

September 15.....

2.3.7 Radiation pressure

- Particles are not the only source of pressure in a star. The radiation field of photons can also exert a pressure.
- We already have an expression for the general pressure in Equation (2.12).
- With a degeneracy factor $g = 2$ (photons have 2 spin states, or polarizations, each with the same energy at fixed frequency), chemical potential (bosons) $\mu_c = 0$, $E = pc$, and $E_j = 0$, the distribution function Equation (2.8) is

$$n(p) = \frac{2}{h^3} \frac{1}{\exp(pc/kT) - 1}. \quad (2.56)$$

- From Equation (2.12), we thus have

$$P_{\text{rad}} = \frac{8\pi c}{3h^3} \int_0^\infty \frac{p^3}{e^{pc/kT} - 1} dp, \quad (2.57)$$

where we made the substitution $v \equiv c$ for photons.

- The integral can be solved (Problem 2.4) to give

$$P_{\text{rad}} = \frac{1}{3} a T^4, \quad (2.58)$$

where $a = 4\sigma/c = 7.5 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$.

- Similarly as before, the energy density

$$u_{\text{rad}} = aT^4 = 3P_{\text{rad}}. \quad (2.59)$$

PROBLEM 2.4: [10 pts]: Carry out the integral in Equation (2.57) to show that

$$a = \frac{8\pi^5}{15} \frac{k^4}{h^3 c^3}. \quad (2.60)$$

Hint: make the substitution $x = pc/kT$.

2.3.8 Density-temperature equation of state landscape

- Putting the previous sections together, the pressure of stellar matter through the equation of state in general is

$$P = P_{\text{ion}} + P_{\text{e}} + P_{\text{rad}}. \quad (2.61)$$

- In some cases, the electron pressure is from degenerate particles. In rare cases, the ions can become degenerate too.
- Not all of these pressure terms contribute equally to the total pressure at any given time, as you saw in Problem 2.3.
- Consider the total gas pressure of an ideal gas as

$$P_{\text{gas}}^{\text{ideal}} = P_{\text{ion}} + P_{\text{e}} = \frac{\rho k_{\text{B}} T}{\mu m_{\text{u}}}. \quad (2.62)$$

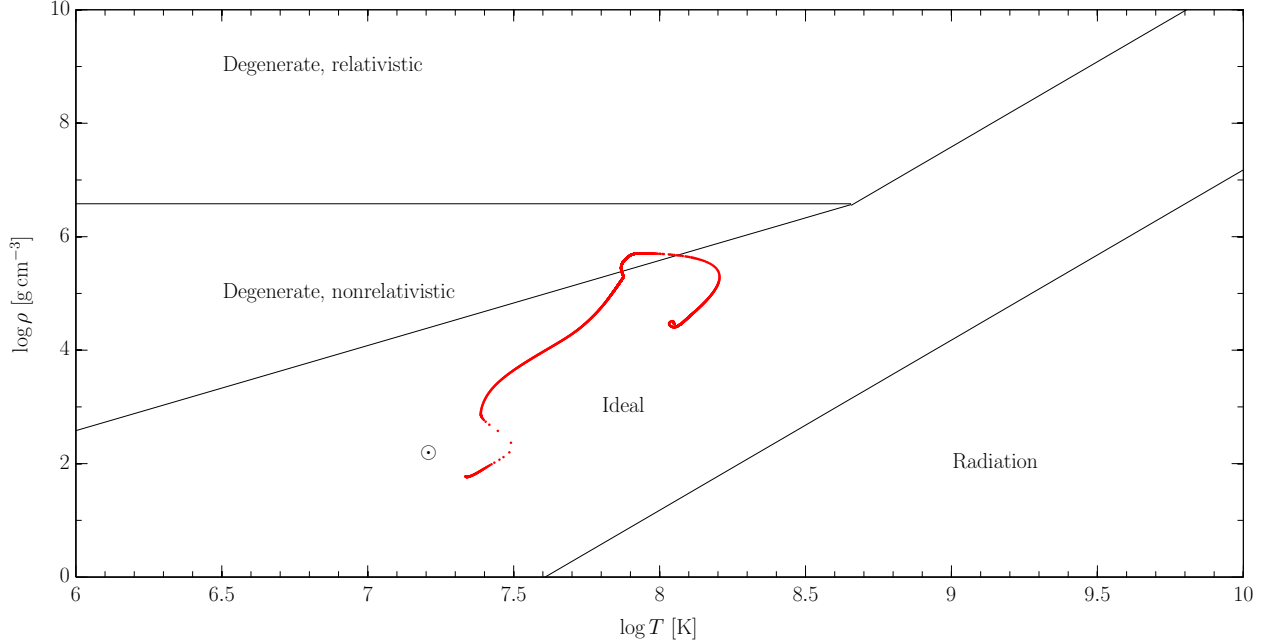


Figure 2.1: Stellar matter under core conditions. These regimes are computed using $\mu_e = 2$ and $\mu = 0.5$. The current location of the Sun is given by its symbol. The red dots are evolution points of a stellar model of $M = 2.15 M_\odot$ through until about 800 Myr. On the red giant branch its core became degenerate, which subsequently relaxed as it initiated He burning. Its final position here is on the horizontal branch.

- It is useful to compare regions where this and P_e^{deg} and P_{rad} compete and transition to one another.
- First consider where an ideal gas transitions to a degenerate nonrelativistic one. Equating Equation (2.51) and Equation (2.62) gives

$$\frac{\rho}{\mu_e^{5/2}} = \left(\frac{k_B}{C\mu m_u} \right)^{3/2} T^{3/2}, \quad (2.63)$$

where C is the constant prefactor in Equation (2.51) .

