

September 17.....

2.3.10 Mixture of ideal gas and radiation: pressure effects

- It should be noted that we have been considering an ideal gas made up of particles only. When radiation is present along with the gas in thermodynamic equilibrium, the photons can cause two changes: (1) a radiation pressure; and (2) ionization effects (see Sec. 2.3.11). In this case, the adiabatic exponents in Eqs. (2.78-2.80) are not constant anymore, nor are they all equal.
- Considering this mixture, the total pressure is

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{\rho k_{\text{B}} T}{\mu m_{\text{u}}} + \frac{1}{3} a T^4, \quad (2.81)$$

and specific internal energy density

$$U = \frac{3}{2} \frac{k_{\text{B}} T}{\mu m_{\text{u}}} + a T^4 V. \quad (2.82)$$

- Since the specific energy depends on volume and temperature $U(T, V)$, quasistatic changes to it in the first law of thermodynamics yields

$$dQ = \left(\frac{\partial U}{\partial T} \right)_V dT + \left(\frac{\partial U}{\partial V} \right)_T dV + P dV. \quad (2.83)$$

- Using the expression for the specific energy and pressure then gives

$$dQ = \left(4aT^3 V + \frac{3}{2} \frac{k_{\text{B}}}{\mu m_{\text{u}}} \right) dT + \left(\frac{4}{3} a T^4 + \frac{k_{\text{B}} T}{\mu m_{\text{u}} V} \right) dV. \quad (2.84)$$

- For an adiabatic process, this equation gives the thermodynamic response to changes in temperature and volume. It can then be nicely rewritten

$$\left(12P_{\text{rad}} + \frac{3}{2} P_{\text{gas}} \right) \frac{dT}{T} + (4P_{\text{rad}} + P_{\text{gas}}) \frac{dV}{V} = 0. \quad (2.85)$$

- To evaluate expressions as in Equations (2.78)-(2.80), it's also useful to have a pressure differential term. Using Equation (2.81) we can write

$$\begin{aligned} dP &= \left(\frac{4}{3} a T^4 + \frac{RT}{\mu V} \right) \frac{dT}{T} - \frac{R}{\mu} \frac{T}{V} \frac{dV}{V}, \\ &= (4P_{\text{rad}} + P_{\text{gas}}) \frac{dT}{T} - P_{\text{gas}} \frac{dV}{V}. \end{aligned} \quad (2.86)$$

- Plugging this into Equation (2.78) gives

$$(4P_{\text{rad}} + P_{\text{gas}}) \frac{dT}{T} + [\Gamma_1 (P_{\text{rad}} + P_{\text{gas}}) - P_{\text{gas}}] \frac{dV}{V} = 0. \quad (2.87)$$

- Comparing Equation (2.85) and Equation (2.87) allows us to solve for Γ_1 . It simplifies things to consider the fractional gas pressure

$$\beta \equiv \frac{P_{\text{gas}}}{P_{\text{gas}} + P_{\text{rad}}}. \quad (2.88)$$

