ME185

Introduction to Continuum Mechanics

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Introduction

This set of notes has been written as part of teaching ME185, an elective senior-year undergraduate course on continuum mechanics in the Department of Mechanical Engineering at the University of California, Berkeley.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Solids and fluids as continuous media

All matter is inherently discontinuous, as it is comprised of distinct building blocks, the molecules. Each molecule consists of a finite number of atoms, which in turn consist of finite numbers of nuclei and electrons.

Many important physical phenomena involve matter in large length and time scales. This is generally the case when matter is considered at length scales much larger than the characteristic length of the atomic spacings and at time scales much larger than the characteristic times of atomic bond vibrations. The preceding characteristic lengths and times can vary considerably depending on the state of the matter (e.g., temperature, precise composition, deformation). However, one may broadly estimate such characteristic lengths and times to be of the order of up to a few Angstroms (1 Å$= 10^{-10}$ m) and a few femtoseconds (1 femtosecond $= 10^{-15}$ sec), respectively. As long as the physical problems of interest occur at length and time scales of several orders of magnitude higher that those noted previously, it is possible to consider matter as a continuous medium, namely to effectively ignore its discrete nature without introducing any even remotely significant errors.

A continuous medium may be conceptually defined as a finite amount of matter whose physical properties are independent of its actual size or the time over which they are measured. As a thought experiment, one may choose to perpetually dissect a continuous medium into smaller pieces. No matter how small it gets, its physical properties remain unaltered.

Mathematical theories developed for continuous media (or "continua") are frequently referred to as "phenomenological", in the sense that they capture the observed physical response without directly accounting for the discrete structure of matter.

Solids and fluids (including both liquids and gases) can be accurately viewed as continuous media in many occasions. Continuum mechanics is concerned with the response of solids and fluids under external loading precisely when they can be viewed as continuous media.

1.2 History of continuum mechanics

Continuum mechanics is a modern discipline that unifies solid and fluid mechanics, two of the oldest and most widely examined disciplines in applied science. It draws on classical scientific developments that go as far back as the Hellenistic-era work of Archimedes (287-212 A.C.) on the law of the lever and on hydrostatics. It is motivated by the imagination and creativity of da Vinci (1452-1519) and the rigid-body experiments of Galileo (1564-1642). It is founded on the laws of motion put forward by Newton (1643-1727), later set on firm theoretical ground by Euler (1707-1783) and further developed and refined by Cauchy (1789-1857).

Continuum mechanics as taught and practiced today emerged in the latter half of the 20th century. This "renaissance" period can be attributed to several factors, such as the flourishing of relevant mathematics disciplines (particularly linear algebra, partial differential equations and differential geometry), the advances in materials and mechanical systems technologies, and the increasing availability (especially since the late 1960s) of high-performance computers. A wave of gifted modern-day mechanicians, such as Rivlin, Truesdell, Ericksen, Naghdi and many others, contributed to the rebirth and consolidation of classical mechanics into this new discipline of continuum mechanics, which emphasizes generality, rigor and abstraction, yet derives its essential features from the physics of material behavior.

Chapter 2

Mathematical preliminaries

2.1 Elements of set theory

This section summarizes a few elementary notions and definitions from set theory. A *set* X is a collection of objects referred to as *elements*. A set can be defined either by the properties of its elements or by merely identifying all elements. For example,

$$X = \{1, 2, 3, 4, 5\} \tag{2.1}$$

or

$$X = \{\text{all integers greater than 0 and less than 6}\} \ . \tag{2.2}$$

Two sets of particular interest in the remainder of the course are:

$$\mathbb{N} \ = \ \{\text{all positive integers}\} \tag{2.3}$$

and

$$\mathbb{R} \ = \ \{\text{all real numbers}\} \tag{2.4}$$

If x is an element of the set X, one writes $x \in X$. If not, one writes $x \notin X$.

Let X, Y be two sets. The set X is a *subset* of the set Y (denoted as $X \subseteq Y$ or $Y \supseteq X$) if every element of X is also an element of Y. The set X is a *proper subset* of the set (denoted as $X \subset Y$ or $Y \supset X$) if every element of X is also an element of Y, but there exists at least one element of Y that does not belong to X.

The *union* of sets X and Y (denoted by $X \cup Y$) is the set which is comprised of all elements of both sets. The *intersection* of sets X and Y (denoted by $X \cap Y$) is a set which includes only the elements common to the two sets. The *empty set* (denoted by $\emptyset$) is a set that contains no elements and is contained in every set, therefore, $X \cup \emptyset = X$.

The *Cartesian product* $X \times Y$ of sets X and Y is a set defined as

$$X \times Y \;=\; \{(x, y) \text{ such that } x \in X, y \in Y\} \;. \tag{2.5}$$

Note that the pair (x, y) in the preceding equation is ordered, i.e., the element (x, y) is, in general, not the same as the element (y, x). The notation X^2, X^3, ..., is used to respectively denote the Cartesian products $X \times X$, $X \times X \times X$, ...

2.2 Vector spaces

Consider a set $\mathcal{V}$ whose members (typically called "points") can be scalars, vectors or functions, visualized in Figure 2.1. Assume that $\mathcal{V}$ is endowed with an addition operation $(+)$ and a scalar multiplication operation $(\cdot)$, which do not necessarily coincide with the classical addition and multiplication for real numbers.

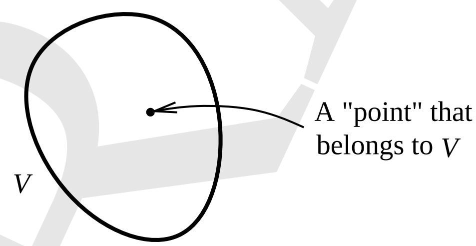

Figure 2.1: *Schematic depiction of a set*

A *linear* (or *vector*) *space* $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is defined by the following properties for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{R}$:

(i) $\alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v} \in \mathcal{V}$ (closure),

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity with respect to $+$),

(iii) $\exists\ 0 \in \mathcal{V} \quad | \quad \mathbf{u} + \mathbf{0} = \mathbf{u}$ (existence of null element),

(iv) $\exists\ -\mathbf{u} \in \mathcal{V} \quad | \quad \mathbf{u} + (-\mathbf{u}) = 0$ (existence of negative element),

(v) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity),

(vi) $(\alpha\beta) \cdot \mathbf{u} = \alpha \cdot (\beta \cdot \mathbf{u})$ (associativity with respect to $\cdot$),

(vii) $(\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$ (distributivity with respect to $\mathbb{R}$),

(viii) $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$ (distributivity with respect to $\mathcal{V}$),

(ix) $1 \cdot \mathbf{u} = \mathbf{u}$ (existence of identity).

Examples:

(1) $\mathcal{V} = P_2 := \{$all second degree polynomials $ax^2 + bx + c\}$ with the standard polynomial addition and scalar multiplication.

It can be trivially verified that $\{P_2, +; \mathbb{R}, \cdot\}$ is a linear function space. P_2 is also "equivalent" to an ordered triad $(a, b, c) \in \mathbb{R}^3$.

(2) Define $\mathcal{V} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with the standard addition and scalar multiplication for vectors. Notice that given $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ as in Figure 2.2,

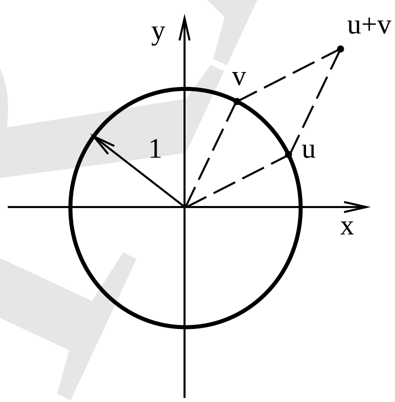

Figure 2.2: *Example of a set that does not form a linear space*

property (i) is violated, i.e., since, in general, for $\alpha = \beta = 1$,

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2\ ,\ y_1 + y_2)\ ,$$

and $(x_1+x_2)^2+(y_1+y_2)^2 \neq 1$. Thus, $\{\mathcal{V},+;\mathbb{R},\cdot\}$ is not a linear space. □

Consider a linear space $\{\mathcal{V},+;\mathbb{R},\cdot\}$ and a subset $\mathcal{U}$ of $\mathcal{V}$. Then $\mathcal{U}$ forms a linear sub-space of $\mathcal{V}$ with respect to the same operations $(+)$ and $(\cdot)$, if, for any $\mathbf{u},\mathbf{v} \in \mathcal{U}$ and $\alpha, \beta, \in \mathbb{R}$

$$\alpha \cdot \mathbf{u} \; + \; \beta \cdot \mathbf{v} \; \in \; \mathcal{U} \; ,$$

i.e., closure is maintained within $\mathcal{U}$.

Example:

(a) Define the set P_n of all algebraic polynomials of degree smaller or equal to $n > 2$ and consider the linear space $\{P_n,+;\mathbb{R},\cdot\}$ with the usual polynomial addition and scalar multiplication. Then, P_2 is a linear subspace of $\{P_n,+;\mathbb{R},\cdot\}$. □

In order to simplify the notation, in the remainder of these notes the symbol $\cdot$ used in scalar multiplication will be omitted.

Let $\mathbf{v}_1 \cdot \mathbf{v}_2, ..., \mathbf{v}_p$ be elements of the vector space $\{\mathcal{V},+;\mathbb{R},\cdot\}$ and assume that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_p\mathbf{v}_p \;=\; 0 \;\Leftrightarrow\; \alpha_1 = \alpha_2 = ... = \alpha_p = 0 \; . \tag{2.6}$$

Then, $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\}$ is a *linearly independent* set in $\mathcal{V}$. The vector space $\{\mathcal{V},+;\mathbb{R},\cdot\}$ is *infinite-dimensional* if, given any $n \in \mathbb{N}$, it contains at least one linearly independent set with $n+1$ elements. If the above statement is not true, then there is an $n \in \mathbb{N}$, such that all linearly independent sets contain at most n elements. In this case, $\{\mathcal{V},+;\mathbb{R},\cdot\}$ is a *finite dimensional* vector space (specifically, n-dimensional).

A *basis* of an n-dimensional vector space $\{\mathcal{V},+;\mathbb{R},\cdot\}$ is defined as any set of n linearly independent vectors. If $\{\mathbf{g}_1,\mathbf{g}_2,...,\mathbf{g}_n\}$ form a basis in $\{\mathcal{V},+;\mathbb{R},\cdot\}$, then given any non-zero $\mathbf{v} \in \mathcal{V}$,

$$\alpha_1\mathbf{g}_1 + \alpha_2\mathbf{g}_2 + \ldots + \alpha_n\mathbf{g}_n + \beta\mathbf{v} \;=\; 0 \;\Leftrightarrow\; \text{ not all } \alpha_1,\ldots,\alpha_n,\beta \text{ equal zero} \; . \tag{2.7}$$

More specifically, $\beta \neq 0$ because otherwise there would be at least one non-zero α_i, $i=1,\ldots,n$, which would have implied that $\{\mathbf{g}_1,\mathbf{g}_2,...,\mathbf{g}_n\}$ are not linearly independent.

Thus, the non-zero vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} \;=\; -\frac{\alpha_1}{\beta}\mathbf{g}_1 - \frac{\alpha_2}{\beta}\mathbf{g}_2 - \ldots - \frac{\alpha_n}{\beta}\mathbf{g}_n \; . \tag{2.8}$$

The above representation of $\mathbf{v}$ in terms of the basis $\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n\}$ is unique. Indeed, if, alternatively,

$$\mathbf{v} \;=\; \gamma_1 \mathbf{g}_1 + \gamma_2 \mathbf{g}_2 + \ldots + \gamma_n \mathbf{g}_n \;, \tag{2.9}$$

then, upon subtracting the preceding two equations from one another, it follows that

$$\mathbf{0} \;=\; \left(\gamma_1 + \frac{\alpha_1}{\beta}\right) \mathbf{g}_1 + \left(\gamma_2 + \frac{\alpha_2}{\beta}\right) \mathbf{g}_2 + \ldots + \left(\gamma_n + \frac{\alpha_n}{\beta}\right) \mathbf{g}_n \;, \tag{2.10}$$

which implies that $\gamma_i = -\dfrac{\alpha_i}{\beta}$, $i = 1, 2, \ldots, n$, since $\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n\}$ are assumed to be a linearly independent.

Of all the vector spaces, attention will be focused here on the particular class of Euclidean vector spaces in which a vector multiplication operation $(\cdot)$ is defined, such that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$,

(x) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity with respect to $\cdot$),

(xi) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity),

(xii) $(\alpha \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha \mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$ (associativity with respect to $\cdot$)

(xiii) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = 0$.

This vector operation is referred to as the *dot-product*. An n-dimensional vector space obeying the above additional rules is referred to as a *Euclidean vector space* and is denoted by E^n.

Example:

The standard dot-product between vectors in $\mathbb{R}^n$ satisfies the above properties. □

The dot-product provide a natural means for defining the *magnitude* of a vector as

$$\|\mathbf{u}\| \;=\; (\mathbf{u} \cdot \mathbf{u})^{1/2} \;. \tag{2.11}$$

Two vectors $\mathbf{u}, \mathbf{v} \in E^n$ are *orthogonal*, if $\mathbf{u} \cdot \mathbf{v} = 0$. A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, ...\}$ is called *orthonormal*, if

$$\mathbf{u}_i \cdot \mathbf{u}_j \;=\; \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \;=\; \delta_{ij} \;, \tag{2.12}$$

where δ_{ij} is called the *Kronecker delta* symbol.

Every orthonormal set $\{\mathbf{e}_1, \mathbf{e}_2 ... \mathbf{e}_k\}$, $k \leq n$ in E^n is linearly independent. This is because, if

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \ldots + \alpha_k \mathbf{e}_k = \mathbf{0} , \tag{2.13}$$

then, upon taking the dot-product of the above equation with $\mathbf{e}_i$, $i = 1, 2, \ldots, k$, and invoking the orthonormality of $\{\mathbf{e}_1, \mathbf{e}_2 ... \mathbf{e}_k\}$,

$$\alpha_1(\mathbf{e}_1 \cdot \mathbf{e}_i) + \alpha_2(\mathbf{e}_2 \cdot \mathbf{e}_i) + \ldots + \alpha_k(\mathbf{e}_k \cdot \mathbf{e}_i) = \alpha_i = 0 . \tag{2.14}$$

Of particular importance to the forthcoming developments is the observation that any vector $\mathbf{v} \in E^n$ can be resolved to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ as

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \ldots + v_n \mathbf{e}_n = \sum_{i=1}^{n} v_i \mathbf{e}_i , \tag{2.15}$$

where $v_i = \mathbf{v} \cdot \mathbf{e}_i$. In this case, v_i denotes the i-th *component* of $\mathbf{v}$ relative to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$.

2.3 Points, vectors and tensors in the Euclidean 3-space

Consider the Cartesian space E^3 with an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. As argued in the previous section, a typical vector $\mathbf{v} \epsilon E^3$ can be written as

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i \ ; \ v_i = \mathbf{v} \cdot \mathbf{e}_i . \tag{2.16}$$

Next, consider points x, y in the *Euclidean point space* $\mathcal{E}^3$, which is the set of all points in the ambient three-dimensional space, when taken to be devoid of the mathematical structure of vector spaces. Also, consider an arbitrary *origin* (or reference point) O in the same space. It is now possible to define vectors $\mathbf{x}, \mathbf{y} \in E^3$, which originate at O and end at points x and y, respectively. In this way, one makes a unique association (to within the specification of O) between points in $\mathcal{E}^3$ and vectors in E^3. Further, it is possible to define a measure of *distance* between x and y, by way of the magnitude of the vector $\mathbf{v} = \mathbf{y} - \mathbf{x}$, namely

$$d(x, y) = |\mathbf{x} - \mathbf{y}| = [(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})]^{1/2} . \tag{2.17}$$

In E^3, one may define the *vector product* of two vectors as an operation with the following properties: for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and any scalar α,

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,

(b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$, or, equivalently $[\mathbf{uvw}] = [\mathbf{vwu}] = [\mathbf{wuv}]$, where $[\mathbf{uvw}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is the *triple product* of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

(c) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta \quad , \quad \cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{(\mathbf{u} \cdot \mathbf{u})^{1/2} (\mathbf{v} \cdot \mathbf{v})^{1/2}}\ , 0 \leq \theta \leq \pi.$

Appealing to property (a), it is readily concluded that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Likewise, properties (a) and (b) can be used to deduce that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$, namely that the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

By definition, for a *right-hand* orthonormal coordinate basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the following relations hold true:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \quad , \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 \quad , \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2\ . \tag{2.18}$$

These relations, together with the implications of property (a)

$$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0} \tag{2.19}$$

and

$$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3 \quad , \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1 \quad , \quad \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2 \tag{2.20}$$

can be expressed compactly as

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \mathbf{e}_k\ , \tag{2.21}$$

where ϵ_{ijk} is the *permutation symbol* defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3),\ (3,2,1), \text{ or } (1,3,2) \\ 0 & \text{otherwise} \end{cases} . \tag{2.22}$$

It follows that

$$\mathbf{u} \times \mathbf{v} = (\sum_{i=1}^{3} u_i \mathbf{e}_i) \times (\sum_{j=1}^{3} v_j \mathbf{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \mathbf{e}_i \times \mathbf{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} u_i v_j e_{ijk} \mathbf{e}_k\ . \tag{2.23}$$

Let $\mathcal{U}$, $\mathcal{V}$ be two sets and define a *mapping* f from $\mathcal{U}$ to $\mathcal{V}$ as a rule that assigns to each point $u \in \mathcal{U}$ a unique point $v = f(u) \in \mathcal{V}$, see Figure 2.3. The usual notation for a mapping is: $f : \mathcal{U} \to \mathcal{V}$, $u \to v = f(u) \in \mathcal{V}$. With reference to the above setting, $\mathcal{U}$ is called the *domain* of f, whereas $\mathcal{V}$ is termed the *range* of f.

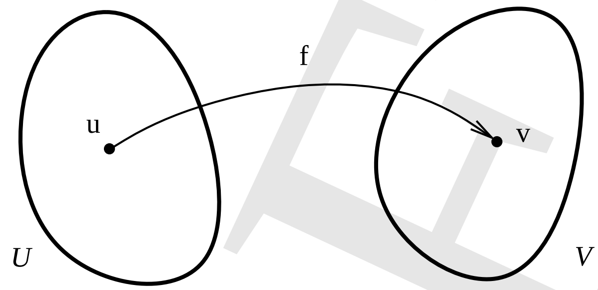

Figure 2.3: *Mapping between two sets*

A mapping $\mathbf{T} : E^3 \to E^3$ is called *linear* if it satisfies the property

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v}) \ , \tag{2.24}$$

for all $\mathbf{u}, \mathbf{v} \in E^3$ and $\alpha, \beta \in \mathbb{N}$. A linear mapping is also referred to as a *tensor*.

Examples:

(1) $\mathbf{T} : E^3 \to E^3$, $\mathbf{T}(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in E^3$. This is called the *identity* tensor, and is typically denoted by $\mathbf{T} = \mathbf{I}$.

(2) $\mathbf{T} : E^3 \to E^3$, $\mathbf{T}(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in E^3$. This is called the *zero* tensor, and is typically denoted by $\mathbf{T} = \mathbf{0}$. □

The *tensor product* between two vectors $\mathbf{v}$ and $\mathbf{w}$ in E^3 is denoted by $\mathbf{v} \otimes \mathbf{w}$ and defined according to the relation

$$(\mathbf{v} \otimes \mathbf{w})\mathbf{u} \ = \ (\mathbf{w} \cdot \mathbf{u})\mathbf{v} \ , \tag{2.25}$$

for any vector $\mathbf{u} \in E^3$. This implies that, under the action of the tensor product $\mathbf{v} \otimes \mathbf{w}$, the vector $\mathbf{u}$ is mapped to the vector $(\mathbf{w} \cdot \mathbf{u})\mathbf{v}$. It can be easily verified that $\mathbf{v} \otimes \mathbf{w}$ is a tensor according to the previously furnished definition. Using the Cartesian components of vectors, one may express the tensor product of $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} \otimes \mathbf{w} \;=\; (\sum_{i=1}^{3} v_i \mathbf{e}_i) \otimes (\sum_{j=1}^{3} w_j \mathbf{e}_j) \;=\; \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_j \mathbf{e}_i \otimes \mathbf{e}_j \; . \tag{2.26}$$

It will be shown shortly that the set of nine tensor products $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$, $i, j = 1, 2, 3$, form a basis for the space $\mathcal{L}(E^3, E^3)$ of all tensors on E^3.

Before proceeding further with the discussion of tensors, it is expedient to introduce a summation convention, which will greatly simplify the component representation of both vectorial and tensorial quantities and their algebra. This originates with A. Einstein, who employed it first in his relativity work. The summation convention has three rules, which, when adapted to the special case of E^3, are as follows:

Rule 1. If an index appears twice in a single component term or in a product term, the summation sign is omitted and summation is automatically assumed from value 1 to 3. Such an index is referred to as *dummy*.

Rule 2. An index which appears once in a single component or in a product expression is not summed and is assumed to attain a single value (1, 2, or 3). Such an index is referred to *free*.

Rule 3. No index can appear more than twice in a single component or in a product term.

Examples:

1. The vector representation $\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i$ is replaced by $\mathbf{u} = u_i \mathbf{e}_i$ and it involves the summation of three terms.

2. The tensor product $\mathbf{u} \otimes \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$ is equivalently written as $\mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$ and it involves the summation of nine terms.

3. The term $u_i v_j$ is a single term with two free indices i and j.

4. It is easy to see that $\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} u_3 = u_j$. This index substitution property is frequently used in component manipulations.

5. A similar index substitution property applies in the case of a two-index quantity, namely $\delta_{ij} a_{ik} = \delta_{1j} a_{1k} + \delta_{2j} a_{2k} + \delta_{3j} a_{3k} = a_{jk}$. □

With the summation convention in place, take a tensor $\mathbf{T} \in \mathcal{L}(E^3, E^3)$ and define its *components* T_{ij}, such that $\mathbf{T}\mathbf{e}_j = T_{ij}\mathbf{e}_i$. It follows that

$$\begin{aligned} (\mathbf{T} - T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{v} &= (\mathbf{T} - T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) v_k \mathbf{e}_k \\ &= \mathbf{T}\mathbf{e}_k v_k - T_{ij} v_k (\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_k \\ &= T_{ik}\mathbf{e}_i v_k - T_{ij} v_k (\mathbf{e}_j \cdot \mathbf{e}_k)\mathbf{e}_i \\ &= T_{ik}\mathbf{e}_i v_k - T_{ij} v_k \delta_{jk} \mathbf{e}_i \\ &= T_{ik}\mathbf{e}_i v_k - T_{ik} v_k \mathbf{e}_i \\ &= \mathbf{0} , \end{aligned} \tag{2.27}$$

hence,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j . \tag{2.28}$$

This derivation demonstates that any tensor $\mathbf{T}$ can be written as a linear combination of the nine tensor product terms $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. The components of the tensor $\mathbf{T}$ relative to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ can be put in matrix form as

$$[T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} . \tag{2.29}$$

The *transpose* $\mathbf{T}^T$ of a tensor $\mathbf{T}$ is defined by the property

$$\mathbf{u} \cdot \mathbf{T}\mathbf{v} = \mathbf{v} \cdot \mathbf{T}^T \mathbf{u} , \tag{2.30}$$

for any vectors $\mathbf{u}$ and $\mathbf{v}$ in E^3. Using components, this implies that

$$u_i T_{ij} v_j = v_i A_{ij} u_j = v_j A_{ji} u_i , \tag{2.31}$$

where A_{ij} are the components of $\mathbf{T}^T$. It follows that

$$u_i (T_{ij} - A_{ji}) v_j = 0 . \tag{2.32}$$

Since u_i and v_j are arbitrary, this implies that $A_{ij} = T_{ji}$, hence the transpose of $\mathbf{T}$ can be written as

$$\mathbf{T}^T \;=\; T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i \;. \tag{2.33}$$

A tensor $\mathbf{T}$ is *symmetric* if $\mathbf{T}^T = \mathbf{T}$ or, when both $\mathbf{T}$ and $\mathbf{T}^T$ are resolved relative to the same basis, $T_{ji} = T_{ij}$. Likewise, a tensor $\mathbf{T}$ is *skew-symmetric* if $\mathbf{T}^T = -\mathbf{T}$ or, again, upon resolving both on the same basis, $T_{ji} = -T_{ij}$. Note that, in this case, $T_{11} = T_{22} = T_{33} = 0$.

Given tensors $\mathbf{T}, \mathbf{S} \in \mathcal{L}\ (E^3, E^3)$, the *multiplication* $\mathbf{TS}$ is defined according to

$$(\mathbf{TS})\mathbf{v} \;=\; \mathbf{T}(\mathbf{Sv}) \;, \tag{2.34}$$

for any $\mathbf{v} \in E^3$.

In component form, this implies that

$$\begin{aligned} (\mathbf{TS})\mathbf{v} \;=\; \mathbf{T}(\mathbf{Sv}) \;&=\; \mathbf{T}[(S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(v_k\mathbf{e}_k)] \\ &=\; \mathbf{T}(S_{ij}v_k\delta_{jk}\mathbf{e}_i) \\ &=\; \mathbf{T}(S_{ij}v_j\mathbf{e}_i) \\ &=\; T_{ki}S_{ij}v_j\mathbf{e}_i \\ &=\; (T_{ki}S_{ij}\mathbf{e}_k \otimes \mathbf{e}_j)(v_l\mathbf{e}_l) \;, \end{aligned} \tag{2.35}$$

which leads to

$$(\mathbf{TS}) \;=\; T_{ki}S_{ij}\mathbf{e}_k \otimes \mathbf{e}_j \;. \tag{2.36}$$

The *trace* $\operatorname{tr}\mathbf{T} : \mathcal{L}(E^3, E^3) \mapsto \mathbb{R}$ of the tensor product of two vectors $\mathbf{u} \otimes \mathbf{v}$ is defined as

$$\operatorname{tr}\mathbf{u} \otimes \mathbf{v} \;=\; \mathbf{u} \cdot \mathbf{v} \;, \tag{2.37}$$

hence, the trace of a tensor $\mathbf{T}$ is deduced from equation (2.37) as

$$\operatorname{tr}\mathbf{T} \;=\; tr(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \;=\; T_{ij}\mathbf{e}_i \cdot \mathbf{e}_j \;=\; T_{ij}\delta_{ij} \;=\; T_{ii} \;. \tag{2.38}$$

The *contraction* (or *inner product*) $\mathbf{T} \cdot \mathbf{S} : \mathcal{L}(E^3, E^3) \times \mathcal{L}(E^3, E^3) \mapsto \mathbb{R}$ of two tensors $\mathbf{T}$ and $\mathbf{S}$ is defined as

$$\mathbf{T} \cdot \mathbf{S} \;=\; \operatorname{tr}(\mathbf{TS}^T) \;. \tag{2.39}$$

Using components,

$$\operatorname{tr}(\mathbf{TS}^T) \;=\; \operatorname{tr}(T_{ki}S_{ji}\mathbf{e}_k \otimes \mathbf{e}_j) \;=\; T_{ki}S_{ji}\mathbf{e}_k \cdot \mathbf{e}_j \;=\; T_{ki}S_{ji}\delta_{kj} \;=\; T_{ki}S_{ki} \;. \tag{2.40}$$

A tensor $\mathbf{T}$ is *invertible* if, for any $\mathbf{w} \in E^3$, the equation

$$\mathbf{T}\mathbf{v} \;=\; \mathbf{w} \tag{2.41}$$

can be uniquely solved for $\mathbf{v}$. Then, one writes

$$\mathbf{v} = \mathbf{T}^{-1}\mathbf{w} \;, \tag{2.42}$$

and $\mathbf{T}^{-1}$ is the *inverse* of $\mathbf{T}$. Clearly, if $\mathbf{T}^{-1}$ exists, then

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{w} - \mathbf{v} \; &= \mathbf{0} \\ &= \mathbf{T}^{-1}(\mathbf{T}\mathbf{v}) - \mathbf{v} \\ &= (\mathbf{T}^{-1}\mathbf{T})\mathbf{v} - \mathbf{v} \\ &= (\mathbf{T}^{-1}\mathbf{T} - \mathbf{I})\mathbf{v} \;, \end{aligned} \tag{2.43}$$

hence $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ and, similarly, $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$.

A tensor $\mathbf{T}$ is *orthogonal* if

$$\mathbf{T}^T \;=\; \mathbf{T}^{-1} \;, \tag{2.44}$$

which implies that

$$\mathbf{T}^T\mathbf{T} \;=\; \mathbf{T}\mathbf{T}^T \;=\; \mathbf{I} \;. \tag{2.45}$$

It can be shown that for any tensors $\mathbf{T}, \mathbf{S} \in \mathcal{L}(E^3, E^3)$,

$$(\mathbf{S} + \mathbf{T})^T \;=\; \mathbf{S}^T + \mathbf{T}^T \quad , \quad (\mathbf{S}\mathbf{T})^T \;=\; \mathbf{T}^T\mathbf{S}^T \;. \tag{2.46}$$

If, further, the tensors $\mathbf{T}$ and $\mathbf{S}$ are invertible, then

$$(\mathbf{S}\mathbf{T})^{-1} \;=\; \mathbf{T}^{-1}\mathbf{S}^{-1} \;. \tag{2.47}$$

2.4 Vector and tensor calculus

Define scalar, vector and tensor functions of a vector variable $\mathbf{x}$ and a real variable t. The scalar functions are of the form

$$\begin{aligned} &\phi_1 : \mathbb{R} \to \mathbb{R} \;,\; t \to \phi = \phi_1(t) \\ &\phi_2 : E^3 \to \mathbb{R} \;,\; \mathbf{x} \to \phi = \phi_2(\mathbf{x}) \\ &\phi_3 : E^3 \times \mathbb{R} \to \mathbb{R} \;,\; (\mathbf{x}, t) \to \phi = \phi_3(\mathbf{x}, t) \;, \end{aligned} \tag{2.48}$$

while the vector and tensor functions are of the form

$$
\begin{aligned}
\mathbf{v}_1 &: \mathbb{R} \to E^3 \ , \ t \to \mathbf{v} = \mathbf{v}_1(t) \\
\mathbf{v}_2 &: E^3 \to E^3 \ , \ \mathbf{x} \to \mathbf{v} = \mathbf{v}_2(\mathbf{x}) \\
\mathbf{v}_3 &: E^3 \times \mathbb{R} \to E^3 \ , \ (\mathbf{x}, t) \to \mathbf{v} = \mathbf{v}_3(\mathbf{x}, t)
\end{aligned}
\tag{2.49}
$$

and

$$
\begin{aligned}
\mathbf{T}_1 &: \mathbb{R} \to \mathcal{L}(E^3, E^3) \ , \ t \to \mathbf{T} = \mathbf{T}_1(t) \\
\mathbf{T}_2 &: E^3 \to \mathcal{L}(E^3, E^3) \ , \ \mathbf{x} \to \mathbf{T} = \mathbf{T}_2(\mathbf{x}) \\
\mathbf{T}_3 &: E^3 \times \mathbb{R} \to \mathcal{L}(E^3, E^3) \ , \ (\mathbf{x}, t) \to \mathbf{T} = \mathbf{T}_3(\mathbf{x}, t) \ ,
\end{aligned}
\tag{2.50}
$$

respectively.

The *gradient* $\operatorname{grad} \phi(\mathbf{x})$ (otherwise denoted as $\nabla \phi(\mathbf{x})$ or $\dfrac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}}$) of a scalar function $\phi = \phi(\mathbf{x})$ is a vector defined by

$$
(\operatorname{grad} \phi(\mathbf{x})) \cdot \mathbf{v} \ = \ \left[\frac{d}{dw} \phi(\mathbf{x} + w\mathbf{v}) \right]_{w=0} \ , \tag{2.51}
$$

for any $\mathbf{v} \in E^3$. Using the chain rule, the right-hand side of equation (2.51) becomes

$$
\left[\frac{d}{dw} \phi(\mathbf{x} + w\mathbf{v}) \right]_{w=0} \ = \ \left[\frac{\partial \phi(\mathbf{x} + w\mathbf{v})}{\partial (x_i + w v_i)} \frac{d(x_i + w v_i)}{dw} \right]_{w=0} \ = \ \frac{\partial \phi(\mathbf{x})}{\partial x_i} v_i \ . \tag{2.52}
$$

Hence, in component form one may write

$$
\operatorname{grad} \phi(\mathbf{x}) \ = \ \frac{\partial \phi(\mathbf{x})}{\partial x_i} \mathbf{e}_i \ . \tag{2.53}
$$

In operator form, this leads to the expression

$$
\operatorname{grad} \ = \ \nabla \ = \ \frac{\partial}{\partial x_i} \mathbf{e}_i \ . \tag{2.54}
$$

<u>Example:</u> Consider the scalar function $\phi(\mathbf{x}) = |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. Its gradient is

$$
\begin{aligned}
\operatorname{grad} \phi \ &= \ \frac{\partial}{\partial \mathbf{x}} (\mathbf{x} \cdot \mathbf{x}) \ = \ \frac{\partial (x_j x_j)}{\partial x_i} \mathbf{e}_i \ = \ \left(\frac{\partial x_j}{\partial x_i} x_j + x_j \frac{\partial x_j}{\partial x_i} \right) \mathbf{e}_i \\
&= \ (\delta_{ij} x_j + x_j \delta_{ij}) \mathbf{e}_i \ = \ 2 x_i \mathbf{e}_i \ = \ 2\mathbf{x} \ .
\end{aligned}
\tag{2.55}
$$

Alternatively, using directly the definition

$$
\begin{aligned}
(\operatorname{grad}\phi)\cdot\mathbf{v} &= \left[\frac{d}{dw}\{(\mathbf{x}+w\mathbf{v})\cdot(\mathbf{x}+u\mathbf{v})\}\right]_{w=0} \\
&= \left[\frac{d}{dw}\{\mathbf{x}\cdot\mathbf{x}+2w\mathbf{x}\cdot\mathbf{v}+w^2\mathbf{v}\cdot\mathbf{v}\}\right]_{w=0} \\
&= [2\mathbf{x}\cdot\mathbf{v}+2w\mathbf{v}\cdot\mathbf{v}]_{w=0} \\
&= 2\mathbf{x}\cdot\mathbf{v}
\end{aligned}
\tag{2.56}
$$

□

The *gradient* $\operatorname{grad}\mathbf{v}(\mathbf{x})$ (otherwise denoted as $\nabla\mathbf{v}(\mathbf{x})$ or $\frac{\partial\mathbf{v}(\mathbf{x})}{\partial\mathbf{x}}$) of a vector function $\mathbf{v}=\mathbf{v}(\mathbf{x})$ is a tensor defined by the relation

$$
(\operatorname{grad}\mathbf{v}(\mathbf{x}))\mathbf{w} = \left[\frac{d}{dw}\mathbf{v}(\mathbf{x}+w\mathbf{w})\right]_{w=0} , \tag{2.57}
$$

for any $\mathbf{w}\in E^3$. Again, using chain rule, the right-hand side of equation (2.57) becomes

$$
\left[\frac{d}{dw}\mathbf{v}(\mathbf{x}+w\mathbf{w})\right]_{w=0} = \left[\frac{\partial v_i(\mathbf{x}+w\mathbf{w})}{\partial(x_j+ww_j)}\frac{d(x_j+ww_j)}{dw}\right]_{w=0}\mathbf{e}_i = \frac{\partial v_i(\mathbf{x})}{\partial x_j}w_j\mathbf{e}_i , \tag{2.58}
$$

hence, using components,

$$
grad(\mathbf{v}(\mathbf{x}) = \frac{\partial v_i(\mathbf{x})}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j . \tag{2.59}
$$

In operator form, this leads to the expression

$$
\operatorname{grad} = \boldsymbol{\nabla} = \frac{\partial}{\partial x_i}\otimes\mathbf{e}_j . \tag{2.60}
$$

Example: Consider the function $\mathbf{v}(\mathbf{x})=\alpha\mathbf{x}$. Its gradient is

$$
\operatorname{grad}\mathbf{v} = \frac{\partial(\alpha\mathbf{x})}{\partial x} = \frac{\partial(\alpha x_i)}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j = \alpha\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = \alpha\mathbf{e}_i\otimes\mathbf{e}_i = \alpha\mathbf{I} , \tag{2.61}
$$

since $(\mathbf{e}_i\otimes\mathbf{e}_i)\mathbf{v}=(\mathbf{v}\cdot\mathbf{e}_i)\mathbf{e}_i=v_i\mathbf{e}_i=\mathbf{v}$. Alternatively, using directly the definition,

$$
(\operatorname{grad}\mathbf{v})\mathbf{w} = \left[\frac{d}{dw}(\mathbf{v}+w\mathbf{w})\right]_{w=0} = \alpha\mathbf{w} , \tag{2.62}
$$

hence $\operatorname{grad}\mathbf{v}=\alpha\mathbf{I}$.

The *divergence* $\operatorname{div}\mathbf{v}(\mathbf{x})$ (otherwise denoted as $\nabla\cdot\mathbf{v}(\mathbf{x})$) of a vector function $\mathbf{v}=\mathbf{v}(\mathbf{x})$ is a scalar defined as

$$
\operatorname{div}\mathbf{v}(\mathbf{x}) = \operatorname{tr}(\operatorname{grad}\mathbf{v}(\mathbf{x})) , \tag{2.63}
$$

on, using components,

$$\operatorname{div}\mathbf{v}(\mathbf{x}) = \operatorname{tr}\left(\frac{\partial v_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right) = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial v_i}{\partial x_j}\delta_{ij} = \frac{\partial v_i}{\partial x_i} = v_{i,i} . \tag{2.64}$$

In operator form, one writes

$$\operatorname{div} = \boldsymbol{\nabla}\cdot = \frac{\partial}{\partial x_i} \cdot \mathbf{e}_i . \tag{2.65}$$

Example: Consider again the function $\mathbf{v}(\mathbf{x}) = \alpha\mathbf{x}$. Its divergence is

$$\operatorname{div}\mathbf{v}(\mathbf{x}) = \frac{\partial(\alpha x_i)}{\partial x_i} = \alpha\frac{\partial x_i}{\partial x_i} = \alpha\delta_{ii} = 3\alpha . \tag{2.66}$$

□

The *divergence* $\operatorname{div}\mathbf{T}(\mathbf{x})$ (otherwise denoted as $\nabla \cdot \mathbf{T}(\mathbf{x})$) of a tensor function $\mathbf{T} = \mathbf{T}(\mathbf{x})$ is a vector defined by the property that

$$(\operatorname{div}\mathbf{T}(\mathbf{x})) \cdot \mathbf{c} = \operatorname{div}\big((\mathbf{T}^T(\mathbf{x}))\mathbf{c}\big) , \tag{2.67}$$

for any *constant* vector $\mathbf{c} \in E^3$.

Using components,

$$\begin{aligned}
\operatorname{div}\mathbf{T}) &= \operatorname{div}(\mathbf{T}^T\mathbf{c}) \\
&= \operatorname{div}[(T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i)(c_k\mathbf{e}_k)] \\
&= \operatorname{div}[T_{ij}c_k\delta_{ik}\mathbf{e}_j] \\
&= \operatorname{div}[T_{ij}c_i\mathbf{e}_j] \\
&= \operatorname{tr}\left[\frac{\partial T_{ij}c_i}{\partial x_k}\mathbf{e}_j \otimes \mathbf{e}_k\right] \\
&= \frac{\partial(T_{ij}c_i)}{\partial x_k}\delta_{jk} \\
&= \frac{\partial(T_{ij}c_i)}{\partial x_j} \\
&= \frac{\partial T_{ij}}{\partial x_j}c_i ,
\end{aligned} \tag{2.68}$$

hence,

$$\operatorname{div}\mathbf{T} = \frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i . \tag{2.69}$$

In the case of the divergence of a tensor, the operator form becomes

$$\operatorname{div} \;=\; \boldsymbol{\nabla}\cdot \;=\; \frac{\partial}{\partial x_i}\mathbf{e}_i\;. \tag{2.70}$$

Finally, the *curl* $\operatorname{curl}\mathbf{v}(\mathbf{x})$ of a vector function $\mathbf{v}(\mathbf{x})$ is a vector defined as

$$\operatorname{curl}\mathbf{v}(\mathbf{x}) \;=\; \nabla\times\mathbf{v}(\mathbf{x})\;, \tag{2.71}$$

which translates using components to

$$\operatorname{curl}\mathbf{v}(\mathbf{x}) \;=\; \left(\frac{\partial}{\partial x_i}\mathbf{e}_i\right)\times(v_j\mathbf{e}_j) \;=\; \frac{\partial v_j}{\partial x_i}\mathbf{e}_i\times\mathbf{e}_j \;=\; \frac{\partial v_j}{\partial x_i}e_{ijk}\mathbf{e}_k \;=\; e_{ijk}\frac{\partial v_k}{\partial x_j}\mathbf{e}_i\;. \tag{2.72}$$

In operator form, the curl is expressed as

$$\operatorname{curl} \;=\; \nabla\times \;=\; \frac{\partial}{\partial x_i}\mathbf{e}_i\times\;\;. \tag{2.73}$$

Chapter 3

Kinematics of deformation

3.1 Bodies, configurations and motions

Let a continuum body $\mathcal{B}$ be defined as a collection of material particles, which, when considered together, endow the body with local (pointwise) physical properties which are independent of its actual size or the time over which they are measured. Also, let a typical such particle be denoted by P, while an arbitrary subset of $\mathcal{B}$ be denoted by $\mathcal{S}$, see Figure 3.1.

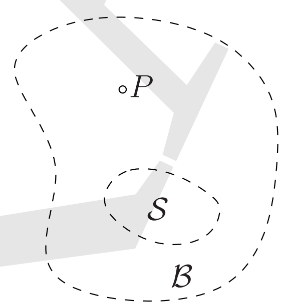

Figure 3.1: *A body $\mathcal{B}$ and its subset $\mathcal{S}$.*

Let x be the point in $\mathcal{E}^3$ occupied by a particle P of the body $\mathcal{B}$ at time t, and let $\mathbf{x}$ be its associated position vector relative to the origin O of an orthonormal basis in the vector space E^3. Then, define by $\bar{\boldsymbol{\chi}} : (P, t) \in \mathcal{B} \times \mathbb{R} \mapsto E^3$ the *motion* of $\mathcal{B}$, which is a differentiable mapping, such that

$$\mathbf{x} \;=\; \bar{\boldsymbol{\chi}}(P, t) \;=\; \bar{\boldsymbol{\chi}}_t(P) \; . \tag{3.1}$$

In the above, $\bar{\boldsymbol{\chi}}_t : \mathcal{B} \mapsto \mathcal{E}^3$ is called the *configuration mapping* of $\mathcal{B}$ at time t. Given $\bar{\boldsymbol{\chi}}$, the body $\mathcal{B}$ may be mapped to its *configuration* $\mathcal{R} = \bar{\boldsymbol{\chi}}_t(\mathcal{B}, t)$ with boundary $\partial\mathcal{R}$ at time t. Likewise, any part $\mathcal{S} \subset \mathcal{B}$ can be mapped to its configuration $\mathcal{P} = \bar{\boldsymbol{\chi}}_t(\mathcal{S}, t)$ with boundary $\partial\mathcal{P}$ at time t, see Figure 3.2. Clearly, $\mathcal{R}$ and $\mathcal{P}$ are point sets in $\mathcal{E}^3$.

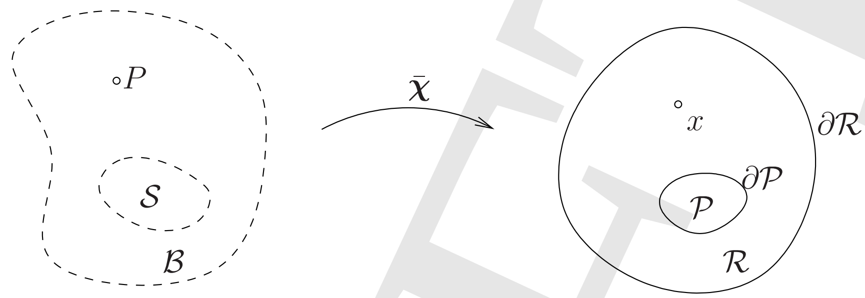

Figure 3.2: *Mapping of a body $\mathcal{B}$ to its configuration at time t.*

The configuration mapping $\bar{\boldsymbol{\chi}}_t$ is assumed to be invertible, which means that, given any point $\mathbf{x} \in \mathcal{P}$,

$$P = \bar{\boldsymbol{\chi}}_t^{-1}(\mathbf{x}) . \tag{3.2}$$

The motion $\bar{\boldsymbol{\chi}}$ of the body is assumed to be twice-differentiable in time. Then, one may define the velocity and acceleration of any particle P at time t according to

$$\mathbf{v} = \frac{\partial \bar{\boldsymbol{\chi}}(P, t)}{\partial t} \quad , \quad \mathbf{a} = \frac{\partial^2 \bar{\boldsymbol{\chi}}(P, t)}{\partial t^2} . \tag{3.3}$$

The mapping $\bar{\boldsymbol{\chi}}$ represents the *material description* of the body motion. This is because the domain of $\bar{\boldsymbol{\chi}}$ consists of the totality of material particles in the body, as well as time. This description, although mathematically proper, is of limited practical use, because there is no direct quantitative way of tracking particles of the body. For this reason, two alternative descriptions of the body motion are introduced below.

Of all configurations in time, select one, say $\mathcal{R}_0 = \bar{\boldsymbol{\chi}}(\mathcal{B}, t_0)$ at a time $t = t_0$, and refer to it as the *reference configuration*. The choice of reference configuration may be arbitrary, although in many practical problems it is guided by the need for mathematical simplicity. Now, denote the point which P occupies at time t_0 as X and let this point be associated with position vector $\mathbf{X}$, namely

$$\mathbf{X} = \bar{\boldsymbol{\chi}}(P, t_0) = \bar{\boldsymbol{\chi}}_{t_0}(P) . \tag{3.4}$$

Thus, one may write

$$\mathbf{x} \;=\; \bar{\boldsymbol{\chi}}(P,t) \;=\; \bar{\boldsymbol{\chi}}(\boldsymbol{\chi}_{t_0}^{-1}(\mathbf{X}),t) \;=\; \boldsymbol{\chi}(\mathbf{X},t)\ . \tag{3.5}$$

The mapping $\boldsymbol{\chi} : E^3 \times \mathbb{R} \mapsto E^3$, where

$$\mathbf{x} \;=\; \boldsymbol{\chi}(\mathbf{X},t) \;=\; \boldsymbol{\chi}_t(\mathbf{X}) \tag{3.6}$$

represents the *referential* or *Lagrangian description* of the body motion. In such a description, it is implicit that a reference configuration is provided. The mapping $\boldsymbol{\chi}_t$ is the *placement* of the body relative to its reference configuration, see Figure 3.3.

