$m/m_\odot$	$r/r_\odot$	$T[K]$
0.000	0.000	1.5×10^7
0.125	0.124	1.2×10^7
0.250	0.170	1.0×10^7
0.375	0.210	8.9×10^6
0.500	0.254	7.7×10^6
0.625	0.306	6.6×10^6
0.750	0.367	5.4×10^6
0.875	0.470	4.2×10^6
1.000	1.000	5.8×10^3

Figure 2.7: Tabulated solar model values. Use in Example Problems 2.2 and 2.3.

September 24 . . . .

- Taking a look at the total energy

$$E_\mathrm{T} = E_\mathrm{G} + U_\mathrm{T} = (1-\zeta)U_\mathrm{T} = \frac{\zeta-1}{\zeta}E_\mathrm{G}. \tag{2.121}$$

- If we want a bound system $E_\mathrm{T} < 0$, we need $\zeta > 1$, or, $\gamma > 4/3$. We'll come back to this later.

- To maintain hydrostatic equilibrium, the star will slowly shrink and expand and these 2 energies will change, making the total energy not constant. Thus, the gas must radiate, so

$$\frac{\mathrm{d}E_\mathrm{T}}{\mathrm{d}t} + L = 0 \tag{2.122}$$

  is always true.

- For an ideal gas,

$$L = (\zeta - 1)\frac{\mathrm{d}U_\mathrm{T}}{\mathrm{d}t} = \frac{\mathrm{d}U_\mathrm{T}}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}E_\mathrm{G}}{\mathrm{d}t}. \tag{2.123}$$

- Therefore, upon contraction for example, half the energy is radiated away, and the other half is used to heat the star.

- Keep in mind that it is the luminosity that is causing the star to shrink. Anything with a finite temperature will radiate, and so $\dot{E}_\mathrm{G} \neq 0$.

EXAMPLE PROBLEM 2.2: Compute the actual gravitational energy of the Sun E_G. To do this, we need to calculate the total work required to disperse the solar matter over distances $r \gg R_\odot$. Use the table of solar model values in Figure 2.7. How does this energy compare to the total solar irradiance ($1.39 \times 10^6 \mathrm{erg\,s^{-1}\,cm^{-2}}$ at 1 AU) or to a solar flare ($\approx 10^{33}$ J)?

Answer: Consider a spherical shell of mass $\mathrm{d}m$ at radius r and the mass interior to r, m. The force acting on this shell from all interior mass is

$$F(r) = -G\frac{m\mathrm{d}m}{r^2}.$$

The work (energy) required to take this shell and carry it to infinity:

$$\begin{aligned}
E_{\mathrm{G}}^{1\,\mathrm{shell}} &= \int F(r')\,\mathrm{d}r' \\
&= -\int_r^\infty \frac{Gm\mathrm{d}m}{r'^2}\mathrm{d}r' \\
&= -\frac{Gm\mathrm{d}m}{r}.
\end{aligned}$$

Now we need to consider removing all shells (all the mass of the Sun) and rewrite it in a form so the table of values can be used:

$$\begin{aligned}
E_G = \int_{\mathrm{all\,shells}} E_{\mathrm{G}}^{1\,\mathrm{shell}} &= -\int_0^{M_\odot} \frac{Gm}{r}\mathrm{d}m \\
&= -\frac{GM_\odot^2}{R_\odot}\int_0^{M_\odot} \frac{m}{M_\odot}\frac{R_\odot}{r}\mathrm{d}\left(\frac{m}{M_\odot}\right) \\
&= -\frac{GM_\odot^2}{R_\odot}\sum_i \frac{m_i}{M_\odot}\frac{R_\odot}{r_i}\Delta\left(\frac{m_i}{M_\odot}\right) \\
&= -\frac{GM_\odot^2}{R_\odot}\cdot 0.125\,[0+1.008+\ldots] \\
&\simeq -1.6476\frac{GM_\odot^2}{R_\odot} \\
&\simeq -6.25\times 10^{48}\,\mathrm{erg}
\end{aligned}$$

Note that we did a similar problem earlier and just integrated without knowing any radial dependence of the quantities, and found $E_G \approx -3.8\times 10^{48}$erg. In this way, we are more precise.
The solar irradiance is $1.39\times 10^6\,\mathrm{erg\,s^{-1}cm^{-2}}$ at $1\,\mathrm{AU} = 1.496\times 10^{13}\,\mathrm{cm}$. So the total irradiance at the solar surface, or luminosity (integrated over the $4\pi R^2$) is $3.87\times 10^{33}\,\mathrm{erg\,s^{-1}}$. This is the energy per second in radiation. So, considering graviational energy only, the Sun can provide $6.25\times 10^{48}/3.87\times 10^{33} = 1.615\times 10^{15}\,\mathrm{s} = 5.12\times 10^7$ year of irradiance. Since the Sun is over 10^9 years old, it must have some other energy source.

