Square Integrable and Orthogonal Functions

Square Integrable Functions; the Space $\mathbf{L^2[a,b]}$ The setting for our discussion here is a finite interval $[a,b]$ and a certain *vector space* of functions on such an interval, generally denoted by $L^2[a,b]$. The space $L^2[a,b]$ includes all functions $f(x)$ possessing a certain type of *regularity*, called *square integrability*, about which we will have little more to say here except that it means the integral $\int_a^b |f(x)|^2 \, dx$ can be unambiguously defined and has a finite value. This space of functions includes all continuous and piecewise continuous functions defined on the interval $[a,b]$ and many other functions as well; functions having infinitely many discontinuities, certain square integrable infinite singularities, etc. It ordinarily does not matter what values are assigned to the function $f(x)$ at points where it may have jump discontinuities. Two functions $f(x)$ and $g(x)$ in $L^2[a,b]$ which agree at all but finitely many points or, for that matter at all but a countably infinite set of points (or even certain uncountable sets of *measure zero*) are regarded as *equivalent*, i.e., not distinguished from each other, in $L^2[a,b]$. Our concern here is not so much with precisely which functions lie in the space as with its algebraic and geometric structure.

For any two functions $f(x)$, $g(x)$ in $L^2[a,b]$ an *inner product* may be defined by

$$\langle f, g \rangle \; = \; \int_a^b \, f(x)\overline{g(x)} \, dx.$$

The conjugation of the second function, $g(x)$, is superfluous when we confine attention to real functions - but this will not always be the case. This inner product is defined in a manner quite analogous to the inner product in the n-dimensional space E^n of complex vectors and has comparable properties. The *norm* or *magnitude* of $f(x) \in L^2[a,b]$ is also defined in a manner analogous to the same notion in E^n:

$$\|f\| \; = \; \sqrt{\langle f, f \rangle} \; = \; \sqrt{\int_a^b \, |f(x)|^2 \, dx},$$

and it can be shown, with much the same argument as in the finite dimensional case, that the *Schwarz inequality*,

$$|\langle f, g\rangle| \le \|f\|\,\|g\|\,,$$

is always satisfied. The *distance* between two functions $f(x)$ and $g(x)$ is defined by $d(f,g) \equiv \|f-g\|$.

The set of functions $L^2[a,b]$ is a *vector space* because it admits functional addition, with functions equivalent to the zero function serving as the additive identity, multiplication of functions by scalars and has the property of being closed under the formation of linear combinations. Suppose $f(x)$ and $g(x)$ lie in $L^2[a,b]$; then

$$\int_a^b |f(x)|^2\,dx < \infty;\ \int_a^b |g(x)|^2\,dx < \infty.$$

For any two scalars α and β we form the linear combination $h(x) = \alpha\,f(x) + \beta\,g(x)$ and we estimate

$$\begin{aligned}
\|h\|^2 &= \|\alpha\,f + \beta\,g\|^2 = \langle \alpha\,f + \beta\,g,\ \alpha\,f + \beta\,g\rangle \\
&= |\alpha|^2\,\|f\|^2 + \alpha\overline{\beta}\,\langle f,g\rangle + \overline{\alpha}\beta\,\langle g,f\rangle + |\beta|^2\,\|g\|^2 \\
&\le |\alpha|^2\,\|f\|^2 + 2\,|\alpha|\,|\beta|\,|\langle f,g\rangle| + |\beta|^2\,\|g\|^2 \\
&\le \text{(Schwarz inequality)} \le (|\alpha|\,\|f\| + |\beta|\,\|g\|)^2 < \infty,
\end{aligned}$$

allowing us to conclude that the linear combination $h(x) = \alpha\,f(x) + \beta\,g(x)$ also lies in the space.

Two functions $f(x)$ and $g(x)$ are *orthogonal* in $L^2[a,b]$ if their inner product is zero. This can happen trivially if one of the functions is the zero function (or equivalent to the zero function, as discussed earlier). Just as in the finite dimensional setting we define a set of functions $f_k(x),\ k \in K$, where K is an arbitary *index set*, to be an *orthogonal set* if

$$\langle f_k, f_j\rangle = 0,\ k \ne j;\ \|f_k\| > 0,\ k \in K.$$

Such a set is further described as *orthonormal* if $\|f_k\| = 1$ for all $k \in K$. The most important instance of such a set for our purposes here is presented in the next example.

