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Singular integral operators with non-smooth kernels on irregular domains

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Abstract. Let $\mathcal{X}$ be a space of homogeneous type. The aims of this paper are as follows.

i) Assuming that T is a bounded linear operator on $L_2(\mathcal{X})$, we give a sufficient condition on the kernel of T so that T is of weak type $(1,1)$, hence bounded on $L_p(\mathcal{X})$ for $1 < p \leq 2$; our condition is weaker than the usual Hörmander integral condition.

ii) Assuming that T is a bounded linear operator on $L_2(\Omega)$ where Ω is a measurable subset of $\mathcal{X}$, we give a sufficient condition on the kernel of T so that T is of weak type $(1,1)$, hence bounded on $L_p(\Omega)$ for $1 < p \leq 2$.

iii) We establish sufficient conditions for the maximal truncated operator T_*, which is defined by $T_* u(x) = \sup_{\varepsilon > 0} |T_\varepsilon u(x)|$, to be L_p bounded, $1 < p < \infty$. Applications include weak $(1,1)$ estimates of certain Riesz transforms, and L_p boundedness of holomorphic functional calculi of linear elliptic operators on irregular domains.

1. Introduction.

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, equipped with a metric d and a measure μ. Let T be a bounded linear operator on

$L_2(\mathcal{X})$ with an associated kernel $k(x, y)$ in the sense that

$$(Tf)(x) = \int_{\mathcal{X}} k(x, y)\, f(y)\, d\mu(y)\,, \tag{1}$$

where $k(x, y)$ is a measurable function, and the above formula holds for each continuous function f with compact support, and for almost all x not in the support of f.

One important result of Calderón–Zygmund operator theory is the well known Hörmander integral condition on the kernel $k(x, y)$, see [Hör], which is a sufficient condition for the operator T to be of weak type $(1, 1)$. It states that T satisfies weak $(1, 1)$ estimates if there exist constants C and $\delta > 1$ so that

$$\int_{d(x,y)\geq\delta d(y_1,y)} |k(x, y) - k(x, y_1)|\, d\mu(x) \leq C\,,$$

for all $y, y_1 \in \mathcal{X}$.

In practice, many operators satisfy the Hörmander integral condition, but there are numerous examples of operators which do not, and certain classes of such operators can be proved to be of weak type $(1, 1)$. See, for example [F], [Ch1], [CR], [Hof], [Se]. However, in these papers, the authors investigate specific classes of operators and do not give sufficient conditions on kernels for general operators to be of weak type $(1, 1)$.

A natural question is whether one can weaken the Hörmander integral condition and still conclude that T is of weak type $(1, 1)$. Although Calderón–Zygmund operator theory is now well established, to our best knowledge, no such condition is known. Our first aim is to give a positive answer to this open question.

There is another limitation of the usual Calderón–Zygmund theory. It is only established for spaces of homogeneous type. The main feature of these spaces is that they satisfy the doubling property. Measurable subsets of $\mathbb{R}^n$ which do not possess any smoothness of their boundaries, do not satisfy the doubling property, hence they are not spaces of homogeneous type. Such measurable sets, however, do appear naturally in partial differential equations. Our second aim is to present a sufficient condition on the kernel of a bounded operator T on $L_2(\Omega)$, where Ω is a measurable subset of a space of homogeneous type, so that T is of weak type $(1, 1)$ on Ω.

The paper is organised as follows. In Section 2, we assume that T is a bounded linear operator on $L_2(\mathcal{X})$, where $\mathcal{X}$ is a space of homogeneous type. We then prove a sufficient condition on the kernel $k(x,y)$ of T so that T is of weak type $(1,1)$ (Theorem 1). Roughly speaking, T is of weak type $(1,1)$ if there exists a class of operators A_t with kernels $a_t(x,y)$, which play the role of approximations to the identity, so that the kernels $k_t(x,y)$ of the composite operators TA_t satisfy the condition

$$\int_{d(x,y)\geq ct^{1/m}} |k(x,y)-k_t(x,y)|\,d\mu(x) \leq C\,,$$

for some positive constants m, c, C, uniformly in $y \in \mathcal{X}$ and $t > 0$. The freedom in choosing A_t is important. In particular circumstances we may require them to commute with T, or we may wish to allow the kernels a_t to be discontinuous.

It is not difficult to check that our condition is a consequence of the Hörmander integral condition (Proposition 1).

In Section 3, we assume that Ω is a measurable subset of a space of homogeneous type with no smoothness on the boundary. We then present a sufficient condition on the kernel $k(x,y)$ which is somewhat stronger than that of Theorem 1, so that the operator T is of weak type $(1,1)$ on Ω (Theorem 2). Our result gives new criteria to investigate the L_p boundedness of singular integrals on measurable sets. The results on Ω are made possible by the fact that no smoothness is required on the kernels $a_t(x,y)$ in Theorem 1.

In Section 4, we extend the results in sections 2 and 3 to establish sufficient conditions on the kernel $k(x,y)$ which ensure the L_p boundedness of the maximal truncated operator T_*, where $T_*u(x) = \sup_{\varepsilon>0}|T_\varepsilon u(x)|$ and

$$T_\varepsilon u(x) = \int_{d(x,y)\geq\varepsilon} k(x,y)\,u(y)\,d\mu(y)\,.$$

Our assumptions on the kernel $k(x,y)$ are somewhat stronger than those used in Theorems 1 and 2, but are essentially weaker than the usual ones on spaces of homogeneous type (Theorem 3). The result is new for measurable subsets of spaces of homogeneous type (Theorem 4).

Applications are given in Section 5. We first establish weak $(1,1)$ estimates for certain Riesz transforms and similar types of operators (Theorem 5). This allows us, for example, to simplify the proof of the L_p boundedness of the Riesz transforms on Lie groups which was given by Saloff–Coste when $1 < p \leq 2$ [SC].

Finally, we prove that every operator L with a bounded holomorphic functional calculus in $L_2(\Omega)$, which generates a semigroup with suitable upper bounds on its heat kernels, also has a bounded holomorphic functional calculus in $L_p(\Omega)$ when $1 < p < \infty$ (Theorem 6). Here Ω is a measurable subset of a space $\mathcal{X}$ of homogeneous type. It is this result which prompted our investigation, so let us outline its background.

In the case when the heat kernels also satisfy Hölder bounds, then this result follows from the usual Calderón–Zygmund theory, because the operators $f(L)$ in the functional calculus satisfy standard Calderón–Zygmund bounds. This is the approach developed by Duong in the case of those elliptic operators having such heat kernels, which are defined by boundary conditions on strongly Lipschitz domains. See his thesis [Du] and also [DM^c]. This method does not work for those elliptic operators whose heat kernels satisfy pointwise bounds but not Hölder bounds. In [DR], Duong and Robinson showed how to proceed in such cases, provided still that the operators are defined on strongly Lipschitz domains. There they proved the first part of Theorem 6 of this paper in the case when Ω is a space of homogeneous type, though the last part, namely the L_p boundedness of the maximal truncated operators, is new. In [AE], Arendt and ter Elst applied this theorem to the Dirichlet problem for certain elliptic operators defined on subsets of $\mathbb{R}^n$ whose boundary has null measure, by extending the functional calculus to that of an operator defined on all of $\mathbb{R}^n$. They asked whether the assumption concerning the null measure of the boundary could be dropped. This is what we do in Theorem 6.

As can be seen, our investigations into removing the assumption of Hölder continuity from the kernels have led to the formulation of general conditions on singular integral operators which are applicable in a variety of situations.

2. Weak (1,1) estimates of singular integral operators.

Let $\mathcal{X}$ be a topological space equipped with a measure μ and a metric d which is a measurable function on $\mathcal{X} \times \mathcal{X}$. We define $\mathcal{X}$ to be *a space of homogeneous type* if the balls $B(x;r) = \{y \in \mathcal{X} : d(x,y) < r\}$ satisfy the doubling property

$$\mu(B(x;2\,r)) \le c\,\mu(B(x;r)) < +\infty\,,$$

for some $c \ge 1$ uniformly for all $x \in \mathcal{X}$ and $r > 0$. A more general definition can be found in [CW, Chapter 3].

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x;\lambda\, r)) \leq c\,\lambda^n \mu(B(x;r))\,,$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$. The parameter n is a measure of the dimension of the space. There also exist c and N, $0 \leq N \leq n$ so that

$$\text{(U)} \qquad \mu(B(y;r)) \leq c\Big(1+\frac{d(x,y)}{r}\Big)^N \mu(B(x;r))$$

uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (U) with $N = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the cases of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

Let T be a bounded linear operator mapping $L_2(\mathcal{X})$ into $L_2(\mathcal{X})$. Assume the operator T is given by a kernel $k(x,y)$ in the sense of (1).

