Linear Systems of Algebraic Equations

QEA

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Linear Systems of Algebraic Equations: Formulation and Definition

As you'll recall from ModSim, one key idea in building models is the step of abstraction: going from some real-world situation to an abstracted model: e.g., a stock-and-flow, a free body diagram, a set of differential equations. There are two important aspects of building such a model: first, deciding what to include or ignore, and second, deciding how to mathematically represent those things you choose to include.

One particularly common kind of mathematical framing is a set of linear algebraic equations, which can be represented by a matrix equation. When systems admit to this type of model, linear algebra techniques can be extremely useful for gaining insight.

A general system of m linear algebraic equations in n unknown variables $x_1, x_2, \ldots, x_n$ takes the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ \ldots &= \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ are known as coefficients and $b_1, b_2, b_3, \ldots, b_m$ are constants. We can write this using matrices and vectors in the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A}$ is the $n \times m$ *coefficient matrix*, $\mathbf{x}$ is the $n \times 1$ unknown vector, and $\mathbf{b}$ is a $m \times 1$ constant vector which is known.

Note that "linear" here means linear in terms of the unknown variables, e.g., if x is an unknown there are only terms like ax, and no terms like $\sin(x)$, x^2, $1/x$, etc. It is often the case that you might have *coefficients* that appear to be non-linear; for example, in solving physics problems, you might have coefficients that depended on trig functions of angles (e.g., $(L\cos\theta)F_x$ is linear in F_x but not linear in θ). Be careful to be clear about what you're solving for when you decide whether something is linear or non-linear.

Linear Systems of Algebraic Equations: Framing

Half of the battle is typically getting your system abstracted to the point that it can be thought of as a system of linear equations. On the following pages are a set of problems. You don't need to solve these problems – you just need to formulate them as linear algebra problems (if you can do so). We'll work through these in a relatively synchronized way, checking in with you as we go...

An Investment Example

At your table, discuss the following example. For each case where you are prompted, decide whether/how you can abstract the system to a mathematical model that can be written as a matrix equation.

Suppose that the following table describes the stock holdings of three of the QEA instructors.

	Apple	IBM	General Mills
Chris	100	100	100
Rebecca	100	200	0
Siddhartan	50	50	200

Also suppose that at a given day the value of the Apple, IBM and General Mill's stock are \$100, \$50 and \$20 respectively.

1. **Here's your first linear algebra formulation question:** What is the total value of the holdings for each professor on the day in question? Can you formulate this as a matrix expression? If so, what is it?

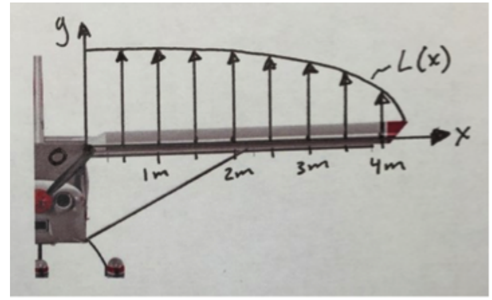
2. Now, suppose that you do not know how many shares of each stock are owned by the instructors. However, you know that the total value of the stocks for each instructor for three consecutive days is as given in the following table

	Day 1	Day 2	Day 3
Chris	\$1500	\$1600	\$1400
Rebecca	\$2600	\$2810	\$2550
Siddhartan	\$950	\$1020	\$1000

You also know that the price of each stock on each of the three days was as follows

	Day 1	Day 2	Day 3
Apple	\$100	\$110	\$100
IBM	\$50	\$50	\$40
General Mills	\$20	\$22	\$30

Now here's the second formulation question: how many stocks of each company does each professor own? Can you formulate this as a matrix equation? If so, what are the matrices/vectors?



