospital 2 has reached its quota. We terminate the algorithm and

$$G(v_2, p) = \{d_2, d_3\} \in D(v_2, p).$$

4.2 EAV Conjecture

Endowed Assignment Valuations are of particular interest to us because they are a large class of gross substitutes valuations and they capture the nuances of large firm/“hospital” behavior well. The following theorem is attributed to Hatfield and Milgrom who synthesized previous results that assignment valuations satisfy gross substitutes and that a valuation with a submodular indirect profit function is substitutable [5]. Following publication of *Matching with Contracts* where EAVs were first introduced, it remained an open question whether the class of Endowed Assignment Valuations encompasses the class of Gross Substitutes Valuations. The idea seemed plausible because of certain properties of EAVs that are related to M-Concavity.

Theorem 8. *Every Endowed Assignment Valuation satisfies the Gross Substitutes property.*

Conjecture 1. *The class of gross substitutes valuations is equal to the class of endowed assignment valuations.*

Renato and Ostrovsky show that the class of Gross Substitutes valuations is strictly larger than the class of Endowed Assignment valuations [10]. Their method of proof is to identify a property that is true of all Endowed Assignment Valuations and then discover a Gross Substitutes Valuation that does not have that property. The property in question is called Strong Exchangeability. Renato and Ostrovsky provide the following definition:

Definition 24. *Valuation function V is strongly exchangeable if for any $\bar{p} \in \mathbb{R}^{|S|}$ and any two sets X and Y that are demanded at $\bar{p}$, there exists a bijective function σ from $X \backslash Y$ to $Y \backslash X$ such that $\forall i \in X \backslash Y$, the bundles $X \cup \{\sigma(i)\} \backslash \{i\}$ and $Y \cup \{i\} \backslash \{\sigma(i)\}$ are also demanded.*

Strong exchangeability requires that if X and Y are both demanded under our valuation function at a given price, then items of each can be swapped out for one another and the resulting sets will still be demanded. This property may look familiar. Compare it to M-Concavity. It is easy to recognize that strongly exchangeable valuations satisfy gross substitutes. The first step in Renato's and Ostrovsky's argument is to show that endowed assignment valuations are always strongly exchangeable. The following is presented as a lemma in their paper [10].

Theorem 9. *Every endowed assignment valuation is strongly exchangeable.*

The fine details of their proof can be found in their original work [10]. We include a sketch of their argument. Consider an endowed assignment valuation

$$V(S) = \max_{d\in S, j\in J} \sum_{d\in S\cup T, j\in J} \alpha_{d,j} z_{dj} - W(T).$$

In this case, T is the endowment of doctors. The valuation function yields an optimal bundle an an optimal matrix z that assigns doctors in $S \cup T$ to positions in J. We suppose that both X and Y are demanded according to this valuation function at some fixed price. This means that $V(X) = V(Y)$ since they are both maximal. Moreover, let z^x be the matrix assigning agents to positions under the assignment that chooses X and let z^y be the matrix assigning agents under the assignment that chooses Y. Now, construct a bipartite graph where nodes are elements of $S \cup J$. For each pair d, j such that $z^x_{dj} = 1$, draw a blue edge connecting the node representing d and the node representing j. Essentially, we are drawing blue edges between doctors and positions to reflect their assignment. Do the same for z^y, but draw these edges in red. Let each edge of the graph have a weight that is equal to α_{dj}, the utility that is contributed by having doctor d in position j under the various assignments. Then the sum of the weights of the blue edges is equal to $V(X)$ and the sum of the weights of the red edges is equal to $V(Y)$. And since $V(X) = V(Y)$, these two sums must also be equal. Since each node in our graph has a degree less than 2, the graph can be decomposed into disjoint cycles and paths. Roughly, this can be seen by picking a node and constructing a path by following along edges until you either get stuck (and have a path) or reach a node that you have already encountered. In this case you stop and have a cycle. Pick $d \in X \backslash Y$. Since d is not a member of Y, it has degree one, and so is the beginning of some path in our graph. Let m be the other end of this path. Suppose $m \notin Y \backslash X$. In this case, our path contains no doctors in $Y \backslash X$, for if it did contain $q \in Y \backslash X$, we arrive at a contradiction, since q has degree 1. This means that each of the doctors that are included in this path are in $X \cup T$. Consider what happens if we swap the colors of the edges along this path. Now the graph represents the bundle $X \backslash \{d\} \cup \{t\}$ for $t \in T$. Swapping the edges will not change their sum, and so the value of this new bundle is equal to the value of X. This is not allowed since the only difference between the new bundle and X is that the new bundle is missing d since t was already contributing utility as part of the endowment. Now we know that m, the other end of the path, is a member of $Y \backslash X$. Switching the colors along this path constructs a representation of the bundles $X \cup \{m\} \backslash \{d\}$ and $Y \cup \{d\} \backslash \{m\}$. But switching the colors leaves the sum of the edge weights unchanged. Therefore, $V(X \cup \{m\} \backslash \{d\}) = V(X) = V(Y) = V(Y \cup \{d\} \backslash \{m\})$. These new sets are also demanded and we have satisfied strong exchangeability.