- You can then show that

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}. \quad (2.89)$$

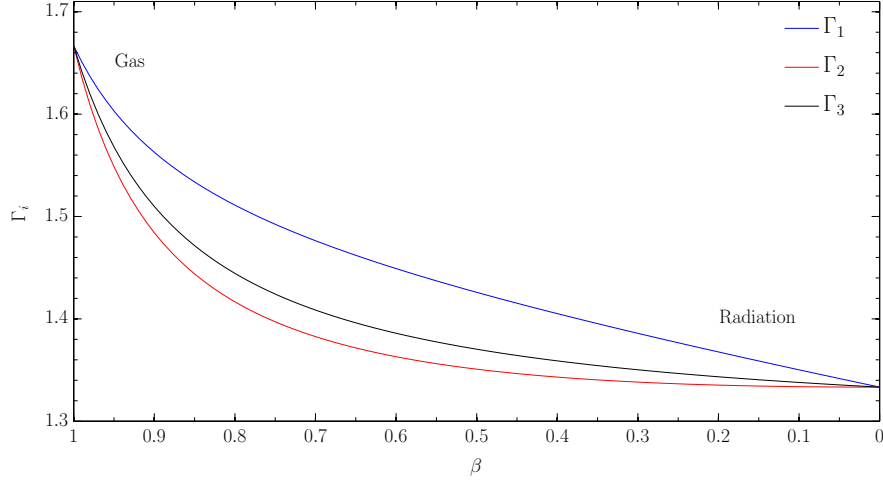


Figure 2.2: The various adiabatic exponents for different mixtures of ideal gas particles and photons.

- For a gas of particles, $\beta = 1$, and therefore $\Gamma_1 = 5/3 = \gamma$, which is what we already found for an ideal gas. For a photon gas, $\beta = 0$ and $\Gamma_1 = 4/3$.
- In a similar fashion,

$$\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}, \quad (2.90)$$

$$\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta}. \quad (2.91)$$

See Figure 2.2 for the dependence of these on β .

- Using the equations we just developed, the specific heats can also be computed:

$$c_V = \left(\frac{dQ}{dT} \right)_V = c_V^0 \frac{8 - 7\beta}{\beta}, \quad (2.92)$$

$$c_P = \left(\frac{dQ}{dT} \right)_P = c_V^0 \frac{32/3 - 8\beta - \beta^2}{\beta^2}, \quad (2.93)$$

$$(2.94)$$

where $c_V^0 = 3/2R/\mu$ is the ideal gas-only value.

- Note that the ratio of specific heats gives

$$\frac{c_P}{c_V} = \frac{\Gamma_1}{\beta}, \quad (2.95)$$

which makes sense in the appropriate limits, reducing to what we found before.

- The same procedure can be carried out for mixtures of some degenerate gas too.

2.3.11 Mixture of ideal gas and radiation: ionization effects

- As mentioned at the beginning of this section, the other consideration is the ionization of the gas that radiation produces, which has profound effects on the thermodynamic state of the gas.
- Let's just consider a hydrogen gas for simplicity in what follows.

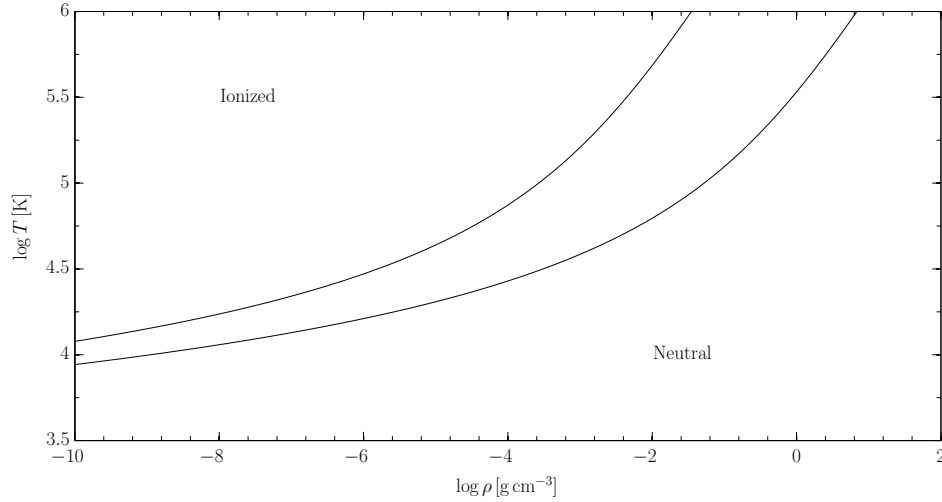


Figure 2.3: Ionization of a pure H gas using Equation (2.99). The lower line represents the state at 50% ionization, while the upper is for 99%.

- In general radiation causes ionization and subsequent recombination:



where $\chi_{\text{H}} = 13.6 \text{ eV}$ is the energy needed to ionize hydrogen. We will only consider the ground state.

