did fundamental work connecting the zeta function to the distribution of prime numbers. And finally just before the close of the nineteenth century J. Hadamard and C.J. de la Vallée Poussin independently proved the prime number theorem using techniques of complex analysis.

Another strand we shall follow is the conformal mapping of domains in the plane and more generally of Riemann surfaces. We shall aim at two poster results: the Riemann mapping theorem and the uniformization theorem for Riemann surfaces. The definitive version of the Riemann mapping theorem, which one finds in all complex analysis textbooks today, was proved by W. Osgood in 1900. The uniformization theorem for Riemann surfaces was proved independently in 1907 by P. Koebe and H. Poincaré, thereby solving Hilbert's 22nd problem from his famous address to the International Mathematical Congress in 1900.

The first quarter of the twentieth century was one of rapid development of the foundations of complex analysis. P. Montel put his finger on the notion of compactness in spaces of meromorphic functions and developed the theory of normal families. P. Fatou and G. Julia used Montel's theorem in their seminal work around 1914-1921 on complex iteration theory. On another front, O. Perron developed in 1923 a powerful method for solving the Dirichlet problem.

By the end of the first quarter of the twentieth century, the complex analysis canon had been established, and nearly all the main results constituting the undergraduate and first-year graduate courses in complex analysis had been obtained. Nevertheless, throughout the twentieth century there has been much exciting progress on the frontiers of research in complex analysis, and meanwhile proofs of the most difficult foundational results have been gradually simplified and clarified. While the complex analysis canon has remained relatively static, the developments at the frontier have led to new perspectives and shifting emphases. For instance, the current research interest in dynamical systems and the advent of computer graphics contributed to elevating the work of Fatou and Julia to a more prominent position.

What lies before you is the distillation of the essential, the useful, and the beautiful, from two centuries of labor. Enjoy!

I

The Complex Plane and Elementary Functions

In this chapter we set the scene and introduce some of the main characters. We begin with the three representations of complex numbers: the Cartesian representation, the polar representation, and the spherical representation. Then we introduce the basic functions encountered in complex analysis: the exponential function, the logarithm function, power functions, and trigonometric functions. We view several concrete functions $w = f(z)$ as mappings from the z-plane to the w-plane, and we consider the problem of describing the inverse functions.

1. Complex Numbers

A **complex number** is an expression of the form $z = x + iy$, where x and y are real numbers. The component x is called the **real part** of z, and y is the **imaginary part** of z. We will denote these by

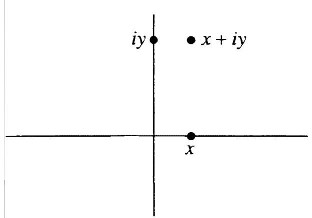
$$\begin{aligned} x &= \operatorname{Re} z, \\ y &= \operatorname{Im} z. \end{aligned}$$

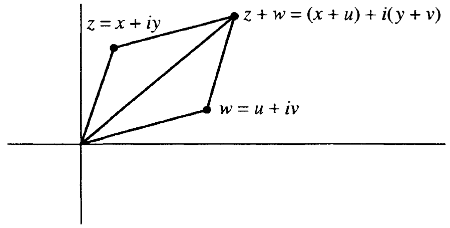
The set of complex numbers forms the **complex plane**, which we denote by $\mathbb{C}$. We denote the set of real numbers by $\mathbb{R}$, and we think of the real numbers as being a subset of the complex plane, consisting of the complex numbers with imaginary part equal to zero.

The correspondence

$$z = x + iy \longleftrightarrow (x, y)$$

is a one-to-one correspondence between complex numbers and points (or vectors) in the Euclidean plane $\mathbb{R}^2$. The real numbers correspond to the x-axis in the Euclidean plane. The complex numbers of the form iy are called **purely imaginary numbers**. They form the **imaginary axis** $i\mathbb{R}$





in the complex plane, which corresponds to the y-axis in the Euclidean plane.