We shall work with a class of integral operators A_t, $t > 0$, which plays the role of approximations to the identity. We assume the operators A_t can be represented by kernels $a_t(x,y)$ in the sense that

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y)\, u(y)\, d\mu(y)\,,$$

for every function $u \in L_2(\mathcal{X}) \cap L_1(\mathcal{X})$, and the kernels $a_t(x,y)$ satisfy the following conditions

$$\text{(2)} \qquad |a_t(x,y)| \leq h_t(x,y)\,,$$

for all $x, y \in \mathcal{X}$ where $h_t(x,y)$ is a function satisfying

$$\text{(3)} \qquad h_t(x,y) = (\mu(B(x;t^{1/m})))^{-1} s(d(x,y)^m\, t^{-1})$$

in which m is a positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\kappa}\, s(r^m) = 0\,,$$

for some $\kappa > N$, where N is the power which appeared in property (U), and n the “dimension” entering the strong homogeneity property.

It then follows that

$$
\begin{aligned}
h_t(x,y) &\le c \min\Big\{\frac{1}{\mu(B(x;t^{1/m}))},\frac{1}{\mu(B(y;t^{1/m}))}\Big\} \\
&\qquad \cdot \Big(1+\frac{d(x,y)}{t^{1/m}}\Big)^N s(d(x,y)^m\, t^{-1}) \qquad (4)\\
&\le c \min\Big\{\frac{1}{\mu(B(x;t^{1/m}))},\frac{1}{\mu(B(y;t^{1/m}))}\Big\} s_1(d(x,y)^m\, t^{-1})\,,
\end{aligned}
$$

where s_1 is a function similar to s with some $\kappa > 0$.

We also note that there exist positive constants c_1 and c_2 so that

$$c_1 \le \int_{\mathcal{X}} h_t(x,y)\, d\mu(x) \le c_2$$

uniformly in t and y.

The existence of such a class of operators A_t in a space of homogeneous type, is not a problem. We can first choose a function s satisfying the decay condition in (3), define h_t as in (3), and let $a_t = h_t$, hence conditions (2) and (3) are automatically satisfied. The kernels a_t then possess the smoothness of the function s.

For any $m > 0$, we can also construct $a_t(x,y)$ with the following additional properties

$$a_t(x,y) = 0\,, \qquad \text{when } d(x,y) \ge c_0\, t^{1/m}\,, \tag{5}$$

$$\int_{\mathcal{X}} a_t(x,y)\, d\mu(x) = 1\,, \tag{6}$$

for all $y \in \mathcal{X}$, $t > 0$. This can be achieved by choosing

$$a_t(x,y) = (\mu(B(y;t^{1/m})))^{-1} \chi_{B(y;t^{1/m})}(x)\,,$$

where $\chi_{B(y;t^{1/m})}$ denotes the characteristic function on the ball $B(y;t^{1/m})$. Then let A_t be the operators which are given by the kernels $a_t(x,y)$.

These operators A_t constructed as above exist in the space $\mathcal{X}$ independently of the operator T. However, for certain operators T, it is useful to construct operators A_t which are related to T. This is of interest since the analysis of A_t, TA_t and A_tT is useful for establishing the boundedness of T in an L_p space. Examples of this are given in Section 5.

The following lemma is needed in the proof of Theorem 1. For its proof, see [DR, Proposition 2.5].

Lemma 1. *Given functions $h_t(x,z)$ which satisfy* (3)*, and $\nu > 0$, there exist positive constants c and θ such that*

$$\sup_{z\in B(y,r)} h_t(x,z) \le c \inf_{z\in B(y,r)} h_{\theta t}(x,z)$$

uniformly for $x, y \in \mathcal{X}$, and $r, t > 0$ with $r^m \le \nu\, t$.

We now present the main result of this section. The proof is based on that used by Duong and Robinson in proving [DR, Theorem 3.1]. It relies upon the idea of Hebisch [He] of using L_2-estimates to obtain weak type $(1,1)$ bounds. Related ideas also appeared earlier in [F].

Theorem 1. *Let T be a bounded linear operator from $L_2(\mathcal{X})$ to $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$. Assume there exists a class of operators A_t, $t > 0$, which satisfy the conditions* (2) *and* (3) *so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1) *and there exist constants C and $c > 0$ so that*

$$\int_{d(x,y)\ge ct^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \le C\,, \tag{7}$$

for all $y \in \mathcal{X}$.

Then the operator T is of weak type $(1,1)$. Hence, T can be extended from $L_2(\mathcal{X}) \cap L_p(\mathcal{X})$ to a bounded operator on $L_p(\mathcal{X})$ for all $1 < p \le 2$.

Proof. We need to prove that T satisfies weak type $(1,1)$ estimates. Boundedness of T on $L_p(\mathcal{X})$ then follows from the Marcinkiewicz interpolation theorem.

Our proof makes use of the Calderón–Zygmund decomposition to decompose an integrable function into "good" and "bad" parts (see, for example, [CW]), then each part is analysed separately.

Given $f \in L_1(\mathcal{X}) \cap L_2(\mathcal{X})$ and $\alpha > \|f\|_1(\mu(\mathcal{X}))^{-1}$, then there exist a constant c independent of f and α, and a decomposition

$$f = g + b = g + \sum_i b_i\ ,$$

so that

a) $|g(x)| \le c\,\alpha$ for almost all $x \in \mathcal{X}$,

b) there exists a sequence of balls Q_i so that the support of each b_i is contained in Q_i and

$$\int |b_i(x)|\, d\mu(x) \le c\,\alpha\,\mu(Q_i)\,,$$

c) $\displaystyle\sum_i \mu(Q_i) \le \frac{c}{\alpha}\int |f(x)|\, d\mu(x)\,,$

d) each point of $\mathcal{X}$ is contained in at most a finite number N of the balls Q_i.

Note that if $\mu(\mathcal{X}) = \infty$, then $\|f\|_1(\mu(\mathcal{X}))^{-1}$ means 0. Besides that, the functions b_i are usually chosen to satisfy $\int b_i\, d\mu(x) = 0$ as well, but we do not need this property.

Conditions b) and c) also imply that $\|b\|_1 \le c\,\|f\|_1$ and hence that $\|g\|_1 \le (1+c)\,\|f\|_1$.

We have

$$\mu(\{x:\ |Tf(x)| > \alpha\}) \\ \le \mu\Big(\Big\{x:\ |Tg(x)| > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{x:\ |Tb(x)| > \frac{\alpha}{2}\Big\}\Big)\,.$$

It is not difficult to check that $g \in L_2(\mathcal{X})$. Using the facts that T is bounded on $L_2(\mathcal{X})$ and that $|g(x)| \le c\,\alpha$, we obtain

$$\mu\Big(\Big\{x:\ |Tg(x)| > \frac{\alpha}{2}\Big\}\Big) \le 4\,\alpha^{-2}\,\|Tg\|_2^2 \le c_1\,\alpha^{-2}\|g\|_2^2 \le \frac{c_2}{\alpha}\,\|f\|_1\,. \tag{8}$$

Concerning the "bad" part $b(x)$, we temporarily fix a b_i whose support is contained in Q_i, then choose $t_i = r_i^m$ where m is the constant appearing in (3) and r_i is the radius of the ball Q_i. We then decompose

$$Tb_i(x) \;=\; TA_{t_i} b_i(x) + (T - TA_{t_i})\, b_i(x)\,.$$

To analyse $TA_{t_i} b_i(x)$, we first estimate the function $A_{t_i} b_i$. Since

$$A_{t_i} b_i(x) = \int_{\mathcal{X}} a_{t_i}(x,y)\, b_i(y)\, d\mu(y)\,,$$

it follows from Lemma 1 that

$$\begin{aligned} |A_{t_i} b_i(x)| &\le \int_{\mathcal{X}} h_{t_i}(x,y)\, |b_i(y)|\, d\mu(y) \\ &\le \|b_i\|_1 \sup_{y\in Q_i} h_{t_i}(x,y) \\ &\le c\,\alpha\, \mu(Q_i) \inf_{y\in Q_i} h_{\theta t_i}(x,y) \\ &\le c\,\alpha \int_{\mathcal{X}} h_{\theta t_i}(x,y)\, \chi_i(y)\, d\mu(y)\,, \end{aligned}$$

where χ_i denotes the characteristic function of the ball Q_i.