Example 1 Let $e_k(x) \equiv e^{i\frac{2\pi k x}{L}}$, $-\infty < k < \infty$; $L \equiv b-a$. Then

$$\langle e_k, e_j \rangle >= \int_a^b e_k(x)\overline{e_j(x)}\,dx = \int_a^b \mathrm{e}^{\frac{i\,2\pi(k-j)x}{L}}\,dx.$$

If $k = j$ we clearly have $\langle e_k, e_j \rangle = L$. Otherwise $\langle e_k, e_j \rangle =$

$$\frac{-iL}{2\pi(k-j)}\,\mathrm{e}^{\frac{i\,2\pi(k-j)x}{L}}\Big|_{x=a}^{x=b} = \frac{-iL}{2\pi(k-j)}\left(\mathrm{e}^{\frac{i\,2\pi(k-j)(a+L)}{L}} - \mathrm{e}^{\frac{i\,2\pi(k-j)a}{L}}\right)$$

$$= \frac{-iL}{2\pi(k-j)}\left(\mathrm{e}^{\frac{i\,2\pi(k-j)a}{L}} - \mathrm{e}^{\frac{i\,2\pi(k-j)a}{L}}\right) = 0.$$

So we conclude that the functions $e_k(x)$ form an orthogonal set in $L^2[a,b]$. Clearly the normalized functions $\frac{1}{\sqrt{L}}\,e_k(x)$ form an orthonormal set in that space.

Example 2 For convenience we take $[a,b] = [0,1]$ and we define $h_{0,0}(x) \equiv 1$, $x \in [0,1]$. Then we let

$$h_{0,1}(x) = \begin{cases} -1, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}.$$

More generally, for an arbitrary positive integer k, we divide the interval $[0,1]$ into 2^k subintervals

$$\mathcal{I}_{k,j} = \begin{cases} [(j-1)2^{-k},\, j\,2^{-k}), \;\; j = 1, 2, ..., 2^k - 1, \\ [1-2^{-k},\, 1], \;\; j = 2^k, \end{cases}$$

and we let

$$h_{k,j}(x) = \begin{cases} 0 \text{ on } \mathcal{I}_{k,\ell}, \;\; \ell \ne j, \\ h_{0,1}(2^k(x-(j-1)2^{-k})) \text{ on } \mathcal{I}_j \end{cases}.$$

Thus each function $h_{k,j}(x)$ has alternate values -1 and 1 in the left and right hand halves of an interval in which one of the functions $h_{k-1,m}$

is constant and is otherwise equal to 0. It is easy in this case to see that the functions $h_{k,j}(x)$ form an orthogonal set in $L^2[0,1]$. These are the so-called *Haar basis functions*. The set of all $h_{k,j}$ with $0 \le k \le K$ contains 2^{K+1} functions; these form an orthogonal basis for the subspace of $L^2[0,1]$ consisting of functions which are constant on the intervals $\mathcal{I}_{K+1,j}, \; j = 1, 2, ..., 2^{K+1}$. For the norms we have

$$\|h_{k,j}\| = \sqrt{\int_0^1 |h_{k,j}(x)|^2 \, dx} = \sqrt{\int_{\mathcal{I}_{k,j}} 1 \, dx} = \sqrt{2^{-k}} = 2^{-k/2}.$$

Returning to the general discussion we suppose now that the set of functions $\{\phi_k(x), \; k \in K\}$ is an orthogonal set in $L^2[a,b]$. For positive integers n let K_n be finite subsets of K with $K_n \subset K_{n+1}$ and $\cup_n K_n = K$. For each n let Φ_n be the subspace of $L^2[a,b]$ spanned by $\phi_k(x), \; k \in K_n$; i.e., Φ_n consists of all linear combinations of functions $\phi_k(x) \in \Phi_n$. An arbitrary function $f(x) \in L^2[a,b]$ will not, in general, lie in Φ_n but we can show there is a function $p_n(x) \in \Phi_n$ which is closest to $f(x)$ relative to the distance function $d(f,g) = \|f - g\|$.

