Unit 3

Energy Transport

Energy liberated in stellar interiors is transferred to the surface by radiation, convection, and conduction.

3.1 Radiation

We are not considering here radiation from a stellar photosphere, only the movement of photons in stellar interiors.

3.1.1 Basics

- The basic idea is that photons emitted in hot regions of a star are absorbed in cooler regions of a star, thus "transferring" energy from hot to cool.

- The "efficiency" of this transfer will depend on the temperature gradient. A very rough approximation of the gradient for the Sun is $T_c/R_\odot \approx 10^{-4}\,\mathrm{K\,cm^{-1}}$.

- The efficiency will also depend on the ability of the photons to travel freely.

- Let's look at that last point. The mean free path of photons can be approximated now as

$$\ell_{\mathrm{ph}} = \frac{1}{\kappa\rho}, \tag{3.1}$$

  where κ is some absorption coefficient (in units of cross section per unit mass) that will be given a physical meaning later

- Typical interior values are $\kappa \approx 1\,\mathrm{cm^2\,g^{-1}}$. A rough mean density is $\rho \approx 1.4\,\mathrm{g\,cm^{-3}}$ (like in the Sun).

- This gives a mean free path of $\ell_{\mathrm{ph}} \approx 1$ cm! Stellar interiors are very opaque.

- Nonetheless, radiative transport occurs by the non-vanishing net flux outward, due to the hotter material below which sets up the gradient.

- Because of the small path, transport can be treated as a **diffusion process** in the interior. (Near the surface, however, this simplification starts to break down).

3.1.2 Diffusion

- Quick and dirty derivation of Fick's Law of diffusion, just to get the point across.
- Consider **particles** diffusing (randomly) in 3D space at some boundary r.
- Let n be the particle number density, $\overline{v}$ be the mean velocity, and ℓ the mean free path, such that $\ell = 1/\sigma n$, with σ the cross section.
- Consider isotropy. Then about 1/3 of the particles will be moving in the $\hat{r}$ direction. About 1/2 of those will be moving in the $-\hat{r}$ direction
- Flux is a quantity (like number of particles or energy) per unit area per unit time.
- From one direction, the particle flux is

$$F_+ = \frac{1}{6} n_{r-\ell} \overline{v}_{r-\ell} \tag{3.2}$$

- From the other direction

$$F_- = \frac{1}{6} n_{r+\ell} \overline{v}_{r+\ell} \tag{3.3}$$

- Net flux

$$F = F_+ - F_- = \frac{1}{6} \overline{v} (n_{r-\ell} - n_{r+\ell}), \tag{3.4}$$

  assuming that $v_{r-\ell} \approx v_{r+\ell} = \overline{v}$.

- If the mean free path does not change on the scale of the density gradient, then

$$\begin{aligned} F &= \frac{1}{6} \overline{v} \left[n_{r-\ell} - n_r - (n_{r+\ell} - n_r) \right] \\ &= \frac{1}{6} \overline{v} \left[-\ell \frac{\mathrm{d}n}{\mathrm{d}r} - \ell \frac{\mathrm{d}n}{\mathrm{d}r} \right] \\ F &= -D \, \nabla_r n, \end{aligned}$$

  where the diffusion coefficient $D = 1/3 \, \overline{v} \, \ell$. This is Fick's Law. Again, if ℓ is large, this fails.

- This is generic. On the left you have a flux (in this case of number of particles) and on the right a gradient of density (in this case number density of particles). Note that the flux is carried from a high concentration to a low concentration of particles.
- But we want to compute the flux of diffusing radiative energy. So we need an energy density.
- For photons, we can just let $\overline{v} = c$, $\ell = \ell_{\mathrm{ph}} = 1/\kappa\rho$, and $n = u$. See Equation (2.58) and note that $u = 3P$ for a relativistic system, as derived previously, which gives

$$u = aT^4. \tag{3.5}$$

- So then the radiative flux F_{rad} is

$$F_{\mathrm{rad}} = -\frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{\mathrm{d}T}{\mathrm{d}r}. \tag{3.6}$$

- The local luminosity at any point passing through a sphere of radius r is $L(r) = 4\pi r^2 F_{\mathrm{rad}}$, so then rearranging we have

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac} \frac{\kappa\rho}{r^2} \frac{L}{T^3}. \tag{3.7}$$

- This is a fundamental equation of stellar structure.

3.1.3 Frequency dependence of radiation

- What we just did was too simple, even in the diffusion approximation. Our answer is in fact integrated over all photon energies.
- In principle, there is a frequency dependence on the flux F_ν since the energy density and the opacity are partitioned in frequency.
- Let us go back to Equation (3.5) and instead consider

$$u_\nu = \frac{4\pi}{c} B_\nu(T), \tag{3.8}$$

where B is the Planck function for a blackbody radiator

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/k_\mathrm{B}T} - 1}. \tag{3.9}$$

This is just from our Bose-Einstein distribution function, Equation (2.56), written in terms of frequency instead of momentum.