Theorem 10. *The EAV conjecture is false.*

Renato and Ostrovsky formulate their counter-example with a graph theoretic interpretation of a valuation. They represent objects as the edges of the complete graph with four nodes. Let S represent the set of goods in this configuration. This configuration is relatively simple and can easily be drawn since $|S| = 6$. The substitutes valuation that is not strongly exchangeable is called r and it is defined so that for $X \subset S$, $r(X)$ is the "size of the largest subset of X containing no cycles"[10]. Then, Renato and Ostrovsky pick two bundles of goods and check against the definition for strong exchangeability. First, they determine two sets of edges that maximize the value of r. Any set that contains more than 3 of the 6 edges will necessarily contain a cycle. So, any subsets that have a value of 3 will be maximizers of r and thus demanded. Let X contain three of the four edges that are not intersected by another edge. Let Y contain the two edges that cross each other and the last of the four edges that is not intersected. Consider edge $i \in X$ where i is an edge connected to a node that is connected to an edge in Y. For strong exchangeability to be satisfied, we must be able to identify an edge of Y that we can trade for i and maintain the same value for r. However, each of the three edges in Y shares a node with an edge of X. So swapping i for any $j \in Y$ will result in a cycle, lowering the value of $r(X) = 3$ to $r(X \backslash \{i\} \cup \{j\}) = 2$. What is left to show is that the function r does satisfy gross substitutes. Renato and Ostrovsky call upon the results that their function r has a special property which is equivalent to being M-concave and M-concavity is equivalent to the gross substitutes condition. Earlier, we discussed the relationship between M-concavity and matroidal structures. Specifically, we said that an M-concave set function on items in N, $f : \mathcal{F} \to \mathbb{R}$ induces a matroid with ground set N and independent sets $\mathcal{F}$. Now, let's try to work in the other direction. A matroid rank function is a function on the independent sets of a matroid that outputs the rank of an independent set.

Definition 25. *Let N be the ground set of a matroid with independent sets in $\mathcal{F}$. The function $f : \mathcal{F} \to \mathbb{R}$ is a matroid rank function if the value of $f(X)$ for $X \in \mathcal{F}$ is equal to the maximum size of an independent subset of X.*

Lemma 4. *Matroid rank functions are M-concave.*

Now, let's consider our function r. We take the ground set $N = \{1, 2, 3, 4, 5, 6\}$ to be the set of edges of the complete graph with four nodes. We take the independent sets to be the subsets of N for which there is no cycle. Then, a matroid rank function f for $(N, \mathcal{F})$, is such that for $X \subset N$, $f(X)$ is equal to the size of the largest independent set in X. We defined independent sets as the subsets of N which contain no cycles. And so, r is consistent with the definition of a matroid rank function.

We conclude that r is a M-concave and thus satisfies gross substitutes. Theorem 10 follows directly since we have identified a gross substitutes valuation that is not an Endowed Assignment Valuation.

4.3 Transversal Matroids

The graph theoretic interpretation of Endowed Assignment Valuations used by Renato and Ostrovsky illuminates the matroidal properties of Endowed Assignment Valuations. Endowed Assignmnet Valuations must have a matroidal structure because they are gross substitutes valuations. In fact, EAV's are closely related to the class of transversal matroids. Recall that the class of Endowed Assignment Valuations are those valuations which can be built up from singleton valuations through the aggregation operation and the endowment operation. We will deconstruct EAV's to recover their graphic representation and then relate this structure to the class of transversal matroids.

