September 22 . . . .

2.5 The Virial Theorem

- The Sun is in hydrostatic equilibrium. It is not necessarily in thermal equilibrium, but let's see.
- First a quick look at numbers. Consider the **thermal** energy (internal energy density) of an ideal monatomic gas
$$u = \frac{3}{2}\frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}}. \tag{2.113}$$
- Integrated over the star $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ and looking at the specific internal energy $\mathrm{d}U = u\mathrm{d}V$, we find
$$U = \int_0^R \left(\frac{3}{2}\frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}} T\right)\rho 4\pi r^2 \mathrm{d}r = \int_0^M \left(\frac{3}{2}\frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}} T\right)\mathrm{d}m = \frac{3}{2}\frac{k_{\mathrm{B}}}{\overline{\mu} m_{\mathrm{u}}}\overline{T} M, \tag{2.114}$$
where the overbars denote average values (we ignore radial dependence of temp. and composition, for now). This is dirty.
- Now consider the gravitational potential energy
$$E_{\mathrm{G}} = -\int_0^M \left(\frac{Gm}{r}\right)\mathrm{d}m = -\frac{GM^2}{R}. \tag{2.115}$$
- Using some typical solar values values: $k_{\mathrm{B}} = 1.38\times10^{-16}\mathrm{erg\,K^{-1}}$, $\overline{\mu} = 0.62$, $m_{\mathrm{u}} = 1.66\times10^{-24}\mathrm{g}$, mean temperature $\overline{T} = 10^7\mathrm{K}$, $M_\odot = 1.99\times10^{33}\mathrm{g}$, $G = 6.67\times10^{-8}\mathrm{dyne\,cm^2\,g^{-2}}$, $R_\odot = 6.96\times10^{10}\mathrm{cm}$. We find that
$$\begin{aligned} U &\approx +4.0\times10^{48}\,\mathrm{erg}, && (2.116)\\ E_G &\approx -3.8\times10^{48}\,\mathrm{erg}. && (2.117)\end{aligned}$$
- Why are these two numbers so close, even the same order of magnitude? This suggests something deeper is happening.
- To find out, let's take the expression for hydrostatic equilibrium, Equation (2.104), multiply by $4\pi r^3$ on both sides and integrate over the whole star:
$$\int_0^R 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = -\int_0^R \rho\frac{Gm}{r^2}4\pi r^3 \mathrm{d}r. \tag{2.118}$$
- Before we proceed, recall from Sec. refsec:eqstate and later that a generalized expression relating pressure and internal energy is
$$P = (\gamma-1)u = (\gamma-1)\rho U, \tag{2.119}$$
which you can prove to yourself works for different types of gases. This will be used below.
- Integrate left side of Equation (2.118) by parts, and set $P(R_\odot) = 0$. This gives
$$\begin{aligned} -\int_0^R 12\pi P r^2 \mathrm{d}r &= -\int_0^R \frac{Gm}{r^2}4\pi\rho r^3 \mathrm{d}r\\ -\int_0^R 12\pi(\gamma-1)U\rho\, r^2 \mathrm{d}r &= -\int_0^R \frac{Gm}{r^2}4\pi\rho r^3 \mathrm{d}r\\ -\int_0^R 3(\gamma-1)U\,4\pi\rho r^2\,\mathrm{d}r &= -\int_0^R \frac{Gm}{r}4\pi\rho r^2 \mathrm{d}r\\ -\int_0^R 3(\gamma-1)U\,\mathrm{d}m &= -\int_0^R \frac{Gm}{r}\,\mathrm{d}m\\ -\zeta U_{\mathrm{T}} &= E_{\mathrm{G}},\end{aligned}$$

$m/m_\odot$	$r/r_\odot$	$T[K]$
0.000	0.000	1.5×10^7
0.125	0.124	1.2×10^7
0.250	0.170	1.0×10^7
0.375	0.210	8.9×10^6
0.500	0.254	7.7×10^6
0.625	0.306	6.6×10^6
0.750	0.367	5.4×10^6
0.875	0.470	4.2×10^6
1.000	1.000	5.8×10^3

Figure 2.7: Tabulated solar model values. Use in Example Problems 2.2 and 2.3.

or

$$E_{\mathrm{G}} + \zeta U_{\mathrm{T}} = 0, \tag{2.120}$$

where we let $\zeta = 3(\gamma - 1)$, and assume that it is constant throughout a star (which is a good approximation in most of the interior). We are also calling U_{T} the total internal energy of the star.

