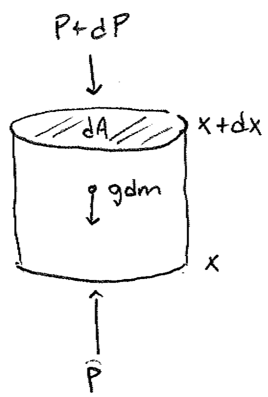
The importance of gravity $\Rightarrow$ Pressure!



$$g = \frac{GM_*}{R_*^2}$$

$G = 6.6742 \times 10^{-8}\ cm^3 g^{-1} s^{-1}$

$M_\odot = 1.9892 \times 10^{33}\ g$

$R_\odot = 6.95997 \times 10^{10}\ cm$

$g_\odot = 2.738 \times 10^4\ cm/s^2$

often expressed $\log(g_\odot) = 4.438 \approx 4.4$

$$\boxed{\log\left(\frac{g_*}{g_\odot}\right) = \log\left(\frac{M_*}{M_\odot}\right) - 2\log\left(\frac{R_*}{R_\odot}\right)}$$

determining $\log(g)$ is one way to get M_*/R_*^2

In hydrostatic equilibrium

$F_{down} = F_{up}$

$(P+dP)dA + gdm = PdA$

defining ρ = mass density = $\dfrac{dm}{dV} = \dfrac{dm}{dxdA}$ $\qquad dm = \rho dx dA$

$\rho = \sum m_i n_i$

$= \sum \dfrac{n_i A_i m_{amu}}{X_i}$

A_i = atomic weight
m_i = mass
n_i = number density [atoms cm^{-3}]
X_i = mass fraction

we have

$(P+dP)dA + g\rho dx dA = PdA$

giving

$$\boxed{\frac{dP}{dx} = -g\rho}$$

$\boxed{g \text{ is important!}}\ \boxed{P \propto g}$

- can assume $g \neq g(x)$ in atmosphere, but not for stellar structure!
- we will see, P and ρ coverned through Equation of state and detailed balancing of particle densities and ionization conditions.
- also, PRESSURE BROADENING of absorption lines (nearest neighbor effects)

$$\frac{dP}{dx} = -\rho g \qquad P = \frac{\rho}{\mu m_{amu}} kT$$

(ignoring radiation)

$\mu m_{amu} = \langle m \rangle_{ave}$ of all particles including electrons.

$\mu \equiv$ mean molecular weight (depends on ionization conditions)

also

$$P = n_{tot} kT = (n_N + n_e) kT$$

$$n_{tot} = \frac{\rho}{\mu m_{amu}}$$

thus

$$\frac{dP}{dx} = -\frac{P \mu m_{amu}}{kT} g$$

$$\frac{dP}{P} = -g \frac{\mu m_{amu}}{kT} dx$$

yielding

$$\boxed{P(x) = P(x_0) \exp\left\{ -\frac{m_{amu}}{k} \int_{x_0}^{x} \frac{\mu g}{T} dx \right\}}$$

general solution

technically $\mu = \mu(x)$ $g = g(x)$ and $T = T(x)$

if they are constant then

$$P(x) = P(x_0) \exp\left\{ -\frac{\mu m_{amu} g}{kT} (x - x_0) \right\} = P(x_0) \exp\left\{ -\frac{(x - x_0)}{H} \right\}$$

idealized solution

where H is scale height $\quad H = \dfrac{kT}{\mu m_{amu} g}$

- atmospheres are not isothermal with physical depth x
- atmosphere have changing μ, which depend upon ionization fractions with physical depth x
- though the gravity is also varying with physical depth x, it can be treated as a constant with little loss of accuracy.

Importance of Pressure consider the mean distance between particles: $\langle r \rangle$

$$n = \frac{1}{\frac{4}{3}\pi \langle r \rangle^3} = \frac{P}{kT} \rightarrow \langle r \rangle = \left[\frac{3kT}{4\pi P} \right]^{1/3} \quad \text{or} \quad \langle r \rangle \propto \left[\frac{T}{P} \right]^{1/3}$$

as P goes up $\langle r \rangle$ decreases! particles influence each other more

Qualitative Overview: Pressure Broadening

$$\langle r \rangle = \left[\frac{3kT}{4\pi P}\right]^{1/3}$$

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \qquad \langle v^2\rangle = \int_0^\infty v^2 f(v)\,dv$$

maxwell-Boltzman speed distribution

$$\langle r \rangle = \left[\frac{m\langle v^2\rangle}{4\pi P}\right]^{1/3}$$

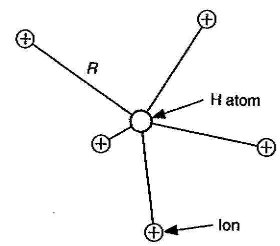


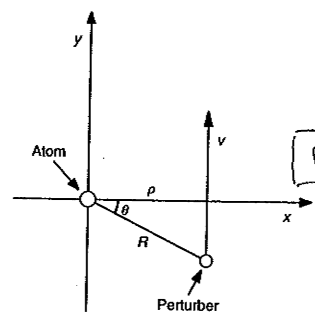
Fig. 11.6. The distribution of ions in space gives a non-zero electric field at the position of the atom. This field causes a perturbation in the energy levels of the atom. Grey

Many methods, but consider "impact" perturbation approach.

1. electric magnetic fields of neighbors influence the frequecies of transitions via

$$\Delta\omega = C_n R^{-n} \qquad \left\{ \begin{aligned} \omega &= 2\pi\nu \\ &= \frac{2\pi c}{\lambda} \end{aligned} \right\}$$

n = integer



$\boxed{\rho = R\cos\theta}$

Fig. 11.3. The perturber passes the atom at an impact distance ρ with a velocity v. The y axis is chosen to be parallel to v. Grey

2. the perturbing ions pass by the target atom with average velocity

$$V = \langle v^2\rangle^{1/2} = \left\{\frac{8kT}{\pi}\left(\frac{1}{M_A} + \frac{1}{M_P}\right)\right\}^{1/2}$$

center of mass; atom; perturber

with closest approach ρ_0 (not density!)

this results in an impulse of duration

$$\tau = \frac{1}{\pi \rho_0^2 n_{ion} V} \qquad \rho_0 = \left[\frac{2\pi C_n}{V}\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta\, d\theta\right]^{1/(n-1)} \quad (n = \text{integer})$$

which causes a shift in the energy of the atom.

{ think of ρ_0 as an average perturbation distance of closest approach. The perturbation goes as $C_n R^{-n}$ integrated over all $\theta = (-\pi/2 \leftrightarrow \pi/2)$ to obtain an impulse time τ (duration). one obtains the integral given to the left }

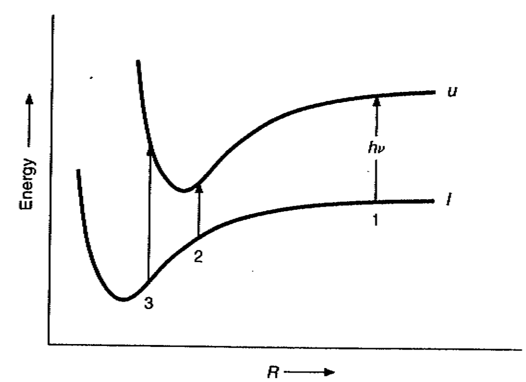


Fig. 11.2. The energies associated with the upper and lower atomic levels on a transition depend on the distance R to the perturber. The transition energy can be either less (2) or greater (3) than the unperturbed value (1). Grey

3. the functional form of the change in the transition energy is of the form

$$\Delta E \sim \frac{1/\pi\tau}{(\omega-\omega_0)^2 + (1/\tau)^2} \qquad \text{(Lorentzian)}$$

which results in significant line broadening in wings of lines $\sim \dfrac{1}{(\omega-\omega_0)^2}$

* the values of n and C_n depend upon the type of perturbation

Pressure Broadening

The value of n (the power index $\Delta\omega = C_n R^{-n}$) applies for the physical effect of the perturbation

$n=2$	Linear Stark (first order electrical)	Effects Hydrogen	Perturbers are protons and electrons
$n=3$	Resonance Broadening	Effects all atoms	Perturbers are like atoms
$n=4$	Quadratic Stark (second order electrical)	Effects non-hydrogen atoms	Perturbers are protons and electrons
$n=6$	van der Waals force	Effects all atoms	Perturbers are all non-like atoms mostly hydrogen perturbing non-hydrogen

$n=2$

Linear Stark Effect: dominates the broadening of hydrogen lines. It requires a slightly different treatment. The broadening is characterized by a narrow Gaussian core and very broad Lorentzian wings due to high velocities (short impulse times) because perturbers are protons and electrons (light particles).

$H\gamma$

example of Linear Stark neutral hydrogen perturbed by electrons

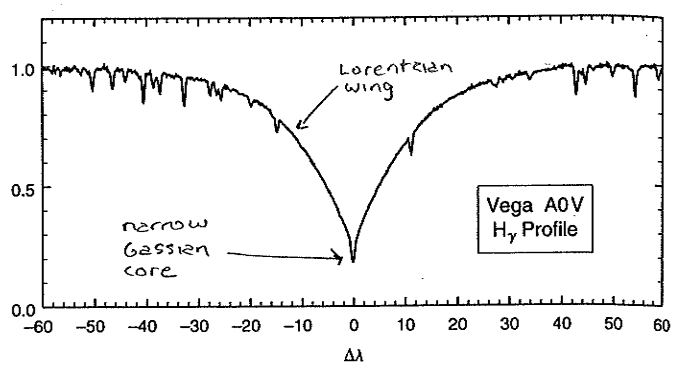


Fig. 11.5. The hydrogen lines in hot stars like Vega are about two orders of magnitude wider than typical metal lines. Taken at Observatoire de Haute-Provence; resolving power of ~16 000. Courtesy of F. Royer. Grey

NOTE. core not saturated - this kind of profile does not look like a damped profile!

recall $\Delta\lambda_D \simeq 0.2\,\text{Å}$! (thermal)

This is $H\gamma$ line, a higher order ($n=2 \rightarrow n=5$) hydrogen line. The Stark Effect dominates these lines. $H\alpha$ is dominated by Resonance broadening ($n=3$)

n=3

Resonance broadening: important mostly for hydrogen - hydrogen interactions. It dominates the lowest order hydrogen lines, particularly the Hα line. It is neglible in comparison to Linear Stark (n=2) for Hβ and higher.

NOTE: Since Linear Stark and Resonance broadening effect hydrogen lines, they are important in stars for which hydrogen is in its first excited state but not ionized. Thus they are important broadening mechanism in A, F, and G stars.