Denoting by M the Hardy-Littlewood maximal operator, we then have for any $u \in L_2(\mathcal{X})$

$$\begin{aligned} |\langle |u|, A_{t_i} b_i\rangle| &\le c\,\alpha \int_{\mathcal{X}}\int_{\mathcal{X}} |u(x)|\, h_{\theta t_i}(x,y)\, \chi_i(y)\, d\mu(y)\, d\mu(x) \\ &\le c\,\alpha\, \langle M\,|u|, \chi_i\rangle\,. \end{aligned}$$

Note that the second inequality follows from properties 3) and 4'). Since the Hardy-Littlewood maximal operator is bounded on $L_2(\mathcal{X})$, (see for example [Ch2]), it follows that

$$\Big\| \sum_i A_{t_i} b_i \Big\|_2 \le c\,\alpha \Big\| \sum_i \chi_i \Big\|_2\,. \tag{9}$$

We now use properties c) and d) of the Calderón–Zygmund decomposition to obtain the estimate

$$\Big\| \sum_i A_{t_i} b_i \Big\|_2 \le c\,\alpha \Big(\sum_i \mu(Q_i) \Big)^{1/2} \le c\,\alpha^{1/2} \|f\|_1^{1/2}\,. \tag{10}$$

Therefore

$$\begin{aligned} \mu\Big(\Big\{x:\ \Big|\sum_i T A_{t_i} b_i(x)\Big| > \frac{\alpha}{4}\Big\}\Big) &\le 16\,\alpha^{-2} \Big\| \sum_i T A_{t_i} b_i \Big\|_2^2 \\ &\le c\,\alpha^{-2} \Big\| \sum_i A_{t_i} b_i \Big\|_2^2 \\ &\le \frac{c}{\alpha} \|f\|_1\,. \end{aligned} \tag{11}$$

On the other hand

$$\mu\Big(\Big\{x:\ \Big|\sum_i (T-TA_{t_i})\, b_i(x)\Big| > \frac{\alpha}{4}\Big\}\Big) \\ \leq \sum_i \mu(B_i) + \sum_i \frac{4}{\alpha}\int_{{}^cB_i} |(T-TA_{t_i})\, b_i(x)|\, d\mu(x)\,,$$

where ${}^cB_i$ denotes the complement of B_i which is the ball with the same centre y_i as that of the ball Q_i in the Calderón–Zygmund decomposition but with radius increased by the factor $(1+c)$, where c is the constant in (7). Because of property c) of the decomposition and the doubling property of $\mathcal{X}$, we have

$$\sum_i \mu(B_i) \leq c\sum_i \mu(Q_i) \leq c\,\alpha^{-1}\|f\|_1\,. \tag{12}$$

By assumption (7), we have

$$\begin{aligned}\int_{{}^cB_i} &|(T-TA_{t_i})\, b_i(x)|\, d\mu(x) \\ &\leq \int_{{}^cB_i} \Big|\int_{\mathcal{X}} k(x,y) - k_{t_i}(x,y)\, b_i(y)\, d\mu(y)\Big|\, d\mu(x) \\ &\leq \int_{\mathcal{X}} |b_i(y)| \Big(\int_{d(x,y)\geq c t_i^{1/m}} |k(x,y)-k_{t_i}(x,y)|\, d\mu(x)\Big)\, d\mu(y) \\ &\leq C\,\|b_i\|_1\,,\end{aligned}$$

because $B(y; c\,t_i^{1/m}) \subset B_i$.

Therefore

$$\sum_i \frac{1}{\alpha}\int_{{}^cB_i} |(T-TA_{t_i})\, b_i(x)|\, d\mu(x) \leq \sum_i \frac{C}{\alpha}\,\|b_i\|_1 \leq \frac{C\,\|f\|_1}{\alpha}\,. \tag{13}$$

Combining the estimates (8), (11), (12) and (13), the theorem is proved.

Remark.

i) It is straightforward from the proof of Theorem 1, that the existence of both the kernels $k(x,y)$ of T and $k_t(x,y)$ of TA_t is not necessary. We only need to assume that the difference operator $T-TA_t$ has an associated kernel so that this kernel (in place of $k(x,y) - k_t(x,y)$) satisfies Condition 7. This remark also applies to Theorem 2.

ii) In Theorem 1, the assumption on boundedness of T on the space $L_2(\mathcal{X})$ can be replaced by boundedness of T on a space $L_{p_o}(\mathcal{X})$ for some $p_0 > 1$. The proof would need only minor changes to show that T is of weak type $(1,1)$, hence bounded on $L_p(\mathcal{X})$ for all $1 < p \leq p_o$.

iii) Theorem 1 and a standard duality argument give the following result.

Let T be a bounded linear operator from $L_2(\mathcal{X})$ into $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$ in the sense of (1). Assume there exists a class of operators B_t whose kernels satisfy the conditions (2) and (3) so that the composite operators B_tT have associated kernels $K_t(x,y)$ in the sense of (1), and there exist constants $c > 0$ and C so that

$$\int_{d(x,y) \geq ct^{1/m}} |k(x,y) - K_t(x,y)| \, d\mu(y) \leq C \,, \tag{14}$$

for all $x \in \mathcal{X}$.

Then the adjoint operator T^* is of weak type $(1,1)$. Hence, T can be extended from $L_2(\mathcal{X}) \cap L_p(\mathcal{X})$ to a bounded operator on $L_p(\mathcal{X})$ for all $2 \leq p < \infty$.

Natural questions about Theorem 1 are how strong is the assumption (7), and what is its relation with the Hörmander integral condition. We shall show that, for suitably chosen A_t, our condition (7) is actually a consequence of the Hörmander integral condition for spaces of homogeneous type.

Proposition 1. *Assume that T has an associated kernel $k(x,y)$ which satisfies the Hörmander integral condition, i.e. there exist constants C and $\delta > 1$, so that*

$$\int_{d(x,y) \geq \delta d(y,z)} |k(x,y) - k(x,z)| \, d\mu(x) \leq C \,,$$

for all $y, z \in \mathcal{X}$. Let A_t be approximations to the identity which are represented by kernels $a_t(x,y)$ satisfying conditions (2), (3), (5) *and* (6).

Then the kernels $k_t(x,y)$ of TA_t satisfy condition (7) *of Theorem* 1. *More precisely, there exist constants c and β so that*

$$\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \leq c\,,$$

for all $y \in \mathcal{X}$.

PROOF. Choose $\delta > 1$ and let $\beta = c_0\delta$ where c_0 is the constant so that $a_t(x,y) = 0$ when $d(x,y) \geq c_0\, t^{1/m}$. Then, for $x, y \in \mathcal{X}$ so that $d(x,y) \geq \beta\, t^{1/m}$,

$$k_t(x,y) = \int_{\mathcal{X}} k(x,z)\, a_t(z,y)\, d\mu(z)\,.$$

For all $y \in \mathcal{X}$,

$$\begin{aligned}
&\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x)\\
&= \int_{d(x,y)\geq \beta t^{1/m}} \Big|k(x,y) - \int_{\mathcal{X}} k(x,z)\, a_t(z,y)\, d\mu(z)\Big|\, d\mu(x)\\
&= \int_{d(x,y)\geq \beta t^{1/m}}\\
&\quad \cdot \Big|k(x,y) \int_{d(z,y)\leq c_0 t^{1/m}} a_t(z,y)\, d\mu(z)\\
&\qquad\qquad - \int_{d(z,y)\leq c_0 t^{1/m}} k(x,z)\, a_t(z,y)\, d\mu(z)\Big|\, d\mu(x)\\
&\leq \sup_{d(z,y)\leq c_0 t^{1/m}} \Big(\int_{d(x,y)\geq \beta t^{1/m}} |k(x,y) - k(x,z)|\, d\mu(x)\Big)\\
&\qquad \cdot \Big(\int_{d(z,y)\leq c_0 t^{1/m}} |a_t(z,y)|\, d\mu(z)\Big)\\
&\leq c_1 \sup_{d(z,y)\leq c_0 t^{1/m}} \Big(\int_{d(x,y)\geq \delta c_0 t^{1/m}} |k(x,y) - k(x,z)|\, d\mu(x)\Big)\\
&\leq c\,.
\end{aligned}$$

Note that the second equality follows from property (6), the second inequality is using the estimate

$$\int_{\mathcal{X}} |a_t(z,y)|\, d\mu(z) \leq \int_{\mathcal{X}} h_t(z,y)\, d\mu(z) \leq c_1\ ,$$

and the last inequality follows from the Hörmander integral condition.

3. Singular integral operators on measurable subsets of a space of homogeneous type.

We assume in this section that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$. An example of Ω is an open domain of the Euclidean space $\mathbb{R}^n$. If Ω possesses certain smoothness on its boundary, for example Lipschitz boundary, then it is a space of homogeneous type and results of Section 2 are applicable. However, a general measurable set Ω needs not satisfy the doubling property, hence it is not a space of homogeneous type. Such a measurable set Ω appears naturally in boundary value problems, for example partial differential equations with Dirichlet boundary conditions.

