Avd. Matematisk statistik

KTH Matematik

TENTAMEN I SF2940 SANNOLIKHETSTEORI/EXAM IN SF2940 PROBABILITY THEORY WEDNESDAY THE 20^{th} OF OCTOBER 2010 2.00 p.m.–7.00 p.m.

Examinator: Timo Koski, tel. 790 71 34, email: tjtkoski@kth.se

Tillåtna hjälpmedel Means of assistance permitted: Appendix 2 in A.Gut: An Intermediate Course in Probability. Formulas for probability theory SF2940. Pocket calculator.

You should define and explain your notation. Your computations and your line of reasoning should be written down so that they are easy to follow. Numerical values should be given with the precision of two decimal points. You may apply results stated in a part of an exam question to another part of the exam question even if you have not solved the first part. The number of exam questions (Uppgift) is six(6).

Each question gives maximum ten (10) points. 30 points will guarantee a passing result. The grade Fx (the exam can completed by extra examination) for those with 27−29 points.

Solutions to the exam questions will be available at
`http://www.math.kth.se/matstat/gru/sf2940/`
starting from Wednesday 20^{th} of October 2010 at 7.05 p.m..

The exam results will be announced at the latest on Friday the 5^{th} of November 2010.

Your exam paper will be retainable at elevexpeditionen during a period of seven weeks after the date of the exam.

Lycka till!

- - - - - - - - - - - - - - - - - - - -

Uppgift 1

The random variable (X, Y) has the joint density

$$f_{X,Y}(x, y) = \begin{cases} 2e^{-x-y} & \text{for } 0 < x < y \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Determine the joint distribution of X and $Y - X$. (4 p)

(b) Show that X and $Y - X$ are independent random variables and find the marginal distributions of X and $Y - X$. (6 p)

Uppgift 2

The random variables $X \in L(a)$ and $Y \in L(a)$, $a > 0$ are independent. The random variable $U \in \mathrm{Be}(p)$, $0 < p < 1$, is independent of X. Let the random variable Z be defined as

$$Z = \begin{cases} 0 & \text{if } U = 1 \\ X & \text{if } U = 0. \end{cases}$$

(a) Find the characteristic function of Z. (4 p)

(b) Suppose that we need to find a random variable W such that

$$Y = W + Z.$$

What must the probability distribution of W be for this to be possible ? (6 p)

Uppgift 3

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $E[X_1] = 5$ and $\mathrm{Var}[X_1] = 9$. Find the limiting distribution of

$$\sqrt{n}\left(\frac{X_1 + X_2 + \ldots + X_n - 5n}{X_1^2 + X_2^2 + \ldots + X_n^2}\right),$$

as $n \to \infty$. Explicate your argument carefully. (10 p)

Uppgift 4

$\mathbf{W} = \{W(t) \mid 0 \le t < \infty\}$ is a Wiener process. $U_0 \in N(0,1)$, independent of the Wiener process. We define the (Ornstein-Uhlenbeck) process $\mathbf{U} = \{U(t) \mid 0 \le t < \infty\}$ by

$$U(t) = e^{-t}U_0 + \frac{1}{\sqrt{2}} \int_0^t e^{-(t-u)} dW(u).$$

(a) What is the distribution of the vector $(U(t-1), U(t))^T$ (^{T} is the vector transpose) ($t > 1$)? . (4 p)

(b) Find the probability

$$P\left(U(t) > E\left[U(t) \mid U(t-1)\right] + 0.9\right).$$

(6 p)

Uppgift 5

$\mathbf{W} = \{W(t) \mid 0 \le t < \infty\}$ is a Wiener process. We consider

$$Y(t) \stackrel{\text{def}}{=} W(t+1) - W(t), \quad t \ge 0.$$

(a) Find the mean function and the autocorrelation function of the process $\{Y(t) \mid t \ge 0\}$. (8 p)

(b) Is the process $\{Y(t) \mid t \ge 0\}$ strictly stationary ? You are expected to justify your answer. (2 p)

