October 6 ....

3.2 Conduction

- Degenerate electrons (in white dwarfs or supergiant cores) are the primary carrier for energy transport in such objects.
- The process is again diffusion of the Fick's Law type

$$F_{\rm cond} = -D_{\rm e}\frac{{\rm d}T}{{\rm d}r}. \tag{3.36}$$

- The diffusion coefficent $D_{\rm e}$ can again be expressed by an "opacity" of sorts, $\kappa_{\rm cond}$, and put into a form similar to Equation (3.6)

$$F_{\rm cond} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\rm cond}\rho}\frac{{\rm d}T}{{\rm d}r}. \tag{3.37}$$

- Assume we can compute $\kappa_{\rm cond}$. Then the total energy flux in the star (so far) would be

$$F_{\rm tot} = F_{\rm rad} + F_{\rm cond} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\rm tot}\rho}\frac{{\rm d}T}{{\rm d}r}, \tag{3.38}$$

  where

$$\frac{1}{\kappa_{\rm tot}} = \frac{1}{\kappa_{\rm R}} + \frac{1}{\kappa_{\rm cond}}. \tag{3.39}$$

- Note again how opacities are not simply "additive."
- Realize that whatever opacity is smaller is the one that contributes most to the total opacity and thus determines the energy flux (or lack thereof).
- For example, in typical non-degenerate stellar matter, $\kappa_{\rm cond}$ is large (so conduction is negligible), and radiative opacities dominate. Think of it as the one with biggest "channel" that lets the heat through.
- From solid-state physics, one can show that

$$\kappa_{\rm cond} \approx 4\times 10^{-8}\frac{\mu_e^2}{\mu_{\rm I}}Z_c^2\left(\frac{T}{\rho}\right)^2. \tag{3.40}$$

- In a white dwarf, one may encounter $\rho \approx 10^6\,{\rm g\,cm^{-3}}$ and $T \approx 10^7$K, made of carbon.
- The radiative opacity in this environment $\kappa_{\rm R} \approx 0.2\,{\rm cm^2\,g^{-1}}$ (Equation (3.21)). With $\mu_e = 2$, $\mu_{\rm I} = 12$, and $Z_c = 6$, we find $\kappa_{\rm cond} \approx 5\times 10^{-5}\,{\rm cm^2\,g^{-1}}$.
- Thus the total opacity is dominated conduction, and the flux is carried out by conduction.

3.3 Convection

Another important carrier of energy from the stellar interior outward is convection.

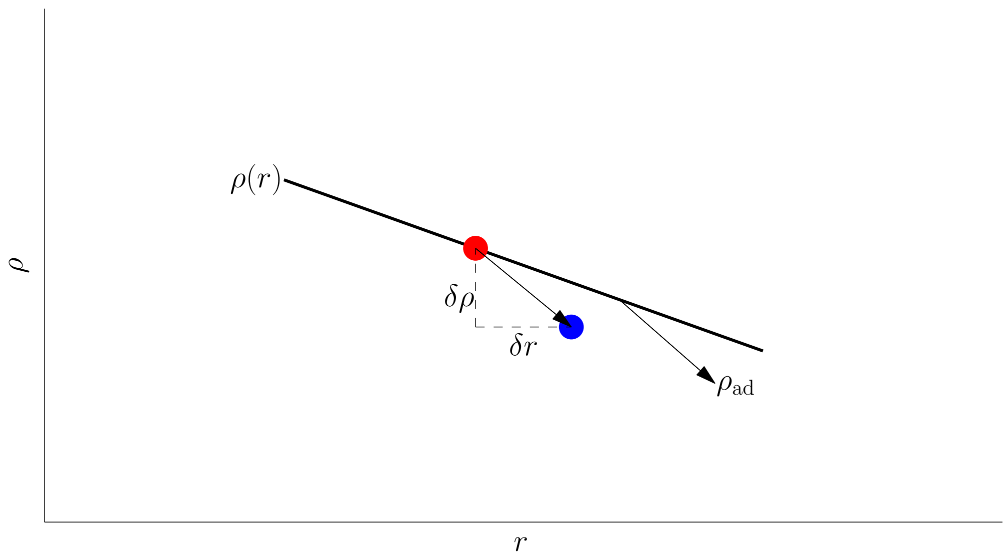

Figure 3.4: Convective instability. The curve $\rho(r)$ denotes the density gradient in some small region of a stellar interior. The arrow is the direction of an adiabat for this material. Take a parcel (red dot) in equilibrium with density ρ, and displace it upwards ($\delta r > 0$) adiabatically. It ends up where the blue dot is. This parcel now has a lower density than the surroundings ($\delta\rho < 0$), and so will continue to rise toward the surface until the conditions change (if they change). The density does not decrease sufficiently fast enough to be stable to convection.

