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- Similarly we can compute the internal energy density from Equation (2.13)

$$u = \frac{3}{2} n k_{\mathrm{B}} T = \frac{3}{2} P; \ [\mathrm{erg\,cm^{-3}}]. \tag{2.36}$$

- Note that the units of pressure and internal energy density are the same.
- From what we saw before with the mean molecular weight, we can also express these quantities as

$$n = \frac{\rho}{\mu m_{\mathrm{u}}}, \tag{2.37}$$

$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}}, \tag{2.38}$$

$$P = \frac{\rho R T}{\mu}, \tag{2.39}$$

where μ is the mean molecular weight, and $R = k_{\mathrm{B}}/m_{\mathrm{u}}$ is the ideal gas constant $R = 8.31 \times 10^7\,\mathrm{erg\,K^{-1}\,mol^{-1}}$.

2.3.4 Completely degenerate gas

- The ideal gas law (because of Maxwell-Boltzmann statistics) breaks down at sufficiently high densities and/or low temperatures.
- Consider the extreme case where $T \to 0$ at fixed density.
- From Equation (2.63), the Maxwell distribution peaks at zero momentum where all the particles want to pile up. They want to be in the lowest energy state, which is zero.
- There's a limit to how close fermions can come, based on the Pauli exclusion principle.
- So instead, we must use Fermi-Dirac statistics instead of Maxwell-Boltzmann
- Consider first the most interesting terms in Equation (2.8)

$$f(p) = \frac{1}{\mathrm{e}^{(E(p)-\mu_c)/kT} + 1}, \tag{2.40}$$

where we've taken a reference energy level $E_j = 0$.

- As $T \to 0$, f goes to 1 or 0 depending on the sign of $E - \mu_{\mathrm{c}}$.
- The function is discontinuous at the Fermi momentum p_{F}, or at energy E_{F}.
- For fermions such as electrons, with spin 1/2, the degeneracy factor in Equation (2.8) $g = 2$.
- The chemical potential is the Fermi energy $\mu_{\mathrm{c}} = E_{\mathrm{F}}$, up to which all the quantum states are filled. This is what is meant by "degeneracy."
- It is convenient to introduce the dimensionless momentum $x = p/mc$ and Fermi momentum $x_{\mathrm{F}} = p_{\mathrm{F}}/mc$.
- The integration to obtain the number density of electrons is therefore

$$n_e = \frac{8\pi}{h^3} \int_0^{p_{\mathrm{F}}} p^2\,\mathrm{d}p = 8\pi \left(\frac{h}{m_e c}\right)^{-3} \int_0^{x_{\mathrm{F}}} x^2\,\mathrm{d}x = \frac{8\pi}{3} \left(\frac{h}{m_e c}\right)^{-3} x_{\mathrm{F}}^3 = 5.865 \times 10^{29}\, x_{\mathrm{F}}^3\ \mathrm{cm^{-3}}. \tag{2.41}$$

Note that $p_{\mathrm{F}} \sim n_e^{1/3}$.

- If we reintroduce the electron mean molecular weight, as in Equation (2.26), we can get this in terms of mass density
$$\frac{\rho}{\mu_e} = \frac{8\pi m_u}{3}\left(\frac{h}{m_e c}\right)^{-3} x_F^3 = 9.74 \times 10^5\, x_F^3 \text{ g cm}^{-3}. \tag{2.42}$$
This is interesting in that it is a way to determine the Fermi momentum or energy if provided a value for ρ/μ_e.

- The electron pressure (the pressure due to degenerate electrons, not ions) from Equation (2.12) is therefore
$$P_e = \frac{8\pi}{3}\frac{m_e^4 c^5}{h^3}\int_0^{x_F} \frac{x^4}{(1+x^2)^{1/2}}\mathrm{d}x = Cf(x), \tag{2.43}$$
where $C = \pi m_e^4 c^5/3h^3 = 6.002 \times 10^{22}\,\text{dyne}\,\text{cm}^{-2}$.