4.3.1 The Structure of the EAV

Suppose that hospital h has positions in J and is choosing over the a set of doctors D. For each $j \in J$ there is an associated singleton valuation $v_j : D \to \mathbb{R}$ such that $v_j(d) = \alpha_d^j$ is the utility doctor d produces in position j.

Definition 26. *Let $v_j : D \to \mathbb{R}$ be a singleton valuation taking values $v_j(d) = \alpha_d^j$, The graph that is dual to the singleton valuation v_j is the weighted bipartite graph $G_j = (D, \{j\}, E_j, v_j)$ where $E_j = e_{dj} = \{d, j\}$ exactly when doctor d is available to work at position j.*

Remark 3. *We are slightly changing our definition of singleton valuations here with the graph theoretic interpretation. Before, we considered that each doctor from the available set was able to work at each position. Now, we allow for some doctors to be "prohibited" from working at some positions. Hatfield and Milgrom mentioned that their formulation of EAVs was able to take into consideration hospitals' affirmative action programs [5]. This is precisely why they were able to make that claim. Additionally, this change opens the door for our valuations to consider the qualifications of applicants. Some positions within the hospital may be reserved for certain types of candidates. Candidates will not have an edge connecting them to positions they are ineligible for.*

Remark 4. *For the time being, we are going to forget that we are working with weighted matroids and graphs. This is to make is easier to discuss the structure of the mathematical objects. When the weights again become relevent, we will add them back in to our notation.*

The aggregation operation merged singleton valuations with assignment valuations in order to construct valuations for hospitals with multiple positions to fill. The analogous operation on our graphs is to merge them.

Definition 27. *Let $v_i : D \to \mathbb{R}$ and $v_k : D \to \mathbb{R}$ be EAVs with dual graphs $G_i = (D, \{i\}, E_i)$ and $G_k = (D, \{k\}, E_k)$. Let V be the Assignment Valuation attained by aggregating v_i and v_k. Let α^V be the $|D| \times |J|$ matrix with entries α_{dj} denoting doctor d's utility in position j. The bipartite graph $G_V = (D, \{i, k\}, E_i \cup E_k, V)$ is the graph that is dual to valuation V.*

From here on, for an Assignment Valuation V, we will call the graph that is dual to the valuation $G_V = (D, \{i, k\}, E_V, V)$. In constructing G_V, we have constructed a set system. There is a set in the set system for each position $j \in J$ and that set contains all of the doctors in D who are allowed to fill position j.

Definition 28. *Let $V : 2^D \to \mathbb{R}$ be an Assignment Valuation with dual graph G_V. Define $\mathcal{S} = \{S_j : j \in J\}$ where $S_j = \{d \in D : \exists e_{dj} \in E_V\}$.*

Let's take a look at a simple example. Earlier, we considered hospital h_1 and its Assignment Valuation v_1 generated by the matrix α^1.

$$\alpha^1 = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 10 & 5 \end{pmatrix}$$

Let $J = \{A, B\}$ denote the positions for doctors within h_1. Let $D = \{a, b, c\}$ denote the set of doctors that are available to h_1. Let $G = (D, J, E_{v_1}, v_1)$ be the bipartite graph with vertices in D and J and edges connecting each $d \in D$ to each $j \in J$. Dual to our graph is a set system $\mathcal{S} = \{S_A, S_B, S_C\}$ where $S_A = S_B = S_C = D$. Here, the doctors in S_A are exactly those doctors who are available to fill position A. In our simple example, all doctors are available to fill all positions. Sets in $\mathcal{S}$ are dual to the edges that are connected to the positions in J. So, S_A contains elements which are dual to each edge in G that is connected to vertex A. Each edge is weighted according to the utility of doctors in each position. For example the edge $\{a, A\}$ has weight 4 because doctor a provides a utility of 4 when working in position A.

We have constructed a weighted graph and a set system to describe the setup of the Assignment Valuation. We did something similar when we established that gross substitutes valuations can be formulated as weighted matroids. We established that for an M-concave function on subsets of N, $f : \mathcal{F} \to \mathbb{R}$, the matroid induced by f is the pair $(N, \mathcal{F})$. Assignment Valuations are gross substitutes valuations. It follows that Assignment Valuations are M-concave and have a matroidal form. We have begun to construct the matroid induced by our Assignment Valuation in our set system $\mathcal{S}$. For an M-concave valuation, the ground set of the induced matroid is the set of doctors to be chosen from and the independent sets are the subsets of doctors that could be chosen. With an Assignment Valuation, the induced matroid has a bit more structure. The Assignment Valuation chooses subsets of doctors and organizes them so that each doctor chosen is matched to a position. Assignment Valuations induce what are called transversal matroids. Transversal matroids are exactly the class of matroids induced by bipartite graphs [11]. The Assignment Valuation will choose a subset of doctors to hire and determine how to allocate those doctors throughout the hospital by identifying a subgraph of G in which edges connect chosen doctors with the position they will work in.

