# ASTR 535 Lecture Notes

Jon Holtzman

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course website: http://astronomy.nmsu.edu/holtz/a535

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## Properties of light, magnitudes, errors, and error analysis

### Light

Wavelength regimes:

- gamma rays
- x-rays
- ultraviolent (UV)
  - near: 900–3500 Å
  - far: 100–900 Å

  The 900 Å break is because of the Lyman limit at 912 Å. This is where neutral hydrogen is ionized, so the universe is largely opaque to wavelengths shorter than this.
- visual (V): 4000–7000 Å (note that 'V' is different from 'optical', which is slightly broader: 3500–10000 Å. The 3500 Å cutoff is due to the Earth's atmosphere being opaque to wavelengths shorter than this).
- IR
  - near: 1–5 $\mu$ (1–10 $\mu$ in online notes)
  - mid: (10–100 $\mu$)
  - far: 5–100 $\mu$ (100–1000 $\mu$)
- sub-mm 500–1000 $\mu$
- microwave
- radio

Quantities of light:

- Intensity $I(\theta, \phi)$ [erg s$^{-1}$ $\nu^{-1}$ $\Omega^{-1}$]: Encapsulates *direction* light is coming from. Also known as radiance.
- Surface Brightness (SB) [erg s$^{-1}$ cm$^{-2}$ $\nu^{-1}$ sterradian$^{-1}$]: amount of energy *received* in a unit surface element per unit time per unit frequency (or wavelength) from a unit solid angle in the direction $(\theta, \phi)$, where $\theta$ is the angle away from the normal to the surface element, and $\phi$ is the azimuthal angle. To calculate SB, just divide the flux by the angle subtended by the object [rad$^2$]. At a larger distance, the flux will be smaller, but so will the angle subtended by the object, so SB is independent of distance unless considering cosmological scales, where the curvature of spacetime has an effect. If SB is per unit *area*, how is that independent of distance?
- Flux (F) [erg s$^{-1}$ cm$^{-2}$ $\nu^{-1}$]: amount of energy passing through a unit surface element in all directions, defined by

$$F_\nu = \int I_\nu \cos(\theta) \mathrm{d}\Omega \tag{1}$$

  where d$\Omega$ is the solid angle element, and the integration is over the entire solid angle. Intensity is per solid angle. All the solid angles that make up a sphere add up (integrate) to get the total flux through that surface. The $\cos(\theta)$ factor is important for, e.g., ISM where light is coming from all directions, but for tiny objects, $\theta$ is negligibly small and can be dropped. Integrates over *all* directions. Also known as irradiance. Can also be per $\lambda$, obviously.
- Luminosity (L) [erg s$^{-1}$]: *intrinsic* energy emitted by the source per second ($\sim$ power). For an isotropically emitting source,

$$L = 4\pi d^2 F \tag{2}$$

  where $d =$ distance to source (so L can only be calculated if the distance is known). Also known as radiant flux.

What to measure for sources:

- Resolved: directly measure surface brightness (intensity) distribution on the sky, usually over some bandpass or wavelength interval.
- Unresolved: measure the flux. Diffraction is the reason stellar surfaces cannot be resolved. Because of this, we cannot measure SB, so we measure flux, integrated over the entire object.

Questions:

- What are the dimensions of the three quantities: luminosity, surface brightness (intensity), and flux?
- How do the three quantities depend on distance to the source?
- To what quantity is apparent magnitude of a star related?
- To what quantity is the absolute magnitude related?

*specific* flux: per unit wavelength (or frequency). Using $\lambda = \frac{c}{\nu} \rightarrow \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} = \frac{-c}{\nu^2}$

$$\begin{aligned} \int F_\nu \mathrm{d}\nu &= -\int F_\lambda \mathrm{d}\lambda \\ F_\nu \mathrm{d}\nu &= -F_\lambda \mathrm{d}\lambda \\ F_\nu &= -F_\lambda \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \\ &= F_\lambda \frac{c}{\nu^2} \\ &= F_\lambda \frac{\lambda^2}{c} \end{aligned}$$

The negative comes from frequency and wavelength increasing in opposite directions. Note that a constant $F_\lambda$ implies a *non*-constant $F_\nu$ and vice versa. For example, a wavelength difference of 100 Å around 1215 Å and 6563 Å corresponds to a frequency difference of 2200 GHz and 71 GHz, respectively.

Units: cgs, magnitudes, Jansky (a flux density unit: 1 Jy = $10^{-26}$ W m$^{-2}$ Hz$^{-1}$)

Terminology of measurements:

- photometry (broad-band flux measurement): SB or flux, integrated over some wavelength range.
- spectroscopy (*relative* measurement of fluxes at different wavelengths): $f(\lambda)$
- spectrophotometry (*absolute* measurement of fluxes at different wavelengths): $f(\lambda)$
- astrometry: concerned with positions of observed flux, not brightness, but direction.
- morphology: intensity as a function of position; often, absolute measurements are unimportant. Deals with *resolved* objects, intensity as function of position.

In general:

- photometry: measure *flux*
- spectroscopy: flux *density* (per unit wavelength, down to the resolution of the spectrograph)

In practice, measure *photon* flux [photons cm$^{-2}$ s$^{-1}$] with a "photon counting device", (rather than energy flux, which is done with bolometers). The monochromatic photon flux is given by the energy flux ($F_\lambda$) divided by the energy per photon ($E_{photon} = \frac{hc}{\lambda}$), or

$$\text{photon flux} = \int F_\lambda \frac{\lambda}{hc} \mathrm{d}\lambda$$

*Have a complete understanding of the difference between intensity, flux, and luminosity and their units. Recognize and understand that these can be specified per unit wavelength or per unit frequency and how to convert between the two.*

## Magnitudes and photometric systems

Magnitudes are related to flux (or SB or L) by:

$$m_1 - m_2 = -2.5 \log \frac{b_1}{b_2}$$

or for a single object:

$$\begin{aligned} m &= -2.5 \log \frac{F}{F_0} \\ &= -2.5 \log F + 2.5 \log F_0 \end{aligned}$$

where the coefficient of proportionality, $F_0$, depends on the definition of the photometric system, and the quantity $2.5 \log F_0$ may be referred to as the photometric system *zeropoint*. (Note that this relationship holds *regardless of what photometric system you are using*. Inverting, we get:

$$F = F_0 \times 10^{-0.4\mathrm{m}}$$

Flux *density*: flux per unit wavelength/frequency (aka *monochromatic* flux as opposed to integrated flux). This can be specified by monochromatic magnitudes:

$$F_\lambda = F_0(\lambda) \times 10^{-0.4\mathrm{m}(\lambda)}$$

although spectra are more often given in flux units than in magnitude units. Note that it is possible that $F_0$ is a function of wavelength.

Since magnitudes are logarithmic, the *difference* between magnitudes corresponds to a *ratio* of fluxes; ratios of magnitudes are generally unphysical. If one is just doing relative measurements of brightness between objects, this can be done without knowledge of $F_0$ (or, equivalently, the system zeropoint); objects that differ in brightness by $\Delta$M have the same ratio of brightness ($10^{-0.4\Delta M}$) regardless of what photometric system they are in. The photometric system definitions and zeropoints are only needed when converting between calibrated magnitudes and fluxes. In other words, if you reference the brightness of A relative to the brightness of B, a magnitude system can be set up with B as the reference source. However, the utility of a system when doing astrophysics generally requires an understanding of the actual fluxes.

*Know how magnitudes are defined, and that <u>relative</u> fluxes can be represented as magnitudes independent of the magnitude system.*

Monday, January 25

Three main types of magnitude systems in use in astronomy:

1. STMAG
2. ABNU
3. UBVRI

The STMAG and ABNU magnitude systems are the simplest. In these systems, the reference flux is just some *constant* value in $F_\lambda$ or $F_\nu$. However, these are not always the most widely used systems in astronomy, because no natural source exists with a flat spectrum.

**STMAG (Space Telescope MAGnitude) system:** defined relative to $F_\lambda$. The reference flux is given by

$$F_{0,\lambda} = 3.60 \times 10^{-9} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Å}^{-1}$$

which is equal to the flux of Vega at $\lambda$ = 5500 Å; hence a star of Vega's brightness at 5500 Å is defined to have m=0 (i.e., if $F = F_0$, then $m = \log(F/F_0) = \log(1) = 0$). Alternatively (using $m = -2.5 \log \frac{F}{F_0}$):

$$m_{\text{STMAG}} = -2.5 \log F_\lambda - 21.1$$

for $F_\lambda$ in cgs units (using cm$^{-2}$ Å$^{-1}$ distinguishes between the collecting area and the wavelength, since both are units of distance. Be careful with units when doing these conversions; as long as the proper flux units come out at the end, the answer should be correct).

**ABNU system:** defined relative to $F_\nu$. The reference flux is given by

$$F_{0,\nu} = 3.63 \times 10^{-20} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} = F_{\nu,Vega}$$

or

$$m_{\text{ABNU}} = -2.5 \log F_\nu - 48.6$$

for $F_\nu$ in cgs units.

Magnitudes usually refer to the *integrated* flux (over a spectral bandpass, not just a single wavelength). The STMAG and ABNU integrated systems are defined relative to sources of *constant* $F_\lambda$ and $F_\nu$, respectively

$$m_{\text{STMAG}} = -2.5 \log \frac{\int F_\lambda \lambda \mathrm{d}\lambda}{\int 3.6 \times 10^{-9} \lambda \mathrm{d}\lambda}$$
$$m_{\text{ABNU}} = -2.5 \log \frac{\int F_\nu \nu^{-1} \mathrm{d}\nu}{\int 3.6 \times 10^{-20} \nu^{-1} \mathrm{d}\nu}$$

These are defined to be the same at 5500 Å. The factors of $\lambda$ and $\nu$ come from the conversion factor $hc/\lambda$ for photon-counting detectors, where $h$ and $c$ cancel in each flux ratio.

Note that these systems differ by more than a constant, because one is defined by units of $F_\lambda$ and the other by $F_\nu$, so the difference betwen the systems is a function of wavelength. (Question: what's the relations between $m_{STMAG}$ and $m_{ABNU}$?)

Note also that, using magnitudes, the measured magnitude is nearly independent of bandpass width, so a broader bandpass does not imply a brighter (smaller) magnitude, which is not the case for fluxes. The reference is being integrated as well, so they cancel.

The standard UBVRI broadband photometric system, as well as several other magnitude systems, however, are not defined for a constant (flat) $F_\lambda$ or $F_\nu$ spectrum; rather, they are defined relative to the spectrum of an A0V star. Most systems are defined (or at least were originally) to have the magnitude of Vega be zero in all bandpasses (VEGAMAGS); if you ever get into this in detail, note that this is not exactly true for the UBVRI system.

For the broadband UBVRI system, we have

$$m_{\text{UBVRI}} \approx -2.5 \log \frac{\int_{UBVRI} F_\lambda(object) \lambda \mathrm{d}\lambda}{\int_{UBVRI} F_\lambda(Vega) \lambda \mathrm{d}\lambda}$$

(as above, the factor of $\lambda$ comes in for photon counting detectors). This gives the magnitude in U, B, V, R, *or* I, by integrating over that same bandpass. The UBVRI filter set had overlapping bandpasses, so there was a switch to interference filter: the ugriz system used by SDSS.

Here is a plot to demonstrate the difference between the different systems.

While it seems that STMAG and ABNU systems are more straightforward, in practice it is difficult to measure absolute fluxes, and much easier to measure relative fluxes between objects. Historically, observations were tied to observations of Vega (or to stars which themselves were tied to Vega), so VEGAMAGs made sense, and the issue of determining physical fluxes boiled down to measuring the physical flux of Vega. Today, in some cases, it may be more accurate to measure the absolute throughput of an instrumental system, and use STMAG or ABNU.

*Know that there are several different magnitude systems in use, and understand how they differ. Know when it is important to know what the magitude system is, and when it isn't.*

**Colors**

In the UBVRI system, the *difference* between magnitudes (e.g. B-V) gives the ratio of the fluxes in different bandpasses ($F_B/F_V$) *relative to the ratio of the fluxes of an A0V star in the same two bandpasses* (for VEGAMAG). Note the typical colors of astronomical objects, which are different for the different photometric systems. Type 'A' stars have color = 0, and have the same SED as Vega. Type 'O' stars have color < 0, while cooler stars have color > 0.

Sloan system: ugriz (g-r=0 indicates constant $F_\nu$).

$$m = -2.5 \log \frac{\int_B F_\lambda \mathrm{d}\lambda}{\int_V F_\lambda \mathrm{d}\lambda}$$

if B-V=0, then $(\mathrm{B/V})_{object}$ is the same as $(\mathrm{B/V})_{Vega}$, or $\left(\frac{\int F_\nu}{\int F_\lambda}\right)_{obj} = \left(\frac{\int F_\nu}{\int F_\lambda}\right)_{Vega}$. An object with B-V=0 has the same spectral *shape* as Vega. Not necessarily the same flux value at each bandpass Keep in mind that the UBVRI system is defined relative to spectrum of an A0V star (not simply a flat spectrum, like the STMAG and ABNU systems).

To know how much observing time is needed, you need to know the flux for object with some magnitude, x. E.g., convert the spectrum of an elliptical galaxy to color; if you know $F$ in one filter, you can get $F$ in another filter.

Questions:

- Which is closer to the UBVRI system, STMAG or ABNU?
- What would typical colors be in a STMAG or ABNU system?