We add complex numbers by adding their real and imaginary parts:

$$(x+iy)+(u+iv) \;=\; (x+u)+i(y+v).$$

Thus $\mathrm{Re}(z+w)=\mathrm{Re}(z)+\mathrm{Re}(w)$, and $\mathrm{Im}(z+w)=\mathrm{Im}(z)+\mathrm{Im}(w)$ for $z,w\in\mathbb{C}$. The addition of complex numbers corresponds to the usual componentwise addition in the Euclidean plane.

The **modulus** of a complex number $z=x+iy$ is the length $\sqrt{x^2+y^2}$ of the corresponding vector (x,y) in the Euclidean plane. The modulus of z is also called the **absolute value** of z, and it is denoted by $|z|$:

$$|z| \;=\; \sqrt{x^2+y^2}.$$

The triangle inequality for vectors in the plane takes the form

$$|z+w| \;\le\; |z|+|w|, \qquad z,w\in\mathbb{C}.$$

By applying the triangle inequality to $z=(z-w)+w$, we obtain $|z|\le|z-w|+|w|$. Subtracting $|w|$, we obtain a very useful inequality,

$$(1.1) \qquad |z-w| \;\ge\; |z|-|w|, \qquad z,w\in\mathbb{C}.$$

Complex numbers can be multiplied, and this is the feature that distinguishes the complex plane $\mathbb{C}$ from the Euclidean plane $\mathbb{R}^2$. Formally, the multiplication is defined by

$$(x+iy)(u+iv) \;=\; xu-yv+i(xv+yu).$$

One can check directly from this definition that the usual laws of algebra hold for complex multiplication:

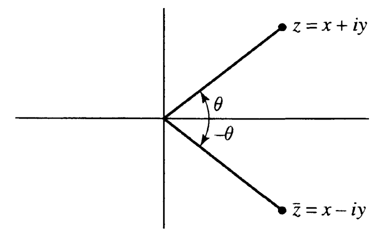
$$\begin{aligned} (z_1z_2)z_3 &= z_1(z_2z_3), && \text{(associative law)}\\ z_1z_2 &= z_2z_1, && \text{(commutative law)}\\ z_1(z_2+z_3) &= z_1z_2+z_1z_3. && \text{(distributive law)} \end{aligned}$$

With respect to algebraic operations, complex numbers behave the same as real numbers. Algebraic manipulations are performed on complex numbers using the usual laws of algebra, together with the special rule $i^2=-1$.

Every complex number $z\ne0$ has a multiplicative inverse $1/z$, which is given explicitly by

$$\frac{1}{z} \;=\; \frac{x-iy}{x^2+y^2}, \qquad z=x+iy\in\mathbb{C},\; z\ne0.$$

Thus for instance, the multiplicative inverse of i is $1/i=-i$.



The **complex conjugate** of a complex number $z=x+iy$ is defined to be $\bar z=x-iy$. Geometrically, $\bar z$ is the reflection of z in the x-axis. If we reflect twice, we return to z,

$$\bar{\bar z} \;=\; z, \qquad z\in\mathbb{C}.$$

Some other useful properties of complex conjugation are

$$\begin{aligned} \overline{z+w} &= \bar z+\bar w, && z,w\in\mathbb{C},\\ \overline{zw} &= \bar z\bar w, && z,w\in\mathbb{C},\\ |z| &= |\bar z|, && z\in\mathbb{C},\\ |z|^2 &= z\bar z, && z\in\mathbb{C}. \end{aligned}$$

Each of these identities can be verified easily using the definition of $\bar z$ and $|z|$. The last formula above allows us to express $1/z$ in terms of the complex conjugate $\bar z$:

$$1/z \;=\; \bar z/|z|^2, \qquad z\in\mathbb{C}, z\ne0.$$

The real and imaginary parts of z can be recovered from z and $\bar z$, by

$$\begin{aligned} \mathrm{Re}\,z &= (z+\bar z)/2, && z\in\mathbb{C},\\ \mathrm{Im}\,z &= (z-\bar z)/2i, && z\in\mathbb{C}. \end{aligned}$$

From $|zw|^2=(zw)(\overline{zw})=(z\bar z)(w\bar w)=|z|^2|w|^2$, we obtain also

$$|zw| \;=\; |z||w|, \qquad z,w\in\mathbb{C}.$$

A **complex polynomial of degree** $n\ge0$ is a function of the form

$$p(z) \;=\; a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0, \qquad z\in\mathbb{C},$$

where $a_0, \dots, a_n$ are complex numbers, and $a_n \neq 0$. A key property of the complex numbers, not enjoyed by the real numbers, is that any polynomial with complex coefficients can be factored as a product of linear factors.