Given a bounded linear operator T on $L_2(\Omega)$ with an associated kernel $k(x,y)$, the question is to find a sufficient condition on $k(x,y)$ for T to be of weak type $(1,1)$. The main problem in this case is the fact that the Calderón–Zygmund theory is not directly applicable. For example, the Calderón–Zygmund decomposition is not valid on Ω, nor is the Hardy-Littlewood maximal operator bounded, as was needed in proving the estimate (9).

A key observation to solve this problem is surprisingly simple. Given a linear operator T which maps $L_p(\Omega)$ into itself for some p, define an associated operator $\widetilde{T}$ on $L_p(\mathcal{X})$ by

$$\widetilde{T}(u)(x) = \begin{cases} T(\chi_\Omega u)(x)\,, & x \in \Omega\,, \\ 0\,, & x \notin \Omega\,, \end{cases}$$

where χ_Ω is the characteristic function on Ω. Then T *is bounded on* $L_p(\Omega)$ *if and only if* $\widetilde{T}$ *is bounded on* $L_p(\mathcal{X})$, also T *is of weak type* $(1,1)$ *on* Ω *if and only if* $\widetilde{T}$ *is of weak type* $(1,1)$ *on* $\mathcal{X}$.

It is straightforward to check the above equivalences, so we leave them to reader. Note that if T has an associated kernel $k(x,y)$ in the sense of (1), then $\widetilde{T}$ also has an associated kernel $\tilde{k}(x,y)$ in the sense of (1), given by

$$\tilde{k}(x,y) = \begin{cases} k(x,y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We can see immediately that the condition that the kernel $k(x,y)$ of T satisfies the Hörmander condition is not sufficient for the kernel $\tilde{k}(x,y)$

of $\widetilde{T}$ to satisfy the Hörmander condition. By using $\widetilde{T}$, what we do is to transform the question of boundedness of T on a measurable set Ω to the boundedness of $\widetilde{T}$ on a better space (of homogeneous type) $\mathcal{X}$, but the kernel of $\widetilde{T}$ could be discontinuous. However, the proof of Theorem 1 makes use of the upper bounds on $a_t(x,y)$ and condition (7), and does not require any continuity assumptions on $k(x,y)$.

From now on, to differentiate between a ball in $\mathcal{X}$ and a ball in Ω, we use the notations $B^{\mathcal{X}}$ and B^{Ω}.

The main theorem of this section is the following.

Theorem 2. *Let T be a bounded linear operator from $L_2(\Omega)$ to $L_2(\Omega)$ with an associated kernel $k(x,y)$ in the sense of* (1). *Assume there exists a class of operators A_t, $t > 0$, with kernels $a_t(x,y)$ defined on $L_2(\Omega)$ so that:*

a)

$$A_t u(x) = \int_{\mathcal{X}} a_t(x,y)\, u(y)\, d\mu(y)\,,$$

for any function $u \in L_2(\Omega) \cap L_1(\Omega)$, and the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)| \leq h_t(x,y)\,, \tag{15}$$

for all $x, y \in \Omega$, where $h_t(x,y)$ is defined on $\mathcal{X} \times \mathcal{X}$ by

$$h_t(x,y) = (\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1} s(d(x,y)^m t^{-1})\,, \tag{16}$$

and s is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\kappa}\, s(r^m) = 0\,,$$

for some $\kappa > 0$ with n the "dimension" entering the strong homogeneity property of $\mathcal{X}$,

b) *the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1) *and there exist constants C and $c > 0$ so that*

$$\int_{d(x,y) \geq ct^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \leq C\,, \tag{17}$$

for all $y \in \Omega$.

Then the operator T is of weak type $(1,1)$. Hence, T can be extended from $L_2(\Omega) \cap L_p(\Omega)$ to a bounded operator on $L_p(\Omega)$ for all $1 < p \leq 2$.

PROOF. First observe that $\widetilde{TA_t} = \widetilde{T}\widetilde{A}_t$ where $\widetilde{T}$, $\widetilde{A_t}$ and $\widetilde{TA_t}$ are defined on $L_2(\mathcal{X})$ as described above. Moreover $\widetilde{T}$ and $\widetilde{TA_t}$ have kernels $\tilde{k}(x,y)$ and $\tilde{k}_t(x,y)$ in the sense of (1), where $\tilde{k}$ was defined above and

$$\tilde{k}_t(x,y) = \begin{cases} k_t(x,y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Further, $\widetilde{A}_t$ is represented by the kernel

$$\tilde{a}_t(x,y) = \begin{cases} a_t(x,y)\,, & \text{when } x \in \Omega \text{ and } y \in \Omega\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

which is readily seen to satisfy conditions (2) and (3) on $\mathcal{X} \times \mathcal{X}$.

Conditions (15), (16), and (17) imply that the operator $\widetilde{T}$ satisfies the hypotheses of Theorem 1, hence it is of weak type $(1,1)$ on $\mathcal{X}$. Therefore, T is of weak type $(1,1)$ on Ω and Theorem 2 is proved.

REMARK. Assume there exist B_t which satisfy (15) and (16) so that the composite operators B_tT satisfy property (14). A standard duality argument shows that the adjoint operator of T is bounded on $L_p(\Omega)$ for $1 < p \leq 2$, hence T is bounded on $L_p(\Omega)$ for $2 \leq p < \infty$.

4. Boundedness of maximal truncated operators on L_p spaces.

4.1 The case of spaces of homogeneous type.

In this subsection, we assume that $\mathcal{X}$ is a space of homogeneous type equipped with a metric d and a measure μ. Let T be a bounded operator on $L_2(\mathcal{X})$ with an associated kernel $k(x,y)$ in the sense of (1). Our aim is to investigate the maximal truncated operator T_* which is defined by

$$T_* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|\,,$$

where T_ε is the truncated singular operator defined by

$$T_\varepsilon f(x) = \int_{d(x,y) \geq \varepsilon} k(x,y)\, f(y)\, d\mu(y)\,.$$

The main result of this section is the following theorem.

Theorem 3. *We assume the following conditions.*

a) *T is a bounded operator on $L_2(\mathcal{X})$ with an associated kernel $k(x, y)$.*

b) *There exists a class of operators A_t which satisfy the conditions (2) and (3) so that the composite operators TA_t have associated kernels $k_t(x, y)$ in the sense of (1). Also assume that there exist constants c_1 and $\delta > 0$ so that*

$$\int_{d(x,y)\geq \delta t^{1/m}} |k(x,y) - k_t(x,y)|\, d\mu(x) \leq c_1\,, \tag{7}$$

for all $y \in \mathcal{X}$.

c) *There exists a class of operators B_t represented by kernels $b_t(x, y)$ which satisfy the upper bounds $h_t(x, y)$ defined by (3), and the composite operators $B_t T$ have kernels $K_t(x, y)$. Also assume that there exist positive constants α, c_2, c_3 and c_4 so that*

$$|K_t(x,y)| \leq c_2\, (\mu(B(x; t^{1/m})))^{-1}\,, \qquad \text{when } d(x,y) \leq c_3\, t^{1/m} \tag{18}$$

and

$$|K_t(x,y) - k(x,y)| \leq c_4\, (\mu(B(x; d(x,y))))^{-1}\, \frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,, \tag{19}$$

when $d(x, y) \geq c_3\, t^{1/m}$. Then T_* is bounded on $L_p(\mathcal{X})$ for all p, $1 < p < \infty$.

PROOF. It follows from conditions (5) and (19), Theorem 1 and a duality argument that T is bounded on $L_p(\mathcal{X})$ for $1 < p < \infty$. Without any loss of generality, we prove the theorem with $c_3 = 1$. For a fixed $\varepsilon > 0$, we write

$$T_\varepsilon u(x) = B_{\varepsilon^m} Tu(x) - (B_{\varepsilon^m} T - T_\varepsilon)\, u(x)\,.$$

Since the class of operators B_t satisfies conditions (2) and (3), we have

$$|B_{\varepsilon^m} Tu(x)| \leq c\, M\,(|Tu(x)|)\,, \tag{20}$$

where M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε.

The kernel of the operator $(B_{\varepsilon^m}T - T_\varepsilon)$ is given by $(K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y))$ where $k_\varepsilon(x,y) = k(x,y)$ if $d(x,y) \geq \varepsilon$ and $k_\varepsilon(x,y) = 0$ otherwise. There are two cases:

Case 1. $d(x,y) < \varepsilon$, then $k_\varepsilon(x,y) = 0$ and it follows from (18) that

$$|K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y)| = |K_{\varepsilon^m}(x,y)| \leq c_2 \, \frac{1}{\mu(B(x;\varepsilon))} \, .$$

Case 2. $d(x,y) \geq \varepsilon$, then $k_\varepsilon(x,y) = k(x,y)$ and it follows from (19) that

$$|K_{\varepsilon^m}(x,y) - k_\varepsilon(x,y)| \leq c_4 \, \frac{1}{\mu(B(x;d(x,y)))} \left(\frac{\varepsilon}{d(x,y)}\right)^\alpha ,$$

for some $\alpha > 0$.

