Unit 1

Energy Generation

1.1 Basics

1.1.1 Energy equilibrium

- Consider a thin layer r, dr in the nuclear burning region of a star. For energy equilibrium, the net flow of energy crossing this regions boundaries should be equal to the energy generation in the region. Let ε be the energy generated per gram of material per second. Then $\rho\,\varepsilon$ is the energy produced per second in each cm^3 of material. The energy generated per second in (r, dr) is

$$\varepsilon \mathrm{d}m = (4\pi r^2 \mathrm{d}r)\rho\varepsilon. \tag{1.1}$$

- This standard relation will appear many times:

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V}, \tag{1.2}$$

or,

$$\mathrm{d}m = 4\pi r^2 \rho\, \mathrm{d}r \tag{1.3}$$

- Consider the luminosity (energy per second) L that carries energy away. In the layer, there should be a balance between energy gains and energy losses (plus and minus denotes outgoing from center or toward center, respectively):

$$\begin{aligned} \varepsilon \mathrm{d}m \quad &+ \quad L_r^+ + L_{r+\mathrm{d}r}^- = L_r^- + L_{r+\mathrm{d}r}^+ && (1.4)\\ \varepsilon \mathrm{d}m \quad &= \quad [L_{r+\mathrm{d}r}^+ - L_{r+\mathrm{d}r}^-] - [L_r^+ - L_r^-] = L_{r+\mathrm{d}r} - L_r && (1.5)\\ \varepsilon \mathrm{d}m/\mathrm{d}r \quad &= \quad \mathrm{d}L/\mathrm{d}r, && (1.6) \end{aligned}$$

by dividing by dr and letting it go to zero (derivative).

- Then finally

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho\, \varepsilon. \tag{1.7}$$

- Sometimes we will use m instead of r as the independent coordinate, and so

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon. \tag{1.8}$$

- Note that in regions where $\varepsilon = 0$, the luminosity is **constant.**

PROBLEM 1.1: [5 pts]: Verify this last statement using `MESA`, by showing one appropriate plot from a model.

PROBLEM 1.2: [10 pts]: What is the average energy generation rate in the Sun, *per unit mass*, ε? Compare this with the average rate of energy production per unit mass in a typical car engine. Also compare to the average energy consumption per unit mass of a typical human being. Do each one in $\mathrm{erg\,s^{-1}\,g^{-1}}$.

1.1.2 Nuclear interactions

- So what causes ε to be finite? For stars, it is the *fusion* of 2 nuclei. In order for charged nuclei to fuse, they have to get close together. This is tricky because of the Coulomb barrier

$$E_\mathrm{C} = \frac{Z_1 Z_2 e^2}{r_0}, \tag{1.9}$$

  where Z_i is the number of protons in the ith nucleus.

- The nucleus has a typical radius of $r_0 \sim 10^{-13}$ cm. The elementary charge $e = 4.8\times10^{-10}$ statcoulomb, where a statcoulomb (statC) is $1\,\mathrm{erg}^{1/2}\,\mathrm{cm}^{1/2}$. Thus

$$E_\mathrm{C} = Z_1 Z_2 \frac{(4.8\times10^{-10})^2}{10^{-13}}[\mathrm{erg}] \approx Z_1 Z_2 \cdot 2\times10^{-6}[\mathrm{erg}] \approx Z_1 Z_2\ [\mathrm{MeV}]. \tag{1.10}$$

  So for 2 protons ($Z = 1$) the Coulomb barrier is about 1 MeV. 1 erg is about 6.24×10^{11} eV.

- In a star the average kinetic energy per particle (as we'll see) is

$$\langle E_\mathrm{kin}\rangle = \frac{3}{2}k_\mathrm{B}T = \frac{3}{2}1.38\times10^{-16}\,T\ [\mathrm{erg}] \approx 1\times10^{-10}\,T\ [\mathrm{MeV}]. \tag{1.11}$$

  For a temperature of 10 million degrees K, this energy is 2×10^{-9} erg, or about 1 keV. This is 3 orders of magnitude *less* than the amount of energy needed to overcome the Coulomb barrier. That's not looking good.

- What about statistically speaking? The energies of the nuclei are distributed by a Maxwell distribution (in the classical sense): $N(E) \sim \exp(-E/kT)$. For energies 1000 times larger than average energies, their number decreases as $N(E) \sim \exp(-1000)$. There are only about 10^{57} particles in the Sun - the chance of any of having such a large energy is nil.

- According to quantum mechanics, there is a finite probability that a particle can *tunnel* through the Coulomb potential barrier and interact with the other particle. At short distances the nuclear force dominates and is attractive. One solves the Schrödinger Equation. This probability is small, but finite.

1.1.3 Nuclear reaction rates

- Here, the goal is to derive a general expression for reaction rates between species.

- Let's assume incoming particles can get through the Coulomb barrier at some probability. Understanding the following will help you appreciate stellar modeling codes later on.

- Consider a target particle A and incoming particles a interacting in a box (in the classical sense). Number densities n_a and n_A. Flux of incoming particles is $n_a v$, where v is the relative velocity.

- The number of reactions in the box in time $\mathrm{d}t$ are

$$N = \sigma v n_a n_A \mathrm{d}V \mathrm{d}t, \tag{1.12}$$

where σ is the cross section.

- The cross section is the number of reactions per unit time per target A, divided by the incident flux of particles a. It has units of cm^2.