Fundamental Theorem of Algebra. *Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization*

$$p(z) \;=\; c(z-z_1)^{m_1}\cdots(z-z_k)^{m_k},$$

where the z_j's are distinct and $m_j \geq 1$. This factorization is unique, up to a permutation of the factors.

We will not prove this theorem now, but we will give several proofs later. Some remarks are in order.

The uniqueness of the factorization is easy to establish. The points $z_1, \dots, z_k$ are uniquely characterized as the **roots** of $p(z)$, or the **zeros** of $p(z)$. These are the points where $p(z) = 0$. The integer m_j is characterized as the unique integer m with the property that $p(z)$ can be factored as $(z-z_j)^m q(z)$ where $q(z)$ is a polynomial satisfying $q(z_j) \neq 0$.

For the proof of the existence of the factorization, one proceeds by induction on the degree n of the polynomial. The crux of the matter is to find a point z_1 such that $p(z_1) = 0$. With a root z_1 in hand, one easily factors $p(z)$ as a product $(z-z_1)q(z)$, where $q(z)$ is a polynomial of degree $n-1$. (See the exercises.) The induction hypothesis allows one to factor $q(z)$ as a product of linear factors, and this yields the factorization of $p(z)$. Thus the fundamental theorem of algebra is equivalent to the statement that every complex polynomial of degree $n \geq 1$ has a zero.

Example. The polynomial $p(x) = x^2+1$ with real coefficients cannot be factored as a product of linear polynomials with real coefficients, since it does not have any real roots. However, the complex polynomial $p(z) = z^2+1$ has the factorization

$$z^2+1 \;=\; (z-i)(z+i),$$

corresponding to the two complex roots $\pm i$ of z^2+1.

Exercises for I.1

1. Identify and sketch the set of points satisfying:
   (a) $|z-1-i| = 1$
   (b) $1 < |2z-6| < 2$
   (c) $|z-1|^2 + |z+1|^2 < 8$
   (d) $|z-1| + |z+1| \leq 2$
   (e) $|z-1| < |z|$
   (f) $0 < \operatorname{Im} z < \pi$
   (g) $-\pi < \operatorname{Re} z < \pi$
   (h) $|\operatorname{Re} z| < |z|$
   (i) $\operatorname{Re}(iz+2) > 0$
   (j) $|z-i|^2 + |z+i|^2 < 2$

2. Verify from the definitions each of the identities (a) $\overline{z+w} = \bar{z} + \bar{w}$, (b) $\overline{zw} = \bar{z}\bar{w}$, (c) $|\bar{z}| = |z|$, (d) $|z|^2 = z\bar{z}$. Draw sketches to illustrate (a) and (c).

3. Show that the equation $|z|^2 - 2\operatorname{Re}(\bar{a}z) + |a|^2 = \rho^2$ represents a circle centered at a with radius ρ.

4. Show that $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$, and sketch the set of points for which equality holds.

5. Show that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$. Show that
$$|z+w|^2 \;=\; |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}).$$
Use this to prove the triangle inequality $|z+w| \leq |z| + |w|$.

6. For fixed $a \in \mathbb{C}$, show that $|z-a|/|1-\bar{a}z| = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$.

7. Fix $\rho > 0$, $\rho \neq 1$, and fix $z_0, z_1 \in \mathbb{C}$. Show that the set of z satisfying $|z-z_0| = \rho|z-z_1|$ is a circle. Sketch it for $\rho = \frac{1}{2}$ and $\rho = 2$, with $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

8. Let $p(z)$ be a polynomial of degree $n \geq 1$ and let $z_0 \in \mathbb{C}$. Show that there is a polynomial $h(z)$ of degree $n-1$ such that $p(z) = (z-z_0)h(z) + p(z_0)$. In particular, if $p(z_0) = 0$, then $p(z) = (z-z_0)h(z)$.