Definition 29. *Let $G = (D, J, E)$ be a bipartite graph with vertices in D and J and edges in E. The transversal matroid induced by G is $M = (D, \mathcal{F})$ with $S \in \mathcal{F}$ if and only if there exists a subgraph $G_S = (S, J_S, E_S) \subset G$ with the following properties:*

(i) $J_S \subset J$

(ii) There exists a bijection $\psi : S \to J_S$ such that $E_S = \{e_{s'j'} : s' \in S, \psi(s') = j'\}$

for all $s \in S$

The independent sets of a transversal matroid are exactly the sets of doctors for which it is feasible for the hospital to hire. For a set of doctors to be independent, there must be a way to match each doctor from that set to a unique position within the hospital. The collection of edges in E_H forms a bijection between elements of S and some subset of positions in J. An edge between a doctor and a position represents a match between the doctor and position in that permutation of possible assignments.

Let's make our example a bit more complicated to illustrate the significance of independent sets in the associated transversal matroid. Suppose that position i is reserved for surgeons. Suppose that doctor a is the only doctor in D who is a surgeon. This means that doctors b and c are ineligible to work in position a. In the graph G_V induced by this valuation, there will be no edges connecting doctors b and c to position i. Let $M_V = (D, \mathcal{F})$ be the transversal matroid induced by this valuation. In this case $\mathcal{F} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $\{b, c\}$ is not independent. There is no subgraph $H_V = (\{b, c\}, \{i, j\}, E_V)$ of G_V with edges forming a bijection between elements of $\{b, c\}$ and $J' \subset \{i, j\}$. The set $\{a, b, c\}$ is also not independent, because three doctors cannot fill only two positions. For the independent sets $\{a, b\}$ and $\{a, c\}$, the subgraphs of G_V are $H_V = (\{a, b\}, \{i, j\}, \{e_{ai}, e_{bj})$ and $H_V = (\{a, c\}, \{i, j\}, \{e_{ai}, e_{cj}\})$. For the singleton independent sets, the subgraphs contain the single edge connecting the doctor to the position it is allowed to fill.

Lemma 5. *Let V be an Assignment Valuation. Let $G = (D, J, E)$ and $M = (D, \mathcal{F})$ be the bipartite graph and transversal matroid induced by V respectively. Then M satisfies the definition of a matroid.*

This lemma is shown by Stallman in *A Gentle Introduction to Matroid Algorithmics* [11]. The only non-obvious property of the transversal matroid is the rule that if $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there exists $x \in X \backslash Y$ such that $Y \cup \{x\} \in \mathcal{F}$. Stallman operates on the graph of an independent set with what he calls an augmenting path to show that for $X, Y \in \mathcal{F}$ with $|X| = |Y| + 1$, there exists $x \in X \backslash Y$ such that $Y \cup \{x\} \in \mathcal{F}$. The property then follows by induction.

It is time to remember that the transversal matroid representing an Endowed

Assignment Valuation is weighted. The weights of the independent sets of the matroid will be equal to the total utility derived by the hospital from the assignment of doctors to positions represented by that set. Recall from the definition of the transversal matroid $M_V = (D, \mathcal{F})$ induced by V, that for each independent set $S \in \mathcal{F}$ there exists a bijection $\psi_S : S \to J_S \subset J$. Each independent set of M_V is a possible assignment of doctors to positions in J. For $S \in \mathcal{F}$ and $s \in S$, $\psi(s) = j$ indicates that under the assignment represented by S, doctor s is selected to work in position j.