- Then the *reaction rate* per unit volume is

$$r_{aA} = \sigma v n_a n_A \tag{1.13}$$

- 3 things to worry about. (1) not all incoming particles have the same energy (velocity); (2) the incoming particles have to actually hit the targets, and the cross section can depend very strongly on energy ($\sigma(E)$); (3) an interaction must take place, so the incoming particles have to get near to the targets (get through the barrier). Let's go through these 3 things.

1. Distribution of energies
   - The v are the relative velocities between incoming and target particles. Assume $f(v)\mathrm{d}v$ denotes the fraction of pairs of particles with speeds between v and $v + \mathrm{d}v$. We should then write

$$r_{aA} = \langle \sigma v \rangle n_a n_A, \tag{1.14}$$

   where

$$\langle \sigma v \rangle = \int_0^\infty \sigma\, v f(v) \mathrm{d}v. \tag{1.15}$$

   - Let's assume that the distribution is Maxwellian. In velocity space then

$$f(v)\mathrm{d}v = 4\pi \left(\frac{m}{2\pi k_\mathrm{B} T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_\mathrm{B}T} \right) v^2 \mathrm{d}v \tag{1.16}$$

   - Detour: What is m in this equation? Note: Z is the number of protons (atomic number); N is the number of neutrons; A is number of nucleons (protons+neutrons, Z+N, atomic weight). So, for example, the total mass of all incoming particles a is

$$m_a = \sum_i m_{a,i} = Z_a m_\mathrm{p} + N_a m_\mathrm{n} = m_\mathrm{u}(Z_a + (A_a - Z_a)) = A_a m_\mathrm{u}, \tag{1.17}$$

   where m_u is the unified atomic mass unit. Anyway, for particles in relative motion, their kinetic energy is

$$E = \frac{1}{2} \frac{m_a m_A}{m_a + m_A} v^2 = \frac{1}{2} A m_\mathrm{u} v^2, \tag{1.18}$$

   where that mass combination is the *reduced* mass and A is the reduced atomic weight

$$A = \frac{A_a A_A}{A_a + A_A}. \tag{1.19}$$

   The m in Equation (1.16) is therefore $m = A m_\mathrm{u}$.
   - Of course it's more convenient to express Equation (1.16) in terms of energy

$$f(v)\mathrm{d}v = \phi(E)\mathrm{d}E = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(k_\mathrm{B}T)^{3/2}} \exp\left(-\frac{E}{k_\mathrm{B}T} \right) \mathrm{d}E. \tag{1.20}$$

   We'll come back to this.

2. So now we have to deal a bit with σ. We first can just consider the geometrical cross section, whose extent is the de Broglie wavelength ($\lambda p = h$)

$$\sigma(E) \propto \pi\lambda^2 \propto p^{-2} \propto E^{-1}. \tag{1.21}$$

3. Finally, the cross section must be a function of the actual probability that the incoming particles indeed tunnel through the Coulomb barrier. Gamow showed that this probability is exponential and proportional to the ratio of Coulomb strength to energy

$$\sigma(E) \propto \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right). \tag{1.22}$$

   Notice that only light nuclei will be able to interact at relatively low temperatures (low v).

- So putting together these last 2 items we can write an expression for the cross section

$$\sigma(E) \equiv \frac{S(E)}{E} \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right). \tag{1.23}$$

  where $S(E)$ is the cross-section factor, describing the energy dependence of the reaction once the nuclei have penetrated the barrier. Hopefully it varies with energy far less than the rest (which is only true for non-resonant reactions; resonant ones are special cases).

- Using the velocity implied from Equation (1.18) as re-expressed in Equation (1.20), we can show that

$$\sigma(E) \equiv \frac{S(E)}{E} \exp(-bE^{-1/2}), \tag{1.24}$$

  where

$$b = 31.291\, Z_1 Z_2\, A^{1/2} \, \left[\mathrm{keV}^{1/2}\right]. \tag{1.25}$$

- Now Equation (1.15) becomes

$$\langle\sigma v\rangle = \left(\frac{8}{m\pi}\right)^{1/2} (k_\mathrm{B}T)^{-3/2} \int_0^\infty S(E)\mathrm{e}^{-bE^{-1/2}}\mathrm{e}^{-E/k_\mathrm{B}T}\,\mathrm{d}E. \tag{1.26}$$

- Note the two competing effects here: the first exponential increases rapidly with energy, since higher energy nuclei have an easier time to tunnel and this increases the cross section. The second exponential decreases rapidly with energy because of the small probability of there being high energy nuclei. This gives a strongly peaked integrand called the Gamow peak (see sketch).

- The maximum in the curve in energy, known as the "Gamow peak," is then

$$E_0 = \left(\frac{bk_\mathrm{B}T}{2}\right)^{2/3} = 1.22042\,\left(Z_1^2 Z_2^2 A T_6^2\right)^{1/3}\ [\mathrm{keV}]. \tag{1.27}$$

PROBLEM 1.3: [10 pts]: First, derive the first equality for E_0 in Equation (1.27). Then show that the constant b in Equation (1.25) is correct. Finally, show that the second equality in Equation (1.27) is correct.

PROBLEM 1.4: [5 pts]: Is the energy at the peak of the Gamow curve still consistent with typical nuclei energies in stellar cores? Compute Equation (1.27) for protons at 20 million K, and compare it to the energy of particles in Equation (1.11), both in keV. What does your comparison qualitatively say about the energies of the particles that will participate in reactions with appreciable cross sections?