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4.2 Homology relations for stars in radiative equilibrium

4.2.1 Basic idea

- Solving the equations of stellar structure is possible, yet difficult.
- There are ways of obtaining useful insights without doing so (like using the first 2 equations to study polytropes).
- Consider we are dealing with stars that are homologous: that a star with mass, say, M is a scaled version of a star of mass M' (if mass doesn't change too rapidly).
- Homologous points on homologous mass shells means that $m/M = m'/M'$.
- This assumption allows us to find relations between one numerical solution and another, without computing more than one solution.
- We will use an ideal gas equation of state.
- One way of doing this is to introduce dimensionless variables

$$\tilde{r} = \frac{r}{R_0}; \quad \tilde{m} = \frac{m}{M_0}; \quad \tilde{L} = \frac{L}{L_0}; \quad \tilde{T} = \frac{T}{T_0}; \quad \tilde{P} = \frac{P}{P_0}. \tag{4.1}$$

- T_0 and P_0 are chosen in such a way so that the basic structure equations are simplified

$$T_0 = \frac{\mu G M_0}{R_0 R_\mathrm{g}}; \qquad P_0 = \frac{G M_0^2}{4\pi R_0^4} \tag{4.2}$$

- The 0 subscripts can be thought of as a reference star.
- Then, hydrostatic equilibrium becomes

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\frac{\tilde{P}}{\tilde{T}}\frac{\tilde{m}}{\tilde{r}^2}. \tag{4.3}$$

- The mass-radius equation is

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = \frac{\tilde{P}}{\tilde{T}}\tilde{r}^2. \tag{4.4}$$

- The temperature gradient for radiative equilibrium is

$$\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{r}} = -C\frac{\tilde{L}}{\tilde{r}^2}\frac{\tilde{P}^{\alpha+1}}{\tilde{T}^{\alpha+\beta+1}}, \tag{4.5}$$

  if an opacity is considered of the form $\kappa_\mathrm{R} = \kappa_0 \rho^\alpha T^{-\beta}$.
- The energy equation is

$$\frac{\mathrm{d}\tilde{L}}{\mathrm{d}\tilde{r}} = D\tilde{P}^2\tilde{T}^{\nu-2}\tilde{r}^2 \tag{4.6}$$

  if the energy is considered to be of the form $\varepsilon = \varepsilon_0 \rho T^\nu$.

- The constants that have appeared are

$$
\begin{aligned}
C &= C_0 \frac{\kappa_0}{\mu^{\beta+4}} \frac{L_0 R_0^{\beta-3\alpha}}{M_0^{\beta+3-\alpha}}, && (4.7)\\
C_0 &= \frac{3}{16\sigma} \left(\frac{R_g}{G}\right)^{\beta+4} \left(\frac{1}{4\pi}\right)^{\alpha+2}, && (4.8)\\
D &= D_0 \varepsilon_0 \mu^{\nu} \frac{M_0^{\nu+2}}{L_0 R_0^{\nu+3}}, && (4.9)\\
D_0 &= \left(\frac{G}{R_g}\right)^{\nu} \frac{1}{4\pi}. && (4.10)
\end{aligned}
$$

- These are the equivalent *dimensionless* equations of stellar structure, along with accompanying boundary conditions.
- C_0 and D_0 are just fundamental constants.
- Note that C and D depend only on M_0, R_0, and L_0.
- The two center boundary conditions, $\tilde{m} = \tilde{L} = 0$ at $\tilde{r} = 0$, imply that there is only one pair of constants C and D that satisfies these conditions.
- Thus, there is only one solution for stars in radiative equilibrium! All those stars have the same C and D.
- All stars are not radiative, there are convection zones, and this messes things up a bit, which will be shown later.
- Therefore, these ideas work well for stars with small convection zones, such as most A, F, G, and some B stars (where we also ignore degeneracy and radiation pressure).
- In what follows, we can study 4 cases: the CNO cycle ($\nu \simeq 20$) and the PP-chain ($\nu \simeq 5$) stars, as well as Kramer's opacities ($\beta = 3.5$, $\alpha = 1$) or electron scattering ($\beta = \alpha = 0$) cases.