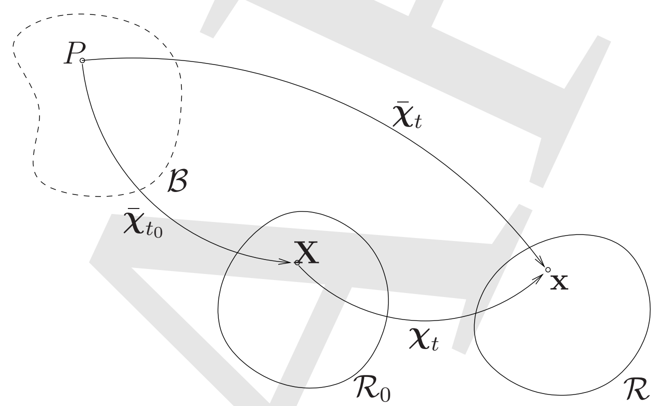

Figure 3.3: *Mapping of a body $\mathcal{B}$ to its reference configuration at time t_0 and its current configuration at time t.*

Assume now that the motion of the body $\mathcal{B}$ is described with reference to the configuration $\mathcal{R}_0$ defined at time $t = t_0$ and let the configuration of $\mathcal{B}$ at time t be termed the *current configuration*. Also, let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be fixed right-hand orthonormal bases associated with the reference and current configuration, respectively. With reference to the preceding bases, one may write the position vectors $\mathbf{X}$ and $\mathbf{x}$ corresponding to the points occupied by the particle P at times t_0 and t as

$$\mathbf{X} \;=\; X_A \mathbf{E}_A \qquad , \qquad \mathbf{x} \;=\; x_i \mathbf{e}_i\ , \tag{3.7}$$

respectively. Hence, resolving all relevant vectors to their respective bases, the motion $\boldsymbol{\chi}$ may be expressed using components as

$$x_i \mathbf{e}_i \;=\; \chi_i(X_A \mathbf{E}_A, t)\mathbf{e}_i\ , \tag{3.8}$$

or, in pure component form,

$$x_i \;=\; \chi_i(X_A, t) \; . \tag{3.9}$$

The velocity and acceleration vectors, expressed in the referential description, take the form

$$\mathbf{v} \;=\; \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \qquad , \qquad \mathbf{a} \;=\; \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \; , \tag{3.10}$$

respectively. Likewise, using the orthonormal bases,

$$\mathbf{v} \;=\; v_i(X_A, t)\mathbf{e}_i \qquad , \qquad \mathbf{a} \;=\; a_i(X_A, t)\mathbf{e}_i \; . \tag{3.11}$$

Scalar, vector and tensor functions can be alternatively expressed using the *spatial* or *Eulerian description*, where the independent variables are the current position vector $\mathbf{x}$ and time t. Indeed, starting, by way of an example, with a scalar function $f = \check{f}(P, t)$, one may write

$$f \;=\; \check{f}(P, t) \;=\; \check{f}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}), t) \;=\; \tilde{f}(\mathbf{x}, t) \; . \tag{3.12}$$

In analogous fashion, one may write

$$f \;=\; \tilde{f}(\mathbf{x}, t) \;=\; \tilde{f}(\boldsymbol{\chi}_t(\mathbf{X}), t) \;=\; \hat{f}(\mathbf{X}, t) \; . \tag{3.13}$$

The above two equations can be combined to write

$$f \;=\; \check{f}(P, t) \;=\; \tilde{f}(\mathbf{x}, t) \;=\; \hat{f}(\mathbf{X}, t) \; . \tag{3.14}$$

$$\begin{tikzcd}
(\mathbf{X}, t) \arrow[r, "\boldsymbol{\chi}"] \arrow[dr, "\hat{\mathbf{v}}"'] & (\mathbf{x}, t) \arrow[d, "\tilde{\mathbf{v}}"] \\
& \mathbf{v}
\end{tikzcd}$$

Figure 3.4: *Schematic depiction of referential and spatial mappings for the velocity* $\mathbf{v}$.

Any function (not necessarily scalar) of space and time can be written equivalently in material, referential or spatial form. Focusing specifically on the referential and spatial descriptions, it is easily seen that the velocity and acceleration vectors can be equivalently expressed as

$$\mathbf{v} \;=\; \hat{\mathbf{v}}(\mathbf{X},t) \;=\; \tilde{\mathbf{v}}(\mathbf{x},t) \qquad , \qquad \mathbf{a} \;=\; \hat{\mathbf{a}}(\mathbf{X},t) \;=\; \tilde{\mathbf{a}}(\mathbf{x},t)\ , \tag{3.15}$$

respectively, see Figure 3.4. In component form, one may write

$$\mathbf{v} \;=\; \hat{v}_i(X_A,t)\mathbf{e}_i \;=\; \tilde{v}_i(x_j,t)\mathbf{e}_i \qquad , \qquad \mathbf{a} \;=\; \hat{a}_i(X_A,t)\mathbf{e}_i \;=\; \tilde{a}_i(x_j,t)\mathbf{e}_i\ . \tag{3.16}$$

Given a function $f = \hat{f}(\mathbf{X},t)$, define the *material time derivative* of f as

$$\dot{f} \;=\; \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\ . \tag{3.17}$$

From the above definition it is clear that the material time derivative of a function is the rate of change of the function when keeping the referential position $\mathbf{X}$ (therefore also the particle P associated with this position) fixed.

If, alternatively, f is expressed in spatial form, i.e., $f = \tilde{f}(\mathbf{x},t)$, then

$$\begin{aligned}\dot{f} \;&=\; \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial \mathbf{x}} \cdot \frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t} \\ &=\; \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial \mathbf{x}} \cdot \mathbf{v} \\ &=\; \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + \operatorname{grad} \tilde{f} \cdot \mathbf{v}\ .\end{aligned} \tag{3.18}$$

The first term on the right-hand side of (3.18) is the *spatial time derivative* of f and corresponds to the rate of change of f for a *fixed* point $\mathbf{x}$ in space. The second term is called the *convective rate of change* of f and is due to the spatial variation of f and its effect on the material time derivative as the material particle which occupies the point $\mathbf{x}$ at time t tends to travel away from $\mathbf{x}$ with velocity $\mathbf{v}$. A similar expression for the material time derivative applies for vector functions. Indeed, take, for example, the velocity $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x},t)$ and write

$$\begin{aligned}\dot{\mathbf{v}} \;&=\; \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial t} + \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial \mathbf{x}} \frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t} \\ &=\; \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial t} + \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial \mathbf{x}} \mathbf{v} \\ &=\; \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial t} + (\operatorname{grad} \tilde{\mathbf{v}})\mathbf{v}\ .\end{aligned} \tag{3.19}$$

A volume/surface/curve which consists of the same material points in all configurations is termed *material*. By way of example, consider a surface in three dimensions, which can be expressed in the form $F(\mathbf{X}) = 0$. This is clearly a material surface, because it contains the same material particles at all times, since its referential representation is independent of time. On the other hand, a surface described by the equation $F(\mathbf{X}, t) = 0$ is generally non material, because the locus of its points contains different material particles at different times. This distinction becomes less apparent when a surface is defined in spatial form, i.e., by an equation $f(\mathbf{x}, t) = 0$. In this case, one may employ *Lagrange's criterion of materiality*, which states the following:

Lagrange's criterion of materiality (1783)

A surface described by the equation $f(\mathbf{x}, t) = 0$ is material if, and only if, $\dot{f} = 0$.

A sketch of the proof is as follows: if the surface is assumed material, then

$$f(\mathbf{x}, t) \;=\; F(\mathbf{X}) = 0 \;, \tag{3.20}$$

hence

$$\dot{f}(\boldsymbol{\chi}, t) \;=\; \dot{F}(\mathbf{X}) = 0 \;. \tag{3.21}$$

Conversely, if the criterion holds, then

$$\dot{f}(\boldsymbol{\chi}, t) \;=\; \dot{f}(\boldsymbol{\chi}(\mathbf{X}, t), t) \;=\; \frac{\partial F}{\partial t}(\mathbf{X}, t) \;=\; 0 \;, \tag{3.22}$$

which implies that $F = F(\mathbf{X})$, hence the surface is material.

A similar analysis applies to curves in Euclidean 3-space. Specifically, a material curve can be be viewed as the intersection of two material surfaces, say $F(\mathbf{X}) = 0$ and $G(\mathbf{X}) = 0$. Switching to the spatial description and expressing these surfaces as

$$F(\mathbf{X}) \;=\; F(\boldsymbol{\chi}_t^{-1}(\mathbf{x})) \;=\; f(\mathbf{x}, t) \;=\; 0 \tag{3.23}$$

and

$$G(\mathbf{X}) \;=\; G(\boldsymbol{\chi}_t^{-1}(\mathbf{x})) \;=\; g(\mathbf{x}, t) \;=\; 0 \;, \tag{3.24}$$

it follows from Lagrange's criterion that a curve is material if $\dot{f} = \dot{g} = 0$. It is easy to show that this is a sufficient, but not a necessary condition for the materiality of a curve.

Some important definitions regarding the nature of the motion $\boldsymbol{\chi}$ follow. First, a motion $\boldsymbol{\chi}$ is *steady* at a point $\mathbf{x}$, if the velocity at that point is independent of time. If this is the case at all points, then $\mathbf{v} = \mathbf{v}(\mathbf{x})$ and the motion is called *steady*. If a motion is not steady, then it is called *unsteady*. A point $\mathbf{x}$ in space where $\mathbf{v}(\mathbf{x}, t) = \mathbf{0}$ at all times is called a *stagnation point*.

Let $\boldsymbol{\chi}$ be the motion of body $\mathcal{B}$ and fix a particle P, which occupies a point $\mathbf{X}$ in the reference configuration. Subsequently, trace its successive placements as a function of time, i.e., fix $\mathbf{X}$ and consider the one-parameter family of placements

$$\mathbf{x} \;=\; \boldsymbol{\chi}(\mathbf{X}, t) \quad , \quad (\mathbf{X} \text{ fixed}) \; . \tag{3.25}$$

The resulting parametric equations (with parameter t) represent in algebraic form the *particle path* of the given particle, see Figure 3.5. Alternatively, one may express the same particle path in differential form as

$$d\mathbf{x} \;=\; \hat{\mathbf{v}}(\mathbf{X}, \tau) d\tau \quad , \quad \mathbf{x}(t_0) \;=\; \mathbf{X} \quad , \quad (\mathbf{X} \text{ fixed}) \; , \tag{3.26}$$

or, equivalently,

$$d\mathbf{y} \;=\; \tilde{\mathbf{v}}(\mathbf{y}, \tau) d\tau \quad , \quad \mathbf{y}(t) \;=\; \mathbf{x} \quad , \quad (\mathbf{x} \text{ fixed}) \; . \tag{3.27}$$

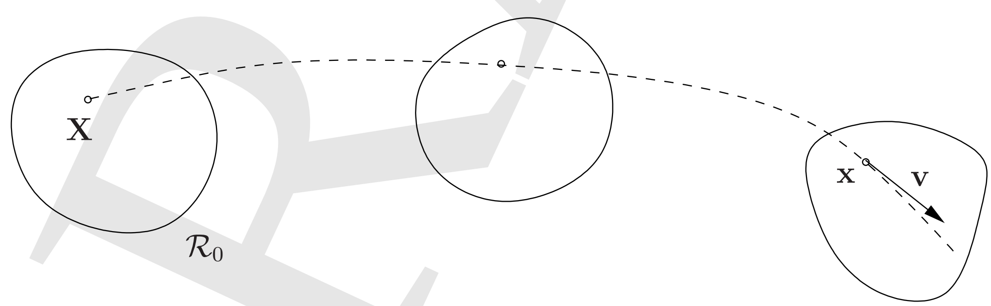

Figure 3.5: *Particle path of a particle which occupies* $\mathbf{X}$ *in the reference configuration.*

Now, let $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t)$ be the velocity field at a given time t. Define a *stream line* through $\mathbf{x}$ at time t as the space curve that passes through $\mathbf{x}$ and is tangent to the velocity field $\mathbf{v}$ at all of its points, i.e., it is defined according to

$$d\mathbf{y} = \tilde{\mathbf{v}}(\mathbf{y}, t) d\tau \quad , \quad \mathbf{y}(\tau_0) \;=\; \mathbf{x} \quad , \quad (t \text{ fixed}) \; , \tag{3.28}$$

where τ is a scalar parameter and τ_0 some arbitrarily chosen value of τ, see Figure 3.6. Using components, the preceding definition becomes

$$\frac{dy_1}{\tilde{v}_1(y_j,t)} \;=\; \frac{dy_2}{\tilde{v}_2(y_j,t)} \;=\; \frac{dy_3}{\tilde{v}_3(y_j,t)} \;=\; d\tau \quad , \quad y_i(\tau_0) = x_i \quad , \quad (t \text{ fixed}) \; . \tag{3.29}$$

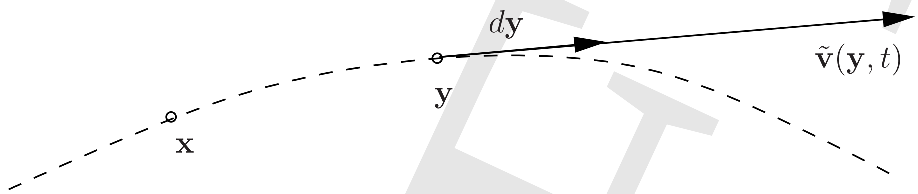

Figure 3.6: *Stream line through point* $\mathbf{x}$ *at time* t.

The *streak line* through a point $\mathbf{x}$ at time t is defined by the equation

$$\mathbf{y} \;=\; \boldsymbol{\chi}(\boldsymbol{\chi}_\tau^{-1}(\mathbf{x}),t) \; , \tag{3.30}$$

where τ is a scalar parameter. In differential form, this line can be expressed as

$$d\mathbf{y} \;=\; \tilde{\mathbf{v}}(\mathbf{y},t)dt \quad , \quad \mathbf{y}(\tau) \;=\; \mathbf{x} \; . \tag{3.31}$$

It is easy to show that the streak line through $\mathbf{x}$ at time t is the locus of placements at time t of all particles that have passed or will pass through $\mathbf{x}$. Equation (3.31) can be derived from (3.30) by noting that $\tilde{\mathbf{v}}(\mathbf{y},t)$ is the velocity at time τ of a particle which at time τ occupies the point $\mathbf{x}$, while at time t it occupies the point $\mathbf{y}$.

Note that given a point $\mathbf{x}$ at time t, then the path of the particle occupying $\mathbf{x}$ at t and the stream line through $\mathbf{x}$ at t have a common tangent. Indeed, this is equivalent to stating that the velocity at time t of the material point associated with $\mathbf{X}$ has the same direction with the velocity of the point that occupies $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$.

In the case of steady motion, the particle path for any particle occupying $\mathbf{x}$ coincides with the stream line and streak line through $\mathbf{x}$ at time t. To argue this property, take a stream line (which is now a fixed curve, since the motion is steady). Consider a material point P which is on the curve at time t. Notice that the velocity of P is always tangent to the stream line, this its path line coincides with the stream line through $\mathbf{x}$. A similar argument can be made for streak lines.

In general, path lines can intersect, since intersection points merely mean that different particles can occupy the same position at different times. However, stream lines do not intersect, except at points where the velocity vanishes, otherwise the velocity at an intersection point would have two different directions.

Example: Consider a motion $\boldsymbol{\chi}$, such that

$$\begin{aligned} \chi_1 &= \chi_1(X_A, t) \;=\; X_1 e^t \\ \chi_2 &= \chi_2(X_A, t) \;=\; X_2 + t X_3 \\ \chi_3 &= \chi_3(X_A, t) \;=\; -t X_2 + \; X_3 \;, \end{aligned} \tag{3.32}$$

with reference to fixed orthonormal system $\{\mathbf{e}_i\}$. Note that $\mathbf{x} = \mathbf{X}$ at time $t = 0$, i.e., the body occupies the reference configuration at time $t = 0$.

The inverse mapping $\boldsymbol{\chi}_t^{-1}$ is easily obtained as

$$\begin{aligned} X_1 &= \chi_{t_1}^{-1}(x_j) \;=\; x_1 e^{-t} \\ X_2 &= \chi_{t_2}^{-1}(x_j) \;=\; \frac{x_2 - t x_3}{1 + t^2} \\ X_3 &= \chi_{t_3}^{-1}(x_j) \;=\; \frac{t x_2 + x_3}{1 + t^2} \;. \end{aligned} \tag{3.33}$$

The velocity field, written in the referential description has components $\hat{v}_i(X_A, t) = \dfrac{\partial \chi_i(X_A, t)}{\partial t}$, i.e.,

$$\begin{aligned} \hat{v}_1(X_A, t) &= X_1 e^t \\ \hat{v}_2(X_A, t) &= X_3 \\ \hat{v}_3(X_A, t) &= -\, X_2 \;, \end{aligned} \tag{3.34}$$

while in the spatial description has components $\tilde{v}_i(\chi_j, t)$ given by

$$\begin{aligned} \tilde{v}_1(\chi_j, t) &= (x_1 e^{-t}) e^t \;=\; x_1 \\ \tilde{v}_2(\chi_j, t) &= \frac{t x_2 + x_3}{1 + t^2} \\ \tilde{v}_3(\chi_j, t) &= -\, \frac{x_2 - t x_3}{1 + t^2} \;. \end{aligned} \tag{3.35}$$

Note that $\mathbf{x} = \mathbf{0}$ is a stagnation point and, also, that the motion is steady on the x_1-axis.

The acceleration in the referential description has components $\hat{a}_i(X_A, t) = \dfrac{\partial^2 \chi_i(X_A, t)}{\partial t^2}$, hence,

$$
\begin{aligned}
\hat{a}_1(X_A, t) &= X_1 e^t \\
\hat{a}_2(X_A, t) &= 0 \\
\hat{a}_3(X_A, t) &= 0 \ ,
\end{aligned}
\tag{3.36}
$$

while in the spatial description the components $\tilde{a}_i(\chi_j, t)$ are given by

$$
\begin{aligned}
\tilde{a}_1(x_j, t) &= x_1 \\
\tilde{a}_2(x_j, t) &= 0 \\
\tilde{a}_3(x_j, t) &= 0 \ .
\end{aligned}
\tag{3.37}
$$

□

3.2 The deformation gradient and other measures of deformation

Consider a body $\mathcal{B}$ which occupies its reference configuration $\mathcal{R}_0$ at time t_0 and the current configuration $\mathcal{R}$ at time t. Also, let $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ be two fixed right-hand orthonormal bases associated with the reference and current configuration, respectively.

Recall that the motion $\boldsymbol{\chi}$ is defined so that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ and consider the deformation of an infinitesimal material line element $d\mathbf{X}$ located at the point $\mathbf{X}$ of the reference configuration. This material element is mapped into another infinitesimal line element $d\mathbf{x}$ at point $\mathbf{x}$ in the current configuration at time t, see Figure 3.7.

It follows from chain rule that

$$
d\mathbf{x} \ = \ \left[\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}\right](\mathbf{X}, t) d\mathbf{X} \ = \ \mathbf{F} d\mathbf{X} \ , \tag{3.38}
$$

where $\mathbf{F}$ is the *deformation gradient* tensor, defined as

$$
\mathbf{F} \ = \ \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} \ . \tag{3.39}
$$

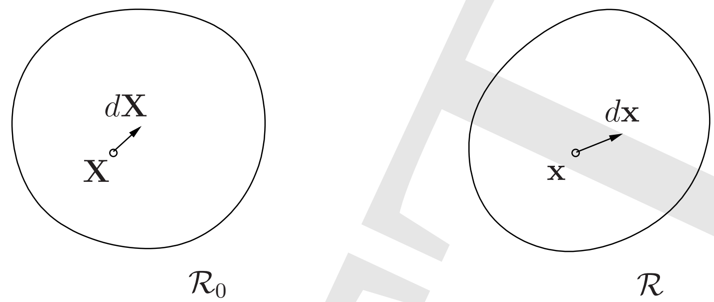

Figure 3.7: *Mapping of an infinitesimal material line elements* $d\mathbf{X}$ *from the reference to the current configuration.*

Using components, the deformation gradient tensor can be expressed as

$$\frac{\partial \chi_i(X_B,t)}{\partial X_A}\mathbf{e}_i \otimes \mathbf{E}_A \;=\; F_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \; . \tag{3.40}$$

It is clear from the above that the deformation gradient is a *two-point* tensor which has one "leg" in the reference configuration and the other in the current configuration. It follows from (3.38) that the deformation gradient $\mathbf{F}$ provides the rule by which infinitesimal line element are mapped from the reference to the current configuration. Using (3.40), one may rewrite (3.38) in component form as

$$dx_i \;=\; F_{iA}dX_A \;=\; \chi_{i,A}dX_A \; . \tag{3.41}$$

Recall now that the motion $\boldsymbol{\chi}$ is assumed invertible for fixed t. Also, recall the inverse function theorem of real analysis, which, in the case of the mapping $\boldsymbol{\chi}$ can be stated as follows: For a fixed time t, let $\boldsymbol{\chi}_t : \; \mathcal{R}_0 \rightarrow \mathcal{R}$ be continuously differentiable (i.e., $\dfrac{\partial \boldsymbol{\chi}_t}{\partial \mathbf{X}}$ exists and is continuous) and consider $\mathbf{X} \in \mathcal{R}_0$, such that $\det \dfrac{\partial \boldsymbol{\chi}_t}{\partial \mathbf{X}}(\mathbf{X}) \neq 0$. Then, there is an open neighborhood $\mathcal{P}_0$ of $\mathbf{X}$ in $\mathcal{R}_0$ and an open neighborhood $\mathcal{P}$ of $\mathcal{R}$, such that $\boldsymbol{\chi}_t(\mathcal{P}_0) = \mathcal{P}$ and $\boldsymbol{\chi}_t$ has a continuously differentiable inverse $\boldsymbol{\chi}_t^{-1}$, so that $\boldsymbol{\chi}_t^{-1}(\mathcal{P}) = \mathcal{P}_0$, as in Figure 3.8. Moreover, for any $\mathbf{x} \in \mathcal{P}$, $\mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x})$ and $\dfrac{\partial \boldsymbol{\chi}_t^{-1}(\mathbf{x})}{\partial \mathbf{x}} = (\mathbf{F}(\mathbf{X},t))^{-1}$[1].

[1] This means that the derivative of the inverse motion with respect to $\mathbf{x}$ is identical to the inverse of the gradient of the motion with respect to $\mathbf{X}$.

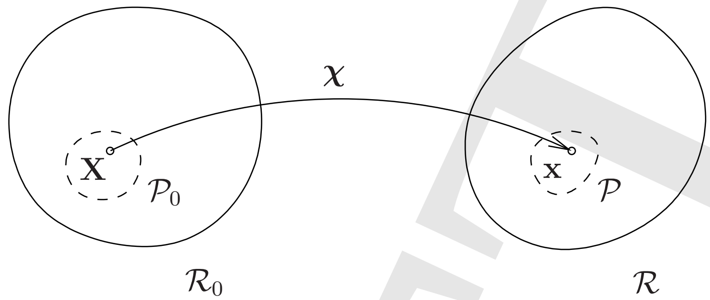

Figure 3.8: *Application of the inverse function theorem to the motion $\boldsymbol{\chi}$ at a fixed time t.*

The inverse function theorem states that the mapping $\boldsymbol{\chi}$ is invertible at a point $\mathbf{X}$ for a fixed time t, if the *Jacobian determinant* $J = \det \dfrac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} = \det \mathbf{F}$ satisfies the condition $J \neq 0$. The condition $\det J \neq 0$ will be later amended to be $\det J > 0$.

Generally, the infinitesimal material line element $d\mathbf{X}$ stretches and rotates to $d\mathbf{x}$ under the action of $\mathbf{F}$. To explore this, write $d\mathbf{X} = \mathbf{M} dS$ and $d\mathbf{x} = \mathbf{m} ds$, where $\mathbf{M}$ and $\mathbf{m}$ are unit vectors (i.e., $\mathbf{M} \cdot \mathbf{M} = \mathbf{m} \cdot \mathbf{m} = 1$) in the direction of $d\mathbf{X}$ and $d\mathbf{x}$, respectively and $dS > 0$ and $ds > 0$ are the infinitesimal lengths of $d\mathbf{X}$ and $d\mathbf{x}$, respectively.

Next, define the *stretch* λ of the infinitesimal material line element $d\mathbf{X}$ as

$$\lambda = \frac{ds}{dS} , \tag{3.42}$$

and note that

$$\begin{aligned} d\mathbf{x} = \mathbf{F} d\mathbf{X} &= \mathbf{F} \mathbf{M} dS \\ &= \mathbf{m} ds , \end{aligned} \tag{3.43}$$

hence

$$\lambda \mathbf{m} = \mathbf{F} \mathbf{M} . \tag{3.44}$$

Since $\det \mathbf{F} \neq 0$, it follows that $\lambda \neq 0$ and, in particular, that $\lambda > 0$, given that $\mathbf{m}$ is chosen to reflect the sense of $\mathbf{x}$. In summary, the preceding arguments imply that $\lambda \in (0, \infty)$.

To determine the value of λ, take the dot-product of each side of (3.44) with itself, which leads to

$$\begin{aligned}(\lambda \mathbf{m}) \cdot (\lambda \mathbf{m}) \ = \ \lambda^2 (\mathbf{m} \cdot \mathbf{m}) \ = \ \lambda^2 \ &= (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M}) \\ &= \mathbf{M} \cdot \mathbf{F}^T (\mathbf{F}\mathbf{M}) \\ &= \mathbf{M} \cdot (\mathbf{F}^T \mathbf{F}) \mathbf{M} \\ &= \mathbf{M} \cdot \mathbf{C}\mathbf{M} \ , \end{aligned} \tag{3.45}$$

where $\mathbf{C}$ is the *right Cauchy-Green deformation tensor*, defined as

$$\mathbf{C} \ = \ \mathbf{F}^T \mathbf{F} \tag{3.46}$$

or, upon using components,

$$C_{AB} \ = \ F_{iA} F_{iB} \ . \tag{3.47}$$

Note that, in general $\mathbf{C} = \mathbf{C}(\mathbf{X}, t)$, since $\mathbf{F} = \mathbf{F}(\mathbf{X}, t)$. Also, it is important to observe that $\mathbf{C}$ is defined with respect to the basis in the reference configuration. Further, $\mathbf{C}$ is a symmetric and positive-definite tensor. The latter means that, $\mathbf{M} \cdot \mathbf{C}\mathbf{M} > 0$ for any $\mathbf{M} \neq \mathbf{0}$ and $\mathbf{M} \cdot \mathbf{C}\mathbf{M} = 0$, if, and only if $\mathbf{M} = \mathbf{0}$, both of which follow clear from (3.45). To summarize the meaning of $\mathbf{C}$, it can be said that, given a direction $\mathbf{M}$ in the reference configuration, $\mathbf{C}$ allows the determination of the stretch λ of an infinitesimal material line element $d\mathbf{X}$ along $\mathbf{M}$ when mapped to the line element $d\mathbf{x}$ in the current configuration.

Recall that the inverse function theorem implies that, since $\det \mathbf{F} \neq 0$,

$$d\mathbf{X} \ = \ \frac{\partial \boldsymbol{\chi}_t^{-1}(\mathbf{x})}{\partial \mathbf{x}} d\mathbf{x} \ = \ \mathbf{F}^{-1} d\mathbf{x} \ . \tag{3.48}$$

Then, one may write, in analogy with the preceding derivation of $\mathbf{C}$, that

$$\begin{aligned} d\mathbf{X} \ = \ \mathbf{F}^{-1} d\mathbf{x} \ &= \mathbf{F}^{-1} \mathbf{m} ds \\ &= \mathbf{M} dS \ , \end{aligned} \tag{3.49}$$

hence

$$\frac{1}{\lambda} \mathbf{M} \ = \ \mathbf{F}^{-1} \mathbf{m} \ . \tag{3.50}$$

Again, taking the dot-products of each side with itself leads to

$$
\begin{aligned}
(\frac{1}{\lambda}\mathbf{M}) \cdot (\frac{1}{\lambda}\mathbf{M}) \;=\; \frac{1}{\lambda^2}(\mathbf{M} \cdot \mathbf{M}) \;=\; \frac{1}{\lambda^2} &= (\mathbf{F}^{-1}\mathbf{m}) \cdot (\mathbf{F}^{-1}\mathbf{m}) \\
&= \mathbf{m} \cdot \mathbf{F}^{-T}(\mathbf{F}^{-1}\mathbf{m}) \\
&= \mathbf{M} \cdot (\mathbf{F}^{-T}\mathbf{F}^{-1})\mathbf{m} \\
&= \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m} \;, \qquad (3.51)
\end{aligned}
$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T = (\mathbf{F}^T)^{-1}$ and $\mathbf{B}$ is the *left Cauchy-Green tensor*, defined as

$$
\mathbf{B} \;=\; \mathbf{F}\mathbf{F}^T \qquad (3.52)
$$

or, using components,

$$
B_{ij} \;=\; F_{iA}F_{jA} \;. \qquad (3.53)
$$

In contrast to $\mathbf{C}$, the tensor $\mathbf{B}$ is defined with respect to the basis in the current configuration. Like $\mathbf{C}$, it is easy to establish that the tensor $\mathbf{B}$ is symmetric and positive-definite. To summarize the meaning of $\mathbf{B}$, it can be said that, given a direction $\mathbf{m}$ in the current configuration, $\mathbf{B}$ allows the determination of the stretch λ of an infinitesimal element $d\mathbf{x}$ along $\mathbf{m}$ which is mapped from an infinitesimal material line element $d\mathbf{X}$ in the reference configuration.

Consider now the difference in the squares of the lengths of the line elements $d\mathbf{X}$ and $d\mathbf{x}$, namely, $ds^2 - dS^2$ and write this difference as

$$
\begin{aligned}
ds^2 - dS^2 &= (d\mathbf{x} \cdot d\mathbf{x}) - (d\mathbf{X} \cdot d\mathbf{X}) \\
&= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{F}d\mathbf{X}) - (d\mathbf{X} \cdot d\mathbf{X}) \\
&= d\mathbf{X} \cdot \mathbf{F}^T(\mathbf{F}d\mathbf{X}) - (d\mathbf{X} \cdot d\mathbf{X}) \\
&= d\mathbf{X} \cdot (\mathbf{C}d\mathbf{X}) - (d\mathbf{X} \cdot d\mathbf{X}) \\
&= d\mathbf{X} \cdot (\mathbf{C} - \mathbf{I})d\mathbf{X} \\
&= d\mathbf{X} \cdot 2\mathbf{E}d\mathbf{X} \;, \qquad (3.54)
\end{aligned}
$$

where

$$
\mathbf{E} \;=\; \frac{1}{2}(\mathbf{C} - \mathbf{I}) \;=\; \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \qquad (3.55)
$$

is the (relative) *Lagrangian strain tensor*. Using components, the preceding equation can be written as

$$E_{AB} \;=\; \frac{1}{2}(F_{iA}F_{iB} - \delta_{AB}) \;, \tag{3.56}$$

which shows that the Lagrangian strain tensor $\mathbf{E}$ is defined with respect to the basis in the reference configuration. In addition, $\mathbf{E}$ is clearly symmetric and satisfies the property that $\mathbf{E} = \mathbf{0}$ when the body undergoes no deformation between the reference and the current configuration.

The difference $ds^2 - dS^2$ may be also written as

$$\begin{aligned} ds^2 - dS^2 &= (d\mathbf{x} \cdot d\mathbf{x}) - (d\mathbf{X} \cdot d\mathbf{X}) \\ &= (d\mathbf{x} \cdot d\mathbf{x}) - (\mathbf{F}^{-1}d\mathbf{x}) \cdot (\mathbf{F}^{-1}d\mathbf{x}) \\ &= (d\mathbf{x} \cdot d\mathbf{x}) - d\mathbf{x} \cdot \mathbf{F}^{-1}(\mathbf{F}^{-1}d\mathbf{x}) \\ &= d\mathbf{x} \cdot d\mathbf{x} - (d\mathbf{x} \cdot \mathbf{B}^{-1}d\mathbf{x}) \\ &= d\mathbf{x} \cdot (\mathbf{i} - \mathbf{B}^{-1})d\mathbf{x} \\ &= d\mathbf{X} \cdot 2\mathbf{e}d\mathbf{X} \;, \end{aligned} \tag{3.57}$$

where

$$\mathbf{e} \;=\; \frac{1}{2}(\mathbf{i} - \mathbf{B}^{-1}) \tag{3.58}$$

is the (relative) *Eulerian strain tensor* or *Almansi tensor*, and $\mathbf{i}$ is the identity tensor in the current configuration. Using components, one may write

$$e_{ij} \;=\; \frac{1}{2}(\delta_{ij} - B_{ij}^{-1}) \;. \tag{3.59}$$

Like $\mathbf{E}$, the tensor $\mathbf{e}$ is symmetric and vanishes when there is no deformation between the current configuration remains undeformed relative to the reference configuration. However, unlike $\mathbf{E}$, the tensor $\mathbf{e}$ is naturally resolved into components on the basis in the current configuration.

While, in general, the infinitesimal material line element $d\mathbf{X}$ is both stretched and rotated due to $\mathbf{F}$, neither $\mathbf{C}$ (or $\mathbf{B}$) nor $\mathbf{E}$ (or $\mathbf{e}$) yield any information regarding the rotation of $d\mathbf{X}$. To extract rotation-related information from $\mathbf{F}$, recall the *polar decomposition theorem*, which states that any invertible tensor $\mathbf{F}$ can be uniquely decomposed into

$$\mathbf{F} \;=\; \mathbf{RU} \;=\; \mathbf{VR} \;, \tag{3.60}$$

where $\mathbf{R}$ is an orthogonal tensor and $\mathbf{U}, \mathbf{V}$ are symmetric positive-definite tensors. In component form, the polar decomposition is expressed as

$$F_{iA} \;=\; R_{iB}U_{BA} \;=\; V_{ij}R_{jA} \; . \tag{3.61}$$

The tensors $(\mathbf{R}, \mathbf{U})$ or $(\mathbf{R}, \mathbf{V})$ are called the *polar factors* of $\mathbf{F}$.

Given the polar decomposition theorem, it follows that

$$\mathbf{C} \;=\; \mathbf{F}^T\mathbf{F} \;=\; (\mathbf{RU})^T(\mathbf{RU}) \;=\; \mathbf{U}^T\mathbf{R}^T\mathbf{RU} \;=\; \mathbf{UU} \;=\; \mathbf{U}^2 \tag{3.62}$$

and, likewise,

$$\mathbf{B} \;=\; \mathbf{FF}^T \;=\; (\mathbf{VR})(\mathbf{VR})^T \;=\; \mathbf{VRR}^T\mathbf{V} \;=\; \mathbf{VV} \;=\; \mathbf{V}^2 \; . \tag{3.63}$$

The tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right stretch tensor* and the *left stretch tensor*, respectively. Given their relations to $\mathbf{C}$ and $\mathbf{B}$, it is clear that these tensors may be used to determine the stretch of the infinitesimal material line element $d\mathbf{X}$, which explains their name. Also, it follows that the component representation of the two tensors is $\mathbf{U} = U_{AB}\mathbf{E}_A \otimes \mathbf{E}_B$ and $\mathbf{V} = V_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, i.e., they are resolved naturally on the bases of the reference and current configuration, respectively. Also note that the relations $\mathbf{U} = \mathbf{C}^{1/2}$ and $\mathbf{V} = \mathbf{B}^{1/2}$ hold true, as it is possible to define the square-root of a symmetric positive-definite tensor.

Of interest now is $\mathbf{R}$, which is a two-point tensor, i.e., $\mathbf{R} = R_{iA}\mathbf{e}_i \otimes \mathbf{E}_A$. Recalling equation (3.38), attempt to provide a physical interpretation of the decomposition theorem, starting from the *right polar decomposition* $\mathbf{F} = \mathbf{RU}$. Using this decomposition, and taking into account (3.38), write

$$d\mathbf{x} \;=\; \mathbf{F}d\mathbf{X} \;=\; (\mathbf{RU})d\mathbf{X} \;=\; \mathbf{R}(\mathbf{U}d\mathbf{X}) \; . \tag{3.64}$$

This suggests that the deformation of $d\mathbf{X}$ takes place in two steps. In the first one, $d\mathbf{X}$ is deformed into $d\mathbf{X}' = \mathbf{U}d\mathbf{X}$, while in the second one, $d\mathbf{X}'$ is further deformed into $\mathbf{R}d\mathbf{X}' = d\mathbf{x}$.

Note that, letting $d\mathbf{X}' = \mathbf{M}' dS'$, where $\mathbf{M}' \cdot \mathbf{M}' = 1$ and dS' is the magnitude of $d\mathbf{X}'$,

$$
\begin{aligned}
d\mathbf{X}' \cdot d\mathbf{X}' \;=\; (\mathbf{M}' dS') \cdot (\mathbf{M}' dS') \; &= dS'^2 \\
&= (\mathbf{U} d\mathbf{X}) \cdot (\mathbf{U} d\mathbf{X}) \\
&= d\mathbf{X} \cdot (\mathbf{C} d\mathbf{X}) \\
&= (\mathbf{M} dS) \cdot (\mathbf{C}\mathbf{M} dS) \\
&= dS^2 \; \mathbf{M} \cdot \mathbf{C}\mathbf{M} \\
&= dS^2 \lambda^2 \; ,
\end{aligned}
\tag{3.65}
$$

which, since $\lambda = \dfrac{ds}{dS}$, implies that $dS' = ds$. Thus, $d\mathbf{X}'$ is stretched to the same differential length as $d\mathbf{x}$ due to the action of $\mathbf{U}$. Subsequently, write

$$
d\mathbf{x} \cdot d\mathbf{x} \;=\; (\mathbf{R} d\mathbf{X}') \cdot (\mathbf{R} d\mathbf{X}') \;=\; d\mathbf{X}' \cdot (\mathbf{R}^T \mathbf{R} d\mathbf{X}') \;=\; d\mathbf{X}' \cdot d\mathbf{X}' \; , \tag{3.66}
$$

which confirms that $\mathbf{R}$ induces a length-preserving transformation on $\mathbf{X}'$. In conclusion, $\mathbf{F} = \mathbf{R}\mathbf{U}$ implies that $d\mathbf{X}$ is first subjected to a stretch $\mathbf{U}$ (possibly accompanied by rotation) to its final length ds, then is reoriented to its final state $d\mathbf{x}$ by $\mathbf{R}$, see Figure 3.9.

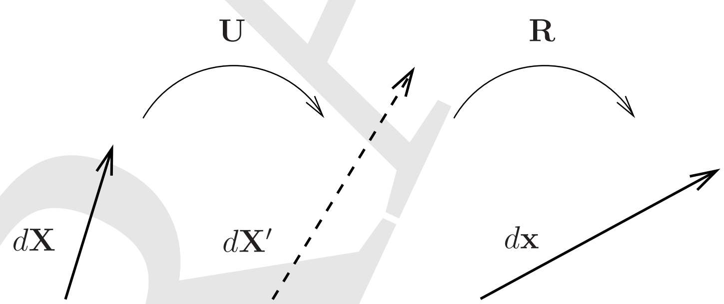

Figure 3.9: *Interpretation of the right polar decomposition.*

Turning attention to the *left polar decomposition* $\mathbf{F} = \mathbf{V}\mathbf{R}$, note that

$$
d\mathbf{x} \;=\; \mathbf{F} d\mathbf{X} \;=\; (\mathbf{V}\mathbf{R}) d\mathbf{X} \;=\; \mathbf{V}(\mathbf{R} d\mathbf{X}) \; . \tag{3.67}
$$

This, again, implies that the deformation of $d\mathbf{X}$ takes place in two steps. Indeed, in the first one, $d\mathbf{X}$ is deformed into $d\mathbf{x}' = \mathbf{R} d\mathbf{X}$, while in the second one, $d\mathbf{x}'$ is mapped into $\mathbf{V} d\mathbf{x}' = d\mathbf{x}$. For the first step, note that

$$
d\mathbf{x}' \cdot d\mathbf{x}' \;=\; (\mathbf{R} d\mathbf{X}) \cdot (\mathbf{R} d\mathbf{X}) \;=\; d\mathbf{X} \cdot (\mathbf{R}^T \mathbf{R} d\mathbf{X}) \;=\; d\mathbf{X} \cdot d\mathbf{X} \; , \tag{3.68}
$$

which means that the mapping from $d\mathbf{X}$ to $d\mathbf{x}'$ is length-preserving. For the second step, write

$$
\begin{aligned}
d\mathbf{x}' \cdot d\mathbf{x}' \;=\; d\mathbf{X} \cdot d\mathbf{X} \;&= dS^2 \\
&= (\mathbf{V}^{-1} d\mathbf{x}) \cdot (\mathbf{V}^{-1} d\mathbf{x}) \\
&= d\mathbf{x} \cdot (\mathbf{B}^{-1} d\mathbf{x}) \\
&= (\mathbf{m} ds) \cdot (\mathbf{B}^{-1} \mathbf{m} ds) \\
&= ds^2 \; \mathbf{m} \cdot \mathbf{B}^{-1} \mathbf{m} \\
&= \frac{1}{\lambda^2} ds^2 \; ,
\end{aligned}
\tag{3.69}
$$

which implies that $\mathbf{V}$ induces the full stretch λ during the mapping of $d\mathbf{x}'$ to $d\mathbf{x}$.

Thus, the left polar decomposition $\mathbf{F} = \mathbf{VR}$ means that the infinitesimal material line element $d\mathbf{X}$ is first subjected to a length-preserving transformation, followed by stretching (with further rotation) to its final state $d\mathbf{x}$, see Figure 3.10.

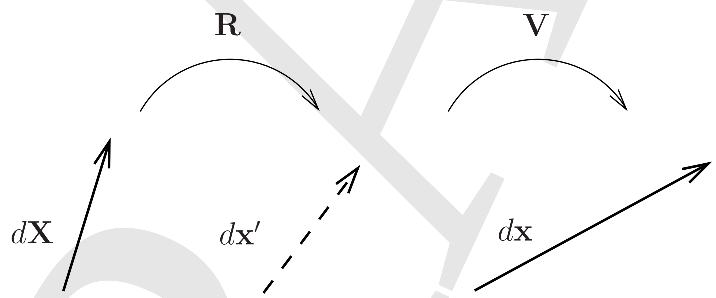

Figure 3.10: *Interpretation of the left polar decomposition.*

It is conceptually desirable to decompose the deformation gradient into a pure rotation and a pure stretch (or vice versa). To investigate such an option, consider first the right polar decomposition of equation (3.66). In this case, for the stretch $\mathbf{U}$ to be pure, the infinitesimal line elements $d\mathbf{X}$ and $d\mathbf{X}'$ need to be parallel, namely

$$
d\mathbf{X}' \;=\; \mathbf{U} d\mathbf{X} \;=\; \lambda d\mathbf{X} \; , \tag{3.70}
$$

or, upon recalling that $d\mathbf{X} \;=\; \mathbf{M} dS$,

$$
\mathbf{UM} \;=\; \lambda \mathbf{M} \; . \tag{3.71}
$$

Equation (3.71) represents a linear eigenvalue problem. The eigenvalues $\lambda_A > 0$ of (3.71) are called the *principal stretches* and the associated eigenvectors $\mathbf{M}_A$ are called the *principal directions*. When λ_A are distinct, one may write

$$\begin{aligned} \mathbf{U}\mathbf{M}_A &= \lambda_{(A)}\mathbf{M}_{(A)} \\ \mathbf{U}\mathbf{M}_B &= \lambda_{(B)}\mathbf{M}_{(B)} , \end{aligned} \tag{3.72}$$

where the parentheses around the subscripts signify that the summation convention is not in use. Upon premultiplying the preceding two equations with $\mathbf{M}_B$ and $\mathbf{M}_A$, respectively, one gets

$$\mathbf{M}_B \cdot (\mathbf{U}\mathbf{M}_A) = \lambda_{(A)}\mathbf{M}_B \cdot \mathbf{M}_{(A)} \tag{3.73}$$

$$\mathbf{M}_A \cdot (\mathbf{U}\mathbf{M}_B) = \lambda_{(B)}\mathbf{M}_A \cdot \mathbf{M}_{(B)} . \tag{3.74}$$

Subtracting the two equations from each other leads to

$$(\lambda_{(A)} - \lambda_{(B)})\mathbf{M}_{(A)} \cdot \mathbf{M}_{(B)} = 0 , \tag{3.75}$$

therefore, since, by assumption, $\lambda_A \neq \lambda_B$, it follows that

$$\mathbf{M}_A \cdot \mathbf{M}_B = \delta_{AB} , \tag{3.76}$$

i.e., that the principal directions are mutually orthogonal and that $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ form an orthonormal basis on E^3.

It turns out that regardless of whether $\mathbf{U}$ has distinct or repeated eigenvalues, the classical *spectral representation theorem* guarantees that there exists an orthonormal basis $\mathbf{M}_A$ of E^3 consisting entirely of eigenvectors of $\mathbf{U}$ and that, if λ_A are the associated eigenvalues,

$$\mathbf{U} = \sum_{A=1}^{3} \lambda_{(A)}\mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)} . \tag{3.77}$$

Thus,

$$
\begin{aligned}
\mathbf{C} \ = \ \mathbf{U}^2 \ &= (\sum_{A=1}^{3} \lambda_{(A)} \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)})(\sum_{B=1}^{3} \lambda_{(B)} \mathbf{M}_{(B)} \otimes \mathbf{M}_{(B)}) \\
&= \sum_{A=1}^{3} \sum_{B=1}^{3} \lambda_{(A)} \lambda_{(B)} (\mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)})(\mathbf{M}_{(B)} \otimes \mathbf{M}_{(B)}) \\
&= \sum_{A=1}^{3} \sum_{B=1}^{3} \lambda_{(A)} \lambda_{(B)} (\mathbf{M}_{(A)} \cdot \mathbf{M}_{(B)})(\mathbf{M}_{(A)} \otimes \mathbf{M}_{(B)}) \\
&= \sum_{A=1}^{3} \lambda_{(A)}^2 \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)}
\end{aligned}
\tag{3.78}
$$

and, more generally,

$$
\mathbf{U}^m \ = \ \sum_{A=1}^{3} \lambda_{(A)}^m \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)} \ , \tag{3.79}
$$

for any integer m.