- To measure the number densities of electrons (n_e), ions (n^+), and neutral H (n^0) in thermodynamic equilibrium, we employ the tools we used in Sec. 2.3.1 using a Boltzmann distribution.
- After taking into account the appropriate degeneracy factors and energy levels and chemical potentials, we can form the ratio n^+n_e/n^0 and obtain the *Saha equation* for a pure hydrogen gas

$$\frac{n^+n_e}{n^0} = \left(\frac{2\pi m_e k_B}{h^2} \right)^{3/2} T^{3/2} e^{-\chi_{\text{H}}/kT}. \quad (2.97)$$

- We constrain the system to have charge neutrality, $n_e = n^+$ and nucleon number density $n^+ + n^0 = n$. Then we define the fraction of ionization

$$y = \frac{n_e}{n} = \frac{n^+}{n}, \quad (2.98)$$

as we did in Equation (2.28).

- Then the Saha equation is

$$\frac{y^2}{1-y} = \frac{1}{n} \left(\frac{2\pi m_e k_B}{h^2} \right)^{3/2} T^{3/2} e^{-\chi_{\text{H}}/kT}. \quad (2.99)$$

- We already see that at high temperatures we expect either collisions or a strong radiation field to ionize the gas $y \rightarrow 1$.
- We recognize for a pure hydrogen gas that $n = \rho/\mu m_{\text{u}}$ with $\mu = 1$, so Equation (2.99) can be solved for a given ionization fraction in terms of temperature and density.
- Figure 2.3 shows the necessary conditions for 50% and 99% ionization.
- Given the number densities, we can compute the pressure and internal energy as in previous cases and then the full thermodynamic set of quantities.

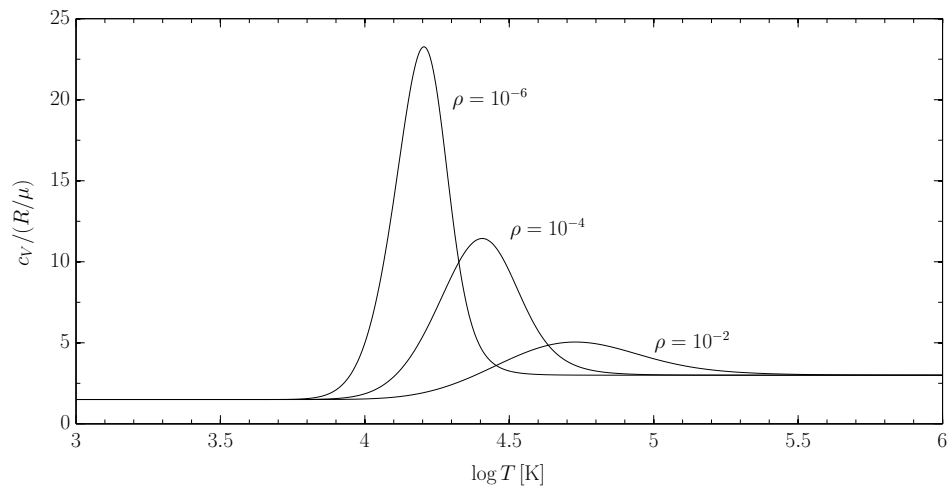


Figure 2.4: The specific heat at constant volume for H at different densities (in g cm^{-3}). The familiar value of $3/2$ is found for lower temperatures before ionization takes place. Full ionization occurs at the highest temperatures where the value reaches 3, which is twice the value of neutral gas because the number of particles per gram is twice as large.