4.2.2 Dependence on mass

- Drop the zero subscripts for now.
- Here we look at $R(M)$, $T(M)$, and $L(M)$.
- Note that the produce of constants CD is independent of luminosity (ignore composition for a moment, assume all stars are the same)

$$CD = \text{const} \frac{R^{\beta-3\alpha} M^{\nu+2}}{R^{\nu+3} M^{\beta-\alpha+3}}. \tag{4.11}$$

- So we have

$$R^{\nu+3-\beta+3\alpha} \propto M^{\nu-1-\beta+\alpha}. \tag{4.12}$$

	$\nu = 5$	$\nu = 20$
e-scattering	$R \propto M^{0.5}$	$R \propto M^{0.83}$
Kramers	$R \propto M^{0.2}$	$R \propto M^{0.73}$

- Just to note, as we saw earlier, degenerate stars shrink with increasing mass, since these are polytropes with $n = 3$ and the exponent is $-1/3$.
- From the ideal gas law and hydrostatic equilibrium the central temperature $T_c \propto M/R$.

	$\nu = 5$	$\nu = 20$
e-scattering	$T_c \propto M^{0.5}$	$T_c \propto M^{0.17}$
Kramers	$T_c \propto M^{0.8}$	$T_c \propto M^{0.27}$

- This can be used along with Equation (4.12) to find the central temperature dependence on mass alone
- So clearly with increasing mass, the central temperature must increase.
- Thus we must go from PP chain to CNO burning in interiors. We can't say anything about effective temperature yet because the photosphere has different opacity laws.
- Finally C can give us the relation for luminosity

$$L \propto \frac{M^{\beta+3-\alpha}}{R^{\beta-3\alpha}} \tag{4.13}$$

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto M^{3.0}$	$L \propto M^{3.0}$
Kramers	$L \propto M^{5.4}$	$L \propto M^{5.14}$

- Note that for electron scattering the luminosity does not depend on the energy generation "type."
- For the CNO cycle values, this best matches A and F stars.
- Note that the mass-luminosity relations follow mostly from C, which describes energy transport (from the $\mathrm{d}T/\mathrm{d}r$ equation).
- Briefly, note from the constant C that we can get an expression for the luminosity in scaled solar values considering a Kramer's opacity:

$$L \simeq 1.4 \times 10^{35} \left(\frac{M}{M_\odot}\right)^{5.5} \frac{1.7}{1+X}\,\frac{0.02}{X} \left(\frac{\mu}{0.62}\right)^{7.5} \left(\frac{R}{R_\odot}\right)^{-0.5} \mathrm{erg\,s^{-1}}. \tag{4.14}$$

- The solar luminosity is about $3.9 \times 10^{33}\,\mathrm{erg\,s^{-1}}$, so this is a bit high, but still impressive.
- What about T_{eff}? Since $T_{\mathrm{eff}}^4 \propto L/R^2$, this is easy.

	$\nu = 5$	$\nu = 20$
e-scattering	$T_{\mathrm{eff}} \propto M^{0.5}$	$T_{\mathrm{eff}} \propto M^{0.34}$
Kramers	$T_{\mathrm{eff}} \propto M^{1.25}$	$T_{\mathrm{eff}} \propto M^{0.92}$

4.2.3 Dependence on T_{eff}

- For a proper H-R diagram we'd like to have $L \propto T_{\mathrm{eff}}^{\gamma}$. We can do that with what we've just found since we have $L \propto R^2 T_{\mathrm{eff}}^4$ and we have both of those as relations to the mass.

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto T_{\mathrm{eff}}^{6.0}$	$L \propto T_{\mathrm{eff}}^{8.9}$
Kramers	$L \propto T_{\mathrm{eff}}^{4.32}$	$L \propto T_{\mathrm{eff}}^{5.6}$

4.2.4 Dependence on mean molecular weight

- We ignore here He abundance which typically enters through κ_0, which contains terms with the number of electrons, etc.
- We multiply C and D but keep the μ dependence now

$$CD = \text{const}\,\frac{\mu^{\nu}}{\mu^{\beta+4}}\frac{R^{\beta-3\alpha}M^{\nu+2}}{R^{\nu+3}M^{\beta-\alpha+3}}. \tag{4.15}$$

- The idea is first to compare stars of the same mass and see how things change with μ.
- The radius can go either way!