*Understand how colors are represented by a difference in magnitude, and recognize how colors expressed in magnitude are related to the shape of the underlying spectrum, with differences for different magnitude systems.*

### UBVRI magnitudes-flux conversion

Wednesday, January 27

To convert Vega-based magnitudes to fluxes, just look up the flux of Vega at the center of the passband. However, if the spectrum of the object differs from that of Vega, this won't be perfectly accurate. Given UBVRI magnitudes of an object in the desired band, filter profiles (e.g. Bessell 1990, PASP 102,1181), and absolute spectrophotometry of Vega (e.g., ), Bohlin & Gilliland 2004, AJ 127, 3508, one can determine the flux.

To estimate the flux of some object in an arbitrary bandpass given just the V magnitude of an object (a common situation used when trying to predict exposures times, see below), estimate the spectral energy distribution (SED). The integral of the SED can be computed over the V bandpass if the filter profiles are known. The scaling is determined by comparing this integral with that of the spectrum of Vega over the same bandpass. The normalized SED is then used to compute the flux in any desired bandpass.

(Some possibly useful references for SEDs are: Bruzual, Persson, Gunn, & Stryker; Hunter, Christian, & Jacoby; Kurucz). Things are certainly simpler in the ABNU or STMAG system, and there has been some movement in this direction: the STScI gives STMAG calibrations for HST instruments, and the SDSS photometric system is close to an ABNU system. However, even when the systems are conceptually well defined, determining the absolute calibration of any photometric system is very difficult in reality, and determining absolute fluxes to the 1% level is very challenging.

As a separate note on magnitudes themselves, note that some people, in particular, the SDSS, have adopted a modified type of magnitudes, called *asinh* magnitudes, which behave like normal (also known as Pogson) magnitudes for brighter objects, but have different behavior for very faint objects (near the detection threshold). See Lupton, Gunn, & Szalay 1999 AJ 118, 1406 for details.

## Observed fluxes and the count equation

If you are measuring flux with an actual instrument, the *intrinsic* photon flux from the source is not trivial to determine from the number of photons that you count. To get the number of photons that you count in an observation, you need to take into account

- The area of your photon collector (telescope)
- Photon losses and gains from the Earth's atmosphere (which change with conditions)
- The efficiency of your collection/detection apparatus (which can change with time).

Generally, the astronomical signal (which might be a flux or a surface brightness, depending on whether the object is resolved) can be written in the form of the ***count equation:***

$$S = Tt \int \frac{F_\lambda}{\left(\frac{hc}{\lambda}\right)} q_\lambda a_\lambda \mathrm{d}\lambda \equiv TtS'$$

- $S$: total number of photons observed (the "signal")
- $S'$: observed flux rate [photons s$^{-1}$ cm$^{-2}$], with all of the real details of the observing system included.
- $a_\lambda$: atmospheric transmission (or *throughput*), the fraction of photons that make it to the detector. A typical value is about 0.9, or $\sim$90% of the photons. Variable in both space and time.
- $q_\lambda$: system efficiency (telescope, instrument, filters, and detector). Can be time-variable.
- $T$: telescope collecting area
- $t$: integration time (total amount of time spent collecting photons).
- $\frac{F_\lambda}{\left(\frac{hc}{\lambda}\right)}$: flux divided by the energy per photon; gives the number of photons per second per square cm.

$T$ and $t$ are the *only* terms that do not depend on the wavelength (or frequency).

Most of this information isn't needed to go backward from $S$ to $F_\lambda$, since the terms can be very difficult to measure precisely. Instead, most observations are performed differentially to a set of other stars of well-known brightness. If the stars of known brightness are observed in the same observation, then the atmospheric term is (approximately) the same for all stars. This is known as *differential photometry.* From the photon flux of the object with known brightness, an "exposure efficiency" or an "effective area" for this exposure can be determined. Equivalently, and more commonly, an *instrumental magnitude* can be calculated:

$$m = -2.5 \log \frac{S}{t}$$

Normalize by the exposure time, $t$, to get counts $\mathrm{s}^{-1}$, (although this is not strictly necessary). Then determine the *zeropoint*, $z$ (which describes the throughput of the system). Adding $z$ to the instrumental magnitude gives the *calibrated* magnitude, $M$ (which is still an *apparent* magnitude).

$$M = m + z$$

Note that in the real world, one has to also consider possible differences between a given experimental setup and the setup used to measure the reference brightnesses, so this is only a first approximation (i.e., the zeropoint may be different for different stars with different spectral properties). If using instrumental mags including exposure time normalization, the zeropoint gives the magnitude of a star that will give 1 count $\mathrm{s}^{-1}$.

$$M = -2.5 \log \frac{S}{t} + z$$

Example: to go from observed to emitted $\rightarrow$, take an additional observation of an object with known brightness (SDSS has list of these). A star that is 10 times fainter than one with g=18 has g=20.5.

$$m = -2.5 \log(qF)$$

$$m = -2.5 \log F - 2.5 \log q$$

the quantity '$-2.5 \log q$' is the zeropoint.

If there is no nearby star, $q$ is still the same, but changes in $a$, effects of the atmosphere, need to be calibrated. This is known as *all-sky*, or absolute, photometry. This requires that the sky is "well-behaved", i.e., the atmospheric throughput as a function of position can be accurately predicted, which requires *photometric* weather (no clouds). Differential photometry can be done in non-photometric weather, hence it is much simpler. It is always possible to obtain differential photometry, then later obtain absolute photometry of the reference stars.

To figure out what the fluxes of the calibrating stars actually are requires understanding all of the terms in the count equation. This is challenging, and often absolute calibration of a system is uncertain to a couple of percent.

It is common to stop with differential photometry when studying variable objects, where only the *change* in brightness is important. In this case, the brightness of the target needs to be referenced relative to another object (or ensemble of objects) in the field that are *non*-variable.

While the count equation isn't usually used for calibration, it is very commonly used for computing the approximate number of photons you will receive from a given source in a given amount of time for a given observational setup. This number is critical to know in order to estimate your expected errors and exposure times in observing proposals, observing runs, etc. Understanding errors in absolutely critical in all sciences, and maybe even more so in astronomy, where objects are faint, photons are scarce, and errors are not at all insignificant. The count equation provides the basis for exposure time calculator (ETC) programs, because it gives $N_{\mathrm{photons}}(t)$. As we will see shortly, this provides the information we need to calculate the *uncertainty* in the measurement as a function of exposure time.

Random note: The field of view (FOV) is usually in arcminute scales.

*Understand the count equation and the terms in it. Understand the distinction between estimating count rates from an understanding of all the terms in the count equaiton vs. measuring the overall throughput (zeropoint) by observing stars of known brightness. Know instrumental magnitudes and zeropoints are.*

## Uncertainties in photon rates

Useful reference: Statistics

For a given rate of *emitted* photons, there's a probability function which gives the rate of *detected* photons. Even assuming 100% detection efficiency, there are still *statistical* uncertainties, and possibly *instrumental* uncertainties.

***General probability distributions and their characteristics***

$p(x)dx \equiv$ probability of event occuring in $x + \mathrm{d}x$:

$$\int p(x)\mathrm{d}x = 1$$

Some definitions relating to values that characterize a distribution:

$$\text{mean} \equiv \mu = \int x p(x)\mathrm{d}x$$

$$\text{variance} \equiv \sigma^2 = \int (x-\mu)^2 p(x)\mathrm{d}x$$

$$\text{standard deviation} \equiv \sigma = \sqrt{\text{variance}}$$

$$\frac{\int_{-\infty}^{x_{median}} p(x)\mathrm{d}x}{\int_{-\infty}^{\infty} p(x)\mathrm{d}x} = \frac{1}{2}$$

mode: most probable value (peak in plot).

Monday, February 1

Note that the geometric interpretation of the above (well-defined) quantities depends on the nature of the distribution. There is a difference between the *sample* quantities and the *population* quantities. The latter apply to the true distribution, while the former are *estimates* of the latter from some finite sample ($N$ measurements) of the population. The sample values approach the true ones as $N$ approaches infinity, but for small samples, they may differ.

The sample quantities are derived from:

$$\text{sample mean: } \bar{x} \equiv \frac{\sum x_i}{N}$$

$$\text{sample variance: } \equiv \frac{\sum (x_i - \bar{x})^2}{N-1} = \frac{\sum x_i^2 - (\sum x_i)^2/N}{N-1}$$

***the binomial distribution***

For astronomical observations, the photon distribution can be derived from the *binomial* distribution:

$$P(x,n,p) = \frac{n! p^x (1-p)^{n-x}}{x!(n-x)!}$$

where

- $x$: number of photons in a single event
- $n$: total number of events
- $p$: probability of observing $x$

under the assumption that all events are independent of each other (e.g., when rolling dice, the result of the second roll is independent of the result of the first). For the binomial distribution:

$$\text{mean} \equiv \mu \equiv \int xp(x)\mathrm{d}x = np$$

$$\text{variance} \equiv \sigma^2 \equiv \int (x-\mu)^2 p(x)\mathrm{d}x = np(1-p)$$

*Understand the concept of probability distribution functions and basic quantities used to describe them: mean, variance, standard deviation, median, and mode. Understand the difference between population quantities and sample quantities.*

***The Poisson distribution***

Unlike a gaussian, a poisson distribution is not symmetric about the mean (you can't detect a negative number of photons); it cuts off at zero.

For photon detection, $n$ is the total number of photons emitted, and $p$ is the probability of detecting a single photon out of the total emitted during the observation. We don't know either of these numbers, but we do know that p<<1 and we can estimate the mean number detected:

$$\mu = np$$

In this limit, the binomial distribution asymtotically approaches the *Poisson* distribution:

$$P(x,\mu) = \frac{\mu^x e^{-\mu}}{x!}$$

From the expressions for the binomial distribution in this limit, the mean of the distribution is $\mu$, and the variance is given by:

$$\sigma^2 = \sum_x [(x-\mu)^2 p(x,\mu)]$$

$$\sigma^2 = np = \mu$$

In other words, the standard deviation, $\sigma$, is the square root of the mean. **This is an *important result*.**

Note that the Poisson distribution is generally the appropriate distribution for *any* sort of counting experiment where a series of events occurs with a known average rate, and are independent of time since the last event.

Plots of what the Poisson distribution looks like for $\mu = 2$, $\mu = 10$, $\mu = 10000$.

*Understand the Poisson distribution and when it applies. Know how the variance/standard deviation of the Poisson distribution is related to the mean of the distribution.*

### *The normal, or Gaussian, distribution*

For large $\mu$, the Poisson distribution is well-approximated around the peak by a Gaussian distribution:

$$P(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

This allows us to use "simple" least squares techniques to fit observational data, which is normally distributed. However, in the *tails of the distribution*, and at *low mean rates*, the Poisson can differ significantly from a Gaussian. In these cases, least-squares may not be appropriate to model observational data, maximum likelihood techniques need to be considered.

*central limit theorem*: if a quantity depends on a number of independent random variables with ANY distribution, the quantity itself will be distributed normally (see statistics texts). In observations, we encounter the normal distribution because *readout* noise is distributed normally. Normal distributions occur often in nature.

*Know what a Gaussian is, including the full functional form. Understand under what circumstances the Poisson distribution is similar to a normal distribution.*

### *Importance of error distribution analysis*

The expected uncertainties in observations need to be understood in order to:

- predict the amount of observing time needed to get uncertainties as small as they need to be.
- Figure out if the scatter in the observed data is consistent with expected uncertainties. If not, then you've either learned some astrophysics or you don't understand something about your observations. This is especially important in astronomy where objects are faint and many projects are pushing down into the noise as far as possible. This can usually only be answered probabilistically. Generally, tests compute the probability

that the observations are consistent with an expected distribution (the null hypothesis). If it is low, the null hypothesis can be rejected.

- interpret your results in the context of a scientific prediction

***Confidence levels***

For example, say we want to know whether some single point is consistent with expectations, e.g., we see a bright point in multiple measurements of a star, and want to know whether the star flared. Say we have a time sequence with known mean and variance, and we obtain a new point, and want to know whether it is consistent with known distribution?

If the form of the probability distribution is known, then you can calculate the probability of getting a measurement more than some observed distance from the mean. In the case where the observed distribution is Gaussian (or approximately so), this is done using the *error function* (sometimes called $erf(x)$), which is the integral of a gaussian from some starting value.

Some simple guidelines to keep in mind follow (the actual situation often requires more sophisticated statistical tests). First, for Gaussian distributions, you can calculate that 68% of the points should fall within $\pm 1\sigma$ from the mean, and 95.3% should fall within $\pm 2\sigma$ from the mean. Thus, if you have a time line of photon fluxes for a star, with $N$ observed points, and a photon noise $\sigma$ on each measurement, you can test whether the number of points deviating more than $2\sigma$ from the mean is much larger than expected. To decide whether any single point is really significantly different, you might want to use more stringent criterion, e.g., $5\sigma$ rather than a $2\sigma$ criterion; a $5\sigma$ has much higher level of significance. On the other hand, if you have far more points in the range $2 - 4\sigma$ brighter or fainter than you would expect, you may also have a significant detection of intensity variations (provided you really understand your uncertainties on the measurements).

Also, note that your observed distribution should be consistent with your uncertainty estimates given the above guidelines. If you have a whole set of points, that all fall within $1\sigma$ of each other, something is wrong with your uncertainty estimates (or perhaps your measurements are correlated with each other).

For a series of measurements, one can calculate the $\chi^2$ statistic, and determine how probable this value is, given the number of points.

$$\chi^2 = \sum [(observed(i) - model(i))^2 / \sigma_i^2]$$

A quick estimate of the consistency of the model with the observed data points can be made using reduced $\chi^2$, defined as $\chi^2$ divided by the *degrees of freedom* (number of data points minus number of parameters).

$$\chi^2 = \sum_N \frac{(y_i - y(x_i))^2}{\sigma_i^2}$$

for $N$ data points. If every point was $1\sigma$ off, $\chi^2 \sim$ number of data points. Can calculate the probability distribution of $\chi^2$. If $\chi^2 = 98$, what's the confidence interval? Will only get 98 one in a million times? Not a good model. Is the data consistent with the model?