Definition 30. *Let V be an EAV on a set of doctors D and set of positions J. Let α^V be the $|D| \times |J|$ matrix with entries $\alpha^V_{dj} = v_j(d)$, the utility of doctor d when working in position j. Let $G_V = (D, J, E_V, w)$ be the weighted bipartite graph induced by valuation V. Let $M_V = (D, \mathcal{F}, W)$ be the weighted transversal matroid induced by valuation V.*

Define $w : E_V \to \mathbb{R}$ such that for each edge $e_{dj} = \{d, j\} \in E_V$, $w(e_{dj}) = \alpha^V_{dj}$.

Define $W : \mathcal{F} \to \mathbb{R}$ such that for each $S \in \mathcal{F}$, $W(S) = \sum_{s \in S} w(e_{s\psi_S})$.

So for each subset of doctors $S \in \mathcal{F}$, $W(S)$ is the utility from selecting the set S of doctors to hire. The assignment valuation maximizes $W(S)$ over the independent sets of the matroid.

Theorem 11. *The matroid induced by an Assignment Valuation is a transversal matroid.*

So far, we have dealt with Assignment Valuations and ignored all those valuations which take into consideration the hospital's endowment of doctors. Closing the set of Assignment Valuations under the endowment operation creates the set of

Endowed Assignment Valuations. Just like adding a position vertex to the bipartite graph was equivalent to the aggregation operation, there is a corresponding matroid transformation that will capture the endowment operation. Stallman provides the following definiton of a matroid contraction [11].

Definition 31. *Let $M = (D, \mathcal{F})$ be a transversal matroid. The contraction of the matroid M with respect to $d \in D$ is the matroid $M' = (D \backslash \{d\}, \mathcal{F}')$ such that $X' \in \mathcal{F}'$ if and only if $X' \cup \{d\} \in \mathcal{F}$.*

Essentially, a contraction on a transversal matroid looks like removing one of the vertices in D from the graph. For an Endowed Assignment Valuation, it corresponds to removing a doctor from consideration in the valuation, perhaps because the position doctor is already employed by the hospital. For Endowed Assignment Valuation $V(S|T)$ where T is the set of doctors already employed by S, the contraction operation removes from S any doctors that belong to T. The contraction operation is closed under the class of matroid valuations since each independent set in $\mathcal{F}'$ is a subset of an independent set in $\mathcal{F}$.

Representing the EAV as a weighted cube is a convenient way to grasp exactly how the contraction operation corresponds to the endowment operation on a transversal matroid. Let $C = \{i_1, ...i_{|D|}\}$ such that each $i_j \in \{0, 1\}$ be the cube of dimension $|D|$ with points of the cube representing subsets of doctors. For example, in the three doctor case, the point $\{1, 1, 0\}$ indicates the bundle of doctors that includes d_1 and d_2 and does not include d_3. Each point on the cube is also given a weight that indicates the utility derived from hiring that particular subset of doctors. The independent sets of the matroid as a cube are the points of the cube. The contraction operation removes a doctor d from the ground set of the matroid. On the cube C, this is equivalent to contracting the cube to the $|D|-1$ cube, where the dimension lost is the one that represents doctor d. Under the endowment operation, we wish to assign a value to set of doctors in the scenario where doctor d is already hired. Then, the contraction of cube C is the set of points on C such that $i_d = 1$. If C' is the contraction of cube C, then $C' = \{i_1, ...i_{d-1}, 1, i_{d+1}, ...i_{|D|}\}$. Now, the endowed assignment valuation on C' evaluates the utility of subsets of doctors under the condition that doctor d is already chosen. The class of transversal matroids is not closed under the contraction operation. So, it is true that the entire class of Assignment Valuations can be described as valuations on a transversal matroid. The class of Endowed Assignment Valuations is slightly larger, containing all transversal matroidal valuations and those matroidal valuations on a matroid that are the contraction of a that are on the contraction of a transversal.

4.4 Counterexample

That EAVs are so closely related to transversal matroids illuminates why the EAV conjecture is false. Recall that the EAV conjecture was that the class of Endowed Assignment Valuations is equal to the class of Gross Substitutes valuations. We understand that the Gross Substitutes property is equivalent to M concavity and the class M concave functions is the class of finite matroid based valuations. Then the relationship between the size of the class of EAVs and the size of the class of Gross Substitutes valuations is similar to the relationship between the size of the set of transversal matroids relative to the set of finite based matroids. Transversal matroids make up only a small subset of the set of all matroids. Therefore, we should not expect EAVs to come close to the size of class of Gross Substitutes valuations.

