October 15 . . . .

3.3.2 Another useful formulation

- Convection can also be understood in terms of entropy.
- For reversible processes, $\mathrm{d}Q = T\mathrm{d}S$ and so

$$T\,\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V. \tag{3.56}$$

- Using standard thermodynamic relations, it can be shown from here that

$$\frac{\mathrm{d}S}{\mathrm{d}r} = c_P(\nabla - \nabla_{\mathrm{ad}})\frac{\mathrm{d}\ln P}{\mathrm{d}r}. \tag{3.57}$$

- So if the star is radiative, $\mathrm{d}S/\mathrm{d}r > 0$ and the entropy increases outward.
- If the star is convective, $\mathrm{d}S/\mathrm{d}r < 0$. If the convection is efficient the gradient is very close to being adiabatic, meaning the entropy is very nearly constant throughout convection zones.

3.3.3 Semiconvection

- Let's return to the Brunt-Väisälä frequency again from Equation (3.49):

$$N^2 = g\left(\frac{1}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right).$$

- We want to rewrite this in a very convenient form, and this time we **will** take into account composition gradients in the gas to be as general as possible. The form is

$$N^2 = \frac{g^2\rho}{P}(\nabla_{\mathrm{ad}} - \nabla + \nabla_\mu), \tag{3.58}$$

where (as some have been defined before)

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}, \qquad \nabla_{\mathrm{ad}} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}}, \qquad \nabla_\mu = \frac{\mathrm{d}\ln \mu}{\mathrm{d}\ln P}. \tag{3.59}$$

> **PROBLEM 3.2:** [5 pts]: Show how you can get from Equation (3.49) to Equation (3.58).

- Ignore the composition gradient for a second. We recover the "standard" stability relation: if the temperature gradient is larger than the adiabatic one (Schwarzchild), the BV frequency becomes complex.
- If it's the reverse, the BV is positive and the medium is stable to convection.
- Now however, we have the possibility that the Schwarzchild criterion is satisfied ($\nabla > \nabla_{\mathrm{ad}}$), yet the medium remains stable because the composition gradient makes it positive again.
- This is the *Ledoux criterion*, and when this is the case we have weak convection, or **semiconvection**.
- This typically would not occur in a convection zone, why? Because convection mixes material and composition gradients are removed.
- But in areas of nuclear burning where gradients do exist, and at the "edges" of convection zones, this situation can arise. Large peaks in the μ-gradient (also caused by g increases) can cause large jumps in N.
- Becomes important for red-giant stars and their gravity modes mixing (boosting frequency) with acoustic modes.

3.3.4 One more useful formulation

- Start with Equation (3.53) and rewrite using ideal gas law and hydrostatic equilibrium:

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = \left(1-\frac{1}{\gamma}\right)\frac{\mu}{\rho R_g}\frac{\mathrm{d}P}{\mathrm{d}r} = -\left(1-\frac{1}{\gamma}\right)\frac{g\mu}{R_g}. \tag{3.60}$$

- Remembering that $\gamma = c_P/c_V$ and $c_P - c_V = R_g/\mu$,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\left(\frac{c_P/c_V-1}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{3.61}$$

$$= -\left(\frac{\frac{c_P-c_V}{c_V}}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{3.62}$$

$$= -\left(\frac{c_P-c_V}{c_P}\right)\frac{g\mu}{R_g}, \tag{3.63}$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\frac{g}{c_P}. \tag{3.64}$$

- This form shows how the parcel of gas is changing as it rises adiabatically and expands
- This can also be derived using energy by considering energy release and work against gravity

3.3.5 Physical conditions for convection onset

- Where in various types of stars does convection occur?
- Consider first energy transport by radiation. Recall Equation (3.33) which is reproduced here:

$$\nabla_{\mathrm{rad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{P\kappa}{T^4}\frac{L}{m},$$

- Another convenient form from substituting an equation of state is

$$\nabla_{\mathrm{rad}} \equiv \frac{3k_{\mathrm{B}}}{16\pi acGm_{\mathrm{u}}}\frac{\kappa}{\mu}\frac{L}{m}\frac{\rho}{T^3}. \tag{3.65}$$

- These are the gradients required to transport all the luminosity (L) by radiation. Keep in mind that $L = L(r)$ and $m = m(r)$.
- The radiative gradient ∇_{rad} is a spatial derivative connecting P and T is consecutive mass shells.
- The adiabtic gradient ∇_{ad} describes the thermal variation of one mass element upon an adiabatic compression.
- Anyway, for instability we can now write

$$\nabla = \nabla_{\mathrm{rad}} > \nabla_{\mathrm{ad}}, \tag{3.66}$$

which, if satisfied, allows for convection to take place.