EXAMPLE PROBLEM 2.3: Suppose the Sun is an ideal monatomic gas in hydrostatic equilibrium. Calculate the internal energy and compare to the gravitational energy, i.e., rederive the Virial theorem. Use the Virial theorem to find the mean mass-weighted temperature and compare to the tabulated value. Hints: Start with hydrostatic equilibrium, but it's convenient to be in terms of $\mathrm{d}P$ and $\mathrm{d}m$. Multiply both sides by $V = 4/3\pi r^3$ and integrate both sides. You will eventually make use of

$$U = \int_0^{M_\odot} c_V T\mathrm{d}m, \tag{2.124}$$

(since $\mathrm{d}U = u\mathrm{d}V$), but you can consider the specific heat constant. Recall $c_V = 3/2(R/\mu)$. Ultimately you will end up with $E_G + 2U = 0$. Finally you want to find the mass-weighted temperature, given as

$$\langle T\rangle = \frac{1}{M_\odot}\int_0^{M_\odot} T\mathrm{d}m. \tag{2.125}$$

Then use Table 2.7 as you have before to find the value.

<u>Answer</u>: Let's start with hydrostatic equilibrium, a balance between pressure changes and gravity, Equation (2.104):

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho m}{r^2}.$$

We also use Equation (2.107) to get

$$\mathrm{d}P = -\frac{Gm}{4\pi r^4}\mathrm{d}m.$$

Now let's multiply both sides by the volume of a sphere:

$$
\begin{aligned}
\text{LHS} &= V\mathrm{d}P = \mathrm{d}(VP) - P\mathrm{d}V \\
\text{RHS} &= -\frac{Gm}{4\pi r^4}\mathrm{d}m\frac{4}{3}\pi r^3 = -\frac{Gm}{3r}\mathrm{d}m.
\end{aligned}
$$

Now integrate both sides from the center to the surface of the Sun. The first term on the LHS vanishes because $P = 0$ at the surface and $V = 0$ at the center. Use $P = \rho RT/\mu = 2/3\rho c_V T$. What's left:

$$
\begin{aligned}
\text{LHS} &= -\int_c^s P\mathrm{d}V \\
&= -\int_c^s \frac{2}{3}\rho c_V T\mathrm{d}V \\
&= -\frac{2}{3}\int_0^{M_\odot} c_V T\mathrm{d}m \\
&= -\frac{2}{3}U \\
\text{RHS} &= -\int_c^s \frac{Gm}{3r}\mathrm{d}m \\
&= \frac{1}{3}E_G.
\end{aligned}
$$

We thus have

$$E_G + 2U = 0.$$

The mass weighted temperature is

$$\langle T\rangle = \frac{1}{M_\odot}\int_0^{m_\odot} T\mathrm{d}m.$$

We just found that the virial theorem gives (for constant c_V)

$$2c_V\int_0^{M_\odot} T\mathrm{d}m = -E_G.$$

Inspecting the above 2 equations shows that

$$2c_V M_\odot\langle T\rangle = -E_G.$$

Then

$$
\begin{aligned}
\langle T\rangle &= -\frac{E_G}{2c_V M_\odot} \\
&= -\frac{E_G\mu m_\mathrm{u}}{3k_\mathrm{B}M_\odot} \\
&= \frac{(6.25\times10^{48}\,\mathrm{erg})(0.62)(1.66\times10^{-24}\,\mathrm{g})}{(3)(1.38\times10^{-16}\,\mathrm{erg\,K^{-1}})(1.99\times10^{33}\,\mathrm{g})} \\
&= 7.81\times10^6\,\mathrm{K}
\end{aligned}
$$

Using the tabulated values we get

$$\langle T\rangle_\mathrm{tab} = \int_0^{M_\odot} T\mathrm{d}\left(\frac{m}{M_\odot}\right) = \sum_i T_i\,\Delta\left(\frac{m_i}{M_\odot}\right) = 6.85\times10^6\,\mathrm{K}.$$

Therefore, we see the Sun is in a very close hydrostatic and Virial equilibrium.