Uppgift 6

Let X_i, $i = 1, 2, \ldots$, be $X_i \in \text{Fs}(p)$ and I.I.D.. Let $N = \{N(t) \mid t \ge 0\}$ be a Poisson process with intensity $\lambda > 0$. The process N is independent of X_i, $i = 1, 2, \ldots,$.
We define a new stochastic process $X = \{X(t) \mid t \ge 0\}$ with

$$X(t) \stackrel{\text{def}}{=} \sum_{i=1}^{N(t)} X_i, \quad X(0) = 0, X(t) = 0 \text{ if } N(t) = 0.$$

We say that $X = \{X(t) \mid t \ge 0\}$ is a **Pólya-Aeppliprocess** or a **Compound Poisson process** with parameters p and λ.

(a) Show that the moment generating function of $X(t)$ is

$$\psi_{X(t)}(s) = e^{\lambda t\left(\frac{e^s p}{1-e^s(1-p)} - 1\right)}, \quad s < -\ln(1-p).$$

(5 p)

(b) Show that

$$E\left[X(t)\right] = \frac{\lambda}{p} \cdot t, \quad \mathrm{Var}\left[X(t)\right] = \frac{\lambda \cdot (2-p)}{p^2} \cdot t.$$

(4 p)

(c) It is being said that a Pólya-Aeppli process is a generalisation of the Poisson process (or that the Poisson process is a special case of Pólya-Aeppli process). Explain what this means. (1 p)

SOLUTIONS TO THE EXAM WEDNESDAY THE 20^{th} OF OCTOBER 2010 02.00 p.m.– 07.00 p.m..

Uppgift 1

The transformation $s = x, t = y - x$ maps the domain $\{0 < x < y\}$ to $\{0 < s, 0 < t\}$, and the inverse is given by $x = s, y = s + t$. The Jacobian is $|d(x, y)/d(s, t)| = 1$.
Since $f_{X,Y}(x, y) = 2e^{-x-y}$ for $0 < x < y$ we get by the theorem of transformation of densities (c.f., the collection of formulas) that the joint density for $(X, Y - X)$ becomes

$$2e^{-2s-t} \text{ for } 0 < s, 0 < t.$$

This density is seen to be factorized as $f_X(s) = 2e^{-2s}$, $s > 0$, and $f_{Y-X}(t) = e^{-t}$, $t > 0$, which shows the claimed independence. In addition it follows that $X \in \text{Exp}(1/2)$ och $Y - X \in \text{Exp}(1)$.

Uppgift 2

(a) From

$$Z = \begin{cases} 0 & \text{om } U = 1 \\ X & \text{om } U = 0. \end{cases}$$

we obtain, as $U \in \text{Be}(p)$ is independent of X that

$$\varphi_Z(t) = E\left[e^{itZ}\right] = EE\left[\left[e^{itZ}|U\right]\right]$$
$$= E\left[e^{it0}|U = 1\right] \cdot p + E\left[e^{itX}|U = 0\right] \cdot (1 - p)$$
$$= p + E\left[e^{itX}\right] \cdot (1 - p),$$

since $p = P(U = 1)$. The Appendix of Gut gives

$$E\left[e^{itX}\right] \cdot (1 - p) = \frac{1 - p}{1 + a^2t^2}.$$

Thus we have found

ANSWER 2(a): $\underline{\varphi_Z(t) = p + \frac{1-p}{1+a^2t^2}.}$

(b) Assume that the random variable W is independent of Z. Then

$$Y = W + Z$$

yields that the characteristic functions satisfy

$$\varphi_Y(t) = \varphi_W(t)\varphi_Z(t)$$

i.e., we can hope to find a solution to

$$\varphi_W(t) = \frac{\varphi_Y(t)}{\varphi_Z(t)}.$$

From (a) and the Appendix of Gut

$$\frac{\varphi_Y(t)}{\varphi_Z(t)} = \frac{\frac{1}{1+a^2t^2}}{p + \frac{1-p}{1+a^2t^2}}$$

$$= \frac{1}{p\cdot(1+a^2t^2)+1-p} = \frac{1}{1+a^2pt^2}$$

This is the characteristic function of $L(\sqrt{p}a)$.