- The function $f(x)$ is
$$f(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1} x. \tag{2.44}$$

- Similarly, the internal energy density from Equation (2.13) is
$$u_e = 8\pi\frac{m_e^4 c^5}{h^3}\int_0^{x_F} x^2\left[(1+x^2)^{1/2} - 1\right]\mathrm{d}x = Cg(x), \tag{2.45}$$
where C is the same as before.

- The function $g(x)$ is
$$g(x) = 8x^3\left[(1+x^2)^{1/2} - 1\right] - f(x). \tag{2.46}$$

- These are very general expressions in terms of x for the pressure and energy density.

IN CLASS WORK

Show that x discriminates between the nonrelativistic regime $x \ll 1$ and the relativistic regime $x \gg 1$. Use Equation (2.10) and Equation (2.11).

Answer: Since $x = p/m_e c$, we need to know what the momentum is. It's not simply $p = mv$. We can compute the velocity and then the momentum.
$$v = \frac{\partial E}{\partial p} = \frac{p/m}{\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}}$$
Solving for p gives
$$p = \frac{mv}{\sqrt{1 - v^2/c^2}},$$
which is expected. Therefore,
$$x = \frac{v/c}{\sqrt{1 - v^2/c^2}}.$$
Another useful form is to solve for the velocity ratio in terms of x:
$$\frac{v^2}{c^2} = \frac{x^2}{1+x^2}.$$
In any case, for electron velocities near the speed of light, x becomes very large.

- Let us first consider the case for **nonrelativistic electrons** where $x \ll 1$. In this limit, to first order

$$\begin{aligned} f(x) &\approx \frac{8}{5}x^5, \\ g(x) &\approx \frac{12}{5}x^5. \end{aligned}$$

- Using Equation (2.43) we thus have

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3}x^5. \tag{2.47}$$

- Relating this through the density in Equation (2.41) to remove x, and then using the electron mean molecular weight in Equation (2.26), we arrive at the final expression

$$P_\mathrm{e} = 1.0036 \times 10^{13} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3} \mathrm{dyne\,cm^{-2}}. \tag{2.48}$$

  This is the equation of state for a fully degenerate nonrelativistic electron gas.

- Carrying out the same exercise for the internal energy, we find

$$u_\mathrm{e} = \frac{3}{2}P_\mathrm{e}. \tag{2.49}$$

- Thus, the equation of state for this nonrelativistic gas has characteristics of an ideal monatomic gas (see Equation (2.36)).

- Let us now consider the case for **relativistic electrons** where $x \gg 1$. In this limit, to first order

$$\begin{aligned} f(x) &\approx 2x^4, \\ g(x) &\approx 6x^4. \end{aligned}$$

- Now, the pressure is

$$P_\mathrm{e} = \frac{2\pi m_e^4 c^5}{3h^3}x^4, \tag{2.50}$$

  which after plugging in constants and introducing n_e and ρ gives

$$P_\mathrm{e} = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} \mathrm{dyne\,cm^{-2}}. \tag{2.51}$$

- Similarly for the energy we have

$$u_\mathrm{e} = 3P_\mathrm{e}. \tag{2.52}$$

- The transition from non- to relativistic states is smooth in x. Note the exponents on the density, which we will come back to these later (polytropes).

PROBLEM 2.3: Find the ratio of the electron degeneracy pressure to the electron ideal gas pressure in the center of the (current) Sun (assuming the center could be degenerate in electrons). Let $T = 15 \times 10^6$ K, $\rho = 150\,\mathrm{g\,cm^{-3}}$, and abundances $X = 0.35$, $Z = 0.02$. Prove that you are using the correct expression for the electron degeneracy pressure.

2.3.5 Partially degenerate gas

- The previous section considered an ideal zero-temperature gas. But if the temperature is finite, then the Fermi-Dirac function is not a simple step function and needs to be avaluated numerically.
- When this is done, the temperature dependence of the equation of state is realized.
- The typical expressions are just expansions of the Fermi function in powers of T.
- We will not spend more time on this right now.