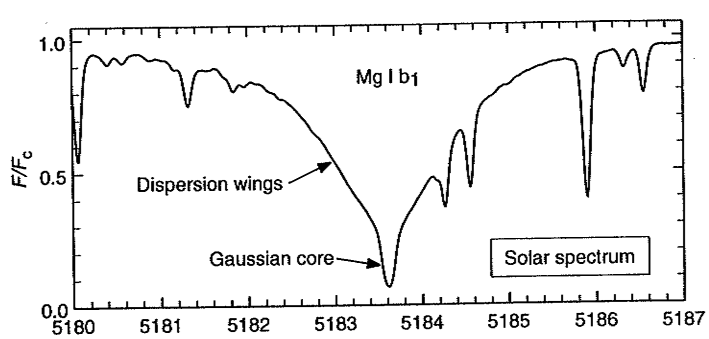
n=4

Quadratic Stark Effect: important for broadening lines in non-hydrogenic atoms (i.e. HeI, etc) due to perturbations by electrons (and somewhat protons). Most effective with high electron density, meaning in highly ionized atmospheres. Thus, important in hot, early type stars.

n=6

van der Waals interactions: important for non-hydrogenic atoms perturbed by neutral hydrogen atoms. It is the dominant source of pressure broadening in cooler-F, G, and hotter K stars. Important for CaII H+K, and NaI H+K, also MgI b and FeI, etc.

log g = 4.4



Gray

Fig. 11.11. In a few cases spectral lines, like this magnesium line in the solar spectrum, clearly show the Gaussian core and the dispersion wings with a relatively sharp transition between the two near $F/F_c \approx 0.3$.

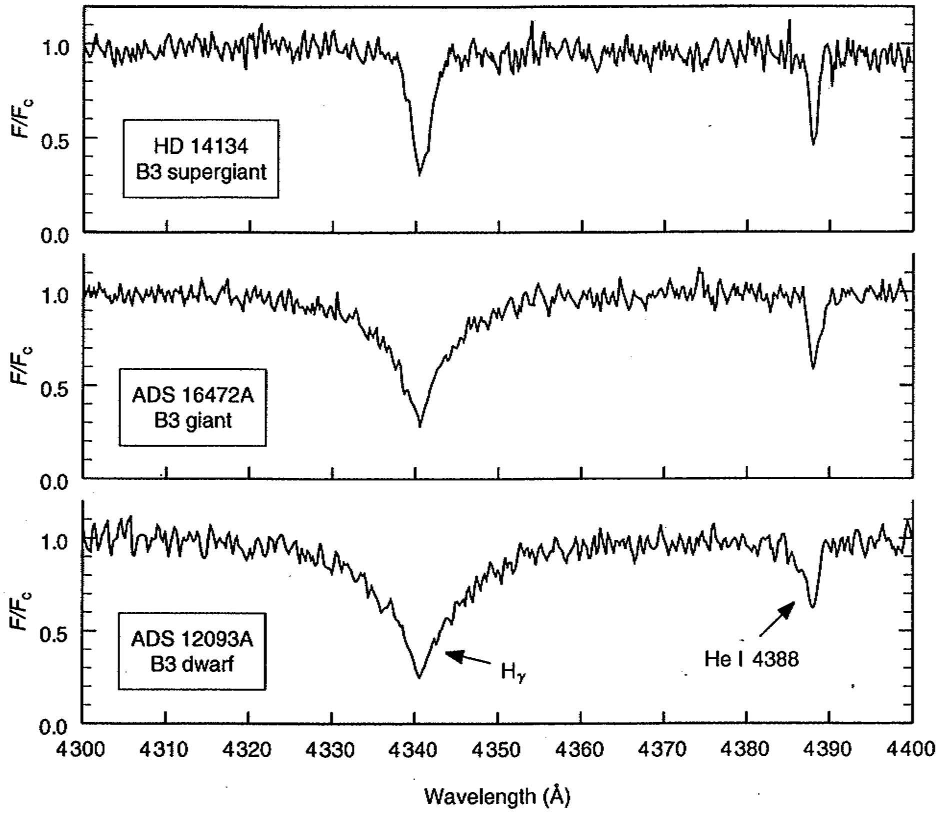
example of vander Waals pressure broadening

metal line perturbed by neutral hydrogen

● PRESSURE BROADENING (*'s of same T)

Hγ

PRESSURE



low log(g)

increasing gravity on surface

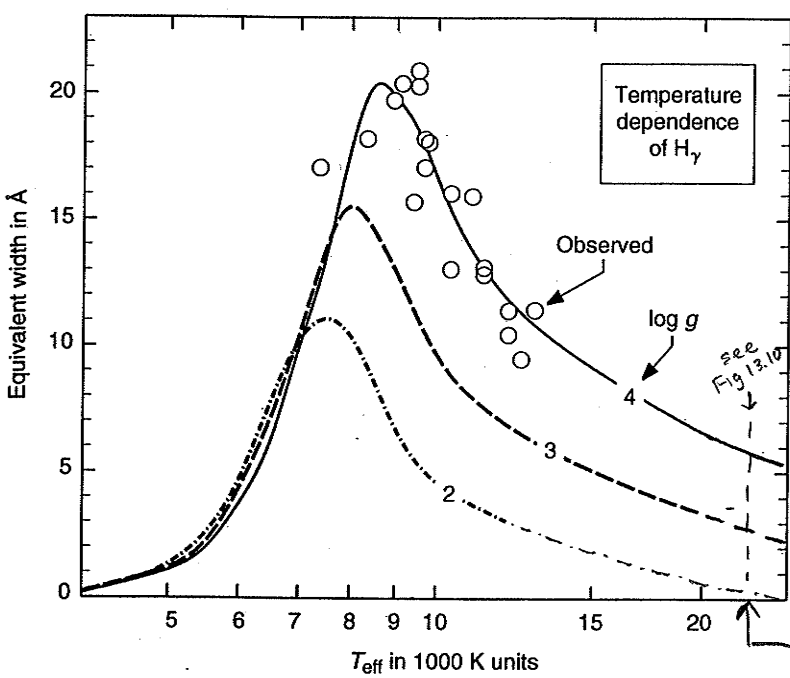
high log(g)

Fig. 13.10. These data of Petrie (1953) for early B stars show the pressure variation in H_γ. The effect arises because of the pressure dependence of the linear Stark effect. (n=2)

● THERMAL + PRESSURE EFFECTS

Hγ

PRESSURE + THERMAL



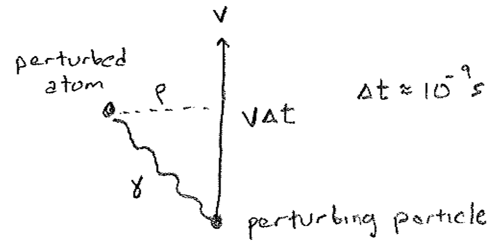
$\log g_\odot = 4.4$

$g \propto \frac{M}{R^2}$

B3 star (for reference in Fig 13.10 above)

Fig. 13.7. The change in equivalent width of H_γ is shown as a function of temperature. Model calculations of Carbon and Gingerich (1969) are shown by the lines (log g indicated). The circles are measured values based on the data of Gray and Evans (1973).

Collisional Broadening (Impact Treatment)



$\Delta t \approx 10^{-9}$ s

γ = photon of perturbing particle's electric field.

ρ = impact parameter $\left\{ \text{closest approach} \right.$

when perturbing particle is at closest approach, $t = 0$

- In the impact approximation (theory), the electric field of a "colliding" particle is treated as a photon originating at the perturbing particle and terminating at the atom being perturbed.

The photon is thus truncated over a path length $c\Delta t$. So, we can treat the photon as being "trapped" in a "box".

A truncated sinisoidal wave has an electric field spectrum

$$E_\nu(\Delta t) = E_0 \,\mathrm{sinc}\left[\pi \Delta t (\nu - \nu_0)\right]$$

where ν_0 is the frequency of the non-truncated photon (the sinisoidal frequency) and the intensity is

$$I_\nu(\Delta t) = E_\nu^*(\Delta t)\cdot E_\nu(\Delta t) = E_0^2 \,\mathrm{sinc}^2\left[\pi \Delta t(\nu - \nu_0)\right]$$

The Fourier transform of $I_\nu(\Delta t)$ provides the frequency power spectrum $F(\nu)$ for a given Δt, yielding

$$F(\nu) = \left[\frac{\sin[\pi \Delta t(\nu-\nu_0)]}{\pi(\nu-\nu_0)}\right]^2 = (\Delta t)^2 \left[\frac{\sin[\pi \Delta t(\nu-\nu_0)]}{\pi\Delta t(\nu-\nu_0)}\right]^2$$

- To obtain the full frequency redistribution function of the effect of many random impulses (or collisions) we must integrate over the impulse times weighted by the probability of a collision occuring in the interval $d\Delta t$:

$$f_p(\nu) = \int_{-\infty}^{\infty} F(\nu)\, dP(\Delta t)$$

where $dP(\Delta t)$ is the probability element in interval Δt

- we need to compute $dP(\Delta t)$ ...

define $P(\Delta t)$ = probability of a collision in Δt

$p(\Delta t)$ = probability of <u>no</u> collision in Δt

where Δt_0 is the average time between collisions

then $p(\Delta t) = 1 - P(\Delta t)$

Collisional Broadening (cont)

in interval $d\Delta t$, the probability of no collision is (to first order)

$$p(\Delta t + d\Delta t) = p(\Delta t) - p(\Delta t)\frac{d\Delta t}{\Delta t_0}$$

the probability of no collision decreases with time

intuitive

comparing to the Taylor expansion

$$p(\Delta t + d\Delta t) = p(\Delta t) + \frac{dp}{d\Delta t}d\Delta t + \cdots$$

we see that

$$\frac{dp}{d\Delta t} = -\frac{p(\Delta t)}{\Delta t_0} \quad \text{or} \quad \frac{dp}{p(\Delta t)} = -\frac{d\Delta t}{\Delta t_0}$$

integrating, we have

$$p(\Delta t) = \exp\left\{-\frac{\Delta t}{\Delta t_0}\right\}$$

From $p(\Delta t) = 1 - P(\Delta t)$

$$P(\Delta t) = 1 - \exp\left\{-\frac{\Delta t}{\Delta t_0}\right\}$$

from which we obtain

$$dP(\Delta t) = \exp\left\{-\frac{\Delta t}{\Delta t_0}\right\}\frac{d\Delta t}{\Delta t_0}$$

- Thus, the integral for $f_p(\nu)$ is

$$f_p(\nu) = 2\int_{-\infty}^{\infty}(\Delta t)^2\left[\frac{\sin[\pi\Delta t(\nu-\nu_0)]}{\pi\Delta t(\nu-\nu_0)}\right]^2\exp\left\{-\frac{\Delta t}{\Delta t_0}\right\}\frac{d\Delta t}{\Delta t_0}$$

which has analytic solution

$$\boxed{f_p(\nu) = \frac{2\,(1/\Delta t_0)}{4\pi^2(\nu-\nu_0)^2 + (1/\Delta t_0)^2} = \frac{1}{\pi}\,\frac{(1/2\pi\Delta t_0)}{(\nu-\nu_0)^2 + (1/2\pi\Delta t_0)^2}}$$

- The frequency redistribution function is a Lorentzian, $L(\nu)$, with $\boxed{\gamma = \frac{1}{2\pi\Delta t_0}}$

note $\int_0^{\infty} f_p(\nu)\,d\nu = \int_0^{\infty} L(\nu)\,d\nu = \frac{1}{\pi}\int_0^{\infty}\frac{\gamma}{(\nu-\nu_0)^2+\gamma^2}\,d\nu = 1$

$f(\nu)$ normalized.

