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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN

THE ACCELERATION OF INTERSTELLAR CLOUDS

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OORT and SPITZER have suggested that at least part of a neutral interstellar cloud near an O or a B star can gain an appreciable speed through a type of rocket action, when the side of the cloud near the star is ionized and heated. The process is considered here in some detail.

When the temperature of the star and the density and temperature of the cloud are given the neutral gas can be accelerated only in a certain range of cases. If the photon intensity of the ionizing radiation lies above a certain limit the gas is ionized so quickly that the neutral cloud experiences no mechanical effects at all. If the intensity lies below another limit the pressure of the newly ionized gas on the neutral gas is too small to hold it back, and the cloud just expands towards the source of the radiation. If the intensity lies between the two critical limits a shock-wave moves ahead into the neutral gas and gives it a velocity away from the source. This last case is the only interesting one.

Quite high compressions and velocities can be produced if the neutral gas contains cooling particles, such as H_2 molecules. For O5 stars the maximum possible speed produced is 37 km/s, the maximum compression 1700 : 1, and the maximum possible mass pushed away 2×10^6 $M_\odot$. For a B2 star these quantities are, respectively, 19 km/s, 420 : 1 and 0.75 $M_\odot$. The weakening of the radiation in the ionized region between the star and the cloud makes the pressure on the cloud diminish with time so that clouds with large masses acquire speeds much below the maximum. The final speeds are about 6 or 7 km/s, for clouds with one solar mass, and 3 or 4 km/s, for clouds with one hundred solar masses.

There are reasons to think that improvements in the theory will lead to higher final speeds. In particular, according to OORT and SPITZER's theory, the layer of ionized gas between the star and the neutral cloud will be rather less extensive in a three-dimensional model than in our one-dimensional case. A greater intensity of radiation than we have allowed for therefore reaches the ionization front in the later stages of the motion.

1. *Introduction.*

OORT and SPITZER[1]) have discussed the possible acceleration of interstellar clouds by ionizing radiation. Their mechanism is, essentially, as follows. The radiation of an O or B star is supposed to fall on one side of the cloud. The gas is ionized and heated and will tend to escape from the cloud in the direction of the star, while the remaining part of the cloud will tend to recoil in the opposite direction and to be compressed. The process is here considered in some detail. It will be shown that this rocket-like action can occur in a certain range of cases, and estimates will be made of the likely compression and increase in velocity.

2. *Conditions at the ionization front.*

We shall first find the conditions which the gas must satisfy near the surface I (or I front) separating the neutral part of the cloud (or H I region) from the ionized part (or H II region). We consider the conditions near a particular point where the ionizing radiation is coming from the side of the H II region and is incident at right angles on the surface I. The radiation contains J photons per cm² per sec with an energy greater than χ_0, the ionization potential of the H atom, and χ_r is the mean energy of these photons. In undiluted stellar radiation there is a relation between J and χ_r which depends only on the type of the star. The relation is different when the radiation is diluted and has passed through an absorbing medium (see section 5).

The hydrogen atoms near the I front will be ionized, and newly formed protons and electrons will stream continually through the front. It will be assumed that the velocity of these particles is perpendicular to the surface. The rate of flow will be J protons and J electrons per cm² per sec.

Such a situation is found to be possible only if J does not lie between two definite values J_R and J_D, which depend on the density and the temperature of the gas in the H I region, and on χ_r. Relations are also found between the densities and temperatures on the two sides of the I front. The effect of impurities in the hydrogen is neglected in this section.

*) Astronomy Department, University of Manchester. The article was written while the author was staying in Leiden.

[1]) To be published in *Ap. J.* See also *B.A.N.* No. 455.

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Let n, n_i = number of H atoms (protons) per cm^3,
ρ, ρ_i = mass density,
U, U_i = gas velocity relative to the I front,
and p, p_i = gas pressure.

Symbols without a suffix refer to the H I region, symbols with the suffix i to the H II region.

Further, let m = mass of the H atom,
and let $I = mJ$.

Since energy χ_0 is used to separate the electron from the proton, an amount of energy $\chi_r - \chi_0$ is left over, on the average, in each ionization. This goes to heat the ionized gas. We put

$$\tfrac{1}{2} m Q^2 = \chi_r - \chi_0 = \text{average excess energy}.$$

Then we have that $n_i U_i = J$, or $\rho_i U_i = I$.
Since $\gamma = {}^5/_3$ in atomic hydrogen gas, whether neutral or ionized, gas crossing the I-front satisfies the equations [cf. COURANT and FRIEDRICHS [1])]

conservation of matter:

$$\rho U = \rho_i U_i = I, \tag{1}$$

conservation of momentum:

$$p + \rho U^2 = p_i + \rho_i U_i^2, \tag{2}$$

conservation of energy:

$$\frac{5p}{2\rho} + \tfrac{1}{2} U^2 + \tfrac{1}{2} Q^2 = \frac{5p_i}{2\rho_i} + \tfrac{1}{2} U_i^2. \tag{3}$$

Let $\rho_i = \varepsilon\rho$. Then, by (1) and (2),

$$p + \rho U^2 = p_i + \frac{1}{\varepsilon}\rho U^2,$$

$$\text{or} \qquad p_i = p - \frac{1}{\varepsilon}\rho U^2 (1 - \varepsilon). \tag{4}$$

By (1), (3) and (4)

$$\frac{5p}{\rho} + U^2 + Q^2 = \frac{5}{\varepsilon\rho}\left[p - \frac{1}{\varepsilon}\rho U^2 (1-\varepsilon)\right] + \frac{U^2}{\varepsilon^2},$$

$$\text{or} \left[\frac{5p}{\rho} + U^2 + Q^2\right]\varepsilon^2 - 5\left[\frac{p}{\rho} + U^2\right]\varepsilon + 4U^2 = 0. \tag{5}$$

The speed of sound in the H I region is $c = \left(\frac{5p}{3\rho}\right)^{\frac{1}{2}}$, and (5) may be written

$$(3c^2 + U^2 + Q^2)\varepsilon^2 - (3c^2 + 5U^2)\varepsilon + 4U^2 = 0. \tag{6}$$

This equation in ε has two positive roots, a pair of coincident roots, or a pair of complex roots according as

$$(3c^2 + 5U^2)^2 \gtrless 16U^2(3c^2 + U^2 + Q^2),$$
$$\text{or} \quad 9(c^2 - U^2)^2 \gtrless 16U^2Q^2. \tag{7}$$

Thus there are real roots if

$$3(c^2 - U^2) \leq -4UQ,$$

i.e. if $$3U^2 - 4UQ - 3c^2 \geq 0, \tag{8}$$

or $$U \geq U_R = \tfrac{1}{3}(2Q + \sqrt{4Q^2 + 9c^2}), \tag{9}$$

or $$J \geq J_R = \tfrac{1}{3} n (2Q + \sqrt{4Q^2 + 9c^2}). \tag{10}$$

There are also real roots if

$$3(c^2 - U^2) \geq 4UQ,$$

i.e. if $$3U^2 + 4UQ - 3c^2 \leq 0, \tag{11}$$

or $$U \leq U_D = \tfrac{1}{3}(-2Q + \sqrt{4Q^2 + 9c^2}), \tag{12}$$

or $$J \leq J_D = \tfrac{1}{3} n (-2Q + \sqrt{4Q^2 + 9c^2}). \tag{13}$$

There are always two distinct critical speeds U_R and U_D. For radiation of a given intensity they correspond to the two critical densities ρ_R and ρ_D, which are given by

$$\left.\begin{aligned} \rho_R &= \frac{I}{U_R} = \frac{3I}{2Q + \sqrt{4Q^2 + 9c^2}}, \\ \rho_D &= \frac{I}{U_D} = \frac{3I}{-2Q + \sqrt{4Q^2 + 9c^2}}. \end{aligned}\right\} \tag{14}$$

(The suffix R stands for "rarefied", and D stands for "dense".)

If the gas density just ahead of the I front equals ρ_R or ρ_D there is only one possible value of ε, and so only one possible value of ρ_i.

If $\rho < \rho_R$ or $\rho > \rho_D$ there are two possible values for ρ_i.

If $\rho_R < \rho < \rho_D$ an I-front cannot be in contact with the H I region.

Conditions at the I-front will be called R-critical if $\rho = \rho_R$, D-critical if $\rho = \rho_D$, R-type if $\rho \leq \rho_R$, D-type if $\rho \geq \rho_D$, and M-type if $\rho_R < \rho < \rho_D$.

Various cases are now considered.

(i) When conditions are R-type, $U \geq U_R > c$, by (9). Thus the I-front advances into the H I region at supersonic speed.

(ii) When conditions are D-type, $U \leq U_D < c$, by (12). Therefore the I-front advances into the H I region at subsonic speed.

(iii) When conditions are R- or D-type, but are not critical, there are two possible values of ε, say ε_1 and ε_2, where $\varepsilon_1 > \varepsilon_2$.

We now show that when $\varepsilon = \varepsilon_1$, the motion of the I-front relative to the gas behind it is subsonic. When $\varepsilon = \varepsilon_2$, the motion is supersonic.

