August 27 . . . .

- Given an $S(E)$, Equation (1.26) can be integrated. A good approximation is to neglect variations of $S(E)$ over the Gamow peak (this doesn't work when there are resonances). We also approximate the rest of the exponential terms by a gaussian function (see Problem 1.5). One finally gets

$$\langle \sigma v \rangle = \frac{8\sqrt{2}}{9\sqrt{3}} \frac{S(E_0)}{\sqrt{mb}} \eta^2 \mathrm{e}^{-\eta}, \tag{1.28}$$

where

$$\eta = \frac{3E_0}{k_{\mathrm{B}}T} = BT_6^{-1/3}, \tag{1.29}$$

and

$$B = 42.487 \left(Z_1^2 Z_2^2 A \right)^{1/3}. \tag{1.30}$$

- This equation determines the temperature dependence of the average reaction rate. The rates decrease with η, and so increasing $Z_1 Z_2$ decreases the rates, as well as with A, since the velocities decrease at fixed energy. And because η varies as $(Z_1 Z_2)^{2/3}$, the temperature sensitivity of the reactions increases quite strongly with nuclear charge. One thing we've left out is electron screening which increases reaction rates by cancelling some of the positive charge.

> **PROBLEM 1.5:** [10 pts]: Show that indeed the exponential terms inside the integral in Equation (1.26) can be approximated as a gaussian function in the vicinity of E_0 when assuming a constant $S(E)$. Thus, show that before integration, the integrand can be expressed as
>
> $$\exp\left(-\frac{3E_0}{k_B T}\right) \exp\left[-\frac{(E - E_0)^2}{2\Delta^2}\right], \tag{1.31}$$
>
> where $\Delta = \sqrt{E_0 k_{\mathrm{B}} T / 3}$. (Hint: Taylor expand the *argument* of the exponentials around the Gamow peak). You are not being asked to carry out the integration to derive Equation (1.28).

- Let's carry on to the actual reaction rates by making a simplification. Let's approximate $\langle \sigma v \rangle$ by its value around some T_0 by

$$\langle \sigma v \rangle \approx \langle \sigma v \rangle_0 \left(\frac{T}{T_0} \right)^n, \tag{1.32}$$

where $\langle \sigma v \rangle_0$ is the value at $T = T_0$.

- The temperature dependence of the cross section is now characterized by

$$n = \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln T} = \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln \eta} \frac{\mathrm{d} \ln \eta}{\mathrm{d} \ln T} = \frac{\eta - 2}{3}, \tag{1.33}$$

evaluated at $T = T_0$, which is indeed the temperature dependence of the entire energy release.

> **PROBLEM 1.6:** [5 pts]: Show that indeed $n = (\eta - 2)/3$ using Equation (1.33).

- Now we have

$$r_{aA} = \langle \sigma v \rangle_0 n_a n_A \left(\frac{T}{T_0} \right)^n. \tag{1.34}$$

If we re-express the number densities in terms of mass fractions X_i,

$$n_i = \frac{\rho}{m_{\mathrm{u}}} \frac{X_i}{A_i}. \tag{1.35}$$

Then

$$r_{aA} = \langle \sigma v \rangle_0 \frac{X_a X_A}{A_a A_A} \frac{\rho^2}{m_{\mathrm{u}}^2} \left(\frac{T}{T_0} \right)^n. \tag{1.36}$$

We'll come back to this.

IN CLASS WORK

Consider 1 hydrogen nucleus and 2 helium nuclei in a $1\,\mathrm{cm}^3$ volume. Using Equation (1.35) compute the number densities of each species, and make sure your answer makes sense.

Answer: The total mass density in this case, with 9 particles, is $\rho = 9\,m_\mathrm{u}\,\mathrm{cm}^{-3}$. For H, we'd have

$$n_\mathrm{H} = 9\frac{1/9}{1}\,\mathrm{cm}^{-3} = 1\,\mathrm{cm}^{-3}.$$

For He,

$$n_\mathrm{He} = 9\frac{8/9}{4}\,\mathrm{cm}^{-3} = 2\,\mathrm{cm}^{-3}.$$

These particle numbers make sense.

