Lecture notes of SW seminar

Characteristic class

① Curvature matrix Ω.

Given smooth v.b. $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, connection is $\nabla^A : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$

s.t. $\nabla^A_X(f\sigma+\tau) = f\nabla^A_X(\sigma) + \nabla^A_X(\tau) + (Xf)\sigma$

$\nabla^A_{fX+\sigma}(\tau) = f\nabla^A_X\tau + \nabla^A_\sigma\tau$

Given ∇^A, we define $d_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$

s.t. $d_A(\sigma)(X) = \nabla^A_X(\sigma)$

Locally, $d_A\sigma = d_A\begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix} = \begin{pmatrix} d\sigma_1 \\ \vdots \\ d\sigma_n\end{pmatrix} + (\omega_{ij})\begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix}$

Connection 2-form Ω: locally, $\Omega_\alpha = d\omega_\alpha + \omega_\alpha\wedge\omega_\alpha$.

② Given $U(m)$-v.b. E, $\mathbb{C}$-rank $= m$, with Her-metric

Given unitary connection $d+\omega_\alpha$. Ω_α.

We see $\left(\frac{i}{2\pi}\Omega_\alpha\right)^k$ is Hermitian. Then $\mathrm{Tr}\left(\frac{i}{2\pi}\Omega_\alpha\right)^k$ is real since Ω_α is skew-Her.

proof. We know ω_α is skew-Her. Then $d\omega_\alpha$ is skew-Her.

$\omega_\alpha\wedge\omega_\alpha(ji) = \sum_{k=1}^n \omega_\alpha(jk)\wedge\omega_\alpha(ki) = \sum_{k=1}^n -\overline{\omega_\alpha(kj)}\wedge(-\overline{\omega_\alpha(ik)})$

$= \sum_{k=1}^n -\overline{\omega_\alpha(ik)\wedge\omega_\alpha(kj)} = -\overline{\omega_\alpha\wedge\omega_\alpha(ij)}$

$\mathrm{Tr}\left[\left(\frac{i}{2\pi}\Omega_\alpha\right)^k\right]$ glue together to form a form $\tau_k(A)$ char. form.

$\qquad\shortparallel$

$\mathrm{Tr}\left[\left(\frac{i}{2\pi}\Omega_\beta\right)^k\right]$ on $U_\alpha\cap U_\beta$

Thm. $d\tau_k(A)=0$. so $[\tau_A(A)] \in H^{2k}(M;\mathbb{R})$ is defined.

$$d\,Tr\left(\frac{i}{2\pi}\Omega_\alpha\right)^k = \left(\frac{i}{2\pi}\right)^k d\left(Tr\,\Omega_\alpha^k\right)$$

$$= \left(\frac{i}{2\pi}\right)^k \left(d\Omega_\alpha (\Omega_\alpha)^{k-1} + \cdots + (\Omega_\alpha)^{k-1}(d\Omega_\alpha)\right)$$

$$\text{(Bianchi id)} = \left(\frac{i}{2\pi}\right)^k Tr\left([\Omega_\alpha,\omega]\,\Omega_\alpha^{k-1} + \cdots + (\Omega_\alpha)^{k-1}[\Omega_\alpha,\omega]\right)$$

And $(\Omega_\alpha\omega - \omega\Omega_\alpha)\Omega_\alpha^{k-1} + \Omega_\alpha(\Omega_\alpha\omega - \omega\Omega_\alpha)\Omega_\alpha^{k-2} + \cdots$

$+ \Omega_\alpha^{k-1}(\Omega_\alpha\omega - \omega\Omega_\alpha)$

$= -\omega\Omega_\alpha^k + \Omega_\alpha^k\omega = [\Omega_\alpha^k,\omega]$

Properties. ① $\tau_k(F^*A) = F^*\tau_k(A)$

$\downarrow$ pull back connection

② $\tau_k(A)$ doesn't depend on choice of d_A & Hermitian metric on E.

Spaces {unitary connection} and {Her. metric} are connected

So we call them $\tau_k(E)$. char. classes of E.