ANSWER 2(b): $\underline{W \in L(\sqrt{p}a) \text{ and independent of } Z.}$

Uppgift 3

We use $\sqrt{n} = \frac{n}{\sqrt{n}}$ and write

$$\sqrt{n}\left(\frac{X_1 + X_2 + \ldots + X_n - 5n}{X_1^2 + X_2^2 + \ldots + X_n^2}\right) = \frac{\frac{1}{\sqrt{n}}\left(X_1 + X_2 + \ldots + X_n - 5n\right)}{\frac{1}{n}\left(X_1^2 + X_2^2 + \ldots + X_n^2\right)}.$$

As we are dealing with independent, identically distributed random variables with $E[X_1] = 5$ and $\operatorname{Var}[X_1] = 9$, the central limit theorem gives that

$$3 \cdot \frac{1}{3\sqrt{n}}\left(X_1 + X_2 + \ldots + X_n - 5n\right) \xrightarrow{d} N(0, 9),$$

as $n \to \infty$. In addition, the law of large numbers gives

$$\frac{1}{n}\left(X_1^2 + X_2^2 + \ldots + X_n^2\right) \xrightarrow{p} E\left[X_1^2\right],$$

where $E[X_1^2] = \operatorname{Var}[X_1] + (E[X_1])^2 = 9 + 25 = 34$. Hence the Slutzky theorem (Theorem 6.5 p. 168 in A. Gut: An Intermediate Course in Probability, 2nd Ed.) gives

$$\sqrt{n}\left(\frac{X_1 + X_2 + \ldots + X_n - 5n}{X_1^2 + X_2^2 + \ldots + X_n^2}\right) \xrightarrow{d} N\left(0, \frac{9}{34^2}\right).$$

ANSWER 3: $\underline{\text{The limiting distribution is } N\left(0, \tfrac{9}{34^2}\right).}$

Uppgift 4

We write

$$U(t) = e^{-t}U_0 + \frac{1}{\sqrt{2}}\int_0^t e^{-(t-u)}dW(u) \tag{1}$$

$$= e^{-t}U_0 + \frac{e^{-t}}{\sqrt{2}}\int_0^t e^{u}dW(u)$$

$$= e^{-t}U_0 + \frac{e^{-t}}{\sqrt{2}}\int_0^{t-1} e^u dW(u) + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u)$$

$$= e^{-1}e^{-(t-1)}U_0 + e^{-1}\frac{e^{-(t-1)}}{\sqrt{2}}\int_0^{t-1} e^u dW(u) + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u)$$

$$= e^{-1}\left[e^{-(t-1)}U_0 + \frac{e^{-(t-1)}}{\sqrt{2}}\int_0^{t-1} e^u dW(u)\right] + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u)$$

$$= e^{-1}U(t-1) + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u). \tag{2}$$

(a) It is clear by construction that $(U(t-1), U(t))^T$ is a bivariate Gaussian random variable. We need the parameters of the distribution. Then from (1)

$$E\left[U(t)\right] = e^{-t}E\left[U_0\right] + \frac{1}{\sqrt{2}}E\left[\int_0^t e^{-(t-u)}dW(u)\right] = 0 + 0 = 0.$$

Since U_0 is independent of the Wiener process, we get

$$\mathrm{Var}\left[U(t)\right] = e^{-2t}\mathrm{Var}\left[U_0\right] + \frac{1}{2}\mathrm{Var}\left[\int_0^t e^{-(t-u)}dW(u)\right]$$

By assumption $\mathrm{Var}\left[U_0\right] = 1$, and by the Collection of formulas

$$\frac{1}{2}\mathrm{Var}\left[\int_0^t e^{-(t-u)}dW(u)\right] = \frac{1}{2}e^{-2t}\mathrm{Var}\left[\int_0^t e^{u}dW(u)\right] = e^{-2t}\frac{1}{2}\int_0^t e^{2u}du$$

$$= e^{-2t}\frac{1}{4}\left[e^{2u}\right]_0^t = \frac{1}{4}e^{-2t}\left[e^{2t}-1\right].$$