Now, attempt a reinterpretation of the right polar decomposition, in light of the discussion of principal stretches and directions. Indeed, when $\mathbf{U}$ applies on infinitesimal material line elements which are aligned with the principal directions $\mathbf{M}_A$, then it subjects them to a pure stretch. Subsequently, the stretched elements are reoriented to their final direction by the action of $\mathbf{R}$, see Figure 3.11.

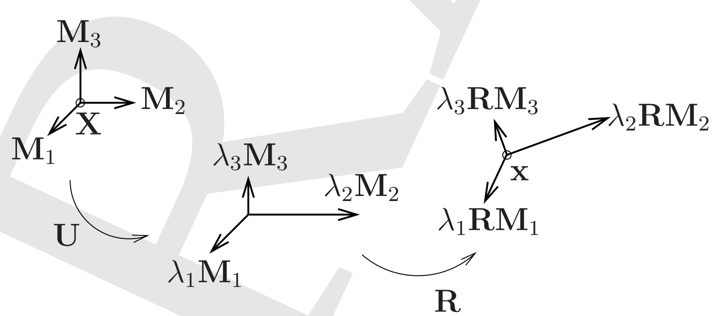

Figure 3.11: *Interpretation of the right polar decomposition relative to the principal directions* $\mathbf{M}_A$ *and associated principal stretches* λ_A*.*

Following an analogous procedure for the left polar decomposition, note for the stretch $\mathbf{V}$ to be pure it is necessary that

$$
d\mathbf{x} \ = \ \mathbf{V} d\mathbf{x}' \ = \ \lambda d\mathbf{x}' \tag{3.80}
$$

or, recalling that $d\mathbf{x}' = \mathbf{RM}dS$,

$$\mathbf{VRM} \;=\; \lambda\mathbf{RM}\,. \tag{3.81}$$

This is, again, a linear eigenvalue problem that can be solved for the principal stretches λ_i and the (rotated) principal directions $\mathbf{m}_i = \mathbf{RM}_i$. Appealing to the spectral representation theorem, one may write

$$\mathbf{V} \;=\; \sum_{i=1}^{3} \lambda_{(i)}(\mathbf{RM}_{(i)}) \otimes (\mathbf{RM}_{(i)}) \;=\; \sum_{i=1}^{3} \lambda_{(i)}\mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)}\;. \tag{3.82}$$

A reinterpretation of the left polar decomposition can be afforded along the lines of the earlier corresponding interpretation of the right decomposition. Specifically, here the infinitesimal material line elements that are aligned with the principal stretches $\mathbf{M}_A$ are first reoriented by $\mathbf{R}$ and subsequently subjected to a pure stretch to their final length by the action of $\mathbf{V}$, see Figure 3.12.

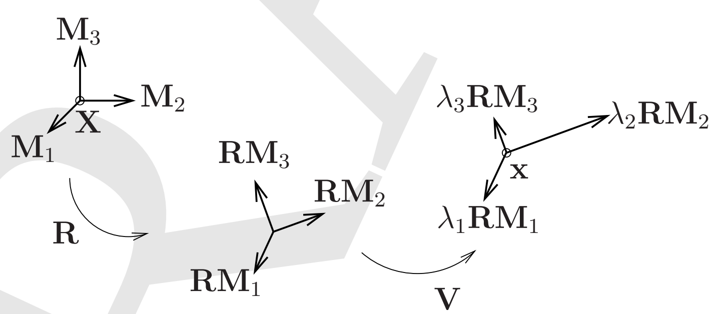

Figure 3.12: *Interpretation of the left polar decomposition relative to the principal directions* $\mathbf{RM}_i$ *and associated principal stretches* λ_i*.*

Returning to the discussion of the polar factor $\mathbf{R}$, recall that it is an orthogonal tensor. This, in turn, implies that $(\det\mathbf{R})^2 = 1$, or $\det\mathbf{R} = \pm 1$. An orthogonal tensor $\mathbf{R}$ is termed *proper* (resp. *improper*) if $\det\mathbf{R} = 1$ (resp. $\det\mathbf{R} = -1$).

Assuming, for now, that $\mathbf{R}$ is proper orthogonal, and invoking elementary properties of determinant, it is seen that

$$\begin{aligned}\mathbf{R}^T\mathbf{R} = \mathbf{I} \Rightarrow & \mathbf{R}^T\mathbf{R} - \mathbf{R}^T = \mathbf{I} - \mathbf{R}^T \\ \Rightarrow & \mathbf{R}^T(\mathbf{R} - \mathbf{I}) = -(\mathbf{R} - \mathbf{I})^T \\ \Rightarrow & \det\mathbf{R}^T \det(\mathbf{R} - \mathbf{I}) = -\det(\mathbf{R} - \mathbf{I})^T \\ \Rightarrow & \det(\mathbf{R} - \mathbf{I}) = -\det(\mathbf{R} - \mathbf{I}) \\ \Rightarrow & \det(\mathbf{R} - \mathbf{I}) = 0 ,\end{aligned} \tag{3.83}$$

so that $\mathbf{R}$ has at least one unit eigenvalue. Denote by $\mathbf{p}$ a unit eigenvector associated with the above eigenvalue (there exist two such unit vectors), and consider two unit vectors $\mathbf{q}$ and $\mathbf{r} = \mathbf{p} \times \mathbf{q}$ that lie on a plane normal to $\mathbf{p}$. It follows that $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ form a right-hand orthonormal basis of E^3 and, thus, $\mathbf{R}$ can be expressed with reference to this basis as

$$\begin{aligned}\mathbf{R} = R_{pp}\mathbf{p}\otimes\mathbf{p} + R_{pq}\mathbf{p}\otimes\mathbf{q} + R_{pr}\mathbf{p}\otimes\mathbf{r} + R_{qp}\mathbf{q}\otimes\mathbf{p} + R_{qq}\mathbf{q}\otimes\mathbf{q} + R_{qr}\mathbf{q}\otimes\mathbf{r} \\ + R_{rp}\mathbf{r}\otimes\mathbf{p} + R_{rq}\mathbf{r}\otimes\mathbf{q} + R_{rr}\mathbf{r}\otimes\mathbf{r} .\end{aligned} \tag{3.84}$$

Note that, since $\mathbf{p}$ is an eigenvector of $\mathbf{R}$,

$$\mathbf{R}\mathbf{p} = \mathbf{p} \Rightarrow R_{pp}\mathbf{p} + R_{qp}\mathbf{q} + R_{rp}\mathbf{r} = \mathbf{p} , \tag{3.85}$$

which implies that

$$R_{pp} = 1 \quad , \quad R_{qp} = R_{rp} = 0 . \tag{3.86}$$

Moreover, given that $\mathbf{R}$ is orthogonal,

$$\mathbf{R}^T\mathbf{p} = \mathbf{R}^{-1}\mathbf{p} = \mathbf{p} \Rightarrow R_{pp}\mathbf{p} + R_{pq}\mathbf{q} + R_{pr}\mathbf{r} = \mathbf{p} , \tag{3.87}$$

therefore

$$R_{pq} = R_{pr} = 0 . \tag{3.88}$$

Taking into account (3.86) and (3.88), the orthogonality of $\mathbf{R}$ can be expressed as

$$\begin{aligned}(\mathbf{p}\otimes\mathbf{p} + R_{qq}\mathbf{q}\otimes\mathbf{q} + R_{qr}\mathbf{r}\otimes\mathbf{q} + R_{rq}\mathbf{q}\otimes\mathbf{r} + R_{rr}\mathbf{r}\otimes\mathbf{r}) \\ (\mathbf{p}\otimes\mathbf{p} + R_{qq}\mathbf{q}\otimes\mathbf{q} + R_{qr}\mathbf{q}\otimes\mathbf{r} + R_{rq}\mathbf{r}\otimes\mathbf{q} + R_{rr}\mathbf{r}\otimes\mathbf{r}) \\ = \mathbf{p}\otimes\mathbf{p} + \mathbf{q}\otimes\mathbf{q} + \mathbf{r}\otimes\mathbf{r} .\end{aligned} \tag{3.89}$$

and, after reducing the terms on the left-hand side,

$$\begin{aligned}\mathbf{p}\otimes\mathbf{p} \;+\; (R_{qq}^2 \;+\; R_{rq}^2)\mathbf{q}\otimes\mathbf{q} \;+\; (R_{rr}^2 \;+\; R_{qr}^2)\mathbf{r}\otimes\mathbf{r} \\ +\; (R_{qq}R_{qr} \;+\; R_{rq}R_{rr})\mathbf{q}\otimes\mathbf{r} \;+\; (R_{rr}R_{rq} \;+\; R_{qr}R_{qq})\mathbf{r}\otimes\mathbf{q} \\ =\; \mathbf{p}\otimes\mathbf{p} \;+\; \mathbf{q}\otimes\mathbf{q} \;+\; \mathbf{r}\otimes\mathbf{r}\;. \end{aligned} \tag{3.90}$$

The above equation implies that

$$R_{qq}^2 \;+\; R_{rq}^2 \;=\; 1\;, \tag{3.91}$$

$$R_{rr}^2 \;+\; R_{qr}^2 \;=\; 1\;, \tag{3.92}$$

$$R_{qq}R_{qr} \;+\; R_{rq}R_{rr} \;=\; 0\;, \tag{3.93}$$

$$R_{rr}R_{rq} \;+\; R_{qr}R_{qq} \;=\; 0\;. \tag{3.94}$$

Equations (3.91) and (3.92) imply that there exist angles θ and ϕ, such that

$$R_{qq} \;=\; \cos\theta \quad , \quad R_{rq} \;=\; \sin\theta\;, \tag{3.95}$$

and

$$R_{rr} \;=\; \cos\phi \quad , \quad R_{qr} \;=\; \sin\phi\;. \tag{3.96}$$

It follows from (3.93) (or (3.94)) that

$$\cos\theta\sin\phi \;+\; \sin\theta\cos\phi \;=\; \sin(\phi \;+\; \theta) \;=\; 0\;, \tag{3.97}$$

thus

$$\phi \;=\; -\theta \;+\; 2k\pi \qquad \text{or} \qquad \phi \;=\; \pi \;-\; \theta \;+\; 2k\pi\;, \tag{3.98}$$

where k is an arbitrary integer. It can be easily seen that the latter choice yields an improper orthogonal tensor $\mathbf{R}$, thus $\phi = -\theta + 2k\pi$, and, consequently,

$$R_{qq} \;=\; \cos\theta\;, \tag{3.99}$$

$$R_{rq} \;=\; \sin\theta\;, \tag{3.100}$$

$$R_{rr} \;=\; \cos\theta\;, \tag{3.101}$$

$$R_{qr} \;=\; -\sin\theta\;. \tag{3.102}$$

From (3.86), (3.88) and (3.99-3.102), it follows that $\mathbf{R}$ can be expressed as

$$\mathbf{R} \;=\; \mathbf{p}\otimes\mathbf{p} \;+\; \cos\theta\,(\mathbf{q}\otimes\mathbf{q} \;+\; \mathbf{r}\otimes\mathbf{r}) \;-\; \sin\theta\,(\mathbf{q}\otimes\mathbf{r} \;-\; \mathbf{r}\otimes\mathbf{q})\;. \tag{3.103}$$

The angle θ that appears in (3.103) can be geometrically interpreted as follows: let an arbitrary vector $\mathbf{x}$ be written in terms of $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$ as

$$\mathbf{x} \;=\; p\mathbf{p} \;+\; q\mathbf{q} \;+\; r\mathbf{r}\;, \tag{3.104}$$

where

$$p \;=\; \mathbf{p}\cdot\mathbf{x} \quad,\quad q \;=\; \mathbf{q}\cdot\mathbf{x} \quad,\quad r \;=\; \mathbf{r}\cdot\mathbf{x}\,, \tag{3.105}$$

and note that

$$\mathbf{R}\mathbf{x} \;=\; p\mathbf{p} + \;(q\cos\theta \;-\; r\sin\theta)\mathbf{q} \;+\; (q\sin\theta \;+\; r\cos\theta)\mathbf{r}\;. \tag{3.106}$$

Equation (3.106) indicates that, under the action of $\mathbf{R}$, the vector $\mathbf{x}$ remains unstretched and it rotates by an angle θ around the $\mathbf{p}$-axis, where θ is assumed positive when directed from $\mathbf{q}$ to $\mathbf{r}$ in the sense of the right-hand rule.

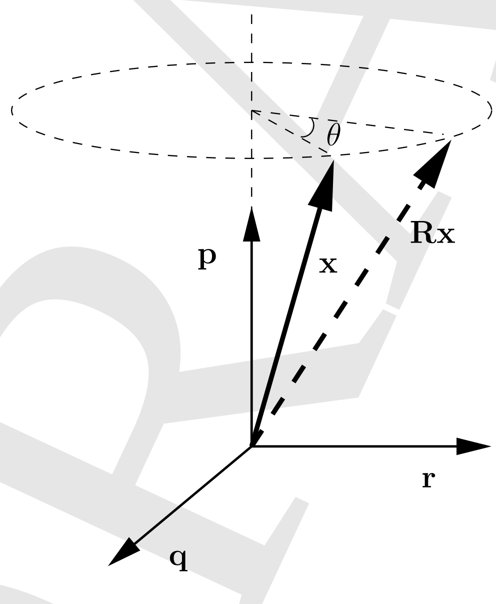

Figure 3.13: *Geometric interpretation of the rotation tensor* $\mathbf{R}$ *by its action on a vector* $\mathbf{x}$.

The representation (3.103) of a proper orthogonal tensor $\mathbf{R}$ is often referred to as the *Rodrigues formula.* If $\mathbf{R}$ is improper orthogonal, the alternative solution in (3.98$_2$ in connection with the negative unit eigenvalue $\mathbf{p}$ implies that, $\mathbf{R}\mathbf{x}$ rotates by an angle θ around the $\mathbf{p}$-axis and is also reflected relative to the origin of the orthonormal basis $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$.

A simple counting check can be now employed to assess the polar decomposition (3.60). Indeed, $\mathbf{F}$ has nine independent components and $\mathbf{U}$ (or $\mathbf{V}$) has six independent components. At the same time, $\mathbf{R}$ has three independent components, for instance two of the three components of the unit eigenvector $\mathbf{p}$ and the angle θ.

Example: Consider a motion $\boldsymbol{\chi}$ defined in component form as

$$
\begin{aligned}
\chi_1 &= \chi_1(X_A, t) &=& (\sqrt{a}\cos\vartheta X_1) - (\sqrt{a}\sin\vartheta)X_2 \\
\chi_2 &= \chi_2(X_A, t) &=& (\sqrt{a}\sin\vartheta)X_1 + (\sqrt{a}\cos\vartheta)X_2 \\
\chi_3 &= \chi_3(X_A, t) &=& X_3 \ ,
\end{aligned}
$$

where $a = a(t) > 0$ and $\vartheta = \vartheta(t)$.

This is clearly a planar motion, specifically independent of X_3.

The components $F_{iA} = \chi_{i,A}$ of the deformation gradient can be easily determined as

$$
[F_{iA}] \ = \ \begin{bmatrix} \sqrt{a}\cos\vartheta & -\sqrt{a}\sin\vartheta & 0 \\ \sqrt{a}\sin\vartheta & \sqrt{a}\cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .
$$

This is clearly a *spatially homogeneous motion*, i.e., the deformation gradient is independent of $\mathbf{X}$. Further, note that $\det(F_{iA}) = a > 0$, hence the motion is always invertible.

The components C_{AB} of $\mathbf{C}$ and the components U_{AB} of $\mathbf{U}$ can be directly determined as

$$
[C_{AB}] \ = \ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

and

$$
[U_{AB}] \ = \ \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .
$$

Also, recall that

$$
\mathbf{CM} = \lambda^2 \mathbf{M} \ ,
$$

which implies that $\lambda_1 = \lambda_2 = \sqrt{a}$ and $\lambda_3 = 1$.

Given that $\mathbf{U}$ is known, one may apply the right polar decomposition to determine the rotation tensor $\mathbf{R}$. Indeed, in this case,

$$[R_{iA}] = \begin{bmatrix} \sqrt{a}\cos\vartheta & -\sqrt{a}\sin\vartheta & 0 \\ \sqrt{a}\sin\vartheta & \sqrt{a}\cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{a}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Note that this motion yield pure stretch for $\vartheta = 2k\pi$, where $k = 0, 1, 2, \ldots$. □

Example: Sphere under homogeneous deformation

Consider the part of a deformable body which occupies a sphere $\mathcal{P}_0$ of radius σ centered at the fixed origin O of $\mathcal{E}^3$. The equation of the surface $\partial\mathcal{P}_0$ of the sphere can be written as

$$\mathbf{\Pi} \cdot \mathbf{\Pi} \;=\; \sigma^2 \;, \tag{3.107}$$

where

$$\mathbf{\Pi} \;=\; \sigma\mathbf{M} \;, \tag{3.108}$$

and $\mathbf{M}\cdot\mathbf{M} = 1$. Assume that the body undergoes a *spatially homogeneous* deformation with deformation gradient $\mathbf{F}(t)$, so that

$$\boldsymbol{\pi} \;=\; \mathbf{F}\mathbf{\Pi} \;, \tag{3.109}$$

where $\boldsymbol{\pi}(t)$ is the image of $\mathbf{\Pi}$ in the current configuration. Recall that $\lambda\mathbf{m} = \mathbf{F}\mathbf{M}$, where $\lambda(t)$ is the stretch of a material line element that lies along $\mathbf{M}$ in the reference configuration,

Figure 3.14: *Spatially homogeneous deformation of a sphere.*

and $\mathbf{m}(t)$ is the unit vector in the direction this material line element at time t, as in the figure above. Then, it follows from (3.107) and (3.109) that

$$\boldsymbol{\pi} \;=\; \sigma\lambda\mathbf{m} \;. \tag{3.110}$$

Also, since $\lambda^2 \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m} = 1$, where $\mathbf{B}(t)$ is the left Cauchy-Green deformation tensor, equation (3.110) leads to

$$\boldsymbol{\pi} \cdot \mathbf{B}^{-1}\boldsymbol{\pi} = \sigma^2 . \tag{3.111}$$

Recalling the spectral representation theorem, the left Cauchy-Green deformation tensor can be resolved as

$$\mathbf{B} = \sum_{i=1}^{3} \lambda_{(i)}^2 \mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} , \tag{3.112}$$

where $(\lambda_i^2, \mathbf{m}_i)$, $i = 1, 2, 3$, are respectively the eigenvalues (squares of the principal stretches) and the associated eigenvectors (principal directions) of $\mathbf{B}$. In addition, note that $\mathbf{m}_i$, $i = 1, 2, 3$, are orthonormal and form a basis of E^3, therefore the position vector $\boldsymbol{\pi}$ can be uniquely resolved in this basis as

$$\boldsymbol{\pi} = \pi_i \mathbf{m}_i . \tag{3.113}$$

Starting from equation (3.112), one may write

$$\mathbf{B}^{-1} = \sum_{i=1}^{3} \lambda_{(i)}^{-2} \mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} , \tag{3.114}$$

and using (3.113) and (3.114), deduce that

$$\boldsymbol{\pi} \cdot \mathbf{B}^{-1}\boldsymbol{\pi} = \lambda_i^{-2}\pi_i^2 . \tag{3.115}$$

It is readily seen from (3.115) that

$$\frac{\pi_1^2}{\lambda_1^2} + \frac{\pi_2^2}{\lambda_2^2} + \frac{\pi_3^2}{\lambda_3^2} = \sigma^2 ,$$

which demonstrates that, under a spatially homogeneous deformation, the spherical region $\mathcal{P}_0$ is deformed into an ellipsoid with principal semi-axes of length $\sigma\lambda_i$ along the principal directions of $\mathbf{B}$, see Figure 3.15. □

Consider now the transformation of an infinitesimal material volume element dV of the reference configuration to its image dv in the current configuration due to the motion $\boldsymbol{\chi}$. The referential volume element is defined as an infinitesimal parallelepiped with sides $d\mathbf{X}^1$, $d\mathbf{X}^2$, and $d\mathbf{X}^3$. Likewise, its spatial counterpart is the infinitesimal parallelepiped with sides $d\mathbf{x}^1$, $d\mathbf{x}^2$, and $d\mathbf{x}^3$, where each $d\mathbf{x}^i$ is the image of $d\mathbf{X}^i$ due to $\boldsymbol{\chi}$, see Figure 3.15.

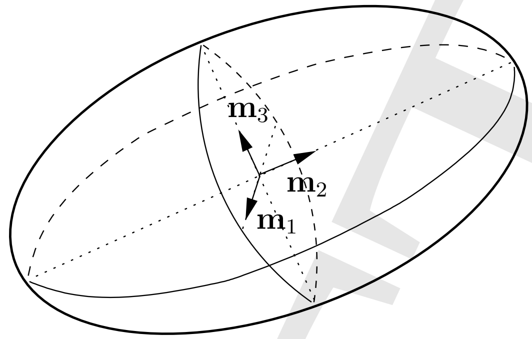

Figure 3.15: *Image of a sphere under homogeneous deformation.*

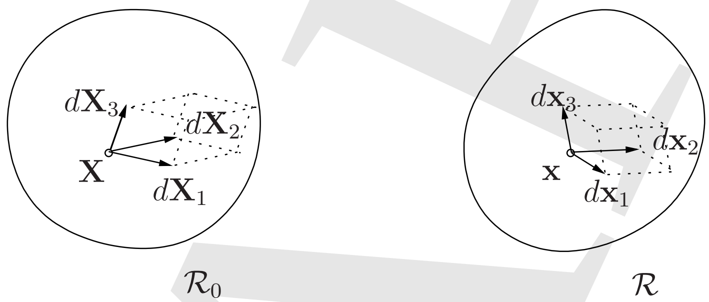

Figure 3.16: *Mapping of an infinitesimal material volume element dV to its image dv in the current configuration.*

First, note that

$$dV \;=\; d\mathbf{X}^1 \cdot (d\mathbf{X}^2 \times d\mathbf{X}^3) \;=\; d\mathbf{X}^2 \cdot (d\mathbf{X}^3 \times d\mathbf{X}^1) \;=\; d\mathbf{X}^3 \cdot (d\mathbf{X}^1 \times d\mathbf{X}^2)\;, \qquad (3.116)$$

where each of the representation of dV in (3.116) corresponds to the triple product $[d\mathbf{X}^1\, d\mathbf{X}^2\, d\mathbf{X}^3]$ of the vectors $d\mathbf{X}^1, d\mathbf{X}^2$ and $d\mathbf{X}^3$. Recalling that the triple product satisfies the property $[d\mathbf{X}^1\, d\mathbf{X}^2\, d\mathbf{X}^3] = \det\,[[dX^1_A], [dX^2_A], [dX^3_A]]$, it follows that in the current configuration

$$\begin{aligned} dv &= d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{x}^3) \\ &= (\mathbf{F}d\mathbf{X}^1) \cdot \big((\mathbf{F}d\mathbf{X}^2) \times \mathbf{F}d\mathbf{X}^3)\big) \\ &= \det\big[[F_{iA}dX^1_A], [F_{iA}dX^2_A], [F_{iA}dX^3_A]\big] \\ &= \det\big[F_{iA}[dX^1_A, dX^2_A, dX^3_A]\big] \\ &= (\det \mathbf{F}) \det\big[[dX^1_A], [dX^2_A], [dX^3_A]\big] \\ &= J dV\;. \end{aligned} \qquad (3.117)$$

A motion for which $dv = dV$ (i.e. $J = 1$) for all infinitesimal material volumes dV at all times is called *isochoric* (or *volume preserving*).

Here, one may argue that if, by convention, $dV > 0$ (which is true as long as $d\mathbf{X}^1, d\mathbf{X}^2$ and $d\mathbf{X}^3$ observe the right-hand rule), then the relative orientation of line elements should be preserved during the motion, i.e., $J > 0$ everywhere and at all time. Indeed, since the motion is assumed smooth, any changes in the sign of J would necessarily imply that there exists a time t at which $J = 0$ at some material point(s), which would violate the assumption the assumption of invertibility. Based on the preceding observation, the Jacobian J will be taken to be positive at all time. Consider next the transformation of an infinitesimal material surface element dA in the reference configuration to its image da in the current configuration. The referential surface element is defined as the parallelogram formed by the infinitesimal material line elements $d\mathbf{X}^1$ and $d\mathbf{X}^2$, such that

$$d\mathbf{A} \;=\; d\mathbf{X}^1 \times d\mathbf{X}^2 \;=\; \mathbf{N} dA \, , \tag{3.118}$$

where $\mathbf{N}$ is the unit normal to the surface element consistently with the right-hand rule, see Figure 3.16. Similarly, in the current configuration, one may write

$$d\mathbf{a} \;=\; d\mathbf{x}^1 \times d\mathbf{x}^2 \;=\; \mathbf{n} da \, , \tag{3.119}$$

where $\mathbf{n}$ is the corresponding unit normal to the surface element defined by $d\mathbf{x}^1$ and $d\mathbf{x}^2$.

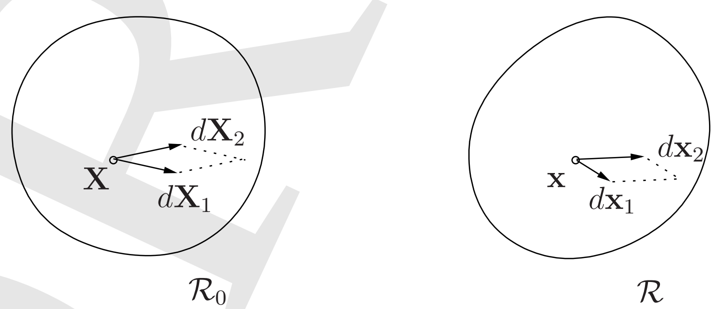

Figure 3.17: *Mapping of an infinitesimal material surface element dA to its image da in the current configuration.*

Let $d\mathbf{X}$ be any infinitesimal material line element, such that $\mathbf{N} \cdot d\mathbf{X} > 0$ and consider the infinitesimal volumes dV and dv formed by $\{d\mathbf{X}^1, d\mathbf{X}^2, d\mathbf{X}\}$ and $\{d\mathbf{x}^1, d\mathbf{x}^2, d\mathbf{x} = \mathbf{F} d\mathbf{X}\}$, respectively. It follows from (3.117) that

$$\begin{aligned} dv &= d\mathbf{x}\cdot(d\mathbf{x}^1\times d\mathbf{x}^2) \;=\; d\mathbf{x}\cdot\mathbf{n}da \;=\; (\mathbf{F}d\mathbf{X})\cdot\mathbf{n}da \\ &= JdV \\ &= Jd\mathbf{X}\cdot(d\mathbf{X}^1\times d\mathbf{X}^2) \;=\; Jd\mathbf{X}\cdot\mathbf{N}dA\;, \end{aligned} \tag{3.120}$$

which implies that

$$(\mathbf{F}d\mathbf{X})\cdot\mathbf{n}da \;=\; Jd\mathbf{X}\cdot\mathbf{N}dA\;, \tag{3.121}$$

for any infinitesimal material line element $d\mathbf{X}$. This, in turn, leads to

$$\mathbf{n}da \;=\; J\mathbf{F}^{-T}\mathbf{N}dA\;. \tag{3.122}$$

Equation (3.122) is known as *Nanson's formula*. Dotting each side in (3.122) with itself and taking square roots yields

$$|da| \;=\; J\sqrt{\mathbf{N}\cdot(\mathbf{C}^{-1}\mathbf{N})}|dA|\;. \tag{3.123}$$

As argued in the case of the infinitesimal volume transformations, if an infinitesimal material line element satisfies $dA > 0$, then $da > 0$ everywhere and at all time, since $J > 0$ and $\mathbf{C}^{-1}$ is positive-definite.

3.3 Velocity gradient and other measures of deformation rate

In this section, interest is focused to measures of the rate at which deformation occurs in the continuum. To this end, define the *spatial velocity gradient tensor* $\mathbf{L}$ as

$$\mathbf{L} \;=\; \operatorname{grad}\mathbf{v} \;=\; \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\;. \tag{3.124}$$

This tensor is naturally defined relative to the basis $\{\mathbf{e}_i\}$ in the current configuration. Next, recall any such tensor can be uniquely decomposed into a symmetric and a skew-symmetric part, so that $\mathbf{L}$ can be written as

$$\mathbf{L} \;=\mathbf{D}+\mathbf{W}\;, \tag{3.125}$$

where

$$\mathbf{D} \;=\; \frac{1}{2}(\mathbf{L}+\mathbf{L}^T) \tag{3.126}$$

is the *rate-of-deformation tensor*, which is symmetric, and

$$\mathbf{W} \;=\; \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \tag{3.127}$$

is the *vorticity* (or *spin*) tensor, which is skew-symmetric.

Consider now the rate of change of the deformation gradient for a fixed particle associated with point $\mathbf{X}$ in the reference configuration. To this end, write the material time derivative of $\mathbf{F}$ as

$$\dot{\mathbf{F}} \;=\; \overline{\left(\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}}\right)}^{\boldsymbol{\cdot}} \;=\; \frac{\partial}{\partial \mathbf{X}} \overline{\boldsymbol{\chi}(\mathbf{X},t)}^{\boldsymbol{\cdot}} \;=\; \frac{\partial \hat{\mathbf{v}}(\mathbf{X},t)}{\partial \mathbf{X}} \;=\; \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial \mathbf{x}} \frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}} \;=\; \mathbf{LF} \;, \tag{3.128}$$

where use is made of the chain rule. Also, in the above derivation the change in the order of differentiation between the derivatives with respect to $\mathbf{X}$ and t is allowed under the assumption that the mixed second derivative $\dfrac{\partial^2 \boldsymbol{\chi}}{\partial \mathbf{X} \partial t}$ is continuous.

Given (3.128), one may express with the aid of the product rule the rate of change of the right Cauchy-Green deformation tensor $\mathbf{C}$ for a fixed particle $\mathbf{X}$ as

$$\dot{\mathbf{C}} \;= \overline{\mathbf{F}^T\mathbf{F}}^{\boldsymbol{\cdot}} \;=\; \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} \;=\; (\mathbf{LF})^T\mathbf{F} + \mathbf{F}^T(\mathbf{LF}) \;=\; \mathbf{F}^T(\mathbf{L}^T + \mathbf{L})\mathbf{F} \;=\; 2\mathbf{F}^T\mathbf{DF} \;. \tag{3.129}$$

Likewise, for the left Cauchy-Green deformation tensor, one may write

$$\dot{\mathbf{B}} \;= \overline{\mathbf{F}\mathbf{F}^T}^{\boldsymbol{\cdot}} \;=\; \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T \;=\; (\mathbf{LF})\mathbf{F}^T + \mathbf{F}(\mathbf{LF})^T \;=\; \mathbf{LB} + \mathbf{BL}^T \;. \tag{3.130}$$

Similar results may be readily obtained for the rates of the Lagrangian and Eulerian strain measures. Specifically, it can be shown that

$$\dot{\mathbf{E}} \;=\; \mathbf{F}^T\mathbf{DF} \tag{3.131}$$

and

$$\dot{\mathbf{e}} \;=\; \frac{1}{2}(\mathbf{B}^{-1}\mathbf{L} + \mathbf{L}^T\mathbf{B}^{-1}) \;. \tag{3.132}$$

Proceed now to discuss physical interpretations for the rate tensors $\mathbf{D}$ and $\mathbf{W}$. Starting from (3.44), take the material time derivatives of both sides to obtain the relation

$$\begin{aligned} \dot{\lambda}\mathbf{m} + \lambda\dot{\mathbf{m}} \;&= \dot{\mathbf{F}}\mathbf{M} + \mathbf{F}\dot{\mathbf{M}} \\ &= \mathbf{LFM} \;=\; \mathbf{L}(\lambda\mathbf{m}) \;=\; \lambda\mathbf{Lm} \;. \end{aligned} \tag{3.133}$$

Note that $\dot{\mathbf{M}} = \mathbf{0}$, since $\mathbf{M}$ is a fixed vector in the fixed reference configuration, hence does not vary with time. Upon taking the dot-product of each side with $\mathbf{m}$, it follows that

$$\dot{\lambda}\mathbf{m} \cdot \mathbf{m} + \lambda\dot{\mathbf{m}} \cdot \mathbf{m} = \lambda(\mathbf{L}\mathbf{m}) \cdot \mathbf{m} \ . \tag{3.134}$$

However, since $\mathbf{m}$ is a unit vector, it follows that $\dot{\mathbf{m}} \cdot \mathbf{m} = 0$, so that the preceding equation simplifies to

$$\dot{\lambda} = \lambda\mathbf{m} \cdot \mathbf{L}\mathbf{m} \ . \tag{3.135}$$

Further, since the skew-symmetric part $\mathbf{W}$ of $\mathbf{L}$ satisfies

$$\mathbf{m} \cdot \mathbf{W}\mathbf{m} = \mathbf{m} \cdot (-\mathbf{W}^T)\mathbf{m} = -\mathbf{m} \cdot \mathbf{W}\mathbf{m} \ , \tag{3.136}$$

hence $\mathbf{m} \cdot \mathbf{W}\mathbf{m} = 0$, one may rewrite (3.135) as

$$\dot{\lambda} = \lambda\mathbf{m} \cdot \mathbf{D}\mathbf{m} \tag{3.137}$$

or, alternatively, as

$$\dot{\overline{\ln \lambda}} = \mathbf{m} \cdot \mathbf{D}\mathbf{m} = \mathbf{D} \cdot (\mathbf{m} \otimes \mathbf{m}) \ . \tag{3.138}$$

Thus, the tensor $\mathbf{D}$ fully determines the material time derivative of the logarithmic stretch $\ln \lambda$ for a material line element along a direction $\mathbf{m}$ in the current configuration.

Before proceeding with the geometric interpretation of $\mathbf{W}$, some preliminary development on skew-symmetric tensors is required. First, note that any skew-symmetric tensor in E^3, such as $\mathbf{W}$, has only only three independent components. This suggests that there exists a one-to-one correspondence between skew-symmetric tensors and vectors in E^3. To establish this correspondence, observe that

$$\mathbf{W} = \frac{1}{2}(\mathbf{W} - \mathbf{W}^T) \ , \tag{3.139}$$

so that, when operating on any vector $\mathbf{z} \in E^3$,

$$\begin{aligned} \mathbf{W}\mathbf{z} &= \frac{1}{2}W_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i)\mathbf{z} \\ &= \frac{1}{2}W_{ij}[(\mathbf{z} \cdot \mathbf{e}_j)\mathbf{e}_i - (\mathbf{z} \cdot \mathbf{e}_i)\mathbf{e}_j] \ . \end{aligned} \tag{3.140}$$

Recalling the standard identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{u})\mathbf{w}$, the preceding equation can be rewritten as

$$\begin{aligned}
\mathbf{W}\mathbf{z} &= \frac{1}{2} W_{ij} [\mathbf{z} \times (\mathbf{e}_i \times \mathbf{e}_j)] \\
&= -\frac{1}{2} W_{ij} [(\mathbf{e}_i \times \mathbf{e}_j) \times \mathbf{z}] \\
&= \frac{1}{2} W_{ji} [(\mathbf{e}_i \times \mathbf{e}_j) \times \mathbf{z}] \\
&= \left[\left(\frac{1}{2} W_{ji} \mathbf{e}_i \right) \times \mathbf{e}_j \right] \times \mathbf{z} \\
&= \mathbf{w} \times \mathbf{z} \ ,
\end{aligned} \tag{3.141}$$

where the vector $\mathbf{w}$ is defined as

$$\mathbf{w} = \frac{1}{2} W_{ji} \mathbf{e}_i \times \mathbf{e}_j \tag{3.142}$$

and is called the *axial vector* of the skew-symmetric tensor $\mathbf{W}$. Using components, one may represent $\mathbf{W}$ in terms of $\mathbf{w}$ and vice-versa. Specifically, starting from (3.142),

$$\mathbf{w} = w_k \mathbf{e}_k = \frac{1}{2} W_{ji} \mathbf{e}_i \times \mathbf{e}_j = \frac{1}{2} W_{ji} e_{ijk} \mathbf{e}_k \ , \tag{3.143}$$

hence,

$$w_k = \frac{1}{2} e_{ijk} W_{ji} \ . \tag{3.144}$$

Conversely, starting from (3.141),

$$W_{ij} z_j \mathbf{e}_i = e_{ijk} w_j z_k \mathbf{e}_i = e_{ikj} w_k z_j \mathbf{e}_i \ , \tag{3.145}$$

so that

$$W_{ij} = e_{ikj} w_k = e_{jik} w_k \ . \tag{3.146}$$

The above relations apply to any skew-symmetric tensor. If $\mathbf{W}$ is specifically the vorticity tensor, then the associated axial vector satisfies the relation

$$
\begin{aligned}
\mathbf{w} &= \frac{1}{4}(v_{j,i} - v_{i,j})\mathbf{e}_i \times \mathbf{e}_j \\
&= \frac{1}{4}(v_{j,i} - v_{i,j})e_{ijk}\mathbf{e}_k \\
&= \frac{1}{4}(e_{ijk}v_{j,i} - e_{ijk}v_{i,j})\mathbf{e}_k \\
&= \frac{1}{4}(e_{ijk}v_{j,i} - e_{jik}v_{i,j})\mathbf{e}_k \\
&= \frac{1}{2}e_{ijk}v_{j,i}\mathbf{e}_k \\
&= \frac{1}{2}\,\mathrm{curl}\,\mathbf{v}\ .
\end{aligned}
\tag{3.147}
$$

In this case, the axial vector $\mathbf{w}$ is called the *vorticity vector*. Also, a motion is termed *irrotational* if $\mathbf{W} = \mathbf{0}$ (or, equivalently, $\mathbf{w} = \mathbf{0}$).

Let $\mathbf{w} = \tilde{\mathbf{w}}(\mathbf{x}, t)$ be the vorticity vector field at a given time t. The *vortex line* through $\mathbf{x}$ at time t is the space curve that passes through $\mathbf{x}$ and is tangent to the vorticity vector field $\tilde{\mathbf{w}}$ at all of its points, hence is defined as

$$
d\mathbf{y} = \tilde{\mathbf{w}}(\mathbf{y}, t)d\tau \quad , \quad \mathbf{y}(\tau_0) \ = \ \mathbf{x} \quad , \quad (t \text{ fixed})\ . \tag{3.148}
$$

For an irrotational motion, any line passing through $\mathbf{x}$ at time t is a vortex line.

Returning to the geometric interpretation of $\mathbf{W}$, take $\bar{\mathbf{m}}$ to be a unit vector that lies along a principal direction of $\mathbf{D}$ in the current configuration, namely

$$
(\mathbf{D} - \gamma\mathbf{i})\bar{\mathbf{m}} \ = \ \mathbf{0}\ . \tag{3.149}
$$

It follows from (3.149) that

$$
(\mathbf{D}\bar{\mathbf{m}}) \cdot \bar{\mathbf{m}} \ = \ \gamma\bar{\mathbf{m}} \cdot \bar{\mathbf{m}} \ = \ \gamma \ = \ \frac{\dot{\bar{\lambda}}}{\bar{\lambda}} \ = \ \dot{\overline{\ln\bar{\lambda}}}, \tag{3.150}
$$

i.e., the eigenvalues of $\mathbf{D}$ are equal to the material time derivatives of the logarithmic stretches $\ln\bar{\lambda}$ of line elements along the eigendirections $\bar{\mathbf{m}}$ in the current configuration.

Recalling the identity (3.133), write

$$
\begin{aligned}
\dot{\mathbf{m}} \ = \ \mathbf{L}\mathbf{m} - \frac{\dot{\lambda}}{\lambda}\mathbf{m} \ &= \ \left(\mathbf{L} - \frac{\dot{\lambda}}{\lambda}\mathbf{i}\right)\mathbf{m} \\
&= \ \left(\mathbf{D} - \frac{\dot{\lambda}}{\lambda}\mathbf{i}\right)\mathbf{m} + \mathbf{W}\mathbf{m}\ ,
\end{aligned}
\tag{3.151}
$$

which holds for any direction $\mathbf{m}$ in the current configuration. Setting in the above equation $\mathbf{m} = \bar{\mathbf{m}}$, it follows that

$$\dot{\bar{\mathbf{m}}} \;=\; \mathbf{W}\bar{\mathbf{m}} \;=\; \mathbf{w} \times \bar{\mathbf{m}} \; . \tag{3.152}$$

Therefore, the material time derivative of a unit vector $\bar{\mathbf{m}}$ along a principal direction of $\mathbf{D}$ is determined by (3.152). Recalling from rigid-body dynamics the formula relating linear to angular velocities, one may conclude that $\mathbf{w}$ plays the role of the angular velocity of a line element which, in the current configuration, lies along the principal direction $\overline{\mathbf{m}}$ of $\mathbf{D}$.

3.4 Superposed rigid-body motions

A *rigid motion* is one in which the distance between any two material points remains constant at all times. Denoting by $\mathbf{X}$ and $\mathbf{Y}$ the position vectors of two material points on the fixed reference configuration, define the distance function as

$$d(\mathbf{X}, \mathbf{Y}) \;=\; |\mathbf{X} - \mathbf{Y}| \; . \tag{3.153}$$

According to the preceding definition, a motion is rigid if, for any points $\mathbf{X}$ and $\mathbf{Y}$,

$$d(\mathbf{X}, \mathbf{Y}) \;=\; d(\boldsymbol{\chi}(\mathbf{X}, t), \boldsymbol{\chi}(\mathbf{Y}, t)) \;=\; d(\mathbf{x}, \mathbf{y}) \; , \tag{3.154}$$

for all times t.

Now, introduce the notion of a superposed rigid-body motion. To this end, take a motion $\boldsymbol{\chi}$ of body $\mathcal{B}$ such that, as usual, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. Then, let another motion $\boldsymbol{\chi}^+$ be defined for $\mathcal{B}$, such that

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{X}, t) \; , \tag{3.155}$$

so that $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ differ by a rigid-body motion. Then, with reference to Figure 3.18, one may write

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{X}, t) \;=\; \boldsymbol{\chi}^+(\boldsymbol{\chi}_t^{-1}(\mathbf{x}), t) \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) \; . \tag{3.156}$$

Said differently,

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}_t^+(\mathbf{X}) \;=\; \bar{\boldsymbol{\chi}}_t^+(\mathbf{x}) \;=\; \bar{\boldsymbol{\chi}}_t^+(\boldsymbol{\chi}_t(\mathbf{X})) \tag{3.157}$$

or, equivalently,

$$\boldsymbol{\chi}_t^+ \;=\; \bar{\boldsymbol{\chi}}_t^+ \circ \boldsymbol{\chi}_t \; . \tag{3.158}$$

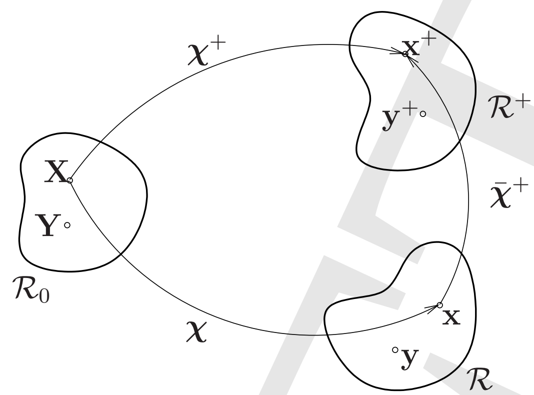

Figure 3.18: *Configurations associated with motions* $\boldsymbol{\chi}$ *and* $\boldsymbol{\chi}^+$ *differing by a superposed rigid motion* $\bar{\boldsymbol{\chi}}^+$*.*

Clearly, the superposed motion $\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)$ is invertible for fixed t, since $\boldsymbol{\chi}^+$ is assumed invertible for fixed t.

Similarly, take a second point $\mathbf{Y}$ in the reference configuration, so that $\mathbf{y} = \boldsymbol{\chi}(\mathbf{Y}, t)$ and write

$$\mathbf{y}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{Y}, t) \;=\; \boldsymbol{\chi}^+(\boldsymbol{\chi}_t^{-1}(\mathbf{y}), t) \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\ . \tag{3.159}$$

Recalling that $\mathcal{R}$ and $\mathcal{R}^+$ differ only by a rigid transformation, one may conclude that

$$\begin{aligned}(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \;&= (\mathbf{x}^+ - \mathbf{y}^+) \cdot (\mathbf{x}^+ - \mathbf{y}^+) \\ &= \left[\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\right] \left[\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\right]\ ,\end{aligned} \tag{3.160}$$

for all $\mathbf{x}$, $\mathbf{y}$ in the region $\mathcal{R}$ at time t. Since $\mathbf{x}$ and $\mathbf{y}$ are chosen independently, one may differentiate equation (3.160) first with respect to $\mathbf{x}$ to get

$$\mathbf{x} - \mathbf{y} \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T (\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t))\ . \tag{3.161}$$

Then, equation (3.161) may be differentiated with respect to $\mathbf{y}$, which leads to

$$\mathbf{i} \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)}{\partial \mathbf{y}}\right]\ . \tag{3.162}$$

Since the motion $\bar{\boldsymbol{\chi}}^+$ is invertible, equation (3.162) can be equivalently written as

$$\left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)}{\partial \mathbf{y}}\right]^{-1}\ . \tag{3.163}$$

Then, the left- and right-hand side should be necessarily functions of time only, hence

$$\left[\frac{\partial \overline{\boldsymbol{\chi}}^+(\mathbf{x},t)}{\partial \mathbf{x}}\right]^T \;=\; \left[\frac{\partial \overline{\boldsymbol{\chi}}^+(\mathbf{y},t)}{\partial \mathbf{y}}\right]^{-1} \;=\; \mathbf{Q}^T(t)\;. \tag{3.164}$$

Since equation (3.164) implies that $\dfrac{\partial \overline{\boldsymbol{\chi}}^+(\mathbf{x},t)}{\partial \mathbf{x}} = \dfrac{\partial \overline{\boldsymbol{\chi}}^+(\mathbf{y},t)}{\partial \mathbf{y}} = \mathbf{Q}(t)$, it follows immediately that $\mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{I}$, i.e., $\mathbf{Q}(t)$ is an orthogonal tensor. Further, note that using the chain rule the deformation gradient $\mathbf{F}^+$ of the motion $\boldsymbol{\chi}^+$ is written as

$$\mathbf{F}^+ \;=\; \frac{\partial \boldsymbol{\chi}^+}{\partial \mathbf{X}} \;=\; \frac{\partial \overline{\boldsymbol{\chi}}^+}{\partial \mathbf{x}}\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \;=\; \mathbf{QF}\;. \tag{3.165}$$

Since, by assumption, both motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ lead to deformation gradients with positive Jacobians, equation (3.165) implies that $\det \mathbf{Q} > 0$, hence $\det \mathbf{Q} = 1$ and $\mathbf{Q}$ is proper orthogonal.