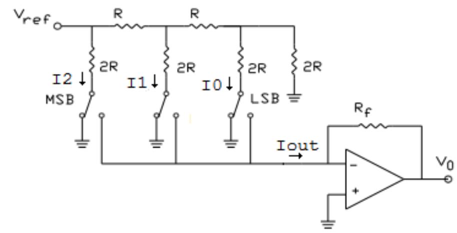
A Mechanical Example

Let's transition to another domain: statics. You might recall this question from the first module:

> An airplane wing experiences a lift force varies as a function of position along the wing according to
>
> $$L(x) = 200\sqrt{1 - \frac{x^2}{17}}$$
>
> where $L(x)$ is the force per unit length along the wing and x is the position along the wing in meters measured from the fuselage. The weight of the wing is 1600 N. Gravity points in the downward vertical direction. Assume the wing is is fixed (no translation, no rotation) to the fuselage at point O (x=0), and that the center of mass of the wing is 2 meters from the fuselage. The wing is attached the the fuselage at its root, and also is supported by a strut that is attached 2 meters along the wing from the fuselage (assume a pin joint where the strut attaches to the wing).
>
> **What forces and moments are applied by the fuselage at the root of the wing, and by the strut at the pin? Assume that the wing to be in static equilibrium.**

AT YOUR TABLE, try to formulate this as a linear algebra problem. Note that the relevant physical idea/model here is statics: $\sum \vec{F} = 0$ and $\sum \vec{M} = 0$ for systems in equilibrium.

An Electrical Example



Here's one more problem to try to formulate:

> The circuit diagram shows a proposal for a simple D to A (digital to analog) circuit: the designer claims that the output voltage should be proportional to the binary input represented by the positions of the three switches (e.g., on-off-off = 100 $= 2^2 = 4$, off-on-on = 011 $= 2^1 + 2^0 = 3$).
>
> Thanks to the magic of op amps, the output voltage $V_O = -R_f I_{out}$, and inverting input of the op amp can be considered to be at ground (not sure if you dealt with this in ISIM).
>
> Given this, is this a reasonable proposal for a digital to analog converter?

NOTE THAT THE RELEVANT PHYSICAL IDEAS/MODELS here are the Kirchoff circuit laws: the sum of currents into any node must be zero, and the sum of voltages around any loop must be zero.

Types of Systems and Types of Solutions

If $\mathbf{b} = \mathbf{0}$ the system of linear algebraic equations is *homogeneous* and if $\mathbf{b} \neq \mathbf{0}$ the system is *non-homogeneous*. Notice that we've already dealt with systems like this before when we were transforming geometrical objects, but in that case we already knew $\mathbf{x}$ and we were simply multiplying by $\mathbf{A}$ in order to get $\mathbf{b}$. Here, we are considering the so-called *inverse* problem, and trying to find $\mathbf{x}$ given $\mathbf{A}$ and $\mathbf{b}$. If $\mathbf{A}$ is square and invertible we've already thought through how to do this: we simply invert the matrix $\mathbf{A}$ and the *solution* becomes

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

However, let's back up and consider some small examples to explore the solution possibilities a little.

Elimination of Variables

In high school you probably learned some basic techniques for solving small linear systems of algebraic equations. Consider the following linear system of algebraic equations

$$\begin{aligned} 2x_1 + 3x_2 &= 6 && (1) \\ 4x_1 + 9x_2 &= 15 && (2) \end{aligned}$$

The basic technique, called *Elimination of Variables*, proceeds as follows: First, solve equation (1) for x_1

$$x_1 = 3 - \frac{3}{2}x_2 \qquad (3)$$

Now substitute this expression for x_1 into equation (2)

$$4(3 - \frac{3}{2}x_2) + 9x_2 = 15$$

Now we simplify this equation

$$\begin{aligned} 12 - 6x_2 + 9x_2 &= 15 \\ \Rightarrow 3x_2 &= 3 \end{aligned}$$

and solve for x_2 to give $x_2 = 1$. Now we substitute this solution back into equation (3) to determine $x_1 = \frac{3}{2}$. The original linear system of algebraic equations therefore has a unique solution, $\mathbf{x} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$.

However, not all linear systems of algebraic equations have a unique solution. For example, the system

$$\begin{aligned} x_1 + 2x_2 &= 1 && (4) \\ 2x_1 + 4x_2 &= 2 && (5) \end{aligned}$$

has an infinite number of solutions because equation (5) is just a multiple of equation (4). Solving equation (4) for x_1 gives

$$x_1 = 1 - 2x_2$$

and choosing an arbitrary value of $x_2 = \alpha$ gives

$$\begin{aligned} x_1 &= 1 - 2\alpha \\ x_2 &= \alpha \end{aligned}$$

or in vector form

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This defines an infinite number of solutions since α is any real number.