Hence

$$\mathrm{Var}\left[U(t)\right] = e^{-2t} + \frac{1}{4} - \frac{1}{4}e^{-2t} = \frac{1}{4} + \frac{3}{4}e^{-2t}$$

The same argument applied to

$$U(t-1) = e^{-(t-1)}U_0 + \frac{e^{-(t-1)}}{\sqrt{2}}\int_0^{t-1} e^u dW(u)$$

gives

$$E\left[U(t-1)\right] = 0, \quad \mathrm{Var}\left[U(t-1)\right] = \frac{1}{4} + \frac{3}{4}e^{-2(t-1)}.$$

Next we need the covariance $E\left[U(t)U(t-1)\right]$, and when we use (2) we get

$$E\left[U(t)U(t-1)\right] = E\left[\left(e^{-1}U(t-1) + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u)\right)U(t-1)\right]$$

$$= E\left[e^{-1}U^2(t-1)\right] + E\left[U(t-1)\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^u dW(u)\right].$$

Here we get from the above that

$$E\left[e^{-1}U^2(t-1)\right] = e^{-1}\mathrm{Var}\left[U(t-1)\right] = e^{-1}\left(\frac{1}{4} + \frac{3}{4}e^{-2(t-1)}\right).$$

Since $U(t-1)$ is measurable with respect to the sigma field $\sigma\left(U_0, W(s) \mid s \leq t-1\right)$ generated by the Wiener process up to time $t-1$ and by the initial value, and the increments of the Wiener process are independent of $\sigma\left(U_0, W(s) \mid s \leq t-1\right)$, we get

$$E\left[U(t-1)\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u)\right] = E\left[U(t-1)\right] E\left[\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u)\right] = 0 \cdot 0.$$

ANSWER 4(a): $\underline{N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \frac{1}{4}+\frac{3}{4}e^{-2(t-1)} & e^{-1}\left(\frac{1}{4}+\frac{3}{4}e^{-2(t-1)}\right) \\ e^{-1}\left(\frac{1}{4}+\frac{3}{4}e^{-2(t-1)}\right) & \frac{1}{4}+\frac{3}{4}e^{-2t} \end{array}\right)\right)}.$

(b) To find the probability

$$P\left(U(t) > E\left[U(t) \mid U(t-1)\right] + 0.9\right)$$

we observe first by (2) that

$$E\left[U(t) \mid U(t-1)\right] = E\left[e^{-1}U(t-1) + \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u) \mid U(t-1)\right]$$

$$= E\left[e^{-1}U(t-1) \mid U(t-1)\right] + E\left[\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u) \mid U(t-1)\right]$$

and since we can take out what is known and since the increments of the Wiener process are independent of $\sigma\left(U_0, W(s) \mid s \leq t-1\right)$ generated by the Wiener process up to time $t-1$ and by the initial value

$$= e^{-1}U(t-1) + E\left[\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u)\right]$$

$$= e^{-1}U(t-1),$$

by a property of the Wiener integral, see the collection of formulas. Thus we see that

$$U(t) - E\left[U(t) \mid U(t-1)\right] = \frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u),$$

and hence

$$P\left(U(t) > E\left[U(t) \mid U(t-1)\right] + 0.9\right) = P\left(U(t) - E\left[U(t) \mid U(t-1)\right] > 0.9\right)$$

$$= P\left(\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u) > 0.9\right).$$

We have that

$$E\left[\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u)\right] = 0$$

and

$$\mathrm{Var}\left[\frac{e^{-t}}{\sqrt{2}}\int_{t-1}^{t} e^{u} dW(u)\right] = \frac{e^{-2t}}{2}\int_{t-1}^{t} e^{2u} du$$

$$= \frac{e^{-2t}}{4}\left[e^{2u}\right]_{t-1}^{t} = \frac{e^{-2t}}{4}\left[e^{2t} - e^{2(t-1)}\right]$$

$$= \frac{1}{4}\left(1 - e^{-2}\right).$$

Thus we have

$$\frac{e^{-t}}{\sqrt{2}} \int_{t-1}^{t} e^{u} dW(u) \in N\left(0, \frac{1}{4}\left(1 - e^{-2}\right)\right).$$