9. Find the polynomial $h(z)$ in the preceding exercise for the following choices of $p(z)$ and z_0: (a) $p(z) = z^2$ and $z_0 = i$, (b) $p(z) = z^3+z^2+z$ and $z_0 = -1$, (c) $p(z) = 1 + z + z^2 + \cdots + z^m$ and $z_0 = -1$.

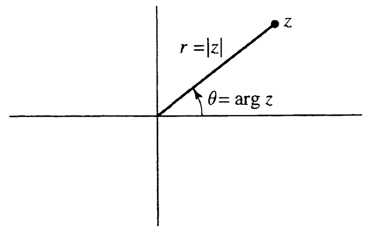
10. Let $q(z)$ be a polynomial of degree $m \geq 1$. Show that any polynomial $p(z)$ can be expressed in the form
$$p(z) \;=\; h(z)q(z) + r(z),$$
where $h(z)$ and $r(z)$ are polynomials and the degree of the remainder $r(z)$ is strictly less than m. *Hint.* Proceed by induction on the degree of $p(z)$. The resulting method is called the **division algorithm**.

11. Find the polynomials $h(z)$ and $r(z)$ in the preceding exercise for $p(z) = z^n$ and $q(z) = z^2 - 1$.

2. Polar Representation

Any point $(x,y) \neq (0,0)$ in the plane can be described by polar coordinates r and θ, where $r = \sqrt{x^2+y^2}$ and θ is the angle subtended by (x,y) and the x-axis. The angle θ is determined only up to adding an integral multiple of 2π. The Cartesian coordinates x, y are recovered from the polar coordinates r, θ by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$



If we write the polar representation in complex notation, we obtain

(2.1) $$z \;=\; x+iy \;=\; r(\cos\theta + i\sin\theta).$$

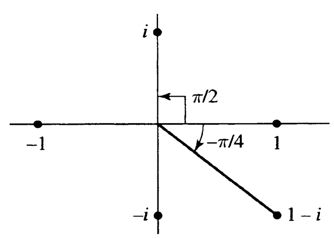
Here $r = |z|$ is the modulus of z. We define the **argument** of z to be the angle θ, and we write

$$\theta \;=\; \arg z.$$

Thus $\arg z$ is a multivalued function, defined for $z \neq 0$. The **principal value of** $\arg z$, denoted by $\operatorname{Arg} z$, is specified rather arbitrarily to be the value of θ that satisfies $-\pi < \theta \le \pi$. The values of $\arg z$ are obtained from $\operatorname{Arg} z$ by adding integral multiples of 2π:

$$\arg z \;=\; \{\operatorname{Arg} z + 2\pi k \,:\, k = 0, \pm 1, \pm 2, \dots\}, \qquad z \neq 0.$$

Example. The principal value of $\arg i$ is $\operatorname{Arg} i = \pi/2$. The principal value of $\arg(1-i)$ is $\operatorname{Arg}(1-i) = -\pi/4$.



It will be convenient to introduce the notation

(2.2) $$e^{i\theta} \;=\; \cos\theta + i\sin\theta.$$

From (2.1) we obtain

$$z \;=\; re^{i\theta}, \qquad r = |z|,\ \theta = \arg z.$$

This representation is called the **polar representation** of z. The sine and cosine functions are 2π-periodic, that is, they satisfy $\sin(\theta + 2\pi m) = \sin\theta$, $\cos(\theta + 2\pi m) = \cos\theta$. Thus the various choices of $\arg z$ yield the same value for $e^{i\theta}$,

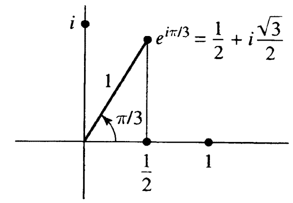
$$e^{i(\theta+2\pi m)} \;=\; e^{i\theta}, \qquad m = 0, \pm 1, \pm 2, \dots .$$