- Also keep in mind that the integrated Planck function

$$B(T) = \int_0^\infty B_\nu(T)\, \mathrm{d}\nu = \frac{ac}{4\pi} T^4. \tag{3.10}$$

- Fick's Law now becomes

$$F_\nu = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{\mathrm{d}B_\nu}{\mathrm{d}r} = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{\mathrm{d}B_\nu}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}r}. \tag{3.11}$$

- The total flux integrated over all frequencies is then

$$F_\mathrm{rad} = \int F_\nu\, \mathrm{d}\nu = -\frac{4\pi}{3} \frac{1}{\rho} \frac{\mathrm{d}T}{\mathrm{d}r} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\, \mathrm{d}\nu. \tag{3.12}$$

- Comparing Equation (3.12) with Equation (3.6), we see that the κ in the latter is

$$\frac{1}{\kappa} = \frac{\pi}{ac} \frac{1}{T^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\, \mathrm{d}\nu. \tag{3.13}$$

- But since

$$\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\, \mathrm{d}\nu = \frac{\mathrm{d}}{\mathrm{d}T} \int_0^\infty B_\nu \mathrm{d}\nu = \frac{\mathrm{d}B}{\mathrm{d}T} = \frac{ac}{\pi} T^3, \tag{3.14}$$

where $B = acT^4/4\pi$ (the integral over all frequencies), we can then define

$$\frac{1}{\kappa_\mathrm{R}} \equiv \frac{1}{\kappa} = \left(\int_0^\infty \frac{1}{\kappa_\nu} \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\, \mathrm{d}\nu \right) \left(\int_0^\infty \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\, \mathrm{d}\nu \right)^{-1}, \tag{3.15}$$

where κ_R is the *Rosseland mean opacity.*

- All this implies is that Equations (3.6) and (3.7) should replace the opacity by the Rosseland mean opacity:

$$F_\mathrm{rad} = -\frac{4ac}{3} \frac{T^3}{\kappa_\mathrm{R} \rho} \frac{\mathrm{d}T}{\mathrm{d}r}, \tag{3.16}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi ac} \frac{\kappa_\mathrm{R} \rho}{r^2} \frac{L}{T^3}. \tag{3.17}$$

- Note that this weighted opacity gives high frequencies more weight than lower ones (as one could find by differentiating).
- Before we go onto using these expressions to understand stellar structure, let's look at a few of the major sources of κ_R.

3.1.4 Opacity sources

- Computing opacities is hard stuff, only a few groups in the world have succeeded (LANL, LLNL). These computations require full quantum-mechanical treatments.
- In general, opacities can be approximated as power laws of the form
$$\kappa = \kappa_0 \rho^n T^{-s} \ \ [\mathrm{cm}^2\,\mathrm{g}^{-1}], \tag{3.18}$$
similar to how we treated energy generation rate and its dependence on these parameters.
- The goal is to compute *total* Rosseland mean opacities from all possible sources, including the following four:

1. Scattering of photons from electrons
   - For relativistic free electrons, this is known as Compton scattering, but for conditions inside most stars, this is Thomspon scattering.
   - For temperatures below about 1 billion Kelvin (thermal energies below electron rest mass energy), the cross section for scattering off electrons is frequency independent.
   - One finds for the opacity
   $$\kappa_\mathrm{e} = \frac{n_\mathrm{e}\sigma_\mathrm{e}}{\rho} \qquad \mathrm{cm}^2\,\mathrm{g}^{-1}, \tag{3.19}$$
   where the cross section is given in the Thompson scattering prescription
   $$\sigma_\mathrm{e} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.6652 \times 10^{-24}\,\mathrm{cm}^2. \tag{3.20}$$
   (Note the classical electron radius buried in there).
   - Scattering off electrons will only occur in a highly ionized gas where a sufficient number of electrons are present. Looking back at Equation (2.26) and considering a gas with inconsequential metals gives
   $$\kappa_e \simeq 0.2(1+X). \tag{3.21}$$
   - If there are many metals or ionization is incomplete then the electron densities have to be computed more carefully.
   - Below 10,000 K hydrogen is not ionized at low pressure and so this opacity does not contribute to the Rosseland mean.

PROBLEM 3.1: [5 pts]: Derive Equation (3.21) in the full ionization case.

2. Free-free absorption
   - A single free electron cannot absorb a photon while still conserving energy and momentum.
   - If an ion is nearby, this absorption is possible however.
   - In this case, it can be shown that the opacity
   $$\kappa_{\mathrm{f-f}} \approx 10^{23}\frac{Z_c^2}{\mu_\mathrm{e}\mu_\mathrm{I}}\rho T^{-3.5}, \tag{3.22}$$
   where Z_c is some average nuclear charge. Typically a quantum-mechanical gaunt factor needs to be included in the coefficient.
   - Opacities of this form that scale as $\kappa \sim \rho T^{-3.5}$, or $n = 1$ and $s = 3.5$, are known as Kramers opacities.
   - The inverse of this process, when an ion changes the momentum of an electron which then emits a photon, is known as Bremsstrahlung emission
   - Note that for reference, in the core of the Sun the product $\rho T^{-3.5} \approx 5 \times 10^{-23}$. So these opacities are of order 1 to 10 or so.