For equation (2) can be written

$$\frac{p}{\rho} + U^2 = \frac{p_i}{\rho} + \frac{\rho_i}{\rho} U_i^2 = \varepsilon\left(\frac{p_i}{\rho_i} + U_i^2\right)$$

$$\text{or} \quad 3c^2 + 5U^2 = \varepsilon(3c_i^2 + 5U_i^2).$$

Now $c_i > U_i$, i.e. the relative motion is subsonic if

$$3c^2 + 5U^2 > 8U_i^2\varepsilon = \frac{8U^2}{\varepsilon},$$

$$\text{or if} \qquad \varepsilon > \frac{8U^2}{3c^2 + 5U^2}. \tag{15}$$

[1]) R. COURANT and K. O. FRIEDRICHS, "Supersonic Flow and Shock Waves", New York, 1948, p. 206.

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It only remains to show that

$$\varepsilon_1 > \frac{8U^2}{3c^2+5U^2}.$$

Now the harmonic mean of ε_1 and ε_2 is

$$\left[\frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)\right]^{-1} = \frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2} = \frac{8U^2}{3c^2+5U^2}, \qquad (16)$$

by equation (6).

Since ε_1 is greater than ε_2 it is also greater than the harmonic mean of ε_1 and ε_2, i.e. $\varepsilon_1 > \frac{8U^2}{3c^2+5U^2}$.

This completes the proof that the relative motion is subsonic when $\varepsilon = \varepsilon_1$. Similarly one can show that the relative motion is supersonic when $\varepsilon = \varepsilon_2$.

(iv) It follows that the motion of the I-front relative to the gas behind it is sonic when conditions are *R*- or *D*-critical.

(v) An I-front cannot advance into an H I region when conditions ahead of it are *M*-type.

(vi) We define the type index α by the equation $3U^2 - 4U\alpha Q - 3c^2 = 0$ for a cloud of neutral gas at a given pressure and density, exposed to radiation of a particular kind. Then $\alpha \geq 1$ when conditions are *R*-type, $\alpha = 1$ when conditions are *R*-critical, $1 > \alpha > -1$ when conditions are *M*-type, $\alpha = -1$ when conditions are *D*-critical and $\alpha \leq -1$ when conditions are *D*-type.

$$\text{But } \alpha = \frac{3}{4UQ}(U^2 - c^2) = \frac{1}{4Q}\left(3U - \frac{3c^2}{U}\right)$$

$$= \frac{1}{4Q}\left(\frac{3I}{\rho} - \frac{5p}{I}\right), \text{ since } c^2 = \frac{5p}{3\rho} \text{ and } U = \frac{I}{\rho}.$$

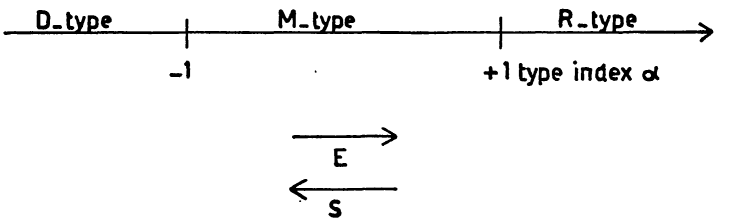
The type index increases if the gas undergoes a change in which both pressure and density decrease and vice versa. Table 1 and Figure 1 show how the condition of the neutral gas can change on passage through a shock wave, in which pressure and density both increase, or through an expansion wave, in which they both decrease.

TABLE 1

Possible changes in the type index.
The changes can be schown schematically as in Figure 1.

Initial condition	Type index	Possible final conditions after passing: through a shock wave	through an expansion wave
		(type index decreases)	(type index increases)
R-type	$\alpha \geq 1$	*R*-type, *R*-critical, *M*-type, *D*-critical, *D*-type	*R*-type
R-critical	$\alpha = 1$	*M*-type, *D*-critical, *D*-type	*R*-type
M-type	$1 > \alpha > -1$	*M*-type, *D*-critical, *D*-type	*M*-type, *R*-critical, *R*-type
D-critical	$\alpha = -1$	*D*-type	*M*-type, *R*-critical, *R*-type
D-type	$\alpha \leq -1$	*D*-type	*D*-type, *D*-critical, *M*-type, *R*-critical, *R*-type

FIGURE 1



To show possible changes in the type index.
Sense of change produced by { an expansion wave: *E* / a shock wave: *S* }

3. *Possible initial motions in a neutral cloud, backed by a vacuum.*

We shall now discuss the initial motions in a one-dimensional model cloud. At first the gas is at rest on the positive side of the plane $x = 0$, which separates it from a vacuum, has a uniform mass ρ_1 of atomic hydrogen per unit volume and a uniform pressure p_1. At time $t = 0$ the gas is released and exposed to uniform and constant plane parallel radiation, travelling in the forward direction, coming from the side of the vacuum and incident perpendicular to the plane $x = 0$. It has intensity J ionizing photons per unit area and unit time, and is such that $\frac{1}{2}mQ^2$ is the average thermal energy liberated in one ionizing process. Impurities in the atomic hydrogen are neglected.

The initial motion depends on the type index of the quiet gas. An I-front must occur somewhere in every case and the motion of the neutral gas has to be such that conditions are not *M*-type just ahead of the I-front. There appear at first to be several motions to suit each particular situation; however, all but one of them can always be eliminated. We use the following two criteria:

(a) In a motion of a particular kind there will always occur, in a certain order, one or more fronts (e.g. ionization, shock or expansion fronts). The motion will not persist if it is such that in a finite time one of the fronts must overtake another front ahead of it.

For example, suppose that the conditions in the quiet gas are *R*-type ($\alpha \geq 1$) and an expansion wave (with front E) precedes the ionization front (or I-front). Then it can be shown that the I-front will overtake the E-front in a finite time and so this type of initial motion can be rejected.

(b) If a disturbance to the given motion can grow, then that motion will not persist.

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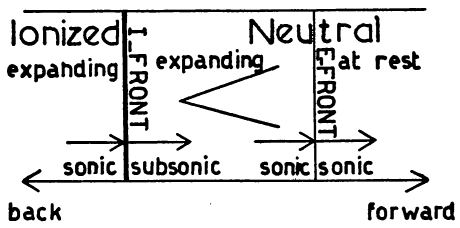
Thus suppose that conditions are D-type in the quiet gas, but not D-critical ($\alpha < -1$). Then the I-front can advance straight into the quiet gas. This motion has an instability in which an expansion wave, with front E, moves into the quiet gas ahead of the I-front, and in which the distance between the E- and I-fronts increases with time. The original motion will therefore not persist.

The discussion may perhaps be easier to understand with the help of the following physical picture of the results which we shall obtain later by means of these two criteria. The type index α can be varied not only by changing p and ρ, but also by changing I while keeping p and ρ fixed. When $I > 0$, α is a continuously increasing function of I. For small I, α is large and negative, and for large I, α is large and positive. Given the pressure and density in the quiet gas and also the value of Q, we can consider the effect of letting I vary from very small to very large values, and so obtain conditions in the quiet gas which are, successively, D-type, D-critical, M-type, R-critical and R-type.

When conditions are extreme D-type, the number of photons falling on the gas per unit area and unit time is very small (i.e. I is very small). The action of the ionizing radiation therefore exerts only a small pressure on the neutral gas at the I-front and the cloud expands in the direction of negative x almost as if it were expanding into a vacuum. As I is increased the pressure on the neutral gas gradually grows until it is sufficient to stop any expansion at all. Conditions are then D-critical. Figures 2 and 3 show typical configurations when conditions in the quiet gas are D-type and D-critical.

FIGURE 2



The motion when conditions are D-type in the quiet gas.

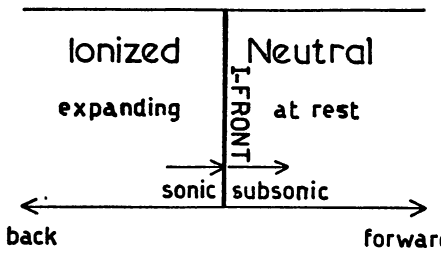
< : indicates direction of decrease of p, ρ and T.

$\xrightarrow{\text{sonic}}$: indicates the sense of motion of the front relative to the gas on that side and whether the relative speed is sonic, sub- or supersonic.

A further increase in I makes conditions in the quiet gas M-type and raises the pressure above that necessary to stop the backward expansion. As a result a shock wave moves into the quiet gas ahead of the I-front.

In these three cases the motion is such as to make

FIGURE 3



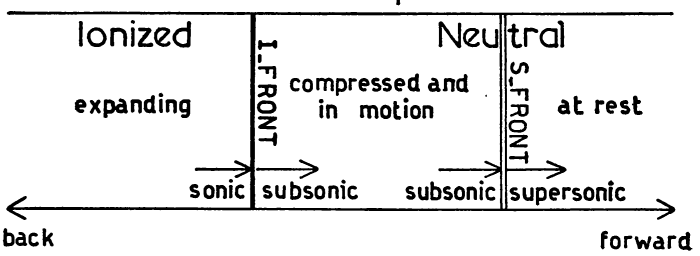
The motion when conditions are D-critical in the quiet gas.

Symbols as in Figure 2.

conditions D-critical at the I-front (cf. CHAPMAN-JOUGUET condition [1])).