Let now, for ease of writing, $Z \equiv \frac{e^{-t}}{\sqrt{2}} \int_{t-1}^{t} e^{u} dW(u)$.

$$P(Z > 0.9) = P\left(\frac{Z}{\sqrt{\frac{1}{4}(1 - e^{-2})}} > \frac{0.9}{\sqrt{\frac{1}{4}(1 - e^{-2})}}\right) = 1 - \Phi\left(\frac{0.9}{\sqrt{\frac{1}{4}(1 - e^{-2})}}\right).$$

ANSWER 4(b): $\underline{P\left(U(t) > E\left[U(t) \mid U(t-1)\right] + 0.9\right) = 1 - \Phi\left(\frac{0.9}{\sqrt{(1-e^{-2})/4}}\right).}$

Uppgift 5

(a)

$$\mu_{\mathbf{Y}}(t) = E\left[Y(t)\right] = E\left[W(t+1) - W(t)\right] = E\left[W(t+1)\right] - E\left[W(t)\right] = 0 - 0 = 0,$$

as the mean function of the Wiener process is $= 0$. The autocorrelation function is given by

$$E\left[Y(t)Y(s)\right] = E\left[W(t+1)W(s+1)\right] - E\left[W(t+1)W(s)\right] - E\left[W(t)W(s+1)\right] + E\left[W(t)W(s)\right]$$

$$= \min(t+1, s+1) - \min(t+1, s) - \min(t, s+1) + \min(t, s). \qquad (3)$$

where we used the autocorrelation/autocovariance of the Wiener process. We must consider two cases.

i) $t \geq s$: Then we have from (3) that

$$E\left[Y(t)Y(s)\right] = s + 1 - s - \min(t, s+1) + s = s + 1 - \min(t, s+1). \qquad (4)$$

Here we must consider two additional cases: (1) $t \geq s+1$ (or $t - s \geq 1$), then we get in (4) that

$$E\left[Y(t)Y(s)\right] = s + 1 - (s+1) = 0.$$

Case (2) $t < s+1$ (or $t - s < 1$), then we get in (4) that

$$E\left[Y(t)Y(s)\right] = s + 1 - t = 1 - (t - s) = 1 - |t - s|.$$

since $t > s$.

ii) $t < s$: Then we have from (3) that

$$E\left[Y(t)Y(s)\right] = t + 1 - \min(t+1, s) - t + t = t + 1 - \min(t+1, s). \qquad (5)$$

Case (1) $s \geq t+1$,(or $s - t \geq 1$) then we get in (5) that

$$E\left[Y(t)Y(s)\right] = t + 1 - (t+1) = 0.$$

Case (2) $s < t+1$(or $s - t < 1$), then we get in (5) that

$$E\left[Y(t)Y(s)\right] = t + 1 - s = 1 - (s - t) = 1 - |t - s|.$$

Note that $(s - t) = |t - s|$, if $t < s$.

When we gather the results in one formula we get the final answer.

$$E\left[Y(t)Y(s)\right] = \begin{cases} 1-|t-s| & |t-s| \leq 1 \\ 0 & |t-s| > 1 \end{cases}$$

This can be expressed even more compactly as follows.