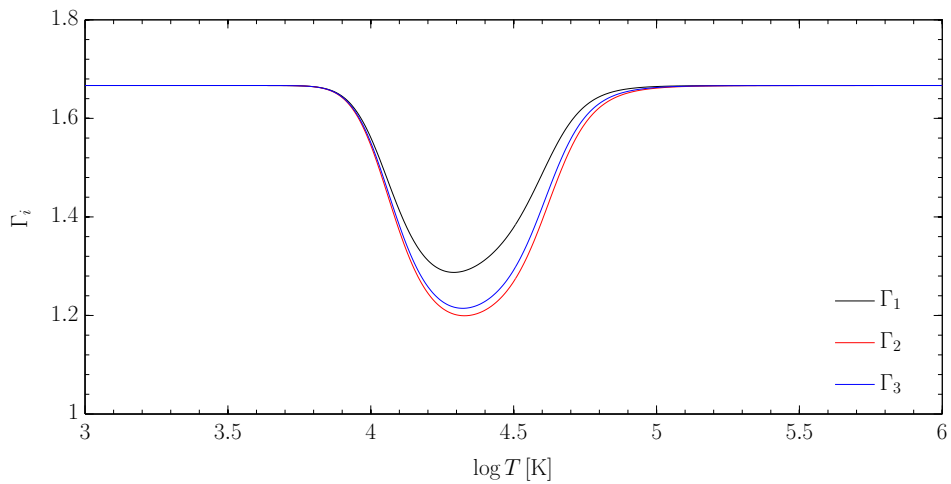


Figure 2.5: Adiabatic exponents for ionized H at a density of $10^{-4} \text{ g cm}^{-3}$.

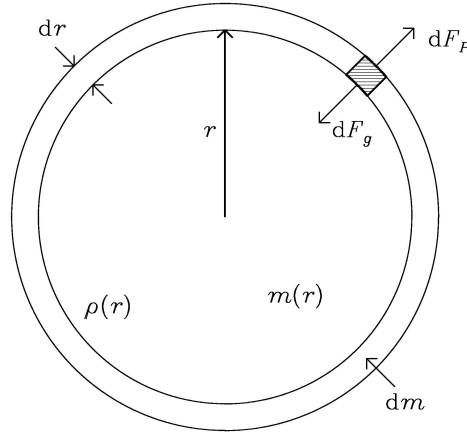


Figure 2.6: Schematic for deriving hydrostatic equilibrium. From [Christensen-Dalsgaard \[2003\]](#).

- Recognize that the pressure

$$P = (n_e + n^+ + n^0)k_B T = (1 + y)N\rho k_B T, \quad (2.100)$$

since $N\rho = n = n^+ + n^0$. N is the total nucleon number (ions plus neutrals) per unit mass, and is independent of density, thus constant.

- The specific internal energy is

$$U = \frac{3}{2}(1 + y)\frac{n}{\rho}k_B T + y\frac{n}{\rho}\chi_H = \frac{3}{2}(1 + y)Nk_B T + yN\chi_H, \quad (2.101)$$

This can be understood since, to completely ionize the gas, we need to add $N\chi_H$ to strip off the electrons, and another $3/2Nk_B T$ to bring the ions up to the ambient temperature.

- From these expressions the specific heats can be computed from the appropriate differentials. A few examples of c_V are shown in Figure 2.4 for several densities.
- Finally, the adiabatic exponents can be computed in similar ways using the prior results and Eqs. (2.78)-(2.80).
- Their values for a density of 10^{-4} across the ionization fraction range is shown in Figure 2.5.
- The Γ_i all take their ideal gas values for full ionization of complete neutrality.
- Where ionization occurs, the values decrease quickly, and then increase again as ionization completes.
- To understand this, consider adiabatic compression of the gas, and let's focus on Γ_3 , which, according to Equation (2.80), relates the temperature and volume.
- Before ionization, we see the value of $5/3$ for the neutral gas, which simply means it is heating up under compression as $T \sim V^{-2/3}$.
- When ionization starts to occur, the value decreases, and the temperature sensitivity on volume is weaker. The energy is used to ionize the gas, instead of heating it up as quickly.