Since $\mathbf{Q}$ is a function of time only, equation $(3.164)_1$ can be directly integrated with respect to $\mathbf{x}$, leading to

$$\mathbf{x}^+ \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{x},t) \;=\; \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)\;, \tag{3.166}$$

where $\mathbf{c}(t)$ is a vector function of time. Equation (3.166) is the general form of the superposed rigid motion $\bar{\boldsymbol{\chi}}$ on the original motion $\boldsymbol{\chi}$.

Given $(3.165)_3$ and recalling the right polar decomposition of $\mathbf{F}$, write

$$\begin{aligned}\mathbf{F}^+ &= \mathbf{R}^+\mathbf{U}^+ \\ &= \mathbf{QF} \;=\; \mathbf{QRU}\;,\end{aligned} \tag{3.167}$$

where $\mathbf{R}$, $\mathbf{R}^+$ are proper orthogonal tensors and $\mathbf{U}$, $\mathbf{U}^+$ are symmetric positive-definite tensors. Since, clearly,

$$(\mathbf{QR})^T(\mathbf{QR}) \;=\; (\mathbf{R}^T\mathbf{Q}^T)(\mathbf{QR}) \;=\; \mathbf{R}^T(\mathbf{Q}^T\mathbf{Q})\mathbf{R} \;=\; \mathbf{R}^T\mathbf{R} \;=\; \mathbf{I} \tag{3.168}$$

and also $\det(\mathbf{QR}) = (\det \mathbf{Q})(\det \mathbf{R}) = 1$, therefore $\mathbf{QR}$ is proper orthogonal, the uniqueness of the polar decomposition, in conjunction with (3.167), necessitates that

$$\mathbf{R}^+ \;=\; \mathbf{QR} \tag{3.169}$$

and

$$\mathbf{U}^+ \;=\; \mathbf{U}\;. \tag{3.170}$$

Similarly, the left decomposition of $\mathbf{F}$ yields

$$\begin{aligned} \mathbf{F}^+ &= \mathbf{V}^+\mathbf{R}^+ \;=\; \mathbf{V}^+(\mathbf{QR}) \\ &= \mathbf{QF} \;=\; \mathbf{Q}(\mathbf{VR}) \;, \end{aligned} \tag{3.171}$$

which implies that

$$\mathbf{V}^+(\mathbf{QR}) \;=\; \mathbf{Q}(\mathbf{VR}) \;, \tag{3.172}$$

hence,

$$\mathbf{V}^+ \;=\; \mathbf{QVQ}^T \;. \tag{3.173}$$

It follows readily from $(3.165)_3$ that

$$\mathbf{C}^+ \;=\; \mathbf{F}^{+^T}\mathbf{F}^+ \;=\; (\mathbf{QF})^T(\mathbf{QF}) \;=\; (\mathbf{F}^T\mathbf{Q}^T)(\mathbf{QF}) \;=\; \mathbf{F}^T\mathbf{F} \;=\; \mathbf{C} \tag{3.174}$$

and

$$\mathbf{B}^+ \;=\; \mathbf{F}^+\mathbf{F}^{+^T} \;=\; (\mathbf{QF})(\mathbf{QF})^T \;=\; (\mathbf{QF})(\mathbf{F}^T\mathbf{Q}^T) \;=\; \mathbf{QBQ}^T \;. \tag{3.175}$$

Likewise,

$$\mathbf{E}^+ \;=\; \frac{1}{2}(\mathbf{C}^+ - \mathbf{I}) \;=\; \frac{1}{2}(\mathbf{C} - \mathbf{I}) \;=\; \mathbf{E} \tag{3.176}$$

and

$$\begin{aligned} \mathbf{e}^+ \;=\; \frac{1}{2}(\mathbf{I} - \mathbf{B}^{+\,-1}) \;&= \frac{1}{2}[\mathbf{I} - (\mathbf{QBQ}^T)^{-1}] \\ &= \frac{1}{2}(\mathbf{I} - \mathbf{Q}^{-T}\mathbf{B}^{-1}\mathbf{Q}^{-1}) \\ &= \frac{1}{2}(\mathbf{I} - \mathbf{QB}^{-1}\mathbf{Q}^T) \\ &= \frac{1}{2}\mathbf{Q}(\mathbf{I} - \mathbf{B}^{-1})\mathbf{Q}^T \\ &= \mathbf{QeQ}^T \;. \end{aligned} \tag{3.177}$$

A vector or tensor is called *objective* if it transforms under superposed rigid motions in the same manner as its natural basis. Physically, this means that the components of an objective tensor relative to its basis are unchanged if both the tensor and its natural basis are transformed due to a rigid motion superposed to the original motion. Specifically, a referential tensor is objective when it does not change under superposed rigid motions, since its natural basis $\{\mathbf{E}_A \otimes \mathbf{E}_B\}$ remains untransformed under superposed rigid motions, as the latter do not affect the reference configuration. Hence, referential tensors such as $\mathbf{C}$, $\mathbf{U}$ and $\mathbf{E}$ are objective. Correspondingly a spatial tensor is objective if it transforms according to $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)\mathbf{Q}^T$. This is because its tensor basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ transforms as $\{(\mathbf{Q}\mathbf{e}_i) \otimes (\mathbf{Q}\mathbf{e}_j)\} = \mathbf{Q}\{\mathbf{e}_i \otimes \mathbf{e}_j\}\mathbf{Q}^T$. Hence, spatial tensors such as $\mathbf{B}$, $\mathbf{V}$ and $\mathbf{e}$ are objective. It is easy to deduce that two-point tensors are objective if they transform as $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)$ or $(\,\cdot\,)^+ = (\,\cdot\,)\mathbf{Q}^T$ depending on whether the first or second leg of the tensor lies in the current configuration, respectively. Analogous conclusions may be drawn for vectors, which, when objective, remain untransformed if they are referential or transform as $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)$ if they are spatial.

Examine next the transformation of the velocity vector $\mathbf{v}$ under a superposed rigid-body motion. In this case

$$\begin{aligned}\mathbf{v}^+ &= \dot{\boldsymbol{\chi}}^+(\mathbf{X}, t) \\ &= \dot{\bar{\boldsymbol{\chi}}}^+(\mathbf{x}, t) \;=\; \overset{\cdot}{\overline{[\mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)]}} \;=\; \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{c}}(t)\;.\end{aligned} \tag{3.178}$$

Since $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$,

$$\overset{\cdot}{\overline{\mathbf{Q}\mathbf{Q}^T}} \;=\; \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T \;=\; \mathbf{0}\;. \tag{3.179}$$

Setting

$$\boldsymbol{\Omega}(t) \;=\; \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)\;, \tag{3.180}$$

it follows from (3.179) that $\boldsymbol{\Omega}$ is skew-symmetric, hence there exists an axial vector $\boldsymbol{\omega}$ such that $\boldsymbol{\Omega}\mathbf{z} = \boldsymbol{\omega} \times \mathbf{z}$, for any $\mathbf{z} \in E^3$. Returning to (3.178), write, with the aid of (3.166) and (3.180),

$$\mathbf{v}^+ \;=\; \boldsymbol{\Omega}\mathbf{Q}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{c}}(t) \;=\; \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}(t)) + \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{c}}(t)\;. \tag{3.181}$$

Invoking the definition of the axial vector $\boldsymbol{\omega}$, one may rewrite (3.181) as

$$\mathbf{v}^+ \;=\; \boldsymbol{\omega} \times (\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}\;. \tag{3.182}$$

Equation (3.181) reveals that the velocity is not an objective vector.

Next, examine how the various tensorial measures of deformation rate transform under superposed rigid motions. Starting with the spatial velocity gradient, write

$$\begin{aligned}
\mathbf{L}^+ \;=\; \frac{\partial \mathbf{v}^+}{\partial \mathbf{x}^+} \;&=\; \frac{\partial}{\partial \mathbf{x}^+}[\boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}] \\
&= \boldsymbol{\Omega} + \frac{\partial(\mathbf{Q}\tilde{\mathbf{v}})}{\partial \mathbf{x}^+} \\
&= \boldsymbol{\Omega} + \frac{\partial(\mathbf{Q}\tilde{\mathbf{v}})}{\partial \mathbf{x}}\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}^+} \\
&= \boldsymbol{\Omega} + \mathbf{Q}\frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}^+}[\mathbf{Q}^T(\mathbf{x}^+ - \mathbf{c})] \\
&= \boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T \;,
\end{aligned} \tag{3.183}$$

where use is made of (3.181), the definition of $\mathbf{L}$ and the chain rule. It follows from (3.183) that $\mathbf{L}$ is not objective. Also, the rate-of-deformation tensor $\mathbf{D}$ transforms according to

$$\begin{aligned}
\mathbf{D}^+ \;&=\; \frac{1}{2}(\mathbf{L}^+ + \mathbf{L}^{+T}) \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T) + \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T)^T \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\Omega}^T) + \mathbf{Q}\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\mathbf{Q}^T \\
&= \mathbf{Q}\mathbf{D}\mathbf{Q}^T \;,
\end{aligned} \tag{3.184}$$

which implies that $\mathbf{D}$ is objective. Turning to the vorticity tensor $\mathbf{W}$, one may write

$$\begin{aligned}
\mathbf{W}^+ \;&=\; \frac{1}{2}(\mathbf{L}^+ - \mathbf{L}^{+T}) \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T) - \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T)^T \\
&= \frac{1}{2}(\boldsymbol{\Omega} - \boldsymbol{\Omega}^T) + \mathbf{Q}\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)\mathbf{Q}^T \\
&= \boldsymbol{\Omega} + \mathbf{Q}\mathbf{W}\mathbf{Q}^T \;,
\end{aligned} \tag{3.185}$$

from where it is seen that $\mathbf{W}$ is not objective.

Objectivity conclusions can be drawn for other kinematic quantities of interest by appealing to results already in place. For instance, infinitesimal material line elements transform as

$$d\mathbf{x}^+ \;=\; \mathbf{F}^+ d\mathbf{X} \;=\; (\mathbf{Q}\mathbf{F})d\mathbf{X} \;=\; \mathbf{Q}(\mathbf{F}d\mathbf{X}) \;=\; \mathbf{Q}d\mathbf{x} \;, \tag{3.186}$$

hence $d\mathbf{x}$ is objective. Similarly, infinitesimal material volume elements transform as

$$dv^+ \;=\; J^+ dV \;= \det(\mathbf{QF}) dV \;=\; (\det \mathbf{Q})(\det \mathbf{F}) dV \;=\; (\det \mathbf{F}) dV \;=\; J dV \;=\; dv \;, \tag{3.187}$$

hence dv is an objective scalar quantity. For infinitesimal material area elements, write

$$\begin{aligned} d\mathbf{a}^+ \;&= \mathbf{n}^+ da^+ \;= J^+ \mathbf{F}^{+\,-T} \mathbf{N} dA \\ &= J(\mathbf{QF})^{-T} \mathbf{N} dA \;=\; J(\mathbf{Q}^{-T}\mathbf{F}^{-T}) \mathbf{N} dA \;=\; J\mathbf{Q}\mathbf{F}^{-T}\mathbf{N} dA \;=\; \mathbf{Qn} da \;=\; \mathbf{Q} d\mathbf{a} \;, \end{aligned} \tag{3.188}$$

hence the vector $d\mathbf{a}$ is objective. Now, taking the dot-product of each side of (3.188) with itself yields

$$(\mathbf{n}^+ da^+) \cdot (\mathbf{n}^+ da^+) \;=\; (\mathbf{Qn} da) \cdot (\mathbf{Qn} da) \;, \tag{3.189}$$

therefore $(da^+)^2 = da^2$, hence also $da^+ = da$ (as long as da is taken positive from the outset) and $\mathbf{n}^+ = \mathbf{Qn}$.

In closing, note that the superposed rigid motion operation $(\,\cdot\,)^+$ commutes with the transposition $(\,\cdot\,)^T$, inversion $(\,\cdot\,)^{-1}$ and material time derivative $\dot{\overline{(\,\cdot\,)}}$ operations.

Chapter 4

Basic physical principles

4.1 The divergence and Stokes' theorems

By way of background, first review the divergence theorem for scalar, vector and tensor functions. To this end, let $\mathcal{P}$ be a bounded closed region with smooth boundary ∂P in the Euclidean point space $\mathcal{E}^3$. Recall that $\mathcal{P}$ is *bounded* if it can be enclosed by a sphere of finite radius, *closed* if it contains its boundary and *smooth* if the normal $\mathbf{n}$ to its boundary is uniquely defined at all boundary points.

Next, define a scalar function $\phi : \mathcal{P} \to \mathbb{R}$, a vector function $\mathbf{v} : \mathcal{P} \to E^3$, and a tensor function $\mathbf{T} : \mathcal{P} \to \mathcal{L}(E^3, E^3)$. All three functions are assumed continuously differentiable. Then, the gradients of ϕ and $\mathbf{v}$ satisfy

$$\int_{\mathcal{P}} \operatorname{grad} \phi \, dv \;=\; \int_{\partial\mathcal{P}} \phi \mathbf{n} \, da \; , \tag{4.1}$$

and

$$\int_{\mathcal{P}} \operatorname{grad} \mathbf{v} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{v} \otimes \mathbf{n} \, da \; . \tag{4.2}$$

In addition, the divergences of $\mathbf{v}$ and $\mathbf{T}$ satisfy

$$\int_{\mathcal{P}} \operatorname{div} \mathbf{v} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{v} \cdot \mathbf{n} \, da \; , \tag{4.3}$$

and

$$\int_{\mathcal{P}} \operatorname{div} \mathbf{T} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n} \, da \; . \tag{4.4}$$

Equation (4.3) expresses the classical divergence theorem. The other three identities can be derived from this theorem. Indeed, the identity (4.4) can be derived by dotting the left-hand side by a constant vector $\mathbf{c}$ and using (4.3) and the definition of the divergence of a tensor. This leads to

$$\begin{aligned}\int_{\mathcal{P}} \operatorname{div} \mathbf{T}\, dv \cdot \mathbf{c} \;&=\; \int_{\mathcal{P}} \operatorname{div} \mathbf{T} \cdot \mathbf{c}\, dv \;=\; \int_{\mathcal{P}} \operatorname{div}(\mathbf{T}^T \mathbf{c})\, dv \;=\; \int_{\partial\mathcal{P}} (\mathbf{T}^T \mathbf{c}) \cdot \mathbf{n}\, da \\ &=\; \int_{\partial\mathcal{P}} (\mathbf{T}\mathbf{n}) \cdot \mathbf{c}\, da \;=\; \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n}\, da \cdot \mathbf{c}\;. \qquad (4.5)\end{aligned}$$

Since $\mathbf{c}$ is chosen arbitrarily, equation (4.4) follows immediately. Equation (4.1) can be deduced from (4.4) by setting $\mathbf{T} = \phi \mathbf{I}$, in which case, for any constant vector $\mathbf{c}$,

$$\begin{aligned}\int_{\mathcal{P}} \operatorname{div}(\phi \mathbf{I})\, dv \cdot \mathbf{c} \;&=\; \int_{\mathcal{P}} \operatorname{div}(\phi \mathbf{c})\, dv \;=\; \int_{\mathcal{P}} \operatorname{tr} \operatorname{grad}(\phi \mathbf{c})\, dv \;=\; \int_{\mathcal{P}} \operatorname{tr}(\mathbf{c} \otimes \operatorname{grad} \phi)\, dv \\ &=\; \int_{\mathcal{P}} \operatorname{grad} \phi \cdot \mathbf{c}\, dv \;=\; \int_{\mathcal{P}} \operatorname{grad} \phi\, dv \cdot \mathbf{c} \;=\; \int_{\mathcal{P}} \operatorname{grad} \phi\, dv \cdot \mathbf{c} \\ &=\; \int_{\partial\mathcal{P}} \phi \mathbf{n}\, da \cdot \mathbf{c}\;, \qquad (4.6)\end{aligned}$$

which, again due to the arbitrariness of $\mathbf{c}$, implies (4.1). Lastly, (4.2) is obtained from (4.1) by taking any constant $\mathbf{c}$ and writing

$$\begin{aligned}\left[\int_{\mathcal{P}} \operatorname{grad} \mathbf{v}\, dv\right]^T \mathbf{c} \;&=\; \int_{\mathcal{P}} (\operatorname{grad} \mathbf{v})^T \mathbf{c}\, dv \;=\; \int_{\mathcal{P}} \operatorname{grad}(\mathbf{v} \cdot \mathbf{c})\, dv \\ &=\; \int_{\partial\mathcal{P}} (\mathbf{v} \cdot \mathbf{c}) \mathbf{n}\, da \;=\; \int_{\partial\mathcal{P}} (\mathbf{n} \otimes \mathbf{v}) \mathbf{c}\, da \;=\; \left[\int_{\partial\mathcal{P}} (\mathbf{n} \otimes \mathbf{v})\, da\right] \mathbf{c}\;, \qquad (4.7)\end{aligned}$$

which proves the identity.

Consider a closed non-intersecting curve $\mathcal{C}$ which is parametrized by a scalar τ, $0 \leq \tau \leq 1$, so that the position vector of a typical point on $\mathcal{C}$ is $\mathbf{c}(\tau)$. Also, let $\mathcal{A}$ be an open surface bounded by $\mathcal{C}$, see Figure 4.1. Clearly, any point on $\mathcal{A}$ possesses two equal and opposite unit vectors, each pointing outward to one of the two sides of the surface. To eliminate the ambiguity, choose one of the sides of the surface and denote its outward unit normal by $\mathbf{n}$. This side is chosen so that $\mathbf{c}(\bar{\tau}) \times \mathbf{c}(\bar{\tau} + d\tau)$ points toward it, for any $\bar{\tau} \in [0, 1)$. If now $\mathbf{v}$ is a continuously differentiable vector field, then Stokes' theorem states that

$$\int_{\mathcal{A}} \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\, dA \;=\; \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{x}\;. \qquad (4.8)$$

This integral on the right-hand side of (4.8) is called the *circulation* of the vector field $\mathbf{v}$ around $\mathcal{C}$. The circulation is the (infinite) sum of the tangential components of $\mathbf{v}$ along $\mathcal{C}$. If $\mathbf{v}$ is identified as the spatial velocity field, then Stokes' theorem states that the circulation of the velocity around $\mathcal{C}$ equals to twice the integral of the normal component of the vorticity vector on any open surface that is bounded by $\mathcal{C}$.

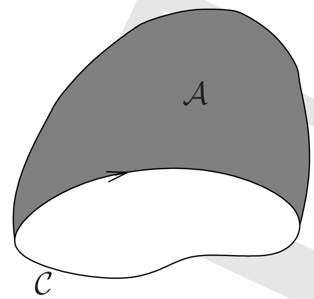

Figure 4.1: *A surface $\mathcal{A}$ bounded by the curve $\mathcal{C}$.*

4.2 The Reynolds' transport theorem

Let $\mathcal{P}$ be a closed and bounded region in $\mathcal{E}^3$ with smooth boundary $\partial\mathcal{P}$ and assume that the particles which occupy this region at time t occupy a closed and bounded region $\mathcal{P}_0$ with smooth boundary $\partial\mathcal{P}_0$ at another time t_0, see Figure 4.2.

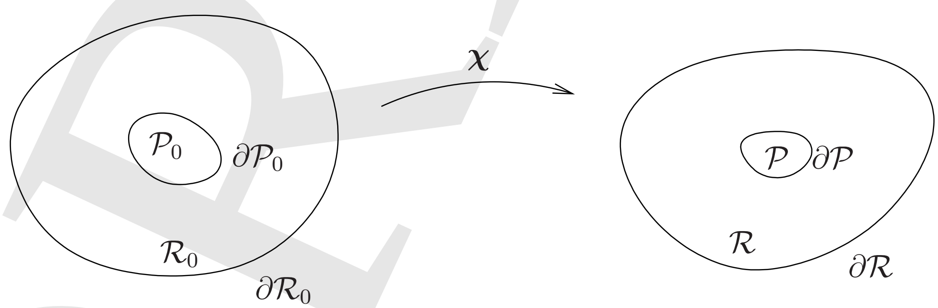

Figure 4.2: *A region $\mathcal{P}$ with boundary $\partial\mathcal{P}$ and its image $\mathcal{P}_0$ with boundary $\partial\mathcal{P}_0$ in the reference configuration.*

Further, let a scalar field ϕ be defined by a referential function $\hat{\phi}$ or a spatial function $\tilde{\phi}$, such that

$$\phi \;=\; \hat{\phi}(\mathbf{X}, t) \;=\; \tilde{\phi}(\mathbf{x}, t)\ . \tag{4.9}$$

Both $\hat{\phi}$ and $\tilde{\phi}$ are assumed continuously differentiable in both of their variables. In the forthcoming discussion of balance laws, it is important to be able to manipulate integrals of the form

$$\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv \ , \tag{4.10}$$

namely, material time derivatives of volume integrals defined in some subset of the current configuration.

Example: Consider the integral in (4.10) for $\phi = 1$. Here, $\frac{d}{dt}\int_{\mathcal{P}} dv = \frac{d}{dt}\operatorname{vol}\{(\mathcal{P})\}$, which is the rate of change at time t of the total volume of the region occupied by the material particles that at time t occupy $\mathcal{P}$. □

In evaluating (4.10), one first notes that the differentiation and integration operations cannot be directly interchanged, because the region $\mathcal{P}$ over which the integral is evaluated is itself a function of time. Therefore, one needs to first map the integral to the (fixed) reference configuration, interchange the differentiation and integration operations, evaluate the derivative of the integrand, and then map the integral back to the current configuration. Taking this approach leads to

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv &= \frac{d}{dt}\int_{\mathcal{P}_0} \hat{\phi} J\, dV && \text{(pull-back to } \mathcal{P}_0\text{)} \\
&= \int_{\mathcal{P}_0} \frac{d}{dt}[\hat{\phi} J]\, dV && \left(\text{interchange } \frac{d}{dt} \text{ and } \int\right) \\
&= \int_{\mathcal{P}_0} \left[\frac{\partial \hat{\phi}}{\partial t} J + \hat{\phi}\frac{\partial J}{\partial t}\right] dV \\
&= \int_{\mathcal{P}_0} (\dot{\phi} J + \hat{\phi} J \operatorname{div} \mathbf{v})\, dV && (\dot{J} = J \operatorname{div} \mathbf{v}) \\
&= \int_{\mathcal{P}_0} (\dot{\phi} + \hat{\phi} \operatorname{div} \mathbf{v}) J\, dV \\
&= \int_{\mathcal{P}} (\dot{\phi} + \tilde{\phi} \operatorname{div} \mathbf{v})\, dv && \text{(push-forward to } \mathcal{P}\text{)} \ .
\end{aligned} \tag{4.11}$$

This result is known as the *Reynolds' transport theorem*. Note that the theorem applies also to vector and tensor functions without any modifications.

To interpret the Reynolds' transport theorem, note that the left-hand side of (4.11) is the rate of change of the integral of ϕ over the region $\mathcal{P}$, when following the set of particles that happen to occupy $\mathcal{P}$ at time t. The right-hand side of (4.11) consists of the sum of two terms. The first one is the integral of the rate of change of ϕ for all particles that happen to occupy $\mathcal{P}$ at time t. The second one is due to the rate of change of the volume occupied by the same particles assuming that $\tilde{\phi}$ remains constant in time for those particles.

The Reynolds' transport theorem can be restated in a number of equivalent forms. To this end, write with the aid of the divergence theorem (4.3),

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv &= \int_{\mathcal{P}} (\dot{\phi} + \phi \operatorname{div} \mathbf{v})\, dv \\ &= \int_{\mathcal{P}} \left[\frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial \tilde{\phi}}{\partial \mathbf{x}} \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v} \right] dv \\ &= \int_{\mathcal{P}} \left[\frac{\partial \tilde{\phi}}{\partial t} + \operatorname{div}(\tilde{\phi}\mathbf{v}) \right] dv \\ &= \int_{\mathcal{P}} \frac{\partial \tilde{\phi}}{\partial t}\, dv + \int_{\partial \mathcal{P}} \tilde{\phi} \mathbf{v} \cdot \mathbf{n}\, da\ . \end{aligned} \tag{4.12}$$

An alternative interpretation of the theorem is now in order. Here, the right-hand side of (4.12) consists of the sum of two terms. The first term is the integral of the rate of change of ϕ at time t for all points that form the fixed region $\mathcal{P}$. The second term is the rate at which the volume of $\mathcal{P}$ weighted by ϕ changes as particles exit $\mathcal{P}$ across $\partial\mathcal{P}$.

Example: Consider again the special case $\tilde{\phi}(\mathbf{x}, t) = 1$, which corresponds to the transport of volume. Here,

$$\frac{d}{dt}\int_{\mathcal{P}} dv \;=\; \int_{\mathcal{P}} \operatorname{div} \mathbf{v}\, dv \;=\; \int_{\partial \mathcal{P}} \mathbf{v} \cdot \mathbf{n}\, da\ . \tag{4.13}$$

This means that the rate of change of the volume occupied by the same material particles equals the boundary integral of the normal component of the velocity $\mathbf{v} \cdot \mathbf{n}$ of $\partial\mathcal{P}$, i.e., the rate at which the volume of $\mathcal{P}$ changes as the particles exit across the boundary $\partial\bar{\mathcal{P}}$ of the fixed region $\bar{\mathcal{P}}$ which equals to $\mathcal{P}$ at time t. □

Starting from (4.12), note that $\displaystyle\int_{\mathcal{P}} \frac{\partial \tilde{\phi}}{\partial t}\, dv = \frac{\partial}{\partial t}\int_{\mathcal{P}} \tilde{\phi}\, dv$, since $\displaystyle\frac{\partial \tilde{\phi}}{\partial t}$ is the rate of change of ϕ for fixed position in space (hence, $\mathcal{P}$ is also fixed in space). Therefore, one may alternatively express (4.12) as

$$\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv \;=\; \frac{\partial}{\partial t}\int_{\mathcal{P}} \tilde{\phi}\, dv + \int_{\partial \mathcal{P}} \tilde{\phi} \mathbf{v} \cdot \mathbf{n}\, da\ . \tag{4.14}$$

4.3 The localization theorem

Another important result with implications in the study of balance laws is presented here by way of background. Let $\tilde{\phi} : \mathcal{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\phi = \tilde{\phi}(\mathbf{x}, t)$, where $\mathcal{R} \subset E^3$. Also, let $\tilde{\phi}$ be continuous function in the spatial argument. Then, assume that

$$\int_{\mathcal{P}} \tilde{\phi}\, dv \;=\; 0 \;, \tag{4.15}$$

for all $\mathcal{P} \subset \mathcal{R}$ at a given time t. The *localization theorem* states that this is true if, and only if, $\tilde{\phi} = 0$ everywhere in $\mathcal{R}$ at time t.

To prove this result, first note that the "if" portion of the theorem is straightforward, since, if $\tilde{\phi} = 0$ in $\mathcal{R}$, then (4.15) holds trivially true for any subset $\mathcal{P}$ of $\mathcal{R}$. To prove the converse, note that continuity of $\tilde{\phi}$ in the spatial argument $\mathbf{x}$ at a point $\mathbf{x}_0 \in \mathcal{R}$ means that for any time t and every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that

$$|\tilde{\phi}(\mathbf{x}, t) - \tilde{\phi}(\mathbf{x}_0, t)| \;<\; \varepsilon \;, \tag{4.16}$$

provided that

$$|\mathbf{x} - \mathbf{x}_0)| \;<\; \delta(\varepsilon) \;. \tag{4.17}$$

Now proceed by contradiction: assume that there exists a point $\mathbf{x}_0 \in \mathcal{R}$, such that, at any fixed time t, $\tilde{\phi}(\mathbf{x}_0, t) = \phi_0 > 0$. Then, invoking continuity of $\tilde{\phi}$, there exists a $\delta = \delta(\frac{\phi_0}{2})$, such that

$$|\tilde{\phi}(\mathbf{x}, t) - \tilde{\phi}(\mathbf{x}_0, t)| \;=\; |\tilde{\phi}(\mathbf{x}, t) - \phi_0| \;<\; \frac{\phi_0}{2} \;, \tag{4.18}$$

whenever

$$|\mathbf{x} - \mathbf{x}_0)| \;<\; \delta(\frac{\phi_0}{2}) \;. \tag{4.19}$$

Now, define the region $\mathcal{P}_\delta$ that consists of all points of $\mathcal{R}$ for which $|\mathbf{x} - \mathbf{x}_0| < \delta(\frac{\phi_0}{2})$. This is a sphere of radius δ in $\mathcal{E}^3$ with volume $\mathrm{vol}(\mathcal{P}_\delta) = \int_{P_\delta} dv > 0$. It follows from (4.18) that $\tilde{\phi}(\mathbf{x}, t) > \frac{\phi_0}{2}$ everywhere in P_δ. This, in turn, implies that

$$\int_{\mathcal{P}_\delta} \tilde{\phi}\, dv \;>\; \int_{\mathcal{P}_\delta} \frac{\phi_0}{2}\, dv \;=\; \frac{\phi_0}{2}\,\mathrm{vol}(\mathcal{P}_\delta) \;>\; 0 \;, \tag{4.20}$$

which constitutes a contradiction. Therefore, the localization theorem holds.

The localization theorem can be also proved with ease for vector and tensor functions.

4.4 Mass and mass density

Consider a body $\mathcal{B}$ and take any arbitrary part $\mathcal{S} \subseteq \mathcal{B}$. Define a set function $m : \mathcal{S} \mapsto \mathbb{R}$ with the following properties:

(i) $m(\mathcal{S}) \geq 0$, for all $\mathcal{S} \subseteq \mathcal{B}$ (i.e., m non-negative)

(ii) $m(\emptyset) = 0$.

(iii) $m(\cup_{i=1}^{\infty} \mathcal{S}_i) = \sum_{i=1}^{\infty} m(\mathcal{S}_i)$, where $\mathcal{S}_i \subset \mathcal{B}$, $i = 1, 2, \ldots$, and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, if $i \neq j$ (i.e., m countably additive).

A function m with the preceding properties is called a *measure* on $\mathcal{B}$. Assume here that there exists such a measure m and refer to $m(\mathcal{B})$ as the *mass* of body $\mathcal{B}$ and $m(\mathcal{S})$ as the mass of the part $\mathcal{S}$ of $\mathcal{B}$. In other words, consider the body as a set of particles with positive mass.

Recall now that at time t the body $\mathcal{B}$ occupies a region $\mathcal{R} \subset \mathcal{E}^3$ and the part $\mathcal{S}$ occupies a region $\mathcal{P}$. Assuming that m is an absolutely continuous measure, it can be established that there exists a unique function $\rho = \rho(\mathbf{x}, t)$, such that, for any function $f = \check{f}(P, t) = \tilde{f}(\mathbf{x}, t)$, one may write

$$\int_{\mathcal{B}} \check{f} \, dm \; = \; \int_{\mathcal{R}} \tilde{f} \rho \, dv \tag{4.21}$$

and

$$\int_{\mathcal{S}} \check{f} \, dm \; = \; \int_{\mathcal{P}} \tilde{f} \rho \, dv \; . \tag{4.22}$$

The function $\rho > 0$ is termed the *mass density*. Its existence is a direct consequence of a classical result in measure theory, known as the Radon-Nikodym theorem.

As a special case, one may consider the function $f = 1$, so that (4.21) and (4.22) reduce to

$$\int_{\mathcal{B}} dm \; = \; \int_{\mathcal{R}} \rho \, dv \; = \; m(\mathcal{B}) \tag{4.23}$$

and

$$\int_{\mathcal{S}} dm \; = \; \int_{\mathcal{P}} \rho \, dv \; = \; m(\mathcal{S}) \; . \tag{4.24}$$

The mass density of a particle P occupying point $\mathbf{x}$ in the current configuration may be defined by a limiting process as

$$\rho \;=\; \lim_{\delta \to 0} \frac{m(\mathcal{S}_\delta)}{\mathrm{vol}(\mathcal{P}_\delta)} \;, \tag{4.25}$$

where $\mathcal{P}_\delta \subset \mathcal{E}^3$ denotes a sphere of radius $\delta > 0$ centered at $\mathbf{x}$ and $\mathcal{S}_\delta$ the part of the body that occupies $\mathcal{P}_\delta$ at time t, see Figure 4.3.

Figure 4.3: *A limiting process used to define the mass density ρ at a point $\mathbf{x}$ in the current configuration.*

An analogous definition of mass density can be furnished in the reference configuration, where, for any function $f = \check{f}(P,t) = \hat{f}(\mathbf{X},t)$,

$$\int_{\mathcal{B}} \check{f}\, dm \;=\; \int_{\mathcal{R}_0} \hat{f} \rho_0 \, dV \tag{4.26}$$

and

$$\int_{\mathcal{S}} \check{f}\, dm \;=\; \int_{\mathcal{P}_0} \hat{f} \rho \, dV \;. \tag{4.27}$$

Here, the *mass density* $\rho_0 = \rho_0(\mathbf{X},t)$ in the reference configuration may be again defined by a limiting process, such that at a given point $\mathbf{X}$,

$$\rho_0 \;=\; \lim_{\delta \to 0} \frac{m(\mathcal{S}_\delta)}{\mathrm{vol}(\mathcal{P}_{0,\delta})} \;, \tag{4.28}$$

where $\mathcal{P}_{0,\delta} \subset \mathcal{E}^3$ denotes a sphere of radius $\delta > 0$ centered at $\mathbf{X}$ and $\mathcal{S}_\delta$ the part of the body that occupies $\mathcal{P}_{0,\delta}$ at time t_0. Also, as in the spatial case, one may write

$$\int_{\mathcal{B}} dm \;=\; \int_{\mathcal{R}_0} \rho_0 \, dV \;=\; m(\mathcal{B}) \tag{4.29}$$

and

$$\int_{\mathcal{S}} dm \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \;=\; m(\mathcal{S}) \;. \tag{4.30}$$

The continuum hypothesis is crucial in establishing the existence of ρ and ρ_0.

4.5 The principle of mass conservation

The principle of mass conservation states that the mass of any material part of the body remains constant at all times, namely that

$$\frac{d}{dt} m(\mathcal{S}) \;=\; 0 \tag{4.31}$$

or, upon recalling (4.24),

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dv \;=\; 0 \;. \tag{4.32}$$

The preceding is an integral form of the principle of mass conservation in the spatial description. Using the Reynolds' transport theorem, the above equation may be written as

$$\int_{\mathcal{P}} [\dot{\rho} + \rho \operatorname{div} \mathbf{v}] \, dv \;=\; 0 \;. \tag{4.33}$$

Assuming that the integrand in (4.33) is continuous and recalling that $\mathcal{S}$ (hence, also $\mathcal{P}$) is arbitrary, it follows from the localization theorem that

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} \;=\; 0 \;. \tag{4.34}$$

Equation (4.34) constitutes the local form of the principle of mass conservation in the spatial description.[1] It can be readily rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v} + \rho \operatorname{div} \mathbf{v} \;=\; 0 \tag{4.35}$$

hence, also as

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{v}) \;=\; 0 \;. \tag{4.36}$$

Example: In a volume-preserving flow of a material with uniform density, conservation of mass reduces to $\dfrac{\partial \rho}{\partial t} = 0$. Hence, recalling (4.14), one may write

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dv \;=\; \int_{\mathcal{P}} \frac{\partial \rho}{\partial t} \, dv + \int_{\partial \mathcal{P}} \rho \mathbf{v} \cdot \mathbf{n} \, da \;=\; \int_{\partial \mathcal{P}} \rho \mathbf{v} \cdot \mathbf{n} \, da \;. \tag{4.37}$$

$\square$

[1] This is also sometimes referred to as the "continuity" equation, with reference to the continuity of fluid flow in a control volume $\mathcal{P}$.

An alternative form of the mass conservation principle can be obtained by recalling equations (4.24) and (4.30), from which it follows that

$$m(\mathcal{S}) \;=\; \int_{\mathcal{P}} \rho \, dv \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \; . \tag{4.38}$$

Recalling also (3.117), one concludes that

$$\int_{\mathcal{P}_0} \rho J \, dV \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \; . \tag{4.39}$$

This is an integral form of the principle of mass conservation in referential description. From it, one finds that

$$\int_{\mathcal{P}_0} (\rho J - \rho_0) \, dV \;=\; 0 \; . \tag{4.40}$$

Taking into account the arbitrariness of $\mathcal{P}_0$, the localization theorem may be invoked to yield a local form of mass conservation in referential description as

$$\rho_0 \;=\; \rho J \; . \tag{4.41}$$

4.6 The principles of linear and angular momentum balance

Once mass conservation is established, the principles of linear and angular momentum are postulated to describe the motion of continua. These two principles originate from the work of Newton and Euler.

By way of background, review Newton's three laws of motion, as postulate for particles in 1687. The first law says that a particle stays at rest or continues at constant velocity unless an external force acts on it. The second law says that the total external force on a particle is proportional to the rate of change of the momentum of the particle. The third law says that every action has an equal and opposite reaction. As Euler recognized, Newton's three laws of motion, while sufficient for the analysis of particles, are not strictly appropriate for the study of rigid and deformable continua. Rather, he postulate a linear momentum balance law (akin to Newton's second law) and an angular momentum balance law. The latter can be easily motivated from the analysis of systems of particles.

To formulate Euler's two balance laws, first define the *linear momentum* of the part of the body that occupies the infinitesimal volume element dv at time t as $dm\mathbf{v}$, where dm is the mass of dv. Also, define the *angular momentum* of the same part relative to the origin of the fixed basis $\{\mathbf{e}_i\}$ as $\mathbf{x} \times (dm\mathbf{v})$, where $\mathbf{x}$ is the position vector associated with the infinitesimal volume element. Similarly, define the linear and angular momenta of the part $\mathcal{S}$ which occupies a region $\mathcal{P}$ at time t as $\int_{\mathcal{S}} \mathbf{v}\, dm$ and $\int_{\mathcal{S}} \mathbf{x} \times \mathbf{v}\, dm$, respectively.

Next, admit the existence of two types of external forces acting on $\mathcal{B}$ at any time t. There are: (a) *body forces* per unit mass (e.g., gravitational, magnetic) $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ which act on the particles which comprise the continuum, and (b) *contact forces* per unit area $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n}) = \mathbf{t}_{(\mathbf{n})}(\mathbf{x}, t)$, which act on the particles which lie on boundary surfaces and depend on the orientation of the surface on which they act through the outward unit normal $\mathbf{n}$ to the surface. The force $\mathbf{t}_{(\mathbf{n})}$ is alternatively referred to as the *stress vector* or the *traction vector*.

The principle of *linear momentum balance* states that the rate of change of linear momentum for any region $\mathcal{P}$ occupied at time t by a part $\mathcal{S}$ of the body equals the total external forces acting on this part. In mathematical terms, this means that

$$\frac{d}{dt} \int_{\mathcal{S}} \mathbf{v}\, dm \;=\; \int_{\mathcal{S}} \mathbf{b}\, dm + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.42}$$

or, equivalently,

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \ . \tag{4.43}$$

Using conservation of mass, the left-hand side of the equation can be written as

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{P}} \rho \mathbf{v}\, dv \;&=\; \int_{\mathcal{P}} \frac{d}{dt}(\rho \mathbf{v})\, dv + \int_{\mathcal{P}} (\rho \mathbf{v}) \operatorname{div} \mathbf{v}\, dv \\
&=\; \int_{\mathcal{P}} (\dot{\rho} \mathbf{v} + \rho \dot{\mathbf{v}})\, dv + \int_{\mathcal{P}} (\rho \mathbf{v}) \operatorname{div} \mathbf{v}\, dv \\
&=\; \int_{\mathcal{P}} [(\dot{\rho} + \rho \operatorname{div} \mathbf{v}) \mathbf{v} + \rho \dot{\mathbf{v}}]\, dv \\
&=\; \int_{\mathcal{P}} \rho \mathbf{a}\, dv \ ,
\end{aligned} \tag{4.44}$$

hence, the principle of linear momentum balance can be also expressed as

$$\int_{\mathcal{P}} \rho \mathbf{a}\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \ . \tag{4.45}$$

The principle of *angular momentum balance* states that the rate of change of angular momentum for any region $\mathcal{P}$ occupied at t by a part $\mathcal{S}$ of the body equals the moment of all external forces acting on this part. Again, this principle can be expressed mathematically as

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{x} \times \mathbf{v}\, dm \;=\; \int_{\mathcal{S}} \mathbf{x} \times \mathbf{b}\, dm + \int_{\partial\mathcal{P}} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}\, da \tag{4.46}$$

or, equivalently,

$$\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}\, da \ . \tag{4.47}$$

Again, appealing to conservation of mass, one may easily rewrite the term on the left-hand side of (4.47) as

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{v}\, dv \;&=\; \int_{\mathcal{P}} \left\{ \frac{d}{dt}(\mathbf{x} \times \rho\mathbf{v}) + (\mathbf{x} \times \rho\mathbf{v})\operatorname{div}\mathbf{v} \right\} dv \\
&= \int_{\mathcal{P}} \left\{ [\dot{\mathbf{x}} \times \rho\mathbf{v} + \mathbf{x} \times \dot{\rho}\mathbf{v} + \mathbf{x} \times \rho\dot{\mathbf{v}}] + (\mathbf{x} \times \rho\mathbf{v}\operatorname{div}\mathbf{v}) \right\} dv \\
&= \int_{\mathcal{P}} \left\{ \mathbf{x} \times (\dot{\rho} + \rho\operatorname{div}\mathbf{v})\mathbf{v} + \mathbf{x} \times \rho\mathbf{a} \right\} dv \\
&= \int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{a}\, dv \ .
\end{aligned} \tag{4.48}$$

As a result, the principle of angular momentum balance may be also written as

$$\int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}\, da \ . \tag{4.49}$$

The preceding two balance laws are also referred to as *Euler's laws.* They are termed "balance" laws because they postulate that there exists a balance between external forces (and their moments) and the rate of change of linear (and angular) momentum. Euler's laws are independent axioms in continuum mechanics.

In the special case where $\mathbf{b} = \mathbf{0}$ in $\mathcal{P}$ and $\mathbf{t}_{(\mathbf{n})} = \mathbf{0}$ on $\partial\mathcal{P}$, then equations (4.43) and (4.47) readily imply that the linear and the angular momentum are conserved quantities in $\mathcal{P}$. Another commonly encountered special case is when the velocity $\mathbf{v}$ vanishes identically. In this case, equations (4.43) and (4.47) imply that the sum of all external forces and the some of all external moments vanish, which gives rise to the classical *equilibrium* equations of statics.

4.7 Stress vector and stress tensor

As in the case of mass balance, it is desirable to obtain local forms of linear and angular momentum balance. Recalling the corresponding integral statements (4.43) and (4.47), it is clear that the Reynolds' transport theorem may be employed in the rate of change of momentum terms to yield volume integrals, while the body force terms are already in the form of volume integral. Therefore, in order to apply the localization theorem, it is essential that the contact form terms (presently written as surface integrals) be transformed into equivalent volume integral terms.

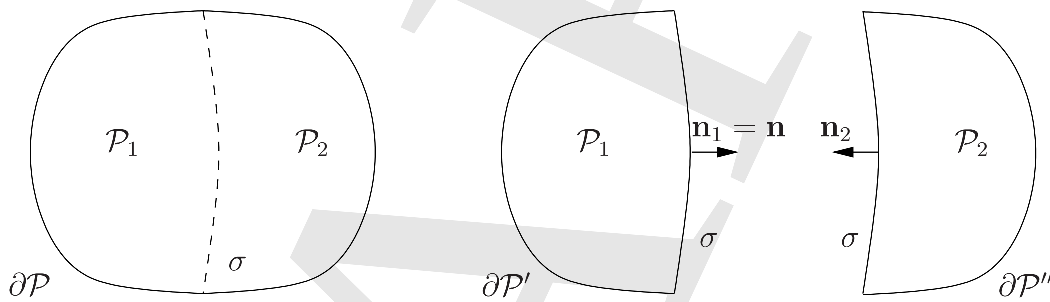

Figure 4.4: *Setting for a derivation of Cauchy's lemma.*

By way of background, consider some properties of the traction vector $\mathbf{t}_{(\mathbf{n})}$. To this end, take an arbitrary region $\mathcal{P} \subseteq \mathcal{R}$ and divide it into two mutually disjoint subregions $\mathcal{P}_1$ and $\mathcal{P}_2$ by means of an arbitrary smooth surface σ, namely $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, see Figure 4.4. Also, note that the boundaries $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$ of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, can be expressed as $\partial\mathcal{P}_1 = \partial\mathcal{P}' \cup \sigma$ and $\partial\mathcal{P}_2 = \partial\mathcal{P}'' \cup \sigma$. Now, enforce linear momentum balance separately in $\mathcal{P}_1$ and $\mathcal{P}_2$ to find that

$$\frac{d}{dt}\int_{\mathcal{P}_1} \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}_1} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_1} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.50}$$

and

$$\frac{d}{dt}\int_{\mathcal{P}_2} \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}_2} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.51}$$

Subsequently, add the two equations together to find that

$$\frac{d}{dt}\int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.52}$$

or

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{P}_1 \cup \partial \mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \;. \tag{4.53}$$

Further, enforce linear momentum balance on the union of $\mathcal{P}_1$ and $\mathcal{P}_2$ to find that

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b}\, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;. \tag{4.54}$$

Subtracting (4.54) from (4.53) leads to

$$\int_{\partial \mathcal{P}_1 \cup \partial \mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;. \tag{4.55}$$

Recalling the decomposition of $\partial \mathcal{P}_1$ and $\partial \mathcal{P}_2$, the preceding equation may be also expressed as

$$\int_{\partial \mathcal{P}' \cup \sigma} \mathbf{t}_{(\mathbf{n})}\, da + \int_{\partial \mathcal{P}'' \cup \sigma} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.56}$$

or

$$\int_{\partial \mathcal{P}' \cup \partial \mathcal{P}''} \mathbf{t}_{(\mathbf{n})}\, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_1)}\, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_2)}\, da \;=\; \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;, \tag{4.57}$$

so, finally,

$$\int_{\sigma} \mathbf{t}_{(\mathbf{n}_1)}\, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_2)}\, da \;=\; \mathbf{0} \;, \tag{4.58}$$

which can be also written as

$$\int_{\sigma} \mathbf{t}_{(\mathbf{n})}\, da + \int_{\sigma} \mathbf{t}_{(-\mathbf{n})}\, da \;=\; \mathbf{0} \;. \tag{4.59}$$

Since σ is an arbitrary surface, assuming that $\mathbf{t}$ depends continuously on $\mathbf{n}$ and $\mathbf{x}$ along σ, the localization theorem yields the condition

$$\mathbf{t}_{(\mathbf{n})} + \mathbf{t}_{(-\mathbf{n})} \;=\; \mathbf{0} \;. \tag{4.60}$$

This result is called *Cauchy's lemma* on $\mathbf{t}_{(\mathbf{n})}$. It states that the stress vectors acting at $\mathbf{x}$ on opposite sides of the same smooth surface are equal and opposite, namely that

$$\mathbf{t}(\mathbf{x}, t; \mathbf{n}) \;=\; -\mathbf{t}(\mathbf{x}, t; -\mathbf{n}) \;. \tag{4.61}$$

It is important to recognize here that in continuum mechanics Cauchy's lemma is not a principle. Rather, it is derivable from linear momentum balance, as above. This is in stark contrast with particle mechanics, where action-reaction is a principle, also known as Newton's Third Law.