A further increase in I makes the rate of ionization

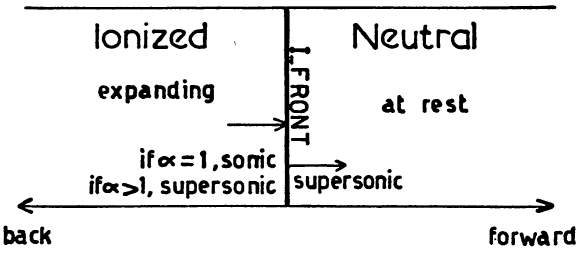
FIGURE 4



The motion when conditions are M-type in the quiet gas.

so large that the I- and S-fronts travel through the gas together. Conditions are then R-critical. For still greater values of I conditions become R-type, and the I-front moves so fast that the shock can no longer keep up with it. The I-front therefore advances straight into the quiet gas, and becomes a weak detonation front.

FIGURE 5



The motion when conditions in the quiet gas are R-type.

We shall now use the two criteria to show that the gas motions described in Figures 2-5 are the ones which do occur, and shall consider first what happens when an expansion front precedes the I-front.

In the most general plane expansion wave the gas moves as though it were bounded on the side of negative x, say, by a plane surface S, initially at $x = 0$. From time $t = 0$ onwards the surface moves so

[1]) R. COURANT and K. O. FRIEDRICHS, "Supersonic Flow and Shock Waves", p. 223.

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that its x co-ordinate is given by $x=\xi(t)$. An expansion wave forms if $\dot{\xi}(t)\left(\equiv d\xi/dt\right) \leq 0$, i.e. if the motion of S allows the gas to expand in the direction of negative x.

The equations of motion of the gas can be written (see, for instance, PACK [1]))

$$\left[\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right]\left(\frac{2c}{\gamma-1}+u\right)=0,$$

$$\left[\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right]\left(\frac{2c}{\gamma-1}-u\right)=0,$$

where γ is the ratio of the specific heats. In the case of a monatomic gas $\gamma = 5/3$, and so in the $x-t$ plane we have that $3c+u=\text{const.}=2r$ on the r-characteristics, whose equations are given by $dx/dt=u+c$; while $3c-u=\text{const.}=2s$ on the s-characteristics, whose equations are given by $dx/dt=u-c$.

When the motion is continuous there is one and only one characteristic of each kind through every point in the $x-t$ plane. The s-characteristics are propagated with a speed smaller than that of the gas. They therefore originate in the quiet gas, where $c=c_1$, and move out into the expanding gas. Thus $3c-u=2s=3c_1$ on every s-characteristic and so at every point in the expanding gas.

But $c+u=\frac{4}{3}r-\frac{2}{3}s=\frac{4}{3}r-c_1$, and this is constant for a given r. The r-characteristics are therefore straight lines, and they originate on the curve $x=\xi(t)$, i.e. on the back surface of the expansion, since they move faster than the gas. We have that $u_0=\dot{\xi}(t_0)$ for the characteristic which originates at the point $(\xi(t_0), t_0)$ and so

$$2r=3c_0+u_0=3c_0-u_0+2u_0$$
$$=3c_1+2\dot{\xi}(t_0),$$

and $c+u=\frac{4}{3}r-c_1=c_1+\frac{4}{3}\dot{\xi}(t_0)$.

This is the gradient of the characteristic through $(\xi(t_0), t_0)$, whose equation is therefore

$$x=\xi(t_0)+(t-t_0)\left[c_1+\tfrac{4}{3}\dot{\xi}(t_0)\right].$$

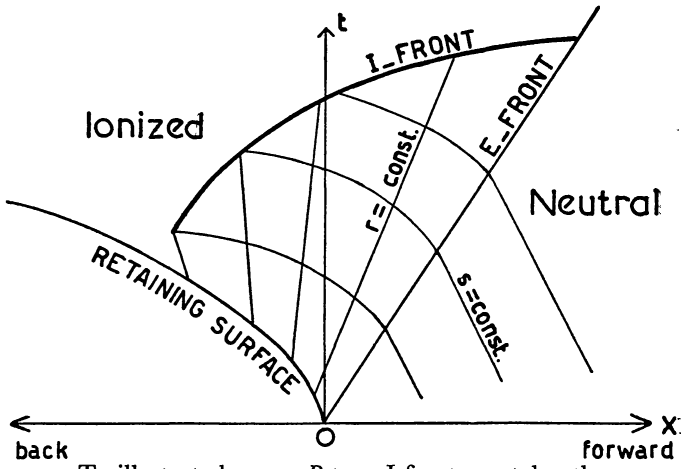
At time t the distance between the r-characteristics with parameters t_0 and $t_0+\delta t_0$ is

$$\delta x=\delta t_0\left[\tfrac{4}{3}(t-t_0)\ddot{\xi}(t_0)-c_1-\tfrac{1}{3}\dot{\xi}(t_0)\right].$$

(Since no two r-characteristics may intersect if the motion is to remain continuous δx must not vanish for any $t>t_0$. The equation can be used to show that this requires $0>\dot{\xi}(t_0)>-3c_1$ and $\ddot{\xi}(t_0)<0$).

(i) We shall now show that an I-front of type R, starting at an arbitrary point in the expansion wave will successively overtake all the r-characteristics and will, in a finite time, reach either the expansion front (the characteristic with parameter $t_0=0$), or a region in the gas where conditions are not R-type. An expansion wave cannot therefore continue to be backed by an I-front of this type.

FIGURE 6



To illustrate how an R-type I-front overtakes the r-characteristics successively.

The I-front advances relative to the gas at a speed

$$U \geq U_R=\tfrac{1}{3}\left[2Q+\sqrt{4Q^2+9c^2}\right]>c+\frac{2Q}{3},$$

and therefore moves forward relative to the r-characteristics at a speed greater than $\frac{2Q}{3}$. It will pass from the characteristic with parameter t_0 to that with parameter $t_0+\delta t_0$ in time δt where

$$\delta t \leq \frac{\delta x}{2Q/3}=\frac{3}{2Q}\left[\tfrac{4}{3}(t-t_0)\ddot{\xi}(t_0)-c_1-\tfrac{1}{3}\dot{\xi}(t_0)\right]\delta t_0,$$

so that

$$\frac{Q}{2}\delta\left(te^{-2\dot{\xi}(t_0)/Q}\right)\leq -e^{-2\dot{\xi}(t_0)/Q}\left[t_0\ddot{\xi}(t_0)+\tfrac{1}{4}\dot{\xi}(t_0)+\tfrac{3}{4}c_1\right]\delta t_0.$$

If an I-front starts at time t from the r-characteristic with parameter t_0 and reaches the r-characteristic with parameter zero at time $t+\tau$ then we have

$$\frac{Q}{2}\int_t^{t+\tau} d\left(te^{-2\dot{\xi}(t_0)/Q}\right)\leq -\int_{t_0}^{0} e^{-2\dot{\xi}(t_0)/Q}\left[t_0\ddot{\xi}(t_0)+\tfrac{1}{4}\dot{\xi}(t_0)+\tfrac{3}{4}c_1\right]dt_0,$$

or

$$\frac{Q}{2}\left[(t+\tau)e^{-2\dot{\xi}(0)/Q}-te^{-2\dot{\xi}(t_0)/Q}\right]\leq \int_0^{t_0} e^{-2\dot{\xi}(t_0)/Q}\left[t_0\ddot{\xi}(t_0)+\tfrac{1}{4}\dot{\xi}(t_0)+\tfrac{3}{4}c_1\right]dt_0.$$

[1]) D. C. PACK, *M.N.* **113**, 43, 1953 (describes also the method of characteristics).

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Since $\dot{\xi}(0) \neq \infty$, τ will be finite, so that the I-front overtakes the E-front in a finite time and cannot continue to back the expansion wave. Thus an I-front cannot follow an expansion wave into the quiet gas when conditions there are R-type. Similarly an R-type I-front following behind a shock wave will successively overtake all the r-characteristics and catch up the shock in a finite time.

Now the I-front in Figure 6 is continually moving into regions of higher density. It may happen that conditions just ahead of it will become M-type before the expansion front is reached. In this case it follows once again that an R-type I-front cannot continue to back the expansion wave.

(ii) When conditions in the quiet gas are D-type there is a possible motion in which an expansion wave precedes the I-front, which is of type D. Here the I-front advances relative to the gas at a positive velocity U where

$$U \leq U_D = \tfrac{1}{3}\left[-2Q + \sqrt{4Q^2 + 9c^2}\right] < c - \frac{2cQ}{3c + 2Q},$$

and so the I-front lags behind the r-characteristics but moves faster than the s-characteristics. The motion of the gas is sketched in Figures 7 and 8. An expansion wave moves ahead into the gas and is followed by a region in which both sets of characteristics are straight lines, so that both u and c, and therefore also p and ρ, have uniform and constant values. This region is followed by the I-front. The E-front will not be overtaken by the I-front.

But if conditions are not D-critical at the I-front a second expansion wave can move into the region just ahead of it and increase still further the type index at the I-front. This disturbance grows; thus the original motion is not stable, and cannot occur unless conditions at the I-front are D-critical (type index $\alpha = -1$). A further expansion ahead of the I-front in that case would make conditions M-type. It will be shown later that this implies that the motion is now stable.