It's also possible that a linear system of algebraic equations has no solution. For example, the system

$$x_1 + 2x_2 = 1 \tag{6}$$

$$2x_1 + 4x_2 = 1 \tag{7}$$

has no solution. Solving equation (7) for x_2 gives

$$x_2 = \frac{1}{4} - \frac{1}{2}x_1$$

and replacing into equation (6) gives

$$x_1 + 2(\frac{1}{4} - \frac{1}{2}x_1) = 1$$

which on simplification gives

$$\frac{1}{2} = 1$$

which hopefully we all agree is incorrect. We assumed that there was a solution, performed elimination and substitution and found a statement that contradicts our assumption: no solution therefore exists.

3. Using the technique of *elimination of variables* described above, determine which values of h and k result in the following system of linear algebraic equations having (a) no solution, (b) a unique solution, and (c) infinitely many solutions?

$$\begin{aligned} x_1 + hx_2 &= 1 \\ 2x_1 + 3x_2 &= k \end{aligned}$$

4. Using the technique of *elimination of variables* described above, determine whether the following linear systems of algebraic equations have zero, one, or infinitely many solutions. If solution(s) exist, determine the actual solution(s).

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 + x_3 &= 2 \\ x_1 - 2x_3 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 + x_3 &= -6 \\ 2x_1 + x_2 - x_3 &= 18 \\ x_1 - 2x_3 &= 4 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 - x_3 &= 10 \\ x_1 - 2x_3 &= 4 \end{aligned}$$

Gaussian Elimination

The basic process of *elimination of variables* can be formalized and is known as Gaussian Elimination. Here will briefly introduce it and you should consult other sources, such as the *Gaussian Elimination* page at WolframMathWorld for more details.

Rather than writing equations, we can cast the LSAE in matrix form and perform *Gaussian Elimination* on the augmented matrix $[\mathbf{A}\ \mathbf{b}]$

$$\begin{bmatrix} 2 & 3 & 6 \\ 4 & 9 & 15 \end{bmatrix}$$

Thinking now in terms of rows, we replace the second row with row 2 - 2 row 1 to give

$$\begin{bmatrix} 2 & 3 & 6 \\ 0 & 3 & 3 \end{bmatrix}$$

This matrix is now in so-called *echelon* form: we can find the solution to the original LSAE by first solving the equation implied by the last row and then back-substituting into the equation implied by the previous row. Unfortunately, there is no method in MATLAB that performs Gaussian Elimination.

5. Set up the augmented matrix for the last three examples and perform *Gaussian Elimination* to reduce the augmented matrix to *echelon form*. Interpret the resulting system and determine the solution(s).

Gauss-Jordan Elimination

Gauss-Jordan elimination is an extension of Gaussian Elimination in order to produce a matrix in *reduced row echelon form*. Here we will briefly introduce it and you should consult other sources, such as the *Gaussian Elimination* page at Wikipedia for more details.

Starting with the matrix in echelon form

$$\begin{bmatrix} 2 & 3 & 6 \\ 0 & 3 & 3 \end{bmatrix}$$

we eliminate the entry in row 1, column 2 by replacing row 1 with row 1 - row 2

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

Finally we divide the first row by 2 and the second row by 3 to give

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{bmatrix}$$

This matrix is now in *reduced two echelon form* and we simply read off the solution. Fortunately, there is an algorithm in MATLAB , *rref*, which will perform Gauss-Jordan elimination for you.

6. Set up the augmented matrix for the last three examples and perform *Gauss-Jordan Elimination* to reduce the augmented matrix to *reduced row echelon form*. Check your answer by using *rref*. Interpret the resulting system and determine the solution(s).

LU Decomposition

The steps used to solve a LSAE using Gaussian Elimination can also be used to *decompose* a matrix into a product of two matrices: a *lower-triangular* matrix **L** and an *upper-triangular* matrix **U**. Here we will briefly introduce it and you should consult other sources, such as the *LU Decomposition* page at WolframMathWorld for more details.