- Equation (2.120) is a generalized Virial theorem, relating the gravitational and internal energies.
- For an ideal gas, $\gamma = 5/3$, $\zeta = 2$, and $E_{\mathrm{G}} = -2U_{\mathrm{T}}$.
- For a photon gas with $\gamma = 4/3$, $E_{\mathrm{G}} = -U_{\mathrm{T}}$.
- Taking a look at the total energy

$$E_{\mathrm{T}} = E_{\mathrm{G}} + U_{\mathrm{T}} = (1 - \zeta)U_{\mathrm{T}} = \frac{\zeta - 1}{\zeta} E_{\mathrm{G}}. \tag{2.121}$$

- If we want a bound system $E_{\mathrm{T}} < 0$, we need $\zeta > 1$, or, $\gamma > 4/3$. We'll come back to this later.
- To maintain hydrostatic equilibrium, the star will slowly shrink and expand and these 2 energies will change, making the total energy not constant. Thus, the gas must radiate, so

$$\frac{\mathrm{d}E_{\mathrm{T}}}{\mathrm{d}t} + L = 0 \tag{2.122}$$

is always true.

- For an ideal gas,

$$L = (\zeta - 1)\frac{\mathrm{d}U_{\mathrm{T}}}{\mathrm{d}t} = \frac{\mathrm{d}U_{\mathrm{T}}}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}E_{\mathrm{G}}}{\mathrm{d}t}. \tag{2.123}$$

- Therefore, upon contraction for example, half the energy is radiated away, and the other half is used to heat the star.
- Keep in mind that it is the luminosity that is causing the star to shrink. Anything with a finite temperature will radiate, and so $\dot{E}_{\mathrm{G}} \neq 0$.

EXAMPLE PROBLEM 2.2: Compute the actual gravitational energy of the Sun E_{G}. To do this, we need to calculate the total work required to disperse the solar matter over distances $r \gg R_\odot$. Use the table of solar model values in Figure 2.7. How does this energy compare to the total solar irradiance ($1.39 \times 10^6 \mathrm{erg\,s^{-1}\,cm^{-2}}$ at 1 AU) or to a solar flare ($\approx 10^{33}$ J)?

Answer: Consider a spherical shell of mass $\mathrm{d}m$ at radius r and the mass interior to r, m. The work (energy) required to take this shell and carry it to infinity:

$$\begin{aligned} E_{\mathrm{G}}^{1\,\mathrm{shell}} &= \int F(r')\mathrm{d}r' && (2.124)\\ &= -\int_r^\infty \frac{Gm\mathrm{d}m}{r'^2}\mathrm{d}r' && (2.125)\\ &= -\frac{Gm\mathrm{d}m}{r}. && (2.126)\end{aligned}$$

Now we need to consider removing all shells (all the mass of the Sun) and rewrite it in a form so the table of values can be used:

$$\begin{aligned} E_G = \int_{\mathrm{all\,shells}} E_{\mathrm{G}}^{1\,\mathrm{shell}} &= -\int_0^{M_\odot} \frac{Gm}{r}\mathrm{d}m && (2.127)\\ &= -\frac{GM_\odot^2}{R_\odot}\int_0^{M_\odot} \frac{m}{M_\odot}\frac{R_\odot}{r}\mathrm{d}\left(\frac{m}{M_\odot}\right) && (2.128)\\ &= -\frac{GM_\odot^2}{R_\odot}\sum_i \frac{m_i}{M_\odot}\frac{R_\odot}{r_i}\Delta\left(\frac{m_i}{M_\odot}\right) && (2.129)\\ &= -\frac{GM_\odot^2}{R_\odot}\cdot 0.125\,[0+1.008+\ldots] && (2.130)\\ &\simeq -1.6476\frac{GM_\odot^2}{R_\odot} && (2.131)\\ &\simeq -6.25\times 10^{48}\mathrm{erg} && (2.132)\end{aligned}$$

Note that we did a similar problem earlier and just integrated without knowing any radial dependence of the quantities, and found $E_G \approx -3.8\times 10^{48}$erg. In this way, we are more precise.
The solar irradiance is $1.39\times 10^6\,\mathrm{erg\,s^{-1}cm^{-2}}$ at $1\,\mathrm{AU} = 1.496\times 10^{13}\,\mathrm{cm}$. So the total irradiance at the solar surface, or luminosity (integrated over the $4\pi R^2$) is $3.87\times 10^{33}\,\mathrm{erg\,s^{-1}}$. This is the energy per second in radiation. So, considering graviational energy only, the Sun can provide $6.25\times 10^{48}/3.87\times 10^{33} = 1.615\times 10^{15}\,\mathrm{s} = 5.12\times 10^7$ year of irradiance. Since the Sun is over 10^9 years old, it must have some other energy source.