At this stage, consider the following problem, originally conceived by Cauchy: take a tetrahedral region $\mathcal{P} \subset \mathcal{R}$, such that three edges are parallel to the axes of $\{\mathbf{e}_i\}$ and meet at a point $\mathbf{x}$, as in Figure 4.5. Let σ_i be the face with unit outward normal $-\mathbf{e}_i$, and σ_0 the (inclined) face with outward unit normal $\mathbf{n}$. Denote by A the area of σ_0, so that the area vector $\mathbf{n}dA$ can be resolved as

$$\mathbf{n}A \;=\; (n_i\mathbf{e}_i)A \;=\; An_i\mathbf{e}_i \;=\; A_i\mathbf{e}_i \;, \tag{4.62}$$

where $A_i = An_i$ is the area of the face σ_i. In addition, the volume V of the tetrahedron can be written as $V = \frac{1}{3}Ah$, where h is the distance of $\mathbf{x}$ from face σ_0.

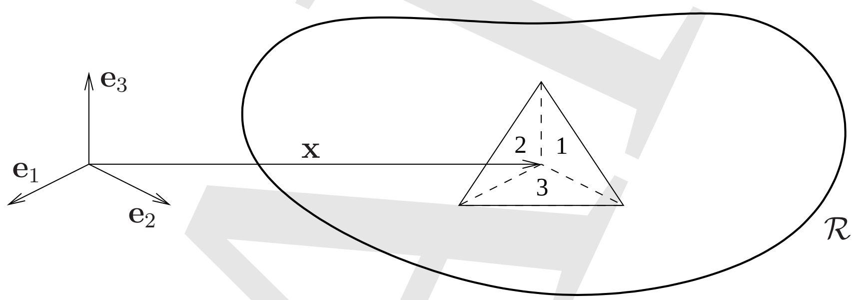

Figure 4.5: *The Cauchy tetrahedron.*

Now apply balance of linear momentum to the tetrahedral region $\mathcal{P}$ in the form of equation (4.45) and concentrate on the surface integral term, which, in this case, becomes

$$\int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \int_{\sigma_1} \mathbf{t}_{(-\mathbf{e}_1)}\, da + \int_{\sigma_2} \mathbf{t}_{(-\mathbf{e}_2)}\, da + \int_{\sigma_3} \mathbf{t}_{(-\mathbf{e}_3)}\, da + \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da \;. \tag{4.63}$$

Upon using Cauchy's lemma, this becomes

$$\int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;=\; -\int_{\sigma_1} \mathbf{t}_{(\mathbf{e}_1)}\, da - \int_{\sigma_2} \mathbf{t}_{(\mathbf{e}_2)}\, da - \int_{\sigma_3} \mathbf{t}_{(\mathbf{e}_3)}\, da + \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da \;. \tag{4.64}$$

Returning now to the balance of linear momentum statement (4.45), it can be written as

$$\int_{\mathcal{P}} \rho(\mathbf{a}-\mathbf{b})\, dv \;=\; \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \int_{\sigma_1} \mathbf{t}_{(\mathbf{e}_1)}\, da - \int_{\sigma_2} \mathbf{t}_{(\mathbf{e}_2)}\, da - \int_{\sigma_3} \mathbf{t}_{(\mathbf{e}_3)}\, da \;. \tag{4.65}$$

Assuming that ρ, $\mathbf{a}$, and $\mathbf{b}$ are bounded, one can obtain an upper-bound estimate for the domain integral on the left-hand side as

$$\left|\int_{\mathcal{P}} \rho(\mathbf{a}-\mathbf{b})\, dv\right| \;\leq\; \int_{\mathcal{P}} |\rho(\mathbf{a}-\mathbf{b})\, dv| \;=\; \int_{\mathcal{P}} K(\mathbf{x},t)\, dv \;=\; K^*V \;=\; K^*\frac{1}{3}Ah \;, \tag{4.66}$$

where $K(\mathbf{x},t) = |\rho(\mathbf{a}-\mathbf{b})|$ and $K^* = K(\mathbf{x}^*,t)$, with $\mathbf{x}^*$ being some interior point of $\mathcal{P}$. The preceding derivation made use of the mean-value theorem for integrals.[2]

Assuming that $\mathbf{t}(\mathbf{e}_i)$ are continuous in $\mathbf{x}$, apply the mean value theorem for integrals component-wise to get

$$\int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \;=\; \mathbf{t}_i^* A_i \;, \tag{4.67}$$

so that summing up all three like equations

$$\sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \;=\; \mathbf{t}_i^* A_i \;=\; \mathbf{t}_i^* A n_i \;. \tag{4.68}$$

Also, for the inclined face

$$\int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \mathbf{t}_{(\mathbf{n})}^* A \;. \tag{4.69}$$

In the preceding two equations, $\mathbf{t}_i^*$ and $\mathbf{t}_{(\mathbf{n})}^*$ are the traction vectors at some interior points of σ_i and σ_0. Recalling from (4.65) and (4.66) that

$$\left| \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \right| \;\leq\; \frac{1}{3} K^* A h \;, \tag{4.70}$$

write

$$\left| \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \right| \;=\; |\mathbf{t}^* A - \mathbf{t}_i^* A n_i| \;=\; A\,|\mathbf{t}^* - \mathbf{t}_i^* n_i| \;\leq\; \frac{1}{3} K^* A h \;, \tag{4.71}$$

which simplifies to

$$|\mathbf{t}_{(\mathbf{n})}^* - \mathbf{t}_i^* n_i| \;\leq\; \frac{1}{3} K^* h \;. \tag{4.72}$$

Now, upon applying the preceding analysis to a sequence of geometrically similar tetrahedra with heights $h_1 > h_2 > \ldots$, where $\lim_{i\to\infty} h_i = 0$, one finds that

$$|\mathbf{t}_{(\mathbf{n})}^* - \mathbf{t}_i^* n_i| \;\leq\; 0 \;, \tag{4.73}$$

where, obviously, all stress vectors are evaluated exactly at $\mathbf{x}$. It follows from (4.73) that at point $\mathbf{x}$

$$\mathbf{t}_{(\mathbf{n})} \;=\; \mathbf{t}_i n_i \;, \tag{4.74}$$

[2] Mean-value Theorem for Integrals: If $\mathcal{P}$ has positive volume ($\mathrm{vol}(\mathcal{P}) > 0$) and is compact and connected, and f is continuous in ε^3, then there exists a point $\mathbf{x}^* \in \mathcal{P}$ for which $\int_{\mathcal{P}} f(\mathbf{x})\, dv = f(\mathbf{x}^*)\,\mathrm{vol}(\mathcal{P})$.

where the superscript has been dropped, since, now, $\mathbf{x}^* = \mathbf{x}$.

With the preceding development in place, define the tensor $\mathbf{T}$ as

$$\mathbf{T} \;=\; \mathbf{t}_i \otimes \mathbf{e}_i \;, \tag{4.75}$$

so that, when operating on $\mathbf{n}$,

$$\mathbf{T}\mathbf{n} \;=\; (\mathbf{t}_i \otimes \mathbf{e}_i)\mathbf{n} \;=\; \mathbf{t}_i(\mathbf{e}_i \cdot \mathbf{n}) = \mathbf{t}_i n_i \;=\; \mathbf{t}_{(\mathbf{n})} \;, \tag{4.76}$$

as seen from (4.74). The tensor $\mathbf{T}$ is called the *Cauchy stress* tensor.

It can be readily seen from (4.75) that

$$\mathbf{T} \;=\; T_{ki}\mathbf{e}_k \otimes \mathbf{e}_i \;=\; \mathbf{t}_i \otimes \mathbf{e}_i \;, \tag{4.77}$$

hence

$$\mathbf{t}_i \;=\; T_{ki}\mathbf{e}_k \;. \tag{4.78}$$

Also, since

$$\mathbf{t}_i \cdot \mathbf{e}_j \;=\; T_{ki}\mathbf{e}_k \cdot \mathbf{e}_j \;=\; T_{ji} \;, \tag{4.79}$$

it is immediately seen that

$$T_{ij} \;=\; \mathbf{e}_i \cdot \mathbf{t}_j \;. \tag{4.80}$$

Note that, from its definition in (4.75), it is clear that the Cauchy stress tensor does not depend on the normal $\mathbf{n}$.

Return now to the integral statement of linear momentum balance and, taking into account (4.76), apply the divergence theorem to the boundary integral term. This leads to

$$\begin{aligned} \int_{\mathcal{P}} \rho\mathbf{a}\, dv \;&=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \\ &=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n}\, da \\ &=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\mathcal{P}} \operatorname{div}\mathbf{T}\, dv \;. \end{aligned} \tag{4.81}$$

It follows from the preceding equation that the condition

$$\int_{\mathcal{P}} (\rho\mathbf{a} - \rho\mathbf{b} - \operatorname{div}\mathbf{T})\, dv \;=\; \mathbf{0} \;, \tag{4.82}$$

holds for an arbitrary area $\mathcal{P}$, which, with the aid of the localization theorem leads to a local form of linear momentum balance in the form

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} \;=\; \rho \mathbf{a} \; . \tag{4.83}$$

An alternative statement of linear momentum balance can be obtained by noting from (4.75) that

$$\begin{aligned} \int_{\mathcal{P}} \rho \mathbf{a} \, dv \;&=\; \int_{\mathcal{P}} \rho \mathbf{b} \, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})} \, da \\ &=\; \int_{\mathcal{P}} \rho \mathbf{b} \, dv + \int_{\partial \mathcal{P}} \mathbf{t}_i n_i \, da \\ &=\; \int_{\mathcal{P}} \rho \mathbf{b} \, dv + \int_{\mathcal{P}} \mathbf{t}_{i,i} \, dv \; . \end{aligned} \tag{4.84}$$

Again, appealing to the localization theorem, this leads to

$$\mathbf{t}_{i,i} + \rho \mathbf{b} \;=\; \rho \mathbf{a} \; . \tag{4.85}$$

Turn attention next to the balance of angular momentum and examine the boundary integral term in (4.49). This can be written as

$$\int_{\partial \mathcal{P}} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} \, da \;=\; \int_{\partial \mathcal{P}} \mathbf{x} \times \mathbf{t}_i n_i \, da \;=\; \int_{\mathcal{P}} (\mathbf{x} \times \mathbf{t}_i)_{,i} \, dv \;=\; \int_{\mathcal{P}} (\mathbf{x}_{,i} \times \mathbf{t}_i + \mathbf{x} \times \mathbf{t}_{i,i}) \, dv \; . \tag{4.86}$$

Substituting the preceding equation into (4.49) yields

$$\int_{\mathcal{P}} (\mathbf{x} \times \rho \mathbf{a}) \, dv \;=\; \int_{\mathcal{P}} (\mathbf{x} \times \rho \mathbf{b}) \, dv + \int_{\mathcal{P}} (\mathbf{x}_{,i} \times \mathbf{t}_i + \mathbf{x} \times \mathbf{t}_{i,i}) \, dv \tag{4.87}$$

or, upon rearranging the terms,

$$\int_{\mathcal{P}} [\mathbf{x} \times (\rho \mathbf{a} - \rho \mathbf{b} - \mathbf{t}_{i,i}) + \mathbf{x}_{,i} \times \mathbf{t}_i] \; dv \;=\; \mathbf{0} \; . \tag{4.88}$$

Recalling the local form of linear momentum balance equation (4.85), the above equation reduces to

$$\int_{\mathcal{P}} \mathbf{x}_{,i} \times \mathbf{t}_i \, dv \;=\; \mathbf{0} \; . \tag{4.89}$$

The localization theorem can be invoked again to conclude that

$$\mathbf{x}_{,i} \times \mathbf{t}_i \;=\; \mathbf{0} \tag{4.90}$$

or

$$\mathbf{e}_i \times \mathbf{t}_i \;=\; \mathbf{0}\;. \tag{4.91}$$

In component form, this condition can be expressed as

$$\mathbf{e}_i \times (T_{ji}\mathbf{e}_j) \;=\; T_{ji}\mathbf{e}_i \times \mathbf{e}_j \;=\; T_{ji}e_{ijk}\mathbf{e}_k \;=\; \mathbf{0}\;, \tag{4.92}$$

which means that $T_{ij} = T_{ji}$, i.e., the Cauchy stress tensor is symmetric (or, said differently, the skew-symmetric tensor $\mathbf{T} - \mathbf{T}^T$ vanishes identically). Hence, angular momentum balance restricts the Cauchy stress tensor to be symmetric.

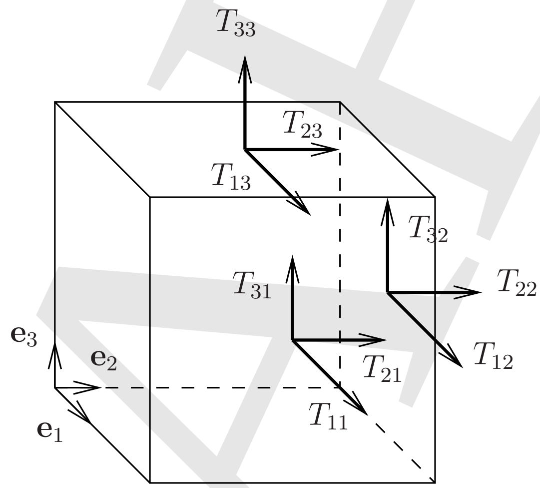

Figure 4.6: *Interpretation of the Cauchy stress components on an orthogonal parallelepiped aligned with the* $\{\mathbf{e}_i\}$*-axes.*

An interpretation of the components of $\mathbf{T}$ on an orthogonal parallelepiped is shown in Figure 4.6. Indeed, recalling (4.78), it follows that

$$\mathbf{t}_1 \;=\; T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3\;, \tag{4.93}$$

which means that T_{i1} is the i-th component of the traction that acts on the plane with outward unit normal $\mathbf{e}_1$. More generally, T_{ij} is the i-th component of the traction that acts on the plane with outward unit normal $\mathbf{e}_j$. The components T_{ij} of the Cauchy stress tensor can be put in matrix form as

$$[T_{ij}] \;=\; \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\;, \tag{4.94}$$

where $[T_{ij}]$ is symmetric. The linear eigenvalue problem

$$(\mathbf{T} - T\mathbf{i})\mathbf{n} \;=\; \mathbf{0} \tag{4.95}$$

yields three real eigenvalues $T_1 \geq T_2 \geq T_3$, which are solutions of the characteristic polynomial equation

$$T^3 - I_T T^2 + II_T T - III_T \;=\; 0 \;, \tag{4.96}$$

where I_T, II_T and III_T are the three principal invariants of the tensor $\mathbf{T}$, defined as

$$\begin{aligned} I_T &= \operatorname{tr}(\mathbf{T}) \\ II_T &= \frac{1}{2}[(\operatorname{tr}(\mathbf{T}))^2 - \operatorname{tr}\mathbf{T}^2] \qquad (4.97) \\ III_T &= \frac{1}{6}[(\operatorname{tr}(\mathbf{T}))^3 - 3\operatorname{tr}\mathbf{T}(\operatorname{tr}\mathbf{T}^2) + 2\operatorname{tr}\mathbf{T}^3] \;=\; \det\mathbf{T}\;. \qquad (4.98) \end{aligned}$$

As is well-known, the associated unit eigenvectors $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}$ and $\mathbf{n}^{(3)}$ of $\mathbf{T}$ are mutually orthogonal provided the eigenvalues are distinct. Also, whether the eigenvalues are distinct or not, there exists a set of mutually orthogonal eigenvectors for $\mathbf{T}$.

The traction vector $\mathbf{t}_{(\mathbf{n})}$ can be generally decomposed into normal and shearing components on the plane of its action. To this end, the *normal traction* (i.e., the projection of $\mathbf{t}_{(\mathbf{n})}$ along $\mathbf{n}$) is given by

$$(\mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n})\mathbf{n} \;=\; (\mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;, \tag{4.99}$$

as in Figure 4.7. Then, the *shearing traction* is equal to

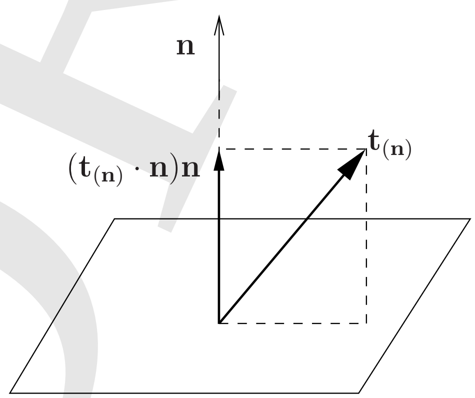

Figure 4.7: *Projection of the traction to its normal and tangential components.*

$$\mathbf{t}_{(\mathbf{n})} - (\mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n})\mathbf{n} \;=\; \mathbf{t}_{(\mathbf{n})} - (\mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;=\; (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \; . \tag{4.100}$$

If $\mathbf{n}$ is a principal direction of $\mathbf{T}$, equations (4.95) and (4.100) imply that

$$(\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;=\; (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{T}\mathbf{n} \;=\; (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})T\mathbf{n} \;=\; \mathbf{0} \; , \tag{4.101}$$

namely that the shearing traction vanishes on the plane with unit normal $\mathbf{n}$.

Next, consider three special forms of the stress tensor that lead to equilibrium states in the absence of body forces.

(a) Hydrostatic pressure

In this state, the stress vector is always pointing in the direction normal to any plane that it is acting on, i.e.,

$$\mathbf{t}_{(\mathbf{n})} \;=\; -p\mathbf{n} \; , \tag{4.102}$$

where p is called the *pressure*. It follows from (4.76) that

$$\mathbf{T} \;=\; -p\mathbf{i} \; . \tag{4.103}$$

(b) Pure tension along the $\mathbf{e}$-axis

Without loss of generality, let $\mathbf{e} = \mathbf{e}_1$. In this case, the traction vectors $\mathbf{t}_i$ are of the form

$$\mathbf{t}_1 \;=\; T\mathbf{e}_1 \quad , \quad \mathbf{t}_2 \;=\; \mathbf{t}_3 \;=\; \mathbf{0} \; . \tag{4.104}$$

Then, it follows from (4.78) that

$$\mathbf{T} \;=\; T(\mathbf{e}_1 \otimes \mathbf{e}_1) \;=\; T(\mathbf{e} \otimes \mathbf{e}) \; . \tag{4.105}$$

(c) Pure shear on the $(\mathbf{e}, \mathbf{k})$-plane

Here, let $\mathbf{e}$ and $\mathbf{k}$ be two orthogonal vectors of unit magnitude and, without loss of generality, set $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{k}$. The tractions $\mathbf{t}_i$ are now given by

$$\mathbf{t}_1 \;=\; T\mathbf{e}_2 \quad , \quad \mathbf{t}_2 \;=\; T\mathbf{e}_1 \quad , \quad \mathbf{t}_3 \;=\; \mathbf{0} \; . \tag{4.106}$$

Appealing, again, to (4.78), it is easily seen that

$$\mathbf{T} \;=\; T(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \;=\; T(\mathbf{e} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{e}) \; . \tag{4.107}$$

It is possible to resolve the stress vector acting on a surface of the current configuration using the geometry of the reference configuration. This is conceivable when, for example, one wishes to measure the internal forces developed in the current configuration per unit area of the reference configuration. To this end, start by letting $d\mathbf{f}$ be the total force acting on the differential area da with outward unit normal $\mathbf{n}$ on the surface ∂P in the current configuration, i.e.,

$$d\mathbf{f} \;=\; \mathbf{t}_{(\mathbf{n})} da \; . \tag{4.108}$$

Also, let dA be the image of da in the reference configuration under $\boldsymbol{\chi}_t^{-1}$ and assume that its outward unit is $\mathbf{N}$. Then, define $\mathbf{p}_{(\mathbf{N})}$ to be the traction vector resulting from resolving the force $d\mathbf{f}$, which acts on ∂P, on the surface ∂P_0, namely,

$$d\mathbf{f} \;=\; \mathbf{p}_{(\mathbf{N})} dA \; . \tag{4.109}$$

Clearly, $\mathbf{t}$ and $\mathbf{p}$ are parallel, since they are both parallel to $d\mathbf{f}$.

Returning to the integral statement of linear momentum balance in (4.45), note that this can be now readily "pulled-back" to the reference configuration, hence taking the form

$$\int_{\mathcal{P}_0} \rho_0 \mathbf{a} \, dV \;=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b} \, dV + \int_{\partial \mathcal{P}_0} \mathbf{p}_{(\mathbf{N})} \, dA \; . \tag{4.110}$$

Upon applying the Cauchy tetrahedron argument to $\mathcal{P}_0$, it is immediately concluded that

$$\mathbf{p}_{(\mathbf{N})} \;=\; \mathbf{p}_A N_A \; , \tag{4.111}$$

where $\mathbf{p}_A$ are the tractions developed in the current configuration, but resolved on the geometry of the reference configuration on surfaces with outward unit normals $\mathbf{E}_A$. Further, let the tensor $\mathbf{P}$ be defined as

$$\mathbf{P} \;=\; \mathbf{p}_A \otimes \mathbf{E}_A \; , \tag{4.112}$$

so that, when operating on $\mathbf{N}$,

$$\mathbf{P}\mathbf{N} \;=\; (\mathbf{p}_A \otimes \mathbf{E}_A)\mathbf{N} \;=\; \mathbf{p}_A N_A \; , \tag{4.113}$$

thus, from (4.111), it is seen that

$$\mathbf{p} \;=\; \mathbf{P}\mathbf{N} \; . \tag{4.114}$$

The tensor $\mathbf{P}$ is called the *first Piola-Kirchhoff stress* tensor and it is naturally unsymmetric, since it has a mixed basis, i.e.,

$$\mathbf{P} \;=\; P_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \; . \tag{4.115}$$

It follows from (4.112) that

$$\mathbf{P} \;=\; \mathbf{p}_A \otimes \mathbf{E}_A \;=\; P_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \; , \tag{4.116}$$

which implies that

$$\mathbf{p}_A \;=\; P_{iA}\mathbf{e}_i \; . \tag{4.117}$$

Further, since

$$\mathbf{p}_A \cdot \mathbf{e}_j \;=\; P_{iA}\mathbf{e}_i \cdot \mathbf{e}_j \;=\; P_{jA} \; , \tag{4.118}$$

it is clear that

$$P_{iA} \;=\; \mathbf{e}_i \cdot \mathbf{p}_A \; . \tag{4.119}$$

Turning attention to the integral statement (4.110), it is concluded with the aid of (4.114) and the divergence theorem that

$$\begin{aligned} \int_{\mathcal{P}_0} \rho_0 \mathbf{a}\, dV \;&=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\, dA \\ &=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{P}\mathbf{N}\, dA \\ &=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} \operatorname{Div} \mathbf{P}\, dA \end{aligned} \tag{4.120}$$

which, upon using the localization theorem, results in

$$\rho_0 \mathbf{a} \;=\; \rho_0 \mathbf{b} + \operatorname{Div} \mathbf{P} \; , \tag{4.121}$$

This is the local form of linear momentum balance in the referential description.

Alternatively, equation (4.113) and the divergence theorem can be invoked to show that

$$\begin{aligned} \int_{\mathcal{P}_0} \rho_0 \mathbf{a}\, dV \;&=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\, dA \\ &=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_A N_A\, dA \\ &=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} \mathbf{p}_{A,A}\, dA \end{aligned} \tag{4.122}$$

from which another version of the referential statement of linear momentum balance can be derived in the form

$$\rho_0 \mathbf{a} \;=\; \rho_0 \mathbf{b} + \mathbf{p}_{A,A} \;, \tag{4.123}$$

Starting from the integral form of angular momentum balance in (4.49) and pulling it back to the reference configuration, one finds that

$$\int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{a}\, dV \;=\; \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\partial \mathcal{P}_0} \mathbf{x} \times \mathbf{p}_{(\mathbf{N})}\, dA \;. \tag{4.124}$$

Using (4.113) and the divergence theorem on the boundary term gives rise to

$$\begin{aligned} \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{a}\, dV \;&=\; \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\partial \mathcal{P}_0} \mathbf{x} \times \mathbf{p}_A N_A\, dA \\ &=\; \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} (\mathbf{x} \times \mathbf{p}_A)_{,A}\, dV \;. \end{aligned} \tag{4.125}$$

Expanding and appropriately rearranging the terms of the above equation leads to

$$\int_{\mathcal{P}_0} \left[\mathbf{x} \times (\rho_0 \mathbf{a} - \rho_0 \mathbf{b} - \mathbf{P}_{A,A}) + \mathbf{x}_{,A} \times \mathbf{p}_A \right] dV \;=\; \mathbf{0} \;. \tag{4.126}$$

Appealing to the local form of linear momentum balance in (4.123) and, subsequently, the localization theorem, one concludes that

$$\mathbf{x}_{,A} \times \mathbf{p}_A \;=\; \mathbf{0} \;. \tag{4.127}$$

With the aid of (4.117) and the chain rule, the preceding equation can be rewritten as

$$\mathbf{x}_{,A} \times \mathbf{p}_A \;=\; F_{iA} \times P_{jA} \mathbf{e}_i \times \mathbf{e}_j \;=\; F_{iA} P_{jA} e_{ijk} \mathbf{e}_k \;=\; \mathbf{0} \;, \tag{4.128}$$

which implies that $\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T$. This is a local form of angular momentum balance in the referential description.

Recalling (4.108) and (4.108), one may conclude with the aid of (4.76), (4.114) and Nanson's formula (3.122) that

$$\mathbf{P}\mathbf{N} dA \;=\; \mathbf{T}\mathbf{n} da \;=\; \mathbf{T} J \mathbf{F}^{-T} \mathbf{N} dA \;, \tag{4.129}$$

so that

$$\mathbf{T} \;=\; \frac{1}{J} \mathbf{P}\mathbf{F}^T \;. \tag{4.130}$$

Clearly, the above relation is consistent with the referential and spatial statements of angular momentum balance, namely (4.130) can be used to derive the local form of angular momentum balance in spatial form from the referential statement and vice-versa. Likewise, it is possible to derive the local linear momentum balance statement in the referential (resp. spatial) form from its corresponding spatial (resp. referential) counterpart.

Note that there is no approximation or any other source of error associated with the use the balance laws in the referential vs. spatial description. Indeed, the invertibility of the motion at any fixed time implies that both forms of the balance laws are completely equivalent.

Other stress tensors beyond the Cauchy and first Piola-Kirchhoff tensors are frequently used. Among them, the most important are the *nominal* stress tensor $\mathbf{\Pi}$ defined as the transpose of the first Piola-Kirchhoff stress, i.e.,

$$\mathbf{\Pi} \;=\; \mathbf{P}^T \;=\; J\mathbf{F}^{-1}\mathbf{T}\;, \tag{4.131}$$

the *Kirchhoff stress* tensor $\boldsymbol{\tau}$, defined as

$$\boldsymbol{\tau} \;=\; J\mathbf{T} \;=\; \mathbf{P}\mathbf{F}^T\;, \tag{4.132}$$

and the *second Piola-Kirchhoff stress* $\mathbf{S}$, defined as

$$\mathbf{S} \;=\; \mathbf{F}^{-1}\mathbf{P} \;=\; J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}\;. \tag{4.133}$$

It is clear from (4.133) that $\mathbf{S}$ has both "legs" in the reference configuration and is symmetric.

All the stress measures defined here are obtainable from one another assuming that the deformation is known.

4.8 The transformation of mechanical fields under superposed rigid-body motions

In this section, the transformation under superposed rigid motions is considered for mechanical fields, such as the stress vectors and tensors. To this end, start with the stress vector $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$, and recalling the general form of the superposed rigid motion in (3.166), write

$\mathbf{t}^+ = \mathbf{t}^+(\mathbf{x}^+, t; \mathbf{n}^+)$. To argue how $\mathbf{t}$ and $\mathbf{t}^+$ are related, first recall that $\mathbf{n}^+ = \mathbf{Q}\mathbf{n}$ and that $\mathbf{t}$ is linear in $\mathbf{n}$, as established in (4.76). Since the two motions give rise to the same deformation, it is then reasonable to *assume* that, under a superposed rigid motion, $\mathbf{t}^+$ will not change in magnitude relative to $\mathbf{t}$ and will have the same orientation relative to $\mathbf{n}^+$ as $\mathbf{t}$ has relative to $\mathbf{n}$. Therefore, it is postulated that

$$\mathbf{t}^+ \;=\; \mathbf{Q}\mathbf{t} \; . \tag{4.134}$$

The above transformation indeed implies that $|\mathbf{t}^+| = |\mathbf{t}|$ and $\mathbf{t}^+ \cdot \mathbf{n}^+ = \mathbf{t} \cdot \mathbf{n}$.

Unlike the transformation of kinematic terms, which is governed purely by geometry, in the case of kinetic terms (both mechanical and thermal) the transformation is governed by the principle of *invariance under superposed rigid motion*. This, effectively, states that the balance laws are invariant under superposed rigid motions, in the sense that their mathematical representation remains unchanged under such motions. To demonstrate this principle, consider the transformation of the Cauchy stress tensor. Taking taking into account (4.76), invariance under superposed rigid motions implies that $\mathbf{t}^+ = \mathbf{T}^+\mathbf{n}^+$. This means that if two motions differ by a mere rigid motion, then the relation between the stress vector $\mathbf{t}$ on a plane with an outward unit normal $\mathbf{n}$ and the Cauchy stress tensor $\mathbf{T}$ is the same with the relation of the stress vector $\mathbf{t}^+$ on a plane with outward unit normal $\mathbf{n}^+$ with the Cauchy stress tensor $\mathbf{T}^+$. Invariance of (4.76) under superposed rigid motions in conjunction with (4.134) implies that

$$\begin{aligned} \mathbf{t}^+ &= \mathbf{Q}\mathbf{t} \;=\; \mathbf{Q}\mathbf{T}\mathbf{n} \\ &= \mathbf{T}^+\mathbf{n}^+ \;=\; \mathbf{T}^+\mathbf{Q}\mathbf{n} \; , \end{aligned} \tag{4.135}$$

from where it is concluded that

$$(\mathbf{Q}\mathbf{T} - \mathbf{T}^+\mathbf{Q})\mathbf{n} \;=\; \mathbf{0} \; , \tag{4.136}$$

hence

$$\mathbf{T}^+ \;=\; \mathbf{Q}\mathbf{T}\mathbf{Q}^T \; . \tag{4.137}$$

Equation (4.137) implies that $\mathbf{T}$ is an objective spatial tensor.

Recall next the relation between the Cauchy and the first Piola-Kirchhoff stress tensor in (4.130). Given that this relation is itself invariant under superposed rigid motions it follows that

$$\mathbf{P}^+ \;=\; J^+\mathbf{T}^+(\mathbf{F}^{-T})^+ \;=\; J(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)(\mathbf{Q}\mathbf{F}^{-T}) \;=\; \mathbf{Q}(J\mathbf{T}\mathbf{F}^{-T}) \;=\; \mathbf{Q}\mathbf{P}\;, \tag{4.138}$$

where the kinematic transformations (3.165) and (3.187) are employed.[3] Equation (4.138) implies that $\mathbf{P}$ is an objective two-point tensor. Similarly, turning to the second Piola-Kirchhoff stress tensor, it follows from (4.133) that

$$\begin{aligned}\mathbf{S}^+ \;=\; J^+(\mathbf{F}^{-1})^+\mathbf{T}^+(\mathbf{F}^{-T})^+ \;&=\; J(\mathbf{F}^{-1}\mathbf{Q}^T)(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)(\mathbf{Q}\mathbf{F}^{-T}) \\ &=\; J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \;=\; \mathbf{S}\;,\end{aligned} \tag{4.139}$$

which implies that $\mathbf{S}$ is an objective referential tensor.

Since (4.139) holds true, it follows also that

$$\dot{\mathbf{S}}^+ \;=\; \dot{\mathbf{S}}\;, \tag{4.140}$$

namely that the rate of $\mathbf{S}$ is objective. However, from (4.137) and the relation (3.180) it can be seen that

$$\begin{aligned}\dot{\mathbf{T}}^+ &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T \\ &= (\boldsymbol{\Omega}\mathbf{Q})\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}(\boldsymbol{\Omega}\mathbf{Q})^T \\ &= \boldsymbol{\Omega}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + (\mathbf{Q}\mathbf{T}\mathbf{Q})^T\boldsymbol{\Omega}^T \\ &= \boldsymbol{\Omega}\mathbf{T}^+ + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T - \mathbf{T}^+\boldsymbol{\Omega}\;,\end{aligned} \tag{4.141}$$

which shows that, unlike $\mathbf{T}$, the spatial tensor $\dot{\mathbf{T}}$ is not objective. A similar conclusion may be drawn for the rate $\dot{\mathbf{P}}$ of the first Piola-Kirchhoff stress tensor.

Invariance under superposed rigid motions applies to the principle of mass conservation. Indeed, using the local referential form (4.41) of this principle gives rise to

$$\begin{aligned}\rho_0 &= \rho^+J^+ \;=\; \rho^+J \\ &= \rho J\;,\end{aligned} \tag{4.142}$$

which imply that

$$\rho^+ \;=\; \rho\;. \tag{4.143}$$

[3] Also, note that, by its definition, $(\mathbf{F}^{-1})^+ = (\mathbf{F}^+)^{-1}$.

Finally, admitting invariance under superposed rigid motions of the local spatial form (4.83) of linear momentum balance leads to

$$\operatorname{div}^+ \mathbf{T}^+ + \rho^+ \mathbf{b}^+ \;=\; \rho^+ \mathbf{a}^+ \; . \tag{4.144}$$

Resorting to components, note that

$$\begin{aligned}
\frac{\partial T^+_{ij}}{\partial x^+_j} &= \frac{\partial (Q_{ik} T_{kl} Q_{jl})}{\partial x_m} \frac{\partial \chi_m}{\partial x^+_j} \\
&= Q_{ik} \frac{\partial T_{kl}}{\partial x_m} Q_{jl} Q_{jm} \\
&= Q_{ik} \frac{\partial T_{kl}}{\partial x_m} \delta_{lm} \\
&= Q_{ik} \frac{\partial T_{kl}}{\partial x_l} \; ,
\end{aligned} \tag{4.145}$$

where it is recognized from (3.166) that $\dfrac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}^+} = \mathbf{Q}^T$, therefore, in components, $\dfrac{\partial x_m}{\partial x^+_j} = Q_{jm}$. Equation (4.145) can be written using direct notation as

$$\operatorname{div}^+ \mathbf{T}^+ \;=\; \mathbf{Q} \operatorname{div} \mathbf{T} \; . \tag{4.146}$$

Using (4.83), (4.144) and (4.146), one concludes that

$$\begin{aligned}
\operatorname{div}^+ \mathbf{T}^+ &= \rho^+ (\mathbf{a}^+ - \mathbf{b}^+) \;=\; \rho(\mathbf{a}^+ - \mathbf{b}^+) \\
&= \mathbf{Q} \operatorname{div} \mathbf{T} \;=\; \mathbf{Q} \rho (\mathbf{a} - \mathbf{b}) \; ,
\end{aligned}$$

from where it follows that

$$\mathbf{a}^+ - \mathbf{b}^+ \;=\; \mathbf{Q}(\mathbf{a} - \mathbf{b}) \; . \tag{4.147}$$

This means that, under superposed rigid motions, the body forces transform as

$$\mathbf{b}^+ \;=\; \mathbf{Q}\mathbf{b} + \mathbf{a}^+ - \mathbf{Q}\mathbf{a} \; . \tag{4.148}$$

4.9 The Theorem of Mechanical Energy Balance

Consider again the body $\mathcal{B}$ in the current configuration $\mathcal{R}$ and take an arbitrary material region $\mathcal{P}$ with smooth boundary $\partial \mathcal{P}$.

Recalling the integral statement of linear momentum balance (4.43), define the rate at which the external body forces $\mathbf{b}$ and surface tractions $\mathbf{t}$ do work in $\mathcal{P}$ and on $\partial\mathcal{P}$, respectively, as

$$R_b(\mathcal{P}) \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv \tag{4.149}$$

and

$$R_c(\mathcal{P}) \;=\; \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\,da\ , \tag{4.150}$$

respectively. Also, define the rate of work done by all external forces as

$$R(\mathcal{P}) \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P})\ . \tag{4.151}$$

In addition, define the total kinetic energy of the material points contained in $\mathcal{P}$ as

$$K(\mathcal{P}) \;=\; \int_{\mathcal{P}} \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho\,dv\ . \tag{4.152}$$

Starting from the local spatial form of linear momentum balance (4.83), one may dot both sides with the velocity $\mathbf{v}$ to obtain

$$\rho\dot{\mathbf{v}}\cdot\mathbf{v} \;=\; \rho\mathbf{b}\cdot\mathbf{v} + \operatorname{div}\mathbf{T}\cdot\mathbf{v}\ . \tag{4.153}$$

Now, note that

$$\begin{aligned}(\operatorname{div}\mathbf{T})\cdot\mathbf{v} \;&=\; \operatorname{div}(\mathbf{T}^T\mathbf{v}) - \mathbf{T}\cdot\operatorname{grad}\mathbf{v}\\ &=\; \operatorname{div}(\mathbf{T}\mathbf{v}) - \mathbf{T}\cdot(\mathbf{D}+\mathbf{W})\\ &=\; \operatorname{div}(\mathbf{T}\mathbf{v}) - \mathbf{T}\cdot\mathbf{D}\ ,\end{aligned} \tag{4.154}$$

where angular momentum balance is invoked in the form of the symmetry of $\mathbf{T}$. Equation (4.154) can be used to rewrite (4.153) as

$$\frac{1}{2}\rho\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) \;=\; \rho\mathbf{b}\cdot\mathbf{v} + \operatorname{div}(\mathbf{T}\mathbf{v}) - \mathbf{T}\cdot\mathbf{D}\ . \tag{4.155}$$

Integrating (4.155) over $\mathcal{P}$ leads to

$$\int_{\mathcal{P}} \frac{1}{2}\rho\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\mathcal{P}} \operatorname{div}(\mathbf{T}\mathbf{v})\,dv - \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv\ . \tag{4.156}$$

or, upon using conservation of mass and the divergence theorem,

$$\frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\,dv + \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\mathcal{P}} (\mathbf{T}\mathbf{v})\cdot\mathbf{n}\,da\ . \tag{4.157}$$

Recalling (4.76), the preceding equation can be rewritten as

$$\frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\,dv + \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\,da\;. \tag{4.158}$$

The second term on the right-hand side of (4.158),

$$S(\mathcal{P}) \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv\;, \tag{4.159}$$

is called the *stress power* and it represents the rate at which the stresses do work in $\mathcal{P}$. Taking into account (4.149), (4.150), (4.152), (4.159) and (4.158), it is seen that

$$\frac{d}{dt}K(\mathcal{P}) + S(\mathcal{P}) \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P}) \;=\; R(\mathcal{P})\;. \tag{4.160}$$

Equation (4.160) (or, equivalently, equation (4.158)) states that, for any region $\mathcal{P}$, the rate of change of the kinetic energy and the stress power are balanced by the rate of work done by the external forces. This is a statement of the *mechanical energy balance theorem*. It is important to emphasize that mechanical energy balance is derivable from the three basic principles of the mechanical theory, namely conservation of mass and balance of linear and angular momentum.

Returning to the stress power term $S(\mathcal{P})$, note that

$$\begin{aligned}
\int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv \;&=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{L}\,dv \\
&=\; \int_{\mathcal{P}} \left(\frac{1}{J}\mathbf{P}\mathbf{F}^T\right)\cdot\mathbf{L}\,dv \\
&=\; \int_{\mathcal{P}_0} (\mathbf{P}\mathbf{F}^T)\cdot\mathbf{L}\,dV \\
&=\; \int_{\mathcal{P}_0} \mathrm{tr}[(\mathbf{P}\mathbf{F}^T)\mathbf{L}^T]\,dV \\
&=\; \int_{\mathcal{P}_0} \mathrm{tr}[\mathbf{P}(\mathbf{L}\mathbf{F})^T]\,dV \\
&=\; \int_{\mathcal{P}_0} \mathrm{tr}[\mathbf{P}\dot{\mathbf{F}}^T]dV \\
&=\; \int_{\mathcal{P}_0} \mathbf{P}\cdot\dot{\mathbf{F}}\,dV\;,
\end{aligned} \tag{4.161}$$

where use is made of (4.130) and (3.128). Further,

$$
\begin{aligned}
\int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv &= \int_{\mathcal{P}_0} \mathbf{P}\cdot\dot{\mathbf{F}}\,dV \\
&= \int_{\mathcal{P}_0} (\mathbf{F}\mathbf{S})\cdot\dot{\mathbf{F}}\,dV \\
&= \int_{\mathcal{P}_0} \mathbf{S}\cdot(\mathbf{F}^T\dot{\mathbf{F}})\,dV \\
&= \int_{\mathcal{P}_0} \mathbf{S}\cdot\frac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F}+\mathbf{F}^T\dot{\mathbf{F}})\,dV \\
&= \int_{\mathcal{P}_0} \mathbf{S}\cdot\dot{\mathbf{E}}\,dV\ ,
\end{aligned}
\tag{4.162}
$$

where, this time use is made of (4.133) and the definition of the Lagrangian strain.

Equations (4.161) and (4.162) reveal that $\mathbf{P}$ is the *work-conjugate* kinetic measure to $\mathbf{F}$ in $\mathcal{P}_0$ and, likewise, $\mathbf{S}$ is work-conjugate to $\mathbf{E}$. These equations appear to leave open the question of work-conjugacy for $\mathbf{T}$.

4.10 The principle of energy balance

The physical principles postulated up to this point are incapable of modeling the interconvertibility of mechanical work and heat. In order to account for this class of (generally coupled) thermomechanical phenomena, one needs to introduce an additional primitive law known as *balance of energy*.

Preliminary to stating the balance of energy, define a scalar field $r = r(\mathbf{x}, t)$ called the *heat supply* per unit mass (or *specific heat supply*), which quantifies the rate at which heat is supplied (or absorbed) by the body. Also, define a scalar field $h = h(\mathbf{x}, t; \mathbf{n}) = h_{(\mathbf{n})}(\mathbf{x}, t)$ called the *heat flux* per unit area across a surface $\partial\mathcal{P}$ with outward unit normal $\mathbf{n}$. Now, given any region $\mathcal{P} \subseteq \mathcal{R}$, define the *total rate of heating* $H(\mathcal{P})$ as

$$
H(\mathcal{P}) = \int_{\mathcal{P}} \rho r\,dv - \int_{\partial\mathcal{P}} h\,da\ , \tag{4.163}
$$

where the negative sign in front of the boundary integral signifies the fact that the heat flux is assumed positive when it exits the region $\mathcal{P}$.