Collisional Broadening (cont)

- Note the identical form to the natural broadening cross section

$$\alpha_n \propto \frac{(\Gamma/4\pi)}{(\nu-\nu_r)^2+(\Gamma/4\pi)^2}$$

so, writing $\gamma_0 = \frac{2}{\Delta t_0}$, the pressure broadening redistribution function is

$$\boxed{f_p(\nu) = \frac{1}{\pi}\frac{(\gamma_0/4\pi)}{(\nu-\nu_0)^2+(\gamma_0/4\pi)^2}}$$

$HWHM = \frac{\gamma_0}{4\pi}$

$FWHM = 2\left(\frac{\gamma_0}{4\pi}\right) = \frac{\gamma_0}{2\pi}$

$AMPLITUDE = \frac{1}{\pi}\left(\frac{\gamma_0}{4\pi}\right) = \frac{\gamma_0}{4\pi^2}$

γ_0 = "damping constant"

- what remains is to determine the value of $\gamma_0 = \frac{2}{\Delta t_0}$

- From standard mean free path calculations

$$\frac{1}{\gamma_0} = \frac{\Delta t_0}{2} = \frac{1}{2}\frac{1}{n_c\langle\sigma\rangle\langle v\rangle}$$

$$\langle\sigma\rangle = \frac{\int_0^\infty \sigma(v)\,dv}{\int_0^\infty dv}$$

where $\sigma(v)$ is the interaction cross-section (where all the physics lurks)
$\langle v\rangle$ is the <u>average</u> (not most probable) velocity of the interaction ← center of mass frame
n_c is number density of perturbing particles ← "colliding"

In actuality

$$\gamma_0 = \frac{2}{\Delta t_0} = 2n_c\int_0^\infty v\,f(v)\,\sigma(v)\,dv$$

the cross section weighted by the Maxwellian velocity distribution

- SIMPLIFIED APPROACH

$$\langle v\rangle = \left\{\frac{8kT}{\pi}\left(\frac{1}{m_a}+\frac{1}{m_c}\right)\right\}^{1/2}$$

center of mass average velocity in T.E.

we now need to compute $\langle\sigma(v)\rangle \equiv \langle\sigma\rangle$, the cross section of the perturbation - this will depend upon the <u>type</u> of electromagnetic perturbation. We designate these types by the subscript n; so we need to find $\underline{\underline{\sigma_n}}$, which will then yield $\underline{\underline{\gamma_n}}$.

Collisional Broadening (cont)

- Classically $\sigma_n = \pi \rho_n^2$ where ρ is the "interaction" impact parameter

  + it is a physical distance [cm], but it includes the interaction physics for the perturbation of type "n".

then

$$\boxed{\gamma_n = 2\pi \rho_n^2 \, n_e \langle v \rangle}$$

we need the "interaction" impact parameter.

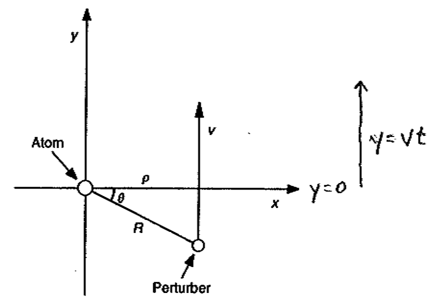


Fig. 11.3. The perturber passes the atom at an impact distance ρ with velocity v. The y axis is chosen to be parallel to v.

Grey

- **FORMS OF THE PERTURBATIONS**

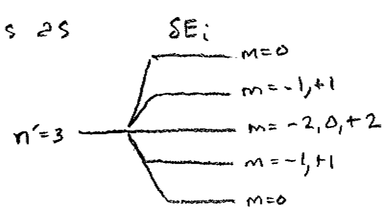
1. Linear Stark Effect ($n=2$)
   - electric dipole interaction, where the dipole electric field scales as R^{-2}, thus $\boxed{n=2}$ and the energy shift of the level in the atom goes as

$$\delta E_i = \frac{h}{2\pi} K_i R^{-2}$$

{ R = distance to perturbing ion/electron }

and the energy shift of transition $i \leftrightarrow j$ goes as

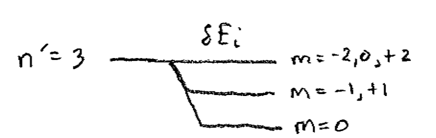
$$\Delta E_{ij} = \frac{h}{2\pi} K_{ij} R^{-2}$$



(odd parity splits)
symmetric broadening

2. Quadratic Stark Effect ($n=4$)
   - electric quadrupole interaction (required when dipole interaction nulls). The electric field scales as R^{-4}, thus $\boxed{n=4}$ and the energy shift goes as

$$\Delta E_{ij} = \frac{h}{2\pi} K_{ij} R^{-4}$$



(even parity splits)
asymmetric broadening

In the Linear Stark Effect, the otherwise energy degenerate levels split symmetrically about the unshifted level. (energy splits). The broadening is due to the magnitude of the splitting. Multiple transitions now occur between these multiple levels, resulting in the broadening. It is symmetric about unshifted line. In the Quadratic Stark Effect the splitting is one sided (to lower energies) relative to the unshifted line - thus, the broadening is asymmetric about the line center.

Collisional Broadening (cont)

3. Resonance Broadening ($n=3$) (like atoms/ions)

- combined Linear stark in resonence with the transition energy in the affected atom/ion. Interaction goes as R^{-3}, thus $\boxed{n=3}$ and

$$\Delta E_{ij} = \frac{h}{2\pi} K_{ij} R^{-3}$$

4. Van der Waals ($n=6$)

- interaction between neutral particles. These atoms have internal electric moments (dipoles, quadrapoles, etc) and they induce dipoles in each other. The induced moment goes as R^{-3} and the interaction of the moments goes as R^{-6}, thus $\boxed{n=6}$. And

$$\Delta E_{ij} = \frac{h}{2\pi} K_{ij} R^{-6}$$

⇒ Recall earlier notes outlining the atom/ions most strongly affected by each of these different forms of the perturbations (REVIEW THEM)

- $\boxed{\text{GETTING THE "INTERACTION" IMPACT PARAMETER } \rho_n}$

recall $\gamma_n = 2\pi \rho_n^2 n_e \langle v \rangle$

and we desire γ_n

ρ_n is obtained from the total energy shifts of the levels, but interms of the frequency shift $\Delta\nu = \frac{\Delta E}{h}$

for a given transition, the total shift (called the phase shift) over the "impact" is

$$\eta_{ij} = \frac{2\pi}{h}\int_{-\infty}^{\infty} \Delta\nu_{ij}\, dt = K_{ij}\int_{-\infty}^{\infty} \frac{dt}{R^n} = K_{ij}\int_{-\infty}^{\infty} \frac{dt}{\left[\rho_n^2 + (\langle v \rangle t)^2\right]^{n/2}}$$

from $R^2 = \rho_n^2 + (\langle v \rangle t)^2$ (see Fig 11.3 Gray) previous page.

re-arranging...

$$\frac{1}{\left[\rho_n^2 + (\langle v \rangle t)^2\right]^{n/2}} = \frac{1}{\left[\rho_n^2\left\{1 + \left(\frac{\langle v \rangle t}{\rho_n}\right)^2\right\}\right]^{n/2}} = \frac{1}{\rho_n^n}\,\frac{1}{\left[1 + \left(\frac{\langle v \rangle t}{\rho_n}\right)^2\right]^{n/2}}$$

we obtain →

Collisional Broadening (cont)

$$\eta_{ij} = \frac{K_{ij}}{\rho_n^n}\int_{-\infty}^{\infty}\frac{dt}{\left[1+\left(\frac{\langle v\rangle t}{\rho_n}\right)^2\right]^{n/2}} = \frac{K_{ij}}{\langle v\rangle \rho_n^{n-1}}\int_{-\infty}^{\infty}\frac{\left(\frac{\langle v\rangle}{\rho_n}\right)}{\left[1+\left(\frac{\langle v\rangle}{\rho_n}\right)^2 t^2\right]^{n/2}}\,dt$$

$$\boxed{I_n = \int_{-\infty}^{\infty}\frac{x}{(1+x^2t^2)^{n/2}}\,dt = \left\{\begin{array}{ll}\pi & n=2\\ 2 & n=3\\ \pi/2 & n=4\\ 3\pi/8 & n=6\end{array}\right\}} \qquad x = \frac{\langle v\rangle}{\rho_n}$$

← this is the integral in Table 11.2 of Grey

so we have

$$\eta_{ij} = \frac{K_{ij}}{\langle v\rangle \rho_n^{n-1}}\cdot I_n \quad \text{or} \quad \boxed{\rho_n^2 = \left\{\frac{K_{ij} I_n}{\eta_{ij}\langle v\rangle}\right\}^{2/n-1}}$$

finally yielding

$$\gamma_n = 2\pi\rho_n^2 n_c\langle v\rangle = 2\pi n_c\left\{\frac{K_{ij}I_n}{\eta_{ij}\langle v\rangle}\right\}^{2/n-1}\langle v\rangle = \underline{2\pi n_c\left\{\frac{K_{ij}I_n}{\eta_{ij}}\right\}^{2/n-1}\langle v\rangle^{\frac{n-3}{n-1}}}$$

invoking

$$\langle v\rangle = \left\{\frac{8kT}{\pi}\left(\frac{1}{m_a}+\frac{1}{m_c}\right)\right\}^{1/2} \quad \text{average velocity}$$

and $P_c = n_c kT \rightarrow n_c = \frac{P_c}{kT}$ where P_c is the partial pressure of the perturbing atom. or electrons

we obtain

$$\boxed{\gamma_n = 2\pi P_c T^{-\frac{n+1}{2(n-1)}}\left\{\frac{K_{ij}I_n}{\eta_{ij}}\right\}^{\frac{2}{n-1}}\left[\frac{8k}{\pi}\left(\frac{1}{m_a}+\frac{1}{m_c}\right)\right]^{\frac{n-3}{2(n-1)}}}$$

$\gamma_n \propto P_c$ the "damping constant" is linear proportional with pressure. ← thus, we call this "pressure broadening"

$\gamma_n \propto T^{-\frac{n+1}{2(n-1)}}$ weakly inverse proportional to T

$n=2$	$n=3$	$n=4$	$n=6$
$\gamma_n \propto T^{-3/2}$	$\gamma_n \propto T^{-1}$	$\gamma_n \propto T^{-5/8}$	$\gamma_n \propto T^{-7/10}$

Pressure Broadening (impact theory parameters)

$\boxed{K_{ij}}$

To fully specify γ_n for a given transition in a given atom/ion we need the values of K_{ij}. They are specific to a given transition within a given atom/ion. They can be computed from Quantum Theory for some cases, but many are measured (but not all!)