Therefore

$$\begin{aligned}
&|(B_{\varepsilon^m}T - T_\varepsilon)u(x)| \\
&\qquad \leq c \int_{d(x,y)\leq\varepsilon} \frac{1}{\mu(B(x;\varepsilon))} \, |u(y)| \, d\mu(y) \\
&\qquad\quad + c \int_{d(x,y)>\varepsilon} \frac{1}{\mu(B(x;d(x,y)))} \left(\frac{\varepsilon}{d(x,y)}\right)^\alpha |u(y)| \, d\mu(y) \\
&\qquad \leq c \, \frac{1}{\mu(B(x;\varepsilon))} \int_{d(x,y)\leq\varepsilon} |u(y)| \, d\mu(y) \\
&\qquad\quad + c \sum_{k=1}^{\infty} \frac{1}{2^{k\alpha}} \, \frac{1}{\mu(B(x;2^{k+1}\varepsilon))} \int_{d(x,y)\leq 2^{k+1}\varepsilon} |u(y)| \, d\mu(y) \\
&\qquad \leq c \, M\left(|u|(x)\right),
\end{aligned}$$

where again M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε. Therefore

$$\sup_{\varepsilon>0} |(B_{\varepsilon^m}T - T_\varepsilon)u(x)| \leq c \, M\left(|u|(x)\right). \tag{21}$$

Combining estimates (20), (21) with boundedness of T and the Hardy-Littlewood maximal operator on $L_p(\mathcal{X})$, we obtain boundedness of T_* on $L_p(\mathcal{X})$, $1 < p < \infty$.

In the next proposition, we show that, for suitably chosen B_t, our condition (19) is a consequence of the Hölder continuity estimates on the kernel.

Proposition 2. *Assume that for some $\alpha > 0$ and $c_1 > 1$, the kernel $k(x,y)$ associated with T satisfies the condition*

$$|k(z,y) - k(x,y)| \le c\,\frac{1}{\mu(B(x;d(x,y))}\Big(\frac{d(x,z)}{d(x,y)}\Big)^{\alpha}, \tag{22}$$

when $d(x,y) \ge c_1\,d(x,z)$. Let B_t be approximations to the identity which are represented by kernels $b_t(x,y)$ which satisfy (3), (5) and $\int_{\mathcal{X}} b_t(x,y)\,d\mu(y) = 1$, for all $x \in \mathcal{X}$, $t > 0$.

Then the kernels $K_t(x,y)$ associated with B_tT satisfy condition (19), i.e. there exists a constant c so that

$$|K_t(x,y) - k(x,y)| \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,,$$

for $d(x,y) \ge c_0\,c_1\,t^{1/m}$ where c_0 is the constant appearing in condition (5).

Proof. Suppose that $d(x,y) \ge c_0\,c_1\,t^{1/m}$. Then

$$\begin{aligned}
&|k(x,y) - K_t(x,y)| \\
&\qquad = \Big|k(x,y) - \int_{\mathcal{X}} k(x,z)\,b_t(z,y)\,d\mu(z)\Big| \\
&\qquad = \Big|k(x,y)\int_{d(z,y)\le c_0 t^{1/m}} b_t(z,y)\,d\mu(z) \\
&\qquad\qquad - \int_{d(z,y)\le c_0 t^{1/m}} k(x,z)\,b_t(z,y)\,d\mu(z)\Big| \\
&\qquad \le \int_{d(z,y)\le c_0 t^{1/m}} |k(x,y) - k(x,z)|\,|b_t(z,y)|\,d\mu(z) \\
&\qquad \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\int_{d(z,y)\le c_0 t^{1/m}} |b_t(z,y)|\,d\mu(z) \\
&\qquad \le c\,(\mu(B(x;d(x,y)))^{-1}\,\frac{t^{\alpha/m}}{d(x,y)^{\alpha}}\,.
\end{aligned}$$

Note that the second equality is using condition (6) and the second inequality follows from (22).

Remark. Propositions 1 and 2 show that conditions (7) and (19) are weaker than the usual conditions which guarantee L_p boundedness of maximal truncated operators. See, for example [St2, Chapter 1]. However, we need the extra assumption (18). In the case of functional calculi of generators of semigroups with suitable heat kernel bounds, condition (18) is satisfied without extra regularity conditions on the kernel of T. See Theorem 6.

4.2. The case of measurable subsets of a space of homogeneous type.

We now assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ as in Section 3. By strengthening the assumptions on the kernel of T in Theorem 3, we can obtain boundedness of maximal truncated operators on L_p spaces as follows.

Theorem 4. *Let T be a bounded operator on $L_2(\Omega)$ with an associated kernel $k(x,y)$. We assume the following conditions.*

a) *There exists a class of operators A_t, $t>0$, represented by kernels $a_t(x,y)$ which satisfy conditions* (15) *and* (16) *so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (1), *and there exist constants c and c_1 so that*

$$\int_{d(x,y)\geq ct^{1/m}} |k(x,y)-k_t(x,y)|\, d\mu(x) \leq c_1 \, , \tag{17}$$

for all $y\in\Omega$.

b) *There exists a class of operators B_t, $t>0$, represented by kernels $b_t(x,y)$ which satisfy conditions* (15) *and* (16) *so that the composite operators B_tT have kernels $K_t(x,y)$, and $K_t(x,y)$ satisfy the following conditions*

$$|K_t(x,y)| \leq c\,(\mu(B^{\mathcal{X}}(x;t^{1/m})))^{-1} \, , \tag{23}$$

for all $x,y\in\Omega$ such that $d(x,y)\leq c_2\, t^{1/m}$,

$$|K_t(x,y)-k(x,y)| \leq c\,(\mu(B^{\mathcal{X}}(x;d(x,y))))^{-1}\frac{t^{\alpha/m}}{d(x,y)^{\alpha}} \, , \tag{24}$$

for all $x, y \in \Omega$ *such that* $d(x,y) \geq c_2\, t^{1/m}$.

Then T_* *is bounded on* $L_p(\Omega)$ *for all* p, $1 < p < \infty$.

PROOF. There is no loss of generality in proving the theorem with $c_2 = 1$.

It follows from Theorem 2 and a duality argument that T is bounded on $L_p(\Omega)$ for $1 < p < \infty$.

Given a function $u \in L_1(\Omega) \cap L_2(\Omega)$ and $\varepsilon > 0$, write

$$T_\varepsilon u(x) = B_{\varepsilon^m} T u(x) + (B_{\varepsilon^m} T - T_\varepsilon)\, u(x)\,.$$

Consider the term $B_{\varepsilon^m} T u(x)$. Let v be the extension of Tu from Ω to $\mathcal{X}$ by putting $v(x) = 0$ for $x \notin \Omega$, then $\|Tu\|_{L_p(\Omega)} = \|v\|_{\mathcal{L}_p(\mathcal{X})}$ for $1 \leq p \leq \infty$. Similarly, let w_ε be the extension of $B_{\varepsilon^m} T u$ from Ω to $\mathcal{X}$ by putting $w(x) = 0$ for $x \notin \Omega$. Since

$$B_{\varepsilon^m} T u(x) = \int_{\mathcal{X}} b_{\varepsilon^m}(x,y)\, Tu(y)\, d\mu(y)$$

and the kernels $b_t(x,y)$ of $B_t(x,y)$ satisfy (15) and (16), we have for $x \in \mathcal{X}$,

$$|w_\varepsilon(x)| \leq c\, M\,(|v|(x))\,, \tag{25}$$

where M is the Hardy-Littlewood maximal operator, and c is a constant independent of ε. This gives

$$\sup_{\varepsilon > 0} |w_\varepsilon(x)| \leq c\, M\,(|v|(x))\,. \tag{26}$$

The next step is to extend u to $\mathcal{X}$ by putting $u(x) = 0$ for $x \notin \Omega$, and denote the extension by $u^{\mathcal{X}}$. Then $\|u\|_{L_p(\Omega)} = \|u^{\mathcal{X}}\|_{L_p(\mathcal{X})}$. It then follows from assumption b) and the argument of (21) that

$$\sup_{\varepsilon > 0} |(B_{\varepsilon^m} T - T_\varepsilon)\, u(x)| \leq c\, M\,(|u^{\mathcal{X}}|(x))\,. \tag{27}$$

Estimates (26) and (27) imply the boundedness of T_* on $L_p(\Omega)$ for all p, $1 < p < \infty$.

A consequence of the boundedness of the maximal truncated operator T_* is pointwise almost everywhere convergence of the limit

$$\lim_{\varepsilon \to 0} T_\varepsilon u(x)\,.$$

More precisely, we have the following lemma.