Next, assume the existence of a scalar function $\varepsilon = \varepsilon(\mathbf{x}, t)$ per unit mass, called the (specific) *internal energy*. This function quantifies all forms of energy stored in the body other than the kinetic energy. Examples of stored energy include strain energy (i.e., energy due to deformation), chemical energy, etc. The internal energy stored in $\mathcal{P}$ is denoted by

$$U(\mathcal{P}) \ = \ \int_{\mathcal{P}} \rho\varepsilon \, dv \ . \tag{4.164}$$

The principle of balance of energy is postulated in the form

$$\frac{d}{dt}\int_{\mathcal{P}} \left[\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho\varepsilon\right] dv \ = \ \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\, da + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h\, da \ . \tag{4.165}$$

This is also sometimes referred to as a statement of the "First Law of Thermodynamics". Equivalently, equation (4.165) can be written as

$$\frac{d}{dt}\left[K(\mathcal{P}) + U(\mathcal{P})\right] \ = \ R(\mathcal{P}) + H(\mathcal{P}) \ . \tag{4.166}$$

Subtracting (4.158) from (4.165) leads to a statement of *thermal energy balance* in the form

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon\, dv \ = \ \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h\, da \ . \tag{4.167}$$

Returning to the heat flux $h = h(\mathbf{x}, t; \mathbf{n})$, and note that one can apply a standard argument to find the exact dependence of h on $\mathbf{n}$, as already done with the stress vector $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$. Indeed, one may apply thermal energy balance to a region $\mathcal{P}$ with boundary $\partial\mathcal{P}$ and to each of two regions $\mathcal{P}_1$ and $\mathcal{P}_2$ with boundaries $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$, where $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}$ and $\mathcal{P}_1 \cup \mathcal{P}_2 = \emptyset$. Also, the boundaries $\partial\mathcal{P}_1 = \partial\mathcal{P}' \cup \sigma$, $\partial\mathcal{P}_2 = \partial\mathcal{P}'' \cup \sigma$ have a common surface σ and $\partial\mathcal{P}' \cup \partial\mathcal{P}'' = \partial\mathcal{P}$. It follows that

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon\, dv \ = \ \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h\, da \ . \tag{4.168}$$

and, also,

$$\frac{d}{dt}\int_{\mathcal{P}_1} \rho\varepsilon\, dv \ = \ \int_{\mathcal{P}_1} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_1} \rho r\, dv - \int_{\partial\mathcal{P}_1} h\, da \tag{4.169}$$

and

$$\frac{d}{dt}\int_{\mathcal{P}_2} \rho\varepsilon\, dv \ = \ \int_{\mathcal{P}_2} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_2} \rho r\, dv - \int_{\partial\mathcal{P}_2} h\, da \tag{4.170}$$

Adding the last two equations leads to

$$\frac{d}{dt}\int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\varepsilon\, dv \ = \ \int_{\mathcal{P}_1\cup\mathcal{P}_2} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho r\, dv - \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h\, da \tag{4.171}$$

or, equivalently,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D} \, dv + \int_{\mathcal{P}} \rho r \, dv - \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h \, da \; . \tag{4.172}$$

Subtracting (4.168) from (4.172) results in

$$\int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h \, da - \int_{\partial\mathcal{P}} h \, da = 0 \; , \tag{4.173}$$

or, equivalently,

$$\int_{\partial\mathcal{P}'\cup\sigma} h \, da + \int_{\partial\mathcal{P}''\cup\sigma} h \, da \;=\; \int_{\partial\mathcal{P}} h \, da \; . \tag{4.174}$$

As in the case of the stress vector, the preceding equation may be expanded to

$$\int_{\partial\mathcal{P}'\cup\partial\mathcal{P}''} h \, da + \int_{\sigma} h_{(\mathbf{n}_1)} \, da + \int_{\sigma} h_{(\mathbf{n}_2)} \, da \;=\; \int_{\partial\mathcal{P}} h \, da \tag{4.175}$$

or

$$\int_{\sigma} (h_{(\mathbf{n})} - h_{(-\mathbf{n})}) da \;=\; 0 \; , \tag{4.176}$$

where $\mathbf{n}_1 = \mathbf{n}$ and $\mathbf{n}_2 = -\mathbf{n}$. Since σ is an arbitrary surface and h is assumed to depend continuously on $\mathbf{n}$ and $\mathbf{x}$ along σ, the localization theorem yields the condition

$$h(\mathbf{n}) \;=\; -h(-\mathbf{n}) \; . \tag{4.177}$$

or, more explicitly,

$$h(\mathbf{x}, t; \mathbf{n}) \;=\; -h(\mathbf{x}, t; -\mathbf{n}) \; . \tag{4.178}$$

This states that the flux of heat exiting a body across a surface with outward unit normal $\mathbf{n}$ at a point $\mathbf{x}$ is equal to the flux of heat entering a neighboring body at the same point across the same surface.

Using the tetrahedron argument in connection with the thermal energy balance equation (4.167) and the flux continuity equation (4.178), gives rise to

$$h \;=\; h_i n_i \tag{4.179}$$

where h_i are the fluxes across the faces of the tetrahedron with outward unit normals $\mathbf{e}_i$. Thus, one may write

$$h \;=\; \mathbf{q}\cdot\mathbf{n} \; , \tag{4.180}$$

where $\mathbf{q}$ is the *heat flux vector* with components $q_i = h_i$.

Now, returning to the integral statement of energy balance in (4.165), use mass conservation to rewrite it as

$$\int_{\mathcal{P}} (\rho\mathbf{v}\cdot\dot{\mathbf{v}} + \rho\dot{\varepsilon})\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\,da + \int_{\mathcal{P}} \rho r\,dv - \int_{\partial\mathcal{P}} h\,da\;. \tag{4.181}$$

Using (4.76) and (4.180), the above equation may be put in the form

$$\int_{\mathcal{P}} (\rho\mathbf{v}\cdot\dot{\mathbf{v}} + \rho\dot{\varepsilon})\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\mathcal{P}} (\mathbf{T}\mathbf{n})\cdot\mathbf{v}\,da + \int_{\mathcal{P}} \rho r\,dv - \int_{\partial\mathcal{P}} \mathbf{q}\cdot\mathbf{n}\,da\;. \tag{4.182}$$

Upon invoking the divergence theorem, it is easily seen that

$$\int_{\partial\mathcal{P}} (\mathbf{T}\mathbf{n})\cdot\mathbf{v}\,da \;=\; \int_{\mathcal{P}} [(\operatorname{div}\mathbf{T})\cdot\mathbf{v} + \mathbf{T}\cdot\mathbf{D}]\,dv \tag{4.183}$$

and

$$\int_{\partial\mathcal{P}} \mathbf{q}\cdot\mathbf{n}\,da \;=\; \int_{\mathcal{P}} \operatorname{div}\mathbf{q}\,dv\;. \tag{4.184}$$

If the last two equations are substituted in (4.182), one finds that

$$\int_{\mathcal{P}} \big[\{\rho\dot{\mathbf{v}} - \rho\mathbf{b} - \operatorname{div}\mathbf{T}\}\cdot\mathbf{v} + \rho\dot{\varepsilon} - \mathbf{T}\cdot\mathbf{D} - \rho r + \operatorname{div}\mathbf{q}\big]\,dv \;=\; \mathbf{0}\;. \tag{4.185}$$

Upon observing linear momentum balance and invoking the localization theorem, the preceding equation gives rise to the local form of energy balance as

$$\rho\dot{\varepsilon} \;=\; \mathbf{T}\cdot\mathbf{D} + \rho r - \operatorname{div}\mathbf{q}\;. \tag{4.186}$$

This equation could be also derived along the same lines from the integral statement of thermal energy balance (4.167).[4]

Regarding invariance under superposed rigid-body motions, it is postulated that

$$\varepsilon^+ \;=\; \varepsilon \tag{4.187}$$

and

$$r^+ \;=\; r \quad , \quad \mathbf{q}^+ \;=\; \mathbf{Q}\mathbf{q}\;. \tag{4.188}$$

[4]The energy equation is frequently quoted in elementary thermodynamics texts as "$dU = \delta Q + \delta W$", where dU corresponds to $\rho\dot{\varepsilon}$, δQ to $\rho r - \operatorname{div}\mathbf{q}$, and δW to $\mathbf{T}\cdot\mathbf{D}$.

Equation $(4.188)_2$ and invariance of the thermal energy balance imply that

$$h^+ \;=\; \mathbf{q}^+ \cdot \mathbf{n}^+ \;=\; (\mathbf{Q}\mathbf{q}) \cdot (\mathbf{Q}\mathbf{n}) \;=\; \mathbf{q} \cdot \mathbf{n} = h \; . \tag{4.189}$$

Example: Consider a rigid heat conductor, where rigidity implies that $\mathbf{F} = \mathbf{R}$ (i.e., $\mathbf{U} = \mathbf{I}$). This means that

$$\begin{aligned} \dot{\mathbf{F}} \;&= \dot{\mathbf{R}} \;=\; \boldsymbol{\Omega}\mathbf{R} \quad (\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T) \\ &= \mathbf{L}\mathbf{F} \;=\; \mathbf{L}\mathbf{R} \; , \end{aligned} \tag{4.190}$$

which implies that $\mathbf{L} = \boldsymbol{\Omega}$, so that $\mathbf{D} = \mathbf{0}$. Further, assume that *Fourier's law* holds, namely that

$$\mathbf{q} \;=\; -k \operatorname{grad} T \; , \tag{4.191}$$

where T is the empirical temperature and $k > 0$ is the (isotropic) conductivity. These conditions imply that the balance of energy (4.186) reduces to

$$\rho\dot{\varepsilon} - \operatorname{div}(k \operatorname{grad} T) - \rho r \;=\; 0 \; . \tag{4.192}$$

Further, assume that the internal energy depends exclusively on T and that this dependence is linear, hence $\dfrac{d\varepsilon}{dt} = c$, where c is termed the *heat capacity*. It follows from (4.192) that

$$\rho c\dot{T} = \operatorname{div}(k \operatorname{grad} T) + \rho r \; , \tag{4.193}$$

which is the classical equation of transient heat conduction. □

4.11 The Green-Naghdi-Rivlin theorem

Assume that the principle of energy balance remains invariant under superposed rigid motions. With reference to (4.165), this means that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{P}^+} \left[\rho^+\varepsilon^+ + \frac{1}{2}\rho^+\mathbf{v}^+ \cdot \mathbf{v}^+\right] dv^+ \\ &\qquad = \int_{\mathcal{P}^+} \rho^+\mathbf{b}^+ \cdot \mathbf{v}^+ \, dv^+ + \int_{\partial\mathcal{P}^+} \mathbf{t}^+ \cdot \mathbf{v}^+ \, da^+ + \int_{\mathcal{P}^+} \rho^+ r^+ \, dv^+ - \int_{\partial\mathcal{P}^+} h^+ \, da^+ \; . \end{aligned} \tag{4.194}$$

Now, choose a special superposed rigid motion, which is a pure *rigid translation*, such that

$$\mathbf{Q} \;=\; \mathbf{I} \quad , \quad \dot{\mathbf{Q}} \;=\; \mathbf{0} \quad , \quad \mathbf{c} \;=\; \mathbf{c}_0 t \; , \tag{4.195}$$

where $\mathbf{c}_0$ is a constant vector in E^3. It follows immediately from (3.178) and (4.195) that

$$\mathbf{v}^+ = \mathbf{v} + \mathbf{c}_0 \quad , \quad \dot{\mathbf{v}}^+ = \dot{\mathbf{v}} . \tag{4.196}$$

Moreover, it is readily concluded from (4.148), (4.134), (4.195) and (4.196) that under this superposed rigid translation

$$\mathbf{b}^+ = \mathbf{b} \quad , \quad \mathbf{t}^+ = \mathbf{t} . \tag{4.197}$$

It follows from (4.143), (4.196) and (4.197) that (4.194) takes the form

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}}\big[\rho\varepsilon + \frac{1}{2}\rho(\mathbf{v}+\mathbf{c}_0)\cdot(\mathbf{v}+\mathbf{c}_0)\big]\,dv \\ = \int_{\mathcal{P}}\rho\mathbf{b}\cdot(\mathbf{v}+\mathbf{c}_0)\,dv + \int_{\partial\mathcal{P}}\mathbf{t}\cdot(\mathbf{v}+\mathbf{c}_0)\,da + \int_{\mathcal{P}}\rho r\,dv - \int_{\partial\mathcal{P}}h\,da .\end{aligned} \tag{4.198}$$

Upon subtracting (4.165) from (4.198), it is concluded that

$$\mathbf{c}_0\cdot\Big[\frac{d}{dt}\int_{\mathcal{P}}\rho\mathbf{v}\,dv - \int_{\mathcal{P}}\rho\mathbf{b}\,dv - \int_{\partial\mathcal{P}}\mathbf{t}\,da\Big] + \frac{1}{2}(\mathbf{c}_0\cdot\mathbf{c}_0)\Big[\frac{d}{dt}\int_{\mathcal{P}}\rho\,dv\Big] = 0 . \tag{4.199}$$

Since $\mathbf{c}_0$ is an arbitrary constant vector, one may set $\mathbf{c}_0 = \gamma\bar{\mathbf{c}}_0$, where γ is an arbitrary non-vanishing scalar constant. Substituting γ with $-\gamma$ in (4.199), it is proved that

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\,dv = 0 \tag{4.200}$$

hence, also,

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\mathbf{v}\,dv = \int_{\mathcal{P}}\rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}}\mathbf{t}\,da . \tag{4.201}$$

This, in turn, means that conservation of mass and balance of linear momentum hold, respectively.

Next, a second special superposed rigid-body motion is chosen, such that

$$\mathbf{Q} = \mathbf{I} \quad , \quad \dot{\mathbf{Q}} = \mathbf{\Omega}_0 \quad , \quad \mathbf{c} = \mathbf{0} , \tag{4.202}$$

where $\mathbf{\Omega}_0$ is a constant skew-symmetric tensor. Given (4.202), it can be easily seen from (3.166) and (3.178) that

$$\mathbf{v}^+ = \mathbf{v} + \mathbf{\Omega}_0\mathbf{x} = \mathbf{v} + \boldsymbol{\omega}_0\times\mathbf{x} \tag{4.203}$$

and

$$\dot{\mathbf{v}}^+ = \dot{\mathbf{v}} + 2\mathbf{\Omega}_0\mathbf{v} + \mathbf{\Omega}_0^2\mathbf{x} = \dot{\mathbf{v}} + \boldsymbol{\omega}_0\times(2\mathbf{v}+\boldsymbol{\omega}_0\times\mathbf{x}) , \tag{4.204}$$

where $\boldsymbol{\omega}_0$ is the (constant) axial vector of $\boldsymbol{\Omega}_0$. Equations (4.203) and (4.204) imply that the superposed motion is a *rigid rotation* with constant angular velocity $\boldsymbol{\omega}_0$ on the original current configuration of the continuum. Taking into account (4.134) (4.148), (4.202) and (4.204), it is established that in this case

$$\mathbf{b}^+ \;=\; \mathbf{b} + 2\boldsymbol{\Omega}_0\mathbf{v} + \boldsymbol{\Omega}_0^2\mathbf{x} \;=\; \mathbf{b} + \boldsymbol{\omega}_0 \times (2\mathbf{v} + \boldsymbol{\omega}_0 \times \mathbf{x}) \tag{4.205}$$

and

$$\mathbf{t}^+ \;=\; \mathbf{t} \; . \tag{4.206}$$

In addition, equations $(4.203)_1$ and $(4.205)_1$ lead to

$$\mathbf{v}^+ \cdot \mathbf{v}^+ \;=\; \mathbf{v}\cdot\mathbf{v} + 2(\boldsymbol{\Omega}_0\mathbf{x})\cdot\mathbf{v} + (\boldsymbol{\Omega}_0\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) \tag{4.207}$$

and

$$\begin{aligned}
\mathbf{b}^+ \cdot \mathbf{v}^+ \;&=\; \mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot(\boldsymbol{\Omega}\mathbf{x}) + 2(\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}\mathbf{x}) + 2(\boldsymbol{\Omega}_0\mathbf{v})\cdot\mathbf{v} \\
&\qquad + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot\mathbf{v} \\
&=\; \mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot(\boldsymbol{\Omega}\mathbf{x}) + (\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) \; ,
\end{aligned} \tag{4.208}$$

where the readily verifiable identities

$$(\boldsymbol{\Omega}_0\mathbf{v})\cdot\mathbf{v} \;=\; 0 \qquad , \qquad (\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot\mathbf{v} \;=\; 0 \tag{4.209}$$

are employed. Similarly, using $(4.203)_1$ and $(4.205)_1$, it is seen that

$$\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}(\mathbf{v}^+\cdot\mathbf{v}^+) \;&=\; \dot{\mathbf{v}}\cdot\mathbf{v} \;+\; \dot{\mathbf{v}}\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + 2(\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + 2(\boldsymbol{\Omega}_0\mathbf{v})\cdot\mathbf{v} \\
&\qquad + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot\mathbf{v} \\
&=\; \dot{\mathbf{v}}\cdot\mathbf{v} + \dot{\mathbf{v}}\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) \; .
\end{aligned} \tag{4.210}$$

Invoking now invariance of the energy equation under the superposed rigid rotation, it can be concluded from (4.194), as well as (4.207) (4.208) and (4.210), that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon\, dv + \int_{\mathcal{P}} \rho\big[\dot{\mathbf{v}}\cdot\mathbf{v} + \dot{\mathbf{v}}\cdot(\boldsymbol{\Omega}_0\mathbf{x}) \;+\; (\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x})\big]\, dv \\
= \int_{\mathcal{P}} \rho\big[\mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0\mathbf{v})\cdot(\boldsymbol{\Omega}_0\mathbf{x}) + (\boldsymbol{\Omega}_0^2\mathbf{x})\cdot(\boldsymbol{\Omega}_0\mathbf{x})\big]\, dv \\
+ \int_{\partial\mathcal{P}} \mathbf{t}\cdot(\mathbf{v} + \boldsymbol{\Omega}_0\mathbf{x})\, da + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h\, da \; .
\end{aligned} \tag{4.211}$$

After subtracting (4.165) from (4.211) and simplifying the resulting equation, it follows that

$$\int_{\mathcal{P}} \rho \dot{\mathbf{v}} \cdot (\boldsymbol{\Omega}_0 \mathbf{x})\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \cdot (\boldsymbol{\Omega}_0 \mathbf{x})\, dv + \int_{\partial \mathcal{P}} \mathbf{t} \cdot (\boldsymbol{\Omega}_0 \mathbf{x})\, da \;. \tag{4.212}$$

Observing that for any given vector $\mathbf{z}$ in E^3

$$\mathbf{z} \cdot (\boldsymbol{\Omega}_0 \mathbf{x}) \;=\; \mathbf{z} \cdot (\boldsymbol{\omega}_0 \times \mathbf{x}) \;=\; \boldsymbol{\omega}_0 \cdot (\mathbf{x} \times \mathbf{z}) \;, \tag{4.213}$$

and recalling that $\boldsymbol{\omega}_0$ is a constant vector, equation (4.212) takes the form

$$\boldsymbol{\omega}_0 \cdot \Big[\int_{\mathcal{P}} \rho \mathbf{x} \times \dot{\mathbf{v}}\, dv - \int_{\mathcal{P}} \rho \mathbf{x} \times \mathbf{b}\, dv - \int_{\partial \mathcal{P}} \mathbf{x} \times \mathbf{t}\, da \Big] \;. \tag{4.214}$$

This, in turn, implies that

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \mathbf{x} \times \mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{x} \times \mathbf{b}\, dv + \int_{\partial \mathcal{P}} \mathbf{x} \times \mathbf{t}\, da \;, \tag{4.215}$$

namely that balance of angular momentum holds.

The preceding analysis shows that the integral forms of conservation of mass, and balance of linear and angular momentum are directly deduced from the integral form of energy balance and the postulate of invariance under superposed rigid-body motions. This remarkable result is referred to as the *Green-Naghdi-Rivlin* theorem.

The Green-Naghdi-Rivlin theorem can be viewed as an implication of the general covariance principle proposed by Einstein. According to this principle, all physical laws should be invariant under any smooth time-dependent coordinate transformation (including, as a special case, rigid time-dependent transformations). This far-reaching principle stems from Einstein's conviction that physical laws are oblivious to specific coordinate systems, hence should be expressed in a covariant manner, i.e., without being restricted by specific choices of coordinate systems. In covariant theories, the energy equation plays a central role, as demonstrated by the Green-Naghdi-Rivlin theorem.

Chapter 5

Infinitesimal deformations

The development of kinematics and kinetics presented up to this point does not require any assumptions on the magnitude of the various measures of deformation. In many realistic circumstances, solids and fluids may undergo "small" (or "infinitesimal") deformations. In these cases, the mathematical representation of kinematic quantities and the associated kinetic quantities, as well as the balance laws, may be simplified substantially.

In this chapter, the special case of infinitesimal deformations is discussed in detail. Preliminary to this discussion, it is instructive to formally define the meaning of "small" or "infinitesimal" changes of a function. To this end, consider a scalar-valued real function $f = f(x)$, which is assumed to possess continuous derivatives up to any desirable order. To analyze this function in the neighborhood of $x = x_0$, one may use a Taylor series expansion at x_0 in the form

$$\begin{aligned} f(x_0 + v) &= f(x_0) + vf'(x_0) + \frac{v^2}{2!}f''(x_0) + \frac{v^3}{3!}f'''(x_0) + \dots \\ &= f(x_0) + vf'(x_0) + \frac{v^2}{2!}f''(\bar{x}) \ , \end{aligned} \tag{5.1}$$

where v is a change to the value of x_0 and $\bar{x} \in (x_0, x_0 + v)$. Denoting by ε the magnitude of the difference between $x_0 + v$ and x_0, i.e., $\varepsilon = |v|$, it follows that as $\varepsilon \to 0$ (i.e., as $v \to 0$), the scalar $f(x_0 + v)$ is satisfactorily approximated by the linear part of the Taylor series expansions in (5.1), namely

$$f(x_0 + v) \ \doteq \ f(x_0) + vf'(x_0) \ . \tag{5.2}$$

Recalling the expansion $(5.1)_2$, one says that $\varepsilon = |v|$ is "small", when the term $\dfrac{v^2}{2!}f''(\bar{x})$ can be neglected in this expansion without appreciable error, i.e. when

$$\left|\frac{v^2}{2!}f''(\bar{x})\right| \;\ll\; |f(x_0+v)| \; , \tag{5.3}$$

assuming that $f(x_0+v) \neq 0$.

5.1 The Gâteaux differential

Linear expansions of the form (5.2) can be readily obtained for vector or tensor functions using the *Gâteaux differential*. In particular, given $\mathfrak{F} = \mathfrak{F}(\mathfrak{X})$, where $\mathfrak{F}$ is a scalar, vector or tensor function of a scalar, vector or tensor variable $\mathfrak{X}$, the Gâteaux differential $D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V})$ of $\mathfrak{F}$ at $\mathfrak{X} = \mathfrak{X}_0$ in the direction $\mathfrak{V}$ is defined as

$$D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) \;=\; \left[\frac{d}{d\omega}\mathfrak{F}(\mathfrak{X}_0 + \omega\mathfrak{V})\right]_{\omega=0} \; , \tag{5.4}$$

where ω is a scalar. Then,

$$\mathfrak{F}(\mathfrak{X}_0 + \mathfrak{V}) \;=\; \mathfrak{F}(\mathfrak{X}_0) + D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) + o(|\mathfrak{V}|^2) \; , \tag{5.5}$$

such that

$$\lim_{|\mathfrak{V}|\to 0} \frac{o(|\mathfrak{V}|^2)}{|\mathfrak{V}|} \;=\; 0 \; . \tag{5.6}$$

The linear part $\mathcal{L}[\mathfrak{F}; \mathfrak{V}]_{\mathfrak{X}_0}$ of $\mathfrak{F}$ at $\mathfrak{X}_0$ in the direction $\mathfrak{V}$ is then defined as

$$\mathcal{L}[\mathfrak{F}; \mathfrak{V}]_{\mathfrak{X}_0} \;=\; \mathfrak{F}(\mathfrak{X}_0) + D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) \; . \tag{5.7}$$

Examples:

(1) Let $\mathfrak{F}(\mathfrak{X}) = f(x) = x^2$. Using the definition in (5.4),

$$\begin{aligned} Df(x_0, v) \;&=\; \left[\frac{d}{d\omega}f(x_0 + \omega v)\right]_{\omega=0} \\ &=\; \left[\frac{d}{d\omega}(x_0 + \omega v)^2\right]_{\omega=0} \\ &=\; \left[\frac{d}{d\omega}(x_0^2 + 2x_0\omega v + \omega^2 v^2)\right]_{\omega=0} \\ &=\; \left[2x_0 v + 2\omega v^2\right]_{\omega=0} \\ &=\; 2x_0 v \; . \end{aligned}$$

Hence,

$$\mathcal{L}[f;v]_{x_0} \;=\; x_0^2 + 2x_0 v \;.$$

$\square$

(2) Let $\mathfrak{F}(\mathfrak{X}) = \mathbf{T}(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$. Using, again, the definition in (5.4),

$$\begin{aligned}
D\mathbf{T}(\mathbf{x}_0, \mathbf{v}) &= \left[\frac{d}{d\omega}\mathbf{T}(\mathbf{x}_0 + \omega\mathbf{v})\right]_{\omega=0} \\
&= \left[\frac{d}{d\omega}\{(\mathbf{x}_0 + \omega\mathbf{v}) \otimes (\mathbf{x}_0 + \omega\mathbf{v})\}\right]_{\omega=0} \\
&= \left[\frac{d}{d\omega}\{\mathbf{x}_0 \otimes \mathbf{x}_0 + \omega(\mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0) + \omega^2 \mathbf{v} \otimes \mathbf{v}\}\right]_{\omega=0} \\
&= \left[(\mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0) + 2\omega \mathbf{v} \otimes \mathbf{v}\right]_{\omega=0} \\
&= \mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0 \;.
\end{aligned}$$

It follows that

$$\mathcal{L}[\mathbf{T};\mathbf{v}]_{\mathbf{x}_0} \;=\; \mathbf{x}_0 \otimes \mathbf{x}_0 + \mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0 \;.$$

$\square$

5.2 Consistent linearization of kinematic and kinetic variables

Start by noting that the position vector $\mathbf{x}$ of a material point P in the current configuration can be written as the sum of the position vector $\mathbf{X}$ of the same point in the reference configuration plus the *displacement* $\mathbf{u}$ of the point from the reference to the current configuration, namely

$$\mathbf{x} \;=\; \mathbf{X} + \mathbf{u} \;. \tag{5.8}$$

As usual, the displacement vector field can be expressed equivalently in referential or spatial form as

$$\mathbf{u} \;=\; \hat{\mathbf{u}}(\mathbf{X}, t) \;=\; \tilde{\mathbf{u}}(\mathbf{x}, t) \;. \tag{5.9}$$

It follows that the deformation gradient can be written as

$$\begin{aligned} \mathbf{F} \;=\; \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \;=\; \frac{\partial(\mathbf{X}+\hat{\mathbf{u}})}{\partial \mathbf{X}} &= \mathbf{I} + \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{X}} \\ &= \mathbf{I} + \mathbf{H}\ , \end{aligned} \tag{5.10}$$

where $\mathbf{H}$ is the *relative displacement gradient* tensor defined by

$$\mathbf{H} \;=\; \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{X}}\ . \tag{5.11}$$

Clearly, $\mathbf{H}$ quantifies the deviation of $\mathbf{F}$ from the identity tensor.

Recalling the discussion in Section 5.1, a linearized counterpart of a given kinematic measure is obtained by first expressing the kinematic measure as a function of $\mathbf{H}$, i.e., as $\mathfrak{F}(\mathbf{H})$ and, then, by expanding $\mathfrak{F}(\mathbf{H})$ around the reference configuration, where $\mathbf{H} = \mathbf{0}$. This leads to

$$\mathfrak{F}(\mathbf{H}) \;=\; \mathfrak{F}(\mathbf{0}) + D\mathfrak{F}(\mathbf{0}, \mathbf{H}) + o(\|\mathbf{H}\|^2)\ , \tag{5.12}$$

where $\|\mathbf{H}\|$ is the *two-norm* of $\mathbf{H}$ defined at a point $\mathbf{X}$ and time t as $\|\mathbf{H}\| \;=\; (\mathbf{H}\cdot\mathbf{H})^{1/2}$. Taking into account (5.7) and (5.12), the linearized counterpart of $\mathfrak{F}(\mathbf{H})$ is the linear part $\mathcal{L}(\mathfrak{F};\mathbf{H})_{\mathbf{0}}$ of $\mathfrak{F}$ around the reference configuration in the direction of $\mathbf{H}$, given by

$$\mathcal{L}(\mathfrak{F};\mathbf{H})_{\mathbf{0}} \;=\; \mathfrak{F}(\mathbf{0}) + D\mathfrak{F}(\mathbf{0}, \mathbf{H})\ . \tag{5.13}$$

Here, a suitable scalar measure of magnitude for the deviation of $\mathbf{F}$ from identity can be defined as

$$\varepsilon \;=\; \varepsilon(t) \;=\; \sup_{\mathbf{X}\in\mathcal{R}_0} ||\mathbf{H}(\mathbf{X},t)||\ , \tag{5.14}$$

where "sup" denotes the least upper bound over all points $\mathbf{X}$ in the reference configuration. Now one may say that the deformations are "small" (of "infinitesimal") at a given time t if ε is small enough so that the term $o(||\mathbf{H}||^2)$ can be neglected when compared with $\mathfrak{F}(\mathbf{H})$.

Now, proceed to obtain infinitesimal counterparts of some standard kinematic fields, starting with the deformation gradient $\mathbf{F}$. To this end, first recall that $\mathbf{F} = \bar{\mathbf{F}}(\mathbf{H}) = \mathbf{I} + \mathbf{H}$. Then, note that the Gâteaux differential

$$\begin{aligned} D\mathbf{F}(\mathbf{0}, \mathbf{H}) \;&=\; \left[\frac{d}{d\omega}\bar{\mathbf{F}}(\mathbf{0} + \omega\mathbf{H})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}(\mathbf{I} + \omega\mathbf{H})\right]_{\omega=0} \\ &= \mathbf{H}\ . \end{aligned} \tag{5.15}$$

Hence,

$$\begin{aligned}\mathcal{L}[\mathbf{F};\mathbf{H}]_{\mathbf{0}} &= \bar{\mathbf{F}}(\mathbf{0}) + D\mathbf{F}(\mathbf{0},\mathbf{H}) \\ &= \mathbf{I} + \mathbf{H}\ .\end{aligned} \tag{5.16}$$

Effectively, (5.16) shows that the linear part of $\mathbf{F}$ is $\mathbf{F}$ itself, which should be obvious from (5.10).

Next, starting from $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$, take the linear part of both sides in the direction of $\mathbf{H}$. This leads to

$$\mathcal{L}[\mathbf{F}\mathbf{F}^{-1};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{F}}(\mathbf{0})\bar{\mathbf{F}}^{-1}(\mathbf{0}) + D(\mathbf{F}\mathbf{F}^{-1})(\mathbf{0},\mathbf{H}) \;=\; \mathcal{L}[\mathbf{I};\mathbf{H}]_{\mathbf{0}} \;=\; \mathbf{I}\ , \tag{5.17}$$

where

$$\begin{aligned}D(\mathbf{F}\mathbf{F}^{-1})(\mathbf{0},\mathbf{H}) &= D\mathbf{F}(\mathbf{0},\mathbf{H})\bar{\mathbf{F}}^{-1}(\mathbf{0}) + \bar{\mathbf{F}}(\mathbf{0})D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \\ &= \mathbf{H} + D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{0}\ .\end{aligned} \tag{5.18}$$

The preceding equation implies that

$$D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \;=\; -\mathbf{H}\ . \tag{5.19}$$

Hence, the linear part of $\mathbf{F}^{-1}$ at $\mathbf{H} = \mathbf{0}$ in the direction $\mathbf{H}$ is

$$\mathcal{L}[\mathbf{F}^{-1};\mathbf{H}]_{\mathbf{0}} \;=\; \mathbf{I} - \mathbf{H}\ . \tag{5.20}$$

Next, consider the linear part of the spatial displacement gradient $\operatorname{grad}\mathbf{u}$. First, observe that, using the chain rule, one finds that

$$\operatorname{grad}\mathbf{u} \;=\; (\operatorname{Grad}\mathbf{u})\mathbf{F}^{-1} \;=(\mathbf{F}-\mathbf{I})\mathbf{F}^{-1} \;=\; \mathbf{I} - \mathbf{F}^{-1}\ , \tag{5.21}$$

therefore

$$\operatorname{grad}\mathbf{u} \;=\; \overline{\operatorname{grad}\mathbf{u}}(\mathbf{H}) \;=\; \mathbf{I} - (\mathbf{I}+\mathbf{H})^{-1}\ . \tag{5.22}$$

This means that, taking into account (5.19),

$$D(\operatorname{grad}\mathbf{u})(\mathbf{0},\mathbf{H}) \;=\; -D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{H}\ . \tag{5.23}$$

As a result,

$$\mathcal{L}[\operatorname{grad}\mathbf{u};\mathbf{H}]_{\mathbf{0}} \;=\; \overline{\operatorname{grad}\mathbf{u}}(\mathbf{0}) + D(\operatorname{grad}\mathbf{u})(\mathbf{0},\mathbf{H}) \;=\; \mathbf{0}+\mathbf{H} \;=\; \mathbf{H}\ . \tag{5.24}$$

The last result shows that the linear part of the spatial displacement gradient grad $\mathbf{u}$ coincides with the referential displacement gradient Grad $\mathbf{u}(= \mathbf{H})$. This, in turn, implies that, within the context of infinitesimal deformations, there is no difference between the partial derivatives of the displacement $\mathbf{u}$ with respect to $\mathbf{X}$ or $\mathbf{x}$. This further implies that the distinction between spatial and referential description of kinematic quantities becomes immaterial in the case of infinitesimal deformations.

To determine the linear part of the right Cauchy-Green deformation tensor $\mathbf{C}$, write

$$\mathbf{C} \;=\; \bar{\mathbf{C}}(\mathbf{H}) \;=\; (\mathbf{I}+\mathbf{H})^T(\mathbf{I}+\mathbf{H}) \;=\; \mathbf{I}+\mathbf{H}+\mathbf{H}^T+\mathbf{H}^T\mathbf{H}\;. \tag{5.25}$$

Then,

$$\begin{aligned} D\mathbf{C}(\mathbf{0},\mathbf{H}) \;&=\; \left[\frac{d}{dw}\bar{\mathbf{C}}(\mathbf{0}+w\mathbf{H})\right]_{w=0} \\ &= \left[\frac{d}{dw}\left\{\mathbf{I}+w(\mathbf{H}+\mathbf{H}^T)+w^2\mathbf{H}^T\mathbf{H}\right\}\right]_{w=0} \\ &= \left[\mathbf{H}+\mathbf{H}^T+2w\mathbf{H}^T\mathbf{H}\right]_{w=0} \\ &= \mathbf{H}+\mathbf{H}^T\;. \end{aligned} \tag{5.26}$$

Consequently, the linear part of $\mathbf{C}$ at $\mathbf{H}=\mathbf{0}$ in the direction $\mathbf{H}$ is

$$\mathcal{L}[\mathbf{C};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{C}}(\mathbf{0})+D\mathbf{C}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{I}+(\mathbf{H}+\mathbf{H}^T)\;. \tag{5.27}$$

Using (5.27), it can be immediately concluded that the linear part of the Lagrangian strain tensor $\mathbf{E}$ is

$$\mathcal{L}\;[\mathbf{E};\mathbf{H}]_{\mathbf{0}} \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T)\;. \tag{5.28}$$

At the same time, the Eulerian strain tensor $\mathbf{e}$ can be written as

$$\mathbf{e} \;=\; \bar{\mathbf{e}}(\mathbf{H}) \;=\; \frac{1}{2}\left(\mathbf{i}-\bar{\mathbf{F}}^{-T}(\mathbf{H})\bar{\mathbf{F}}^{-1}(\mathbf{H})\right)\;, \tag{5.29}$$

hence its Gâteaux differential is given, with the aid of (5.19), by

$$D\mathbf{e}(\mathbf{0},\mathbf{H}) \;=\; -\frac{1}{2}\left(D\mathbf{F}^{-T}(\mathbf{0},\mathbf{H})+D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H})\right) \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T)\;. \tag{5.30}$$

This means that the linear part of $\mathbf{e}$ is equal to

$$\mathcal{L}[\mathbf{e};\mathbf{H}]_{\mathbf{0}} \;= \bar{\mathbf{e}}(\mathbf{0})+D\mathbf{e}(\mathbf{0},\mathbf{H}) \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T)\;. \tag{5.31}$$

It is clear from (5.28) and (5.31) that the linear parts of the Lagrangian and Eulerian strain tensors coincide. Hence, under the assumption of infinitesimal deformations, the distinction between the two strains ceases to exist and one writes that

$$\mathcal{L}[\mathbf{E};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\mathbf{e};\mathbf{H}]_{\mathbf{0}} \;=\; \boldsymbol{\varepsilon}\;, \tag{5.32}$$

where $\boldsymbol{\varepsilon}$ is the classical *infinitesimal strain tensor*, with components $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

Proceed next with the linearization of the right stretch tensor $\mathbf{U}$. To this end, write

$$\mathbf{U}^2 \;=\; \mathbf{C} \;=\; \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}\;, \tag{5.33}$$

so that, with the aid of (5.26),

$$\begin{aligned} D\mathbf{U}^2(\mathbf{0},\mathbf{H}) \;=\; D\mathbf{U}(\mathbf{0},\mathbf{H})\bar{\mathbf{U}}(\mathbf{0}) + \bar{\mathbf{U}}(\mathbf{0})D\mathbf{U}(\mathbf{0},\mathbf{H}) &= 2D\mathbf{U}(\mathbf{0},\mathbf{H}) \\ &= \mathbf{H} + \mathbf{H}^T\;, \end{aligned} \tag{5.34}$$

hence,

$$\mathcal{L}\left[\mathbf{U};\mathbf{H}\right]_{\mathbf{0}} \;=\; \bar{\mathbf{U}}(\mathbf{0}) + D\mathbf{U}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{I} + \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)\;. \tag{5.35}$$

Repeating the procedure used earlier in this section to determine the Gâteaux differential of $\mathbf{F}^{-1}$, one easily finds that

$$D\mathbf{U}^{-1}(\mathbf{0},\mathbf{H}) \;=\; -\frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \tag{5.36}$$

and

$$\mathcal{L}\left[\mathbf{U}^{-1};\mathbf{H}\right]_{\mathbf{0}} \;=\; \mathbf{I} - \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)\;. \tag{5.37}$$

It is now possible to determine the linear part of the rotation tensor $\mathbf{R}$, written as

$$\mathbf{R} \;=\; \bar{\mathbf{R}}(\mathbf{H}) \;=\; \bar{\mathbf{F}}(\mathbf{H})\bar{\mathbf{U}}^{-1}(\mathbf{H})\;, \tag{5.38}$$

by first obtaining the Gâteaux differential as

$$D\mathbf{R}(\mathbf{0},\mathbf{H}) \;=\; D\mathbf{F}(\mathbf{0},\mathbf{H})\bar{\mathbf{U}}^{-1}(\mathbf{0}) + \bar{\mathbf{F}}(\mathbf{0})D\mathbf{U}^{-1}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{H} - \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \;=\; \frac{1}{2}(\mathbf{H} - \mathbf{H}^T)\;, \tag{5.39}$$

and then writing

$$\mathcal{L}\left[\mathbf{R};\mathbf{H}\right]_{\mathbf{0}} \;=\; \bar{\mathbf{R}}(\mathbf{0}) + D\mathbf{R}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{I} + \frac{1}{2}(\mathbf{H} - \mathbf{H}^T)\;. \tag{5.40}$$

When $\mathbf{H}$ is small, the tensor

$$\boldsymbol{\omega} \;=\; \frac{1}{2}(\mathbf{H}-\mathbf{H}^T) \tag{5.41}$$

is called the *infinitesimal rotation* tensor and has components $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$.

Next, derive the linear part of the Jacobian J of the deformation gradient. To this end, observe that

$$\begin{aligned}
D(\det \mathbf{F})(\mathbf{0},\mathbf{H}) \;&=\; \left[\frac{d}{d\omega}\det \bar{\mathbf{F}}(\omega\mathbf{H})\right]_{\omega=0} \\
&=\; \left[\frac{d}{d\omega}\det(\mathbf{I}+\omega\mathbf{H})\right]_{\omega=0} \\
&=\; \left[\frac{d}{d\omega}\det\{\omega(\mathbf{H}-(-\frac{1}{\omega})\mathbf{I})\}\right]_{\omega=0} \\
&=\; \left[\frac{d}{d\omega}[\omega^3\left\{-(-\frac{1}{\omega})^3 + I_H(-\frac{1}{\omega})^2 - II_H(-\frac{1}{\omega}) + III_H\right\}]\right]_{\omega=0} \\
&=\; \left[\frac{d}{d\omega}\left\{1+\omega I_H + \omega^2 II_H + \omega^3 III_H\right\}\right]_{\omega=0} \\
&= I_H \;=\; \operatorname{tr}\mathbf{H}\;,
\end{aligned} \tag{5.42}$$

where I_H, II_H, and III_H are the three principal invariants of $\mathbf{H}$. This, in turn, leads to

$$\mathcal{L}[\det\mathbf{F};\mathbf{H}]_{\mathbf{0}} \;=\; \det\bar{\mathbf{F}}(\mathbf{0}) + D(\det\mathbf{F})(\mathbf{0},\mathbf{H}) \;=\; 1+\operatorname{tr}\mathbf{H} \;=\; 1+\operatorname{tr}\boldsymbol{\varepsilon}\;. \tag{5.43}$$

The balance laws are also subject to linearization. For instance, the conservation of mass statement (4.41) can be linearized to yield

$$\mathcal{L}[\rho_0;\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\rho J;\mathbf{H}]_{\mathbf{0}}\;. \tag{5.44}$$

This means that

$$\rho_0 \;=\; \bar{\rho}(\mathbf{0})\bar{J}(\mathbf{0}) + D\rho(\mathbf{0},\mathbf{H})\bar{J}(\mathbf{0}) + \bar{\rho}(\mathbf{0})DJ(\mathbf{0},\mathbf{H})\;. \tag{5.45}$$

Since conservation of mass is assumed to hold in all configurations therefore also including the reference configuration, it follows that

$$\rho_0 \;=\; \bar{\rho}(\mathbf{0})\bar{J}(\mathbf{0}) \;=\; \bar{\rho}(\mathbf{0})\;, \tag{5.46}$$

thus equation (5.45), with the aid of (5.42) results in

$$D\rho(\mathbf{0},\mathbf{H}) + \bar{\rho}(\mathbf{0})\operatorname{tr}\boldsymbol{\varepsilon} \;=\; 0\;, \tag{5.47}$$

or, equivalently,

$$D\rho(\mathbf{0}, \mathbf{H}) \;=\; -\varrho_0 \operatorname{tr} \boldsymbol{\varepsilon} \;. \tag{5.48}$$

The linear part of the mass density relative to the reference configuration now takes the form

$$\mathcal{L}[\rho; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\rho}(\mathbf{0}) + D\rho(\mathbf{0}, \mathbf{H}) \;=\; \rho_0(1 - \operatorname{tr} \boldsymbol{\varepsilon}) \;. \tag{5.49}$$

Equation (5.49) reveals that the linearized mass density does not coincide with the mass density of the reference configuration.

Chapter 6

Mechanical constitutive theories

6.1 General requirements

The balance laws furnish a total of seven equations (one from mass balance, three from linear momentum balance and three from angular momentum balance) to determine thirteen unknowns, namely the mass density ρ, the position $\mathbf{x}$ (or velocity $\mathbf{v}$) and the stress tensor (e.g., the Cauchy stress $\mathbf{T}$). Clearly, without additional equations this system lacks closure. The latter is established by constitutive equations, which relate the stresses to the mass density and the kinematic variables.

Before accounting for any possible restrictions or reductions, a general constitutive equation for the Cauchy stress may be written as

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{x}, \mathbf{v}, \ldots, \mathbf{F}, \dot{\mathbf{F}}, \ldots, \mathrm{Grad}\mathbf{F}, \ldots, \rho, \dot{\rho}, \ldots) \tag{6.1}$$

or, in rate form, as

$$\dot{\mathbf{T}} \;=\; \hat{\dot{\mathbf{T}}}(\mathbf{x}, \mathbf{v}, \ldots, \mathbf{F}, \dot{\mathbf{F}}, \ldots, \mathrm{Grad}\mathbf{F}, \ldots, \rho, \dot{\rho}, \ldots) \;. \tag{6.2}$$

Analogous general functional representations may be written for other stress measures.

A number of restrictions may be placed to the preceding equations on mathematical or physical grounds. Some of these restrictions appear to be universally agreed upon, while others tend to be less uniformly accepted. Three of these restrictions are reviewed below.

First, all constitutive laws need to be dimensionally consistent. This simply means that the units of the left- and right-hand side of (6.1) or (6.2) should be the same. For example, when applied to a constitutive law of the form $\dot{\mathbf{T}} = \alpha\mathbf{B}$, this would necessitate that the parameter α have units of stress (since $\mathbf{B}$ is unitless). Second, the same laws need to be consistent in their tensorial representation. This means that the right-hand side of (6.1) and (6.2) should have a tensorial representation in terms of the Eulerian basis. This is to maintain consistency with the left-hand sides, which are naturally resolved on this basis. This restriction would disallow, for example, a constitutive law of the form $\mathbf{T} = \beta\mathbf{F}$, where β is a constant.

The third source of restrictions is the postulate of invariance under superposed rigid motions (also referred to as objectivity), which is assumed to apply to constitutive laws just as it applies to balance laws. This postulate requires the functions $\hat{\mathbf{T}}$ and $\hat{\dot{\mathbf{T}}}$ in (6.1) and (6.2) to remain unaltered under superposed rigid motions. By way of example, applying the postulate of invariance under superposed rigid motions to the constitutive law

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{F}) \tag{6.3}$$

necessitates that

$$\mathbf{T}^{+} \;=\; \hat{\mathbf{T}}(\mathbf{F}^{+})\;. \tag{6.4}$$

Taking into account (3.165) and (4.137), equations (6.3) and (6.4) lead to

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^{T} \;=\; \hat{\mathbf{T}}(\mathbf{QF})\;, \tag{6.5}$$

for all proper orthogonal tensors $\mathbf{Q}$. Equation (6.5) places a restriction on the choice of $\hat{\mathbf{T}}$. An identical restriction is placed on the function $\hat{\dot{\mathbf{T}}}$, where, for example,

$$\mathring{\mathbf{T}} \;=\; \hat{\mathbf{T}}(\mathbf{F})\;, \tag{6.6}$$

and $\mathring{\mathbf{T}}$ is the *Jaumann* rate of the Cauchy stress tensor. The same conclusion is reached when using any other objective rate of the Cauchy stress tensor in (6.6).

6.2 Inviscid fluids

An inviscid fluid is defined by the property that the stress vector $\mathbf{t}$ acting on any surface is always opposite to the outward normal $\mathbf{n}$ to the surface, regardless of whether the fluid is stationary or flowing. Said differently, an inviscid fluid cannot sustain shearing tractions under any circumstances. This means that

$$\mathbf{t} \;=\; \mathbf{T}\mathbf{n} \;=\; -p\mathbf{n} \;, \tag{6.7}$$

hence

$$\mathbf{T} \;=\; -p\mathbf{i} \;, \tag{6.8}$$

see Figure 6.1.

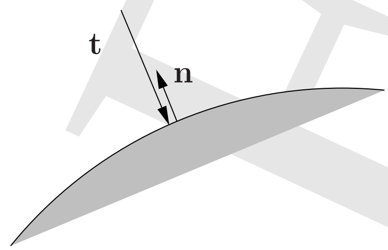

Figure 6.1: *Traction acting on a surface of an inviscid fluid.*

On physical grounds, one may assume that the pressure p depends on the density ρ, i.e.,

$$\mathbf{T} \;=\; -p(\rho)\mathbf{i} \;. \tag{6.9}$$

This constitutive relation defines a special class of inviscid fluids referred to as *elastic fluids*.

