Optimisation Coursework

February 2020

1

$$\begin{aligned} \text{Minimise} \quad & x_1 - 3x_2 \\ \text{Subject to} \quad & x_1 - x_2 \leq 1 \\ & x_2 - x_2 \geq -1 \\ & 2x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We want to rewrite this in Standard Equational Form.
First, we convert to a maximisation problem. So we are maximising

$x_1 - 3x_2$

subject to the same constraints as before. We then add slack variables x_3, x_4, x_5 to remove the inequalities from the constraints:

$x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq 1$
$2x_1 - x_2 \leq 3$
$x_1, x_2 \geq 0$

Which becomes:

$x_1 - x_2 + x_3 = 1$
$-x_1 + x_2 + x_4 = 1$
$2x_1 - x_2 + x_5 = 3$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

This forms the Tableau

$$T^1 = \left(\begin{array}{c|ccccc|c} 1 & 1 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 & 0 & 1 & 3 \end{array}\right)$$

$B = \{3, 4, 5\}$

The only $k \in N$ where $c_k \leq 0$ is 2, so we select column 2 for entering.
$t = min\{-, 1, -\} = 1$, which is found at row index 2. Therefore we choose $T^1_{2,2}$ as the pivot, and get

$$T^2 = \left(\begin{array}{c|ccccc|c} 1 & -2 & 0 & 0 & 3 & 0 & 3 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}\right)$$

$B = \{2, 3, 5\}$

Again, there is only one entering option, so we choose column 1 for entering. $t = min\{-, -, 4\} = 4$, which is found at row index 3. Therefore we pivot on $T^2_{3,1}$ to get

$$T^3 = \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & 0 & 5 & 2 & 11 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \end{array}\right)$$

$B = \{1, 2, 3\}$

The top row of T^3 is non-negative, so we have a feasible basic solution of $(4, 5, 2, 0, 0)$ with value 11.

The original problem was one of minimisation, and we have maximised the negative value of the minimum.
Therefore the minimum value is -11, where $x_1 = 4, x_2 = 5$

2

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -3 & -1 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \quad c = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We use these to create the Tableau

$$T^1 = \left(\begin{array}{c|cccccc|c} 1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & -3 & -1 & 0 & 0 & 1 & 6 \end{array}\right)$$

Using Bland's rule, we choose to enter at column 1. $t = min\{-, 2, 3\} = 2$, at row index 2. So we pivot on $T^1_{2,1}$ to get

$$T^2 = \left(\begin{array}{c|cccccc|c} 1 & 0 & -3 & 1 & 1 & 2 & 0 & 4 \\ \hline 0 & 0 & -1 & 1 & 1 & 2 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & 0 & -2 & 1 & 2 \end{array}\right)$$

$B = \{1, 4, 6\}$

The only $k \in N$ where $c_k \leq 0$ is 2, however $A_2 \leq \mathbf{0}$, so the solution is unbounded. The basic solution at this point is $(2, 0, 0, 5, 0, 2)$. Our solution can be written as

$$x'_B = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} - t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$x'_N = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

so combining these, our certificate of unboundedness is $\begin{pmatrix} 2 \\ 0 \\ 0 \\ 5 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

3

The objective function is the total value of all new mixed paint created. So using variables r, g, b, k to represent red, green, blue and black respectively we have the objective function:

$$10r + 15g + 25b + 25k$$

For red, green, blue and black, we then add constraints representing the recipes used to create the new colours, and to ensure that the paints are mixed in the correct ratios. Note that for each primary colour we have four separate variables. For instance cyan is represented by c_r, c_g, c_b, c_k. This is because the cyan paint used to make red cannot then be used to make any other colours, and we need to explicitly state this.

We also have constraints specifying the maximum amounts of cyan, magenta and yellow paints that we are allowed to use. Finally, all variables must have values ≥ 0 (this constraint is omitted from the list below as you can specify this in PuLP when initialising each variable). In total we add the constraints:

$$y_r + m_r = r$$
$$y_r = m_r$$
$$y_g + c_g = g$$
$$y_g = c_g$$
$$m_b + c_b = b$$
$$m_b = c_b$$
$$y_k + m_k + c_k = k$$
$$y_k = m_k$$
$$y_k = c_k$$
$$m_k = c_k$$
$$y_r + y_g + y_b + y_k \leq 10$$
$$m_r + m_g + m_b + m_k \leq 5$$
$$c_r + c_g + c_b + c_k \leq 11$$

PuLP gives the results $r = 0, g = 12, b = 2, k = 12$ (to see the full results, run the python program). In total, this uses all 26 gallons of paint that we had at the beginning, which makes sense as we are aiming to maximise the value of the new paint produced. The program has also prioritised higher value paints - no red paint has been mixed since it is the least valuable.

4

a

We use variables A, B, C, D, E, F to represent the six different items. These are all integers taking a value from $\{0, 1\}$

The objective function is the total value of all items that we choose to take, which is written as:

$$60A + 70B + 40C + 70D + 17E + 100F$$

There is also a single constraint. This specifies that the total weight of all the items we take does not exceed 20kg:

$$6A + 7B + 4C + 9D + 3E + 8F \leq 20$$

PuLP gives the solution $A = 0, B = 1, C = 1, D = 0, E = 0, F = 1$. The total weight of the three items it chooses to take is 19kg, close to the maximum allowed. The values per kg for each item are:

$A : 10$
$B : 10$
$C : 10$
$D : 7.78$
$E : 5.67$
$F : 12.5$

The solution to the LP picks three of the items with the highest value per kg, which makes sense in context.

b

We add the additional constraint:

$D \geq C$

which ensures that if $C = 1$, D cannot be 0, i.e. if we take C, we have to take D as well.

The solution changes to $A = 0, B = 1, C = 0, D = 0, E = 1, F = 1$, with both items C and D not being chosen.

c

We add an additional variable X, which is an integer that can take any non-negative value. X represents the number of kg over the 20kg limit the luggage weighs. The objective function then becomes:

$$60A + 70B + 40C + 70D + 17E + 100F - 15X$$

To ensure that the correct value of X is set, we modify the weight constraint by removing the inequality and adding X as a slack variable:

$$6A + 7B + 4C + 9D + 3E + 8F - 20 - X = 0$$

The optimal solution becomes to $A = 1, B = 1, C = 0, D = 0, E = 0, F = 1$, with a total weight of 21kg. The only change to the solution to part 4a is that instead of taking item C and being under the weight limit by 1, we now take item A instead and are over the weight limit by 1. This incurs a cost of £15, but allows us to increase the value of the luggage by £20, which is an overall improvement of £5. It is not worth going any further over the limit by taking more items, since no item is worth more than £15/kg, which means the fine outweighs the value of any additional items.