$\boxed{\eta_{ij}}$

Also required is the value η_{ij}, the total phase shift of the "photon in a box" over the duration of the impact. The "interaction" impact parameter is proportional to $\eta_{ij}^{-2/n-1}$. This has remained an arbitrary parameter of impact theory. $\eta_{ij} = 1$ radian is nominally adopted.

COLLISIONAL THEORY is one of many approaches, most are more sophisticated and better account for the spatial distribution of a "sea" of perturbing particles as well as polarization of the gas (The Debye shielding) An entire course could be dedicated to Pressure broadening.

- example: NaI $D_2 D_1$ lines

(n=4, Quadratic Stark)

NaI D_2 $\lambda 5890$ $\log K_{ij} = -15.17$

NaI D_1 $\lambda 5896$ $\log K_{ij} = -15.33$

(n=6, van der Waals)

$$K_{ij} = 3\times10^{-29}\left[\left(\chi_I - \chi_\ell - \frac{hc}{\lambda}\right)^{-2} - \left(\chi_I - \chi_\ell\right)^{-2}\right]$$

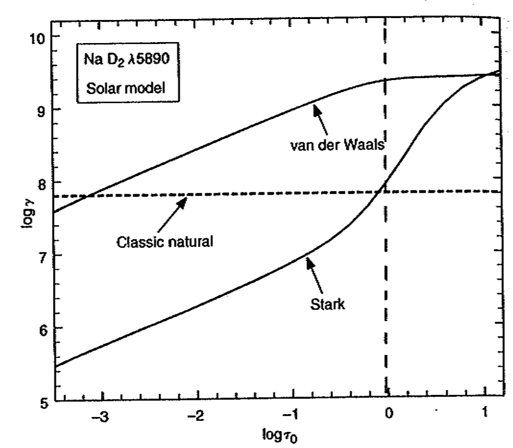
(from Unsöld 1955)

χ_I = ionization potential

χ_ℓ = excitation potential (lower state)

λ = transition wavelength.

Temp, Pressure $\longrightarrow$ increasing



$\approx 6.28\times10^{7}\ s^{-1}$

Fig. 11.4. Damping constants for the Na I D_2 line are shown as a function of depth in a solar model. The van der Waals damping constant is computed using Eq. (11.9). The Stark damping constant comes from Eq. (11.27). For omparison, the natural radiation damping according to Eq. (11.13) is shown.

- Figure of Solar Photosphere γ_4 and γ_6 for NaI D_2

$\log \tau_0 = 0$, $\tau_0 = 1$ is roughly depth where photons escape to space from the sun's photosphere

$\left\{ \begin{array}{l} \text{Van der Waals dominates NaI lines} \\ \text{Quadratic Stark negligible until } \tau_0 = 1 \end{array} \right\}$

Mihalas' 1 page overview (p281)

he uses "p" for our "n" and uses "Cp" for our "Kij"

9-3 Collision Broadening: Classical Impact Theory

THE WEISSKOPF APPROXIMATION

The simplest classical impact theory has its origins in an analysis by Lorentz, who considered the atom to be a radiating oscillator that suffers changes in phase during encounters with perturbing particles. It is assumed that the collisions occur between the radiating atom and a single perturber, one at a time. The collisions are assumed to occur essentially instantaneously, so that the wavetrain suffers an instantaneous phase dislocation that, in effect, terminates it. During the time between collisions the atom is assumed to be unperturbed. Thus suppose that the time between two successive collisions is T, and that in this interval the radiator emits a monochromatic wavetrain $f(t) = \exp(i\omega_0 t)$. The Fourier transform of this *finite* wavetrain is

(see notes below)

$$F(\omega, T) = \int_0^T e^{i(\omega_0 - \omega)t}\, dt = \frac{\exp[i(\omega - \omega_0)T] - 1}{i(\omega - \omega_0)} \tag{9-46}$$

The energy spectrum $E(\omega, T)$ of this wavetrain is given by substitution of $F(\omega, T)$ into equation (9-3).

In general there is not a *unique* time interval between collisions; rather, these intervals are distributed probabilistically about some mean value. If the collisions occur as a result of a random-walk process, and the *mean time between collisions* is τ, then the probability that the interval between two successive collisions lies on the range $(T, T + dT)$ is

$$W(T)\, dT = e^{-T/\tau}(dT/\tau) \tag{9-47}$$

Hence averaging over all collision times T, we obtain a mean energy spectrum

$$E(\omega) \equiv \langle E(\omega, T)\rangle_T = (2\pi)^{-1} \int_0^\infty F^*(\omega, T)F(\omega, T)W(T)\, dT \tag{9-48}$$

Computation of the integral, with normalization, yields

$$E(\omega) = \frac{(1/\pi\tau)}{(\omega - \omega_0)^2 + (1/\tau)^2} = \frac{(\Gamma/2\pi)}{(\omega - \omega_0)^2 + (\Gamma/2)^2} \tag{9-49}$$

Exercise 9-3: Derive equation (9-49).

The collision broadening theory described above again yields a Lorentz profile (a result of assuming that all the collisions are distinct), with a damping parameter $\Gamma = 2/\tau$. To complete the theory we must obtain an estimate of τ. As was done for the radiation-damped oscillator, we take the profile of an ensemble of randomly phased oscillators, continuously created, to be proportional to the energy spectrum of a single oscillator [averaged over all times, as given in equation (9-49)]. If *both* radiation and collision damping occur, with widths Γ_R and Γ_C respectively, and are assumed to be *completely uncorrelated*, then the profile is a convolution of the two Lorentz profiles. By an analysis similar to that leading to equation (9-24), one may readily show that the combined profile is again Lorentzian with a total width $\Gamma = \Gamma_R + \Gamma_C$. The effects of Doppler broadening can be taken into account as in §9-2, by using a Voigt profile with the appropriate total damping width.

We must now calculate the mean collision time τ. If the radiating atoms and perturbing particles have atomic weights A_r and A_p respectively, and both have a Maxwellian velocity distribution at a temperature T, then their average relative velocity is

$$v = \langle v^2\rangle^{\frac{1}{2}} = [(8kT/\pi m_H)(A_r^{-1} + A_p^{-1})]^{\frac{1}{2}} \tag{9-50}$$

Assuming that the effective impact parameter of the collisions responsible for the broadening is ρ_0, we then have

$$\tau^{-1} = \pi\rho_0^2 N v \tag{9-51}$$

and

$$\Gamma = 2\pi\rho_0^2 N v \tag{9-52}$$

where N is the perturber density. We must now determine ρ_0.

Following Weisskopf (661) we assume that (a) the perturber is a *classical particle*; (b) the perturber moves with constant velocity past the atom on a *straight-line path* with *impact parameter* ρ; (c) the *interaction* between atom and perturber is described approximately by

$$\Delta\omega = C_p/r^p \tag{9-53}$$

where $r(t) = (\rho^2 + v^2t^2)^{\frac{1}{2}}$, $t = 0$ occurring at the point of closest approach; and (d) *no transitions* in the atom are produced by the action of the perturber. The validity of these assumptions will be considered later. The form of the interaction in equation (9-53) is only approximate but holds over a fairly wide range of distances. The value of the exponent p depends upon the nature of the interaction. Values of astrophysical interest and the interaction they represent are as follows: $p = 2$, *linear Stark effect* (hydrogen + charged particle); $p = 3$, *resonance broadening* (atom A + atom A); $p = 4$, *quadratic Stark effect* (nonhydrogenic atom + charged particle); $p = 6$, *van der Waals force* (atom A + atom B). The interaction constant C_p must be calculated from quantum theory or measured by experiment.

The *phase shift* induced by the perturbation is

$$\eta(t) = \int_{-\infty}^{t} \Delta\omega(t')\, dt' = C_p \int_{-\infty}^{t} (\rho^2 + v^2t'^2)^{-p/2}\, dt' \tag{9-54}$$

The *total phase shift* $\eta(\rho) \equiv \eta(t = \infty)$ is found directly to be

$$\eta(\rho) = C_p\psi_p/v\rho^{p-1} \tag{9-55}$$

where

$$\psi_p = \pi^{\frac{1}{2}}\Gamma[\tfrac{1}{2}(p - 1)]/\Gamma(\tfrac{1}{2}p) \quad \text{our } I_n \tag{9-56}$$

Here Γ denotes the usual gamma function; for $p = (2, 3, 4, 6)$ one finds $\psi_p = (\pi, 2, \pi/2, 3\pi/8)$.

We now assume that only those collisions that produce a total phase shift greater than some critical value η_0 are effective in broadening the line. The effective impact parameter for such collisions is thus

$$\rho_0 = (C_p\psi_p/\eta_0 v)^{1/(p-1)} \tag{9-57}$$

and the corresponding value for the damping constant is

$$\Gamma = 2\pi N v (C_p\psi_p/\eta_0 v)^{2/(p-1)} \tag{9-58}$$

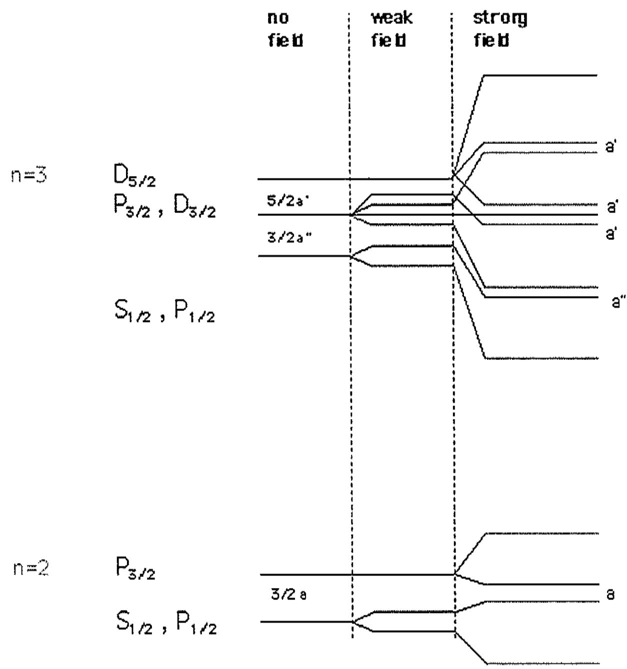
Weisskopf arbitrarily adopted $\eta_0 = 1$ as the critical phase shift; with this choice we obtain the *Weisskopf radius* ρ_W from equation (9-57) and the Weisskopf damping parameter Γ_W from equation (9-58).

