September 8 . . . .

2.3.2 Mean molecular weight

- Before we start applying this machinery, let's take a brief detour here, since the mean molecular weight μ is important to understand.
- Stellar interiors have a mixture of atoms of different elements and various ionizations.
- Consider the mean mass $\overline{m}$ per particle

$$\overline{m} = \frac{\sum_i n_{i,\mathrm{I}} m_{i,\mathrm{I}} + n_\mathrm{e} m_\mathrm{e}}{\sum_i n_{i,\mathrm{I}} + n_\mathrm{e}} \approx \frac{\sum_i n_{i,\mathrm{I}} m_{i,\mathrm{I}}}{\sum_i n_{i,\mathrm{I}} + n_\mathrm{e}}, \tag{2.14}$$

  where $n_{i,\mathrm{I}}$ is the ion number density of ion i, $m_{i,\mathrm{I}}$ is its mass, and n_e and m_e are the numbers and mass of the electron (and then we ignore the electron mass).
- The mass of the ith ion is approximately its number of protons and neutrons (A_i) times the amu, or $m_{i,\mathrm{I}} = A_i m_\mathrm{u}$.
- So then we define

$$\mu = \frac{\overline{m}}{m_\mathrm{u}} = \frac{\sum_i n_{i,\mathrm{I}} A_i}{\sum_i n_{i,\mathrm{I}} + n_\mathrm{e}}. \tag{2.15}$$

  This can be interpreted as the average mass per particle (ion, electron, etc.) in units of the amu.
- Note that the total particle number density in the gas is

$$n = n_e + n_\mathrm{I} = n_e + \sum_i n_{i,\mathrm{I}} = \sum_i (1 + Z_i) n_{i,\mathrm{I}}, \tag{2.16}$$

  since one ionized atom contributes 1 nucleus plus Z_i electrons. The total $n_\mathrm{e} = \sum_i n_{i,\mathrm{I}} Z_i$.
- In general though, the electron density (or level of ionization) is complicated and derived from the Saha equation.
- But to be more useful, it's easier to express the number densities in terms of mass fractions X_i, where $\sum_i X_i = 1$.
- The number densities we looked at earlier are for some species i are

$$n_i = \frac{\rho}{m_\mathrm{u}} \frac{X_i}{A_i}. \tag{2.17}$$

  Think of this as the mass per unit volume of species i (ρX_i), over the mass of 1 ion of species i ($m_\mathrm{u} A_i$).
- So using this, we now have

$$\mu = \frac{\sum_i \frac{\rho}{m_\mathrm{u}} X_i}{\sum_i \frac{\rho X_i}{m_\mathrm{u} A_i} + n_e}, \tag{2.18}$$

  or

$$\mu = \frac{\sum_i \frac{\rho}{m_\mathrm{u}} X_i}{\sum_i \frac{\rho X_i}{m_\mathrm{u} A_i}(1 + Z_i)}. \tag{2.19}$$

- For example, for a neutral gas we have

$$\mu = \frac{\sum_i X_i}{\sum_i X_i / A_i} = \left(\sum_i \frac{X_i}{A_i} \right)^{-1} \approx \left(X + \frac{Y}{4} + \frac{Z}{\overline{A}_i} \right)^{-1}, \tag{2.20}$$

  where it is standard to write mass fractions X for hydrogen, Y for helium, and Z for everything else (metals), where $X + Y + Z = 1$.

- $\overline{A}_i$ is an average over metals, which at solar composition is about 15.5.
- For a fully ionized gas

$$\mu^{-1} \approx \sum_i \frac{X_i}{A_i}(1+Z_i) \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z, \tag{2.21}$$

or

$$\mu \approx \frac{4}{3+5X-Z}, \tag{2.22}$$

where for metals we usually approximate $(Z_i+1)/A_i \approx 1/2$ (roughly equal number of protons and neutrons). We've eliminated Y in this expression through $Y = 1 - X - Z$.

IN CLASS WORK

Compute the mean molecular weight for (1) the ionized solar photosphere, where we have 90% hydrogen, 9% helium, and 1% heavy elements, (2) the ionized solar interior where 71% hydrogen, 27% helium, and 2% heavy elements, (3) completely ionized hydrogen, (4) completely ionized helium, and finally (5) neutral gas at the solar interior abundance.