EXAMPLE PROBLEM 2.3: Suppose the Sun is an ideal monatomic gas in hydrostatic equilibrium. Calculate the internal energy and compare to the gravitational energy, i.e., rederive the Virial theorem. Use the Virial theorem to find the mean mass-weighted temperature and compare to the tabulated value. Hints: Start with hydrostatic equilibrium, but it's convenient to be in terms of $\mathrm{d}P$ and $\mathrm{d}m$. Multiply both sides by $V = 4/3\pi r^3$ and integrate both sides. You will eventually make use of

$$U = \int_0^{M_\odot} c_V T\mathrm{d}m, \tag{2.133}$$

(since $\mathrm{d}U = u\mathrm{d}V$), but you can consider the specific heat constant. Recall $c_V = 3/2(R/\mu)$. Ultimately you will end up with $E_G + 2U = 0$. Finally you want to find the mass-weighted temperature, given as

$$\langle T\rangle = \frac{1}{M_\odot}\int_0^{M_\odot} T\mathrm{d}m. \tag{2.134}$$

Then use Table 2.7 as you have before to find the value.

Answer: Let's start with hydrostatic equilibrium, a balance between pressure changes and gravity, Equation (2.104):

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}.$$

We also use Equation (2.107) to get

$$\mathrm{d}P = -\frac{Gm}{4\pi r^4}\mathrm{d}m.$$

Now let's multiply both sides by the volume of a sphere:

$$\begin{aligned}\text{LHS} &= V\mathrm{d}P = \mathrm{d}(VP) - P\mathrm{d}V \\ \text{RHS} &= -\frac{Gm}{4\pi r^4}\mathrm{d}m\frac{4}{3}\pi r^3 = -\frac{Gm}{3r}\mathrm{d}m.\end{aligned}$$

Now integrate both sides from the center to the surface of the Sun. The first term on the LHS vanishes because $P = 0$ at the surface and $V = 0$ at the center. Use $P = \rho RT/\mu = 2/3\rho c_V T$. What's left:

$$\begin{aligned}\text{LHS} &= -\int_c^s P\mathrm{d}V \\ &= -\int_c^s \frac{2}{3}\rho c_V T\mathrm{d}V \\ &= -\frac{2}{3}\int_0^{M_\odot} c_V T\mathrm{d}m \\ &= -\frac{2}{3}U \\ \text{RHS} &= -\int_c^s \frac{Gm}{3r}\mathrm{d}m \\ &= \frac{1}{3}E_G.\end{aligned}$$

We thus have

$$E_G + 2U = 0.$$

The mass weighted temperature is

$$\langle T\rangle = \frac{1}{M_\odot}\int_0^{m_\odot} T\mathrm{d}m.$$

We just found that the virial theorem gives (for constant c_V)

$$2c_V\int_0^{M_\odot} T\mathrm{d}m = -E_G.$$

Inspecting the above 2 equations shows that

$$2c_V M_\odot\langle T\rangle = -E_G.$$

Then

$$\begin{aligned}\langle T\rangle &= -\frac{E_G}{2c_V M_\odot} \\ &= -\frac{E_G\mu m_\mathrm{u}}{3k_\mathrm{B}M_\odot} \\ &= \frac{(6.25\times10^{48}\,\mathrm{erg})(0.62)(1.66\times10^{-24}\,\mathrm{g})}{(3)(1.38\times10^{-16}\,\mathrm{erg\,K^{-1}})(1.99\times10^{33}\,\mathrm{g})} \\ &= 7.81\times10^6\,\mathrm{K}\end{aligned}$$

Using the tabulated values we get

$$\langle T\rangle_\mathrm{tab} = \int_0^{M_\odot} T\mathrm{d}\left(\frac{m}{M_\odot}\right) = \sum_i T_i\,\Delta\left(\frac{m_i}{M_\odot}\right) = 6.85\times10^6\,\mathrm{K}.$$

Therefore, we see the Sun is in a very close hydrostatic and Virial equilibrium.