If C_p is given, the theory described above yields a definite value for Γ, and the results are found to be of the right order of magnitude. Yet there remain serious defects in it. (a) The choice $\eta_0 = 1$ is arbitrary, and there is no means of determining a priori the correct value of η_0 to be used. (b) The theory does not account for the collisions that produce small phase shifts even though the number of such collisions increases as ρ^2. (c) The theory fails to predict the existence of a *line shift*; as will be shown below this failure arises from the omission of weak collisions, as mentioned in (b).

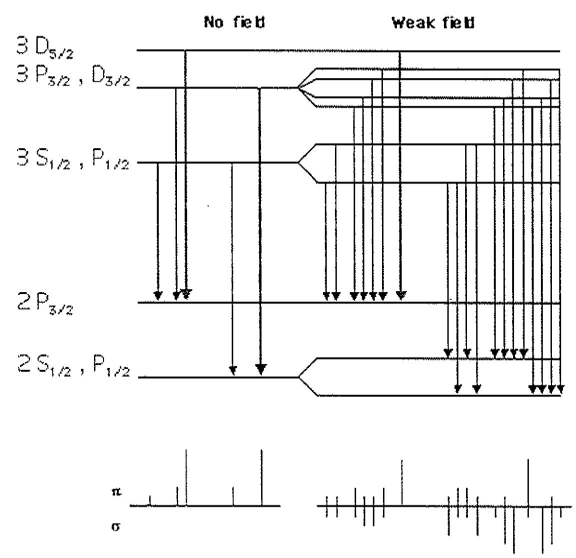
Given that Classical Impact Theory is the simplest view, more comphensive theories have been developed. See Mihalas for how deep this rabbit hole goes. (don't take the red pill!)

- NOTE that this theory invokes a classical view of the atom, which is treated like a "radiating oscillator".
- Eq. 9-46 and 9-48 are full complex notation; in our treatment we invoked the real part of these expressions.

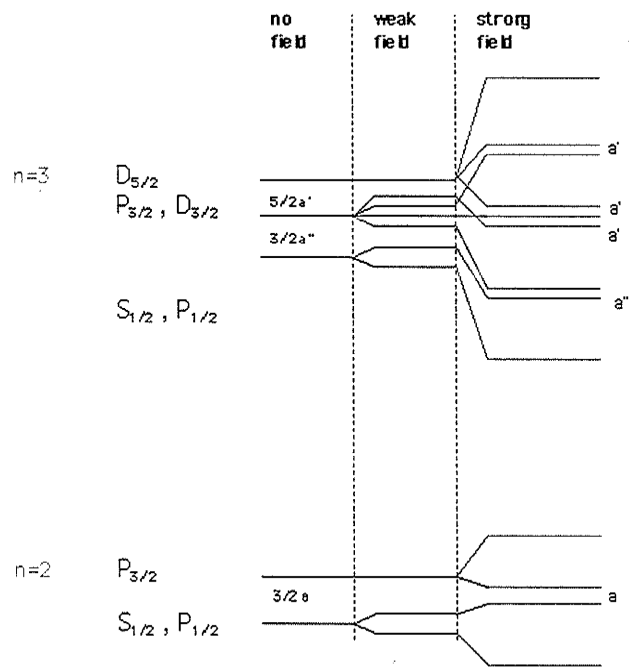
Stark effect of Hydrogen for n=3 and n=2



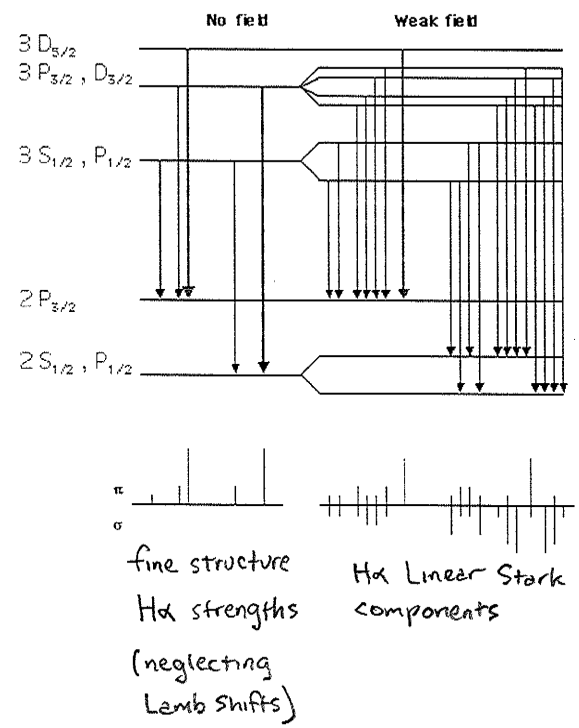
Fine structure and
Weak field Stark effect for Hydrogen Hα



Stark effect of Hydrogen for n=3 and n=2



Fine structure and
Weak field Stark effect for Hydrogen Hα



Summarizing. getting total cross section

$$\alpha(\lambda) = \alpha_n(\lambda) * f_p(\Delta\lambda) * f_{th}(\Delta\lambda) * f_t(\Delta\lambda)$$

1. $\alpha_n(\lambda)$ = natural broadening (Lorentzian)
2. $f_p(\Delta\lambda)$ = pressure broadening (Lorentzian)
3. $f_{th}(\Delta\lambda)$ = thermal broadening (Gaussian)
4. $f_t(\Delta\lambda)$ = turbulent broadening (Gaussian)

with

$$\int_0^\infty \alpha(\lambda)d\lambda = \frac{\pi e^2}{m_e c}\frac{\lambda^2}{c}f \qquad \text{or} \qquad \int_0^\infty \alpha(\nu)d\nu = \frac{\pi e^2}{m_e c}f$$

natural ①

$$\alpha_n(\lambda) = \frac{e^2}{m_e c^2}\lambda_r^2 f \frac{(\Gamma\lambda_r^2/4\pi c)}{(\lambda-\lambda_r)^2 + (\Gamma\lambda_r^2/4\pi c)^2}$$

f = oscillator strength
Γ = natural damping constant
λ_r = transition wavelength.

pressure ②

$$f_p^{(n)}(\Delta\lambda) = \frac{1}{\pi}\frac{(\gamma_n\lambda_r^2/4\pi c)}{(\lambda-\lambda_r)^2 + (\gamma_n\lambda_r^2/4\pi c)^2}$$

γ_n = damping constant for R^{-n} mechanism.

the convolution of a Lorentzian with a Lorentzian is a Lorentzian with damping constant equal to the sum of the individual damping constants.

$$\alpha_n(\lambda) * f_p^{(2)}(\Delta\lambda) * f_p^{(3)}(\Delta\lambda) * f_p^{(4)}(\Delta\lambda) * f_p^{(6)}(\Delta\lambda) = \frac{e^2}{m_e c^2}\lambda_r^2 f \frac{(\Gamma_t\lambda_r^2/4\pi c)}{(\lambda-\lambda_r)^2 + (\Gamma_t\lambda_r^2/4\pi c)}$$

$$\Gamma_t = \Gamma + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_6$$

recall that $\gamma_n \propto P_c T^{-\frac{n+1}{2(n-1)}}$ so you need to know $P_c = n_c kT$ and T
c = colliding perturbing particle.

Total Cross Section

thermal

$$③ \quad f_{th}(\Delta\lambda) = \frac{1}{\pi^{1/2}\Delta\lambda_{th}} \exp\left\{-\frac{(\lambda-\lambda_r)^2}{\Delta\lambda_{th}^2}\right\}$$

turbulent

$$④ \quad f_t(\Delta\lambda) = \frac{1}{\pi^{1/2}\Delta\lambda_t} \exp\left\{-\frac{(\lambda-\lambda_r)^2}{\Delta\lambda_{th}^2}\right\}$$

the convolution of Gaussians with Gaussians is a Gaussian with width given by the sum of the individual widths

$$f(\Delta\lambda) = f_{th}(\Delta\lambda) * f_t(\Delta\lambda) = \frac{1}{\pi^{1/2}\Delta\lambda_D} \exp\left\{-\frac{(\lambda-\lambda_r)^2}{\Delta\lambda_D^2}\right\}$$

$$\Delta\lambda_D^2 = \Delta\lambda_{th}^2 + \Delta\lambda_t^2$$

TOTAL $\alpha(\lambda)$

is the convolution of a Lorentzian w/ Γ_t and a Gaussian w/ $\Delta\lambda_D$

$\alpha(\lambda) = \frac{\pi e^2}{m_e c^2} \lambda_r^2 f \cdot U(a,b)$	total cross section
$U(a,b) = \frac{1}{\pi^{1/2}\Delta\lambda_D} H(a,b)$	Voigt function
$H(a,b) = \int_{-\infty}^{\infty} \frac{\exp(-t^2)}{(a-t)^2 + b^2}\, dt$	Hjerting function
$a = \frac{\lambda - \lambda_r}{\Delta\lambda_D} \qquad b = \frac{1}{\Delta\lambda_D} \frac{\Gamma_t \lambda_r^2}{4\pi c}$	(physics is in $\Delta\lambda_D$ and Γ_t)

all the physics lurks in $\Delta\lambda_D$ (thermal + turbulent broadening)

and Γ_t (natural + pressure broadening) returning to

$$I_\lambda = I_\lambda^\circ \exp\{-\tau_\lambda\} = I_\lambda^\circ \exp\{-N\alpha(\lambda)\}$$

we obtain the <u>absorption profile</u> as a function of unknown parameters

N = column density of absorber — if Hα line, the absorber is neutral hydrogen in $n=2$ excited state.

$\Delta\lambda_D$ = Doppler width
- thermal broadening: T
- turbulent broadening: v_t

$\Gamma_t = \Gamma + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_6$ — the total damping constant
- $\gamma_n \propto P_c T^{-\frac{n+1}{2(n-1)}}$

so, we fit the absorption profile, <u>observed</u> as $I_\lambda / I_\lambda^\circ$ to the <u>model</u> $\exp\{-N\alpha(\lambda)\}$ to obtain the free parameters.

$$T, v_t, P_c$$

the model can be simplified for a given line if the dominant or two most dominant form of pressure broadening are used (not all 4).

LEAST SQUARES FITTING: χ^2

let
$\lambda_i = \lambda$ in pixel i
$y_i = I_\lambda / I_\lambda^\circ$ in pixel i
σ_i = uncertainty in y_i

$\Big\}$ quantities measured in the data.

let $\bar{a}$ = the vector of parameters to be determined.

$a_1 = N \quad a_2 = T \quad a_3 = v_t \quad a_4 = P_c$

$n = 4$ # of free parameters

then the function/model is

$$f_i = f(\lambda_i; \bar{a}) = \exp\{-a_1\alpha(\lambda_i, a_2, a_3, a_4)\}$$

χ^2 is then minimized

$$\boxed{\chi^2 = \sum_{i=1}^{M}\left[\frac{y_i - f_i}{\sigma_i}\right]^2}$$

for a "good fit" should obtain

$$\chi_\nu^2 = \frac{\chi^2}{\nu} \approx 1 \quad (\text{reduced } \chi^2)$$

where $\nu = M - n$ is the degrees of freedom; M = # of functions (pixels)
n = # of free parameters

Is Rotational Broadening Important? YES!