In Gaussian Elimination we execute a set of row operations. In our previous example, we replaced row 2 with the result of row 2 - 2 row 1. This action can be neatly represented in terms of a matrix operation. Let's multiply the original matrix equation $\mathbf{Ax} = \mathbf{b}$ with the transformation matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

to form $\mathbf{MAx} = \mathbf{Mb}$. Note that this transformation leaves row 1 of **A** unchanged, and it replaces the row 2 with row 2 - 2 row 1. The product **MA** is therefore an *upper-triangular* matrix **U**

$$\mathbf{U} = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

and the LSAE is now expressed as $\mathbf{Ux} = \mathbf{Mb}$. If we now multiply this expression by $\mathbf{M}^{-1}$ we obtain

$$\mathbf{M}^{-1}\mathbf{Ux} = \mathbf{b}$$

The inverse of **M** is just

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

which is just a *lower-triangular* matrix **L**. The LSAE now reads

$$\mathbf{LUx} = \mathbf{b}$$

We have therefore *decomposed* the original matrix **A** into the product of **L** and **U**,

$$\mathbf{A} = \mathbf{LU}$$

How does this help, you might be asking? First of all, knowing the decomposition of **A** into **LU** allows us to solve the original LSAE $\mathbf{Ax} = \mathbf{b}$. Here is how.

Let's define a new vector $\mathbf{y} = \mathbf{Lx}$. Then the original LSAE can be expressed as

$$\mathbf{Ly} = \mathbf{b}$$

which is easy to solve by *forward-substitution* because **L** is *lower-triangular*,

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

and the solution for $\mathbf{y}$ is $y_1 = 6$, $y_2 = 3$. We can now solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$ using *forward-substitution* because **U** is *upper-triangular*,

$$\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

and the solution for $\mathbf{x}$ is $x_1 = 1$, $x_2 = 3/2$.

Second of all, and more importantly, knowing the decomposition of **A** into **LU** allows us to solve any LSAE involving **A**. Need to solve the LSAE with a different **b**? No problem, just use the **LU** decomposition that you already computed and away you go. No need to redo all the steps of *Gaussian Elimination* just because **b** changed. Need to solve a LSAE for lots of different **b**'s? No problem, just use the **LU** decomposition that you already computed and away you go. Finally, if you want to compute the inverse or determinant of a matrix this is easy too using LU decomposition as we show next. Fortunately, there is an algorithm in MATLAB, *lu*, which does this for you.

Please note that sometimes you will need to perform a row swap in order to proceed with *Gaussian Elimination*. This changes the nature of LU decomposition. Check out a source on LU decomposition for more details.

7. Consider the appropriate matrix for the last three examples and perform *LU Decomposition*. Check your answer by using *lu*.

Determinant

The basic algorithm for computing a determinant of **A** is to first perform LU decomposition, and make use of the following property:

> The determinant of an upper-triangular or lower-triangular matrix is just the product of the diagonal entries.

We already met the property another property of determinants, namely that the determinant of a product is just the product of the determinants. Therefore, $det(\mathbf{A}) = det(\mathbf{L})det(\mathbf{U})$, each of which is just the product of the diagonal entries.

8. Consider the appropriate matrix for the last three examples and find the determinant using the LU decomposition previously determined. Check your answer using *det* in MATLAB.

Inverse

The basic algorithm for computing the inverse of **A** is to first perform LU decomposition, and make use of the following idea. **B** is the inverse of **A** if it satisfies the following property

$$\mathbf{AB} = \mathbf{I}$$

The columns of **B** are just the solutions of a LSAE with a different **b**. For example, in the two by two case we can solve

$$\mathbf{Ax} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and then

$$\mathbf{Ax} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and if we fill the columns of **B** with the solution to these LSAE we will have constructed the inverse. Since we already have the LU decomposition of A we simply solve each case using the technique already presented.

For example, the first column of **B** is determined as follows: First we solve $\mathbf{Ly} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to give $y_1 = 1$ and $y_2 = -2$. Now we solve $\mathbf{Ux} = \mathbf{y}$

$$\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and the solution for **x** is $x_1 = 3/2$, $x_2 = -2/3$. This is the entries in the first column of the inverse. Repeating this process for $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will give the second column of the inverse which now reads

$$\mathbf{A}^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -2/3 & 1/3 \end{bmatrix}$$

9. Consider the appropriate matrix for the last three examples and find the inverse using the LU decomposition previously determined. Check your answer using *inv* in MATLAB.

The \Operator in MATLAB

In MATLAB the main tool for solving linear systems of algebraic equations employs the backslash operator. For example, if A is a matrix and b is a vector then the following command

```
>> x = A\b
```

will find a solution to the LSAE $\mathbf{Ax} = \mathbf{b}$.