2.4 Hydrostatic Equilibrium

2.4.1 Derivation

- Consider Figure 2.6. Take a thin mass element in a star of thickness dr and surface dA at radius r (and thus of mass $dm = \rho dr dA$) from the center.
- The gravitational force on that mass element is

$$d\mathbf{F}_g = -\frac{G[\rho(r)drdA]m(r)}{r^2}\hat{\mathbf{r}}, \quad (2.102)$$

directed radially inward.

- In equilibrium, this force is balanced by an outward pressure force acting at r and $r + dr$ ($P=F/A$)

$$dF_P = [P(r + dr) - P(r)] dA = -dF_g = \frac{dP}{dr} dr dA, \quad (2.103)$$

by Taylor expansion.

- Equating the forces gives

$$\frac{dP}{dr} = -\frac{G\rho(r)m(r)}{r^2}. \quad (2.104)$$

- We will commonly see this written in vector form as

$$\nabla_{\mathbf{r}} P = \rho \mathbf{g}. \quad (2.105)$$

- Note that the mass element in the thin shell can be expressed as

$$dm = 4\pi r^2 \rho dr, \quad (2.106)$$

or

$$\frac{dm}{dr} = 4\pi \rho r^2. \quad (2.107)$$

- Thus the mass as a function of radius is found by

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'. \quad (2.108)$$

IN CLASS WORK

Estimate the central pressure of the Sun from the equation of hydrostatic equilibrium. Compare to the tabulated values. Try to put all final expressions in terms of scaled solar values as we have been doing.

Answer: The simplest thing one can do is ignore the derivatives in the equilibrium expression and assume the density is constant:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \quad (2.109)$$

$$\frac{P_c}{R} = \frac{3}{4\pi} \frac{GM^2}{R^5} \quad (2.110)$$

$$P_c = \frac{3}{4\pi} \frac{GM_\odot^2}{R_\odot^4}, \quad (2.111)$$

where we've replaced all values by the gross solar ones. For a general star we can show

$$P_c \approx 2.69 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{ dyne cm}^{-2}.$$

We know that $1 \text{ dyne cm}^{-2} = 0.1 \text{ N m}^{-2} [\text{Pa}]$, so we are only off in pressure by about 2 orders of magnitude when compared to the tabulated value of $2.3 \times 10^{17} \text{ dyne cm}^{-2}$. Not so bad actually. The tabulated value is roughly

$$P_c \approx \frac{261}{4\pi} \frac{GM_\odot^2}{R_\odot^4} \quad (\text{tabulated})$$

PROBLEM 2.6: [10 pts]: (a) Do the same type of calculation as for the central pressure to find the central temperature and compare to the table value. Use the same mass fractions as in Problem 2.3. (b) Then, using the expressions you now have for the gas pressure and temperature, show that we can ignore the radiation pressure for the Sun in the core. I.e., show that $P_R/P_G \approx c(M/M_\odot)^2$, where c is a small number. Try to put all final expressions in terms of scaled solar values as we have been doing.

PROBLEM 2.7: [5 pts]: In Equation (2.111) we found a cheap and dirty estimate of the central pressure. Now, using Equations (2.104) and (2.107), find a **lower bound** for the pressure at the center of the Sun. First find an expression for dP/dm and integrate from core to surface. Making a simple assumption in the integrand allows you to argue this is really a lower limit. (You may want to read Problem 2.8 before you try this one). Compare again to the previous result of the in class problem by expressing your final answer in terms of

$$P_c = \text{const} \times \frac{GM_\odot^2}{R_\odot^4}, \quad (2.112)$$

where the constant is really the key quantity in your computation.

PROBLEM 2.8: [5 pts]: Let's improve the lower limit now (i.e., make it a bit larger). All that is needed is to assume a mean density that is a decreasing function of r such as

$$\overline{\rho(r)} = \frac{m}{4\pi r^3/3}.$$

Use this at the right step in Problem 2.7 to get a new lower limit, again expressed as

$$P_c = \text{const} \times \frac{GM_\odot^2}{R_\odot^4}.$$