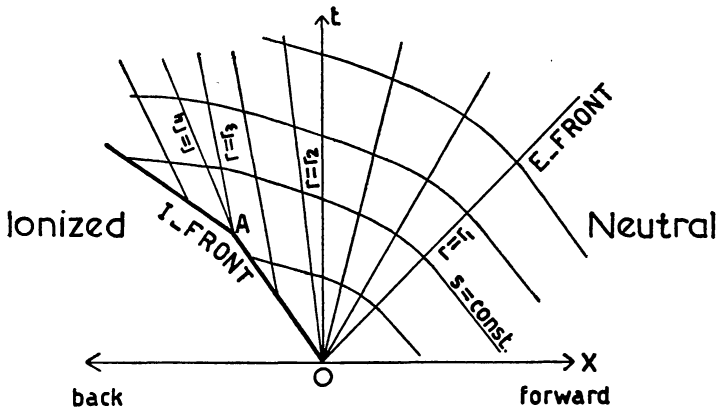
Some other cases are now treated.

(iii) A shock wave cannot continue to precede the I-front into the quiet gas when conditions there are D-type. For the type index of the gas decreases on passage through a shock. Thus conditions just ahead of the I-front would be D-type but not critical. An expansion wave would then precede the I-front. But a shock wave moves at subsonic speed relative to the gas behind it while an E-front advances at sonic speed. The expansion wave therefore overtakes the shock front and this type of motion cannot continue.

(iv) If conditions in the neutral gas are M-type, it cannot be in contact with an I-front. Therefore either an expansion wave or a shock wave precedes the I-front. But an expansion increases the type index of the gas, so that an E-front can here only be followed by an I-front of type R. According to (i) such a system cannot continue. It follows that a shock wave (or S-front) precedes the I-front into the quiet gas when conditions there are M-type. The shock compresses the gas and conditions are D-type just ahead of the I-front.

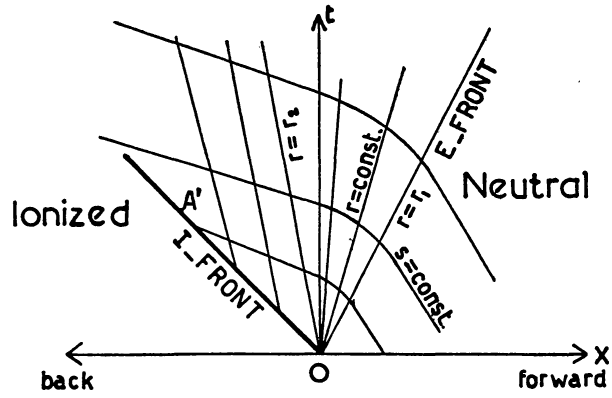
There are now three regions: region 1 ahead of the S-front, region 2 between the S- and I-fronts and region i behind the I-front.

FIGURE 7



To show the instability of the motion when conditions are not D-critical at the I-front. From O to A the I-front moves so that conditions are D-type but not critical just ahead of it. There is an expansion between the characteristics r_1 and r_2 followed by a region of constant density between r_2 and r_3. However a disturbance occurs at A in which a further expansion develops between the characteristics r_3 and r_4, followed again by a region of constant density, in which conditions are once more D-type. The disturbance can grow and so the previous motion is unstable.

FIGURE 8



To show the stable type of motion when conditions are D-type in the quiet gas.
The I-front moves so that there is an expansion between the characteristics r_1 and r_2 followed by a region of constant density, which is such that conditions are D-critical at the I-front. No further expansion can start at any point A′ for this would make conditions M-type at the I-front (see section (vii) below).

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Let n_1, n_2, n_i = number of heavy particles per cm^3 (H^+ ions or H atoms),
ρ_1, ρ_2, ρ_i = gas density,
p_1, p_2, p_i = gas pressure,
and c_1, c_2, c_i = speed of sound,
in the three regions.

Let V = velocity of advance of the S-front into the quiet gas,
v_2 = gas velocity between the S and I-fronts,
U_2 = velocity of advance of the I-front into the gas ahead of it,

and let
$$\eta = \frac{\rho_1}{\rho_2}, \quad \varepsilon = \frac{\rho_i}{\rho_2}.$$

The equations at the S-front are (see COURANT and FRIEDRICHS [1]))

conservation of matter: $\rho_1 V = \rho_2 (V - v_2)$, (17)

momentum: $p_1 + \rho_1 V^2 = p_2 + \rho_2 (V - v_2)^2$, (18)

energy: $\frac{5 p_1}{2 \rho_1} + \frac{1}{2} V^2 = \frac{5 p_2}{2 \rho_2} + \frac{1}{2}(V - v_2)^2$. (19)

By (17) $V - v_2 = \eta V$

and by (18) $p_1 + \rho_1 V^2 = p_2 + \eta \rho_1 V^2$,

or $p_2 = p_1 + \rho_1 V^2 (1 - \eta)$. (20)

By (19)
$$5 p_2 = \rho_2 \left[\frac{5 p_1}{\rho_1} + V^2 - (V - v_2)^2 \right] = \frac{\rho_1}{\eta} \left[\frac{5 p_1}{\rho_1} + V^2 (1 - \eta^2) \right]. \quad (21)$$

Elimination of p_2 gives
$$\frac{5 p_1}{\eta} + \frac{\rho_1 V^2 (1 - \eta^2)}{\eta} = 5 p_1 + 5 \rho_1 V^2 (1 - \eta),$$
or
$$\frac{5 p_1 (1 - \eta)}{\eta} = \frac{\rho_1 V^2}{\eta} (1 - \eta)(4\eta - 1).$$

Then, either $\eta = 1$, in which case there is no shock,

or since $\eta \neq 0$,
$$V^2 = \frac{5 p_1}{\rho_1} \frac{1}{4\eta - 1} = \frac{3 c_1^2}{4\eta - 1}. \quad (22)$$

Equation (22) shows the well-known result that $\eta > \frac{1}{4}$.

If now $U_1 = I/\rho_1$ and $U_2 = I/\rho_2$, then $U_2 = \eta U_1$, and this is the velocity of advance of the I-front into region 2. The distance between the S- and I-fronts increases with time if the I-front travels less fast than the S-front,

i.e. if $U_2 + v_2 < V$

or $U_2 < V - v_2$

or $\eta U_1 < \eta V$

or $U_1^2 < V^2 = \frac{3 c_1^2}{4\eta - 1}$, by (22),

or
$$4\eta - 1 < \frac{3 c_1^2}{U_1^2}. \quad (23)$$

The shock wave continues in existence only if this relation holds. It does, in fact, do so for all possible values of η when $U_1 < U_{R,1}$, the R-critical speed for region 1. This includes the case when conditions are M-type in region 1.

Now $U_{R,1} = \frac{1}{3}[2Q + \sqrt{4Q^2 + 9c_1^2}]$ and so $U_1 < U_{R,1}$ is equivalent to $(3U_1 - 2Q)^2 < 4Q^2 + 9c_1^2$

or
$$3U_1^2 - 4U_1 Q - 3c_1^2 < 0. \quad (24)$$

Conditions are D-type ahead of the I-front, and by relation (ii) of section 2
$$3U_2^2 + 4U_2 Q - 3c_2^2 \leq 0. \quad (25)$$

Since $c_1^2 = \frac{5 p_1}{3 \rho_1}$, &c., it follows from (19) that
$$3c_2^2 = 3c_1^2 + V^2 - (V - v_2)^2 = 3c_1^2 + V^2(1 - \eta^2) = 3c_1^2 \left[1 + \frac{1 - \eta^2}{4\eta - 1} \right], \text{ by (22)},$$

or
$$c_2^2 = c_1^2 \left(\frac{4\eta - \eta^2}{4\eta - 1} \right). \quad (26)$$

With $U_2 = \eta U_1$ and with relation (26) the condition (25) can be written
$$f(\eta) = 3\eta^2 U_1^2 + 4\eta U_1 Q - 3c_1^2 \left(\frac{4\eta - \eta^2}{4\eta - 1} \right) \leq 0. \quad (27)$$

Now $f(1) = 3U_1^2 + 4U_1 Q - 3c_1^2 > 0$, if conditions in the quiet gas are not D-type and
$$f(\tfrac{1}{4}) = -\infty,$$
so that there is always a range of values of η between 1 and $\frac{1}{4}$ which satisfy the inequality (27).

Since $\eta \neq 0$ the inequality can be re-arranged as
$$3U_1^2(1 + \eta)(4\eta - 1) - (3U_1^2 - 4U_1 Q - 3c_1^2)(4\eta - 1) - 9c_1^2(1 + \eta) \leq 0,$$
or
$$4\eta - 1 \leq \frac{3c_1^2}{U_1^2} + \frac{4\eta - 1}{3U_1^2(1 + \eta)} (3U_1^2 - 4U_1 Q - 3c_1^2). \quad (28)$$

But $4\eta - 1 > 0$, so that if $3U_1^2 - 4U_1 Q - 3c_1^2 < 0$, then also $4\eta - 1 < \frac{3c_1^2}{U_1^2}$ for all possible shock waves.

