ASTR 565
Computer Problem 2.1: Polytropes

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1 Introduction

Polytropes are gaseous spheres in hydrostatic equilibrium whose pressure and density at any point along the radius are related by:

$$P = K\rho^{(n+1)/n} \tag{1}$$

where K is a constant and n is the polytropic index. The structure of a polytrope can be modeled using the *Lane-Emden equation*:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{2}$$

where ξ is a dimensionless quantity that represents distance from the center of the sphere. This relationship is a simplification following from the combination of hydrostatic equilibrium and equation (1). [1]

2 Methods

Exact solutions only exist for $n = 0$, 1, and 5, so solutions for other values must be approximated using numerical techniques. Since the *Lane-Emden equation* is a second-order differential equation, the best approach was to first express it as two first-order differential equations. Replacing ξ with x and θ with y, we have:

$$y' = z \tag{3}$$

and

$$z' = -\left(\frac{2}{x}\right)z - y^n \tag{4}$$

where z is a new variable equal to dy/dx.

To approximate y (θ) as a function of x (ξ), the *Runge-Kutta* method[2] was used to the fourth-order. Given boundary conditions at the center: $x_o = 0$ and $y_o = 1$, and at the surface: $y_1 = 0$, y was

approximated by stepping from the center(0) to the surface(1) using the following equations:

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\ k_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5) \end{aligned}$$

The first four correspond to the "fourth order" part of the approximation and were calculated for every step h through x to generate the next value for y (with the exception of $O(h^5)$ in the last equation, which is a margin of error that was not included here). This process was repeated until y reached the surface boundary condition. Similar equations were used to approximate z (dy/dx), replacing k with l.

3 Results

Returning to the original symbols, the values for ξ_1 (ξ at the surface) and other parameters for a polytropic index of $n = 4.5$ are listed in table 1.

Table 1: Values for polytropic index $n = 4.5$

n	ξ_1	ρ_c/ρ	N_n	W_n	Θ_n	$\rho_c[g\,cm^{-3}]$	$P_c[dyne\,cm^{-2}]$	$T_c[K]$
4.5	31.841	6187.500	0.658	4917.415	3.329	8718.704	5.535e19	4.742e7

The resulting approximation for θ is plotted in figure 1, along with θ^n, θ^{n+1}, and q. $\theta^n = \rho/\rho_c$, where ρ_c is the density at the center of the star. Therefore, θ^n for any value of n is always equal to 1 at the center, and 0 at the surface, where the density is assumed to fall to 0, as shown in figure 1. This was how the boundary conditions were determined for the *Runge-Kutta* method. The mass distribution is represented by q, which can be expressed as:

$$q = \frac{m}{M} = \left(\xi^2 \frac{d\theta}{d\xi}\right)\left(\xi_1^2 \frac{d\theta}{d\xi_1}\right)^{-1} \tag{5}$$

and increases from 0 to 1 in figure 1.

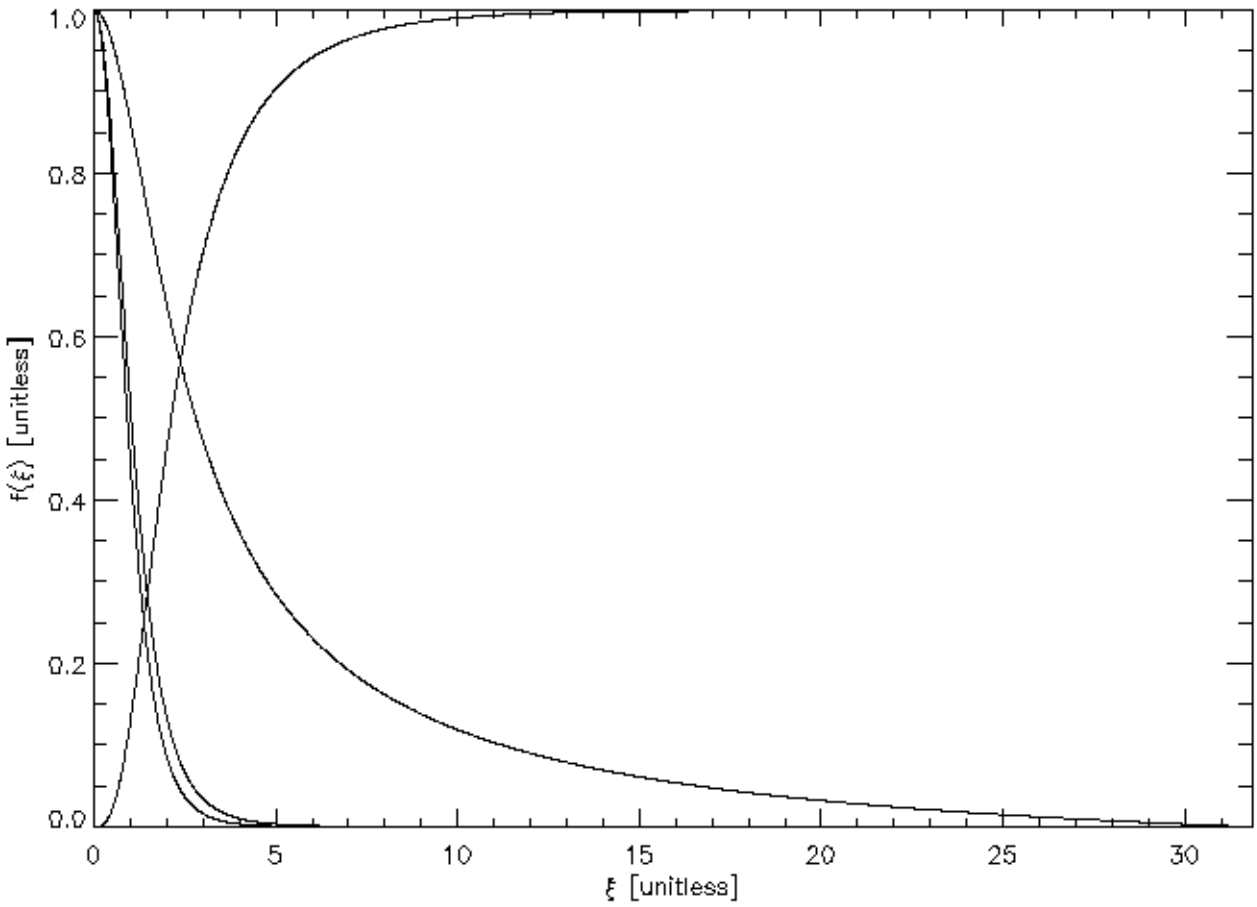

Figure 1: This plot shows θ (red), θ^n (green), θ^{n+1} (blue), and q (purple) as functions of ξ.

References

[1] Donald D. Clayton. *Principles of Stellar Evolution and Nucleosynthesis.* McGraw-Hill, Inc., 1968.

[2] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical Recipes: The Art of Scientific Computing.* Cambridge University Press, 3rd edition, 2007.