1.1.4 Energy release in nuclear reactions

- Just recall some definitions (ignoring electrons):
  - Atomic number Z: number of protons in a nucleus. $Z_\mathrm{H} = 1$, $Z_\mathrm{He} = 2$, etc. Always an integer.
  - Mass number A: number of protons Z and neutrons N in a nucleus. Always an integer.
  - Atomic mass: The true mass of a single atom (single isotope). A number very nearly equal to the mass number A when expressed in amu. Can be greater or less than A.
  - Atomic weight: The averaged mass over all isotopes of an element (typically what is given in a periodic table). Usually greater than A.
  - Atomic mass unit, amu: 1/12 the mass of neutral carbon 12.
- Consider the reaction of *nuclei*
$$a + A \longrightarrow y + Y. \tag{1.37}$$
  Conservation of energy requires
$$E_{a,A} + (m_a + m_A)c^2 = E_{y,Y} + (m_y + m_Y)c^2, \tag{1.38}$$
  where the $E_{i,I}$ on each side is the kinetic energy of the center-of-mass of each system. Also included are the rest mass energies of each species.
- We can rewrite Equation (1.38) as
$$E_{y,Y} = E_{a,A} + Q, \tag{1.39}$$
  where
$$Q = c^2[m_a + m_A - m_y - m_Y]. \tag{1.40}$$
  The quantity Q can be interpreted as the energy released in any reaction, or, the increase in energy for each reaction.
- Note that these reactions are taking place among *nuclei*. However, since charge is conserved, we may replace the nuclear masses implied in the above equations with the atomic masses, because the same number of electron rest masses will be added to both sides of the equation. A small error in the neglected electron binding energy is introduced (of a few eV), but the great convenience in using atomic masses is well worth it.
- Also conserved in these reactions is the number of *nucleons*, and it is convenient to remove their contribution as it will not change the energy budget. The way to do this is to consider that the mass number is the nearest integer to the exact mass of an atom in atomic mass units.

- So consider an atom with Z protons, N neutrons, and $A = Z + N$ nucleons, with atomic mass m. We can derive what's known as the *mass excess* or *mass defect* Δm (which has units of energy):

$$\begin{aligned}\Delta m &= (m - (Z+N)m_{\mathrm{u}})c^2, \\ &= (m - Am_{\mathrm{u}})c^2 \\ &= [m(\mathrm{amu}) - A]\, c^2 m_{\mathrm{u}},\end{aligned}$$

or finally,

$$\Delta m = 931.494\,\mathrm{MeV/c^2}\,[m(\mathrm{amu}) - A]\,c^2 \quad [\mathrm{MeV}], \tag{1.41}$$

where $m(\mathrm{amu})$ is the atomic mass of the nucleon in question in amu, and $1m_{\mathrm{u}} = 931.5\,\mathrm{MeV/c^2}$.

- Note, mass excess is really just the difference between the atomic mass of an element (which is usually a number with a very small decimal addition) and its mass number (which is always an integer, A).

- Mass excesses are given in the table in Figure 1.5. As an example of using the above expression to come up with these values, take ^{4}He. Its atomic mass $m = 4.002602$. It has 4 nucleons. So

$$\Delta m = (931.494)(4.002602 - 4) = 2.44\,\mathrm{MeV}. \tag{1.42}$$

(slightly different than the table because of updated atomic masses).

- We can therefore write for the energy release of some reaction in terms of mass excesses:

$$Q = [\Delta m(a) + \Delta m(A) - \Delta m(y) - \Delta m(Y)]. \tag{1.43}$$

- In general, the energy is released as kinetic energy to the resultant particles (as implied in Equation (1.39)), and sometimes in photons (and neutrinos, which isn't too important in the energy budget for normal reactions). The energy gets redistributed in the gas through collisions and the absorption of photons. The details don't matter because of energy equilibrium, but what matters is the total amount of heat added to the gas.

- Back to Equation (1.36), the energy generation rate per gram is the reaction rate multiplied by the energy for each reaction divided by density, so

$$\varepsilon_{aA} = \frac{r_{aA}Q_{aA}}{\rho}, \quad [\mathrm{erg\,g^{-1}\,s^{-1}}], \tag{1.44}$$

$$\varepsilon_{aA} = Q_{aA}\langle\sigma v\rangle_0 \frac{X_a X_A}{A_a A_A}\frac{\rho}{m_{\mathrm{u}}^2}\left(\frac{T}{T_0}\right)^n. \tag{1.45}$$

- One can in general write

$$\varepsilon = \varepsilon_0 \rho T^n, \tag{1.46}$$

for any reaction, absorbing most of the constant terms in ε_0.

1.1.5 Binding energy

- Do not confuse mass excess with nuclear binding energy, since they are very similar.

- The nuclear binding energy is the energy required to separate a stable nucleus into its constituent parts, as depicted in Figure 1.1. Note the following definitions:
  - $1\,\mathrm{amu} = 931.494\,\mathrm{MeV}/c^2 = m_{\mathrm{u}}$
  - $m_{\mathrm{p}} = 1.007825$ amu
  - $m_{\mathrm{n}} = 1.00867$ amu
  - $m_{\mathrm{e}} = 0.0005486$ amu

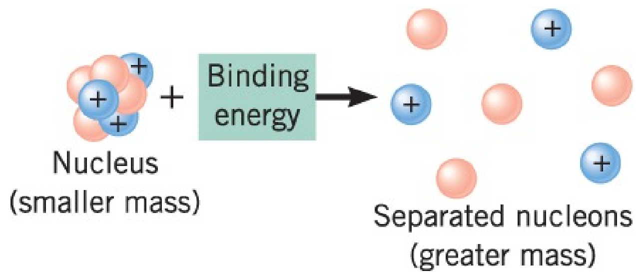

Figure 1.1: Illustration of the concept of binding energy of a nucleus. Typically, a nucleus has a lower energy than if its particles were free. Source of Figure 1.1: http://staff.orecity.k12.or.us/les.sitton/Nuclear/313.htm.