Wednesday, February 3

**Noise equation: how do we predict expected uncertainties?**

*Signal-to-noise*

Shape parameterized by stddev which $= \sqrt{\mu} \rightarrow$ Fundamental! e.g. source emitting 1000 photons has uncertainty of 100, etc. (telescope, brightness of source, etc. don't matter… only number of photons, no matter how you got them).

$$\frac{S}{N} = \frac{S}{\sqrt{S}} = \sqrt{S}$$

How does $S/N$ change for a given object?

Astronomers often describe uncertainties in terms of the fractional error, e.g. the amplitude of the uncertainty divided by the amplitude of the quantity being measured (the signal); often, the inverse of this, referred to as the signal-to-noise ratio, is used. Given an estimate the number of photons expected from an object in an observation, we can calulate the signal-to-noise ratio:

$$\frac{S}{N} = \frac{S}{\sqrt{\sigma^2}}$$

which is the inverse of the predicted fractional error ($N/S$).

Consider an object with observed photon flux, $S'$ [cm$^{-2}$ s$^{-1}$], leading to a signal, $S = S'Tt$. In the simplest case (i.e. perfect instruments), the only noise source is Poisson statistics from the source, in which case:

$$\sigma^2 = S = S'Tt$$

$$\frac{S}{N} = \sqrt{S} = \sqrt{S'Tt}$$

In other words, $S/N$ increases as the square root of the object brightness, telescope area, efficiency, or exposure time. Note that $S$ is directly observable, so one can calculate $S/N$ for an observation without knowing the telescope area or exposure time. We've just broken $S$ down so that you can specifically see the dependence on telescope area and/or exposure time.

*Background noise*

Rayleigh scattering

A more realistic case includes the noise contributed from Poisson statistics of "background" light, $B'$, which has units of flux per area on the *sky*, not the detector (i.e. a surface brightness); note that this is also usually given in magnitudes (more on the physical nature of this later).

$$B' = \int \frac{B_\lambda}{\frac{hc}{\lambda}} q_\lambda \mathrm{d}\lambda$$

$B_\lambda$ is the sky brightness at $\lambda \ldots$? And $B'$ is the observed flux from the background. So $B$ vs. $B'$ is the same as $S$ vs. $S'$.

The amount of background in our measurement depends on how much sky area we observe. Say we just use an aperture with area, $A$, so the total observed background counts is

$$AB = AB'Tt$$

Again, $B'Tt$ is the directly observable quantity, but we split it into the quantities on which it depends to understand what factors are important in determining $S/N$.

The total number of photons observed, $O$, is a combination of photons from the source ($S$), and phtons from the background ($AB$):

$$O = S + AB = (S' + AB')Tt$$

The variance of the total observed counts, from Poisson statistics, is:

$$\sigma_O^2 = O = S + AB = (S' + AB')Tt$$

To get the desired signal from the object only, we will need to measure separately the total signal and the background signal to estimate:

$$S \equiv S'Tt = O - A < B >$$

where $< B >$ is some estimate we have obtained of the background surface brightness. The noise in the measurement is

$$\sigma_S^2 \approx \sigma_O^2 = S + AB = (S' + AB')Tt$$

where the approximation is accurate if the background is determined to high accuracy, which one can do if one measures the background over a large area, thus getting a large number of background counts (with correspondingly small fractional error in the measurement).

This leads to a common form of the *noise equation*:

$$\frac{S}{N} = \frac{S}{\sqrt{S + AB}}$$

Breaking out the dependence on exposure time and telescope area, this is:

$$\frac{S}{N} = \frac{S'}{\sqrt{S' + AB'}}\sqrt{T}\sqrt{t}$$

Limiting regimes:

- *signal-limited* case: $S' \gg B'A$, the background goes away and we get

$$\frac{S}{N} = \sqrt{S} = \sqrt{S'tT}$$

- *background-limited* case: $B'A \gg S'$ and

$$\frac{S}{N} = \frac{S}{\sqrt{BA}} = \frac{S'}{\sqrt{B'A}}\sqrt{tT}$$

starlight dominates by factor of 1000 for $m_* = 12.5$ and $m_{BG} = 20$ ($\Delta m = 7.5 = -2.5 \log \frac{b_{BG}}{b_*} \rightarrow 10^3$)

$\odot$: little circle around star to reduce sky background noise. Depends on how blurry the star is; determined by seeing.

- Dust scattering in the plane of the solar system
- Diffraction limit of space telescope $\sim$ 20th of an arcsecond. ($\sim$ angular resolution).

As one goes to fainter objects, the S/N drops, and it drops faster when you're background limited. This illustrates the importance of dark-sky sites, and also the importance of good image quality.

Consider two telescopes of collecting area, $T_1$ and $T_2$. If we observe for the same exposure time on each and want to know how much fainter we can see with the larger telescope at a given $S/N$, we find this for a signal-limited case:

$$S_2 = \frac{T_1}{T_2} S_1$$

but find *this* for the background-limited case:

$$S_2 = \sqrt{\frac{T_1}{T_2}} S_1$$

*Instrumental noise*

Most common:readout noise. Gaussian, not poisson. $\mu = 0$, $\sigma =$ readout noise (depends on detector, e.g. 10 electrons ($\sim$ photons create photoelectrons). Get a smear of 10 no matter how many photons are coming in. Represented by range around zero with no exposure time. Mean is zero, so can't subtract it out, it just blurs what you're looking at. B - background per square arcsecond, A - square arcsecond. Number of pixels over which it's spread:

- sharper image
- 10 arcsec all in one pixel

Can change magnification of optics, but there is a balance. Might have two stars five arcsec apart. Maxiumum resolution is not always the best! readout noise is very important when the background is very low. $N_{pix}\sigma_{rn}^2$ compared to $BA$. rn more important for spectroscopy applications than imaging.

In addition to the errors from Poisson statistics (statistical noise), there may be additional terms from instrumental errors. A common example of this that is applicable for CCD detectors is readout noise, which is additive *noise* (with zero mean) that comes from the detector and is independent of signal level. For a detector whose readout noise is characterized by $\sigma_{rn}$,

$$\frac{S}{N} = \frac{S}{\sqrt{S + BA_{pix} + \sigma_{rn}^2}}$$

if a measurement is made in a single pixel. If an object is spread over $N_{pix}$ pixels, then

$$\boxed{\frac{S}{N} = \frac{S}{\sqrt{S + BA + N_{pix}\sigma_{rn}^2}}}$$

This is the key equation! For large $\sigma_{rn}$, the behavior is the same as the background-limited case. This makes it clear that if you have readout noise, image quality (and/or proper optics to keep an object from covering too many pixels) is important for maximizing $S/N$. It is also clear that it is critical to have minimum read-noise for low background applications (e.g., spectroscopy).

There are other possible additional terms in the noise equation, arising from things like dark current, digitization noise, errors in sky determination, errors from photometric technique, etc. (we'll discuss some of these later on), but in most applications, the three sources discussed so far - signal noise, background noise, and readout noise - are the dominant noise sources.

Note the applications where one is likely to be signal dominated, background dominated, and readout noise dominated.

## Review

Three sources of uncertainty:

1. Possion statistics from source: $\sigma^2 = S$; $\sigma = \sqrt{S}$. $S$ is the same as $\mu$.
2. Poisson statistics from sky/background: $< B >$ is mean background rate.
3. Non-poisson statistics from instrumentation: aka readout noise. $\mu = 0$ (no exposure time). Smear around $\mu$. $\sigma = \sigma_{rn}$ per pixel. $\sigma^2 = N_{pix}\sigma_{rn}^2$; $\sigma = \sqrt{N_{pix}\sigma_{rn}^2}$

Noise, $N = \sqrt{S + BA + N_{pix}\sigma_{rn}^2}$ (source + sky + instrument).

**Error propagation**

Monday, February 8

So now we know how to estimate uncertainties of observed count rates. Let's say we want to make some calculations (e.g., calibration, unit conversion, averaging, conversion to magnitudes, calculation of colors, etc.) using these observations: we need to be able to estimate the uncertainties in the calculated quantities that depend on our measured quantities.

Total signal: $S = S + B- < B > +O$. There are uncertainties from all four terms, but $< B >\sim 0$ ($\sigma_{<B>} = 0$), so there are actually only three. $S + BA$: total inside aperture; $< B > A$: total outside aperture (with mean background).

Consider what happens if you have several independently measured known quantities with known error distributions and you combine these into some new quantity: we wish to know what the error is in the new quantity.

$$x = f(u, v, \ldots)$$

The question is, what is $\sigma_x$ if we know $\sigma_u$, $\sigma_v$, etc.? In other words, how do the individual uncertainties *propogate* onto the uncertainty of the resultant quantity?

As long as errors are small:

$$x_i- < x > \sim \ (u_i- < u >)\left(\frac{\partial x}{\partial u}\right) + (v_i- < v >)\left(\frac{\partial x}{\partial v}\right) + \ldots$$

where $var_i- < var >$ is the uncertainty, and the derivatives are the amount by which the input quantity depends on other quantities.

Possible class example involving S=100, B=10, A=10, and $\sigma_{rn} = 10\ldots$? The *sample variance* (by definition, see above notes) is

$$\sigma_x^2 = lim(N \to \infty)\frac{1}{N}\sum(x_i- < x >)^2$$

$$= \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2 + 2\sigma_{uv}^2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \ldots$$

The last term is the *covariance*, which relates to whether errors are *correlated*. Covarience is the dgree to which the uncertainty in one quantity might be correlated (or not) with another quantity.

$$\sigma_{uv}^2 = lim(N \to \infty)\frac{1}{N}\sum(u_i- < u >)(v_i- < v >)$$

If errors are uncorrelated, then $\sigma_{uv} = 0$ because there's equal chance of getting opposite signs on $v_i$ for any given $u_i$. When working out errors, make sure to consider whether there are correlated errors. If there are, you *may* be able to reformulate quantities so that they have independent errors: this can be very useful.

**Understanding these terms is very important!**

Examples for *uncorrelated* errrors:

- addition/subtraction:
$$x = u + v$$
$$\sigma_x^2 = \sigma_u^2 \left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2 \left(\frac{\partial x}{\partial v}\right)^2$$
Both derivatives = 1, so
$$\sigma_x^2 = \sigma_u^2 + \sigma_v^2$$
In this case, errors are said to *add in quadrature*.
- multiplication/division:
$$x = uv$$
$$\sigma_x^2 = v^2\sigma_u^2 + u^2\sigma_v^2$$
$$\frac{\sigma_x^2}{v^2u^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2}$$
- logs:
$$x = \ln u$$
$$\sigma_x^2 = \frac{\sigma_u^2}{u^2}$$
Note that when dealing with logarithmic quantities, errors in the log correspond to *fractional* errors in the raw quantity.

E.g. $x = 10u \rightarrow \sigma_x = 10\sigma_u$

$N \equiv \sigma_S$

Reasons for adding in quadrature: important! Also don't overestimate errors. This is just as bad as underestimating them.

*Distribution of resultant errors*

When propagating errors, even though you can calculate the variances in the new variables, the distribution of errors in the new variables is not, in general, the same as the distribution of errors in the original variables, e.g. if errors in individual variables are normally distributed, errors in output variable is not necessarily.

When two variables are added, however, the output is normally distributed.

**Determining sample parameters: averaging measurements**

We've covered errors in single measurements ($\sigma_i$). Next we turn to averaging measurements. Say we have multiple observations and want the best estimate of the mean and variance of the population, e.g. multiple measurements of stellar brightness. Here we'll define the best estimate of the mean as the value which maximizes the likelihood that our estimate equals the true parent population mean.

For equal errors (simplest case), this estimate just gives our normal expression for the sample mean:

$$\bar{x} = \frac{\sum x_i}{N}$$

Using error propagation, the estimate of the error in the sample mean is given by:

$$\sigma_{\bar{x}}^2 = \sum \frac{\sigma_i^2}{N^2} = \frac{\sigma^2}{N}$$

But what if errors on each observation aren't equal, say for example we have observations made with several different exposure times? Then the optimal determination of the mean is using a:

$$\text{weighted mean} = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}}$$

which can be derived from maximum liklihood. large weight means small error. The estimated error in this value is given by:

$$\sigma_\mu^2 = \sum \frac{\frac{\sigma_i^2}{\sigma_i^4}}{(\sum \frac{1}{\sigma_i^2})^2} = \frac{1}{\sum \frac{1}{\sigma_i^2}}$$

where the $\sigma_i$'s are individual weights/errors (people often talk about the *weight* of an observation, i.e. $\frac{1}{\sigma_i^2}$: large weight means small error.

This is a standard result for determining sample means from a set of observations with different weights.

However, there can sometimes be a subtlety in applying this formula, which has to do with the question: how do we go about choosing the weights/errors, $\sigma_i$? We know we can *estimate* $\sigma$ using Poisson statistics for a given count rate, but remember that this is a sample variance (which may be based on a single observation) not the true population variance. This can lead to biases.

Consider observations of a star made on three nights, with measurements of 40, 50, and 60 photons. It's clear that the mean observation is 50 photons. However, beware of the being trapped by your error *estimates*. From each observation alone, you would estimate errors of $\sqrt{40}$, $\sqrt{50}$, and $\sqrt{60}$. If you plug these error estimates into a computation of the weighted mean, you'll get a mean rate of $< x >= 48.64$.

Imagine 10,000 measurements; 50 is the *true* population mean. $\sigma$ is really $\sqrt{50}$ for all 3 observations above... estimation of true uncertainties. Avoid by: revised mean: $50 \rightarrow 48.64 \ldots \sqrt{48.64}$. Something other than Poisson stats, some measurements have more weight than others, same uncertainty across all measurements.