The S-front will separate from the I-front for all permissible values of η. In particular η may be such that conditions are D-critical at the I-front. This type of motion can occur.

(v) If conditions in region 1 are R-type a shock

[1]) R. COURANT and K. O. FRIEDRICHS, "Supersonic Flow and Shock-Waves", p. 129.

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wave may again precede the I-front, but it will not continue to do so if conditions are D-critical at the I-front. This can be shown by means of the working in section (iv).

When conditions are D-critical at the I-front equality holds in (25), and therefore also in (28), and so

$$4\eta - 1 = \frac{3c_1^2}{U_1^2} + \frac{4\eta - 1}{1+\eta}(3U_1^2 - 4U_1Q - 3c_1^2). \quad (29)$$

When conditions are R-type in region 1 the expression in brackets on the R.H.S. is positive, and so

$$4\eta - 1 > \frac{3c_1^2}{U_1^2}.$$

Condition (23) is then not satisfied, and the I-front will overtake the S-front.

(vi) We now combine the results of (ii) and (v). It has been shown that a shock-wave can precede the I-front into the quiet gas when conditions there are R-type. However if conditions are D-critical at the I-front then the I-front will overtake the S-front. The shock-wave can only continue to exist if conditions at the I-front are D-type, but not critical.

But the result of (ii) shows that an expansion wave moves off at sonic speed ahead of the I-front in this case, and will overtake the shock, which moves at subsonic speed relative to the gas behind it. We conclude that a shock wave cannot continue to precede the I-front into the quiet gas if conditions there are R-type.

(vii) When conditions in the quiet gas are D-type an expansion wave precedes the I-front, conditions just ahead of which are D-critical. A further expansion ahead of the I-front would make conditions there M-type and would therefore be followed and overtaken by a shock. On the other hand a new compression ahead of the I-front would make conditions there D-type again, and would be overtaken by a further expansion wave. Neither type of disturbance can be amplified and the motion is stable.

(viii) When conditions in the quiet gas are R-type, the I-front must advance straight into the quiet gas, for neither an expansion nor a shock wave can remain ahead of it. There are two possible values of ε, say ε_1 and ε_2 ($\varepsilon_1 > \varepsilon_2$), which satisfy all the conditions imposed so far. As was shown in section 2 the ionized gas moves at subsonic speed relative to the I-front when $\varepsilon = \varepsilon_1$. Thus every r- and s-characteristic in the ionized gas intersects the I-front [1]). But the conditions just behind the I-front are fixed in terms of I and Q, and of p_1 and ρ_1 in the quiet gas. Therefore

$$3c - u = 2s = 3c_i - u_i, \quad (30)$$

and

$$3c + u = 2r = 3c_i + u_i, \quad (31)$$

where c_i and u_i are the boundary values, at the I-front, of the sonic speed and the gas velocity in the ionized gas. Equations (30) and (31) hold throughout the ionized gas and so

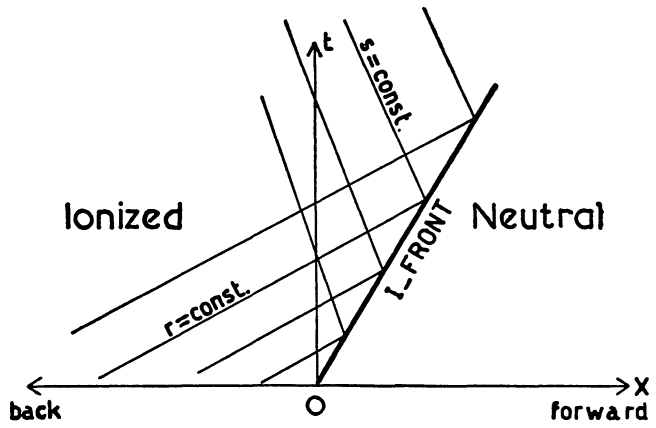
$$u = u_i \text{ and } c = c_i \quad (32)$$

everywhere behind the I-front. This implies that the density and pressure are constant everywhere behind the I-front. But the ionized gas is backed by a vacuum, so that this situation is impossible; thus we must have $\varepsilon = \varepsilon_2$. Conditions are then supersonic behind the I-front and not all the r-characteristics intersect it (this a weak detonation) [2]).

[1]) It is assumed that the motion in the ionized gas is adiabatic. This is approximately true in the initial stages of the motion.

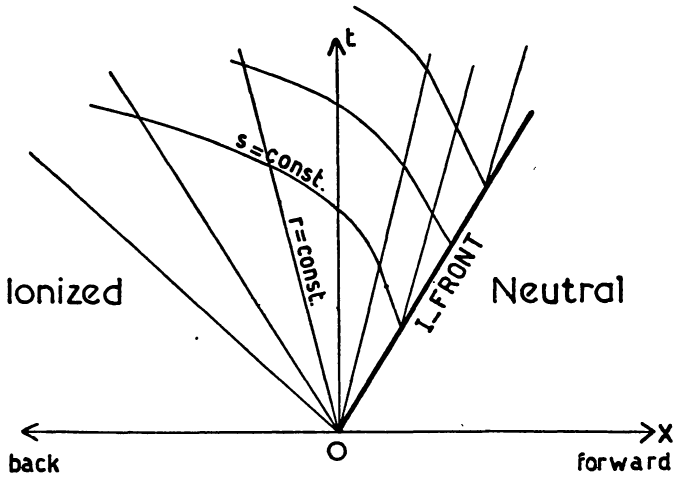
[2]) R. Courant and K. O. Friedrichs, "Supersonic Flow and Shock Waves", p. 215.

Figure 9



When $\varepsilon = \varepsilon_1$ the ionized gas moves at subsonic speed relative to the I-front, which is met by every r- and s-characteristic. The density and pressure in the ionized gas are therefore constant. This can only occur if the ionized gas is being pushed from the rear.

Figure 10



When $\varepsilon = \varepsilon_2$ the I-front moves at supersonic speed relative to the gas behind it. Some of the r-characteristics meet the I-front at O, where conditions are indeterminate. The gas density and pressure need not remain constant throughout the ionized region.

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To sum up:

a) If conditions in the quiet gas are D-type an expansion wave precedes the I-front. Conditions are D-critical at the I-front.

b) If conditions in the quiet gas are M-type a shock wave precedes the I-front. Conditions are D-critical at the I-front.

c) If conditions in the quiet gas are R-type a shock wave cannot continue to precede the I-front.

d) An expansion wave cannot continue to precede the I-front into the quiet gas if conditions there are M- or R-type.

e) If conditions in the quiet gas are R-type, the I-front itself advances into the quiet gas. Conditions are supersonic behind the I-front.

(Many improvements in section 3 were made after discussions with Prof. J. M. BURGERS.)

4. *The effect of cooling particles.*

So far we have neglected the impurities which the atomic hydrogen gas may contain. These include hydrogen molecules and solid grains, both of which are able to radiate heat which is given to them by the surrounding gas. They therefore act as cooling particles in the gas. There is reason to believe that the H_2 molecule is the principal cooling particle in neutral interstellar gas clouds. SPITZER [1]) gives some formulae which describe its rate of radiation at different temperatures, and finds that the rate of cooling produced is quite rapid at high temperatures but becomes slower as the temperature falls.

The cooling effect in the neutral gas can be neglected unless there is heating by a shock-wave. An expansion wave, moving into the neutral gas, itself produces cooling and the action of the cooling particles is not important.

Now the H atoms are generally much more abundant than the H_2 molecules: thus the rate of development of a shock is so great that the cooling can be ignored in the initial motion, and the gas behind the shock may be treated as though it were adiabatic. One might adopt some physically reasonable law to describe the cooling of the gas as the motion develops. But then conditions become non-adiabatic between the I- and S-fronts and the equations of motion are intractable. It is best to simplify matters and suppose that the cooling particles are so effective that the gas crossing the S-front is, almost at once, cooled to a temperature T_c independent of the shock strength. The assumption becomes more apt in the later stages as the S- and I-fronts draw further apart. At any rate it is an extreme case and can be compared with the cases discussed in section 3.

The symbols to be used are those of section 3, part (iv). The equations holding at the S-front are equation of continuity: $\rho_1 V = \rho_2 (V - v_2)$, (33)

conservation of momentum:

$$p_1 + \rho_1 V^2 = p_2 + \rho_2 (V - v_2)^2. \tag{34}$$

The cooling particles dissipate energy at the S-front and the third equation is simply

$$\frac{p_2}{\rho_2} = \frac{k T_c}{m}. \tag{35}$$

Let $\rho_1 = \eta \rho_2$, then, by (33) and (34)

$$\frac{p_1}{\rho_1} + V^2 = \frac{1}{\eta}\left[\frac{p_2}{\rho_2} + \eta^2 V^2\right]$$

or

$$V^2 (1 - \eta) = \frac{3}{5}\left(\frac{c_2^2}{\eta} - c_1^2\right)$$

or

$$V^2 = \frac{3}{5}\frac{c_2^2 - \eta c_1^2}{\eta (1 - \eta)}, \tag{36}$$

where $c_s = \left(\frac{5 p_s}{3 \rho_s}\right)^{\frac{1}{2}}$ = sonic velocity in region s ($s = 1$ or 2).