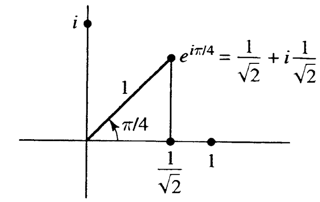
Example. Some common complex exponentials are

$$e^{i\pi} = -1, \quad e^{i\pi/2} = i, \quad e^{i\pi/3} = \frac{1+\sqrt{3}i}{2}, \quad e^{i\pi/4} = \frac{1+i}{\sqrt{2}}.$$

Also note that

$$e^{2\pi m i} \;=\; 1, \qquad m = 0, \pm 1, \pm 2, \dots .$$





Several useful identities satisfied by the exponential function are

(2.3) $$|e^{i\theta}| \;=\; 1,$$

(2.4) $$\overline{e^{i\theta}} \;=\; e^{-i\theta},$$

(2.5) $$1/e^{i\theta} \;=\; e^{-i\theta}.$$

The identity (2.3) is equivalent to the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, while (2.4) follows from $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

One of the most important properties of the exponential function is the **addition formula**

(2.6) $$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}, \qquad -\infty < \theta,\ \varphi < \infty.$$

In view of the definition (2.2), this is equivalent to

$$\cos(\theta+\varphi) + i\sin(\theta+\varphi) \;=\; (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi).$$

Multiplying out the right-hand side and equating real and imaginary parts, we obtain the equivalent pair of identities

(2.7) $$\begin{cases} \cos(\theta+\varphi) \;=\; \cos\theta\cos\varphi - \sin\theta\sin\varphi, \\ \sin(\theta+\varphi) \;=\; \cos\theta\sin\varphi + \sin\theta\cos\varphi, \end{cases}$$

which are the addition formulae for sine and cosine. Thus the addition formula (2.6) for the complex exponential is a compact form of the addition formulae (2.7) for the sine and cosine functions, and it is much easier to remember!

The properties (2.4), (2.5), (2.6) of the exponential function correspond respectively to the following properties of the argument function:

(2.8) $$\arg \bar{z} \;=\; -\arg z,$$

(2.9) $$\arg(1/z) \;=\; -\arg z,$$

(2.10) $$\arg(z_1 z_2) \;=\; \arg z_1 + \arg z_2,$$

where each formula is understood to hold modulo adding integral multiples of 2π. To establish (2.8) and (2.9), note that if the polar representation of z is $re^{i\theta}$, then the polar representation of $\bar{z}$ is $re^{-i\theta}$, and that of $1/z$ is $(1/r)e^{-i\theta}$. For (2.10), write $z_1 = r_1e^{i\varphi_1}$, $z_2 = r_2e^{i\theta_2}$, and use the addition formula to obtain the polar form of z_1z_2,

$$z_1z_2 \;=\; r_1r_2e^{i\theta_1}e^{i\theta_2} \;=\; r_1r_2e^{i(\theta_1+\theta_2)}.$$

The addition formula (2.6) can be used to derive formulae for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. Write

$$\cos(n\theta) + i\sin(n\theta) \;=\; e^{in\theta} \;=\; (e^{i\theta})^n \;=\; (\cos\theta + i\sin\theta)^n,$$

expand the right-hand side, and equate real and imaginary parts. This yields expressions for $\cos(n\theta)$ and $\sin(n\theta)$ that are polynomials in $\cos\theta$ and $\sin\theta$. These identities are known as **de Moivre's formulae**. For instance, by equating $\cos(3\theta) + i\sin(3\theta)$ to

$$(\cos\theta + i\sin\theta)^3 \;=\; \cos^3\theta - 3\cos\theta\sin\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

and taking real and imiginary parts, we obtain

$$\begin{aligned}\cos(3\theta) &= \mathrm{Re}(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta,\\ \sin(3\theta) &= \mathrm{Im}(\cos\theta + i\sin\theta)^3 = 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