It is instructive here to take an alternative path for the derivation of (6.9). In particular, suppose that one starts from the more general constitutive assumption

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho) \;. \tag{6.10}$$

Upon invoking invariance under superposed rigid motions, it follows that

$$\mathbf{T}^{+} \;=\; \hat{\mathbf{T}}(\rho^{+}) \;, \tag{6.11}$$

which, with the aid of (4.137) and (4.143) leads to

$$\mathbf{Q}\hat{\mathbf{T}}(\rho)\mathbf{Q}^{T} \;=\; \hat{\mathbf{T}}(\rho) \;, \tag{6.12}$$

for all proper orthogonal $\mathbf{Q}$. Furthermore, substituting $-\mathbf{Q}$ for $\mathbf{Q}$ in (6.12), it is clear that (6.12) holds for all improper orthogonal tensors $\mathbf{Q}$ as well, hence it holds for all orthogonal tensors. A tensor function $\hat{\mathbf{T}}(\phi)$ of a scalar variable is termed *isotropic* when

$$\mathbf{Q}\hat{\mathbf{T}}(\phi)\mathbf{Q}^{T} \;=\; \hat{\mathbf{T}}(\phi) \;, \tag{6.13}$$

for all orthogonal $\mathbf{Q}$. This condition may be interpreted as meaning that the components of the tensor function remain unaltered when resolved on any two orthonormal bases. Clearly, the constitutive function $\hat{\mathbf{T}}$ in (6.10) is isotropic.

The *representation theorem for isotropic tensor functions of a scalar variable* states that a tensor function of a scalar variable is isotropic if, and only if, it is a scalar multiple of the identity tensor. In the case of $\hat{\mathbf{T}}$ in (6.10), this immediately leads to the constitutive equation (6.9).

To prove the preceding representation theorem, first note that the sufficiency argument is trivial. The necessity argument can be made by setting

$$\mathbf{Q} \;=\; \mathbf{Q}_1 \;=\; \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2 \;, \tag{6.14}$$

which, recalling the Rodrigues formula (3.103), corresponds to $\mathbf{p} \;=\; \mathbf{e}_1$, $\mathbf{q} = \mathbf{e}_2$, $\mathbf{r} = \mathbf{e}_3$, and $\theta = \pi/2$, i.e., to a rigid rotation of $\pi/2$ with respect to the axis of $\mathbf{e}_1$. It is easy to verify that, in this case, equation (6.13) yields

$$\begin{bmatrix} T_{11} & -T_{13} & T_{12} \\ -T_{31} & T_{33} & -T_{32} \\ T_{21} & -T_{23} & T_{22} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} . \tag{6.15}$$

This, in turn, means that

$$T_{22} \;=\; T_{33} \quad , \quad T_{12} \;=\; T_{21} = T_{13} = T_{31} \;=\; 0 \quad , \quad T_{23} \;= -T_{32} \; . \tag{6.16}$$

Next, set

$$\mathbf{Q} \;=\; \mathbf{Q}_2 \;=\; \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3 \;, \tag{6.17}$$

which corresponds to $\mathbf{p} = \mathbf{e}_2$, $\mathbf{q} = \mathbf{e}_3$, $\mathbf{r} = \mathbf{e}_1$, and $\theta = \pi/2$. This is a rigid rotation of $\pi/2$ with respect to the axis of $\mathbf{e}_2$. Again, upon using this rotation in (6.13), it follows that

$$\begin{bmatrix} T_{33} & T_{32} & -T_{31} \\ T_{23} & T_{22} & -T_{21} \\ -T_{13} & -T_{12} & T_{11} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} , \tag{6.18}$$

which leads to

$$T_{11} \;=\; T_{33} \qquad , \qquad T_{32} \;=\; T_{12} \;=\; -T_{12} \qquad , \qquad T_{23} \;=\; T_{21} \; . \tag{6.19}$$

One may combine the results in (6.16) and (6.19) to deduce that

$$\mathbf{T} \;=\; T\,\mathbf{I}\;, \tag{6.20}$$

where $T = T_{11} = T_{22} = T_{33}$, which completes the proof.

Invariance under superposed rigid motions may be also used to exclude certain functional dependencies in the constitutive assumption for $\mathbf{T}$. Indeed, assume that

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{x}, \rho)\;, \tag{6.21}$$

namely that the Cauchy stress tensor depends explicitly on the position. Invariance of $\hat{\mathbf{T}}$ under superposed rigid motions implies that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\mathbf{x}^+, \rho^+) \tag{6.22}$$

hence

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{x}, \rho)\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{x} + \mathbf{c}, \rho)\;, \tag{6.23}$$

for all proper orthogonal tensors $\mathbf{Q}$ and vectors $\mathbf{c}$. Now, choose a superposed rigid motion, such that $\mathbf{Q} = \mathbf{I}$ and $\mathbf{c} = \mathbf{c}_0$, where $\mathbf{c}_0$ is constant. It follows from (6.23) that

$$\hat{\mathbf{T}}(\mathbf{x}, \rho) \;=\; \hat{\mathbf{T}}(\mathbf{x} + \mathbf{c}_0, \rho)\;. \tag{6.24}$$

However, given that $\mathbf{c}_0$ is arbitrary, the condition (6.24) can be met only if $\hat{\mathbf{T}}$ is altogether independent of $\mathbf{x}$.

A similar derivation can be followed to argue that a constitutive law of the form

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{v}, \rho) \tag{6.25}$$

violates invariance under superposed rigid motions. Indeed, in this case, invariance implies that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\mathbf{v}^+, \rho^+)\;, \tag{6.26}$$

which readily translates to

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{v}, \rho)\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{\Omega}\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}, \rho)\;. \tag{6.27}$$

Now, choose $\mathbf{Q} = \mathbf{I}$, $\mathbf{\Omega} = \mathbf{0}$ and $\mathbf{c} = \mathbf{c}_0 t$, where $\mathbf{c}_0$ is, again, a constant. It follows that for this particular choice of a superposed motion

$$\hat{\mathbf{T}}(\mathbf{v}, \rho) \;=\; \hat{\mathbf{T}}(\mathbf{v} + \mathbf{c}_0, \rho) \;, \tag{6.28}$$

which leads to the elimination of $\mathbf{v}$ as an argument in $\hat{\mathbf{T}}$.

Returning to the balance laws for the elastic fluid, note that angular momentum balance is satisfied automatically by the constitutive equation (6.8) and the non-trivial equations that govern its motion are written in Eulerian form (which is suitable for fluids) as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} \; &= 0 \\ - \operatorname{grad} p(\rho) + \rho \mathbf{b} \; &= \rho \mathbf{a} \end{aligned} \tag{6.29}$$

or, upon expressing the acceleration in Eulerian form in terms of the velocities

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} \; &= 0 \\ - \operatorname{grad} p(\rho) + \rho \mathbf{b} \; &= \rho \Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v}\Big) \;. \end{aligned} \tag{6.30}$$

Equations $(6.30)_2$ are referred to as the *compressible Euler equations*. Equations (6.30) form a set of four coupled non-linear partial differential equations in $\mathbf{x}$ and t, which, subject to the specification of suitable initial and boundary conditions and a pressure law $p = p(\rho)$, can be solved for $\rho(\mathbf{x}, t)$ and $\tilde{\mathbf{v}}(\mathbf{x}, t)$.

Recall the definition of an isochoric (or volume-preserving) motion, and note that for such a motion conservation of mass leads to $\rho_0(\mathbf{X}) = \rho(\mathbf{x}, t)$ for all time. Then, upon appealing to the local statement of mass conservation (4.34), it is seen that $\operatorname{div} \mathbf{v} = 0$ for all isochoric motions.

A material is called *incompressible* if it can only undergo isochoric motions. If the inviscid fluid is assumed incompressible, then the constitutive equation (6.9) loses its meaning, because the function $p(\rho)$ does not make sense as the density ρ is not a variable quantity. Instead, the constitutive equation $\mathbf{T} = -p\mathbf{i}$ holds with p being the unknown. In summary, the governing equations for an incompressible inviscid fluid (ofter also referred to as an *ideal fluid*) are

$$\begin{aligned} \operatorname{div} \mathbf{v} \; &= 0 \\ - \operatorname{grad} p + \rho_0 \mathbf{b} \; &= \rho_0 \Big(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v}\Big) \;, \end{aligned} \tag{6.31}$$

where now the unknowns are p and $\mathbf{v}$.

Notice that if a set $(p, \mathbf{v})$ satisfies the above equations, then so does another set of the form $(p + c, \mathbf{v})$, where c is a constant. This suggests that the pressure field in an incompressible elastic fluid is not uniquely determined by the equations of motion. The indeterminacy is broken by specifying the value of the pressure on some part of the boundary of the domain. This point is illustrated by way of an example: consider a ball composed of an ideal fluid, which is in equilibrium under uniform time-independent pressure p. The same "motion" of the ball can be also sustained by any pressure field $p+c$, where c is a constant, see Figure 6.2.

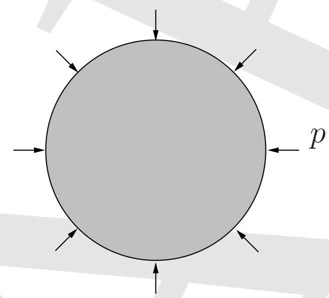

Figure 6.2: *A ball of ideal fluid in equilibrium under uniform pressure.*

6.3 Viscous fluids

All real fluids exhibit some viscosity, i.e., some ability to sustain shearing forces. It is easy to conclude on physical grounds that the resistance to shearing is related to the gradient of the velocity. Therefore, it is sensible to postulate a general constitute law for viscous fluids in the form

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{L}) \tag{6.32}$$

or, recalling the unique additive decomposition of $\mathbf{L}$ in (3.125), more generally as

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \;. \tag{6.33}$$

The explicit dependence of the Cauchy stress tensor on $\mathbf{W}$ can be excluded by invoking invariance under superposed rigid-body motions. Indeed, invariance requires that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\rho^+, \mathbf{D}^+, \mathbf{W}^+) \;. \tag{6.34}$$

Now, consider a special superposed rigid motion for which $\mathbf{Q}(t) = \mathbf{I}$, $\dot{\mathbf{Q}}(t) = \mathbf{\Omega}_0$ ($\mathbf{\Omega}_0$ being constant skew-symmetric tensor), $\mathbf{c}(t) = \mathbf{0}$ and $\dot{\mathbf{c}}(t) = \mathbf{0}$. This is a superposed rigid rotation with constant angular velocity defined by the skew-symmetric tensor $\mathbf{\Omega}_0$ (or, equivalently, its axial vector $\boldsymbol{\omega}_0$). Recalling (3.184), (3.185) and (4.143), equation (6.34) takes the form

$$\mathbf{Q}\hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\rho, \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \mathbf{\Omega}) \;, \tag{6.35}$$

for all proper orthogonal $\mathbf{Q}$. Given the special form of the chosen superposed rigid motion, equation (6.35) leads to

$$\hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}\mathbf{Q}^T, \mathbf{W} + \mathbf{\Omega}_0) \;, \tag{6.36}$$

which needs to hold for any skew-symmetric tensor $\mathbf{\Omega}_0$. This implies that the constitutive function $\hat{\mathbf{T}}$ cannot depend on $\mathbf{W}$, thus it reduces to

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}) \;. \tag{6.37}$$

Invariance under superposed rigid motions for the constitutive function in (6.37) gives rise to the condition

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\rho^+, \mathbf{D}^+) \;, \tag{6.38}$$

which, in turn, necessitates that

$$\mathbf{Q}\hat{\mathbf{T}}(\rho, \mathbf{D})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\rho, \mathbf{Q}\mathbf{D}\mathbf{Q}^T) \;, \tag{6.39}$$

for all proper orthogonal tensors $\mathbf{Q}$. In fact, since both sides of (6.39) are even functions of $\mathbf{Q}$, it is clear that (6.39) must hold for all orthogonal tensors $\mathbf{Q}$.

Neglecting, for a moment, the dependence of $\hat{\mathbf{T}}$ on ρ in equation (6.39), note that a tensor function $\hat{\mathbf{T}}$ of a tensor variable $\mathbf{S}$ is called *isotropic* if

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{S})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{S}\mathbf{Q}^T) \;, \tag{6.40}$$

for all orthogonal tensors $\mathbf{Q}$. It can be proved following the process used earlier for isotropic tensor functions of a scalar variable that a tensor function $\hat{\mathbf{T}}$ of a tensor variable $\mathbf{S}$ is isotropic if, and only if, it can be written in the form

$$\hat{\mathbf{T}}(\mathbf{S}) \;=\; a_0\mathbf{I} + a_1\mathbf{S} + a_2\mathbf{S}^2 \;, \tag{6.41}$$

where a_0, a_1, and a_2 are scalar functions of the three principal invariants I_S, II_S and III_S of $\mathbf{S}$, i.e.,

$$\begin{aligned} a_0 &= \hat{a}_0(I_S, II_S, III_S) \\ a_1 &= \hat{a}_1(I_S, II_S, III_S) \; . \\ a_2 &= \hat{a}_2(I_S, II_S, III_S) \end{aligned} \tag{6.42}$$

The above result is known as the *first representation theorem for isotropic tensors*. Using this theorem, it is readily concluded that the Cauchy stress for a fluid that obeys the general constitutive law (6.37) is of the form

$$\mathbf{T} \;=\; a_0\mathbf{i} + a_1\mathbf{D} + a_2\mathbf{D}^2 \;, \tag{6.43}$$

where a_0, a_1 and a_2 are functions of I_D, II_D, III_D and ρ. The preceding equation characterizes what is known as the *Reiner-Rivlin fluid*. Materials that obey (6.43) are also generally referred to as *non-Newtonian* fluids.

At this stage, introduce a physically plausible assumption by way of which the Cauchy stress $\mathbf{T}$ reduces to hydrostatic pressure $-p(\rho)\mathbf{i}$ when $\mathbf{D} = 0$. Then, one may rewrite the constitutive function (6.43) as

$$\mathbf{T} \;=\; (-p(\rho) + a_0^*)\mathbf{i} + a_1\mathbf{D} + a_2\mathbf{D}^2 \;, \tag{6.44}$$

where, in general, $a_0^* = \hat{a}_0^*(\rho, I_D, II_D, III_D)$. Clearly, when $a_0^* = a_1 = a_2 = 0$, the viscous fluid degenerates to an inviscid one.

From the above general class of viscous fluids, consider the sub-class of those which are linear in $\mathbf{D}$. To preserve linearity in $\mathbf{D}$, the constitutive function in (6.44) is reduced to

$$\mathbf{T} \;=\; (-p(\rho) + a_0^*)\mathbf{I} + a_1\mathbf{D} \;, \tag{6.45}$$

where, $a_0^* = \lambda I_D$ and $a_1 = 2\mu$, where λ and μ depend, in general, on ρ. This means that the Cauchy stress tensors takes the form

$$\mathbf{T} \;=\; -p(\rho)\mathbf{i} + \lambda \operatorname{tr}\mathbf{D}\mathbf{i} + 2\mu\mathbf{D} \;. \tag{6.46}$$

Viscous fluids that obey (6.46) are referred to as *Newtonian viscous fluids* or *linear viscous fluids*. The functions λ and μ are called the *viscosity coefficients*.

With the constitutive equation (6.46) in place, consider the balance laws for the Newtonian viscous fluid. Clearly, angular momentum balance is satisfied at the outset, since $\mathbf{T}$ in (6.46) is symmetric. Conservation of mass and linear momentum balance can be expressed as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} \; &= 0 \\ \operatorname{div}(-p(\rho)\mathbf{i} + \lambda \operatorname{tr} \mathbf{D}\mathbf{i} + 2\mu\mathbf{D}) + \rho\mathbf{b} \; &= \rho\dot{\mathbf{v}} \; . \end{aligned} \tag{6.47}$$

Assuming that either λ and μ are independent of ρ (which is very common) or that ρ is spatially homogeneous, one may write balance take the form

$$\begin{aligned} \operatorname{div}(-p(\rho)\mathbf{i} + \lambda \operatorname{tr} \mathbf{D}\mathbf{i} + 2\mu\mathbf{D}) \; &= \; -\operatorname{grad} p + \lambda \operatorname{grad} \operatorname{div} \mathbf{v} + \mu(\operatorname{div} \operatorname{grad} \mathbf{v} + \operatorname{grad} \operatorname{div} \mathbf{v}) \\ &= \; -\operatorname{grad} p + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} + \mu \operatorname{div} \operatorname{grad} \mathbf{v} \; . \end{aligned} \tag{6.48}$$

This implies that for this special case the governing equations of motions (6.47) can be recast as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} \; &= 0 \\ -\operatorname{grad} p + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v} + \mu \operatorname{div} \operatorname{grad} \mathbf{v} + \rho\mathbf{b} \; &= \rho\dot{\mathbf{v}} \; . \end{aligned} \tag{6.49}$$

Equations $(6.49)_2$ are known as the *Navier-Stokes equations* for the compressible Newtonian viscous fluid. As in the case of the compressible inviscid fluid, there are four coupled non-linear partial differential equations in (6.49) and four unknowns, namely the velocity $\mathbf{v}$ and the mass density ρ.

If the Newtonian viscous fluid is incompressible, the Cauchy stress is given by

$$\mathbf{T} \; = \; -p\mathbf{i} + 2\mu\mathbf{D} \; , \tag{6.50}$$

hence, the governing equations of (6.49) become

$$\begin{aligned} \operatorname{div} \mathbf{v} \; &= 0 \\ -\operatorname{grad} p + \mu \operatorname{div} \operatorname{grad} \mathbf{v} + \rho\mathbf{b} \; &= \rho\dot{\mathbf{v}} \; . \end{aligned} \tag{6.51}$$

The former equation is a local statement of the constraint of incompressibility, while the latter is the reduced statement of linear momentum balance that reflects incompressibility. As in the inviscid case, the four unknowns now are the velocity $\mathbf{v}$ and the pressure p.

The Navier-Stokes equations (compressible or incompressible) are non-linear in $\mathbf{v}$ due to the acceleration term, which is expanded in the form $\dot{\mathbf{v}} = \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{x}} \mathbf{v}$. In the special case of very slow and nearly steady flow, referred to as *creeping flow* or *Stokes flow*, the acceleration term may be ignored, giving rise to a system of linear partial differential equations.

6.4 Non-linearly elastic solid

Recalling the definition of stress power in the mechanical energy balance theorem of equation (4.158), define the *non-linearly elastic solid* by admitting the existence of a *strain energy function* $\Psi = \hat{\Psi}(\mathbf{F})$ per unit mass, such that

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \dot{\Psi} \; . \tag{6.52}$$

It follows that the stress power in the region $\mathcal{P}$ takes the form

$$\int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \;=\; \int_{\mathcal{P}} \rho \dot{\Psi} \, dv \;=\; \frac{d}{dt} \int_{\mathcal{P}} \rho \Psi \, dv \;=\; \frac{d}{dt} W(\mathcal{P}) \; , \tag{6.53}$$

where $W(\mathcal{P}) = \int_{\mathcal{P}} \rho \Psi \, dv$ is the total *strain energy* of the material occupying the region $\mathcal{P}$. As a result, the mechanical energy balance theorem for this class of materials takes the form

$$\frac{d}{dt} [K(\mathcal{P}) + W(\mathcal{P})] \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P}) \;=\; R(\mathcal{P}) \; . \tag{6.54}$$

In words, equation (6.54) states that the rate of change of the kinetic and strain energy (which together comprise the total internal energy of the non-linearly elastic material) equals the rate of work done by the external forces.

Recall that the strain energy function at a given time depends exclusively on the deformation gradient. With the aid of the chain rule, this leads to

$$\dot{\Psi} \;=\; \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} \; , \tag{6.55}$$

so that, upon recalling (6.52),

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \dot{\Psi} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot (\mathbf{L} \mathbf{F}) \; , \tag{6.56}$$

which, in turn, leads to

$$\mathbf{T} \cdot \mathbf{L} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot (\mathbf{L}\mathbf{F}) \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \cdot \mathbf{L} \; . \tag{6.57}$$

The preceding equation can be also written as

$$(\mathbf{T} - \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T) \cdot \mathbf{L} \;=\; 0 \; . \tag{6.58}$$

Observing that this equation holds for any $\mathbf{L}$, it is immediately concluded that

$$\mathbf{T} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \; . \tag{6.59}$$

Upon enforcing angular momentum balance, equation (6.59) leads to

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \;=\; \mathbf{F} \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \right)^T \; . \tag{6.60}$$

This places a restriction on the form of the strain energy function $\hat{\Psi}$. Instead of explicitly enforcing this restriction, one may simply write the Cauchy stress as

$$\mathbf{T} \;=\; \frac{1}{2} \rho \left[\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T + \mathbf{F} \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \right)^T \right] \; . \tag{6.61}$$

Alternative expressions for the strain energy of the non-linearly elastic solid can be obtained by invoking invariance under superposed rigid motions. Specifically, invariance of the function $\hat{\Psi}$ implies that

$$\begin{aligned} \Psi^+ \;&=\; \hat{\Psi}(\mathbf{F}^+) \;=\; \hat{\Psi}(\mathbf{Q}\mathbf{F}) \\ &=\; \Psi \;=\; \hat{\Psi}(\mathbf{F}) \; , \end{aligned} \tag{6.62}$$

for all proper orthogonal tensors $\mathbf{Q}$. Selecting $\mathbf{Q} = \mathbf{R}^T$, where $\mathbf{R}$ is the rotation stemming from the polar decomposition of $\mathbf{F}$, it follows from (6.62) that

$$\hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{Q}\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{R}^T\mathbf{R}\mathbf{U}) \;=\; \hat{\Psi}(\mathbf{U}) \; . \tag{6.63}$$

Therefore, one may write

$$\Psi \;=\; \hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{U}) \;=\; \bar{\Psi}(\mathbf{C}) \;=\; \check{\Psi}(\mathbf{E}) \; , \tag{6.64}$$

by merely exploiting the one-to-one relations between tensors **U**, **C** and **E**. Then, the material time derivative of Ψ can be expressed as

$$\dot{\Psi} \;=\; \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} \;=\; \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot (2\mathbf{F}^T \mathbf{D} \mathbf{F}) \;, \tag{6.65}$$

where (3.129) is invoked. It follows from (6.52) that

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot (2\mathbf{F}^T \mathbf{D} \mathbf{F}) \;=\; 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \cdot \mathbf{D} \;, \tag{6.66}$$

which readily leads to

$$\left(\mathbf{T} - 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \right) \cdot \mathbf{D} \;=\; 0 \;. \tag{6.67}$$

Given the arbitrariness of **D**, it follows that

$$\mathbf{T} \;=\; 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \;. \tag{6.68}$$

Using an analogous procedure, one may also derive a constitutive equation for the Cauchy stress in terms of the strain energy function $\check{\Psi}$ as

$$\mathbf{T} \;=\; \rho \mathbf{F} \frac{\partial \check{\Psi}}{\partial \mathbf{E}} \mathbf{F}^T \;. \tag{6.69}$$

Now, consider a body made of non-linearly elastic material that undergoes a special motion $\boldsymbol{\chi}$, for which there exist times t_1 and $t_2 (> t_1)$, such that

$$\begin{aligned} \mathbf{x} \;&=\; \boldsymbol{\chi}(\mathbf{X}, t_1) \;=\; \boldsymbol{\chi}(\mathbf{X}, t_2) \\ \mathbf{v} \;&=\; \dot{\boldsymbol{\chi}}(\mathbf{X}, t_1) \;=\; \dot{\boldsymbol{\chi}}(\mathbf{X}, t_2) \;. \end{aligned} \tag{6.70}$$

for all **X**. This motion is referred to as a *closed cycle*. Recall next the theorem of mechanical energy balance in the form of (6.54) and integrate this equation in time between t_1 and t_2 to find that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t_2} \;=\; \int_{t_1}^{t_2} [R_b(\mathcal{P}) + R_c(\mathcal{P})] \, dt \;. \tag{6.71}$$

However, since the motion is a closed cycle, it is immediately concluded from (6.70) that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t_2} \;=\; \left[\int_{\mathcal{P}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \rho \hat{\Psi}(\mathbf{F}) \, dv \right]_{t_1}^{t_2} \;=\; 0 \;, \tag{6.72}$$

thus, also

$$\int_{t_1}^{t_2} [R_b(\mathcal{P}) + R_c(\mathcal{P})] \, dt \;=\; \int_{t_1}^{t_2} \left[\int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da \right] dt \;=\; 0 \;. \tag{6.73}$$

This proves that the work done on a non-linearly elastic solid by the external forces during a closed cycle is equal to zero.

Equation (6.54) further implies that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t} \;=\; \int_{t_1}^{t} \left[\int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da \right] dt \; . \tag{6.74}$$

This means that the work done by the external forces taking the body from its configuration at time t_1 to time $t (> t_1)$ depends only on the end states at t and t_1 and not on the path connecting these two states. This is the sense in which the non-linearly elastic material is characterized as *path-independent*.

The preceding class of non-linearly elastic materials for which there exists a strain energy function $\hat{\Psi}$ is sometimes referred to as *Green-elastic* or *hyperelastic* materials. A more general class of non-linearly elastic materials is defined by the constitutive relation

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{F}) \; . \tag{6.75}$$

Such materials are called *Cauchy-elastic* and, in general, do not satisfy the condition of worklessness in a closed cycle. Recalling the constitutive equation (6.61), it is clear that any Green-elastic material is also Cauchy-elastic.

Figure 6.3: *Orthogonal transformation of the reference configuration.*

The concept of material symmetry is now introduced for the class of Cauchy-elastic materials. To this end, let P be a material particle that occupies the point $\mathbf{X}$ in the reference configuration. Also, take an infinitesimal volume element $\mathcal{P}_0$ which contains $\mathbf{X}$ in the reference configuration. Since the material is assumed to be Cauchy-elastic, it follows that the Cauchy stress tensor for P at time t is given by (6.75). Now, subject the reference configuration to an orthogonal transformation characterized by the orthogonal tensor $\mathbf{Q}$, see

Figure 6.3. This transforms the region $\mathcal{P}_0$ to $\mathcal{P}_0'$. Note, however, that the stress at a point is independent of the specific choice of reference configuration. Hence, when expressed in terms of the deformation relative to the orthogonally transformed reference configuration, the Cauchy stress at point P is, in general, given by

$$\mathbf{T} \;=\; \hat{\mathbf{T}}'(\mathbf{FQ}) \; . \tag{6.76}$$

The preceding analysis shows that the constitutive law depends, in general, on the choice of reference configuration. For this reason, one may choose, at the expense of added notational burden, to formally write (6.75) and (6.76) as

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \; . \tag{6.77}$$

and

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0'}(\mathbf{FQ}) \; , \tag{6.78}$$

respectively.

By way of background, recall here that a *group* $\mathcal{G}$ is a set together with an operation $*$, such that the following properties hold for all elements a, b, c of the set:

(i) $a * b$ belongs to the set (closure),

(ii) $(a * b) * c = a * (b * c)$ (associativity),

(iii) There exists an element i, such that $i * a = a * i = a$ (existence of identity),

(iv) For every a, there exists an element $-a$, such that $a * (-a) = (-a) * a = i$ (existence of inverse).

It is easy to confirm that the set of all orthogonal transformations $\mathbf{Q}$ of the original reference configuration forms a group under the usual tensor multiplication, called the *orthogonal group* or $O(3)$. In this group, the identity element is the identity tensor $\mathbf{I}$ and the inverse element is the inverse (or transpose) $\mathbf{Q}^{-1}$ (or $\mathbf{Q}^T$) of any given element $\mathbf{Q}$. The subgroup[1]

[1] A subset of the group set together with the group operation is called a *subgroup* if it satisfies the closure property within the subset.

$\mathcal{G}_{\mathcal{P}_0} \subseteq O(3)$ is called a *symmetry group* for the Cauchy-elastic material with respect to the reference configuration $\mathcal{P}_0$ if

$$\hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{Q}) \;, \tag{6.79}$$

for all $\mathbf{Q} \in \mathcal{G}_{\mathcal{P}_0}$. Physically, equation (6.79) identifies orthogonal transformation $\mathbf{Q}$ which produce the same stress at P under two different loading cases. The first one subjects the reference configuration to a deformation gradient $\mathbf{F}$. The second one subjects the reference configuration to an orthogonal transformation $\mathbf{Q}$ and then to the deformation gradient $\mathbf{F}$. If the stress in both cases is the same, then the orthogonal transformation characterizes the material symmetry of the body in the neighborhood of P relative to the reference configuration $\mathcal{P}_0$.

If equation (6.79) holds for all $\mathbf{Q} \in O(3)$, then the Cauchy-elastic material is termed *isotropic* relative to the configuration $\mathcal{P}_0$. An isotropic material is insensitive to any orthogonal transformation of its reference configuration. Choosing $\mathbf{Q} = \mathbf{R}^T$ and recalling the left polar decomposition of the deformation gradient, equation (6.79) implies that

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{R}^T) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}\mathbf{R}\mathbf{R}^T) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}) \;. \tag{6.80}$$

In addition, invariance of $\hat{\mathbf{T}}_{\mathcal{P}_0}$ under superposed rigid motions leads to the condition

$$\mathbf{Q}\hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{Q}\mathbf{V}\mathbf{Q}^T) \;, \tag{6.81}$$

for all proper orthogonal tensors $\mathbf{Q}$ (hence, given that (6.80) is quadratic in $\mathbf{Q}$, all orthogonal $\mathbf{Q}$). This means that $\hat{\mathbf{T}}_{\mathcal{P}_0}$ is an isotropic tensor function of $\mathbf{V}$. Invoking the first representation theorem for isotropic tensor functions, it follows that

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}) \;=\; a_0\mathbf{i} + a_1\mathbf{V} + a_2\mathbf{V}^2 \;, \tag{6.82}$$

where a_0, a_1, and a_2 are functions of the three principal invariants of $\mathbf{V}$. An alternative representation of the Cauchy stress of a Cauchy-elastic material is

$$\mathbf{T} \;=\; \bar{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{B}) \;=\; b_0\mathbf{i} + b_1\mathbf{B} + b_2\mathbf{B}^2 \;, \tag{6.83}$$

where, now, b_0, b_1, and b_2 are functions of the three principal invariants of $\mathbf{V}$. This can be trivially derived by noting that $\bar{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{B}) = \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V})$.

6.5 Linearly elastic solid

In this section, a formal procedure is followed to obtain the equations of motion and the constitutive equations for a linearly elastic solid. To this end, start by writing the linearized version of linear momentum balance as

$$\mathcal{L}[\operatorname{div}\mathbf{T};\mathbf{H}]_{\mathbf{0}} + \mathcal{L}[\rho\mathbf{b};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\rho\mathbf{a};\mathbf{H}]_{\mathbf{0}} \; . \tag{6.84}$$

Now, proceed by making three assumptions:

First, let the reference configuration be stress-free, i.e., assume that if $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}) = \bar{\mathbf{T}}(\mathbf{H})$, then

$$\hat{\mathbf{T}}(\mathbf{I}) = \bar{\mathbf{T}}(\mathbf{0}) \;=\; \mathbf{0} \; . \tag{6.85}$$

It follows that

$$\mathcal{L}[\mathbf{T};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{T}}(\mathbf{0}) + D\bar{\mathbf{T}}(\mathbf{0},\mathbf{H}) \;=\; D\bar{\mathbf{T}}(\mathbf{0},\mathbf{H}) \; , \tag{6.86}$$

where

$$D\bar{\mathbf{T}}(\mathbf{0},\mathbf{H}) \;=\; \left[\frac{d}{d\omega}\bar{\mathbf{T}}(\mathbf{0}+\omega\mathbf{H})\right]_{\omega=0} \;=\; \mathbb{C}\mathbf{H} \; . \tag{6.87}$$

The quantity $\mathbb{C}$ is called the *elasticity tensor*. This is a fourth-order tensor that can be resolved in components as

$$\mathbb{C} \;=\; C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \; . \tag{6.88}$$

The product $\mathbb{C}\mathbf{H}$ in (6.87) can be written explicitly as

$$\begin{aligned} \mathbb{C}\mathbf{H} &= (C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(H_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) \\ &= C_{ijkl}H_{mn}\mathbf{e}_i \otimes \mathbf{e}_j[(\mathbf{e}_k \otimes \mathbf{e}_l)\cdot(\mathbf{e}_m \otimes \mathbf{e}_n)] \\ &= C_{ijkl}H_{mn}\mathbf{e}_i \otimes \mathbf{e}_j[\delta_{km}\delta_{ln}] \\ &= C_{ijkl}H_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \; . \end{aligned} \tag{6.89}$$

At this state, note that invariance under superposed rigid motions implies that

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{F}) \; , \tag{6.90}$$

for all proper orthogonal tensors $\mathbf{Q}$. Taking into account (6.85), it is concluded from the above equation that

$$\hat{\mathbf{T}}(\mathbf{Q}(t_0)) \;=\; \mathbf{0} \; , \tag{6.91}$$

namely that any rigid rotation results in no stress. Hence, one may choose a special such rotation for which $\mathbf{Q}(t_0) = \mathbf{I}$ (hence, also $\mathbf{H}(t_0) = \mathbf{0}$) and $\dot{\mathbf{Q}}(t_0) = \mathbf{\Omega}_0$ (hence, also $\dot{\mathbf{H}} = \mathbf{\Omega}_0$), where $\mathbf{\Omega}_0$ is a constant skew-symmetric tensor. In this case, one may write that

$$\left[\frac{d}{d\omega}\bar{\mathbf{T}}(\mathbf{0} + \omega\dot{\mathbf{H}})\right]_{\omega=0} \;=\; D\bar{\mathbf{T}}(\mathbf{0}, \dot{\mathbf{H}}) \;=\; \mathbb{C}\dot{\mathbf{H}} \;=\; \mathbb{C}\mathbf{\Omega}_0 \;=\; \mathbf{0}\ . \tag{6.92}$$

Since $\mathbf{\Omega}_0$ is an arbitrarily chosen skew-tensor, this means that $\mathbb{C}\mathbf{\Omega} = \mathbf{0}$ for any skew-symmetric tensor $\mathbf{\Omega}$. Recalling (6.86), (6.87) and that $\mathbf{H} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, it follows that

$$\mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \;=\; \mathbb{C}(\boldsymbol{\varepsilon} + \boldsymbol{\omega}) \;=\; \mathbb{C}\boldsymbol{\varepsilon} \;=\; \boldsymbol{\sigma}\ . \tag{6.93}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor of the theory of linear elasticity.

Since the distinction between partial derivatives with respect to $\mathbf{X}$ and $\mathbf{x}$ disappears in the infinitesimal case, it is clear that so does the distinction between the "Div" and "div" operators. Therefore,

$$\mathcal{L}[\operatorname{div}\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\operatorname{Div}\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \;=\; \operatorname{Div}\mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \;=\; \operatorname{Div}\boldsymbol{\sigma}\ . \tag{6.94}$$

By way of a second assumption, write

$$\mathcal{L}[\rho\mathbf{a}; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\rho}(\mathbf{0})\bar{\mathbf{a}}(\mathbf{0}) + [D\rho(\mathbf{0}, \mathbf{H})]\bar{\mathbf{a}}(\mathbf{0}) + \bar{\rho}(\mathbf{0})[D\mathbf{a}(\mathbf{0}, \mathbf{H})]\ . \tag{6.95}$$

With reference to (6.95), assume that in the linearized theory the acceleration $\mathbf{a}$ satisfies $\bar{\mathbf{a}}(\mathbf{0}) = \mathbf{0}$ (i.e., there is no acceleration in the reference configuration) and also that $[D\mathbf{a}(\mathbf{0}, \mathbf{H})] = \mathbf{a}$ (i.e., the acceleration is linear in $\mathbf{H}$). It follows from (6.95) and the preceding assumptions that

$$\mathcal{L}[\rho\mathbf{a}; \mathbf{H}]_{\mathbf{0}} \;=\; \rho_0\mathbf{a}\ . \tag{6.96}$$

Finally, noting first that

$$\mathcal{L}[\rho\mathbf{b}; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\rho}(\mathbf{0})\bar{\mathbf{b}}(\mathbf{0}) + [D\rho(\mathbf{0}, \mathbf{H})]\bar{\mathbf{b}}(\mathbf{0}) + \rho(\mathbf{0})[D\mathbf{b}(\mathbf{0}, \mathbf{H})] \tag{6.97}$$

assume that in the linearized theory $\bar{\mathbf{b}}(\mathbf{0}) = \mathbf{0}$ (which, in view of the earlier two assumptions, is tantamount to admitting that equilibrium holds in the reference configuration) and also that $[D\mathbf{b}(\mathbf{0}, \mathbf{H})] = \mathbf{b}$ (i.e., the body force is linear in $\mathbf{H}$). With (6.97) and the preceding assumptions on $\mathbf{b}$ in place, it is easily seen that

$$\mathcal{L}[\rho\mathbf{b}; \mathbf{H}]_{\mathbf{0}} \;=\; \rho_0\mathbf{b}\ . \tag{6.98}$$

Taking into account (6.94), (6.96) and (6.98), equation (6.84) reduces to

$$\operatorname{Div} \boldsymbol{\sigma} + \rho_0 \mathbf{b} \;=\; \rho_0 \mathbf{a} \; . \tag{6.99}$$

Of course, neither $\mathbf{a}$ nor $\mathbf{b}$ are *explicit* functions of the deformation. The acceleration $\mathbf{a}$ depends on the deformation implicitly in the sense that the latter is obtained from the motion $\boldsymbol{\chi}$ whose second time derivative equals $\mathbf{a}$. On the other hand, $\mathbf{b}$ depends on the motion (and deformation) implicitly through the balance of linear momentum.

In the context of linear elasticity, all measures of stress coincide, namely the distinction between the Cauchy stress $\mathbf{T}$ and other stress tensors such as $\mathbf{P}, \mathbf{S}$, etc, disappears. To see this point, recall the relation between $\mathbf{T}$ and $\mathbf{P}$ in (4.130) and take the linear part of both sides to conclude that

$$\mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}\left[\frac{1}{J}\mathbf{P}\mathbf{F}^T; \mathbf{H}\right]_{\mathbf{0}} \; , \tag{6.100}$$

which, in light of (6.93), implies that

$$\boldsymbol{\sigma} \;=\; \frac{1}{\bar{J}(\mathbf{0})}\bar{\mathbf{P}}(\mathbf{0})\bar{\mathbf{F}}^T(\mathbf{0}) + \left[D\frac{1}{J}(\mathbf{0}, \mathbf{H})\right]\bar{\mathbf{P}}(\mathbf{0})\bar{\mathbf{F}}^T(\mathbf{0}) + \frac{1}{\bar{J}(\mathbf{0})}[D\mathbf{P}(\mathbf{0}, \mathbf{H})]\bar{\mathbf{F}}^T(\mathbf{0}) + \frac{1}{\bar{J}(\mathbf{0})}\bar{\mathbf{P}}(\mathbf{0})[D\mathbf{F}^T(\mathbf{0}, \mathbf{H})] \; . \tag{6.101}$$

Recalling that the reference configuration is assumed stress-free, it follows from the above equation that

$$\boldsymbol{\sigma} \;=\; D\mathbf{P}(\mathbf{0}, \mathbf{H}) \; , \tag{6.102}$$

which implies further that

$$\mathcal{L}[\mathbf{P}; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{P}}(\mathbf{0}) + D\mathbf{P}(\mathbf{0}, \mathbf{H}) \;=\; \boldsymbol{\sigma} \; , \tag{6.103}$$

hence,

$$\mathcal{L}[\mathbf{P}; \mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} \; . \tag{6.104}$$

Similar derivations apply to deduce equivalency of $\mathbf{T}$ to other stress tensors in the infinitesimal theory.

Return now to the constitutive law (6.93) and write it in component form as

$$\sigma_{ij} \;= C_{ijkl}\varepsilon_{kl} \; . \tag{6.105}$$

In general, the fourth-order elasticity tensor possesses $3 \times 3 \times 3 \times 3 = 81$ material constants C_{ijkl}. However, since balance of angular momentum implies that $\sigma_{ij} = \sigma ji$ and also, by the definition of $\boldsymbol{\varepsilon}$ in (5.32), $\varepsilon_{ij} = \varepsilon_{ji}$, it follows that

$$C_{ijkl} \;=\; C_{jikl} \;=\; C_{ijlk} \;=\; C_{jilk} \;, \tag{6.106}$$

which readily implies that only $6 \times 6 = 36$ of these components are independent.[2]

Next, note that in the infinitesimal theory, equation (6.93) may be derived from a strain energy function $W = W(\boldsymbol{\varepsilon})$ per unit volume as

$$\boldsymbol{\sigma} \;=\; \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \;, \tag{6.107}$$

where

$$W \;=\; \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon} \;. \tag{6.108}$$

It follows from (6.108) and (6.107) that

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \;=\; \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \;=\; C_{ijkl} \;, \tag{6.109}$$

which, in turn, implies that $C_{ijkl} = C_{klij}$. The preceding identity reduces the number of independent material parameters from 36 to $36 - \frac{36}{2} + 3 = 21$.[3]

The number of independent material parameters can be further reduced by material symmetry. To see this point, recall the constitutive equations (6.82) or (6.83) for the isotropic non-linearly elastic solid and note that a corresponding equation can be easily deduced in terms of the Almansi strain $\mathbf{e}$, namely

$$\mathbf{T} \;=\; \tilde{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{e}) \;=\; c_0 \mathbf{i} + c_1 \mathbf{e} + c_2 \mathbf{e}^2 \;, \tag{6.110}$$

where c_0, c_1, and c_2 are function of the three principal invariants I_e, II_e and III_e of $\mathbf{e}$. Recalling $(5.32)_2$ and noting that, by assumption, the reference configuration is stress-free, it is easy to see that the linearization of (6.110) leads to

$$\boldsymbol{\sigma} \;=\; c_0^* I_\varepsilon \mathbf{I} + c_1 \boldsymbol{\varepsilon} \;, \tag{6.111}$$

[2] To see this, take each pair (i,j) or (k,l) and use (6.106) to conclude that only 6 combinations of each pair are independent.

[3] To see this, write the 36 parameters as a 6×6 matrix and argue that only the terms on and above (or below) the major diagonal are independent.

where c_0^* and c_1 are constants. Setting $c_0^* = \lambda$ and $c_1 = 2\mu$, one may rewrite the preceding equation as

$$\boldsymbol{\sigma} \;=\; \lambda \operatorname{tr} \varepsilon \mathbf{I} + 2\mu\boldsymbol{\varepsilon} \; . \tag{6.112}$$

The parameters λ and μ are known as the *Lamé constants*. These are related to the *Young's modulus* E and the *Poisson's ratio* ν by

$$\lambda \;=\; \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu \;=\; \frac{E}{2(1+\nu)} \tag{6.113}$$

and, inversely,

$$E \;=\; \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad , \quad \nu \;=\; \frac{\lambda}{2(\lambda+\mu)} \; . \tag{6.114}$$

6.6 Viscoelastic solid

Most materials exhibit some memory effects, i.e., their current state of stress depends not only on the current state of deformation, but also on the deformation history.

Consider first a broad class of materials with memory, whose Cauchy stress is given by

$$\mathbf{T}(\mathbf{X}, t) \;=\; \hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{F}(\mathbf{X}, \tau)]; \mathbf{X}) \; . \tag{6.115}$$

This means that the Cauchy stress at time t of some material particle P which occupies point $\mathbf{X}$ in the reference configuration depends on the history of the deformation gradient of that point up to (and including) time t. Materials that satisfy the constitutive law (6.115) are called *simple*.

Invoking invariance under superposed rigid motions and suppressing, in the interest of brevity, the explicit dependence of functions on $\mathbf{X}$, it is concluded that

$$\mathbf{Q}(t)\hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{F}(\tau)])\mathbf{Q}^T(t) \;=\; \hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{Q}(\tau)\mathbf{F}(\tau)]) \; , \tag{6.116}$$

for all proper orthogonal $\mathbf{Q}(\tau)$, where $-\infty < \tau \le t$. Choosing $\mathbf{Q}(\tau) = \mathbf{R}^T(\tau)$, for all $\tau \in (-\infty, t]$, it follows that

$$\mathbf{R}^T(t)\hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{F}(\tau)])\mathbf{R}(t) \;=\; \hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{U}(\tau)]) \; , \tag{6.117}$$

where, as usual, $\mathbf{F}(\tau) = \mathbf{R}(\tau)\mathbf{U}(\tau)$. Equation (6.117) can be readily rewritten as

$$\mathbf{T}(t) \;=\; \mathbf{R}(t)\hat{\mathbf{T}}(\underset{\tau \le t}{\mathfrak{H}}[\mathbf{U}(\tau)])\mathbf{R}^T(t) \; . \tag{6.118}$$

or, equivalently,

$$\mathbf{T}(t) \;=\; \mathbf{F}(t)\mathbf{U}^{-1}(t)\hat{\mathbf{T}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{U}(\tau)])\mathbf{U}^{-1}(t)\mathbf{F}^T(t) \;. \tag{6.119}$$

Upon recalling $(4.133)_2$, this means that

$$\mathbf{S}(t) \;=\; J(t)\mathbf{U}^{-1}(t)\hat{\mathbf{T}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{U}(\tau)])\mathbf{U}^{-1}(t) \;=\; \hat{\mathbf{S}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{U}(\tau)]) \;. \tag{6.120}$$

As usual, one may alternatively write

$$\mathbf{S}(t) \;=\; \bar{\mathbf{S}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{C}(\tau)]) \;=\; \check{\mathbf{S}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{E}(\tau)]) \;. \tag{6.121}$$

Now, proceed by distinguishing between the past ($\tau < t$) and the present ($\tau = t$) in the preceding constitutive laws. To this end, define

$$\mathbf{E}_t(s) \;=\; \mathbf{E}(t-s) - \mathbf{E}(t) \;, \tag{6.122}$$

where, obviously, $\mathbf{E}_t(0) = \mathbf{0}$. Clearly, for any given time t, the variable s is probing the history of the Lagrangian strain looking further in the past as s increases.