Lemma 2. *Assume that the operator T satisfies the conditions of Theorem 3. Let $1 < p < \infty$, and assume that $\lim_{\varepsilon\to 0} T_\varepsilon u(x)$ exists almost everywhere for every u in a dense subspace of $L_p(\Omega)$, then $\lim_{\varepsilon\to 0} T_\varepsilon u(x)$ exists almost everywhere for every function $u \in L_p(\Omega)$.*

Proof. Lemma 2 follows from a standard argument of proving the existence of almost everywhere pointwise limits as a consequence of the corresponding maximal inequality. See, for example [St1, p. 45].

5. Applications: Riesz transforms and holomorphic functional calculi of elliptic operators.

5.1 Definitions.

We first give some preliminary definitions. References are [Mc], [CDMcY], [ADMc].

For $0 \le \omega < \nu < \pi$, define the closed sector in the complex plane $\mathbb{C}$

$$S_\omega = \{\zeta \in \mathbb{C} : \ |\arg \zeta| \le \omega\} \cup \{0\}$$

and denote its interior by S_ω^0.

We employ the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0.

$$H_\infty(S_\nu^0) = \{f \in H(S_\nu^0) : \ \|f\|_\infty < \infty\},$$

where $\|f\|_\infty = \sup\{|f(\zeta)| : \zeta \in S_\nu^0\}$,

$$\Psi(S_\nu^0) = \{\psi \in H(S_\nu^0) : \text{ exists } s > 0,\ |\psi(\zeta)| \le C\,|\zeta|^s\,(1 + |\zeta|^{2s})^{-1}\}$$

and

$$F(S_\nu^0) = \{f \in H(S_\nu^0) : \text{ exists } s > 0,\ |f(\zeta)| \le C\,(|\zeta|^{-s} + |\zeta|^s)\},$$

so that

$$\Psi(S_\nu^0) \subset H_\infty(S_\nu^0) \subset F(S_\nu^0).$$

Let $0 \le \omega < \pi$. A closed operator L in $L_p(\mathcal{X})$ is said to be of type ω if $\sigma(L) \subset S_\omega$ and, for each $\nu > \omega$, there exists C_ν such that

$$\|(L - \zeta I)^{-1}\| \le C_\nu\,|\zeta|^{-1}, \qquad \zeta \notin S_\nu\,.$$

By the Hille–Yoshida Theorem, an operator L of type ω with $\omega < \pi/2$ is the generator of a bounded holomorphic semigroup e^{-zL} on the sector S^0_ν with $\nu = \pi/2 - \omega$.

Suppose that L is a one–one operator of type ω with dense domain and dense range in $L_p(\mathcal{X})$. We can define a functional calculus of L as follows.

If $\psi \in \Psi(S^0_\nu)$, then

$$\psi(L) = \frac{1}{2\pi i}\int_\gamma (L - \zeta I)^{-1}\,\psi(\zeta)\,d\zeta\,, \tag{28}$$

where γ is the contour $\{\zeta = r\,e^{\pm i\theta} : r \geq 0\}$ parametrised clockwise around S_ω, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(\mathcal{X})$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$.

Let $f \in F(S^0_\nu)$, so that for some c and k, $|f(\zeta)| \leq c\,(|\zeta|^k + |\zeta|^{-k})$ for every $\zeta \in S^0_\nu$. Let

$$\psi(\zeta) = \Big(\frac{\zeta}{(1+\zeta)^2}\Big)^{k+1}.$$

Then ψ, $f\,\psi \in \Psi(S^0_\nu)$ and $\psi(L)$ is one–one. So $(f\,\psi)(L)$ is a bounded operator on $\mathcal{X}$, and $\psi(L)^{-1}$ is a closed operator in $\mathcal{X}$. Define $f(L)$ by

$$f(L) = (\psi(L))^{-1}\,(f\,\psi)(L)\,. \tag{29}$$

An important feature of this functional calculus is the following Convergence Lemma.

Convergence Lemma. *Let $0 \leq \omega < \nu \leq \pi$. Let L be an operator of type ω which is one–one with dense domain and range. Let $\{f_\alpha\}$ be a uniformly bounded net in $H_\infty(S^0_\nu)$, which converges to $f \in H_\infty(S^0_\nu)$ uniformly on compact subsets of S^0_ν, such that $\{f_\alpha(L)\}$ is a uniformly bounded net in $\mathcal{L}(\mathcal{X})$. Then $f(L) \in \mathcal{L}(\mathcal{X})$, $f_\alpha(L)u \longrightarrow f(L)u$ for all $u \in \mathcal{X}$, and $\|f(L)\| \leq \sup_\alpha \|f_\alpha(L)\|$.*

For the proof of the Convergence Lemma, see [M^c], or [ADM^c].

5.2. L_p boundedness of Riesz Transforms.

In this subsection, we assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ in Section 3. Let L be a linear operator of type ω on $L_2(\Omega)$ with $\omega < \pi/2$, so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\mathrm{Arg}\,(z)| < \theta$, $\theta = \pi/2 - \omega$. Assume that for real $t > 0$, the distribution kernels $a_t(x,y)$ of e^{-tL} belong to $L_\infty(\Omega \times \Omega)$ and satisfy the estimate

$$|a_t(x,y)| \leq h_t(x,y)\,,$$

for $x, y \in \Omega$ where h_t is defined on $\mathcal{X} \times \mathcal{X}$ by (3) and.

For $0 \leq \alpha < 1$, $\nu > 0$, define $F_\alpha(S_\nu^0)$ as follows

$$F_\alpha(S_\nu^0) = \{f \in H(S_\nu^0) : \text{ exists } c,\ |f(\zeta)| \leq C\,|\zeta|^{-\alpha}\}\,.$$

Assume that $g \in F_\alpha(S_\nu^0)$ for some $\nu > \pi/2$, and that D is a densely defined linear operator on $L_2(\Omega)$ which possesses the following two properties:

a) $Dg(L)$ is bounded on $L_2(\Omega)$,

b) the function Da_t, $t > 0$, obtained by the action of D on $a_t(x,y)$ with respect to the variable x, satisfies the estimate

$$|Da_t(x,y)| \leq c\,t^{-\alpha} h_t(x,y)\,,$$

for all $x, y \in \Omega$.

The main result of this section is the following theorem.

Theorem 5. *Under the above assumptions* a) *and* b)*, the operator $Dg(L)$ is of weak type $(1,1)$, hence it can be extended to a bounded operator on $L_p(\Omega)$ for $1 < p \leq 2$.*

Before giving the proof of the theorem, we give examples which are applications of our results. Some specific operators which satisfy the assumptions of Theorem 5 are as follows.

i) Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra. Assume that

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i$$

as a vector space, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j, and $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^{m} j \dim(\mathfrak{g}_j),$$

where $\dim(\mathfrak{g}_j)$ denotes the dimension of $\mathfrak{g}_j$.

Consider any finite basis $\{X_k\}$ of $\mathfrak{g}_1$. Each X_k can be identified with a unique left invariant vector field on G. Define

$$L = -\sum_k X_k^2 .$$

Then the *sub-Laplacian* L is a left invariant second order differential operator, which is a non-negative self-adjoint operator in $L_2(G)$. The Banach spaces $L_p(G)$ are defined with respect to Haar measure.

Note that G is a Lie group of polynomial growth, hence it is a space of homogeneous type. Consider the Riesz transforms $X_k L^{-1/2}$ which are special cases of our operator $Dg(L)$ when $D = X_k$ and $g(L) = L^{-1/2}$. It is not difficult to check that $X_k L^{-1/2}$ is bounded on $L_2(G)$. Gaussian upper bounds on heat kernels and their space derivatives are well known (see, for example [SC]), hence our condition b) is satisfied with $\alpha = 1/2$. It follows that the Riesz transforms $X_k L^{-1/2}$ are bounded on $L_p(G)$ for $1 < p \leq 2$ and are of weak type $(1,1)$. Thus we have simplified the proof of the L_p boundedness of the Riesz transforms given by Saloff-Coste [SC], because we have not used the smoothness of the heat kernels in the variable y.

In the same setting of G, we can also consider the case when L is a $2m$-th order strongly elliptic operator with constant coefficients (plus a sufficiently large constant c_0), and $D = X_{i_1} X_{i_2} \cdots X_{i_n}$ for some $n < 2m$. Then the operator $DL^{-n/2m}$ is bounded on $L_2(G)$. The condition (b) is also satisfied with $\alpha = n/2m$. See, for example, [R, Chapter 4]. Our theorem then shows that $DL^{-n/2m}$ can be extended to a bounded operator on each $L_p(G)$ for $1 < p \leq 2$.

ii) Let $\mathcal{X}$ be a complete Riemannian manifold with non-negative Ricci curvature, L the Laplace-Beltrami operator, and D a vector field. Then the Riesz transform $DL^{-1/2}$ is bounded on $L_2(\mathcal{X})$. Upper bounds on the heat kernels and their derivatives can be found, for example, in

[CLY], [Da2]. Thus the assumptions of Theorem 5 are satisfied (with $\alpha = 1/2$), so $DL^{-1/2}$ can be extended to a bounded operator on each $L_p(\mathcal{X})$ for $1 < p \leq 2$.