2.6 Polytropes

2.6.1 Motivation and derivation

- So far we've collected 3 equations for stellar structure, collected here for convenience

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi\rho r^2,\\ \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{G\rho m}{r^2},\\ \frac{\mathrm{d}L}{\mathrm{d}r} &= 4\pi\rho r^2\,\varepsilon.\end{aligned}$$

- There are several others that we need to full-out model a real star. But even now we can get some very important insights on stellar structure.
- Ignore the 3rd equation for now. The first 2 equations have 3 unknowns and cannot be solved simultaneously as they stand.
- First law of thermodynamics

$$\frac{\mathrm{d}Q}{\mathrm{d}T} = \frac{\mathrm{d}U}{\mathrm{d}T} + P\frac{\mathrm{d}V}{\mathrm{d}T} = C = c_V + P\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_P. \tag{2.126}$$

- Ideal gas: $P = RT/V\mu$, where again $V = (1/\rho)$ is the specific volume and recall that $c_P - c_V = R/\mu$. We then find after manipulation

$$\frac{\mathrm{d}T}{T} + \frac{1}{n}\frac{\mathrm{d}V}{V} = 0, \tag{2.127}$$

  where n is the polytropic index $n = (c_V - C)/(c_P - c_V)$.
- We can eliminate the temperature from this to get pressure and density:

$$\frac{\mathrm{d}P}{P} = \left(1 + \frac{1}{n}\right)\frac{\mathrm{d}\rho}{\rho}, \tag{2.128}$$

  which, when integrated, gives

$$P = \mathrm{const} \times \rho^{(1+1/n)}. \tag{2.129}$$

- A system where pressure and density are related as $P = K\rho^{1+1/n}$ is called a polytrope.
- For an adiabatic change $C = 0$, $n = 1/(\gamma - 1)$, and

$$P = K\rho^{\gamma}, \tag{2.130}$$

  where $\gamma = c_P/c_V$.
- This is very useful because we can now get radial profiles of $P(r), T(r), m(r)$ and $\rho(r)$.

2.6.2 Lane-Emden equation

- Take hydrostatic equilibrium, divide by density, multiply by r^2, and use the mass gradient equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{2.131}$$

- Consider the polytropic equation of state

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 K\gamma\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{2.132}$$

- Use polytropic index n and let the density be rewritten as a unitless quantity θ by

$$\frac{\rho}{\rho_c} = \theta^n, \tag{2.133}$$

where ρ_c is the central density of a model.

- Then

$$\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n. \tag{2.134}$$

- Let the coefficient (of units distance squared) be α^2, where

$$\alpha = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2} = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}. \tag{2.135}$$

(You might want to prove to yourself the above is true and the unit is distance).

- We then scale the radial coordinate

$$r = \alpha\xi. \tag{2.136}$$

- Then

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{2.137}$$

This is known as the Lane-Emden equation.

- Boundary conditions at the center must satisfy (noting Equation (2.133))

$$\theta(\xi) = 1 \quad \text{at} \quad \xi = 0, \tag{2.138}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = 0 \quad \text{at} \quad \xi = 0. \tag{2.139}$$

- Define the surface as

$$\theta(\xi) = 0 \quad \text{at} \quad \xi = \xi_1. \tag{2.140}$$

2.6.3 Polytrope solutions

- In what follows, a subscript n denotes the label of the polytropic index, whereas the supercript n is the quantity raised to the nth power.
- Assume a solution $\theta_n(\xi)$ can be found to Equation (2.137) for a given index n.
- Then the radius of the model is

$$R = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2}\xi_1 = \alpha\xi_1. \tag{2.141}$$

- The mass interior to $m(\xi)$ is

$$m(\xi) = \int_0^{\alpha\xi} 4\pi r'^2 \rho\, \mathrm{d}r' = 4\pi\alpha^3\rho_c \int_0^{\xi} \xi'^2\theta^n\, \mathrm{d}\xi' \tag{2.142}$$

$$= -4\pi\alpha^3\rho_c \int_0^{\xi} \frac{\mathrm{d}}{\mathrm{d}\xi'}\left(\xi'^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi'}\right) \mathrm{d}\xi' \tag{2.143}$$

$$= -4\pi\alpha^3\rho_c\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \tag{2.144}$$

$$m(\xi) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\, \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}. \tag{2.145}$$

- The total mass is

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}. \tag{2.146}$$

- Inspecting the mass and radius relations, the constant K is

$$K = NGM^{(n-1)/n}R^{(3-n)/n}, \tag{2.147}$$

where

$$N = \frac{(4\pi)^{1/n}}{n+1}\left[-\xi_1^{(n+1)/(n-1)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}\right]^{(1-n)/n}. \tag{2.148}$$

- Mean density of model

$$\overline{\rho} = \frac{M}{V} = -\frac{3}{\xi_1}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}\rho_c. \tag{2.149}$$

- Central density is then

$$\rho_c = -\frac{\xi_1}{3}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}^{-1}\frac{M}{4/3\pi R^3}. \tag{2.150}$$

- Central pressure

$$P_c = K\rho_c^{(n+1)/n}, \tag{2.151}$$

or

$$P_c = W_n\frac{GM^2}{R^4}, \tag{2.152}$$

where

$$W_n = \left[4\pi(n+1)\left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_1}^2\right]^{-1}. \tag{2.153}$$

Note that this is the coefficient we were computing in some simple example problems (cf. Equation (2.111)).