	$\nu = 5$	$\nu = 20$
e-scattering	$R \propto \mu^{1/8}$	$R \propto \mu^{0.7}$
Kramers	$R \propto \mu^{-1/3}$	$R \propto \mu^{0.6}$

- What about luminosity? We can use constant C again

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto \mu^{4.0}$	$L \propto \mu^{4.0}$
Kramers	$L \propto \mu^{7.67}$	$L \propto \mu^{7.2}$

- We see that the luminosity increases steeply with increasing μ, more than with mass!
- A higher central temperature is required to increase the central pressure to balance the heavier material, hence higher luminosity. This will be a critical point for understanding different phases of stellar evolution.
- The T_{eff} effect on molecular weight is not insignificant

	$\nu = 5$	$\nu = 20$
e-scattering	$T_{\text{eff}} \propto \mu^{0.94}$	$T_{\text{eff}} \propto \mu^{0.65}$
Kramers	$T_{\text{eff}} \propto \mu^{2.1}$	$T_{\text{eff}} \propto \mu^{1.5}$

- Stars move up and to the left of the HR diagram as the mean molecular weight increases.

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto T_{\text{eff}}^{4.25}$	$L \propto T_{\text{eff}}^{6.2}$
Kramers	$L \propto T_{\text{eff}}^{3.65}$	$L \propto T_{\text{eff}}^{4.8}$

- At the same time, stars have a less-steep HR curve for the same M and different μ than for the same μ and different M.
- So, we see that stars are hotter and more luminous with increasing μ. The question is are they above or below the main sequence of H-rich stars?
- Firstly, the luminosity increases more slowly for He stars than H stars for a given temperature.
- Secondly, for increasing μ, T_{eff} increases more steeply with L than for H-rich stars.
- Therefore, it turns out that He-rich stars, for a given mass, have a higher L and a higher T_{eff} than a "solar" main sequence, and thus falls "below" it. See Figure 4.1.

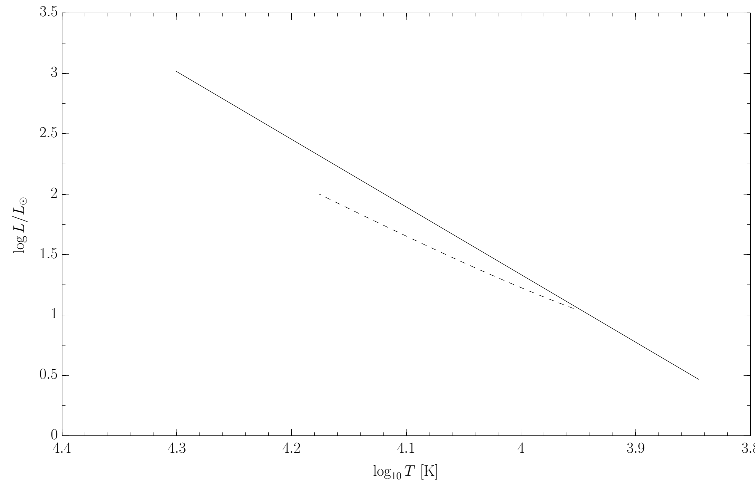

Figure 4.1: The solid line represents the main sequence for stars of various mass but fixed μ in the CNO cycle, so $L \propto \mu^{5.6}$. The dashed line is the evolutionary track for a star with a given mass but increasing μ, so $L \propto \mu^{4.8}$.

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4.2.5 Dependence on heavy metal abundances

- Let's look at lower-mass stars and their dependence on Z.
- This is only approximate because the depth of the outer convection zone depends on Z, but for homologous stars this will still give the right trend.
- The metals appear mainly through the κ_0 term in the bound-free processes in deep interiors where only metals still have electrons.
- Furthermore there is no dependence of ε_0 on Z in the PP-chain reactions.
- So considering $\beta = 3.5$ and $\alpha = 1$ we have for the product CD

$$CD = \text{const}\, \frac{Z R^{0.5} M^{\nu+2}}{R^{\nu+3} M^{5.5}}. \tag{4.16}$$

Kramers, $\nu = 5$	$R \propto Z^{0.13}$	$L \propto Z^{-1.1}$	$T_{\text{eff}} \propto Z^{-0.34}$	$L \propto T_{\text{eff}}^{3.24}$

- For stars of a given mass the luminosity increases with decreasing metals.
- The lifetime of a metal-poor star is shorter than for a metal-rich star: it burns through its fuel faster.
- The opacity decreases with smaller Z and radiation escapes more easily.
- For stars with the same mass but different Z, $L \propto T_{\text{eff}}^{3.24}$, which is less steep than stars for the same Z but different mass, $L \propto T_{\text{eff}}^{4.3}$, and less steep than one for changing mean molecular weight too.
- Stars with few metals, like Population II ones, would have a main sequence **below** the solar abundance one (for low mass).