The gas between the S- and I-fronts moves with a uniform velocity v_2. As before conditions will be D-critical at the I-front, and, in the notation of section 3, part (ii),

$$\rho_2 = \rho_D = \frac{3I}{(4Q^2 + 9c_2^2)^{\frac{1}{2}} - 2Q}. \tag{37}$$

(It is still assumed that I remains constant.)

Now the temperature T_c will be much smaller than T_*, the temperature of the ionizing star. Since also $Q^2 > 2kT_*/m$ it follows that $Q^2 \gg c_2^2$ and

$$\rho_2 \approx \frac{3I}{2Q\left[\left(1 + 9c_2^2/8Q^2\right) - 1\right]} = \frac{4QI}{3c_2^2} = \frac{4QIm}{5kT_c}, \tag{38}$$

and

$$\eta = \frac{\rho_1}{\rho_2} \approx \frac{5\rho_1 k T_c}{4QIm} = \frac{5kT_c}{4QU_1 m}. \tag{39}$$

A numerical example shows the order of magnitude of η. A shock can occur only if $U_1 < U_R \approx 4Q/3$. The extreme value of η is therefore

$$\eta_{min} = \frac{15kT_c}{16mQ^2}.$$

If $T_c = 100$ deg K and $Q = 3 \times 10^6$ cm/sec, then $\eta_{min} = 9 \times 10^{-4}$, and the gas behind the S-front is 1100 times as dense as the gas in front; η is in fact usually small compared with unity. Then (36) may be written

$$V^2 = \frac{3c_2^2}{5\eta} = \frac{kT_c}{\eta m} = \frac{4QU_1}{5}, \tag{40}$$

and also

$$v_2 = V(1 - \eta) \approx V. \tag{41}$$

[1]) L. SPITZER JR. *Ap. J.* **109**, 337, 1948. See also F. D. KAHN, Paper at Liège Symposium, 1954.

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It may be shown that the pressure in the gas just ahead of the I-front and the density of the gas just behind it are both independent of conditions in the undisturbed gas.

For, by (38),

$$p_2 = \rho_2 \frac{k T_c}{m} \approx \frac{4 Q I}{5}. \tag{42}$$

Further, conditions are D-critical at the I-front and equation (5), of section 2, has equal roots, and

$\varepsilon = \dfrac{8 U_2^2}{3 c_2^2 + 5 U_2^2}$. (If conditions were D-type, but not critical, an expansion wave would advance into region 2 and overtake the S-front.)

But $U_2 = \dfrac{I}{\rho_2}$ so that, by (38), $U_2 = \dfrac{5 k T_c}{4 Q m} = \dfrac{3 c_2^2}{4 Q}$

and $$\varepsilon = \frac{8\left(9 c_2^4/16 Q^2\right)}{3 c_2^2 + 5\left(9 c_2^4/16 Q^2\right)} \approx \frac{3 c_2^2}{2 Q^2},$$

and $$\rho_i = \rho_2 \varepsilon = \frac{3 \rho_2 c_2^2}{2 Q^2} = \frac{5 p_2}{2 Q^2} = \frac{2 I}{Q}, \tag{43}$$

which can be written $$n_i = \frac{2 J}{Q}. \tag{44}$$

This completes the description of all those cases which are of interest when the radiation remains constant at the I-front. We shall refer to these as the "steady radiation cases".

5. *Unsteady radiation cases.*

We consider a uniform cloud of neutral matter containing n_1 H atoms per cm³. The cloud is at rest, and a plane surface separates one side of it from a vacuum. At time $t = 0$ the cloud is released and is exposed to the radiation of a star of radius a and surface temperature T_*. The star is at a distance $r\,(\gg a)$ from the cloud along a line normal to the surface, so that its radiation is effectively a plane parallel beam incident at right angles on the plane of separation.

If conditions in the cloud are R-type an ionization front will advance into the quiet gas, and an H II region will grow between it and the star. Since ionized hydrogen has a finite recombination coefficient there will be an increasing number of H atoms between the star and the I-front, which will weaken the radiation incident there. Eventually the weakening will be sufficient to make conditions M-type in the quiet gas. A shock-wave then advances into the quiet gas and so some neutral gas ahead of the I-front will be compressed and pushed in the direction away from the star. This is the rocket effect of Oort and Spitzer.

The radiation continues to weaken at the I-front so that the push gets progressively less powerful. – In this section we shall find the order of magnitude of the compressions and velocities which can be obtained.

For simplicity we shall consider only the case in which conditions at the surface of the cloud are R-critical at time $t = 0$. It is not likely that other cases will give essentially different results: they are just harder to work out.

Thus, at time $t = 0$, ionizing radiation with an intensity J_0 photons per cm² per sec falls on the plane face of a uniform cloud, containing n_1 H atoms per cm³, and also some cooling particles. The average photon has an energy $\frac{1}{2} m Q_0^2$ in excess of χ_0, the ionization potential of the H atom. Conditions are R-critical, and so

$$J_0 \approx \frac{4 n_1 Q_0}{3}. \tag{45}$$

An S-front advances into the gas, followed by an I-front. The incident radiation is weakened by the H atoms behind the I-front. The absorption coefficient of the H atom in the Lyman continuum varies as ν^{-3}, where ν is the frequency of the radiation absorbed, and photons of low frequency are removed more easily than photons of high frequency. The average energy of the photons remaining increases as their number decreases, and there will be a relation between J, the number of photons incident per cm² per sec, and $\frac{1}{2} m Q^2$, the average amount of thermal energy liberated at one ionization.

The relation can be worked out but is too complex to use. The approximation given here is good enough to illustrate the effect.

Consider an example in which the photons are removed in order of increasing frequency. In the radiation reaching the I-front all those photons have been removed whose frequency lies between ν_0, the Lyman limit, and ν_i say, where $\nu_i > \nu_0$. All photons with a higher frequency are still present.

The number of photons incident, per cm² per sec, is

$$J = \frac{8 \pi \delta}{c^2} \int_{\nu_i}^{\infty} \nu^2 e^{-h\nu/kT_*} d\nu$$
$$= \frac{8 \pi \delta}{c^2} (k T_*/h)^3 (\alpha_i^2 + 2\alpha_i + 2) e^{-\alpha_i},$$

where $\alpha_i = h\nu_i/k T_* = \chi_i/k T_*$, say, and $\delta = a^2/4r^2$ = dilution factor of the radiation.

If $\chi_0 = h\nu_0$ and $\alpha_0 = \chi_0/k T_*$ then

$$\frac{J}{J_0} = \frac{\alpha_i^2 + 2\alpha_i + 2}{\alpha_0^2 + 2\alpha_0 + 2} e^{-(\alpha_i - \alpha_0)}$$

and

$$\log_e \frac{J}{J_0} = \log_e \left(\frac{\alpha_i^2 + 2\alpha_i + 2}{\alpha_0^2 + 2\alpha_0 + 2}\right) - (\alpha_i - \alpha_0).$$

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In the cases which are of interest here $\alpha_\circ > 2$, and α_i lies between $\alpha_\circ$ and $6\alpha_\circ$. It is, then, a reasonable approximation to write

$$\log_e\left(\frac{\alpha_i^2+2\alpha_i+2}{\alpha_\circ^2+2\alpha_\circ+2}\right)-(\alpha_i-\alpha_\circ)\approx-\frac{2\alpha_\circ^2}{2\alpha_\circ^2+2\alpha_\circ+1}(\alpha_i-\alpha_\circ)=-\frac{\alpha_i-\alpha_\circ}{L_*},\text{ say,}$$

and so
$$\log_e\frac{J}{J_\circ}=-\frac{\chi_i-\chi_\circ}{L_*kT_*}. \tag{46}$$

The average energy of the photons reaching the I-front may be shown to be

$$E=\tfrac{1}{2}mQ^2+\chi_\circ=\frac{8\pi\delta}{Jc^2}\int_{\nu_i}^{\infty}h\nu^3e^{-h\nu/kT_*}d\nu$$

$$=\chi_i+kT_*\left(1+\frac{2\alpha_i+4}{\alpha_i^2+2\alpha_i+2}\right),$$

and
$$\tfrac{1}{2}mQ^2=\chi_i-\chi_\circ+kT_*\left(1+\frac{2\alpha_i+4}{\alpha_i^2+2\alpha_i+2}\right)$$
$$\approx\chi_i-\chi_\circ+kT_*. \tag{47}$$

By (46)
$$\chi_i-\chi_\circ=-L_*kT_*\log_e\frac{J}{J_\circ},$$

and so
$$Q^2=\frac{2kT_*}{m}\left(1-L_*\log_e\frac{J}{J_\circ}\right). \tag{48}$$

Now
$$L_*=1+\frac{2\alpha_\circ+1}{2\alpha_\circ^2}=1+\frac{(2\chi_\circ+kT_*)kT_*}{2\chi_\circ^2},$$

so that L_* varies from 1.6, for a star with $T_*=80000$ degK, to 1.06, for $T_*=10000$ degK.

