Study Guide on Logarithms and Derivatives

1 Logarithms

A logarithm is the inverse operation of exponentiation. If $a^b = c$, then $\log_a(c) = b$. In other words, the logarithm tells you the exponent to which a base a must be raised to yield c.

1.1 Common Types of Logarithms

- **Natural Logarithm:** $\log_e(x)$, often written as $\ln(x)$, where the base is the mathematical constant e (approximately 2.718).
- **Common Logarithm:** $\log_{10}(x)$, where the base is 10.

1.2 Logarithmic Rules

- **Product Rule:**
  $$\log_b(xy) = \log_b(x) + \log_b(y)$$
  The logarithm of a product is the sum of the logarithms.
- **Quotient Rule:**
  $$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$
  The logarithm of a quotient is the difference of the logarithms.
- **Power Rule:**
  $$\log_b(x^n) = n\log_b(x)$$
  The logarithm of a power is the exponent times the logarithm of the base.
- **Change of Base Formula:**
  $$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$$
  This formula allows you to convert logarithms between different bases.

1.3 Key Properties of Logarithms

- $\log_b(1) = 0$, because $b^0 = 1$.
- $\log_b(b) = 1$, because $b^1 = b$.

2 Derivatives

A derivative measures the rate of change of a function with respect to one of its variables. For a function $f(x)$, the derivative, denoted as $f'(x)$ or $\frac{d}{dx}f(x)$, gives the slope of the tangent line at any point x.

2.1 Basic Derivative Rules

- **Power Rule:**
$$\frac{d}{dx}[x^n] = nx^{n-1}$$
Example: $\frac{d}{dx}[x^3] = 3x^2$.
- **Constant Rule:**
$$\frac{d}{dx}[c] = 0$$
The derivative of a constant is 0.
- **Sum and Difference Rule:**
$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$
The derivative of a sum (or difference) is the sum (or difference) of the derivatives.
- **Product Rule:**
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$
The derivative of a product is the derivative of the first function times the second function, plus the first function times the derivative of the second function.
- **Quotient Rule:**
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$
The derivative of a quotient is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the denominator squared.
- **Chain Rule:**
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$
The chain rule is used when differentiating a composite function.

2.2 Derivatives of Common Functions

- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[a^x] = a^x \ln(a)$

3 Solved Problems

3.1 Problem 1: Solving Logarithmic Equations

Solve for x:
$$\log_2(x^2 - 3x) = 4$$

3.1.1 Solution:

Rewrite the equation using the definition of a logarithm:
$$x^2 - 3x = 2^4 = 16$$
Solve the quadratic equation:
$$x^2 - 3x - 16 = 0$$
Factor or use the quadratic formula:
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-16)}}{2(1)} = \frac{3 \pm \sqrt{9 + 64}}{2} = \frac{3 \pm \sqrt{73}}{2}$$
Thus, the solutions are:
$$x = \frac{3 + \sqrt{73}}{2}, \quad x = \frac{3 - \sqrt{73}}{2}$$

3.2 Problem 2: Derivative Using Product Rule

Find the derivative of $f(x) = x^3 \ln(x)$.

3.2.1 Solution:

We apply the product rule, where $f(x) = x^3$ and $g(x) = \ln(x)$:

$$\begin{aligned}\frac{d}{dx}[x^3 \ln(x)] &= \frac{d}{dx}[x^3] \cdot \ln(x) + x^3 \cdot \frac{d}{dx}[\ln(x)] \\ &= 3x^2 \ln(x) + x^3 \cdot \frac{1}{x} \\ &= 3x^2 \ln(x) + x^2\end{aligned}$$

3.3 Problem 3: Derivative Using Chain Rule

Find the derivative of $f(x) = \ln(3x^2 + 1)$.

3.3.1 Solution:

This is a composite function, so we use the chain rule. Let $u = 3x^2 + 1$, so $f(x) = \ln(u)$:

$$\begin{aligned}\frac{d}{dx}[\ln(3x^2 + 1)] &= \frac{1}{3x^2 + 1} \cdot \frac{d}{dx}[3x^2 + 1] \\ &= \frac{1}{3x^2 + 1} \cdot 6x = \frac{6x}{3x^2 + 1}\end{aligned}$$

3.4 Problem 4: Using Logarithmic Differentiation

Find the derivative of $f(x) = x^x$.

3.4.1 Solution:

To differentiate x^x, we first take the natural logarithm of both sides:

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

Now differentiate both sides:

$$\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} f'(x), \quad \frac{d}{dx}[x \ln(x)] = \ln(x) + 1$$

Thus, the derivative is:

$$\frac{f'(x)}{f(x)} = \ln(x) + 1$$

Multiply both sides by $f(x) = x^x$:

$$f'(x) = x^x(\ln(x) + 1)$$

4 Summary

- Logarithms help simplify complex multiplications and divisions into additions and subtractions. Use the logarithmic rules (product, quotient, power) to break down complex expressions.
- Derivatives are used to find rates of change and slopes of functions. The power rule, product rule, quotient rule, and chain rule are essential tools for differentiating various types of functions.
- Both logarithms and derivatives are foundational tools in calculus, making them highly useful in solving real-world problems, including those in machine learning and optimization.