Using the individual estimates of the variances, we'll bias values to lower rates, since these will have estimated higher $S/N$.

Note that it's pretty obvious from this example that you should just weight all observations equally. However, note that this certainly isn't always the right thing to do. For example, consider the situation in which you have three exposures of different exposure times. Here you probably want to give the longer exposures higher weight (at least, if they are signal or background limited). In this case, you again don't want to use the individual error estimates or you'll introduce a bias. There is a simple solution here also: just weight the observations by the exposure time. However, while this works fine for Poisson errors (variances proportional to count rate), it isn't strictly correct if there are instrumental errors as well which don't scale with exposure time. For example, the presence of readout noise can have this effect; if all exposures are readout noise dominated, then one would want to weight them equally, if readout noise dominates the shorter but not the longer exposures, once would want to weight the longer exposures even higher than expected for the exposure time ratios! The only way to properly average measurements in this case is to estimate a sample mean, then use this value scaled to the appropriate exposure times as the input for the Poisson errors.

Another subtlety: averaging counts and converting to magnitudes is not the same as averaging magnitudes.

Wednesday, February 10

*Can you split exposures?*

Although from $S/N$ considerations, one can determine the required number of counts you need (exposure time) to do your science, when observing, one must also consider the question of whether this time should be collected in single or in multiple exposures, i.e. how long individual exposures should be. There are several reasons why one might imagine that it is nicer to have a sequence of shorter exposures rather than one single longer exposure (e.g., tracking, monitoring of photometric conditions, cosmic ray rejection, saturation issues), so we need to consider whether doing this results in poorer $S/N$.

Consider the object with photon flux $S'$, background surface brightness $B'$, and detector with readout noise $\sigma_{rn}$. A single short exposure of time $t$ has a variance:

$$\sigma_S^2 = S'Tt + B'ATt + N_{pix}\sigma_{rn}^2$$

If $N$ exposures are summed, the resulting variance will be

$$\sigma_N^2 = N\sigma_S^2$$

If a single long exposure of length Nt is taken, we get

$$\sigma_L^2 = S'TNt + B'ATNt + N_{pix}\sigma_{rn}^2$$

The ratio of the noises, or the inverse ratio of the S/N (since the total signal measured is the same in both cases), is

$$\frac{\sigma_N}{\sigma_L} = \sqrt{\frac{S'TNt + B'ATNt + NN_{pix}\sigma_{rn}^2}{S'TNt + B'ATNt + N_{pix}\sigma_{rn}^2}}$$

The only difference is in the readout noise term. In the signal- or background-limited regimes, exposures can be added with no loss of $S/N$. However, if readout noise is significant, then splitting exposures leads to reduced $S/N$.

If the desired $S/N$ reguires one hour of exposure, is it better to have, e.g. 3600 1-second exposure, 1800 2-second exposures, 300 12-second exposures… Pros of lots of short exposures:

- avoid saturation
- tracking
- cosmic rays
- variable object
- monitor atmospheric throughput variation

cons of lots of exposures:

- data processing
- high data quantity
- can't see object
- readout time penalty

If tracking is bad, image can be blurry, and would get more background. If you're signal-limited though, this doesn't matter. How do you figure out the minimum amount of time to expose? Need to be above the readout noise. If [rn?] = 10, then $S_{min} = 100$ ($10^2$). $\sigma_{rn}$ is not dependent on $t$. "Readout noise limited".

**Random errors vs systematic errors**

So far, we've been discussing *random* errors. There is an additional, usually more troublesome, type of errors known as *systematic* errors. These don't occur randomly but rather are correlated with some, possibly unknown, variable relating to your observations.

EXAMPLE : flat fielding

EXAMPLE : WFPC2 CTE

Note also that in some cases, systematic errors can masquerade as random errors in your test observations (or be missing altogether if you don't take data in exactly the same way), but actually be systematic in your science observations.

EXAMPLE: flat fielding, subpixel QE variations.

Note that error analysis from expected random errors may be the only clue you get to discovering systematic errors. To discover systematic errors, plot residuals vs. everything

Important to deal with random errors correctly; don't overestimate error bars. Flat fielding: uniform brightness → non-uniformity in sensitivity. To what % accuracy is something "flat"?

Monday, February 15

zeropoint, $z$, bundles count equation into one number for a *particular bandpass*. This is an empirical measurement of the count equation [counts]. Use the count equation to derive the zeropoint. If the error bars are smaller than the scatter of the data points themselves, there's a problem. $\chi^2$ would be much [larger?] than one. A possible issue is the pixel senstivity isn't uniform across a ccd. Stars at the bottom of the chip could be fainter than the ones at the top. CCDs in space telescope. (CCDs are read by moving charge through chip $\rightarrow$ *grating*.) Something about charge-transfer efficiency… observation scatter larger than expected scatter. See handwritten notes for diagrams and more on this.

**Important to know the equation for error propagation!**

# Effects of the earth's atmosphere

Four effects:

1. Emission
2. Absorption
3. Refraction (shifting apparent direction of incoming light)
4. Seeing (degrading the coherence of incoming light, leading to degradation of image quality when collecting light over a large aperture)

## Night sky emission

(brightness as function of wavelength)

Sources (many of which are emission line sources, not continuum sources, as shown in this plot):

- **air glow:** thermally excited molecules (by sunlight). Emission takes time, so this continues into the night. OH lines. Lines are pretty bright.
- **zodiacal light:** (outside of our atmosphere), caused by dust in the solar system, and emits in the optical regime. Brightness: $m_V \sim 22.2$ - 23.5 [magnitudes arcsec$^{-2}$], depending on ecliptic latitude (and to a lesser degree on ecliptic longitude) Note that this exists even in Earth orbit. When brightness of observed objects drop to 21-22 mag, go from signal-limited to background-limited.
- **sunlight:** Brightness is small away from twilight, depending on distance sun is below horizon. Note that there are Different definitions of twilight:
  - civil (6 degrees)
  - nautical (12 degrees; "pretty dang dark"),
  - astronomical (18 degrees; "dark as it's gonna get").

  By astronomical twilight, there is essentially no contribution from sunlight, but much useful observing can be done before this.
- **moonlight:** Brightness is variable; can be very bright. $\sim$ 10 times brighter at full moon (compared to what?).
- **aurorae:** line emission
- **light pollution:** Strong (bright) in distinct lines
- **unresolved stars and galaxies:** (outside of our solar system)
- **thermal emission:** IR emission from sky, telescope, and dome. $T_{atm} \rightarrow$ blackbody emission.

**Optical:**

For broadband work, for example in the V band, $m_{sky} \sim 22$ mag/arcsec at good site, so $S/N$ becomes background limited around m=22 for good image quality, and m=20 for poorer image quality. Consequently, image quality matters for faint objects. Moonlight is very significant, hence faint optical imaging requires dark time. Readout noise of zodiacal light ($\mathrm{m_{sky}}$) really important. Be careful when splitting exposures; can become readoutnoise limited. Consider the minimum exposure time needed (context?)

*Optical spectroscopy:* sky emission is generally not much of a problem (except around lines) so long as moon is down (or work on bright objects); low dispersion observations can be background-limited for long exposures, but at higher dispersion or shorter exposures, spectroscopy is often readout-noise limited.

**IR:**

Most ($\sim 90\%$) of the emission in the near-IR is from vibrational and rotational transition emission lines of the OH molecule, the so-called "OH forest." For broadband work:

- H band: $m \sim 13.5$ mag arcsec$^{-1}$
- K band: $m \sim 12.5$ mag arcsec$^{-1}$.

($\sim$ surface brightness); So for all except bright objects, we're background limited. This leads to some fundamental differences in data acquisition and analysis between the near-IR and the optical. For infrared spectra, it's harder to estimate $S/N$: depends on where your feature is located. Moonlight is not very significant, hence much IR work is done in bright time.

Wednesday, February 17

**Unsorted notes**: IR surface brightness $\sim$ 12.13 mag (1000$\times$ brighter than optical). Optical surface brightness $\sim$ 21-22 mag. Surface brightness varies and fluctuates, esp OH lines. Local conditions in atmosphere scales on the order of degrees. Movie: OH lines "dancing around". 6000Å $\rightarrow$ 9000Å brighter. Floor set by zodiacal light with added stuff by moonlight.

Farther in the IR (5 $\mu$ +), thermal emission from the sky dominates and is extremely bright. In fact, when working at wavelengths with thermal background, the exposure time is often limited by the time it takes to saturate the detector with background (sky) light, in seconds or less.

Sky brightness from most sources *varies* with time and position in the sky in an *irregular* fashion. Consqeuently, it's essentially impossible to estimate the sky *a priori*: sky must be determined from your observations, and if your observations don't distinguish object from sky, you'd better measure sky close by in location and in time: especially critical in the IR. See some IR movies (really important!); spectral movie from ESO/Paranal see here.

## Transmission of atmosphere

**More unorganized notes:** Extinction: mostly from scattering and aerosols. SHOULD REVIEW EXTINCTION FROM ISM! Mean extinction curve (individual observatories) subtracted for optical spectroscopy. A and B bands (nomenclature from Fraunhofer - discovered solar liens $\sim$ 1800s. Absorption lines were labeled alphabetically in $\nu$. A and B bands were actually from $O_2$. C from H$\alpha$, D from sodium (the "sodium D line"), and H and K were two lines of ionized calcium (the CaII H and K lines)). Water vapor is the dominant absorber in the IR. Windows in Earth's atmosphere: YJHK are the filters that fit inside these windows, which are carved out by water absorption. This defines the bandpasses. But water vapor changes too, so absorption strength changes a lot. The bandpass itself changes! High peaks [from some video or plot?] are oqaque; transparent toward the bottom. Still molecular absorption in these windows that could show up in spectrum, e.g. the H-band. Have to correct for this. More atmosphere = more light lost. How much more? Airmass is defined locally:

$$\text{airmass} = \frac{\text{amt of air looking at}}{\text{amt of air through zenith}}$$

So looking at zenith, no matter where you are, airmass = 1.

Earth's atmosphere doesn't transmit 100% of light. Various things contribute to the absorption of light:

- scattering, e.g., Rayleigh scattering off molecules.
- aerosols: scattering off larger particles (e.g. natural aerosols like fog, forest exudates, and geyser steam, and artificial aerosols like haze, dust, particulate air pollutants and smoke). Particle size $\sim 1\ \mu m$.
- variety of molecules:
  - ozone
  - $H_2O$
  - $O_2$
  - $CO_2$

- $N_2O$
- $CH_4$

All of these have a $\lambda$ dependency.

All are functions of wavelength, time to some extent, and position in sky.

**Sources of extinction**

In the optical part of the spectrum, extinction is a roughly smooth function of wavelength and arises from a combination of ozone, Rayleigh scattering, and aerosols, as shown in this plot. The optical extinction can vary from night to night or season to season, as shown in this plot. Of course, this is showing the variation over a set of photometric nights; if there are clouds, then the level of variation is much higher. Because of this variation, you must determine the amount of extinction on each night separately if you want accuracy better than a few percent (even for photometric nights). Generally, the *shape* of the extinction curve as a function of wavelength probably varies less than the amplitude at any given wavelength. Because of this, one commonly uses *mean extinction coefficients* when doing spectroscopy where one often only cares about *relative* fluxes. To first order, the extinction from clouds is "gray", i.e. not a function of wavelength, so relative fluxes can be obtained even with some clouds present.

There is significant molecular absorption in the far-red part of the optical spectrum, in particular, note the A (7600) and B (6800) atmospheric bands from $O_2$. (Historical note: The 'A' and 'B' nomenclature comes from the Fraunhofer lines, which were labeled alphabetically with increasing (decreasing?) frequency when first discovered in the sun's spectrum in the 1800s. Originally they were not recognized as absorption lines. C is H$\alpha$, D is the sodium D line, CaII H and K lines, etc.

In the infrared, the extinction does not vary so smoothly with wavelength because of the effect of molecular absorption. In fact, significant absorption bands define the so-called infrared windows (yJHKLM), as shown in the near IR in this plot. At longer wavelengths, the broad absoprtion band behavior continues, as shown in this plot. In this figure, $transmission = f(b_\lambda l)$ where $l$ is path length (units of airmass):

The L band is at 3.5 $\mu$, M band at 5 $\mu$.

Note that even within the IR "windows", there can still be significant telluric (terrestrial) absorption features, e.g. from $CO_2$, $H_2O$, and $CH_4$. When doing

| $b_\lambda l$ | $f$ |
|---|---|
| -3 | 1 |
| -2 | 0.97 |
| -1 | 0.83 |
| 0 | 0.5 |
| 1 | 0.111 |
| 2 | 0.000 |

IR spectroscopy, one needs to be aware of these and possibly attempt to correct for them, taking care not to confuse them with real stellar features.

### Airmass and zenith distance dependence

Clearly, if the light has to pass through a larger path in the Earth's atmosphere, more light will be scattered/absorbed; hence one expects the least amount of absorption directly overhead (zenith), increasing as one looks down towards the horizon.