- Examining this condition, we see that convection may occur if
  - L/m is large. Thinks of this as $L/m \sim \mathrm{d}L/\mathrm{d}m = \varepsilon$ at small m (near the core). The energy generation rate is huge in massive stars in the cores, so many massive stars have convective cores.

  - κ is large. This is satisfied in the outer parts of less-massive stars, or with low surface temperatures and ionization zones of hydrogen (Sun).
  - ρ/T^3 is large. Happens typically in the outer parts of relatively cool stars. In fact, this ratio increases rapidly as the effective temperatures go down.
  - $\nabla_{\rm ad} = 1 - 1/\gamma$ is small. Satisfied in the ionization zone of hydrogen, in outer parts of cool stars where γ gets small (increased specific heat).

- $n = 1.5$ polytropes are good models for convective regions, as this corresponds to $\gamma = 5/3$.

3.3.6 Mixing length theory

- How does the convection actually transfer the energy?
- We have previously shown how convection can take place by considering a blob displaced from its equilibrium position.
- This blob rises with a velocity up to a point where a new hydrodynamic instability sets in, whereby the motion becomes turbulent and the blob dissolves, depositing its heat in the surroundings.
- The exact details of this process are still unknown and are a subject of much research. For solar convection, the most sophisticated numerical simulations carried out on the fastest computers take an order of magnitude more computation time than the solar "time" they are trying to simulate, and that is just for a small section of the Sun.
- In comparison, computing the solar model over the Sun's lifetime, with a simple model of convection, takes a couple minutes.
- Remember that the radiative flux $L/4\pi r^2$ is

$$F_{\rm rad} = \frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{{\rm d}T}{{\rm d}r} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla,$$

  using hydrostatic equilibrium. ∇ is the actual stratification.
- Neither $F_{\rm rad}$ or ∇ are really known, since radiation only carries some of the star's flux.
- But we did note that the temperature gradient required to carry **all** of the stellar luminosity is $\nabla_{\rm rad}$. If this includes convection, the total flux then is

$$F_{\rm tot} = F_{\rm rad} + F_{\rm conv} = \frac{4acG}{3}\frac{mT^4}{\kappa P r^2}\nabla_{\rm rad}. \tag{3.67}$$

- A very reasonable convective flux for a parcel in pressure equilibrium is its heat content multiplied by the mass flux:

$$F_{\rm conv} = \rho v c_P \Delta T. \tag{3.68}$$

- All that's left to do is find ρ, v and ΔT of the blob.
- $\Delta T = T_i - T$ is the excess heat of a rising parcel of gas with respect to its surroundings.
- Assume a parcel moves a distance ℓ before dissolving into the background material, so that at any given time, a typical parcel will have moved $\ell/2$. The temperature difference between the parcel and the gas is

$$\Delta T = T_i - T = \left(\frac{{\rm d}T_i}{{\rm d}r} - \frac{{\rm d}T}{{\rm d}r}\right)\frac{\ell}{2}. \tag{3.69}$$

- Multiply by the pressure scale height $H_P = -\left(\mathrm{d}\ln P/\mathrm{d}r\right)^{-1}$ and divide by T

$$\frac{\Delta T}{T} = (\nabla - \nabla_i)\frac{\ell}{2H_P}. \tag{3.70}$$

- If the parcel remains in pressure equilibrium, assuming a general equation of state, then changes in its density are

$$\frac{\mathrm{d}\rho}{\rho} = -\delta\frac{\mathrm{d}T}{T} = -\delta(\nabla - \nabla_i)\frac{\ell}{2H_p}, \tag{3.71}$$

where

$$\delta = -\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln T}. \tag{3.72}$$