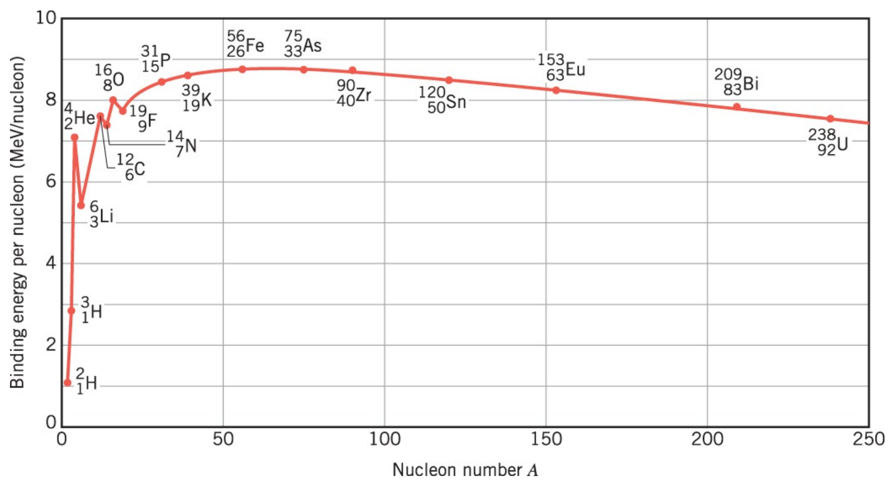

Figure 1.2: The binding energy of isotopes per nucleon. After iron, energy must be used to fuse its nucleons. Source of Figure 1.1: http://staff.orecity.k12.or.us/les.sitton/Nuclear/313.htm.

- Electron binding energy is usually referred to as *ionization energy*.
- Binding energy for neutral atoms is

$$E_{\rm bind} = 931.494\,{\rm MeV/c^2}\,(Zm_{\rm p} + Nm_{\rm n} + Zm_{\rm e} - m)\,c^2 \qquad [{\rm MeV}], \tag{1.47}$$

  where m is the atomic mass of the element or isotope in question. Often you'll see the binding energy per nucleon, $E_{\rm bind}/A$. Electrons can be ignored when talking about nuclei. This is plotted in Fig. 1.2.

- The atomic mass of ^{4}He is 4.0026 amu. It's binding energy per nucleon is $931.494 * (2m_{\rm p} + 2m_{\rm n} - 4.0026)/4 = 7.07$ MeV/nucleon.
- The large binding energy of ^{4}He is important in stellar physics, as we'll see.

1.2 Hydrogen burning

- Historically, considering a star of H and He, all major 2-particle interactions produce unstable nuclei

$$\begin{aligned} {\rm p + p} &\longrightarrow {}^2{\rm He\ (unstable)} \longrightarrow {\rm p + p} \\ {\rm p} + {}^4\,{\rm He} &\longrightarrow {}^5{\rm Li\ (unstable)} \longrightarrow {\rm p} + {}^4\,{\rm He} \\ {}^4{\rm He} + {}^4\,{\rm He} &\longrightarrow {}^8{\rm Be\ (unstable)} \longrightarrow {}^4{\rm He} + {}^4\,{\rm He}. \end{aligned} \tag{1.48}$$

- It was Hans Bethe who first showed that the weak force plays a role in all this, in the form of beta decay (see below).
- Anyway, the general idea of hydrogen fusion is always

  $$4^1\mathrm{H} \longrightarrow^4 \mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e}, \tag{1.49}$$

  where 2 positrons are needed to keep charge conserved, and 2 electron neutrinos conserve lepton number (from the 2 anti-lepton positrons). This does not happen all at once, but along certain "paths," see below.
- The atomic mass of H is 1.007852 amu and of ^{4}He is 4.002603 amu. So 0.0288 mass units are lost. The mass fraction that is turned into energy is thus $0.0288/4 = 0.007$, or 0.7%.
- Using Δmc^2, we have $(0.0288)(931.494\,\mathrm{MeV}/c^2)c^2 = 26.8\,\mathrm{MeV}$.
- Using the mass excesses in the units of MeV we can compute the energy liberated in yet another way

  $$Q = 4(7.289) - 2.4248 - 2(0.263) = 26.21\,\mathrm{MeV}, \tag{1.50}$$

  where this time we take into account the energy carried away by the neutrinos (see table in Fig. 1.3). Note, as mentioned before, that the values are from atomic mass excesses, not *nuclear* mass excesses, so electrons are implicitly in there. That is why we do not take into account the ~ 0.5 MeV from the positrons.