3.3.1 The convective instability

- Consider in what follows an ideal gas.
- Assume a blob of gas of density ρ and pressure P at point r. It is in equilibrium with its surroundings also then of density ρ and pressure P.
- Let's displace the blob, or perturb it vertically into the medium (at $r + \delta r$) which now has density ρ' and pressure P', which we know are less than the unprimed quantities. What happens to the blob?
- Let ρ^* be the density of the blob. If $\rho^* < \rho'$ then the blob will be buoyant and continue rising: this is unstable. If $\rho^* > \rho'$ then the blob will return to its original position and there is no instablility. So how do ρ^* and ρ' compare?
- Two physically-motivated assumptions: (1) The pressure imbalances are quickly removed by acoustic waves (on the dynamical time scale), so that the pressure of the blob is also P'. (2) Heat is exchanged on the thermal timescale, which is long, so this is an adiabatic displacement.
- We know for an adiabatic displacement that $P/\rho^\gamma = \text{const}$ (Equation (2.77)). Comparing at bottom and top we can show

$$\rho^* = \rho \left(\frac{P'}{P}\right)^{1/\gamma}. \tag{3.41}$$

- Let's expand the pressure and density about point r to first order:

$$P' = P(r + \delta r) = P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \ldots \tag{3.42}$$

$$\rho' = \rho(r + \delta r) = \rho(r) + \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r + \ldots \tag{3.43}$$

- Substitute Equations (3.42)-(3.43) into (3.41) and expand (binomial):

$$\rho^* = \rho + \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r. \tag{3.44}$$

- For an instability to occur, $\rho^* - \rho' < 0$, or

$$\rho^* - \rho' = \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r < 0. \tag{3.45}$$

- So, an instability occurs if

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} < \frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{3.46}$$

  where we introduced the adiabatic gradient

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} = \frac{1}{\Gamma}\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{3.47}$$

  where we've denoted $\gamma = \Gamma$ in the adiabatic case.

- This can be interpreted as the density gradient resulting from adiabatic motion in the given pressure gradient.

- Since the gradient of pressure is always negative (hydrostatic equilibrium), instability occurs when the density does not decrease sufficiently rapidly compared to the adiabatic case.

- See Figure 3.4 for a schematic of this.

- Note that it is convention to express Equation (3.46) as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} < \frac{1}{\Gamma_1}. \tag{3.48}$$

  (Note since we've divided by a negative number, $\mathrm{d}\ln P/\mathrm{d}r$, the inequality changes). For a fully ionized ideal gas, the RHS is $3/5$.

- Let's now consider the force per unit volume acting on the displaced blob. That force (buoyancy and gravitational) is $F = -(\rho^* - \rho')g$, since g acts downwards.

IN CLASS WORK

Use this force in Newton's second law and derive a simple equation of motion for the displacement δr. Show that a characterstic frequency N comes out

$$N^2 = g\left(\frac{1}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = \frac{g}{\rho*}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right], \tag{3.49}$$

called the Brunt-Väisälä frequency. Examine the solutions of the equation of motion based on the possible values of N in the stable or unstable condition.

Answer:
From Newton's second law

$$\rho^*\frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} = -(\rho^* - \rho')g.$$

If we plug in Equation (3.45) we get the equation of motion

$$\frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} + N^2\delta r = 0,$$

where N is the Brunt-Väisälä frequency given above (Equation (3.3.1)).
A general solution to this equation is $\delta r \propto \mathrm{e}^{\mathrm{i}Nt}$.
If the medium is stable to convection, we know that $N^2 > 0$. When this is the case, the solution is thus sinusoidal and the blob δr oscillates about a given point (gravity/buoyancy waves).
In the other case, $N^2 < 0$ and so N is imaginary: $N \to \mathrm{i}N$. The the solution goes as $\delta r \propto \mathrm{e}^{-Nt} + \mathrm{e}^{Nt}$. This solution describes an exponentially growing parcel, in other words, a convective instability.

- It is convenient to look at the stability criterion in terms of temperature gradients instead.
- Using Eq. (3.43) and the ideal gas law $P = \rho RT/\mu$ (ignore gradients in mean molecular weights FOR NOW) we can show that

$$\rho' = \rho + \frac{\rho}{p}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r. \tag{3.50}$$

- For instability, again, we then require that $\delta\rho < 0$, or

$$\rho^* - \rho' = \left(\frac{1}{\gamma} - 1\right)\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r < 0. \tag{3.51}$$

- Now the instability condition becomes

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} > \frac{\mathrm{d}T}{\mathrm{d}r}, \tag{3.52}$$

  where the adiabatic temperature gradient is given by

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{3.53}$$

- This says that if the temperature gradient decreases too steeply out through the star there will be convection.
- To simplify in analogy with what we did before (Eq. 3.48), it is convention to write the inequality as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{3.54}$$

- We reintroduce
$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}, \qquad \nabla_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2},$$
so that the instability condition is
$$\nabla > \nabla_{\mathrm{ad}}. \tag{3.55}$$
Again, the RHS is 2/5 in the ionized ideal case.
- This is known as the **Schwarzschild criterion**. The inequality is also referred to as superadiabaticity.