Definition of airmass: path length that light takes through atmosphere relative to length at zenith: X $\equiv$ 1 vertically (at $z = 0$). Given the zenith distance, $z$, which can be computed from:

$$\sec z = (\sin\phi \sin\delta + \cos\phi \cos\delta \cos h)^{-1}$$

where $\phi$ is the latitude of the observatory location, $\delta$ is the declination, and $h$ is the hour angle ($h =$ local sidereal time - $\alpha$, where $\alpha$ is the right ascenstion), we have

$$X \sim \sec z$$

which is exactly true in the case of a plane parallel atmosphere. Since the earth's atmosphere is not a plane, the plane parallel approximation breaks down for larger airmasses. For $X > 2$, a more precise formula is needed, the following gives a higher order approximation:

$$X = \sec z - 0.0018167(\sec z - 1) - 0.002875(\sec z - 1)^2 - 0.0008083(\sec z - 1)^3$$

### How much light is lost going through the atmosphere?

Consider a thin sheet of atmosphere, with incident flux $F$, and outgoing flux $F + \mathrm{d}F$. Let the thin sheet have opacity $\kappa = N\sigma$, where $N$ is the number density of absorbers/scatterers, and $\sigma$ is the cross-section/absorber-scatterer.

$$\mathrm{d}F = -\kappa F \mathrm{d}x$$

$$F = F_{\mathrm{top}} e^{-\int \kappa \mathrm{d}x} \equiv F_{\mathrm{top}} e^{-\tau}$$

where $\tau$ is the *optical depth* of the atmosphere, which parameterizes how much light is lost. Density and cross-section determine the fraction of light that is lost (opacity). For example, $2\kappa \rightarrow e^{-\tau}$ more absorption (not twice as much). Light lost $\sim e^{-\tau}$. Total $\tau$ is increased by the airmass.

If the optical depth through the atmosphere is just proportional to the physical path length (true if same atmospheric structure is sampled in different directions), then

$$\tau(X) \sim \tau_0 X$$

where $\tau_0$ is the optical depth at the zenith.

$$F = F_{\mathrm{top}} e^{-\tau_0 X}$$

Expressing things in magnitudes, we have:

$$m = m_0 + 1.086\tau_0 X$$

Define the *extinction coefficient* $k_\lambda$ [magnitudes of absorption per airmass, I think]:

$$m_0 = m + k_\lambda X$$

$$k_\lambda \equiv -1.086\tau_0$$

($m_0$ - top of atmosphere) so the amount of light lost in magnitudes can be specified by a set of extinction coefficients. Note by this definition, the extinction coefficient will be negative; others may use the opposite sign convention (e.g. defining $m_0 = m - k_\lambda X$). Of course, use of the scaling of $\tau$ or $k$ with airmass assumes *photometric weather*.

We will talk later about some details of determining extinction coefficient, but the basic idea is that you can determine the extinction by monitoring the brightness of a star (or a set of stars of known brightness) at a range of different airmasses. This needs to be done as a function of wavelength, i.e., for each filter you observe in.

Instrumental mag - what you measure, given equipment, sky conditions, etc. (as opposed to official, recorded magnitude, I think). $k_\lambda$ has to be measured in each bandpass as it's $\lambda$-dependent.

Random thoughts: what's the difference between error, standard deviation, uncertainty, etc.? Stddev is a way to "describe the uncertainty". Errors are something we "can control", like instrumental stuff.

## Atmospheric Refraction

Monday, February 22

Amount of refraction depends on where you're looking zenith: = 0.

The direction of light as it passes through the atmosphere is also changed because of refraction since the index of refraction changes through the atmosphere. The amount of change is characterized by Snell's law:

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

Let $z_0$ be the true zenith distance (relative to how much star moves), $z$ be the observed zenith distance, $z_n$ by the observed zenith distance at layer $n$ in the atmosphere, $\mu$ be the index of refraction at the surface, and $\mu_n$ be the index of refraction at layer $n$. At the top of the atmosphere:

$$\frac{\sin z_0}{\sin z_N} = \frac{\mu_N}{1}$$

At each infinitessimal layer:

$$\frac{\sin z_n}{\sin z_{n-1}} = \frac{\mu_{n-1}}{\mu_n}$$

and so on for each layer down to the lowest layer:

$$\frac{\sin z_1}{\sin z} = \frac{\mu}{\mu_1}$$

Multiply these to get:

$$\sin z_0 = \mu \sin z$$

from which we can see that the refraction depends *only on the index of refraction near the earth's surface.*

We define astronomical refraction, $r$, to be the angular amount that the object is displaced by the refraction of the Earth's atmosphere:

$$\sin(z + r) = \mu \sin z$$

In cases where $r$ is small (pretty much always):

$$\sin z + r\cos z = \mu \sin z$$

$$r = (\mu - 1)\tan z$$

$$\approx R \tan z$$

where we have defined $R$, known as the "constant of refraction".

A typical value of the index of refraction is $\mu \sim 1.00029$, which gives $R = 60$ arcsec (red light).

The direction of refraction is that a star apparently moves *towards* the zenith. Consequently in most cases, star moves in both RA and DEC:

$$r_\alpha = r \sin q$$

$$r_\delta = r \cos q$$

where $q$ is the *parallactic* angle, the angle between $N$ and the zenith:

$$\sin q = \cos \phi \frac{\sin h}{\sin z}$$

Note that the expression for $r$ is only accurate for small zenith distances ($z <\sim 45$). At larger $z$, can't use plane parallel approximation. Observers have empirically found:

$$r = A \tan z + B \tan^3 z$$

$$A = (\mu - 1) + B$$

$$B \sim -0.07''$$

but these vary with time, so for precise measurements, you would have to determine $A$ and $B$ on your specific night of observations.

Why is it important to understand refraction? Clearly, it's relevant for pointing a telescope, but this is generally always automatically handled in the telescope pointing software. If you're just taking images, then the stars are just a tiny bit moved relative to each other, but who really cares? One key issue today is the use of multiobject spectrographs, where slits or fibers need to be placed on objects to accuracies of a fraction of an arcsec. For small fields, refraction isn't too much of an issue, but for large fields, it can be ... note SDSS plates.

The other *extremely* important effect of refraction arises because the index of refraction varies with wavelength, so the astronomical refraction also depends on wavelength: This gives rise to the phenomenon of *atmospheric dispersion*, or *differential refraction*. Because of the variation of index of refraction with wavelenth, every object actually appears as a little spectrum with the blue end towards the zenith. The spread in object position is proportional to tan $z$.

| $\lambda$ | R |
|---|---|
| 3000 | 63.4 |
| 4000 | 61.4 |
| 5000 | 60.6 |
| 6000 | 60.2 |
| 7000 | 59.9 |
| 10000 | 59.6 |
| 40000 | 59.3 |

Measuring rotation curve of galaxy. Not at parallactic angle? Not getting all the light, just at wavelength used to position the slit (?) Instrument: atmospheric dispersion corrector. No rainbow straight up above gets bigger as you move down. Two prisms that move relative to each other (APO doesn't have this.)

1. emission, 2. transmission, 3. refraction, 4. seeing:theory and practice. Adaptive optics - deformable mirrors that move really fast.

This effect is critical to understand for spectroscopy when using a slit or a fiber, since the location of an object in the sky depends on the wavelength. If you point to the location at one wavelength, you can miss another wavelength (depending on how big your fiber is) and the relative amount of flux you collect will be a function of wavelength, something you may want to take into account if you're interested in the relative flux (continuum shape) over a broad wavelength range. Note the consequent importance of the relation between the orientation slit orientation and the parallactic angle: a slit aligned with the parallactic angle will not lose light as a function of wavelength, but otherwise it will. However, for a slit at the parallactic angle, be careful about matching up flux at different wavelengths for extended objects.

## Seeing: theory and practice

References: Coulson, ARAA 23,19; Beckers, ARAA 31, 13; Schroeder 16.II.

Generally, a perfect astronomical optical system will make a perfect (diffraction-limited) image for an incoming plane wavefront of light. The plane will focus to a nice point. The Earth's atmosphere is turbulent and variations in the index of refraction cause the plane wavefront from distant objects to be distorted. These cause several astronomical effects:

- scintillation, which is amplitude variations, which typically vary over scales of cm: generally very small for larger apertures. "twinkling", small aperture (e.g. eye) prevents from getting below a fraction of a % in brightness... something. Planets don't twinkle because they're bigger
- seeing: positional changes and image quality changes. The effect of seeing depends on aperture size: for small apertures, one sees a diffraction pattern moving around, while for large apertures, one sees a set of diffraction patterns (speckles) moving around on scale of $\sim 1$ arcsec. These observations imply:
  - local wavefront curvatures flat on scales of small apertures
  - instantaneous slopes vary by $\sim$ an arcsec

The time variation scales are several milliseconds and up.

star actually moves around: A/B, where A is the aperture size and B is the size of wiggles in the wavefront. Comparing 8" telescope to 4-m telescope (See handwritten diagrams). Both telescopes produced the same end resutl, but how it got there is different for each one. Difference? Aperture size compared to length of [curve] $\sim$ 6-8 inches (size scale of wavefront pieces) approx. same as waveplane front.

Wednesday, February 24

Cells in the atmosphere each have a different index of refraction. All we have are statistical descriptions of turbulence. Difference of squares: average over all points. E.g. at 1 cm rms [something] = x, at 0.5 cm rms = y, etc.

The effect of seeing can be derived from theories of atmospheric turbulence, worked out originally by Kolmogorov, Tatarski, Fried. Here are some pertinent results, without derivation:

A turbulent field can be described statistically by a *structure function*:

$$D_N(x) == < |N(r+x) - N(r)|^2 >$$

where $x$ is the separation of points, $N$ is any variable (e.g. temperature, index of refraction, etc.), and $r$ is position. This describes how the "quantity" changes as a function of separation between the two points. Quantity of... turbulence? Translate N to phase change of wave (determine image quality).

The structure function of a velocity field can be described as the difference between the values of the flow velocity between two points with coordinates $r$ and $r + x$. See this paper.

Komogorov turbulence gives:

$$D_n(x) = C_n^2 x^{2/3}$$

where $C_n$ is the refractive index structure constant. From this, one can derive the phase structure function at the telescope aperture:

$$D_\phi(x) = 6.88 \left(\frac{x}{r_0}\right)^{5/3}$$

where the coherence length $r_0$ (also known as the Fried parameter) is:

$$r_0 = 0.185 \left(\lambda^{6/5}\right) \left(\cos^{3/5} z\right) \left[\int \left(C_n^2 \mathrm{d}h\right)\right]^{-3/5}$$

where $z$ is the zenith angle and $\lambda$ is the wavelength. Using optics theory, one can convert $D_\phi$ into an image shape. Bigger $r_0$ is better; integral is over the height of the atmosphere. $r_0$ same size as telescope...?

Physically, $r_0$ is (roughly) proportional to the image size from seeing:

$$d \sim \lambda / r_0$$

as compared with the image size from diffraction-limited images:

$$d \sim \lambda / D$$

Seeing dominates when $r_0 < D$; a larger $r_0$ means better seeing.

Seeing is more important than diffraction at shorter wavelengths (and for larger apertures); diffraction is more important at longer wavelengths (and for smaller apertures); the effects of diffraction and seeing cross over in the IR for most astronomical-sized telescopes (∼5 microns for 4m); the crossover falls at a shorter wavelength for smaller telescope or better seeing.

Image size from seeing is almost the same, regardless of $\lambda$ (*little* better toward the IR, but not much). Diffraction swamps seeing at *really* long wavelengths. Bigger telescopes are seeing-limited. $r_0$ has $\lambda$ in it, hence the lack of $\lambda$ dependence. Crossover in the micron regime.

The meat of $r_0$ is in $\int (C_n^2 \mathrm{d}h)$; as you might expect, this varies from site to site and also in time. At most sites, there seems to be three regimes of "surface layer" (wind-surface interactions and manmade seeing), "planetary boundary layer" (influenced by diurnal heating), and "free atmosphere" (10 km is tropopause: high wind shears), as seen in this plot. A typical astronomical site has $r_0 \sim 10$ cm at 5000Å.

We also want to consider the coherence of the same turbulence pattern over the sky: this coherence angle is called the *isoplanatic angle*, and the region over which the turbulence pattern is the same is called the *isoplanatic patch*. This is relevant to adaptive optics, where we will try to correct for the differences across the telescope aperture; if we do a single correction, how large a field of view will be corrected?

$$\theta \sim 0.314 r_0 / H$$

where $H$ is the average distance of the seeing layer:

$$H = \sec z \left[ \frac{\int \left( C_n^2 h^{5/3} \mathrm{d}h \right)}{\int \left( C_n^2 \mathrm{d}h \right)} \right]^{3/5}$$

For $r_0 = 10$ cm, $H \sim 5000$ m, $\theta \sim 1.3$ arcsec. In the IR, $r_0 = 70$ cm, $H \sim 5000$ m, $\theta \sim 9$ arcsec.

Note however, that the "isoplanatic patch for image motion" (not wavefront) is $\sim 0.3D/H$. For $D = 4$ m, $H \sim 5000$ m, $\theta_{kin} = 50$ arcsec. This is relevant for low-order atmosphereic correction, i.e. tip-tilt, where one is doing *partial* correction of the effect of the atmosphere.

Want the isoplanatic patch to be big. Area where wiggles, wavefronts are the same. Smaller patch if $H$ (height in atmosphere) is bigger. $H$ is bigger at top of atm than at bottom. Take out seeing entirely: correction in one area doesn't work in another area. Space telescopes get wider FOV. Adaptive optics: multiple corrections for different angles (complicated). Better in IR (bigger $r_0$). 1/2' or 1' patch. Most adaptive optics in NIR. Partial correction: coherent over larger angular scale, but not down to diffraction-limited. Wobbly wavelength front; can correct *some* spatial area. Measure *mean* tilt and correct for that → *active* optics (or tip-tilt system).

### Other sources of seeing

Although the Earth's atmosphere provides a limit on the quality of images that can be obtained, at many observatories, there are other factors that can dominate the image quality budget. These have been recognized over the past several decades to be significant effects.