- Since the density difference implies a buoyancy force $-g(\rho_i - \rho)$, there is work that goes into moving the parcel

$$-g(\rho_i - \rho)\frac{\ell}{2} = g\rho\delta(\nabla - \nabla_i)\frac{\ell^2}{4H_p}. \tag{3.73}$$

- Suppose half of the work goes into the kinetic energy of the parcel, and the other half goes into the surroundings, moving it aside. The velocity is then

$$v^2 = \frac{\ell^2}{4}\frac{g\delta(\nabla - \nabla_i)}{H_p}. \tag{3.74}$$

- We get for the convective flux then that

$$F_{\mathrm{conv}} = \rho c_p T (g\delta)^{1/2}\frac{\ell^2}{4}H_p^{-3/2}(\nabla - \nabla_i)^{3/2}. \tag{3.75}$$

- It is possible to compute ∇_i, because as the parcel moves its internal energy will change because of radiative losses and adiabatic expansion/contraction. One finds

$$\frac{\nabla_i - \nabla_{\mathrm{ad}}}{\nabla - \nabla_i} = \frac{6acT^3}{\kappa\rho^2 c_p v\ell}. \tag{3.76}$$

- The overal problem can now be solved. Five new equations for the 5 unknowns F_{rad}, F_{conv}, v, ∇, and ∇_i. The *mixing-length parameter* $\alpha = \ell/H_P$ is used as a free parameter, chosen to match observations.
- Solutions can be found in the literature. Limiting cases are interesting. In dense regions of a star, $\nabla \to \nabla_{\mathrm{ad}}$. A gradient just over the adiabatic limit is all that is necessary to transport all of the luminosity.
- Near stellar photospheres, $\nabla \to \nabla_{\mathrm{rad}}$, and convection is ineffective so $F \to F_{\mathrm{rad}}$.
- In between these limits, the mixing-length equations need to be solved.
- Near the surface, convection does not transport energy efficiently, since convective flux is proportional to the the density which is quite low.
- It is also proportional to the mixing length which has to be small. So the only way for convection to dominate is if δ can get large. This means it has to be very far from adiabatic.
- We know this because we see *granulation*, which typifies the size of convective elements, about $10^{-3}R_\odot$.
- The convective time scale

$$t_{\mathrm{conv}} \simeq \left[\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right]^{-1/2} t_{\mathrm{dyn}}. \tag{3.77}$$

- This timescale is a dynamical time scale, but since the gravitational acceleration is reduced because of the difference in density that comes into all of this, the timescale is increased by that pre-factor.
- That prefactor in the Sun is of order 10^{-6}, and we find convective time scales $t_{\mathrm{conv}} \simeq 0.2$ yr.
- This is still small compared to evolution. Matter mixes over such a time scale in the convection zone.

3.3.7 Convective overshoot

- We briefy mention this here because it has very important implications for stellar modeling.
- The parcels we consider unstable to convection will eventually reach a stable layer.
- However the momentum causes them to overshoot the boundary into this stable layer.
- For example, this is why we see granulation in the solar photosphere (which is stable).
- One important consequence of this overshoot is the deposition of convectively mixed material into stable regions, which can affect the thermal structure as well as evolutionary properties.

3.3.8 Depth of outer convection zones

- We've seen where convection tends to set in, but how large are these regions and what do they depend on?
- To first order the depth depends on $T_{\rm eff}$, since this determines where ionization layers are.
- Stars with lower surface temperatures achieve H ionzization at deeper depths with higher pressure.
- So for cool main-sequence stars, outer convection zones can extend deep, even to the center.
- For hotter stars, up to about 8000K, ionization occurs higher and higher in the atmosphere, and the convective shells of these stars can be very thin.
- The depth can also depend on **chemical abundances**.
- Consider a star with relatively high He abundance.
- The larger mean molecular weight corresponds to lower pressures at a given layer.
- He ionization, which can cause a convective instability, thus happens at a deeper layer, or higher temperature.
- Convection zones will be deeper for He-rich stars.
- For metal-poor stars, the opacity is reduced and therefore the radiative temperature gradient gets reduced, reducing the convective instability.
- Metal-poor stars will have shallower convection zones than metal-rich ones.