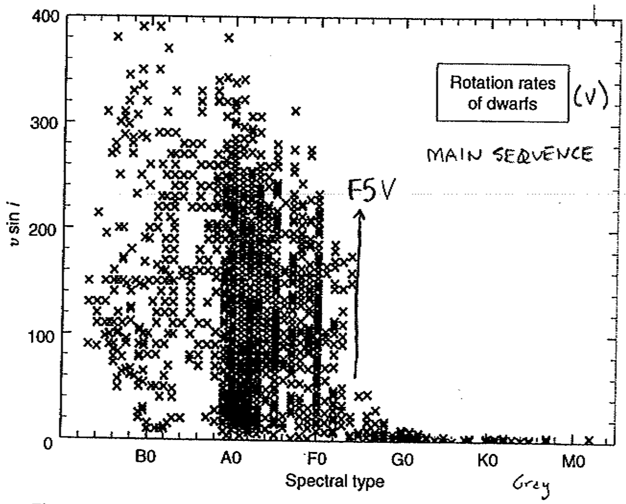


Fig. 18.21. Rapid rotation is normal for hot stars; slow rotation is seen for cool stars. The transition occurs near F5. Data from: Slettebak *et al.* (1975), Conti and Ebbets (1977), Soderblom (1982), Gray (1984b), Halbedel (1996), Penny (1996), Fekel (1997, 2003), and Royer *et al.* (2002b).

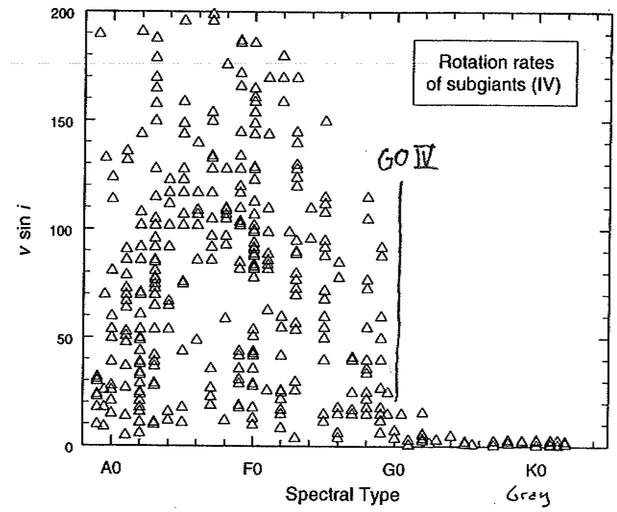


Fig. 18.23. Subgiants show a wide range of rotation rates for stars hotter than G0 IV, but slow rotation for cooler stars. Data from: Uesugi and Fukuda (1982), Gray and Nagar (1985), Fekel (1997, 2003), and Royer *et al.* (2002b).

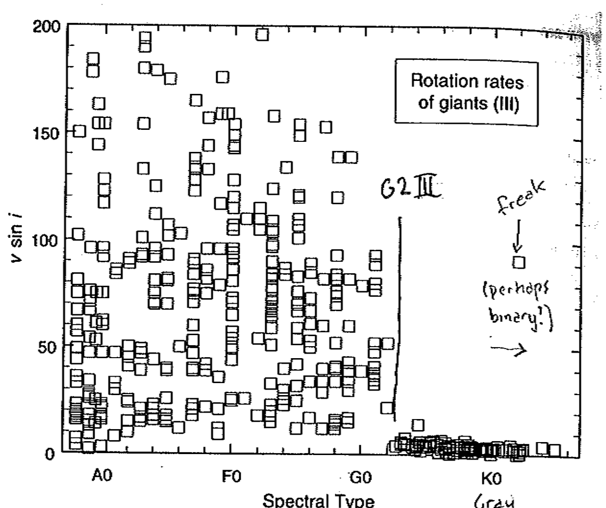


Fig. 18.24. Giants mimic the lower luminosity classes, but the drop in rotation occurs at G2 III. From G2 III to K2 III, rotation is a single-valued function of spectral type as given by Eq. (18.25). Data from: Alschuler (1975), Gray (1982c, 1989a), Hoffleit and Jaschek (1982), Royer *et al.* (2002b), and Fekel (2003).

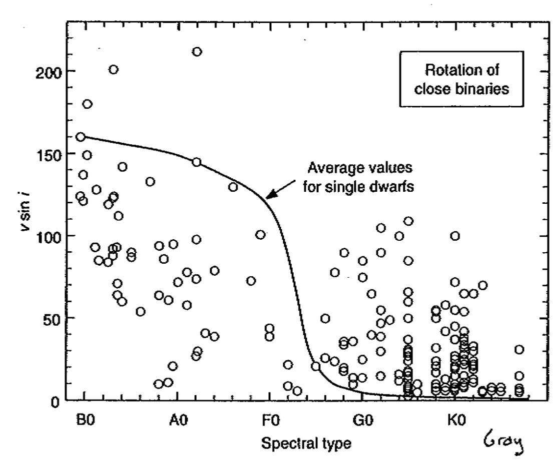


Fig. 18.27. Interacting binaries exchange orbital and rotational angular momentum through tidal coupling. Hot binaries are slowed down by this process and rotate more slowly than single stars, while cool binaries are sped up and rotate more rapidly than their single counterparts.

Binaries! { early types slowed
late types sped up

fast rotator } V = F5 main sequence
slow rotator } IV = G0 sub giants
transitions } III = G2 giants

- late type stars rotate slowly – exclusively
- early type stars rotate over a large range of velocities, up to 200–400 km/s

Limb Darkening:

→ for a fixed optical depth, say $\tau = 1$, we see a given/fixed physical depth into the star.

the physical depth, in the simple geometric case goes as $x = \tau \cos\theta$ where θ is the angle define in Fig 9.2b.

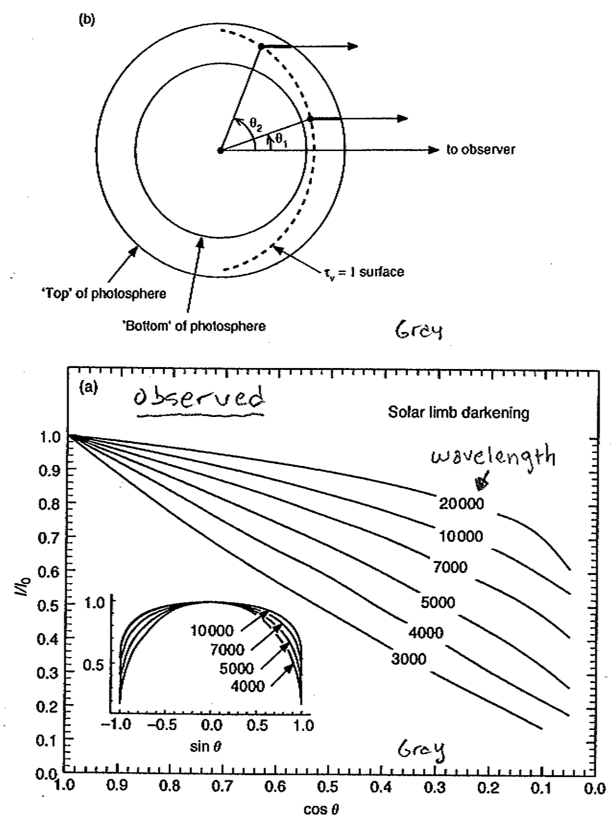


Fig. 9.2. (a) The limb darkening observations of the Sun are shown for several wavelengths. The darkening is nearly linear with cos θ. The small inset graph shows the limb darkening as a function of position across the disk, i.e., sinθ. Data from Pierce (2000). (b) A schematic illustration of the cause of limb darkening is shown. The top and bottom of the photosphere are indicated by the outer and inner circles. (The real solar photosphere is ~25 times thinner than the lines used to draw the circles!) Unit optical depth into the photosphere corresponds to different heights in the photosphere depending on θ. The dashed curve indicates the surface of unit optical depth from the top of the photosphere. Radiation seen at disk position sinθ_2 is characteristic of the higher cooler layers compared to the radiation seen at disk position sinθ_1.

The result is that the intensity decreases toward the limbs of the disk.

$$F_\lambda = F_\lambda^\circ (a + b\cos\theta) \quad \left\{ \text{Eddington-Barbier relation} \right.$$

we define $\mu = \cos\theta$

- this simple linear model predicts that the flux decreases linearly with $\cos\theta$ toward the limb

← Examples of Limb Darkening

- note that the Eddington-Barbier relation is a better approximation for bluer wavelengths

- note that a and b are wavelength dependent and are determined empirically.

Rotational Broadening

The observed intensity will be

$$I_\lambda = I_\lambda(0) * \Phi_\lambda(\Delta\lambda)$$

the intensity that corresponds to zero rotation convolved with the rotational broadening function $\Phi(\Delta\lambda)$, where

$$\int_{-\infty}^{\infty} \Phi_\lambda(\Delta\lambda)\, d(\Delta\lambda) = 1 \quad \leftarrow \text{ conserves Equivalent Width}$$

we obtain $\Phi(\Delta\lambda)$ from

$$\Phi_\lambda(\Delta\lambda)\, d(\Delta\lambda) = A \frac{I(p)}{I(0)}\, d(\Delta\lambda)$$

($\Delta\lambda$ is a function of p) - shown below

plane cut at x for Fig 3.4 defining r

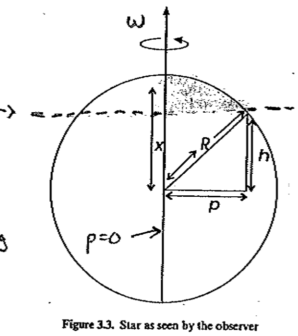


Figure 3.3. Star as seen by the observer

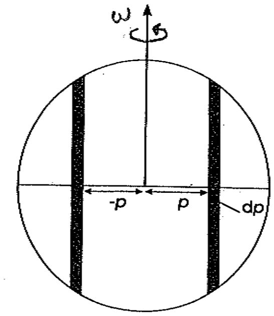


Figure 3.6.

Emerson

Figures 3.3 and 3.6

Face-on view of rotating star defining the coordinates p, x, and h. Star is assumed to be oriented with rotation axis perpendicular to line of sight to observer. i.e., inclination $i = 0°$.

note that $h = \sqrt{R^2 - p^2} = R\sqrt{1 - \frac{p^2}{R^2}}$

and x is the spatial variable in the h direction

$I(0)$ = intensity at $p = 0$ = $F_0\, dp\, 2R$

F_0 = flux of surface, which we assume is constant for this example.