This is a very powerful operation, and it employs a variety of algorithms depending on the structure and properties of A. So go ahead and use it, but just be careful.

Building Bridges

Systems of linear equations often come up in engineering when evaluating the strength and stability of structures under load.

What's a truss?

A *truss* is a simplified model of a structure. It consists of a collection of straight, rigid elements or sections that are long compared to the dimensions of their cross-section. Sections are connected only at their ends through frictionless, pin joints (Remember them? They can only constrain translation but not rotation, i.e., they can only apply force but not moments to a section). This means that sections of a truss are either in tension or compression (axial forces along its length). In analyzing trusses it is often assumed that the weight of the sections (dead load) is relatively small, and can therefore be neglected.

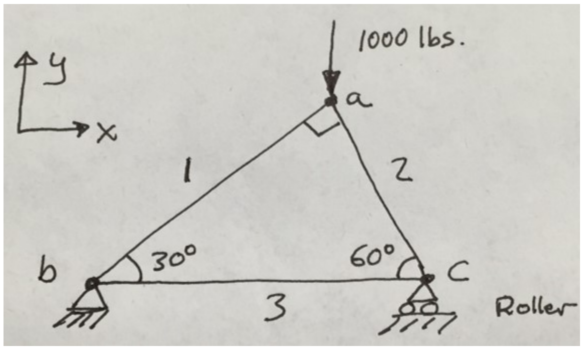
The Method of Joints

The method of joints is a classic technique for determining the forces acting on all of the sections of a truss that is in static equilibrium. Here are the steps:

- Draw a free body diagram for every pin in the truss. Note that the forces acting at the pin have to be in the directions implied by the things the pin is attached to!
- Write out the equations of static equilibrium, $\sum \vec{F} = 0$, for every one of the pins. Note that some of your forces will be known forces (e.g., external loads), and some will be unknown reaction forces.
- Express these equations in the matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- Evaluate whether the system is statically determinate or not. Note the connection to types of solutions to linear equations here: if you look at the form of $\mathbf{A}$, you should be able to tell whether the system is statically determinate!

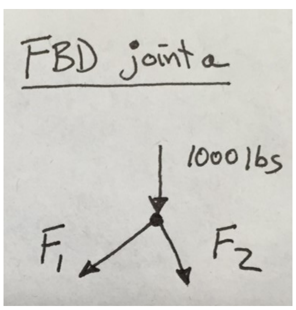
Here's an example of how this plays out:

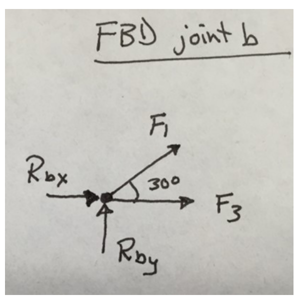
Consider a planar, triangular truss that is supported and loaded as shown. Determine the force (tension or compression) in each section.

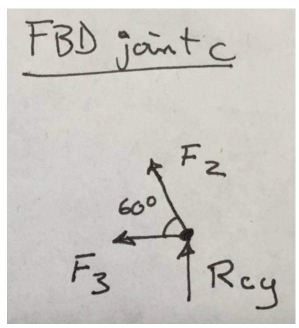


- **Draw a free body diagram** for each joint a, b, and c.

  Note that the pin joint, b, cannot move in space so the ground must apply unknown reaction forces in both the x and y directions. The pin joint, c, is attached to a roller so it is free to move in the x direction but cannot move in the y direction. Therefore, the ground only applies a reaction force (unknown) in the y direction. For the entire truss, there is one known applied external force (1000 lbs) and six unknown axial and reaction forces (F_1, F_2, F_3, R_{bx}, R_{by}, and R_{cy}).







- **Write the equations of equilibrium** for each joint.