Next, rewrite $(6.121)_2$ as

$$\mathbf{S}(t) \;=\; \check{\mathbf{S}}(\underset{\tau\le t}{\mathfrak{H}}[\mathbf{E}(\tau)]) \;=\; \check{\mathbf{S}}(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]) \;. \tag{6.123}$$

Then, define the *elastic response function* $\mathbf{S}^e$ as

$$\mathbf{S}^e(\mathbf{E}(t)) \;=\; \check{\mathbf{S}}(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{0},\mathbf{E}(t)]) \;. \tag{6.124}$$

and the *memory response function* $\mathbf{S}^m$ as

$$\mathbf{S}^m(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]) \;=\; \check{\mathbf{S}}(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]) \;-\; \check{\mathbf{S}}(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{0},\mathbf{E}(t)]) \;. \tag{6.125}$$

In summary, the stress response is given by

$$\mathbf{S}(t) \;=\; \mathbf{S}^e(\mathbf{E}(t)) \;+\; \mathbf{S}^m(\underset{s\ge 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]) \;. \tag{6.126}$$

The first term on the right-hand side of (6.126) is the stress which depends exclusively on the present state of the Lagrangian strain. The second term on the right-hand side of (6.126) contains all the dependency of the stress response on past Lagrangian strain states.

Note that, by definition, the stress during a state deformation, i.e., which $\mathbf{E}(t) = \mathbf{E}_0$, where $\mathbf{E}_0$ is a constant, is equal to $\mathbf{S}(t) = \mathbf{S}^e(\mathbf{E}_0)$, or, equivalently, $\mathbf{S}^m(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{0}, \mathbf{E}(t)]) = \mathbf{0}$.

All viscoelastic materials can be described by the constitutive equation (6.126). For such materials, $\mathbf{S}^m$ is rate-dependent (i.e., it depends on the rate $\dot{\mathbf{E}}$ of Lagrangian strain) and also exhibits *fading memory*. The latter means that the impact on the stress at time t of the deformation at time $t - s$ ($s > 0$) diminishes as $s \rightarrow \infty$. This condition can be expressed mathematically as

$$\lim_{\delta\rightarrow\infty} \mathbf{S}^m(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t^\delta(s), \mathbf{E}(t)]) \;=\; \mathbf{0}\;, \tag{6.127}$$

where

$$\mathbf{E}_t^\delta(s) \;=\; \begin{cases} 0 & \text{if } 0 \leq s < \delta \\ \mathbf{E}^t(s-\delta) & \text{if } \delta \leq s < \infty \end{cases} \tag{6.128}$$

is the *static continuation* of $\mathbf{E}_t(s)$ by δ. With reference to Figure 6.4, it is seen that the static

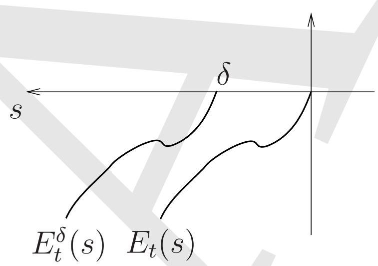

Figure 6.4: *Static continuation $E_t^\delta(s)$ of $E_t(s)$ by δ.*

continuation is a rigid translation of the argument $\mathbf{E}_t(s)$ of the memory response function $\mathbf{S}^m$ by δ. Therefore, the fading memory condition (6.127) implies that, as time elapses, the effect of earlier Lagrangian strain states on $\mathbf{S}^m$ continuously diminishes and, ultimately, disappears altogether. The condition (6.127) is often referred to as the *relaxation property*. This is because it implies that any time-dependent Lagrangian strain process which reaches a steady-state results in memory response which ultimately relaxes to zero memory stress (plus, possibly, elastic stress), see Figure 6.5.

Under special regularity conditions, the memory response function bS^m can be reduced to the linear functional form

$$\mathbf{S}^m(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]) \;=\; \int_0^\infty \boldsymbol{\mathcal{L}}(\mathbf{E}(t), s)\mathbf{E}_t(s)\, dS\;, \tag{6.129}$$

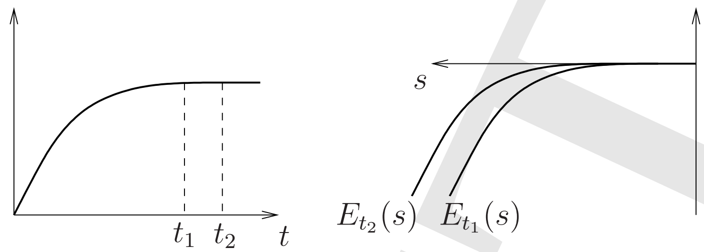

Figure 6.5: *An interpretation of relaxation*

where $\mathcal{L}(\mathbf{E}(t), s)$ is a fourth-order tensor function of $\mathbf{E}(t)$ and s. Of course, $\mathcal{L}(\mathbf{E}(t), s)$ needs to be chosen so that $\mathbf{S}^m$ satisfy the relaxation property (6.127).

Upon Taylor expansion of $\mathbf{E}_t(s)$ around $t-s$, one finds that

$$\mathbf{E}_t(s) \; = \mathbf{E}(t-s) - \mathbf{E}(t) \; = \; -s\dot{\mathbf{E}}(t-s) + o(s^2) \; . \tag{6.130}$$

Ignoring the second-order term in (6.130), which is tantamount to neglecting long-term memory effects, one may substitute $\mathbf{E}_t(s)$ in (6.129) to find that

$$\mathbf{S}^m(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]) \; = \; \int_0^\infty \mathcal{L}(\mathbf{E}(t), s)\{-s\dot{\mathbf{E}}(t-s)\}\, ds \; = \; \int_0^\infty \bar{\mathcal{L}}(\mathbf{E}(t), s)\dot{\mathbf{E}}(t-s)\, ds \; . \tag{6.131}$$

Alternatively, upon Taylor expansion of $\mathbf{E}_t(s)$ around t, one finds that

$$\mathbf{E}_t(s) \; = \mathbf{E}(t-s) - \mathbf{E}(t) \; = \; -s\dot{\mathbf{E}}(t) + o(s^2) \; , \tag{6.132}$$

which leads to

$$\begin{aligned} \mathbf{S}^m(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]) \; &= \; \int_0^\infty \mathcal{L}(\mathbf{E}(t), s)\{-s\dot{\mathbf{E}}(t)\}\, ds \; = \; \left[-\int_0^\infty \mathcal{L}(\mathbf{E}(t), s)s\, ds\right]\dot{\mathbf{E}}(t) \\ &= \boldsymbol{\mathcal{M}}(\mathbf{E}(t))\dot{\mathbf{E}}(t) \; , \end{aligned} \tag{6.133}$$

where

$$\boldsymbol{\mathcal{M}}(\mathbf{E}(t)) \; = \; -\int_0^\infty \mathcal{L}(\mathbf{E}(t), s)s\, ds \; . \tag{6.134}$$

Now, attempt to reconcile the general constitutive framework developed here with the classical one-dimensional viscoelasticity models of Kelvin-Voigt and Maxwell. Start with the Kelvin-Voigt model, which is comprised of a spring and a dashpot in parallel, and let the spring constant be E and the dashpot constant be η. It follows that the uniaxial stress σ is related to the uniaxial strain ε by

$$\sigma \;=\; E\varepsilon + \eta\dot{\varepsilon} \; . \tag{6.135}$$

Clearly, this law is a simple reduction of (6.126), where the memory response is obtained from a one-dimensional counterpart of (6.133).

Next, consider the Maxwell model of a spring and dashpot in series with material properties as in the Kelvin-Vogt model. In this case, the constitutive law becomes

$$\dot{\varepsilon} \;=\; \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \; , \tag{6.136}$$

and take the accompanying initial condition to be $\sigma(0) = 0$. The general solution of (6.136) is

$$\sigma(t) \;=\; c(t)e^{-\frac{E}{\eta}t} \; , \tag{6.137}$$

which, upon substituting into (6.136) leads to

$$\dot{c}(t) \;=\; Ee^{\frac{E}{\eta}t}\dot{\varepsilon}(t) \; . \tag{6.138}$$

This, in turn, may be integrated to yield

$$c(t) \;=\; c(0) + \int_0^t Ee^{\frac{E}{\eta}\tau}\dot{\varepsilon}(\tau)\, d\tau \; . \tag{6.139}$$

Noting that the initial condition results in $c(0) = 0$, one may write that

$$\begin{aligned} \sigma(t) \;&=\; \left[\int_0^t Ee^{\frac{E}{\eta}\tau}\dot{\varepsilon}(\tau)\, d\tau\right] e^{-\frac{E}{\eta}t} \;=\; \int_0^t Ee^{\frac{E}{\eta}(\tau - t)}\dot{\varepsilon}(\tau)\, d\tau \\ &=\; \int_0^t Ee^{-\frac{E}{\eta}s}\dot{\varepsilon}(t-s)\, ds \; . \end{aligned} \tag{6.140}$$

Clearly, the stress response of the Maxwell model falls within the constitutive framework of (6.126), where the elastic response function vanishes identically and the memory response can be deduced from (6.131).

Chapter 7

Boundary- and Initial/boundary-value Problems

7.1 Incompressible Newtonian viscous fluid

Parts of this chapter include material taken from the continuum mechanics notes of the late Professor P.M. Naghdi.

7.1.1 Gravity-driven flow down an inclined plane

Consider an incompressible Newtonian viscous fluid under steady flow down an inclined plan due to gravity, see Figure 7.1. Let the pressure of the free surface be p_0 and assume that the fluid region has constant depth h.

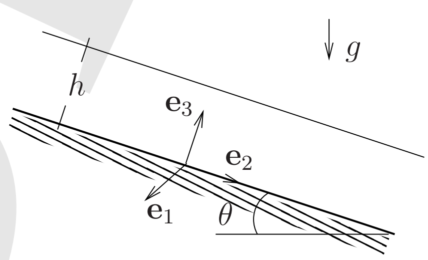

Figure 7.1: *Flow down an inclined plane*

Assume that the velocity and pressure fields are of the form

$$\mathbf{v} \;=\; \tilde{v}(x_2, x_3)\mathbf{e}_2 \tag{7.1}$$

and

$$p \;=\; \tilde{p}(x_1, x_2, x_3) \;, \tag{7.2}$$

respectively. Incompressibility implies that

$$\operatorname{div} \mathbf{v} \;=\; \frac{\partial \tilde{\mathbf{v}}}{\partial x_2} \;=\; 0 \;, \tag{7.3}$$

which means that the velocity field is independent of x_2, namely that

$$\mathbf{v} \;=\; \bar{v}(x_3)\mathbf{e}_2 \;. \tag{7.4}$$

Given the reduced velocity field in (7.4), the rate-of-deformation tensor has components

$$[D_{ij}] \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\frac{\partial \bar{v}}{\partial x_3} \\ 0 & \frac{1}{2}\frac{\partial \bar{v}}{\partial x_3} & 0 \end{bmatrix} \;. \tag{7.5}$$

Recalling the constitutive equation (6.50), the Cauchy stress is given by

$$[T_{ij}] \;=\; \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu\frac{\partial \bar{v}}{\partial x_3} \\ 0 & \mu\frac{\partial \bar{v}}{\partial x_3} & -p \end{bmatrix} \;. \tag{7.6}$$

Note that gravity induces a body force per unit mass equal to

$$\mathbf{b} \;=\; g(\sin\theta \mathbf{e}_2 - \cos\theta \mathbf{e}_3) \;, \tag{7.7}$$

where g is the gravitational constant. Given (7.2), (7.4), (7.6) and (7.7), the equations of linear momentum balance assume the form

$$\begin{aligned} -\frac{\partial \tilde{p}}{\partial x_1} \;&= 0 \\ -\frac{\partial \tilde{p}}{\partial x_2} + \mu\frac{d^2\bar{v}}{dx_3^2} + \rho g \sin\theta \;&= \rho\left[\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial x_k}v_k\right] \;. \\ -\frac{\partial \tilde{p}}{\partial x_3} - \rho g \cos\theta \;&= 0 \end{aligned} \tag{7.8}$$

It follows from $(7.8)_{1,3}$ that

$$p = \bar{p}(x_2, x_3) = -\rho g \cos\theta x_3 + f(x_2) \ . \tag{7.9}$$

Imposing the boundary condition on the free surface leads to

$$\bar{p}(x_2, h) = -\rho g \cos\theta h + f(x_2) = p_0 \ , \tag{7.10}$$

which implies that the function $f(x_2)$ is constant and equal to

$$f(x_2) = p_0 + \rho g \cos\theta h \ . \tag{7.11}$$

Substituting this equation to (7.9) results in an expression for the pressure as

$$p = p_0 + \rho g \cos\theta(h - x_3) \ . \tag{7.12}$$

Substituting the pressure from (7.12) into the remaining momentum balance equation $(7.8)_2$ yields

$$\mu \frac{d^2\bar{v}}{dx_3^2} + \rho g \sin\theta = 0 \ , \tag{7.13}$$

which may be integrated to

$$\bar{v}(x_3) = \frac{-\rho g \sin\theta}{2\mu} x_3^2 + c_1 x + c_2 \ . \tag{7.14}$$

Enforcing the boundary conditions $\bar{v}(0) = 0$ (no slip condition on the solid-fluid interface) and $T_{23}(h) = 0$ (no shear stress on the free surface), gives $c_2 = 0$ and $c_1 = \dfrac{\rho g h \sin\theta}{\mu}$, which, when substituted into (7.14) yield

$$\bar{v}(x_3) = \frac{\rho g \sin\theta}{\mu} x_3 (h - \frac{x_3}{2}) \ . \tag{7.15}$$

It is seen from (7.15) that the velocity distribution is parabolic along x_3 and attains a maximum value of $v_{max} = \dfrac{\rho g \sin\theta}{2\mu} h^2$ on the free surface.

7.1.2 Couette flow

This is the steady flow between two concentric cylinder of radii R_o (outer cylinder) and R_i (inner cylinder) rotating with constant angular velocities ω_o (outer cylinder) and ω_i (inner

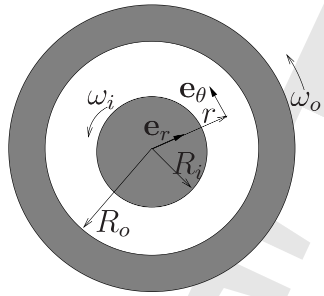

Figure 7.2: *Couette flow*

cylinder), see Figure 7.2. The fluid is assumed Newtonian and incompressible. Also, the effect of body forces is neglected.

The problem lends itself naturally to analysis using cylindrical polar coordinates with basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The velocity and pressure fields are assumed axisymmetric and, using the cylindrical polar coordinate representation, can be expressed as

$$\mathbf{v} \ = \ \bar{\mathbf{v}}(r)\mathbf{e}_\theta \tag{7.16}$$

and

$$p \ = \ \bar{p}(r) \ . \tag{7.17}$$

Taking into account (A.9), the spatial velocity gradient can be written as

$$\mathbf{L} \ = \ \frac{d\bar{v}(r)}{dr}\mathbf{e}_\theta \otimes \mathbf{e}_r - \frac{\bar{v}}{r}\mathbf{e}_r \otimes \mathbf{e}_\theta \ , \tag{7.18}$$

so that

$$\mathbf{D} \ = \ \frac{r}{2}\frac{d}{dr}\left(\frac{\bar{v}(r)}{r}\right)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \ . \tag{7.19}$$

It follows from (7.19) that

$$\mathbf{T} \ = \ -p\mathbf{i} + \mu r\frac{d}{dr}\left(\frac{\bar{v}(r)}{r}\right)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \ . \tag{7.20}$$

Given the form of the velocity in (7.16), it is clear that $\operatorname{div}\mathbf{v} = 0$, hence the incompressibility condition is satisfied at the outset. Also, the acceleration becomes

$$\mathbf{a} \ = \ \left(\frac{d\bar{v}(r)}{dr}\mathbf{e}_\theta \otimes \mathbf{e}_r - \bar{v}\frac{1}{r}\mathbf{e}_r \otimes \mathbf{e}\theta\right)\bar{v}\mathbf{e}_\theta \ = \ -\frac{\bar{v}^2}{r}\mathbf{e}_r \ . \tag{7.21}$$

Taking into account (7.20) and (7.21) the linear momentum balance equations in the r- and θ-directions become

$$\begin{aligned} -\frac{dp}{dr} &= -\rho\frac{\bar{v}^2(r)}{dr} \\ \mu\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{v}{r}\right)\right] + 2\mu\frac{d}{dr}\left(\frac{v}{r}\right) &= 0 \,, \end{aligned} \tag{7.22}$$

respectively.

The second of the above equations may be integrated twice to give

$$\bar{v}(r) = c_1 r + \frac{c_2}{r} \,. \tag{7.23}$$

The integration constants c_1 and c_2 can be determined by imposing the boundary conditions $\bar{v}(R_i) = \omega_i R_i$ and $\bar{v}(R_o) = \omega_o R_o$. Upon determining these constants, the velocity of the flow takes the form

$$\bar{v}(r) = \omega_o R_o \frac{\frac{R_o}{R_i}\left(\frac{r}{R_i} - \frac{R_i}{r}\right) + \frac{\omega_i}{\omega_o}\left(\frac{R_o}{r} - \frac{r}{R_o}\right)}{\left(\frac{R_o}{R_i}\right)^2 - 1} \,. \tag{7.24}$$

Finally, integrating equation $(7.22)_1$ and using a pressure boundary condition such as, e.g., $\bar{p}(R_0) = p_0$, leads to an expression for the pressure $\bar{p}(r)$.

7.1.3 Poiseuille flow

This is the flow of an incompressible Newtonian viscous fluid through a straight cylindrical pipe of radius R. Assuming that the pipe is aligned with the $\mathbf{e}_3$-axis, assume that the velocity of the fluid of the general form

$$\mathbf{v} = v(r)\mathbf{e}_z \,, \tag{7.25}$$

while the pressure is

$$p = \bar{p}(r,z) \,. \tag{7.26}$$

Clearly, the assumed velocity field satisfies the incompressibility condition at the outset.

Taking again into account (A.9), the velocity gradient for this flow is

$$\mathbf{L} = \frac{\partial v}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r \,, \tag{7.27}$$

hence the rate-of-deformation tensor is

$$\mathbf{D} \;=\; \frac{1}{2}\frac{\partial v}{\partial r}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r)\ . \tag{7.28}$$

Then, the Cauchy becomes

$$\mathbf{T} \;=\; -p\mathbf{i} + \mu\frac{\partial v}{\partial r}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r)\ . \tag{7.29}$$

Noting that the acceleration vanishes identically, the equations of linear momentum balance in the absence of body force take the form

$$\begin{aligned} -\frac{\partial p}{\partial r} &= 0 \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} &= 0\ , \\ -\frac{\partial p}{\partial z} + \mu\frac{d^2 v}{dr^2} + \frac{\mu}{r}\frac{dv}{dr} &= 0 \end{aligned} \tag{7.30}$$

where (A.11) is employed. Equation $(7.30)_2$ is satisfied identically due to the assumption (7.26), while equation $(7.30)_1$ implies that $p = p(z)$. However, given that $\mathbf{v}$ depends only on r, equation $(7.30)_3$ requires that

$$\frac{dp}{dz} \;=\; c\ , \tag{7.31}$$

where c is a constant. Upon integrating $(7.30)_3$ in r, one finds that

$$v \;=\; \frac{cr^2}{4\mu} + c_1 \ln r + c_2\ , \tag{7.32}$$

where c_1 and c_2 are also constants. Admitting that the solution should remain finite at $r = 0$ and imposing the no-slip condition $v(R) = 0$, it follows that

$$v(r) \;=\; \frac{c}{4\mu}(r^2 - R_0^2)\ . \tag{7.33}$$

Two additional boundary conditions are necessary (either a velocity boundary condition on one end and a pressure boundary condition on another or pressure boundary conditions on both ends of the pipe) in order to fully determine the velocity and pressure field.

7.2 Compressible Newtonian viscous fluids

7.2.1 Stokes' First Problem

See Problem 5 in Homework 11.

7.2.2 Stokes' Second Problem

Consider the semi-infinite domain $\mathcal{R} = \{(x_1, x_2, x_3) \mid x_3 > 0\}$, which contains a Newtonian viscous fluid, see Figure 7.3. The fluid is subjected to a periodic motion of its boundary $x_3 = 0$ in the form

$$\mathbf{v}_p \ = \ U \cos \omega t \mathbf{e}_1 \ , \tag{7.34}$$

where U and ω are given positive constants. In addition, the body force is zero everywhere in the domain.

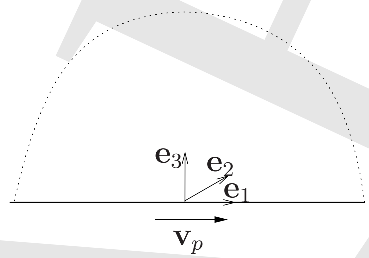

Figure 7.3: *Semi-infinite domain for Stokes' Second Problem*

Adopting a semi-inverse approach, assume a general form of the solution as

$$\mathbf{v} \ = \ f(x_3) \cos{(\omega t - \alpha x_3)} \mathbf{e}_1 \ , \tag{7.35}$$

where the function f and the constant α are to be determined. The solution (7.35) assumes that the velocity field varies along x_3 and is also phase-shifted by αx_3 relative to the boundary velocity. Further, note that the assumed solution renders the motion isochoric.

In view of (7.35), the acceleration field is

$$\mathbf{a} \ = \ -\omega f \sin{(\omega t - \alpha x_3)} \mathbf{e}_1 \ . \tag{7.36}$$

Further, mass conservation implies that $\dot{\rho} = 0$, i.e., if the fluid is initially homogeneous (which it is assumed to be), then it remains homogeneous for all time.

The only non-vanishing components of the rate-of-deformation tensor are

$$D_{23} \ = \ D_{32} \ = \ \frac{1}{2} \left[\frac{df}{dx} \cos{(\omega t - \alpha x_3)} + \alpha f \sin{(\omega t - \alpha x_3)} \right] \ . \tag{7.37}$$

Recalling (6.46), with $\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = 0$, and noting that p and μ are necessarily constant, since the mass density is homogeneous, it follows that

$$[T_{ij}] \;=\; \begin{bmatrix} -p & 0 & T_{13} \\ 0 & -p & 0 \\ T_{31} & 0 & -p \end{bmatrix} , \tag{7.38}$$

where

$$T_{13} \;=\; T_{31} \;=\; \mu\left[\frac{df}{dx}\cos\left(\omega t - \alpha x_3\right) + \alpha f \sin\left(\omega t - \alpha x_3\right)\right] . \tag{7.39}$$

Taking into account (7.36) and (7.38) it is easy to see that the linear momentum balance equations in the $\mathbf{e}_2$ and $\mathbf{e}_3$ directions hold identically. In the $\mathbf{e}_1$ direction, the momentum balance equation reduces to

$$\frac{\partial T_{13}}{\partial x_3} \;=\; \rho a_1 \tag{7.40}$$

or

$$\begin{aligned} \mu\left[\frac{d^2 f}{dx_3^2}\cos\left(\omega t - \alpha x_3\right) + 2\alpha\frac{df}{dx_3}\sin\left(\omega t - \alpha x_3\right) - \alpha^2 f\cos\left(\omega t - \alpha x_3\right)\right] \\ \;=\; \rho\omega f \sin\left(\omega t - \alpha x_3\right) . \end{aligned} \tag{7.41}$$

The preceding equation can be also written as

$$\mu\left[\frac{d^2 f}{dx_3^2} - \alpha^2 f\right]\cos\left(\omega t - \alpha x_3\right) + \left[2\mu\alpha\frac{df}{dx_3} + \rho\omega f\right]\sin\left(\omega t - \alpha x_3\right) \;=\; 0 . \tag{7.42}$$

For this equation to be satisfied identically for all x_3 and t, it is necessary and sufficient that

$$\frac{d^2 f}{dx_3^2} - \alpha^2 f \;=\; 0 \tag{7.43}$$

and

$$2\mu\alpha\frac{df}{dx_3} + \rho\omega f \;=\; 0 . \tag{7.44}$$

These two equations can be directly integrated to give

$$f(x_3) \;=\; c_1 e^{\alpha x_3} + c_2 e^{-\alpha x_3} \tag{7.45}$$

and

$$f(x_3) \;=\; c_3 e^{-\frac{\rho\omega}{2\mu\alpha}x_3} , \tag{7.46}$$

respectively. To reconcile the two solutions, one needs to take

$$f(x_3) \ = \ c e^{-\frac{\rho\omega}{2\mu\alpha}x_3} \ , \tag{7.47}$$

where $\alpha = \sqrt{\dfrac{\rho\omega}{2\mu}}$. With this expression in place, the velocity field of equation (7.35) takes the form

$$\mathbf{v} \ = \ c e^{-\frac{\rho\omega}{2\mu\alpha}x_3} \cos{(\omega t - \alpha x_3)}\mathbf{e}_1 \ . \tag{7.48}$$

Applying the boundary condition $\mathbf{v}(0) = U \cos \omega t \mathbf{e}_1$ leads to $c = U$, so that, finally,

$$\mathbf{v} \ = \ U e^{-\frac{\rho\omega}{2\mu\alpha}x_3} \cos{(\omega t - \alpha x_3)}\mathbf{e}_1 \ . \tag{7.49}$$

In this problem, the pressure p is constitutively specified, yet is here constant throughout the semi-infinite domain owing to the homogeneity of the mass density.

7.3 Linear elastic solids

Consider a homogeneous and isotropic linearly elastic solid, and recall that its stress-strain relation is given by (6.112). Taking the trace of both sides on (6.112) and assuming that $\lambda + \frac{2}{3}\mu$ is non-zero, it is easily seen that

$$\operatorname{tr} \boldsymbol{\varepsilon} \ = \ \frac{1}{3\lambda + 2\mu} \operatorname{tr} \boldsymbol{\sigma} \ , \tag{7.50}$$

hence

$$\boldsymbol{\varepsilon} \ = \ \frac{1}{2\mu}\left[\boldsymbol{\sigma} - \frac{\lambda}{3\lambda + 2\mu} \operatorname{tr} \boldsymbol{\sigma}\mathbf{I}\right] \tag{7.51}$$

or, using the elastic materials parameters E and ν,

$$\boldsymbol{\varepsilon} \ = \ \frac{1}{E}\left[(1+\nu)\boldsymbol{\sigma} - \nu \operatorname{tr} \boldsymbol{\sigma}\mathbf{I}\right] \ . \tag{7.52}$$

where (6.113) is used.

7.3.1 Simple tension and simple shear

In the case of simple tension along the $\mathbf{e}_3$-axis, $\sigma_{33} \neq 0$, while all other components $\sigma_{ij} = 0$. It follows from (7.52) that

$$\varepsilon_{33} \ = \ \frac{\sigma_{33}}{E} \quad , \quad \varepsilon_{11} \ = \ \varepsilon_{22} \ = \ -\frac{\nu\sigma_{33}}{E} \ , \tag{7.53}$$

while all shearing components of strain vanish. A simple tension experiment can be used to determine the material constants E and ν as

$$E = \frac{\sigma_{33}}{\varepsilon_{33}} \quad , \quad \nu = -\frac{\varepsilon_{11}}{\varepsilon_{33}} \; . \tag{7.54}$$

In the case of simple shear on the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$, the only non-zero components of stress is $\sigma_{23} = \sigma_{32}$. Recalling (7.52), it follows that

$$\varepsilon_{12} = \frac{\sigma_{12}}{2\mu} \; , \tag{7.55}$$

while all other strain components vanish. The elastic constant μ can be experimentally measured by arguing that 2ε is the change in the angle between the axes $\mathbf{e}_1$ and $\mathbf{e}_2$.

7.3.2 Uniform hydrostatic pressure

Suppose that a homogeneous and isotropic linearly elastic solid is subjected to a uniform hydrostatic pressure $\boldsymbol{\sigma} = -p\mathbf{I}$. Taking into account (7.51), it follows that

$$\boldsymbol{\varepsilon} = -p\frac{1}{3\lambda + 2\mu}\mathbf{I} = -p\frac{1}{3K}\mathbf{I} \; , \tag{7.56}$$

where $K = \dfrac{3\lambda + 2\mu}{3} = \dfrac{E}{3(1-2\nu)}$ is the *bulk modulus* of elasticity. Equation (7.56) can be used in an experiment to determine the bulk modulus by noting that $\operatorname{tr}\boldsymbol{\varepsilon} = -\frac{p}{K}$ is the infinitesimal change of volume due to the hydrostatic pressure p.

7.3.3 Saint-Venant torsion of a circular cylinder

Consider an isotropic homogeneous linearly elastic cylinder under quasistatic conditions. The cylinder has length L and radius R and is fixed on the one end ($x_3 = 0$), while at the opposite end ($x_3 = L$) it is subjected to a resultant moment $M\mathbf{e}_3$ relative to the point with coordinates $(0, 0, L)$. Also, the lateral sides of the cylinder are assumed traction-free.

Due to symmetry, it is assumed that the cross-section remains circular and that plane sections of constant x_3 remain plane after the deformation. With these assumptions in place, assume that the displacement of the cylinder may be written as

$$\mathbf{u} = \alpha x_3 r \mathbf{e}_\theta \; , \tag{7.57}$$

where α is the angle of twist per unit x_3-length. Recalling that $\mathbf{e}_\theta = -\mathbf{e}_1 \frac{x_2}{r} + \mathbf{e}_2 \frac{x_1}{r}$, one may rewrite the displacement as

$$\mathbf{u} \;=\; \alpha(-x_2 x_3 \mathbf{e}_1 + x_1 x_3 \mathbf{e}_2)\ . \tag{7.58}$$

It follows that the infinitesimal strain tensor has components

$$[\varepsilon_{ij}] \;=\; \begin{bmatrix} 0 & 0 & -\frac{1}{2}\alpha x_2 \\ 0 & 0 & \frac{1}{2}\alpha x_1 \\ -\frac{1}{2}\alpha x_2 & \frac{1}{2}\alpha x_1 & 0 \end{bmatrix} . \tag{7.59}$$

Hence, the stress tensor has components

$$[\sigma_{ij}] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} . \tag{7.60}$$

It can be readily demonstrated that, in the absence of body forces, all the equilibrium equations are satisfied identically. Further, for the lateral surfaces, the tractions vanish, since

$$[t_i] \;=\; [\sigma_{ij}][n_j] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \frac{1}{r} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{7.61}$$

On the other hand, the traction at $x = L$ is

$$[t_i] \;=\; [\sigma_{ij}][n_j] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;=\; \mu\alpha \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} , \tag{7.62}$$

so that the resultant force is given by

$$\begin{aligned} \int_{x_3=L} [t_i]\, dA \;&=\; \mu\alpha \int_0^{2\pi} \int_0^R r \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} r\, dr d\theta \\ &=\; \mu\alpha \frac{R^3}{3} \int_0^{2\pi} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} d\theta \;=\; \mu\alpha \frac{R^3}{3} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}_0^{2\pi} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \end{aligned} \tag{7.63}$$

Moreover, the resultant moment with respect to $(0, 0, L)$ equals

$$\begin{aligned} M\mathbf{e}_3 &= \int_{x_3=L} (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \times \mu\alpha(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2)\, dA \\ &= \mu\alpha \int_{x_3=L} (x_1^2 + x_2^2)\, dA \;=\; \mu\alpha \int_0^{2\pi}\int_0^R r^2\, r dr d\theta \;=\; \mu\alpha I \ , \end{aligned} \tag{7.64}$$

where $I = \dfrac{\pi R^4}{2}$ is the *polar moment of inertia* of the circular cross-section.

7.4 Non-linearly elastic solids

7.4.1 Rivlin's cube

Consider a unit cube made of a homogeneous, isotropic and incompressible non-linearly elastic material. First, observe that the pressure p is work-conjugate to the volume change in that

$$\mathbf{T} \cdot \mathbf{D} \;=\; (\mathbf{T}' + \frac{1}{3}\operatorname{tr}\mathbf{T}\,\mathbf{i}) \cdot (\mathbf{D}' + \frac{1}{3}\operatorname{tr}\mathbf{D}\,\mathbf{i}) \;=\; \mathbf{T}' \cdot \mathbf{D}' + (-p)\operatorname{div}\mathbf{v} \ , \tag{7.65}$$

where $\mathbf{T}'$ and $\mathbf{D}'$ are the deviatoric parts of $\mathbf{T}$ and $\mathbf{D}$, respectively. Next, recall the general form of the constitutive equations for isotropic non-linearly elastic materials in (6.83) and, assuming, as a special case, $b_2 = 0$, write

$$\mathbf{T} \;=\; -p\mathbf{i} + b_1\mathbf{B} \ , \tag{7.66}$$

where $b_1 (> 0)$ is a constant, and p is a Lagrange multiplier to be determined upon enforcing the incompressibility constraint.

Returning to the unit cube, assume that its edges are aligned with the common orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the reference and current configuration, and is loaded by three pairs of equal and opposite tensile forces, all of equal magnitude, and distributed uniformly on each face.

Taking into account (4.130) and (7.66), one may write

$$\mathbf{P} \;= J(-p\mathbf{i} + b_1\mathbf{B})\mathbf{F}^{-T} \;=\; -p\mathbf{F}^{-T} + b_1\mathbf{F} \ , \tag{7.67}$$

where $J = 1$ due to the assumption of incompressibility. The tractions, when resolved on the geometry of the reference configuration, satisfy

$$\mathbf{P}\mathbf{e}_i \;=\; c\mathbf{e}_i \ , \tag{7.68}$$

where $c > 0$ is the magnitude of the tractions per unit area in the reference configuration. Note that c is the same for all faces of the cube, since, by assumption, the force on each face is constant and uniform. Therefore, one may also write that

$$\mathbf{P} \;=\; c(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \; . \tag{7.69}$$

Solutions of the form

$$\mathbf{F} \;=\; \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \tag{7.70}$$

are sought for this boundary-value problem, subject to the condition $\lambda_1\lambda_2\lambda_3 = 1$. It is clear from (7.68) that the equilibrium equations are satisfied identically in the absence of body forces. Returning to the constitutive equations, combine (7.67) and (7.69) to conclude that

$$c \;=\; -\frac{p}{\lambda_i} + b_1\lambda_i \; , \tag{7.71}$$

or

$$b_1\lambda_i^2 \;=\; c\lambda_i + p \; , \tag{7.72}$$

where $i = 1, 2, 3$. Eliminating the pressure in the preceding equation leads to

$$b_1(\lambda_i^2 - \lambda_j^2) \;=\; c(\lambda_i - \lambda_j) \; , \tag{7.73}$$

where $i \neq j$. This, in turn, means that

$$\begin{aligned} \lambda_1 \;&=\; \lambda_2 \quad \text{or} \quad b_1(\lambda_1 + \lambda_2) \;=\; c \\ \lambda_2 \;&=\; \lambda_3 \quad \text{or} \quad b_1(\lambda_2 + \lambda_3) \;=\; c \; , \\ \lambda_3 \;&=\; \lambda_1 \quad \text{or} \quad b_1(\lambda_3 + \lambda_1) \;=\; c \end{aligned} \tag{7.74}$$

subject to $\lambda_1\lambda_2\lambda_3 = 1$, where not all conditions can be chosen from the same column in (7.74).

One solution of (7.74) is obviously

$$\lambda_1 \;=\; \lambda_2 \;=\; \lambda_3 \;=\; 1 \; . \tag{7.75}$$

This corresponds to the cube remaining rigid under the influence of the tensile load. Another possible set of solutions is $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_2 = \lambda_3 \neq \lambda_1$ and $\lambda_3 = \lambda_1 \neq \lambda_2$. Explore one of these solutions, say $\lambda_1 = \lambda_2 \neq \lambda_3$ by setting $\lambda_3 = \lambda$ and noting from (7.74) that

$$\lambda_2 + \lambda_3 \;=\; \lambda_3 + \lambda_1 \;=\; \frac{c}{b_1} \;= \eta \; , \tag{7.76}$$

so that

$$\lambda_1\lambda_2\lambda_3 \;=\; (\eta-\lambda)^2\lambda \;=\; 1\;. \tag{7.77}$$

The preceding equation may be rewritten as

$$f(\lambda) \;=\; \lambda^3 - 2\eta\lambda^2 + \eta^2\lambda - 1 \;=\; 0\;. \tag{7.78}$$

To examine the roots of $f(\lambda) = 0$, note that

$$f'(\lambda) \;=\; 3\lambda^2 - 4\eta\lambda + \eta^2 \quad , \quad f''(\lambda) \;=\; 6\lambda - 4\eta\;, \tag{7.79}$$

hence the extrema of f occur at

$$\lambda_{1,2} \;=\; \begin{cases} \dfrac{\eta}{3} & \text{where } f''(\dfrac{\eta}{3}) = -2\eta < 0 \quad \text{(maximum)} \\ \eta & \text{where } f''(\eta) = \eta > 0 \qquad \text{(minimum)} \end{cases} \tag{7.80}$$

and are equal to

$$f(\frac{\eta}{3}) \;=\; \frac{4}{27}\eta^3 - 1 \quad , \quad f(\eta) \;=\; -1\;. \tag{7.81}$$

It is also obvious from the definition of $f(\lambda)$ that $f(0) = -1$, $f(-\infty) = -\infty$, and $f(\infty) = \infty$. The plot in Figure 7.4 depicts the essential features of $f(\lambda)$.

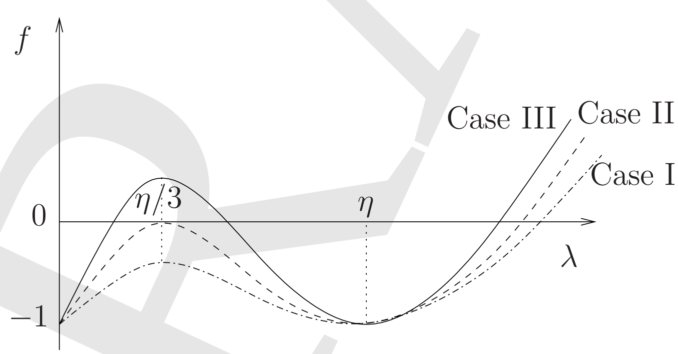

Figure 7.4: *Function $f(\lambda)$ in Rivlin's cube*

In summary, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is always a solution. Further,

1. If $\frac{4}{27}\eta^3 < 1$, then there are no additional solutions (Case I).

2. If $\frac{4}{27}\eta^3 = 1$, then there is one set of three additional solutions corresponding to $\lambda = \dfrac{\eta}{3}$ (Case II).

3. If $\frac{4}{27}\eta^3 > 1$, then there are two sets of three additional solutions corresponding to the two roots of $f(\lambda)$ which are smaller than η (Case III).

Note that the root $\lambda = \lambda_3 > \eta$ is inadmissible because it leads to $\lambda_1 = \lambda_2 = \eta - \lambda < 0$.

7.5 Multiscale problems

It is sometimes desirable to relate the theory of continuous media to theories of particle mechanics. This is, for example, the case, when one wishes to analyze metals and semiconductors at very small length and time scales, at which the continuum assumption is not unequivocally satisfied. In such cases, multiscale analyses offer a means for relating kinematic and kinetic information between the continuum and the discrete system.

7.5.1 The virial theorem

The virial theorem is a central result in the study of continua whose constitutive behavior is derived from an underlying microscale particle system. Preliminary to the derivation of the theorem, recall that the mean Cauchy stress $\mathbf{T}$ in a material region $\mathcal{P}$ can be written as

$$(\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \;=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da - \int_{\mathcal{P}} \mathrm{div}\,\mathbf{T}\otimes\mathbf{x}\,dv \;. \tag{7.82}$$

Taking into account (4.83), the preceding equation may be rewritten as

$$\begin{aligned}(\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \;&=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da + \int_{\mathcal{P}} \rho(\mathbf{b}-\mathbf{a})\otimes\mathbf{x}\,dv & (7.83)\\ &=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da + \int_{\mathcal{P}} \rho\mathbf{b}\otimes\mathbf{x}\,dv - \frac{d}{dt}\int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\mathbf{x}\,dv + \int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}}\,dv \;. & (7.84)\end{aligned}$$

Define next the (long) *time-average* $\langle\phi\rangle$ of a time-dependent quantity $\phi = \phi(t)$ as

$$\langle\phi\rangle \;=\; \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\phi(t)\,dt \;. \tag{7.85}$$

and note that

$$\left\langle \frac{d}{dt}\int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\mathbf{x}\,dv\right\rangle \;=\; \lim_{T\to\infty}\frac{1}{T}\left[\int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\mathbf{x}\,dv\,\Big|_{t=t_0+T} - \int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\mathbf{x}\,dv\,\Big|_{t=t_0}\right] \;=\; \mathbf{0}\;. \tag{7.86}$$

The preceding time-average vanishes due to the assumed boundedness of the integral $\int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x}\, dv$ throughout time. Using (7.86), the time-averaged counterpart of the mean-stress formula (7.83) is

$$\langle (\mathrm{vol}\,\mathcal{P}) \bar{\mathbf{T}} \rangle \;=\; \Big\langle \int_{\partial \mathcal{P}} \mathbf{t} \otimes \mathbf{x}\, da \Big\rangle + \Big\langle \int_{\mathcal{P}} \rho \mathbf{b} \otimes \mathbf{x}\, dv \Big\rangle + \Big\langle \int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \dot{\mathbf{x}}\, dv \Big\rangle \;. \tag{7.87}$$

Turn attention now to a system of N particles whose motion is governed by Newton's Second Law, namely

$$m^\alpha \ddot{\mathbf{x}}^\alpha \;=\; \mathbf{f}^\alpha \;, \tag{7.88}$$

where m^α and $\mathbf{x}^\alpha$ are the mass and the current position of particle α, while $\mathbf{f}^\alpha$ is the total force acting on particle α. It is easy to show that the above equation leads to

$$m^\alpha \frac{d}{dt}(\dot{\mathbf{x}}^\alpha \otimes \mathbf{x}^\alpha) - m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \;=\; \mathbf{f}^\alpha \otimes \mathbf{x}^\alpha \;. \tag{7.89}$$

Taking time averages of (7.89) for the totality of the particles and assuming boundedness of the term $\sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \mathbf{x}^\alpha$, it is seen that

$$-\Big\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \Big\rangle \;=\; \Big\langle \sum_{\alpha=1}^N \mathbf{f}^\alpha \otimes \mathbf{x}^\alpha \Big\rangle \;. \tag{7.90}$$

Recognizing that, in each particle, the total force $\mathbf{f}^\alpha$ is comprised of an internal part $\mathbf{f}^{ext,\alpha}$ (due to inter-particle potentials) and an external part $\mathbf{f}^{int,\alpha}$, the preceding equation may be rewritten as

$$-\Big\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \Big\rangle \;=\; \Big\langle \sum_{\alpha=1}^N \mathbf{f}^{int,\alpha} \otimes \mathbf{x}^\alpha \Big\rangle + \Big\langle \sum_{\alpha=1}^N \mathbf{f}^{ext,\alpha} \otimes \mathbf{x}^\alpha \Big\rangle \;. \tag{7.91}$$

Upon ignoring the body forces in the continuum problem and comparing (7.87) to (7.91), it can be argued that there is a one-to-one correspondence between the three terms in each statement. Specifically, for the mean stress, one may argue that

$$\langle (\mathrm{vol}\,\mathcal{P}) \bar{\mathbf{T}} \rangle \;\doteq\; -\Big\langle \sum_{\alpha=1}^N \mathbf{f}^{int,\alpha} \otimes \mathbf{x}^\alpha \Big\rangle \;, \tag{7.92}$$

which leads to an estimate of the mean Cauchy stress in terms of the underlying particle system dynamics as

$$\langle (\mathrm{vol}\,\mathcal{P}) \bar{\mathbf{T}} \rangle \;\doteq\; \Big\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \Big\rangle + \Big\langle \sum_{\alpha=1}^N \mathbf{f}^{ext,\alpha} \otimes \mathbf{x}^\alpha \Big\rangle \;. \tag{7.93}$$

Equation (7.93) is a statement of the virial theorem.

7.6 Expansion of the universe

APPENDIX A

A.1 Cylindrical polar coordinate system

The basis vectors of the cylindrical polar coordinate system are $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, where

$$
\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta \\
\mathbf{e}_\theta &= -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta \ , \qquad (A.1) \\
\mathbf{e}_z &= \mathbf{e}_3 \qquad (A.2)
\end{aligned}
$$

Here, θ is the angle formed between the $\mathbf{e}_1$ and $\mathbf{e}_r$ vectors. Conversely,

$$
\begin{aligned}
\mathbf{e}_1 &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \\
\mathbf{e}_2 &= \mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta \ . \qquad (A.3) \\
\mathbf{e}_z &= \mathbf{e}_3 \qquad (A.4)
\end{aligned}
$$

Further, one may easily conclude that

$$
r = \sqrt{x_1^2 + x_2^2} \quad , \quad \theta = \arctan\frac{x_2}{x_1} \quad , \quad z = x_3 \qquad (A.5)
$$

and, conversely,

$$
x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad , \quad x_3 = z \ . \qquad (A.6)
$$

Using chain rule, one may express the gradient of a scalar function f in polar coordinates as

$$
\operatorname{grad} f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z = \left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z\right) f \ . \qquad (A.7)
$$

When the gradient operator grad is applied on a vector function $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$, then

$$
\operatorname{grad}^T \mathbf{v} = \left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z\right) \otimes \mathbf{v} \ , \qquad (A.8)
$$

or, upon expanding,

$$
\begin{aligned}
\operatorname{grad} \mathbf{v} = {} & \frac{\partial v_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \frac{\partial v_z}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \\
& \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\mathbf{e}_z \otimes \mathbf{e}_\theta + \\
& \frac{\partial v_r}{\partial z}\mathbf{e}_r \otimes \mathbf{e}_z + \frac{\partial v_\theta}{\partial z}\mathbf{e}_\theta \otimes \mathbf{e}_z + \frac{\partial v_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z \ . \qquad (A.9)
\end{aligned}
$$

Also, given a symmetric tensor function

$$\begin{aligned}\mathbf{T} \;=\; & T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{rz}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \\ & T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\theta z}(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta) + T_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \;, \end{aligned} \tag{A.10}$$

its divergence can be expanded to

$$\begin{aligned}\operatorname{div}\mathbf{T} \;=\; & \left[\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z}\right]\mathbf{e}_r + \\ & \left[\frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z}\right]\mathbf{e}_\theta + \\ & \left[\frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z}\right]\mathbf{e}_z \;. \end{aligned} \tag{A.11}$$