We move into another field of study to develop a proof of that the EAV conjecture is false that does not require the intermediate step of comparing definitions to an intermediate property (strong exchangeability in the case of Ostrovsky and Leme). The study of combinatorial auctions is interested in describing bidding behavior in an auction for differentiated goods. A bid is another word for valuation. Types of bids are characterized mathematically, often starting with unit demand functions and building more complicated valuations through an aggregation operation. The construction of different types of bids is analogous to the construction of Assignment Valuations and Endowed Assignment Valuations. Of particular interest is the OR and XOR bid. We are interested in the class of OR of XOR bids of singleton valuations. We will call this class of bid the OXS bid. XOR and OR are operations that are performed on valuations in order to construct a more complicated class of valuations. Bidding language differs slightly from the demand language we are used to.

Definition 32. *Let D be a set of items. A singleton bid is a pair (S, p) where $S \subset D$ and $p \in \mathbb{R}$ where p represents the utility from having set S.*

An XOR bid on a set of singleton bids chooses the singleton bid that provides the highest utility and chooses that particular set of items to purchase. The XOR bid is equivalent to the valuation that determines the utility produced by a single position in an EAV. The XOR bid on singletons is a singleton bid itself, outputting a single item and its corresponding utility.

Definition 33. *Let D be a set of items. An XOR bid on a set of bids $(S_1, p_1), ..., (S_n, p_n)$ is (S_i, p_i) such that $p_i \geq p_j$ for all $j \leq n$. The value of the bid is p_i and the set of items purchased is S_i.*

The OR operation is equivalent to the aggregation operation on singleton valuations which gave us the class of assignment valuations. We have a collection of XOR bids, and apply the OR operation so that the value of the aggregated bid is equal to the sum of the XOR singleton bids.

Definition 34. *Let D be a set of items. Let (S_1, p_1) and (S_2, p_2) be bids. Then The OR operation on (S_1, p_1) and (S_2, p_2) creates the bid $(S_1 \cup S_2, p_1 + p_2)$.*

The OR of XOR operation on singleton valuation takes a collection of singleton XOR bids, maximizes their value, and then applies the OR operation to aggregate the total value of the bid as the sum of value of the maximized XOR bids. It is for this reason that the class of OR of XOR bids, which we call OXS bids, is equivalent to the class of Assignment Valuations and is the class of valuations based on transversal matroids. The XOR and OR operations on bids construct different classes of bids/valuations. For example, the XOR of OR operation on singleton valuations reverses the role of OR and XOR in our above construction so that a collection of bids is first aggregated and then maximized over. Lehmann, Lehmann, and Nisan were interested in the relationship between different classes of bids [6]. In particular, they compared the size of the OXS class to the class of Gross Substitutes valuations. We have seen Lemme's example of a Gross Substitutes valuation that is not an EAV. We also have some insight as to why OXS is contained by the class of Endowed Assignmnet Valuations. Indeed, Lehmann, Lehmann, and Nisan demonstrate that the class of OXS valuations is contained by the the class of Gross substitutes valuations through a clever example of a Gross Substitutes valuation that is not in OXS [6]. As we have seen with our work on transversals, the relationship between OXS and EAVs is close. Here, we adapt Lehmann's, Lehmann's, and Nisan's counterexample to the language of Endowed Assignment Valuations to provide an alternate proof that the EAV conjecture is false.

Let $D = \{a, b, c, d\}$. Let $V : 2^D \to \mathbb{R}$ be a valuation such that $V(\{x\}) = 10$ $\forall x \in D$, $V(\{a, c\}) = V(\{b, d\}) = 15$, and $V(S) = 19$ for all other subsets of D. In this instance, we ignore the endowment aspect of the EAV. It is as if we treated each of these values as if the endowment were the empty set. Whether V is an EAV comes down to whether V is an Assignment Valuation. For V to be an Assignment Valuation, there must exist a set of positions J and $|D| \times |J|$ matrix α such that $V(S) = \max_z \sum_{x \in S, j \in J} \alpha_{xj} z_{xj}$.

Suppose matrix α does exist. A few things must be true of α for it to induce valuation V.

1. For each $x \in D$, there must exist $j_x^{10} \in J$ such that $\alpha_{xj} = 10$ $\forall x \in D$. This is necessary for the value of singletons to each be equal to 10.