A complex number z is an **nth root** of w if $z^n = w$. Thus the nth roots of w are precisely the zeros of the polynomial $z^n - w$ of degree n. Since this polynomial has degree n, w has at most n nth roots. If $w \neq 0$, then w has exactly n nth roots, and these are determined as follows. First express w in polar form,

$$w \;=\; \rho e^{i\varphi}.$$

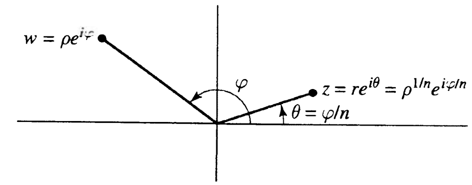
The equation $z^n = w$ becomes

$$r^n e^{in\theta} \;=\; \rho e^{i\varphi}.$$

Thus $r^n = \rho$ and $n\theta = \varphi + 2\pi k$ for some integer k. This leads to the explicit solutions

$$\begin{aligned} r &= \rho^{1/n},\\ \theta &= \frac{\varphi}{n} + \frac{2\pi k}{n}, \qquad k = 0, 1, 2, \dots, n-1,\end{aligned}$$

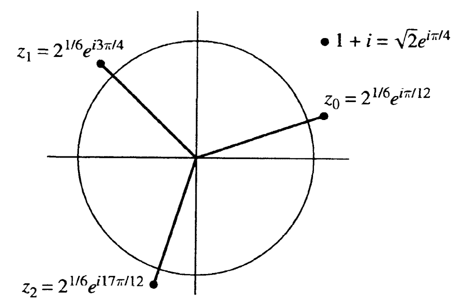
where we take the usual positive root of ρ. Since these n roots are distinct, and there are at most n nth roots, this list includes all the nth roots of w. Other values of k do not give different roots, since any other integer k leads to a value of θ that is obtained from the above list by adding an integral multiple of 2π. Graphically, the roots are distributed in equal arcs on the circle centered at 0 of radius $|w|^{1/n}$.



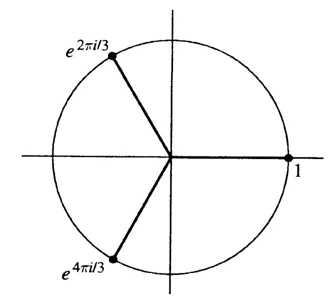
Example. To find and plot the square roots of $4i$, first express $4i$ in polar form $\rho e^{i\varphi}$. Here $\rho = |4i| = 4$ and $\varphi = \arg(4i) = \pi/2$. One root is given by $\sqrt{\rho}\, e^{i\varphi/2} = 2e^{i\pi/4}$. The other is $2e^{i(\pi/4+\pi)} = -2e^{i\pi/4}$. In Cartesian form, the roots are $\sqrt{4i} = \pm(\sqrt{2} + \sqrt{2}i)$.

Example. To find and plot the cube roots of $1+i$, express $1+i$ in polar form as $\sqrt{2}\, e^{i\pi/4}$. The polar form of the three cube roots is given by

$$2^{1/6}e^{i(\pi/12 + 2k\pi/3)}, \qquad k = 0, 1, 2.$$



cube roots of $1 + i$



cube roots of unity

The nth roots of 1 are also called the **nth roots of unity**. They are given explicitly by

$$\omega_k \;=\; e^{2\pi ik/n}, \qquad 0 \le k \le n-1.$$

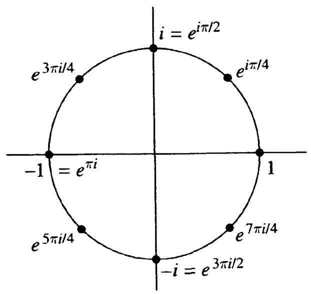
Graphically, they are situated at equal intervals around the unit circle in the complex plane. Thus the two square roots of unity are $e^0 = 1$ and $e^{i\pi} = -1$.