Proposition 1 *The function $p_n(x)$ is characterized by the condition: $\langle f - p_n, \, \phi \rangle = 0$ for all $\phi \in \Phi_n$. Moreover,*

$$\|f\|^2 = \|f - p_n\|^2 + \|p_n\|^2 .$$

Proof We assume p_n is the closest point to f lying in Φ_n and suppose the condition does not hold; i.e., we assume there is $\phi \in \Phi_n$ such that $\langle f - p_n, \, \phi \rangle \neq 0$. For scalar ϵ we compute

$$\|f - (p_n + \epsilon\,\phi)\|^2 = \langle f - p_n - \epsilon\,\phi, \, f - p_n - \epsilon\,\phi \rangle$$

$$= \|f - p_n\|^2 - \epsilon \, \langle \phi, \, f - p_n \rangle - \overline{\epsilon} \, \langle f - p_n, \, \phi \rangle + \epsilon^2 \, \|\phi\|^2 .$$

We let $\epsilon = \alpha \, \langle f - p_n, \, \phi \rangle$ for $\alpha > 0$. Then $\|f - (p_n + \epsilon\,\phi)\|^2 =$

$$\|f - p_n\|^2 - 2\alpha \, |\langle f - p_n, \, \phi \rangle|^2 + \alpha^2 \, |\langle f - p_n, \, \phi \rangle|^2 \, \|\phi\|^2 .$$

This number is $< \|f - p_n\|^2$ if α is small and positive, contradicting the definition of p_n. Thus $\langle f - p_n,\, \phi\rangle = 0,\ \ \phi \in \Phi_n$.

Now suppose $p_n \in \Phi_n$ has the indicated property. Let p be any other point in Φ_n. Then $\phi \equiv p_n - p \in \Phi_n$. and we compute

$$\|f - p\|^2 = \langle f - p_n + p_n - p,\, f - p_n + p_n - p\rangle$$

$$= \langle f - p_n + \phi,\, f - p_n + \phi\rangle = \|f - p_n\|^2 + \|\phi\|^2 + 2\,\mathrm{Re}\,\langle f - p_n,\, \phi\rangle$$

$$= \|f - p_n\|^2 + \|\phi\|^2 \geq \|f - p_n\|^2$$

and we see that equality holds if and only if $\|\phi\| = 0$, i.e., if and only if $p = p_n$. Taking $\phi = p_n$ we have the second result in the proposition and the proof is complete.

Now we want to identify the function $p_n(x)$; we will do this by relating it to the functions $\phi_j(x)$. First of all, since $\phi_j \in \Phi_n,\ \ j \in K_n$, we have

$$\langle f - p_n,\, \phi_j\rangle = 0,\ \ j \in K_n.$$

Since $p_n \in \Phi_n$, which is the span of the functions $\phi_k(x),\ \ k \in K_n$,

$$p_n(x) = \sum_{k \in K_n} c_k\, \phi_k(x)$$

for some coefficients c_k. Substituting we have

$$0 = \left\langle f - \sum_{k \in K_n} c_k\, \phi_k,\ \phi_j \right\rangle =$$

$$\langle f,\, \phi_j\rangle - \sum_{k \in K_n} c_k\, \langle \phi_k,\, \phi_j\rangle = \langle f,\, \phi_j\rangle - c_j\, \|\phi_j\|^2$$

and we conclude that

$$c_j = \frac{1}{\|\phi_j\|^2}\,\langle f,\, \phi_j\rangle\,; \qquad p_n(x) = \sum_{k \in K_n} \frac{\langle f, \phi_k\rangle}{\|\phi_k\|^2}\,\phi_k.$$

Proposition 2 *With $f(x)$ and $p_n(x)$ in $L^2[a,b]$ related as described above we have*

$$\|p_n\|^2 \;=\; \sum_{k \,\in\, K_n} |c_k|^2 \; \|\phi_k\|^2 \;=\; \sum_{k \,\in\, K_n} \frac{|\langle f,\, \phi_k\rangle|^2}{\|\phi_k\|^2} \le \|f\|^2$$

with equality holding just in case $p_n \;=\; f$ in $L^2[a,b]$. Further, the infinite series

$$\sum_{k \,\in\, K} |c_k|^2 \; \|\phi_k\|^2 \;=\; \sum_{k=1}^{\infty} \frac{|\langle f,\, \phi_k\rangle|^2}{\|\phi_k\|^2}$$

is convergent and satisfies Bessel's inequality

$$\sum_{k \,\in\, K} |c_k|^2 \; \|\phi_k\|^2 \le \|f\|^2 .$$

with equality holding just in case $f(x)$ has the expansion

$$f(x) \;=\; \sum_{k \,\in\, K} c_k \, \phi_k(x)$$

in the sense that

$$\lim_{n \to \infty} \|f \;-\; p_n\| \;=\; \lim_{n \to \infty} \left\| f \;-\; \sum_{k \,\in\, K} c_k \, \phi_k(x) \right\| \;=\; 0.$$