Answer: (1) For the photosphere we can write

$$\mu^{-1} = 0.9\frac{2}{1} + 0.09\frac{3}{4} + 0.01\frac{1}{2} = 1.8725,$$

or $\mu \approx 0.53$.
(2) For the interior we can write

$$\mu^{-1} = 0.71\frac{2}{1} + 0.27\frac{3}{4} + 0.02\frac{1}{2} = 1.63,$$

or $\mu \approx 0.61$.
(3) For hydrogen, we will take $X = Z = A = 1$, and find then that $1/\mu = 2$.
(4) For helium, $X = Z = 0$ and $Y = 1$, so $\mu = 4/3$.
(5) For a neutral gas, we have

$$\mu^{-1} = 0.71 + 0.27\frac{1}{4} + 0.02\frac{1}{15.5} = 0.779,$$

or $\mu \approx 1.28$.

- From the above, we can also consider separately the mean molecular weight for ions and electrons.
- For ions, define μ_{I} as

$$n_{\mathrm{I}} = \frac{\rho}{\mu_{\mathrm{I}} m_{\mathrm{u}}}. \tag{2.23}$$

Recall that

$$n_{\mathrm{I}} = \sum_i n_{i,\mathrm{I}} = \frac{\rho}{m_{\mathrm{u}}} \sum_i \frac{X_i}{A_i}. \tag{2.24}$$

So that

$$\mu_{\mathrm{I}} = \left(\sum_i \frac{X_i}{A_i}\right)^{-1}. \tag{2.25}$$

- This result should make sense, since above in Equation (2.20) we did not consider electrons.
- For electrons it's a bit harder since not all electrons need be free. But we will still define the *mean molecular weight per election* μ_{e}:

$$n_{\mathrm{e}} = \frac{\rho}{\mu_{\mathrm{e}} m_{\mathrm{u}}} \tag{2.26}$$

- Fully ionized, each atom contributes Z_i electrons. If an ion is partially ionized, we can consider the fraction $y_i Z_i$. (To compute the proper fraction of ionization of a gas (n_e), one needs to use the *Saha equation*).

- As before then

$$n_e = \sum_i n_{e,i} = \sum_i n_{i,\mathrm{I}} y_i Z_i = \frac{\rho}{m_\mathrm{u}} \sum_i \left(\frac{X_i}{A_i}\right) y_i Z_i, \tag{2.27}$$

which defines

$$\mu_e = \left(\sum_i \frac{X_i y_i Z_i}{A_i}\right)^{-1}. \tag{2.28}$$

- So finally

$$n = n_e + n_\mathrm{I} = \frac{\rho}{\mu m_\mathrm{u}}, \tag{2.29}$$

where

$$\mu = \left(\frac{1}{\mu_\mathrm{I}} + \frac{1}{\mu_e}\right)^{-1}. \tag{2.30}$$

IN CLASS WORK

Compute an expression for μ_e in the deep stellar interior as a function only of X. Ignore metals.

<u>Answer</u>: Fully ionized case. We can write

$$\begin{aligned}
\mu_\mathrm{e} &\approx \left(\frac{1}{1}X + \frac{2}{4}Y\right)^{-1} \\
&= \left(X + \frac{1}{2}(1-X)\right)^{-1} \\
&= \left(\frac{X+1}{2}\right)^{-1} = \frac{2}{1+X}.
\end{aligned}$$

This should make sense. For a full H gas, the mean mass of particles per number of electrons (1/1) is 1. For a He gas ($X = 0$), we have a mass of 4 divided by 2 electrons, or $\mu_e = 2$.

2.3.3 Ideal monatomic gas

- As a first demonstration, we consider a gas of single species nonrelativistic particles. We will be using Equation (2.8).

- Their energy is $E = p^2/2m$. Consider one energy level $E_j = E_0$.

- For this system, the chemical potential goes to negative infinity, so the exponential term is large, and the ± 1 term can be safely ignored.

- Thus,

$$n(p) = \frac{g}{h^3} e^{-p^2/2mkT} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}, \tag{2.31}$$

and so the number density is

$$n = \frac{4\pi g}{h^3} \int_0^\infty p^2 e^{-p^2/2mkT} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}\, \mathrm{d}p. \tag{2.32}$$

- This gives an expression we will use

$$n = \frac{(2\pi m k_\mathrm{B} T)^{3/2} g}{h^3} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}. \tag{2.33}$$

(Note that this also defines μ_c in terms of density.)

- Returning to the definition of gas pressure in Equation (2.12), we can compute the integral to find

$$P = g \frac{4\pi}{h^3} \frac{\pi^{1/2}}{8m} (2 m k_\mathrm{B} T)^{5/2} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}. \tag{2.34}$$

- Using the generalized number density from Equation (2.33), this gives what you thought it would

$$P = n k_\mathrm{B} T \ [\mathrm{dyne\,cm^{-2}}]. \tag{2.35}$$

This is the equation of state for an ideal gas.