- For large densities or low temperatures, i.e., when $\rho T^{-3/2} > \text{const}$, the gas is dominated by degenerate pressure. This is shown by the line of a slope of 1.5 in Figure 2.1.
- When electron speeds become relativistic, Equation (2.54) becomes appropriate, and when equated with the ideal gas pressure yields

$$\frac{\rho}{\mu_e^4} = \left(\frac{k_B}{D\mu m_u} \right)^3 T^3, \quad (2.64)$$

where D is the constant prefactor in Equation (2.54).

- This is shown in Figure 2.1 with the line of slope 3 at high temperature and density.
- In the degenerate regime, the transition from non- to relativistic is found by equating Equation (2.51) and Equation (2.54), which is independent of temperature (since these are completely degenerate systems):

$$\frac{\rho}{\mu_e} = \left(\frac{D}{C} \right)^3. \quad (2.65)$$

This is given in the figure by the horizontal line.

- Finally, we can determine where radiation pressure starts to exceed ideal gas pressure. We use Equation (2.58) to find

$$\frac{\rho}{\mu} = \frac{1}{3} \frac{a m_u}{k_B} T^3. \quad (2.66)$$

- In Figure 2.1 this is shown by the line at the bottom right of slope 3.

2.3.9 Thermodynamics of an ideal gas

- Here we consider quasistatic changes to the state of a nondegenerate gas.
- As already stated, the internal energy per unit volume of an ideal gas is

$$u = \frac{3}{2} n k_B T = \frac{3}{2} \frac{\rho k_B T}{\mu m_p} = \frac{3}{2} P. \quad (2.67)$$

- Therefore the average energy per particle is $3/2 k_B T$. Example problem 2.1 arrived at this in a slightly different way.
- We define the specific volume V as the volume corresponding to unit mass, $V = \text{volume}/\text{mass} = 1/\rho$. The specific internal energy is the internal energy per unit mass $U = u/\rho$:

$$U = \frac{3}{2} \frac{k_B T}{\mu m_u}. \quad (2.68)$$

- Remember that the first law of thermodynamics tells us that we can (slowly) change the internal energy of gas by adding heat or doing work:

$$dU = dQ + dW, \quad (2.69)$$

where U is the specific internal energy of the matter, V is the specific volume it occupies, and dQ is some amount of heat added to it.

- The work done is to contract or expand it, so $dW = -PdV$.
- The more proper form for our use is

$$dQ = dU + PdV. \quad (2.70)$$

This heat partly changes the internal energy of the matter and also potentially changes the volume.

- Keeping the volume constant the first law of thermodynamics becomes

$$c_V = \left(\frac{dQ}{dT} \right)_V = \frac{dU}{dT} = \frac{3}{2} \frac{k_B}{\mu m_u} = \frac{3}{2} \frac{R}{\mu}. \quad (2.71)$$

This is the specific heat at constant volume.

- Now rewrite the first law

$$dQ = dU + PdV + VdP - VdP = dU - VdP + d(PV), \quad (2.72)$$

$$d(PV) = d(Nk_B T) = \frac{k_B}{\mu m_u} dT, \quad (2.73)$$

remembering that $N = nV = n/\rho = \rho/(\mu m_u \rho) = 1/\mu m_u$.

- Then

$$dQ = dU - VdP + \frac{k_B}{\mu m_p} dT = \frac{5}{2} \frac{k_B}{\mu m_u} dT - VdP, \quad (2.74)$$

by using Equation (2.68).

- Therefore the specific heat at constant pressure is

$$c_P = \frac{5}{2} \frac{k_B}{\mu m_u}. \quad (2.75)$$

Note that $c_P - c_V = R/\mu$.

- Note also the ratio of specific heats, $\gamma = c_P/c_V$, which for an ideal gas $\gamma = 5/3$ since the specific heats are constants.
- An **adiabatic process** is one in which no heat is added to the gas ($dQ = 0$).
- In this special case, we can find expressions relating changes in P and V . Using the above expressions we can show

$$c_V \left(\frac{dP}{P} + \frac{dV}{V} \right) = -\frac{k_B}{\mu m_p} \frac{dV}{V} = (c_V - c_P) \frac{dV}{V}. \quad (2.76)$$

- Finally we see that

$$\frac{dP}{P} = -\gamma \frac{dV}{V} = \gamma \frac{d\rho}{\rho} = -\frac{\gamma}{1-\gamma} \frac{dT}{T}. \quad (2.77)$$

- Since γ is constant in this case, such equations can be readily integrated to yield relations such as $PV^\gamma = \text{const.}$
- Using the ideal gas law we can also write

$$\left(\frac{\partial \ln P}{\partial \ln \rho} \right)_s = \gamma \equiv \Gamma_1 \quad (2.78)$$

$$\left(\frac{\partial \ln P}{\partial \ln T} \right)_s = \frac{\gamma}{\gamma - 1} \equiv \frac{\Gamma_2}{\Gamma_2 - 1} \quad (2.79)$$

$$\left(\frac{\partial \ln T}{\partial \ln \rho} \right)_s = \gamma - 1 \equiv \Gamma_3 - 1. \quad (2.80)$$

- The s means adiabatic, or at constant entropy, where $dQ = TdS$.

PROBLEM 2.5: [5 pts]: Derive the 2 equations (2.76) and (2.77).