- Dome seeing: arises from turbulence pattern around the dome and the interface between inside the dome and outside the dome. Even small temperature differences can lead to significant image degradataion. $T_{out} = T$, $T_{in} = T + \Delta T$. Could open dome early, climate control during the day, get rid of it completely (roll-off roof!). This tends to be worse for smaller domes.

- Mirror seeing: arises from turbulence right above the surface of the mirror, which can arise if the mirror temperature differs from that of the air above it. Thinner mirrors are better; less mass, or thermal something-or-other.
- Wind shake of the telescope. Maybe getting rid of the dome isn't such a good idea... Good idea to put telescopes on mountain on edge facing the wind. As the wind goes up over where the telescope is, the turbulence happens later, on the other side, as it flows down the less-steep side of the mountain. Minimizeing turbulence from wind laminar flow, or something like that.
- Poor telescope tracking.
- Design, quality, and alignment of the telescope optics: (In general, however, telescope design is done such that the image degradation from telescope design is significantly smaller than that arising from seeing).
- (lack of) effect of clouds. Cause decreased *transparency*, but don't actaully make the images blurry. The seeing can atually be better with thin haze.

**What does seeing cause the image to look like?**

Monday, February 29

The "quality" of an image can be described in many different ways. The overall shape of the distribution of light from a point source is specified by the *point spread function*. Diffraction gives a basic limit to the quality of the PSF, but seeing, aberrations, or image motion add to structure/broadening of the PSF. For a good ground-based telescope, seeing is generally the dominant component of the PSF. PSF provides a full description... 2D function.

Probably the most common way of describing the seeing is by specifying the full-width-half-maximum (FWHM) of the image, which may be estimated either by direct inspection or by fitting a function (usually a Gaussian); note the correspondence of FWHM to $\sigma$ of a gaussian: FWHM $= 2.354\sigma$. $\mathrm{FWHM} = 2\sigma\sqrt{2\ln 2}$ However, beware of effects of sampling of the PSF: you're really getting the PSF integrated over pixels, not the PSF. Remember that the FWHM doesn't fully specify a PSF, and one should always consider how applicable the quantity is.

Another way of describing the quality of an image is to specify its *modulation transfer function* (MTF). The MTF and PSF are a Fourier transform pair, thin gaussian ←FT→ fat gaussian so the MTF gives the *power* in an image on various *spatial* scales. Turbulence theory makes a prediction for the MTF from seeing:

$$MTF(\nu) = exp\left[-3.44\left(\frac{\lambda\nu}{r_o}\right)^{5/3}\right]$$

where $\nu$ is the spatial frequency, $\xi = \frac{1}{\lambda}$ [cycles per cm]. (Aside: wavenumber $k = \frac{2\pi}{\lambda} = 2\pi\xi$)

- MTF: power as function of spatial frequency
- PSF: intensity as function of position

Note that a gaussian goes as $e^{\left(-\nu^2\right)}$, so this is close to a gaussian. The shape of seeing-limited images is roughly Gaussian in core but has more extended wings. This is relevant because the seeing is often described by fitting a Gaussian to a stellar profile. Note that the stellar profile is the same for all stars, and doesn't depend on the brightness of the star. They all get lost in the noise eventually… sooner for fainter stars, which look smaller. In case the wings are needed, fitting to a Gaussian won't work; a potentially better empirical fitting function is a Moffat function:

$$I = p_1\left[1 + \frac{\left(x-p_2\right)^2}{p_4^2} + \frac{\left(y-p_3\right)^2}{p_5^2}\right]^{-p_6}$$

"Nothing special, six parameters, more accurate."

Ways of characterizing the PSF:

- *encircled energy* as a function of radius, or at some specified radius. The encircled energy is just the cumulative integral of the PSF. Encircled energy requirements are often used for specifying optical tolerances.
- Strehl ratio: more commonly used in adaptive optics applications not for "normal observing". The Strehl ratio is the ratio between the peak amplitude of the PSF and the peak amplitude expected in the presence of diffraction only. With normal atmospheric seeing, the Strehl ratio is *very* low. However, the Strehl ratio is often used when discussing the

performance of adaptive optics systems. Ratio of how good the actual image is to the best that could be obtained (pure diffraction-limited). A really good adaptive optics system would have a Strehl ratio of about 1/2. Seeing of a big telescope is nowhere close to diffraction-limited... fraction of a %. From Nancy's solutions: the Strehl ratio is the ratio of the amoutn of light actually delivered by the optical system into the Airy disk to the theoretical maximum at the diffraction limit.

# Astronomical Optics

Because astronomical sources are faint, we need to collect light. We use telescopes/cameras to make images of astronomical sources. Example: a 20th magnitude star gives $\sim 0.01$ photons/s/cm$^2$ at 5000Å through a 1000Å filter. However, using a 4-meter telescope gives 1200 photons/s.
In other words, need a bigger collecting area to get lots of photons. The bandpass and wavelength information given here is irrelevant.

## Single surface optics and definitions

By definition, an optical system collects light and (usually) makes images. This requires the bending of light rays, which is accomplished using *curved surfaces*:

- lenses (refraction)
- mirrors (reflection)

Building large telescopes with lenses is difficult, but the instruments often have lenses.

The operation of refractive optical systems is given by Snell's law of refraction:

$$n \sin i = n' \sin i'$$

where $n$ and $n'$ are the indices of refraction and $i$ and $i'$ are the angles of incidence relative to the normal to the surface. For reflection:

$$i' = -i$$

An optical element takes a source at $s$ and makes an image at $s'$. The image can be *real* or *virtual*. A real image exists at some point in space; a virtual image is formed where light rays apparently emanate from or converge to, but at a location where no light actually appears. For example, in a Cassegrain telescope, the image formed by the primary is virtual, because the secondary intercepts the light and redirects it before light gets to the focus of the primary (see figure 1).

The image will not necessarily be a perfect image: all rays regardless of height at the surface (the lens or mirror), $y$, might not cross at the same point. This is the subject of aberrations, which we will get into in a while. For a "smooth" surface, the amount of aberration will depend on how much the different rays differ in $y$, which depends on the shape of the surface. We define:

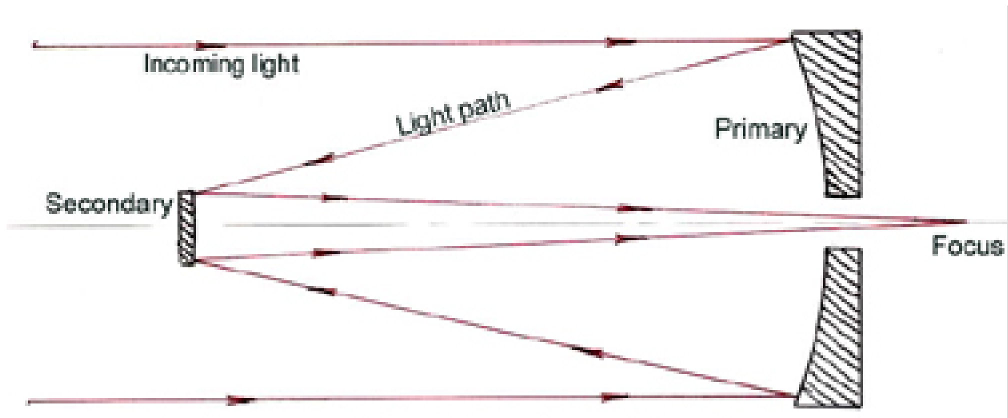


Figure 1: Diagram showing the optical arrangement of the Cassegrain.

- *paraxial* rays are near the center of the aperture.
- *marginal* rays are on the edge of the aperture.
- the *chief* ray passes through the center of the aperture.

To define nominal (unaberrated) quantities, we consider the *paraxial regime*, a small region near the optical axis surrounding the chief ray. In this regime, all angles are small, aberrations vanish, and a surface can be wholly specified by its radius of curvature, $R$.

The *field angle* (FA) gives the angle formed between the chief ray from an object and the z-axis. Note that paraxial does not necessarily mean a field angle of zero; one can have an object at a field angle and still consider the paraxial approximation.

For the time being, we are ignoring *diffraction* and considering *geometric* optics, (what you get from diffraction as wavelength tends to 0). For nonzero wavelength, geometric optics applies as scales $x > \sim \lambda$.

We can derive the basic relation between the object location at $s$ and the image location at $s'$ as a function of a surface where the index of refraction changes (Schroeder, chapter 2).

$$\frac{n'}{s'} - \frac{n}{s} = \frac{(n' - n)}{R}$$

The points at $s$ and $s'$ are called *conjugate*. If either $s$ or $s'$ is at infinity (true for astronomical sources at $s$), the other distance is defined as the *focal length*, $f$, of the optical element. For $s = \infty$, $f = s'$.

We can define the quantity on the right side of the equation, which depends only the the surface parameters (not the image or object locations), as the *power*, $P$, of the surface:

$$P \equiv \frac{(n' - n)}{R} = \frac{n'}{f'} = \frac{n}{f}$$

We can make a similar derivation for the case of reflection:

$$\frac{1}{s'} + \frac{1}{s} = \frac{2}{R}$$

This shows that the focal length for a mirror is given by $R/2$.

Note that one can treat reflection by considering refraction with $n' = -n$, and get the same result:

$$\frac{n'}{s'} + \frac{n'}{s} = \frac{(n' + n')}{R}$$

The *focal ratio* (or *F-number*) is defined as

$$f/ = \frac{f}{A}$$

where A is the aperture diameter and $f/$ denotes the focal ratio. (Note that, for example, $f/10$ means a focal ratio of 10; $f$ is not some variable being divided by 10). The focal ratio gives the beam "width"; systems with a small focal ratio have a short focal length compared with $A$, and hence the imcoming beam to the image is wide. Systems with small focal ratios are called "fast" systems; systems with large focal ratios are called "slow" systems. Wednesday, March 2

The *magnification* of a system gives the ratio of the image height to the object height:

$$\frac{h'}{h} = \frac{s' - R}{s - R} = \frac{ns'}{n's}$$

The magnification is negative for inverted images, and also for reflection (n' = - n). Magnification is an important quantity for *multi-element* systems.

The *plate scale* is defined as the "motion" of an image for a given incident angle of parallel beam from infinity. From a consideration of the chief rays for objects on-axis and at field angle $\alpha$, we get:

$$\tan \alpha \approx \alpha = \frac{x}{f}$$

or

$$\text{scale} \equiv \frac{\alpha}{x} = \frac{1}{f}$$

In other words, the scale, in units of angular motion per physical motion in the focal plane, is given by 1/f. For a fixed aperture diameter, systems with a small focal ratio (smaller focal length) have a larger scale, i.e. more light in a patch of fixed physical size: hence, these are "faster" systems.

## Multi-surface systems

To combine surfaces, the image from the first surface becomes the source for the second surface, and so on for each surface in the system. The basic parameters of multi-surface systems can generally be described by equivalent single-surface parameters. For example, the effective focal length of a multi-surface system can be defined as the focal length of some equivalent single-surface system. The *effective* focal length is the focal length of the first element multiplied by the magnification of each subsequent element. The two systems (single and multi) are equivalent in the paraxial approximation ONLY.

### The two-surface lens

Consider a lens in air ($n \sim 1$). The first surface gives:

$$\frac{n}{s'_1} - \frac{1}{s_1} = \frac{n-1}{R_1} = P_1$$

The second surface gives:

$$\frac{1}{s'_2} - \frac{n}{s_2} = \frac{n-1}{R_2} = P_2$$

but we have $s_2 = s'_1 - d$ (remember we have to use the plane of the second surface to measure distances for the second surface).

After some algebra, we find the effective focal length (from center of lens):

$$P = \frac{1}{f'} = P_1 + P_2 - \frac{d}{n} P_1 P_2$$

$$P = \frac{(n-1)}{R_1} + \frac{(1-n)}{R_2} - \frac{d}{n} \frac{(n-1)(1-n)}{R_1 R_2}$$

From this, we derive the *thin lens* formula:

$$P = \frac{1}{f'} = \frac{(n-1)}{R_1} + \frac{(1-n)}{R_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f'} = \frac{1}{f_1} + \frac{1}{f_2}$$

**plane-parallel plate**

Zero power, but moves image laterally:

$$\Delta = d\left[1 - \left(\frac{1}{n}\right)\right]$$

Application to filters: variation of focus.

**Two-mirror telescopes**

In astronomy, most telescopes are two-mirror telescopes of Newtonian, Cassegrain, or Gregorian design. All 3 types have a concave primary. The Newtonian has a flat secondary, the Cassegrain a convex secondary, and the Gregorian a concave secondary. The Cassegrain is the most common for research astronomy; it is more compact than a Gregorian and allows for magnification by the secondary. Basic parameters are outlined here. Each of these telescope types defines a *family* of telescopes with different first-order performances. From the usage/instrumentation point of view, important quantities are:

- the diameter of the primary, which defines the light collecting power
- the scale of the telescope, which is related to the focal length of the primary and the magnification of the secondary:

  $$f_{eff} = f_1 m$$

  (alternatively, the focal ratio of the telescope, which gives the effective focal length with the diameter)
- the back focal distance, which is the distance of the focal plane behind the telescope.

From the design point of view, we need to specify:

- the radii of curvature of the mirrors
- the separation between the mirrors

The relation between the usage and design parameters can be derived from simple geometry. Some basic definitions:

- ratio of focal lengths, $\rho$:
$$\rho = \frac{R_2}{R_1} = \frac{f_2}{f_1}$$
- magnification of the secondary, $m$ (be aware that $s_2'$ is negative for a Cassegrain):
$$m = -\frac{s_2'}{s_2}$$
- *back focal distance*, the distance from the primary vertex to the focal plane (often expressed in units of the primary focal length, or primary diameter):
$$f_1\beta = D\eta$$
- primary focal ratio, $F1$:
$$F_1 = \frac{f_1}{D}$$
- ratio of marginal ray heights, $k$ (directly related to separation of mirrors):
$$k = \frac{y_2}{y_1}$$

Using some geometry, we can derive some basic relations between these quantities, in particular:
$$\rho = \frac{mk}{(m-1)}$$
and
$$(1+\beta) = k(m+1)$$
Usually, $f_1$ is limited by technology/cost. Then choose $m$ to match desired scale. $k$ is related to separation of mirrors, and is a compromise between making telescope shorter and blocking out more light vs. longer and blocking less light; in either case, the focal plane has to be kept behind the primary.