We now proceed to prove Theorem 5. The following off-diagonal estimate is proved in [DR, Proposition 2.1].

Lemma 3. *Let $h_t(x, y)$ be given by (3), then for each δ, $0 < \delta < \kappa - N*$, where $\kappa, N*$ are the constants in (4), there exists $c > 0$ so that*

$$\int_{d(x,y) \geq r} h_t(x, y)\, d\mu(x) \leq c\,(1 + r^m\, t^{-1})^{-\delta}\,,$$

uniformly for all $r \geq 0$, $t > 0$ and $y \in \mathcal{X}$.

PROOF OF THEOREM 5. Observe that, for each positive integer k, the powers of the resolvent $(L - \lambda\, I)^{-k}$ are given by

$$(L - \lambda\, I)^{-k} = c_k \int_0^\infty t^{k-1}\, e^{\lambda t}\, e^{-tL}\, dt\,,$$

when $\lambda < 0$. Therefore, the operators $(L - \lambda\, I)^{-k}$ are represented by kernels $g_\lambda^k(x, y)$ where

$$g_\lambda^k(x, y) = c_k \int_0^\infty t^{k-1}\, e^{\lambda t}\, a_t(x, y)\, dt\,.$$

It follows from this representation and the estimates (3) and (4) on the heat kernels that for a sufficiently large integer k, the kernels $g_\lambda^k(x, y)$ possess upper bounds which are similar to those of the heat kernels. More specifically, there exist $h_t(x, y)$ satisfying (3) and (4), possibly with a different function s, so that

$$|\lambda^k g_\lambda^k(x, y)| \leq h_t(x, y)\,,$$

for all $x, y \in \Omega$, where $t = 1/|\lambda|$.

Similarly, we also have the bound

$$|D\lambda^k g_\lambda^k(x, y)| \leq c\, t^{-\alpha}\, h_t(x, y)\,,$$

for all $x, y \in \Omega$.

Choose the class of operators $A_t = (t\,L + I)^{-k}$. By Theorem 2, it suffices to show that condition (17) is satisfied. The kernels $(k(x,y) - k_t(x,y))$ in condition (17) are associated with operators $Dg(L)(I-(t\,L+I)^{-k})$. Let $g(L)(I-(t\,L-I)^{-k}) = f(L)$ where $f(z) = g(z)\,(1-(t\,z+1)^{-k})$. Using the upper bounds on $g(z)$, we see that f belongs to the class $\Psi(S_\nu^0)$.

We next represent the operator $f(L)$ by using the semigroup e^{-zL}. By (28), $f(L)$ (acting on $L_2(\mathcal{X})$) is given by

$$f(L) = \frac{1}{2\pi i}\int_\gamma (L-\lambda\,I)^{-1} f(\lambda)\,d\lambda\,,$$

where the contour $\gamma = \gamma_+ \cup \gamma_-$ is given by $\gamma_+(t) = t\,e^{i\nu}$ for $t \geq 0$ and $\gamma_-(t) = -t\,e^{-i\nu}$ for $t \leq 0$, and $\nu > \pi/2$.

For $\lambda \in \gamma$, substitute

$$(L-\lambda\,I)^{-1} = \int_0^\infty e^{\lambda s}\,e^{-sL}\,ds\,.$$

Changing the order of integration gives

$$f(L) = \int_0^\infty e^{-sL}\,n(s)\,ds\,, \tag{30}$$

where

$$n(s) = \frac{1}{2\pi i}\int_\gamma e^{\lambda s}\,f(\lambda)\,d\lambda\,. \tag{31}$$

Consequently, the kernel $k_f(x,y)$ of $f(L)$ is given by

$$k_f(x,y) = \int_0^\infty a_s(x,y)\,n(s)\,ds\,. \tag{32}$$

It follows from the upper bound on $g(z)$ and assumption b) that

$$\begin{aligned}
&\int_{d(x,y)\geq ct^{1/m}} |k(x,y) - k_t(x,y)|\,d\mu(x)\\
&\leq c\int_0^\infty s^{-\alpha}\Big(\int_0^\infty |\lambda^{-\alpha}\,e^{s\lambda}\,(1-(t\,\lambda+1)^{-k})|\\
&\qquad\qquad \cdot\Big(\int_{d(x,y)\geq ct^{1/m}} h_s(x,y)\,d\mu(x)\Big)\,d|\lambda|\Big)\,ds\\
&\leq c\int_0^\infty s^{-\alpha}\Big(\int_0^\infty |\lambda^{-\alpha}\,e^{s\lambda}\,(1-(t\,\lambda+1)^{-k})|\,(1+t\,s^{-1})^{-\delta}\,d|\lambda|\Big)\,ds\,,
\end{aligned}$$

by Lemma 3. Observe that $|1-(t\,\lambda+1)^{-k}|\le c$ and $|1-(t\,\lambda+1)^{-k}|\le c\,t\,|\lambda|$ when $t\,|\lambda|\le 1$. We then split the integral on the right hand side into two parts, I_1 and I_2, corresponding to integration over $t\,|\lambda|>1$ and $t\,|\lambda|\le 1$. Then

$$I_1 \le \int_0^\infty s^{-\alpha}\int_{1/t}^\infty v^{-\alpha}\,e^{-\beta s v}\,(1+t\,s^{-1})^{-\delta}\,dv\,ds$$

with $\beta>0$. Changing variables $t\,v\longrightarrow v$ and $s/t\longrightarrow s$, and choosing a positive $\varepsilon<\delta$, we have

$$\begin{aligned}
I_1 &\le c\int_0^\infty s^{-\alpha}\Big(\int_1^\infty v^{-\alpha}\,e^{-\beta s v}\,(1+s^{-1})^{-\delta}\,dv\Big)\,ds\\
&= c\int_0^\infty \frac{s^\delta}{(1+s)^\delta}\Big(\int_1^\infty \frac{1}{s^{1+\varepsilon}}\,\frac{1}{v^{1+\varepsilon}}\,(s\,v)^{1+\varepsilon-\alpha}\,e^{-\beta s v}\,dv\Big)\,ds\\
&\le c\int_0^\infty \frac{s^\delta}{(1+s)^\delta\,s^{1+\varepsilon}}\Big(\int_1^\infty \frac{1}{v^{1+\varepsilon}}\,dv\Big)\,ds\\
&\qquad \text{(since } (s\,v)^{1+\varepsilon-\alpha}\,e^{-\beta s v} \text{ is bounded)}\\
&\le C\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &\le c\int_0^\infty s^{-\alpha}\int_0^{1/t} v^{-\alpha}\,t\,v\,e^{-\beta s v}\,dv\,(1+t\,s^{-1})^{-\delta}\,dv\,ds\\
&\le c\int_0^\infty s^{-\alpha}\,(1+s^{-1})^{-\delta}\int_0^1 v^{1-\alpha}\,e^{-\beta s v}\,dv\,ds\\
&\le c\int_0^1 dv\int_0^\infty w^{-\alpha}\,e^{-\beta w}\,dw\\
&\le C\,.
\end{aligned}$$

Therefore, condition (17) is satisfied and Theorem 5 follows from Theorem 2.

REMARKS.

α) In the assumption b), we do not assume any regularity of $a_t(x,y)$ in the variable y.

β) The theorem is still true for $g \in F_\alpha(S^0_\nu)$ with $\nu < \pi/2$ if the upper bounds on $a_t(x,y)$ in condition b) hold for all complex $t \in S^0_\theta$ with $\theta > \pi/2 - \nu$. This can be achieved by first choosing $\mu = \nu - \varepsilon$, (with ε to be specified later), using the formula

$$f(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda\, I)^{-1} f(\lambda)\, d\lambda \,,$$

where the contour $\gamma = \gamma_+ \cup \gamma^-$ is given by $\gamma_+(t) = t\, e^{i\mu}$ for $t \geq 0$ and $\gamma_-(t) = -t\, e^{-i\mu}$ for $t \leq 0$.

We then substitute

$$(L - \lambda\, I)^{-1} = \int_\Gamma e^{\lambda z}\, e^{-zL}\, dz \,,$$

for $\lambda \in \gamma_+$, where Γ is given by $\Gamma(t) = t\, e^{i(\theta-\varepsilon)}$ for $t \geq 0$, and ε is chosen sufficiently small so that $(\theta + \mu - \varepsilon) > \pi/2$. We also have similar expression for $\lambda \in \gamma_-$. Thus we obtain a similar representation of $f(L)$ to that of (30), and the rest of the proof is the same as before.