  For each joint, $\sum F_x = 0$ and $\sum F_y = 0$. Thus we have the following six equations:

$$
\begin{aligned}
-1000 - F_1 \sin(30) - F_3 \sin(60) &= 0 \\
-F_1 \cos(30) + F_3 \cos(60) &= 0 \\
R_{bx} + F_2 + F_1 \cos(30) &= 0 \\
R_{by} + F_1 \sin(30) &= 0 \\
-F_2 - F_3 \cos(60) &= 0 \\
R_{cy} + F_3 \sin(60) &= 0
\end{aligned}
$$

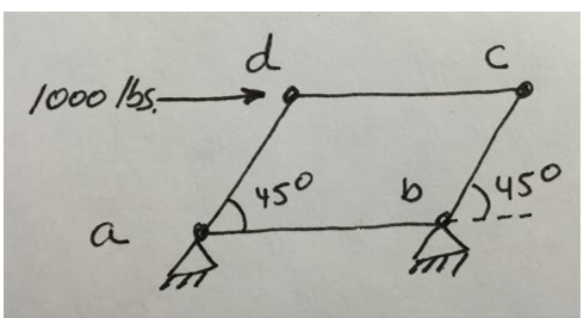
- **Translate the equations into matrix form.** These equations can be written as $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} -\sin 30 & 0 & -\sin 60 & 0 & 0 & 0 \\ -\cos 30 & 0 & \cos 60 & 0 & 0 & 0 \\ \cos 30 & 1 & 0 & 1 & 0 & 0 \\ \sin 30 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -\cos 60 & 0 & 0 & 0 \\ 0 & 0 & \sin 60 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ R_{bx} \\ R_{by} \\ R_{cy} \end{bmatrix}$$

.

Questions: Building Bridges

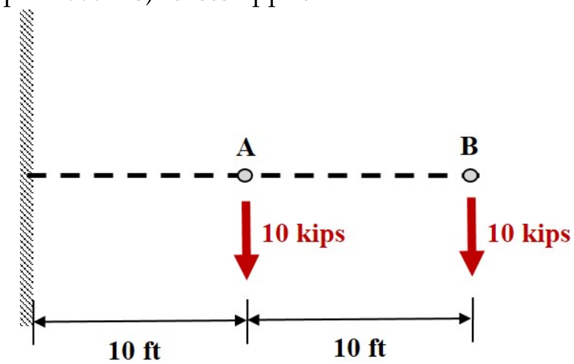
10. Is this problem determinate or indeterminate? Why?
11. If it is determinate, solve it using MATLAB in a couple of ways:
    (a) Solve it using `rref` in MATLAB.
    (b) Solve it using the `inv` in MATLAB.
    (c) Solve it using `A \b` in MATLAB.
12. Interpret the solution. In particular, discuss the meanings of the signs and magnitudes of the various forces. What sections are in tension? In compression? Does this make sense?
13. Now, for the triangle truss above, change the constraint at joint c from a pin on a roller to a pin fixed in space.
    (a) How do the equations of equilibrium change?
    (b) Write them in form the $\mathbf{Ax} = \mathbf{b}$.
    (c) Try solving for $\mathbf{x}$ using `rref`, `inv`, and `\`. How do they compare?
14. Finally, consider the the parallelogram-shaped, simple truss constrained as shown.



Joints a, b, c, and d are frictionless pins. What will happen when a load of 1000 lbs. is applied in the horizontal direction at joint d? Write the equations of equilibrium, and solve using our three main techniques. Are the results consistent with the behavior you predict?

Design Problem: Building Bridges

Now we'll apply the idea of the method of joints to the design of a structure. You have the task of hanging a sign off of a vertical wall. The sign is represented as two 10 kip (1 kip = 1000 lbs) forces applied



to two pin joints located at points A and B.

Design a 2D (in the plane of the page) truss that will support the sign based upon the following constraints:

- The truss is to be constructed from straight, perfectly rigid sections (e.g., rods or bars).
- A section has (only) two pin joint at each of its ends.
- Sections can only be connected at pin joints. They cannot be connected anywhere else along its length.
- Sections can only be connected to the wall by either a pin joint or a roller joint (can roll freely parallel to the wall but must stay in contact with the wall)
- The tension within any section cannot exceed 45 kips.
- All pin joints are frictionless.
- The weight of the rods can be neglected.

In particular, you should to the following:

15. Specify the **geometry** (e.g., position of all pin/roller joints in space) of your truss. Specify how the structure is attached to the wall. The answer to this question should be a diagram showing the different sections, and the different connection points drawn as rollers or pins. The diagram should have lengths and angles defined.

16. Using the method of joints to determine the force within each section. Note that the force in each section will depend both on the overall configuration and on your choice of angles, lengths, etc. Be clear about which sections are in tension and which are in compression.

17. Verify that the maximum tension design constraint has been met (and if it hasn't, change your design appropriately).