

Homework 4 - Theory

Lecture: Prof. Qiang Liu

1. Assume X is a discrete random variable that takes values in $\{1, 2, 3\}$, with probability defined by

$$\begin{aligned}\Pr(X = 1) &= \theta_1 \\ \Pr(X = 2) &= 2\theta_1 \\ \Pr(X = 3) &= \theta_2,\end{aligned}$$

where $\theta = [\theta_1, \theta_2]$ is an unknown parameter to be estimated.

Now assume we observe a sequence $D := \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$ that is *independent and identically distributed (i.i.d.)* from the distribution. We assume the number of observations of the values: 1, 2, 3 in D are s_1, s_2, s_3 , respectively.

- (a) **[5 points]** To ensure that $\Pr(X = i)$ is a valid probability mass function, what constraint should we put on $\theta = [\theta_1, \theta_2]$? Write your answers quantitatively as expressions that include θ_1 and θ_2 .
- (b) **[5 points]** Write down the joint probability of the data sequence

$$\Pr(D \mid \theta) = \Pr\left(\left\{x^{(1)}, \dots, x^{(n)}\right\} \mid \theta\right),$$

and the log probability $\log \Pr(D \mid \theta)$.

- (c) **[5 points]** Calculate the maximum likelihood estimation $\hat{\theta}$ of θ based on the sequence D .
2. **[10 points]** Let $\{x^{(1)}, \dots, x^{(n)}\}$ be an *i.i.d.* sample from an exponential distribution, whose the density function is defined as

$$f(x \mid \beta) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right), \quad \text{for } 0 \leq x < \infty.$$

Please find the maximum likelihood estimator (MLE) of the parameter β . Show your work.

3. (a) **[10 points]** Assume that you want to investigate the proportion (θ) of defective items manufactured at a production line. You take a random sample of 30 items and found 5 of them were defective. Assume the prior of θ is a uniform distribution on $[0, 1]$. Please compute the posterior of θ . It is sufficient to write down the posterior density function upto a normalization constant that does not depend on θ .
- (b) **[10 points]** Assume an observation $D := \{x^{(1)}, \dots, x^{(n)}\}$ is *i.i.d.* drawn from a Gaussian distribution $\mathcal{N}(\mu, 1)$, with an unknown mean μ and a variance of 1. Assume the prior distribution of μ is $\mathcal{N}(0, 1)$. Please derive the posterior distribution $p(\mu \mid D)$ of μ given data D .