Proof Using the orthogonality property of the $\phi_k(x)$ we readily compute

$$\|p_n\|^2 \;=\; \left\langle \sum_{k \,\in\, K_n} c_k \, \phi_k, \; \sum_{k \,\in\, K_n} c_k \, \phi_k \right\rangle \;=\; \sum_{k \,\in\, K_n} |c_k|^2 \; \|\phi_k\|^2 .$$

Since we have already seen that $\|f\|^2 \;=\; \|p_n\|^2 \;+\; \|f - p_n\|^2$ the first claimed inequality follows and it is clear that equality can hold if and only if $\|f - p_n\| \;=\; 0$, which is the same thing as saying that $p_n \;=\; f$ in $L^2[a,b]$. Since the infinite series indicated has non-negative terms and its partial sums are bounded above by $\|f\|^2$ the standard results on infinite series apply to show convergence. Because $\Phi_{n+1} \supseteq \Phi_n$ the norms

$\|f - p_n\|$ are decreasing and, being bounded below by zero, have a non-negative limit as $n \to \infty$. Using this in $\|f\|^2 = \|p_n\|^2 + \|f - p_n\|^2$ we see that

$$\|f\|^2 = \lim_{n \to \infty} \|p_n\|^2 = \sum_{k \in K} |c_k|^2 \|\phi_k\|^2$$

just in case $\lim_{n \to \infty} \|f - p_n\| = 0$, completing the proof.

Completeness of Orthogonal Sets of Functions

For a given orthogonal set $\{ \phi_k \mid k \in K\} \in L^2[a,b]$ the question as to whether, for all $f \in L^2[a,b]$, it is true that $\lim_{n \to \infty} \|f - p_n\| = 0$ is to a large extent beyond the scope of our present treatment. If this is true we say that the sequence $\{ \phi_k \}$ is *complete* in the space $L^2[a,b]$. An important result is the following.

Proposition 3 *The orthogonal set $\{ \phi_k \mid k \in K\}$ is complete in $L^2[a,b]$ if and only if, for $g \in L^2[a,b]$, the condition $\langle g, \phi_j \rangle = 0, \; j \in K$, implies $g(x) = 0$ in that space.*

Proof If the indicated set is not complete then there is a function $f(x) \in L^2[a,b]$ such that $f(x)$ is not equal to the corresponding infinite series in the $\phi_k(x)$; i.e.,

$$g(x) \equiv f(x) - \sum_{k \in K} c_k \, \phi_k(x) \neq 0, \;\; c_k = \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2}, \;\; k \in K.$$

The fact that the series converges in $L^2[a,b]$ implies that we can form inner products term by term. So for $j \in K$ we have

$$\langle g, \phi_j \rangle = \langle f, \phi_j \rangle - \sum_{k \in K} c_k \, \langle \phi_k, \phi_j \rangle = \langle f, \phi_j \rangle - c_j \, \|\phi_j\|^2 = 0,$$

the last identity following from $c_j = \langle f, \phi_j \rangle / \|\phi_j\|^2$.

On the other hand, if such a there is a non-zero function $g(x)$ such that $\langle g, \phi_j \rangle = 0$ for all j then the coefficients of the series corresponding to $g(x)$ are $c_j = \langle g, \phi_j \rangle / \|\phi_j\|^2 = 0, \;\; j \in K$, so that the series is identically equal to zero and therefore could not converge to the non-zero function $g(x)$. This completes the proof.

Example 3 Let K consist of the non-negative integers and, for $k = 0, 1, 2, ...,$ let $\phi_k(x) = \cos kx, \;\; x \in [-\pi, \pi]$. This set of functions *is not complete* in $L^2([-\pi, \pi])$ because every odd function $g(x)$ on $[-\pi, \pi]$ has the property

$$\langle \, g \, , \phi_k \, \rangle \; = \; \int_{-\pi}^{\pi} g(x) \; \cos kx \, dx \; =$$

$$\int_0^{\pi} (g(x) + g(-x)) \; \cos kx \, dx \; = \; 0.$$