$I(p)$ = intensity at $p \neq 0$ = $F_0\, dp\, 2h$

$$\therefore \quad \frac{I(p)}{I(0)} = \frac{h}{R} = \sqrt{1 - \frac{p^2}{R^2}}$$

- assume a solid rotator with angular velocity ω
  - at the equator $V(R) \equiv V_R = \omega R$
  - at arbitrary r $\quad v(r) = \omega r$

  see Figs 3.3, 3.4

$$\Big\} \text{ yielding } \frac{v(r)}{V_R} = \frac{r}{R} \rightarrow v(r) = V_R\left(\frac{r}{R}\right)$$

- but, we observe the line of sight velocity, not $v(r)$.

$r = R\sqrt{1 - \frac{x^2}{R^2}}$

from Fig 3.4, an arbitrary observed point "o" has

$$V_{los}(r) = V_{los}(p) = v(r)\cos(90 - \theta) = v(r)\sin\theta$$

where $\sin\theta = \frac{p}{r}$

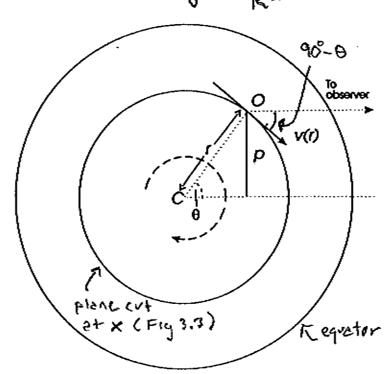


Figure 3.4. Star viewed along the rotation axis

Emerson

(looking down along rotation axis)

$$\therefore \quad V_{los}(p) = v(r)\cdot\left(\frac{p}{r}\right) = V_R\left(\frac{r}{R}\right)\left(\frac{p}{r}\right) = V_R\left(\frac{p}{R}\right)$$

Rotational Broadening (cont).

recall:

from $V_{los}(p) = V_R\left(\frac{p}{R}\right) \rightarrow \frac{p}{R} = \frac{V_{los}(p)}{V_R}$

$\left\{ \frac{p}{R} \text{ appears in } \frac{I(p)}{I(0)} = \sqrt{1 - \frac{p^2}{R^2}} \right\}$

- incorporating inclination of rotation axis.

we have so far assumed $i = 0$, if we allow for the star to be inclined wrt to the observer,

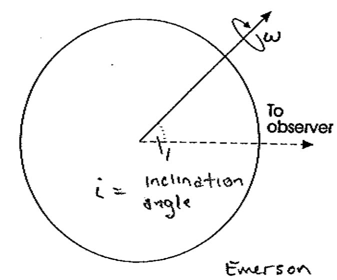


Figure 3.5.

side view of star, observer to the right. all velocities observed (radially) are reduced by factor $\sin i$

$$V_{los}(p) = V_R\left(\frac{p}{R}\right)\sin i$$

yielding $\frac{p}{R} = \frac{V_{los}(p)}{V_R \sin i}$

(recall that $V_{los}(p)$ is observed)

- returning to

$$\frac{I(p)}{I(0)} = \sqrt{1 - \frac{p^2}{R^2}} = \sqrt{1 - \left(\frac{V_{los}(p)}{V_R \sin i}\right)^2}$$

from Doppler relation, we have

$$\frac{\Delta\lambda}{\lambda} = \frac{V_{los}(p)}{c} \rightarrow V_{los}(p) = \left(\frac{c}{\lambda}\right)\Delta\lambda$$

$$\frac{\Delta\lambda_{max}}{\lambda} = \frac{V_R \sin i}{c} \rightarrow V_R \sin i = \left(\frac{c}{\lambda}\right)\Delta\lambda_{max}$$

yielding

$$\underset{\substack{\uparrow \\ p=0}}{\frac{I(p)}{I(0)}} = \underset{\substack{\uparrow \\ \Delta\lambda=0}}{\frac{I(\Delta\lambda)}{I(0)}} = \sqrt{1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2}$$

so that

$$\Phi_\lambda(\Delta\lambda)\, d(\Delta\lambda) = A\frac{I(p)}{I(0)}\, d(\Delta\lambda) = A\frac{I(\Delta\lambda)}{I(0)}\, d(\Delta\lambda) = A\sqrt{1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2}\, d(\Delta\lambda)$$

we need to integrate to obtain the normalization constant A

Rotational Broadening (cont).

normalizing,

$$\int_{-\infty}^{\infty} \Phi_\lambda(\Delta\lambda)\,d(\Delta\lambda) = 1 = A\,\Delta\lambda_{max} \int_{-\infty}^{\infty} \left[1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2\right]^{1/2} d\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)$$

$\left\{ \text{multiply and divide by } \Delta\lambda_{max} \right\}$

yields $A = \dfrac{2}{\pi\,\Delta\lambda_{max}^2}$

so that

$$\boxed{\Phi_\lambda(\Delta\lambda) = \frac{2}{\pi\,\Delta\lambda_{max}} \sqrt{1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2} = \frac{2}{\pi}\left(\frac{1}{\lambda}\right)\frac{c}{V_R \sin i}\left[1 - \left(\frac{c}{V_R \sin i}\right)^2\left(\frac{\Delta\lambda}{\lambda}\right)^2\right]^{1/2}}$$

recalling that $I_\lambda = I_\lambda(0) * \Phi(\Delta\lambda)$

the FWHM of $\Phi(\Delta\lambda)$ is $\sqrt{3}\,\Delta\lambda_{max} = \sqrt{3}\left(\frac{\lambda}{c}\right)V_R \sin i$

ASSUMPTIONS:
1. solid rotator
2. uniform disk intensity (both are ideal)
3. uniform line formation across disk

Relaxing Uniform Disk Intensity

- Limb Darkening
  - the temperature cools going from deeper layers to shallower layers
  - toward the limbs, observer doesn't see as deep so limb intensity is lower than central disk intensity.

limb darkening law $F_\nu = F_0(a + b\mu')$ $\quad F_0$ = disk center flux

where $\mu' = \cos\theta'$ (see Fig 3.7a)

geometrically $\cos\theta' = \dfrac{r}{R} = \dfrac{\sqrt{R^2 - (p^2 + x^2)}}{R} = \sqrt{1 - \dfrac{p^2 + x^2}{R^2}}$

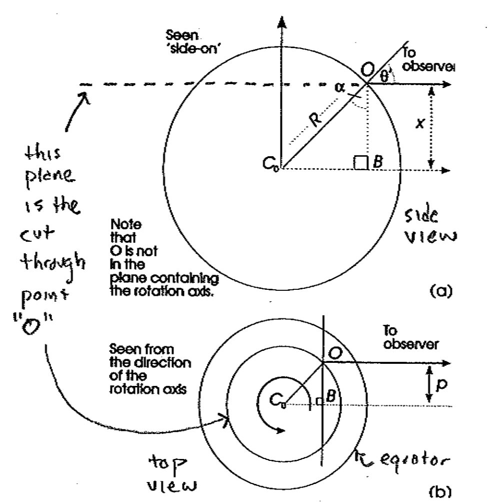


Figure 3.7.

Emerson

Rotational Broadening w/ Limb Darkening (cont)

- generalizing

$$I_\lambda(\Delta\lambda)\,d(\Delta\lambda) = F_0 I(p)\,dp \int_{-h(p)}^{h(p)} (a + b\mu')\,dx$$

recall $h(p) = R\left[1 - \frac{p^2}{R^2}\right]^{1/2}$

thus $h(0) = R$

$$I(\Delta\lambda)\,d(\Delta\lambda) = F_0 I(p)\,dp\left[2ah(p) + b\int_{-h(p)}^{h(p)}\left[1 - \frac{p^2+x^2}{R^2}\right]^{1/2} dx\right]$$

$$1 - \frac{p^2+x^2}{R^2} = 1 - \frac{p^2}{R^2} - \frac{x^2}{R^2} = h^2(p) - \frac{x^2}{R^2}$$

the integral can be written and solved via

$$bR\int_{-h(p)}^{h(p)}\left[h^2(p) - \frac{x^2}{R^2}\right]^{1/2}\frac{dx}{R} = 2bR\,h^2(p)\cdot\frac{\pi}{4}$$

yielding

$$I(\Delta\lambda)\,d(\Delta\lambda) = F_0 I(p)\,dp\left\{2aR\sqrt{1 - \frac{p^2}{R^2}} + 2bR\left[1 - \frac{p^2}{R^2}\right]\frac{\pi}{4}\right\}$$

recall

$$\Phi_\lambda(\Delta\lambda)\,d(\Delta\lambda) = A\frac{I(p)}{I(0)}\,d(\Delta\lambda)$$

we need $\frac{I(p)}{I(0)}$

thus

$$\frac{I(p)}{I(0)} = \frac{2aR\sqrt{1 - p^2/R^2} + 2bR\left[1 - p^2/R^2\right]\frac{\pi}{4}}{2aR + 2bR\left(\frac{\pi}{4}\right)}$$

$p=0 \longrightarrow I(0)$

$$\frac{I(p)}{I(0)} = \frac{\sqrt{1 - p^2/R^2} + \frac{\pi}{4}\frac{b}{a}\left[1 - p^2/R^2\right]}{1 + \frac{\pi}{4}\frac{b}{a}}$$

normalizing to get $\Phi_\lambda(\Delta\lambda)$ while using $p^2/R^2 = \frac{\Delta\lambda}{\Delta\lambda_{max}}$ $\quad \Delta\lambda_{max} = \lambda\frac{v_R \sin i}{c}$

$$\int_{-\infty}^{\infty}\Phi_\lambda(\Delta\lambda)\,d(\Delta\lambda) = 1 = A\,\Delta\lambda_{max}\int_{-\infty}^{\infty}\frac{\sqrt{1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2} + \frac{\pi}{4}\frac{b}{a}\left[1 - \left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2\right]}{1 + \frac{\pi}{4}\frac{b}{a}}\,d\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)$$

Rotational Broadening (cont)

yields

$$\boxed{\Phi_\lambda(\Delta\lambda) = \frac{1}{\pi\,\Delta\lambda_{max}}\left[\frac{2\sqrt{1-\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2} + \frac{\pi}{2}\frac{b}{a}\left[1-\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2\right]}{1+\frac{2}{3}\frac{b}{a}}\right]}$$ (Eq. 18.14)

recall that a and b are fixed for a given λ

recall that $\boxed{\Delta\lambda_{max} = \lambda\,\frac{V_R \sin i}{c}}$

→ note that for $b=0$ (no limb darkening) the previous expression is returned

NOTE: since Limb Darkening is more complex than the Eddington-Barbier relation, these functions are not precisly correct.