ANSWER 5(a): $\underline{\mu_{\mathbf{Y}}(t) = 0,\ R_{\mathbf{Y}}(h) = \max\left(0, 1-|h|\right), h = t-s.}$

(b) Is the process $\{Y(t) \mid t \geq 0\}$ strictly stationary ? Since the mean function is a constant ($= 0$) and the autocorrelation $E\left[Y(t)Y(s)\right]$ is only a function of the time difference $t-s$, we can say that the process satisfies the conditions required to be called weakly stationary. In addition, since the Wiener process is a Gaussian process, and $\{Y(t) \mid t \geq 0\}$ is a linear transformation of the Wiener process, $\{Y(t) \mid t \geq 0\}$ is a Gaussian process. Weakly stationary Gaussian processes are strictly stationary. To be quite precise with definitions, we should perhaps define, by reflection, $\{Y(t) \mid t \geq 0\}$ for negative t, too.

Uppgift 6

(a) We use the following result, which depends on the assumption that N is independent of the I.I.D. X_i, $i = 1, 2, \ldots,$; the moment generating function of $X(t)$ is

$$\psi_{X(t)}(s) = g_{N(t)}\left(\psi_X(s)\right),$$

where $g_{N(t)}$ is the probability generating function of $N(t)$ and $\psi_X(s)$ is the moment generating function of any X_i.

The moment generating function of $X_i \in \mathrm{Fs}\,(p)$ is

$$\psi_{X(t)}(s) = E\left[e^{sX(t)}\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1}e^{sk}$$

$$= pe^s \sum_{k=1}^{\infty}(1-p)^{k-1}e^{s(k-1)} = pe^s \sum_{k=0}^{\infty}((1-p)e^s)^k$$

$$= pe^s \frac{1}{1-(1-p)e^s}$$

which requires that $(1-p)e^s < 1$. From the collection of formulas we know that

$$g_{N(t)}(s) = e^{\lambda t(s-1)},$$

as $N(t) \in \mathrm{Po}(\lambda t)$. This gives

$$\psi_{X(t)}(s) = g_{N(t)}\left(\psi_X(s)\right) = e^{\lambda t(\psi_{X(t)}(s)-1)}$$

$$= e^{\lambda t(pe^s \frac{1}{1-(1-p)e^s} - 1)},$$

as was to be shown.

(b) We are asked to show that

$$E\left[X(t)\right] = \frac{\lambda}{p} \cdot t, \quad \mathrm{Var}\left[X(t)\right] = \frac{\lambda \cdot (2-p)}{p^2} \cdot t.$$

This can be done by evaluating $E\left[X(t)\right] = \frac{d}{ds}\psi_{X(t)}(s) \mid_{s=0}$ and $E\left[X^2(t)\right] = \frac{d^2}{ds^2}\psi_{X(t)}(s) \mid_{s=0}$. An easier way is to recall/derive the result in Theorem 6.2 p. 81 in Gut, which gives

$$E\left[X(t)\right] = E\left[N(t)\right] E\left[X\right] = \lambda t \cdot \frac{1}{p},$$

since $N(t) \in \mathrm{Po}(\lambda t)$ and $X \in \mathrm{Fs}\,(p)$, see Appendix B of Gut. Again, the result in Theorem 6.2 p. 81 in Gut and Appendix B of Gut yield

$$\begin{aligned}\mathrm{Var}\left[X(t)\right] &= E\left[N(t)\right] \mathrm{Var}\left[X\right] + \left(E\left[X\right]\right)^2 \mathrm{Var}\left[N(t)\right] \\ &= \lambda t \frac{1-p}{p^2} + \frac{1}{p^2}\lambda t \\ &= \lambda t \left(\frac{1-p+1}{p^2}\right) = \lambda t \left(\frac{2-p}{p^2}\right),\end{aligned}$$

as was to be proved.

(c) If we take $p = 1$ with $X_i \in \mathrm{Fs}\,(p)$, then for every i

$$P\left(X_i = 1\right) = 1.$$

Then with probability one,

$$X(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i=1}^{N(t)} 1 = N(t).$$

Thus $X(t) = N(t)$ for $p = 1$, and therefore the Pólya-Aeppli process is a generalization of the Poisson process and, the Poisson process is a special case of the Pólya-Aeppli process. Note also that for $p = 1$ $E\left[X(t)\right] = \lambda t$ and $\mathrm{Var}\left[X(t)\right] = \lambda t$, as they should.