2. For all $x \in D$ and $j \neq j_x^{10}$, $\alpha_{xj} \leq 5$. To see why this is true, without loss of generality, let $x = a$ and suppose $\alpha_{aj} > 5$. Then $V(\{a, c\}) \geq 10 + \alpha_{aj} > 10 + 5$ since it is possible to assign a to position j_a^{10} and c to position j.

However, $V(\{a, b\}) = 19$. This means that there exist $j, k \in J$ such that $\alpha_{aj} + \alpha_{bk} = 19$. According to our two conclusions from before, this cannot be. If $\alpha_{aj} = 10$, then $\alpha_{bk} \leq 5$ and vice versa. If neither α_{aj} nor α_{bk} is equal to 10, then their sum cannot be greater than 10.

Lehmann, Lehmann, and Nisan provide a proof-sketch to illustrate that the valuation V has the gross substitutes property. We will confirm that V satisfies the definition of an M concave function. Recall that for a valuation to be M concave, for all X, Y and $i \in X \backslash Y$, one of the following must be true:

1. $V(X) + V(Y) \leq V(X \backslash \{i\}) + V(Y \cup \{i\})$

2. $V(X) + V(Y) \leq V(X \backslash \{i\} \cup \{j\}) + V(Y \cup \{i\} \backslash \{j\})$.

Whenever X and Y are singletons and whenever $|X| = |Y| = 3$, property 2 is satisfied with equality because all sets of size 1 have the same value and all sets of size 3 have the same value. Whenever $|X| = 3$ pr $|X| = 4$ and $|Y| = 1$, property 2 is satisfied because the left hand side will always be equal to 29 while the right hand side is at least 30. The rest of the cases can be broken down into the following.

- $|X| = 2$ and $|Y| = 1$

  Let $X = \{a, b\}$, $Y = \{a\}$, and $i = b$.

$$V(\{a, b\}) + V(\{a\}) = V(\{a\}) + V(\{a, b\})$$

$$19 + 10 = 10 + 19.$$

  Let $X = \{a, b\}$, $Y = \{c\}$, and $i = a$, $j = c$.

$$V(\{a, b\}) + V(\{c\}) = V(\{b, c\} + V(\{a\})$$

$$19 + 10 = 19 + 10.$$

  Let $X = \{a, c\}$, $Y = \{a\}$, and $i = c$.

$$V(\{a, c\}) + V(\{a\}) = V(\{a\}) + V(\{a, c\}).$$

$$15 + 10 = 10 + 15.$$

Let $X = \{a, c\}$, $Y = \{b\}$, and $i = c$.

$$V(\{a, c\}) + V(\{b\}) < V(\{a\}) + V(\{a, b\}).$$

$$15 + 10 < 10 + 19.$$

- $|X| = 2$ and $|Y| = 2$

  Let $X = \{a, b\}$, $Y = \{c, d\}$, and $i = b$, $j = d$.

$$V(\{a, b\}) + V(\{c, d\}) < V(\{a, d\}) + V(\{c, b\}).$$

$$19 + 19 = 19 + 19.$$

  Let $X = \{a, c\}$, $Y = \{b, d\}$, and $i = c$, $j = b$.

$$V(\{a, c\}) + V(\{b, d\}) < V(\{a, b\}) + V(\{b, c\}).$$

$$15 + 15 < 19 + 19.$$

- $|X| = 3$ and $|Y| = 2$

  Let $X = \{a, b, c\}$, $Y = \{a, b\}$, and $i = c$.

$$V(\{a, b, c\}) + V(\{a, b\}) = V(\{a, b\}) + V(\{a, b, c\}).$$

$$19 + 19 = 19 + 19.$$

  Let $X = \{a, b, c\}$, $Y = \{a, c\}$, and $i = b$.

$$V(\{a, b, c\}) + V(\{a, c\}) = V(\{a, c\}) + V(\{a, b, c\}).$$

$$19 + 15 = 15 + 19.$$

Let $X = \{a, b, c\}$, $Y = \{d\}$, and $i = a$.

$$V(\{a, b, c\}) + V(\{d\}) < V(\{b, c\}) + V(\{a, d\}).$$

$$19 + 10 < 19 + 19.$$

Let $X = \{a, b, c\}$, $Y = \{d\}$, and $i = b$.

$$V(\{a, b, c\}) + V(\{d\}) < V(\{a, c\}) + V(\{b, d\}).$$

$$19 + 10 < 15 + 15.$$

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