The procedure for finding the nth roots of $w \neq 0$ can be rephrased in terms of the nth roots of unity. We express $w = \rho e^{i\varphi/n}$ in polar form as above. One root is given by $z_0 = \rho^{1/n}e^{i\varphi/n}$. The others are found by multiplying z_0 by the nth roots of unity:

$$z_k \;=\; z_0\omega_k \;=\; \rho^{1/n}e^{i\varphi/n}e^{2\pi ik/n}, \qquad 0 \le k \le n-1.$$

Exercises for I.2

1. Express all values of the following expressions in both polar and cartesian coordinates, and plot them.



The eight eighth roots of unity

(a) $\sqrt{i}$ (c) $\sqrt[4]{-1}$ (e) $(-8)^{1/3}$ (g) $(1+i)^8$

(b) $\sqrt{i-1}$ (d) $\sqrt[4]{i}$ (f) $(3-4i)^{1/8}$ (h) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{25}$

2. Sketch the following sets:
   (a) $|\arg z| < \pi/4$ (c) $|z| = \arg z$
   (b) $0 < \arg(z-1-i) < \pi/3$ (d) $\log|z| = -2\arg z$

3. For a fixed complex number b, sketch the curve $\{e^{i\theta} + be^{-i\theta} : 0 \le \theta \le 2\pi\}$. Differentiate between the cases $|b| < 1$, $|b| = 1$ and $|b| > 1$. *Hint.* First consider the case $b > 0$, and then reduce the general case to this case by a rotation.

4. For which n is i an nth root of unity?

5. For $n \ge 1$, show that
   (a) $1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1-z), \qquad z \ne 1,$
   (b) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin(n+\frac{1}{2})\theta}{2\sin\theta/2}.$

6. Fix $n \ge 1$. Show that the nth roots of unity $\omega_0, \dots, \omega_{n-1}$ satisfy:
   (a) $(z-\omega_0)(z-\omega_1)\cdots(z-\omega_{n-1}) = z^n - 1,$
   (b) $\omega_0 + \cdots + \omega_{n-1} = 0,$
   (c) $\omega_0 \cdots \omega_{n-1} = (-1)^{n-1},$
   (d) $\displaystyle\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \le k \le n-1, \\ n, & k = n. \end{cases}$

7. Fix $R > 1$ and $n \ge 1$, $m \ge 0$. Show that

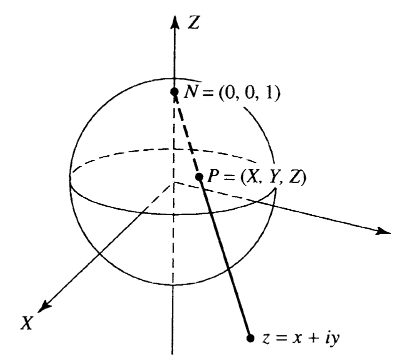
$$\left|\frac{z^m}{z^n+1}\right| \le \frac{R^m}{R^n-1}, \qquad |z| = R.$$

Sketch the set where equality holds. *Hint.* See (1.1).

8. Show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$ using de Moivre's formulae. Find formulae for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

3. Stereographic Projection

The **extended complex plane** is the complex plane together with the point at infinity. We denote the extended complex plane by $\mathbb{C}^*$, so that $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. One way to visualize the extended complex plane is through stereographic projection. This is a function, or map, from the unit sphere in three-dimensional Euclidean space $\mathbb{R}^3$ to the extended complex plane, which is defined as follows. If $P = (X, Y, Z)$ is any point of the unit sphere other than the north pole $N = (0,0,1)$, we draw a straight line through N and P, and we define the **stereographic projection** of P to be the point $z = x + iy \sim (x, y, 0)$ where the straight line meets the coordinate plane $Z = 0$. The stereographic projection of the north pole N is defined to be ∞, the point at infinity.



An explicit formula for the stereographic projection is derived as follows. We represent the line through P and N parametrically by $N + t(P - N)$, $-\infty < t < \infty$. The line meets the (x,y)-plane at a point $(x, y, 0)$ that satisfies

$$\begin{aligned}(x,y,0) &= (0,0,1) + t[(X,Y,Z) - (0,0,1)] \\ &= (tX, tY, 1 + t(Z-1))\end{aligned}$$

for some parameter value t. Equating the third components, we obtain $0 = 1 + t(Z-1)$, which allows us to solve for the parameter value t,

$$t = 1/(1-Z).$$