- So pressure throughout model is

$$P = P_c\theta^{n+1}. \tag{2.154}$$

- For the temperature, assume an ideal gas with μ constant, then

$$T = T_c\theta, \tag{2.155}$$

where

$$T_c = \Theta\frac{GM\mu m_\mathrm{u}}{k_\mathrm{B}R}, \tag{2.156}$$

and

$$\Theta = \left[-(n+1)\xi_1 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)_{\xi=\xi_1} \right]^{-1}. \tag{2.157}$$

Again here, this coefficient is a number obtained through simpler means using only hydrostatic equilibrium (cf. Problem 2.6).

- The distribution of mass in a polytrope can be obtained easily (see Equation (2.145) and Equation (2.146))

$$q = \frac{m}{M} = \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right) \left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)^{-1}_{\xi=\xi_1}. \tag{2.158}$$

- Only 3 analytical solutions of the Lane-Emden equation are possible:

$$n = 0 \quad ; \qquad \theta_0 = 1 - \frac{1}{6}\xi^2, \tag{2.159}$$

$$n = 1 \quad ; \qquad \theta_1 = \frac{\sin \xi}{\xi}, \tag{2.160}$$

$$n = 5 \quad ; \qquad \theta_5 = (1 + \frac{1}{3}\xi^2)^{-1/2}. \tag{2.161}$$

COMPUTER PROBLEM 2.1: [35 points]: Solve the Lane-Emden Equation (2.137) numerically with the two boundary conditions using your method of choice for your assigned index n.

What to do

- To solve the 2nd-order nonlinear differential equation, which is a difficult task to do as is, the first thing you will need to do is to make a substitution to generate 2 first-order differential equations, as can always be done. A suitable choice may be to let $y = \theta$ and $z = \mathrm{d}\theta/\mathrm{d}\xi = y'$. You'll then be able to have an equation for y' and z'.
- Next you need to choose your solver. You can treat this as a boundary-value problem, in which case a method like Newton-Rhapson can be used. Fancy software like `IDL` and `MATLAB` have built-in boundary value algorithms, but can be tricky to implement in some cases. Perhaps a better option is to use a "shooting method," which treats the problem as an initial-value problem. You "shoot" from the center, say, and work your way out to the end of the model using an integrator. A common and very handy integrator is Runge-Kutta. I'd suggest this method (shooting) because you don't need any sophisticated algorithms and it works! But it's your choice. A 4th-order Runge-Kutta is sufficient for this problem if you choose to do so. There's lots of information on this in *Numerical Recipes*, for example.
- You have to be a little careful about how you treat the origin since there is a possible divergence in your equation(s) (as should be apparent already). If one expands the Lane-Emden equation about the origin, it can be shown that

$$\theta_n(\xi) \simeq 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \ldots \tag{2.162}$$

  You can use this to set your first values for y and z (if you choose to solve the problem with this class of methods) by taking a very small, but finite starting ξ. Then just run it until you cross the first zero in θ_n. Make sure your grid is sufficiently fine so the solution is smooth.

What to hand in

(a) You will solve for a polytrope of given index n, assigned in the table. You will provide the instructor the values (including 3 decimal places!) you find for the table columns for your n only. For the last 3 columns in the table, compute those values for 1 solar mass, 1 solar radius, and composition $X = 0.7$ and $Z = 0.02$. We'll fill the table in together when finished. Also provide a copy of your code. [15 pts]

(b) Plot the following quantities on a single full-page plot with appropriate labels and clearly distinguishable lines: $\theta_n(\xi)$, $\theta_n^n(\xi)$, $\theta_n^{n+1}(\xi)$, and $q(\xi)$. What do each of the 4 quantities that you are showing here physically correspond to? [10 pts]

(c) Derive those first 4 terms of the expansion in Equation (2.162). Hint: First show or explain that if $\theta(\xi)$ is a solution of the Lane-Emden equation, then so must $\theta(-\xi)$. This motivates you to try a solution of the form below of even powers which you can plug into the Lane-Emden equation:

$$\theta(\xi) = C_0 + C_2\xi^2 + C_4\xi^4 + C_6\xi^6 + \dots \tag{2.163}$$

Don't forget to use any boundary conditions you may need. [10 pts]

n	Name	ξ_1	$\rho_c/\overline{\rho}$	N_n	W_n	Θ_n	ρ_c [g cm^{-3}]	P_c [dyne cm^{-2}]	T_c [K]
0.0	–								
0.5	Katie								
1.0	–								
1.5	Caitlyn								
2.0	Drew								
2.5	Jeremy								
2.75	Gavin								
3.0	Jodi								
3.25	Amber								
3.5	Emma								
4.0	Ethan								
4.5	Laurel								
5.0	–								