One final thing to note is how we focus a Cassegrain telescope. Most instruments are placed at a fixed location behind the primary. Ideally, this will be at the back focal distance, and everything should be set as designed. However, sometimes the instrument may not be exactly at the correct back focal distance, or it might move slightly because of thermal expansion/contraction. In this case, focussing is usually then done by moving the secondary mirror.

The amount of image motion for a given secondary motion is given by:

$$\frac{d\beta}{dk} = \frac{d}{dk}k(m+1) - 1$$

Working through the relations above, this gives:

$$\frac{d\beta}{dk} = m^2 + 1$$

so the amount of focal plane motion ($f_1 d\beta$) for a given amount of secondary motion ($f_1 dk$) depends on the magnification of the system.

If you move the secondary you change $k$. Since $\rho$ is fixed by the mirror shapes, it's also clear that you change the magnification as you move the secondary; this is expected since you are changing the system focal length, $f = mf_1$. So it's possible that a given instrument could have a slightly varying scale if its position is not perfectly fixed relative to the primary.

Note that even if the instrument is at exactly the back focal distance, movement of the secondary is required to account for mechanical changing of spacing between the primary and secondary as a result of thermal expansion/contraction.

### Definitions for multi-surface system: stops and pupils

Monday, March 7, 2016

- aperture stop: determines the amount of light reaching an image (usually the primary mirror).
- field stop: determines the angular size of the field. This is usually the detector, but for a large enough detector, it could be the secondary.
- pupil: location where rays from all fiel angles fill the same aperture.
- entrance pupil: image of aperture stop as seen from source object (usually the primary).
- exit pupil: image of aperture stop formed by all subsequent optical elements.

In a two-mirror telescope, the location of the exit pupil is where the image of the primary is formed by the secondary. This can be calculated using $s = d$

as the object distance (where $d$ is the separation of the mirrors), then with the reflection equation, we can solve for $s'$ which gives the location of the exit pupil relative to the secondary mirror. If one defines the quantity $\delta$, such that $f_1\delta$ is the distance between the exit pupil and the focal plane, then (algebra not shown):

$$\delta = \frac{m^2 k}{m+k-1} = \frac{m^2(1+\beta)}{m^2+\beta}$$

This pupil is generallynot accessible, so if one needs access to a pupil, additional optics are used.

The exit pupil is an important concept. When we discuss aberrations, it is the total wavefront error at the exit pupil which gives the system aberration. Pupils are important for aberration compensation. They can also be used to put light at a location that is independent of pointing errors.

## Aberrations

### Surface requirements for unaberrated images

Next we consider non-paraxial rays. We first consider what surface is required to make an unaberrated image.

We can derive the surface using Fermat's principle. Fermat's principle states that light travels in the path such that infinitessimally small variations in the path doesn't change the travel time to first order: $\mathrm{d}t/\mathrm{d}l$ is a minimum. For a single surface, this reduces to the statement that light travels the path which takes the least time. An alternate way of stating Fermat's principle is that the *optical path length* is unchanged to first order for a small change in path. The OPL is given by:

$$OPL = \int c\mathrm{d}t = \int \frac{c}{v} v\mathrm{d}t = \int n\mathrm{d}s$$

Fermat's principle has a physical interpretation when one considers the wave nature of light. It is clear that around a stationary point of the optical path light, the maximum amount of light can be accumulated over different paths with a minimum of destructive interference. By the wave theory, light travels over all possible paths, but the light coming over the "wrong" paths destructively interferes, and only the light coming over the "right" path constructively interferes.

Fermat's principle can be used to derive the basic laws of reflection and refraction (Snell's law).

Now consider a perfect imaging system that takes all rays from an object and makes them all converge to an object. Since Fermat's principle says the only paths taken will be those for which the OPL is minimally changed for small changes in path, the only way a perfect image will be formed is when all optical path lengths along a surface between an image and object point are the same - otherwise the light doesn't get to this point.

Instead of using Fermat's principle, we could solve for the parameters of a perfect surface using analytic geometry, but this would require an inspired guess for the correct functional form of the surface.

We find that the perfect surface depends on the situation: whether the light comes from a source at finite or infinite distance, and whether the mirror is concave or convex. We consider the various cases now, quoting the results without actually doing the geometry. In all cases, consider the z-axis to be the optical axis, with the y-axis running perpendicular. We want to know the shape of the surface, $y(z)$, that gives a perfect image.

***Concave mirror with one conjugate at infinity***

Sample application: primary mirror of telescope looking at stars.

Fermat's principle gives:

$$y^2 = 2R_z$$

where $R = 2f$, the radius of curvature at the mirror vertex. This equation is that of a parabola. Note, however, that a prabola makes a perfect image only for on-axis images (field angle = 0).

***Concave mirror with both conjugates at finite distances***

Sample application: Gregorian secondary looking at image formed by primary.

For a concave mirror with both conjugates finite, we get an ellipse. Again, this is perfect only for field angle = 0.

$$\frac{(z-a)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 - \frac{2zb^2}{a} + \frac{z^2b^2}{a^2} = 0$$

where

$$a = \frac{s+s'}{2}$$

$$b = \sqrt{(ss')}$$

$$R = \frac{ss'}{s+s'} = \frac{2b^2}{a}$$

***Convex mirror with both conjugates at finite distance***

Sample application: Cassegrain secondary looking at image formed by primary.

For a convex mirror with both conjugates finite, we get a hyperbola:

$$\frac{(z-a)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y^2 + \frac{2zb^2}{a} - \frac{z^2b^2}{a^2} = 0$$

where

$$a = \frac{s+s'}{2}$$

$$b^2 = \sqrt{(-ss')}$$

($s$ is negative)

$$R = -\frac{2b^2}{a}$$

***Convex mirror with one conjugate at infinity***

For a convex mirror with one conjugate at infinity, we get a parabola.

***2D to 3D***

Note that in all cases we've considered a one-dimension surface. We can generalize to 2D surfaces by rotating around the z-axis; for the equations, simply replace $y^2$ with $(x^2 + y^2)$.

***Conic sections***

As you may recall from analytic geometry, all of these figures are *conic sections*, and it is possible to describe all of these figures with a single equation:

$$\rho^2 - 2R_z + (1+K)z^2 = 0$$

where $\rho^2 = x^2 + y^2$ and $R$ is the radius of curvature at the mirror vertex, $K$ is called the conic constant ($K = -e^2$, where $e$ is the eccentricity for an ellipse, $e(b,a)$).

- $K > 0$ gives a prolate ellipsoid

- $K = 0$ gives a sphere
- $-1 < K < 0$ gives an oblate ellipsoid
- $K = -1$ gives a paraboloid
- $K < -1$ gives a hyperboloid

**Aberrations: general description and low-order aberrations**

?, March ?, 2016

Now consider what happens for surfaces that are not perfect, e.g. for the cases considered above for $FA \neq 0$ (since only a sphere is symmetric for all field angles), or for field angle 0 for a conic surface which doesn't give a perfect image?

You get *aberrations*; the light from all locations in the aperture does not land at any common point.

One can consider aberrations in either of two ways:

- All rays don't land at a common point.
- Wavefront deviates from a spherical wavefront.

These two descriptions are equivalent. The former refers to *transverse* aberrations, which give the distance by which the rays miss the paraxial focus, or the *angular* aberration, which is the angle by which the rays deviate from the perfect ray which will hit paraxial focus. For the latter, one discusses the wavefront error, i.e., the deviation of the wavefront from a spherical wavefront as a function of location in the exit pupil.

In general, the angular and transverse aberrations can be determined from the optical path difference between a given ray and that of a spherical wavefront. The relations are given by:

$$\text{angular aberration} = \frac{\mathrm{d}2\Delta z}{\mathrm{d}\rho}$$

$$\text{transverse aberration} = s' \frac{\mathrm{d}\Delta z}{\mathrm{d}\rho}$$

If the aberrations are not symmetric in the pupil, then define angular and transverse $x$ and $y$ aberrations separately by taking derivatives with respect to $x$ or $y$ instead of $\rho$.

***Spherical aberration***

First, consider the axisymmetric case of looking at an object on axis (field angle equal zero) with an optical element that is a conic section. We can consider where rays land as $f(\rho)$, and derive the effective focal length, $f_e(\rho)$, for an arbitrary conic section:

$$z_0 = \frac{\rho}{\tan(2\phi)} = \frac{\rho(1-(\tan\phi)^2)}{2\tan\phi}$$

$$\tan\phi = \frac{\mathrm{d}z}{\mathrm{d}\rho}$$

from conic sections: ... a crapload of equations.

Note that $f_e$ is independent of $z$ only for $K=-1$, a parabola. Also note that $\Delta f$ is symmetric with respect to $\rho$.

We define spherical aberration as the aberration resulting from $K \neq -1$. Rays from different radial positions in the entrance aperture focus at different locations. It is an aberration which is present on axis as seen here. Spherical aberration is symmetric in the pupil. There is no location in space where all rays focus at a point. Note that the behavior (image size) as a function of focal position is not symmetric. One can define several criteria for where the "best focus" might be, leading to the terminology paraxial focus, marginal focus, diffraction focus, and the circle of least confusion.

The asymmetric nature of spherical aberration as a function of focal position distinguishes it from other aberrations and is a useful diagnostic for whether a system has this aberration. This is shown in this figure which shows a sequence of images at different focal positions in the presence of spherical aberration. We define *transverse spherical aberration* (TSA) as the image size at paraxial focus. This is not the location of the minimum image size.

$$\frac{\mathrm{TSA}}{\Delta f} = \frac{\rho}{f - z(\rho)}$$

$$\mathrm{TSA} = 1(1+K)\frac{\rho^3}{2R^2} - 3(1+K)(3+K)\frac{\rho^5}{8R^4} + \dots$$

The difference in angle between the "perfect" ray from the parabola and the actual ray is called the *angular aberration*, in this case *angular spherical aberration* (ASA).

$$\mathrm{ASA} = 2(\phi_P - \phi) \approx \frac{d}{d\rho}(2\Delta z) \approx -(1+K)\frac{\rho^3}{R^3}$$

where $2\Delta z$ gives the optical path difference between the two rays.

This is simply related to the transverse aberration:

$$\text{TSA} = \frac{R}{2}\text{ASA}$$

We can also consider aberration as the difference between our wavefront and a spherical wavefront, which in this case is the wavefront given by a parabolic surface.

$$\Delta z = z_{\text{parabola}} - z(K) = -\frac{\rho^4}{8R^3}(1+K) + \ldots$$

This result can be generalized to any sort of aberration: the angular and transverse aberrations can be determined from the optical path difference between a given ray and that of a spherical wavefront. The relations are given by:

$$\text{angular aberration} = \frac{d(2\Delta z)}{d\rho}$$

$$\text{transverse aberration} = s'\frac{d(2\Delta z)}{d\rho}$$

If the aberrations are not symmetric in the pupil, then we could define angular and transverse $x$ and $y$ aberrations separation by taking derivatives with respect to $x$ or $y$ instead of$\rho$.

***General aberration description***

We can describe deviations from a spherical wavefront generally. Since all we care about are optical path differences, we write an expression for the optical path difference between an arbitrary ray and the chief ray, and in doing this, we can also include the possibility of an off-axis image, and get

$$OPD = OPL - OPL(chiefray)$$

$$OPD = A_0 y + A_1 y^2 + A_1' x^2 + A_2 y^3 + A_2' x^2 y + A_3 \rho^4$$

where we've kept terms only to fourth order and chosen our coordinate system such that the object lies in the y-z plane. The coefficients, $A$, depend on lots of things, such as $(\theta, K, n, R, s, s')$.

Note that rays along the y-axis are called *tangential* rays, while rays along the x-axis are called *sagittal* rays.

Analytically, people generally restrict themselves to talking about *third-order* aberrations, which are fourth-order (in powers of $x, y, \rho$, or $\theta$) in the optical path difference, because of the derivative we take to get transverse or angular aberrations. In the third-order limit, one finds that $A2 = A2'$, and $A1 = -A1'$. Working out the geometry, we find for a mirror that: Craploads more equations.

From the general expression, we can derive the angular or the transverse aberrations in either the $x$ or $y$ direction. Considering the aberrations in the two separate directions, we find:

$$bleh$$

$$bleh$$

The first term is proportional to $\theta^2 y$ and is called *astigmatism*. The second term is proportional to $\theta(x^2 + 3y^2)$ and is called *coma*. The final term, proportional to $y\rho^2$ is *spherical aberration*, which we've already discussed (note for spherical, $AA_x = AA_y$ and in fact the $AA$ in any direction is equal, hence the aberration is circularly symmetric).

***Astigmatism***

For astigmatism, rays from opposite sides of the pupil focus in different locations relative to the paraxial rays. At the paraxial focus, we end up with a circular image. As you move away from this image location, you move towards the tangential focus in one direction and the sagittal focus in the other direction. At either of these locations, the astigmatic image looks like a elongated ellipse. Astigmatism goes as $\theta^2$, and consequently looks the same for opposite field angles. Astigmatism is characterized in the image plane by the *transverse* or *angular* astigmatism (TAS or AAS), which refer to the height of the marginal rays at the paraxial focus. Astigmatism is symmetric around zero field angle.