The rate of absorption of photons in a given volume of the H II region equals the rate of recombination of H^+ ions and electrons, and therefore depends on the temperature and density of the gas there. SPITZER [1]) has discussed the heating and cooling of H II regions and has found that the temperature there will generally be about 8000 degK. We shall adopt this value. The density is not easy to determine because, once again, the motion of the gas is non-adiabatic. We therefore assume that the density of H^+ ions is uniform at any given time and equal to $2J/Q$, its value just behind the I-front. We shall use the formulae, found in section 4 for the steady radiation case, and see later how long they can reasonably be applied.

Let γ be the recombination coefficient of protons and electrons. (In this case γ (8000 degK) $=5\times10^{-13}$ cm^3 sec^{-1}). There are γn_i^2 recombinations per cm^3 per sec, and the number of ionizing photons removed from the incident radiation on passing through the H II region is

$$J_\circ-J=\int\gamma n_i^2\,ds,$$

integrated through the region. This estimate for J neglects the contribution of photons, emitted on the recombination of ions and electrons to form H atoms in the ground state. STRÖMGREN [2]) has made the same approximation in his first paper on H II regions and has given reasons to justify it.

The total number of ions in a column of unit cross-sectional area is $N=\int n_i ds$ and so

$$J_\circ-J=\gamma Nn_i=2\gamma NJ/Q$$

or
$$J_\circ=J(1+2\gamma N/Q). \tag{49}$$

The rate of formation of new ions per unit area is

$$J=\frac{dN}{dt}. \tag{50}$$

Putting $Q_\circ^2=\dfrac{2kT_*}{m}$ one can write (48) as

$$Q=Q_\circ\left(1-L_*\log_e\frac{J}{J_\circ}\right)^{\frac{1}{2}}. \tag{51}$$

This last equation is only approximately true, for there is an approximation in (47). In fact one can see from (47) that, if $\chi_\circ/kT_*=2$, which corresponds to $T_*=80000$ degK, then

$$Q_\circ^2=\frac{2kT_*}{m}(1+\tfrac{4}{5})$$

and so
$$Q_\circ=1.3\left(\frac{2kT_*}{m}\right)^{\frac{1}{2}}.$$

The highest value of T_* which needs to be considered is about 80000 degK, and so $Q_\circ$ is never out by more than about 30 per cent.

An approximate solution of the equations (49), (50) and (51) can be made as follows. First solve (49) and (50) with $Q=Q_\circ$.

Then
$$J_\circ=\frac{dN}{dt}(1+2\gamma N/Q_\circ)$$

$$J_\circ t=N+\gamma N^2/Q_\circ$$

$$(1+4\gamma J_\circ t/Q_\circ)^{\frac{1}{2}}=1+2\gamma N/Q_\circ$$

and
$$J=\frac{J_\circ}{(1+4\gamma J_\circ t/Q_\circ)^{\frac{1}{2}}}. \tag{52}$$

For large values of t

$$J\approx\frac{1}{2}\left(\frac{J_\circ Q_\circ}{\gamma t}\right)^{\frac{1}{2}}.$$

Conditions are R-critical at time $t=0$ and so $J_\circ=\dfrac{4n_1Q_\circ}{3}$.

Thus
$$J\approx\frac{n_1Q_\circ}{(3\gamma n_1t)^{\frac{1}{2}}}=\frac{J_\circ}{4}\left(\frac{3}{\gamma n_1t}\right)^{\frac{1}{2}}. \tag{53}$$

[1]) L. SPITZER JR, *Henry Norris Russell lecture*, *Ap. J.* **120**, 1, 1954.
[2]) B. STRÖMGREN, *Ap. J.* **89**, 526, 1938.

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This is always an overestimate but becomes quite close when $\gamma n_1 t \gg \frac{16}{3}$.

With $n_1 = 100$ atoms/cm^3, $\frac{J}{J_\circ} = \frac{1}{50}$ when $t = 10^{13}$ sec. This shows the order of magnitude by which J can vary in a reasonable length of time. According to (51) the value Q corresponding to this ratio of J to $J_\circ$ differs from $Q_\circ$ by a factor

$$(1 + L_* \log_e 50)^{\frac{1}{2}} \approx (1 + 4L_*)^{\frac{1}{2}}.$$

The largest likely value of L_* is about 1.6 and so the second estimate Q differs from $Q_\circ$, the first estimate, by a factor of 2.6, at most. If $Q_\circ$, in equation (52), is replaced by the second estimate of Q, the value of J is increased at most by a factor 1.6. The third estimate of Q then differs from the second by a factor smaller than

$$\left[\frac{1 + 1.6 \log_e (50 \times 1.6)}{1 + 1.6 \log_e 50}\right]^{\frac{1}{2}} = 1.05.$$ This is negligible. To a good approximation, then,

$$\begin{aligned} Q &= Q_\circ \left[1 - L_* \log_e \left\{\tfrac{1}{4}\left(\frac{3}{\gamma n_1 t}\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}} \\ &= Q_\circ \left[1 + 0.83 L_* + \frac{L_*}{2} \log_e \tau\right]^{\frac{1}{2}} \quad (\text{where } \tau = \gamma n_1 t) \\ &= Q_\circ \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2}\right]^{\frac{1}{2}}. \end{aligned} \tag{54}$$

One can write (53) as

$$J = \frac{n_1 Q_\circ}{\sqrt{3\tau}} \tag{55}$$

and

$$U_1 = \frac{J}{n_1} = \frac{Q_\circ}{\sqrt{3\tau}}. \tag{56}$$

The pressure in the gas just ahead of the I-front is

$$\begin{aligned} p_2 &= \frac{4QI}{5} = \frac{4QJm}{5} = \\ &= \frac{4Q_\circ^2 \rho_1}{5\sqrt{3\tau}} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2}\right]^{\frac{1}{2}} \\ &= \frac{8}{5\sqrt{3}} \frac{\rho_1 k T_*}{m} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{2}} \\ &\approx \frac{\rho_1 k T_*}{m} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{2}}. \end{aligned} \tag{57}$$

The temperature here is T_c and so the density is

$$\rho_2 = \frac{p_2}{k T_c / m} = \rho_1 \frac{T_*}{T_c} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{2}}. \tag{58}$$

Further, by (40), $V^2 = \frac{4QU_1}{5}$

$$= \frac{8}{5\sqrt{3}} \frac{k T_*}{m} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{2}}$$

and so

$$v_2 \approx V \approx \left(\frac{k T_*}{m}\right)^{\frac{1}{2}} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{4}}. \tag{59}$$

One can now decide how long the formulae derived for the steady radiation case remain a good approximation. In the unsteady radiation case the pressure $4QI/5$ at the back of region 2 decreases with time. Sets of expansion waves are therefore made to travel forward towards the S-front and these will tend to reduce and make uniform the pressure throughout the region. They will keep conditions fairly uniform if they can cross the region in a time short compared with that during which the pressure at the I-front changes significantly. If so the pressure and density just behind the S-front will at any instant be similar to that just ahead of the I-front, and the approximation to the steady radiation case is justified. (In the steady radiation case both pressure and density are uniform throughout region 2.)

Now by (57)

$$\log_e p_2 = \text{const.} + \tfrac{1}{2} \log_e (2 + 1.66 L_* + L_* \log_e \tau) - \tfrac{1}{2} \log_e \tau$$

and on differentiation with respect to t,

$$\frac{1}{p_2} \frac{dp_2}{dt} = \frac{1}{2\tau} \left[\frac{1}{2 + 1.66 L_* + L_* \log_e \tau} - 1\right] \frac{d\tau}{dt}$$

and for $\tau > 1$

$$\left|\frac{1}{p_2} \frac{dp_2}{dt}\right| < \frac{1}{2\tau} \frac{d\tau}{dt} = \frac{1}{2t}. \tag{60}$$

The proportional rate of change of the pressure at the back of region 2 is less than $\frac{1}{2t}$ at time t.

But the S-front moves at a speed ηV relative to the gas in the region 2 and, at time t, its distance from the I-front will be of the order $\eta V t$. An expansion wave moves through this distance in a time $\frac{\eta V t}{c_2}$; the state of region 2 is approximately uniform if p_2 changes by only a small amount in this time, i.e. if

$$\frac{1}{2t} \frac{\eta V t}{c_2} = \frac{\eta V}{2 c_2} \ll 1. \tag{61}$$

By (39) and (40) of section 4

$$\begin{aligned} \eta V &= \frac{5 k T_c}{4 Q U_1 m} \left(\frac{4 Q U_1}{5}\right)^{\frac{1}{2}} = \left(\frac{5 k T_c}{3m}\right)^{\frac{1}{2}} \left(\frac{3 k T_c}{4 m Q U_1}\right)^{\frac{1}{2}} \\ &= c_2 \left(\frac{3 k T_c}{4 m Q U_1}\right)^{\frac{1}{2}}. \end{aligned}$$

By (54) and (56)

$$\begin{aligned} Q U_1 &= Q_\circ \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{2}\right]^{\frac{1}{2}} \frac{Q_\circ}{\sqrt{3\tau}} \\ &= \frac{2 k T_*}{m} \left[\frac{2 + 1.66 L_* + L_* \log_e \tau}{6\tau}\right]^{\frac{1}{2}} \end{aligned}$$

and so

$$\eta V = c_2 \left[\frac{3}{8} \frac{T_c}{T_*} \left(\frac{6\tau}{2 + 1.66 L_* + L_* \log_e \tau}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

or

$$\frac{\eta V}{2 c_2} = \left(\frac{3}{32} \frac{T_c}{T_*}\right)^{\frac{1}{2}} \left(\frac{6\tau}{2 + 1.66 L_* + L_* \log_e \tau}\right)^{\frac{1}{4}}. \tag{62}$$

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Now $L_* > 1$ always, and the formula is valid, in any case, only when $\tau \gg 1$. Then $\dfrac{6\tau}{2 + 1.66L_* + L_* \log_e \tau} < 2\tau$

and so $$\frac{\eta V}{2c_2} < \left(\frac{3\sqrt{2}}{32}\right)^{\frac{1}{2}} \left(\frac{T_c}{T_*}\right)^{\frac{1}{2}} \tau^{\frac{1}{4}} = 0.35 \left(\frac{T_c}{T_*}\right)^{\frac{1}{2}} \tau^{\frac{1}{4}}.$$

The steady radiation solution is a good approximation as long as $\dfrac{\eta V}{2c_2} \ll 1$ and so certainly as long as $\left(\dfrac{T_c}{T_*}\right)^{\frac{1}{2}} \tau^{\frac{1}{4}} \ll 1$, or $\tau \ll \dfrac{T_*^2}{T_c^2}$.