γ) The pointwise bound in condition (b) can be replaced by a weaker condition on the L_2 norm with a suitable weight of Da_t (with respect to x variable). See [CD].

5.3 Holomorphic functional calculi of elliptic operators.

We again assume that Ω is a measurable subset of a space of homogeneous type $(\mathcal{X}, d, \mu)$ as in Section 3.

Let L be a linear operator of type ω on $L_2(\Omega)$ with $\omega < \pi/2$, hence L generates a holomorphic semigroup e^{-zL}, $0 \leq |\mathrm{Arg}\,(z)| < \pi/2 - \omega$.

Theorem 6. *Assume the following two conditions.*

a) *The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}\,(z)| < \pi/2 - \omega$, is represented by kernels $a_z(x,y)$ which satisfy, for all $\theta > \omega$, an estimate*

$$|a_z(x,y)| \leq c_\theta\, h_{|z|}(x,y) \,,$$

for $x, y \in \Omega$ and $|\mathrm{Arg}\,(z)| < \pi/2 - \theta$, where h_t is defined on $\mathcal{X} \times \mathcal{X}$ by (3).

b) *The operator L has a bounded holomorphic functional calculus in $L_2(\Omega)$. That is, for any $\nu > \omega$ and $f \in H_\infty(S^0_\nu)$, the operator $f(L)$ satisfies*

$$\|f(T)\|_2 \leq c_\nu \, \|f\|_\infty \, .$$

Then the operator L has a bounded holomorphic functional calculus in $L_p(\Omega)$, $1 < p < \infty$, that is,

$$\|f(L)\|_p \leq c_{p,\nu} \, \|f\|_\infty \, ,$$

for all $f \in H_\infty(S^0_\nu)$.

When $p = 1$, the operator $f(L)$ is of weak-type $(1,1)$.

If we denote $T = f(L)$, then the maximal truncated operator T_* is bounded on $L_p(\Omega)$ for all p, $1 < p < \infty$.

Proof. Given $\pi/2 > \nu > \omega$, choose θ and μ such that $\omega < \theta < \mu < \nu$. For $f \in \Psi(S^0_\nu)$, represent the operator $f(L)$ by using the semigroup e^{-zL} as before. This gives

$$f(L) = \int_{\Gamma_+} e^{-zL} \, n_+(z) \, dz + \int_{\Gamma_-} e^{-zL} \, n_-(z) \, dz \, ,$$

where we choose the contour $\Gamma_+(s) = s \, e^{i(\pi/2-\theta)}$ for $s \geq 0$ and $\Gamma_-(s) = -s \, e^{-i(\pi/2-\theta)}$ for $s \leq 0$. The functions $n_\pm(z)$ are given by

$$n_\pm = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\lambda z} \, f(\lambda) \, d\lambda \, ,$$

where $\gamma_+(s) = s \, e^{i\mu}$ for $t \geq 0$ and $\gamma_-(t) = -t \, e^{-i\mu}$ for $t \leq 0$.

This implies the bound

$$|n_\pm(z)| \leq c \, \|f\|_\infty \, |z|^{-1} \, .$$

Consequently, the kernel $k_f(x,y)$ of $f(L)$ is given by

$$k_f(x,y) = \int_{\Gamma_+} a_z(x,y) \, n_+(z) \, dz + \int_{\Gamma_-} a_z(x,y) \, n_-(z) \, dz \, .$$

Choose operators $A_t = e^{-tL}$. Using the upper bounds on the heat kernels and Lemma 3, similar estimates to the terms I_1 and I_2 in the proof of Theorem 5 shows that condition (17) of Theorem 2 is satisfied. Therefore, $f(L)$ is bounded on $L_p(\Omega)$. The Convergence Lemma then allows us to extend L_p boundedness of $f(L)$ to all $f \in H_\infty(S_\nu^0)$, hence the operator L has a bounded holomorphic function calculus in $L_p(\Omega)$. Although the extension of the weak type (1,1) estimates from $f(L)$ for $f \in \Psi(S_\nu^0)$ to $f(L)$ for $f \in H_\infty(S_\nu^0)$ does not follow from the Convergence Lemma, it is not difficult. See for example [ADM$^{\mathrm{c}}$, Lecture 7, Section N].

To prove the L_p boundedness of the maximal truncated operator T_*, first choose $B_t = A_t = e^{-tL}$. We then just need to verify conditions (23) and (24) of Theorem 4.

To verify (23), we use the commutative property of functional calculus:

$$e^{-tL} f(L) = e^{-tL/2} f(L)\, e^{-tL/2}\,.$$

Since e^{-tL} maps $L_1(\Omega)$ into $L_1(\Omega)$ with the operator norm less than a constant, and e^{-tL} maps $L_1(\Omega)$ into $L_\infty(\Omega)$ with the operator norm less than $(\mu(B^{\mathcal{X}}(x;t^{-1/m})))^{-1}$, interpolation and duality gives

$$\begin{aligned}\|e^{-tL/2}\|_{L_1(\Omega)\to L_2(\Omega)} &= \|e^{-tL/2}\|_{L_2(\Omega)\to L_\infty(\Omega)}\\ &\le c\,(\mu(B^{\mathcal{X}}(x;t^{-1/m})))^{-1/2}\,.\end{aligned}$$

These estimates, combined with the fact that $f(L)$ is bounded on $L_2(\Omega)$, imply condition (23).

The proof of (24) is straightforward. Consider $d(x,y) \ge c\,t^{1/m}$, we have

$$|k(x,y) - k_t(x,y)| \le c \int_0^\infty |h_z(x,y)|\, d|z| \int_0^\infty |\lambda^{-\alpha}\, e^{z\lambda}\,(1 - e^{-t\lambda})\,|d|\lambda|\,.$$

Observe that $|1 - e^{-t\lambda}| \le c$ since $\mathrm{Re}\,(\lambda) \ge 0$ and $|1 - e^{-t\lambda}| \le c\,t\,|\lambda| \le c\,(t\,|\lambda|)^\alpha$ for $0 < \alpha < 1$ when $t\,|\lambda| \ge 1$. We then split the integral on the left hand side into two parts, I_1 and I_2, corresponding to integration over $t\,|\lambda| > 1$ and $t\,|\lambda| \le 1$. Using the heat kernel bounds and elementary integration, similar estimates to those of I_1 and I_2 of Theorem 5 show that

$$|K_t(x,y) - k(x,y)| \le c\,(\mu(B^{\mathcal{X}}(x;d(x,y)))^{-1}\,\frac{t^{\beta/m}}{d(x,y)^\beta}\ ,$$

for some $\beta > 0$, $x, y \in \Omega$, $d(x,y) \ge t^{1/m}$. We leave details of these estimates to reader.

REMARKS.

a) Condition a) of Theorem 6 can be replaced by a more general condition as follows. Assume that L is an operator of type ω and that there exists a positive integer k so that the kernels $g_\lambda^k(x,y)$ of the power of the resolvents $(\lambda L - I)^{-k}$ satisfy the following estimate

$$|g_\lambda^k(x,y)| \leq (\mu(B^{\mathcal{X}}(x;|\lambda|^{-1/m})))^{-1}\, s(d(x;y)^m|\lambda|)\,,$$

where s is a function which satisfies the decay condition in (3). The proof under this assumption is still the same as that of Theorem 6, with the operators $(\lambda L - I)^{-k}$ replacing the semigroup e^{-zL}. The advantage of this assumption is that the operator L can be of type ω with $\omega > \pi/2$, or of type ω on a double sector.

b) When $\mathcal{X}$ is a space of homogeneous type, the result on boundedness of holomorphic functional calculi of Theorem 6 was first presented in [DR, Theorem 3.1]. Note that the Hörmander integral condition is not applicable when we have no control on smoothness of heat kernels in the space variables.

c) Heat kernel bounds are known for a large class of elliptic and subelliptic operators. Also see [AM^{c}T], [A] for recent results on heat kernel bounds for second order elliptic operators with non-smooth coefficients.

d) Theorem 6 gives new results when Ω is a measurable set with no smoothness on its boundary. An example of an operator on such a domain, which possesses Gaussian bounds on its heat kernels, is the Laplacian on an open subset of Euclidean space $\mathbb{R}^n$ subject to Dirichlet boundary conditions. Gaussian upper bounds can be obtained in this case by a simple argument using the comparison principle. More general operators on open domains of $\mathbb{R}^n$ which possess Gaussian bounds can be found in [Da1] and [AE]. Indeed Theorem 6 can be applied to prove the general statement of Theorem 5.5 in [AE] on boundedness of holomorphic functional calculi in L_p spaces, without the assumption that the boundary has null measure.

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