This figure shows the rays in the presence of astigmatism. This figure shows the behavior of astigmatism as one passes through paraxial focus.

***Coma***

For coma, rays from opposite sides of the pupil focus at the same focal distance. However, the tangential rays focus at a different location than the sagittal rays, and neither of these focus at the paraxial focus. The net effect is to make an image that vaguely looks like a comet, hence the name coma. Coma goes as $\theta$, so the direction of the comet flips sign for opposite field angles. Coma is characterized by either the *tangential* or *sagittal transverse/angular* *coma* (TTC, TSC, ATC, ASC) which describe the height/angle of either the tangential or sagittal marginal rays at the paraxial focus: $TTC = 3TSC$.

This figure shows the rays in the presence of coma. This figure shows the behavior of coma as one passes through paraxial focus. In fact, there are two more third-order aberrations: *distortion* and *field curvature*. Neither affects image quality, only location (unless you are forced to use a flat image plane). Field curvature gives a curved focal plane: if imaging onto a flat detector, this will lead to focus deviations as one goes off-axis. Distortion affects the location of images in the focal plance, and goes as $\theta^3$. The amount of field curvature and distortion can be derived from the aberration coefficients and the mirror parameters.

We can also determine the relevant coefficients for a surface with a displaced stop (Schroeder p 77), or for a surface with a decentered pupil (Schroeder p89-90); it's just more geometry and algebra. With all these realtions, we can determine the optical path differences for an entire system: for a multi-surface system, we just add the OPD's as we go from surface to surface. The final aberrations can be determined from the system OPD.

### Aberration compensation and different telescope types

Wednesday, March 30 (I think)

Using the techniques above, we can write expressions for the system aberrations as a function of the surface figures (and field angles). If we give ourselves the freedom to choose surface figures, we can eliminate one (or more) aberrations.

For example, given a conic constant of the primary mirror, we can use the aberration relations to determine $K_2$ such that spherical aberration is zero; this will give us perfect images on-axis. We find that:

$$K_2 = \left(\frac{m+1}{m-1}\right)^2 + \frac{m^3}{k\,(m-1)^3}\,(K_1+1)$$

satisfies this criterion. If we set the primary to be a parabola ($K_1 = -1$), this gives the conic constant of the secondary we must use to avoid spherical aberration. This type of telescope is called a *classical* telescope. Using the aberration relations, we can determine the amount of astigmatism and coma for such telescopes, and we find that coma gives significantly larger aberrations than astigmatism.

If we allow ourselves the freedom to choose both $K_1$ and $K_2$, we can eliminate both spherical aberration and coma. Designs of this sort are called *aplanatic*. The relevant expression, in terms of the magnification and back focal distance (we could use the relations discussed earlier to present these in terms of other paraxial parameters), is:

$$K_1 = -1 - \frac{2(1+\beta)}{m^2(m-\beta)}$$

We can only eliminate two aberrations with two mirrors, so even this telescope will be left with astigmatism.

There are two different classes of two-mirror telescopes that allow for freedom in the shape of both mirrors: Cassegrain telescopes and Gregorian telescopes (Newtonians have a flat secondary). For the classical telescope with a parabolic primary, the Cassegrain secondary is hyperbolic, whereas for a Gregorian it is ellipsoidal (because of the appropriate conic sections derived above for convex and concave mirrors with finite conjugates). For the aplanatic design, the Cassegrain telescope has two hyperbolic mirrors, while the Gregorian telescope has two ellipsoidal mirrors. An aplanatic Cassegrain telescope is called a *Ritchey-Chretien* telescope.

The following table gives some characteristics of "typical" telescopes. Aberrations are given at a field angle of 18 arc-min in units of arc-seconds. Coma is given in terms of tangential coma.

**Characteristics of Two-Mirror Telescopes**

| Parameter | CC | CG | RC | AG |
|---|---|---|---|---|
| m | 4.00 | -4.00 | 4.00 | -4.00 |
| k | 0.25 | -0.417 | 0.25 | -0.417 |
| 1-k | 0.75 | 1.417 | 0.75 | 1.417 |
| mk | 1.000 | 1.667 | 1.000 | 1.667 |
| ATC | 2.03 | 2.03 | 0.00 | 0.00 |
| AAS | 0.92 | 0.92 | 1.03 | 0.80 |
| ADI | 0.079 | 0.061 | 0.075 | 0.056 |
| $\kappa_m R_1$ | 7.25 | -4.75 | 7.625 | -5.175 |
| $\kappa_P R_1$ | 4.00 | -8.00 | 4.00 | -8.00 |

The image quality is clearly better for the aplanatic designs than for the classical designs, as expected because coma dominates off-axis in the classical design. In the aplanatic design, the Gregorian is slightly better. However, when considerations other than just optical quality are considered, the Cassegrain usually is favored: for the same primary mirror, the Cassegrain is considerably shorter and thus it is less costly to build an enclosure and telescope structure. To keep the physical length the same, the Gregorian would have to have a faster primary mirror, which are more difficult (i.e. costly) to fabricate, and which will result in a greater sensitivity to alignment errors. Both types of telescopes have a *curved* focal plane.

## Sources of aberrations

So far, we have been discussing aberrations which arise from the optical design of a system when we have a limited number of elements. However, it is important to realize that aberrations can arise from other sources as well. These other sources can give additional third-order aberrations, as well as higher order aberrations. Some possible sources include:

- misfigured or imperfectly figured optics : rarely is an element made exactly to specification!
- misalignments. If the mirrors in a multiple-element system are not perfectly aligned, aberrations will result. These can be derived (third-order) from the aberration expressions for decentered elements. For two mirror systems, one finds that decentering or tilting the secondary introduces a constant amount of coma over the field. Coma dominates astigmatism for a misaligned telescope.
- mechanical/support problems. When the mirrors are mounted in mirror cells the weight of the mirror is distributed over some support structures. Because the mirrors are not infinitely stiff, some distortion of the mirror shape will occur. Generally, such distortion will probably change as a function of which way the telescope is pointing. Separate from this, becuase the telescope structure itself is not perfectly stiff, one expects some flexure which gives a different secondary (mis)alignment as a function of where one is pointing. Finally, one might expect the spacing between the primary and secondary to vary with temperature, if the telescope structure is made of materials which have non-zero coefficients of expansion.
- chromatic aberration. Generally, we've only been discussing mirrors since this is what is used in telescopes. However, astronomers often put additional optics (e.g., cameras or spectrographs) behind telescopes which may use refractive elements rather than mirrors. There are aber-

ration relations for refractive elements just as we've discussed, but these have additional dependences on the indices of refraction of the optical elements. For most refractive elements, the index of refraction varies with wavelength, so one will get wavelength-dependent aberrations, called chromatic aberrations. These can be minimized by good choices of materials or by using combinations of different materials for different elements; however, it is an additional source of aberration.
- seeing. The earth's atmosphere introduces optical path differences between the rays across the aperture of the telescope. This is generally the **dominant** source of image degradation from a ground-based telescope. Consequently, one builds telescopes in good sites, and as far as design and other sources of image degradation are concerned, one is generally only interested in getting these errors small when compared with the smallest expected seeing errors.

## Ray tracing

For a fully general calculation of image quality, one does not wish to be limited to third-order aberrations, nor does one often wish to work out all of the relations for the complex set of aberrations which result from all of the sources of aberration mentioned above. Real world situations also have to deal with *vignetting* in optical systems, in which certain rays may be blocked by something and never reach the image plane (e.g., in a two-mirror telescope, the central rays are blocked by the secondary).

Because of these and other considerations, analysis of optical systems is usually done using *ray tracing*, in which the parameters of an optical system are entered into a computer, and the computer calculates the expected images on the basis of geometric optics. Many programs exist with many features: one can produce *spot* diagrams which show the location of rays from across the aperture at an image plane (or any other location), plots of transverse aberrations, plots of optical path differences, etc., etc.

(Demo ray trace program. Start with on-axis object, single mirror. Where is focus? What will image look like with spherical mirror? What do we need to do to make it perfect? How does it depend on aperture size? Now how do off-axis images look like? spot diagrams, through focus, ray fan, opd plots, etc. Now introduce second mirror. What determines where focus will be? Magnification? What shape to make a perfect on-axis image? What do off-axis images look like? How do we make them better? Now how is performance?

Real 3.5m and 1m prescriptions. Issue: guider.)

## Physical (diffraction) optics

Up until now, we have avoided considering the wave nature of light which introduces *diffraction* from interference of light coming from different parts of the aperture. Because of diffraction, images of a point source will be slightly blurred. From simple geometric arguments, we can estimate the size of the blur introduced from diffraction:

We find that:

$$\theta \sim \frac{\lambda}{D}$$

Using this, we find that the diffraction blur is smaller than the blur introduced by seeing for $D > 0.2$ meters at 5500 Å, even for the excellent seeing conditions of 0.5 arcsecond images. However, the study of diffraction has become important recently because of several reasons: 1) the existence of the Hubble Space Telescope, which is diffraction limited (no seeing), 2) the increasing use of infrared observations, where diffraction is more important than in the optical, and 3) the development of adaptive optics, which attempts to remove some of the distortions caused by seeing. Consequently, it's now worthwhile to understand some details about diffraction.

To work out in detail the shape of the images formed from diffraction involves understanding wave propagation. Basically, one integrates over all of the source points in the aperture (or exit pupil for an optical system), determining the contribution of each point at each place in the image plane. The contributions are all summed taking into account phase differences at each image point, which causes reinforcment at some points and cancellation at others. The expression which sums all of the individual source points is called the *diffraction integral*. When the details are worked out, one finds that the intensity in the image plane is related to the intensity and phase at the exit pupil. In fact the wavefront is described at any plane by the *optical transfer function*, which gives the intensity and phase of the wave at all locations in that plane. The OTF at the pupil plane and at the image plane are a Fourier transform pair. Consequently, we can determine the light distribution in the image plane by taking the Fourier transform of the pupil plane; the light distribution, or point spread function, is just the modulus-squared of the OTF at the image plane. Symbolically, we have

$\mathrm{PSF} = |\mathrm{FT}(\mathrm{OTF}(\mathrm{pupil}))|^2$

where $FT$ represents a Fourier transform, and

$$OTF(pupil) = P(x, y) \exp ik\phi(x, y)$$

$P(x, y)$ is the *pupil function*, which gives the transmission properties of the pupil, and usually consists of ones and zeros for locations where light is either transmitted or blocked (e.g., for a circular lens, the pupil function is unity within the radius of lens, and zero outside; for a typical telescope the pupil function includes obscuration by the secondary and secondary support structure). $\phi$ is the phase in the pupil. More relevantly, $\phi$ can be taken to be the optical path difference in the pupil with some fiducial phase, since only OPDs matter, not the absolute phase. Finally the wavenumber $k$ is just $\frac{2\pi}{\lambda}$.

For the simple case of a plane wave with no phase errors, the diffraction integral can be solved analytically. The result for a circular aperture with a central obscuration, when the fractional radius of the obscuration is given by $\epsilon$, the expression for the PSF is:

$$\text{PSF} \propto \left[ \frac{2J_1(v)}{v} \text{ - } \epsilon^2 \frac{2J_1(\epsilon v)}{\epsilon v} \right]^2$$

$$\text{v} = \frac{\pi r}{\lambda F}$$

where $J_1$ is a first order Bessel function, $r$ is the distance in the image plane, $\lambda$ is the wavelength, and $F$ is the focal ratio ($F = f/D$).

This expression gives the so-called *Airy pattern* which has a central disk surrounded by concentric dark and bright rings. The radius of the first dark ring is at the physical distance $r = 1.22\lambda F$, or alternatively, the angular distance $\alpha = 1.22\lambda/D$. This gives the size of the *Airy disk*.

For more complex cases, the diffraction integral is solved numerically by doing a Fourier transform. The pupil function is often more complex than a simple circle, because there are often additional items which block light in the pupil, such as the support structures for the secondary mirror.

This figure shows the Airy pattern, both without obscurations, and with a central obscuration and spiders in a setup typical of a telescope.

In addition, there may be phase errors in the exit pupil, because of the existence of any one of the sources of aberration discussed above. For general use, $\phi$ is often expressed as an series, where the expansion is over a set of orthogonal polynomials for the aperture which is being used. For circular apertures with (or without) a central obscuration (the case most often found in astronomy), the appropriate polynomials are called *Zernike* polynomials. The lowest order terms are just uniform slopes of phase across the pupil, called tilt, and simply correspond to motion in the image plane. The next terms correspond to the expressions for the OPD which we found above for focus, astigmatism, coma, and spherical aberration, generalized to allow any orientation of the phase errors in the pupil. Higher order terms correspond to higher order aberrations.

This figure shows the form of some of the low order Zernike terms: the first corresponds to focus aberration, the next two to astigmatism, the next two to coma, the next two to trefoil aberration, and the last to spherical aberration.

A wonderful example of the application of all of this stuff was in the diagnosis of spherical aberration in the Hubble Space Telescope, which has been corrected in subsequent instruments in the telescope, which introduce spherical aberration of the opposite sign. To perform this correction, however, required and accurate understanding of the amplitude of the aberration. This was derived from analysis of on-orbit images, as shown in this figure. Note that it is possible in some cases to try to recover the phase errors from analysis of images. This is called phase retrieval. There are several ways of trying to do this, some of which are complex, so we won't go into them, but it's good to know that it is possible. But an accurate amplitude of spherical aberration was derived from these images. This derived value was later found to correspond almost exactly to the error expected from an error which was made in the testing facility for the HST primary mirror, and the agreement of these two values allowed the construction of new corrective optics to proceed...

## Adaptive optics

Monday, April 4