With the typical values $T_* = 25000$ degK,
$T_c = 100$ degK,

the approximation remains good as long as

$$\gamma n_1 t = \tau \ll 6.3 \times 10^4,$$

or $t \ll \dfrac{6.3 \times 10^4}{\gamma n_1} = 3 \times 10^{14}$ sec $= 10^7$ years, if $n_1 =$ 200 H atoms/cm³.

This length of time is about the expected life-time of the exciting star.

An approximation can be used to simplify expression (59).

We have
$$v_2 \approx \left(\frac{kT_*}{m}\right)^{\frac{1}{2}} \left[\frac{2 + 1.66L_* + L_* \log_e \tau}{2\tau}\right]^{\frac{1}{4}}$$
$$\approx \left(\frac{2kT_*}{m}\right)^{\frac{1}{2}} \left[\frac{2 + 1.66L_* + L_* \log_e \tau}{8\tau}\right]^{\frac{1}{4}}$$
$$= Q_0 \left[\frac{2 + 1.66L_* + L_* \log_e \tau}{8\tau}\right]^{\frac{1}{4}}. \quad (63)$$

The lowest value of L_* is 1.0 and that of $\log_e \tau$ zero. The coefficient of Q_0 in (63) then equals $0.83\tau^{-\frac{1}{4}}$.

The largest likely value of L_* is 1.6 and that of τ is 3×10^5, giving 12.4 for the maximum of $\log_e \tau$ and $1.35\tau^{-\frac{1}{4}}$ for the maximum value of the coefficient of Q_0. It is quite a good approximation to put

$$v_2 = 1.1\, Q_0 \tau^{-\frac{1}{4}} = \frac{1.1\, Q_0}{(\gamma n_1 t)^{\frac{1}{4}}}. \quad (64)$$

When time t has elapsed the cloud, and the shock front, have travelled a distance

$$s_2 = \frac{1.5\, Q_0 t^{\frac{3}{4}}}{(\gamma n_1)^{\frac{1}{4}}} = \frac{1.5\, Q_0 \tau^{\frac{3}{4}}}{\gamma n_1}. \quad (65)$$

TABLE 2

To show typical accelerations and compressions produced by stars of different spectral classes, and the typical masses of neutral gas affected.

1	2	3	4	5	6		7
Type of star	Temperature T_* (degK)	Radius a (cm)	$\frac{1}{4\pi} \times$ photon output (sec⁻¹)	Q_0 (cm sec⁻¹)	Radius of H II region (cm)	(pc)	Photon intensity at half radius (cm⁻²sec⁻¹)
O5	80 000	1.8×10^{12}	1.2×10^{50}	3.6×10^6	2×10^{20}	66	1.1×10^{10}
O7	50 000	1.0×10^{12}	4.7×10^{48}	2.9×10^6	6.8×10^{19}	23	4.0×10^9
O9	32 000	6.2×10^{11}	2.9×10^{47}	2.3×10^6	2.7×10^{19}	9	1.6×10^9
B0	25 000	4.6×10^{11}	1.4×10^{46}	2.1×10^6	9.6×10^{18}	3.2	6.0×10^8
B2	20 000	3.7×10^{11}	1.2×10^{45}	1.8×10^6	4.0×10^{18}	1.3	2.4×10^8

1	8	9	10	11	12	13	14
					Final speed for		
Type of star	Critical density at half radius (cm⁻³)	Maximum compression $(\eta \max)^{-1}$	Maximum speed (v_2)max (km/s)	Maximum mass accelerated ($M_\odot$)	maximum mass (km/s)	$M_\odot$ (km/s)	100 $M_\odot$ (km/s)
O5	2250	1,700	37	2×10^6	1.4	7.0	4.0
O7	1000	1,050	30	3.3×10^4	2.1	7.0	4.0
O9	500	650	24	1.0×10^3	2.5	5.5	3.0
B0	225	520	22	22.5	4.0	5.6	
B2	100	420	19	0.75	5.5		

Notes.

Column 4: The photon output referred to is the number of photons *beyond* the *Lyman* limit emitted by the star per second.

Column 6: For a density of 10 H⁺ ions per cm³ in the H II region.

Column 7: The weakening of the radiation by absorption in the H II region is allowed for.

Column 8: The critical density is such as to make conditions *R*-critical at the cloud surface.

Column 11: This is the mass contained in a cube whose side is half the radius of the H II region, with a density of H atoms equal to that given in column 8. (N.B.: Mass of a typical interstellar cloud = 60 $M_\odot$.)

Columns 12, 13, 14: The mass of gas is supposed contained in a cube initially. Gas density as in column 8.

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Some representative values are given in the table for typical stars of different spectral classes. The temperatures for the different spectral classes resemble those taken by STRÖMGREN [1]), and the relation between typical radii and temperatures is derived from some formulae for main sequence stars found empirically by PARENAGO and MASSEVICH [2]).

Each star is surrounded by an H II region, for which a particle density of 10 H^+ ions per cm^3 has been adopted. The radiation from the star can accelerate a neutral cloud only if this lies inside the H II region. The values in the table are given for a cloud whose inner surface lies, initially, at a distance from th star equal to half the radius of the H II region. The critical value of the density, given in column 8, is such as to make the conditions just *R*-critical at the cloud surface. The values are rather high, but if the densities were lower the part of the cloud near the star would just become a part of the general H II region and would not be accelerated. The more remote parts of such a cloud may be pushed away by the rocket action but only if the ionized part of the cloud concerned is extensive enough and weakens the ionizing radiation sufficiently.

The critical values given for the cloud density are large enough to permit us to treat the background H II region as though it were a vacuum, at least as far as dynamical effects are concerned.

The maximum velocities and compressions given will occur soon after the beginning of the motion when the cooling particles have begun to dispose of the extra thermal energy but before the ionizing radiation has been seriously weakened in the newly formed ionized region. The rate of cooling should be quite rapid with the high gas densities expected. The order of magnitude of the initial velocities and compressions is satisfactory.

The entry in column 11 is to show only very roughly how much matter can possibly be pushed away. It is seen that stars later than type B0 will not be effective in accelerating large masses though they can compress smaller masses quite well. The final velocities are worked out on the assumption that the shock-front has not yet reached the far boundary of the undisturbed neutral cloud.

In every case the cloud is supposed to have the shape of a cube, initially. Clouds which are more flattened and present a larger area to the star can clearly be given greater speeds but will hardly be recognisable as units when they have been flattened still more by the rocket effect. The velocities are found from equations (64) and (65). They are of the expected order of magnitude for clouds of one to a hundred solar masses, though perhaps too small by a factor of 1.5 or 2.5. Larger masses will generally acquire smaller speeds. But there are two reasons why the final speeds given in the table may be too low.

Firstly it is possible that the shock front passes through the far end of the unaccelerated cloud (region 1). After this has happened there is no pressure acting on the front of the compressed cloud (region 2) and the gas will expand forwards, away from the exciting star. An expansion wave therefore travels back through the compressed gas towards the I-front. When it reaches there the gas density ahead of the I-front falls, and conditions there will no longer be *D*-critical, but become *M*-type. A second shock then advances into the neutral gas. In this way the neutral gas can be accelerated two, or even more times.

Secondly we have considered only a one-dimensional model in this paper, and have therefore not allowed for the fact that the ionized gas formed behind the I-front can expand into three dimensions, instead of only one. If this is allowed for the density of protons and electrons in the ionized region will be found to be somewhat smaller. This reduces the rate of recombination there, and the absorption of ionizing photons will be less important. Thus, finally, the radiation falling on the I-front will be more intense and the pressure at the back of region 2 greater than we have found.

A further calculation is needed to decide how much can be gained by making these two improvements.

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[1]) B. STRÖMGREN, *Ap. J.* **108**, 242, 1948.
[2]) P. P. PARENAGO and A. G. MASSEVICH, *Sternberg Publ.* No. 20, 1951.

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