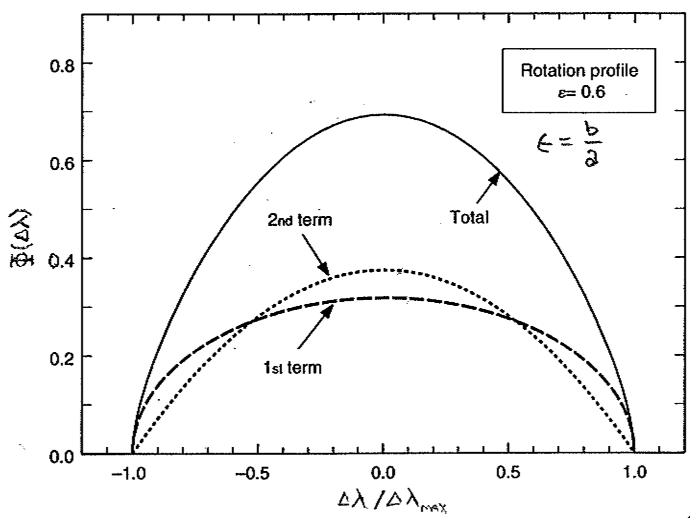


Fig. 18.5. The rotation profile, Eq. (18.14), is shown by the solid line for ε = 0.6 and labeled "total." It is the sum of the "1st term" and "2nd term" curves. $= \frac{b}{a}$

Gray (labels modified)

$\boxed{\text{* Gray cells } \Phi(\Delta\lambda) \rightarrow G(\Delta)}$

also,

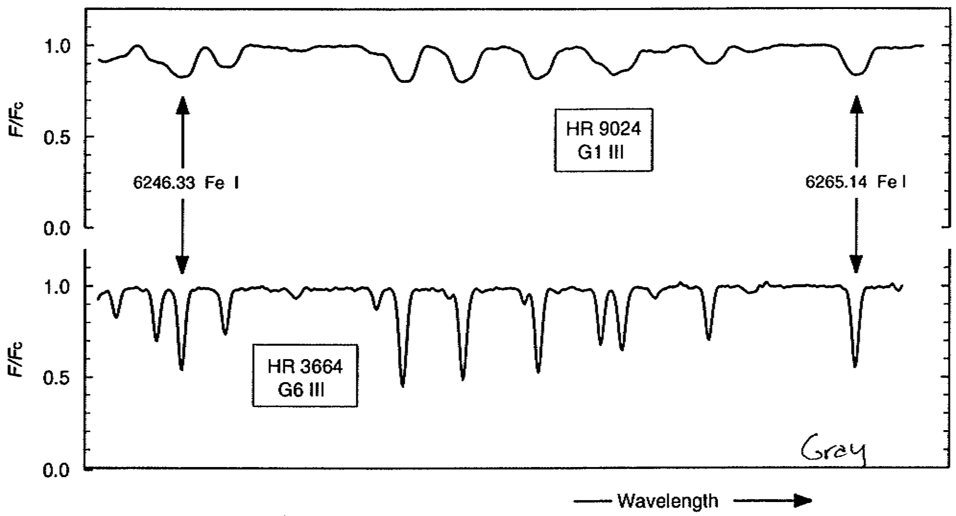
$$\frac{\Delta\lambda}{\Delta\lambda_{max}} = \frac{V_{los}}{V_R \sin i}$$

so x-axis of Fig 18.5 can be viewed as either

$$\text{1st Term} = \frac{2}{\pi\left(1+\frac{2}{3}\frac{b}{a}\right)\Delta\lambda_{max}}\sqrt{1-\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)}$$ weighted uniform flux

$$\text{2nd Term} = \frac{b}{a}\,\frac{1}{2\Delta\lambda_{max}}\cdot\frac{1}{\left(1+\frac{2}{3}\frac{b}{a}\right)}\cdot\left[1-\left(\frac{\Delta\lambda}{\Delta\lambda_{max}}\right)^2\right]$$ weighted correction for limb darkening

- since $\Phi_\lambda(\Delta\lambda)$ is normalized to unit area, equivalent width are <u>not</u> modified!



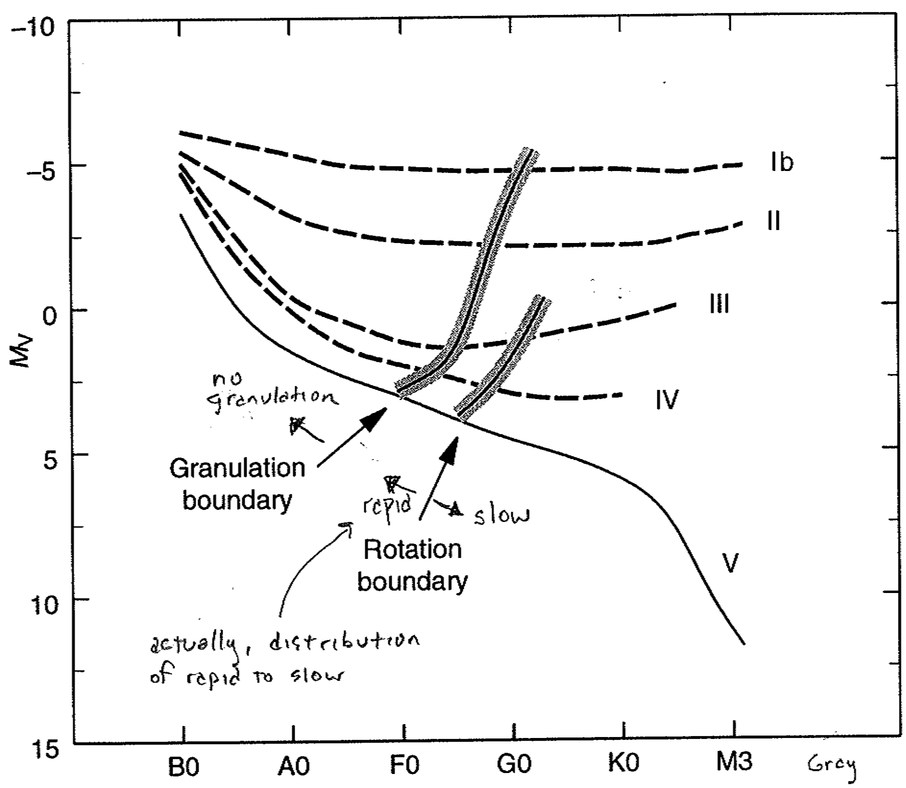
rapid rotation

slow rotation

Grey

Fig. 18.7. These two G giants illustrate the Doppler broadening of the line profiles by rotation. HR 3664 shows low rotation, comparable to the macro-turbulence broadening, of a few km/s, while HR 9024 shows rotational broadening that is substantially larger. Data taken at the Elginfield Observatory.

- Regions on HR diagram of slow/fast rotators and granulation on surface (convective envelope)



no granulation

rapid

slow

actually, distribution of rapid to slow

Grey

Fig. 17.18. The granulation boundary is the locus of vertical bisectors. The bisectors of the solar type, the "(" shape, are located to the right of the granulation boundary. The bisectors showing the reversed shape, ")", are found to left of the boundary. The rotation boundary will be discussed in the next chapter, but generally stars on the hot side rotate rapidly, while those on the cool side rotate slowly.

Effects of Instrument

- the end result of a spectrum passing through a spectrograph is the spectral "purity" is degraded. Any wavelength λ is "blurred" by an amount $\Delta\lambda$,

Resolution $R = \frac{\lambda}{\Delta\lambda}$ where $\Delta\lambda$ is the FWHM of the Istrumental Spread Function

$$\Phi_{ISF}(\Delta\lambda) = \frac{1}{\sqrt{2\pi}\,\sigma_{ISF}} \exp\left\{-\frac{(\lambda'-\lambda)^2}{2\sigma_{ISF}^2}\right\} \qquad \Delta\lambda_{ISF} = 2.35\,\sigma_{ISF} \qquad \Delta\lambda = \lambda' - \lambda \qquad \lambda = \text{incoming wavelength}$$

The ISF is usually modeled as a Gaussian, but it can also be an irregular function with broader/narrower or asymmetric!

ISFs of five different R values are shown in Fig 6.10b.

Convolving

$$\boxed{I_\lambda = \Phi_{ISF}(\Delta\lambda) * I^0_\lambda \exp\{-\tau_\lambda\}}$$

yields the profile captured by the dectector.

NOTE: equivalent width is unchanged by spectral resolution

$\exp\{-\tau_\lambda\}$ $\quad$ $\Phi_{ISF}(\Delta\lambda)$ $\quad$ $\Phi_{ISF}(\Delta\lambda) * \exp\{-\tau_\lambda\}$

the spectrograph ISF blurs the intrisic profile via convolution of the ISF...

intrinsic line profile coming into the spectrograph

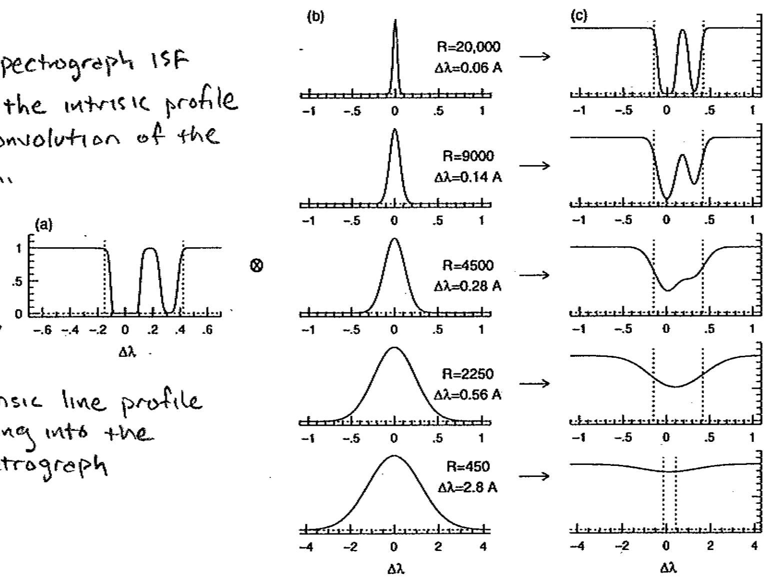


Figure 6.10: Illustrations of the reduction in spectral purity via convolution of the ISF for five somewhat typical spectroscopic resolutions, $R = 450$, 2250, 4500, 9000, and 20,000 (increasing from the lower to the upper panels). (*a*) The observed flux illustrating the intrinsic Lyα profile entering the spectrograph in relative wavelength units, centered at 1215 Å. (*b*) The Gaussian ISF profiles for the five resolutions, with $\Delta\lambda$ (FWHM) labeled for each. Note the different scale of each panel. (*c*) The apparent flux profiles for each resolution over the same relative wavelength scales. Vertical lines at $\Delta\lambda = -0.15$ and 0.42 provide wavelength range of the intrinsic profile. (Churchill)

so, in our χ^2 model, the function $f_i = f(\lambda_i; \bar{a})$ must be convolved with the ISF.

$$\boxed{\chi^2 = \sum_{i=1}^{M} \left[\frac{y_i - \{\Phi_{ISF}(\Delta\lambda) * f_i\}}{\sigma_i}\right]^2}$$

otherwise the parameters, $\bar{a}$, will not be correct!

This makes life more complicated. You have to have an accurate ISF.