Chern character: $ch(E) = \left[Tr\left(\exp\left(\frac{i}{2\pi}\Omega_\alpha\right)\right)\right]$

$$= rk(E) + \tau_1 + \frac{1}{2!}\tau_2 + \cdots + \frac{1}{k!}\tau_k + \cdots$$

Thm. $ch(E_1+E_2) = ch(E_1) + ch(E_2)$

$ch(E_1\otimes E_2) = ch(E_1)\,ch(E_2)$

$E_i \sim d_{A_i}$, $\quad E_1\oplus E_2 \sim d_{A_1\oplus A_2}$

$$(d_{A_1\oplus A_2})^2(\sigma_1\oplus\sigma_2) = d_{A_1}^2\sigma_1 \oplus d_{A_2}^2\sigma_2$$

$$\left(\frac{i}{2\pi}\Omega^{A_1\oplus A_2}\right)^k = \begin{pmatrix} \left(\left(\frac{i}{2\pi}\right)^k\Omega^{A_1}\right)^k & \\ & \left(\left(\frac{i}{2\pi}\right)^k\Omega^{A_2}\right)^k \end{pmatrix}$$

$$\tau_k(E_1 \oplus E_2) = \tau_k(E_1) + \tau_k(E_2)$$

$$d_{A_1 \otimes A_2}(\sigma_1 \otimes \sigma_2) = d_{A_1}\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes d_{A_1}\sigma_2$$

$$d^2_{A_1 \otimes A_2}(\sigma_1 \otimes \sigma_2) = d^2_{A_1}\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes d^2_{A_1}\sigma_2 \quad \text{(skew derivation)}$$

$$(d_{A_1 \otimes A_2})^{2k}(\sigma_1 \otimes \sigma_2) = \sum \binom{k}{j} (d^{2j}_{A_1}\sigma_1) \otimes (d^{2k-2j}_{A_2}\sigma_2)$$

$$\frac{1}{k!}(d_{A_1 \otimes A_2})^{2k}(\sigma_1 \otimes \sigma_2 = \sum \frac{1}{j!}(d^{2j}_{A_1}\sigma_1) \otimes \frac{1}{k-j}(d^{2k-2j}_{A_2}\sigma_2)$$

Then $\exp((d_{A_1 \otimes A_2})^2) = \exp((d^2_{A_1})) \exp((d_{A_2})^2)$

Chern classes.

$c_1(E) := \tau_1(E)$, $\quad c_2(E) := \frac{1}{2}[\tau_1(E)^2 - \tau_2(E)]$

$c_1(E^*) = -c_1(E)$, $\quad c_2(E^*) = c_2(E)$

Thom form

$c_1(E)$: $\mathbb{C} \hookrightarrow E$ geometric zeroes ~ section.
$\downarrow$
S

$\mathbb{C} \to L$ $\quad t_1, t_2 : \pi^{-1}(U) \to \mathbb{R}$
$\downarrow$
$M \supset U$ open $\quad t_i(a_j e^j(p)) = a_i$

frame $e_1, e_2 = ie_1$

Thom form : $\Phi_\alpha = \frac{1}{2\pi} e^{-(t_1^2+t_2^2)}[\Omega_{12} + 2(dt_1 + \omega_{12}t_2)\wedge(dt_2 - \omega_{12}t_1)]$

Def. ~ ω_{12}, Ω_{12}. $= \frac{1}{2\pi} e^{-(t_1^2+t_2^2)}[\Omega_{12} + 2dt_1\wedge dt_2 + \omega_{12}\wedge d(t_1^2 + t_2^2)]$

$\mathbb{C} \to L$ $\quad e_1$: unit length section of E
$\downarrow$
M $\quad e_2 = ie_1$, $\quad \omega_{12}$: $de_1 = (-i\omega_{12})e_1 = -e_2\omega_{12}$

$F_A = \Omega_{12} = d\omega_{12}$

Property ~ Φ

1. closed 2. rapidly decreasing 3. integral over $\forall$ fiber is 1.

Δ We can replace Φ by

$\eta(t_1^2+t_2^2)\Omega_{12} - 2\eta'(t_1^2+t_2^2)(dt_1 + \omega_{12}t_2)\wedge(dt_2 - \omega_{12}t_1)]$

η: $[0,\infty) \to \mathbb{R}$ smooth, $\eta(0) = \frac{1}{2\pi}$, $\eta(u) \to 0$ rapidly

$\sigma: M \to L$ $\not\sim$ $\sigma^*(\Phi)$ since $\forall$ 2 sections homotopic.

$[\sigma^*(\Phi)] = c_1(L) = \frac{1}{2\pi}[\Omega_{12}]$ $\quad \sigma(\Sigma)$: transverse intersection with 0 section at p.

$\mathbb{C} \to L$ ~ unitary connection

$\sigma \uparrow\downarrow$
Σ nondgnrt zero $p_1, \cdots, p_k$

$\sum \omega(\sigma, p_i) = \frac{1}{2\pi}\int_\Sigma F_A = \langle c_1(L), [\Sigma] \rangle$

$\sigma_s = s\sigma$ as $s \to \infty$ $\quad t_i = t_i \circ \sigma$

$\sigma_s^* \Phi = \frac{1}{2\pi} e^{-s^2(\sigma_1^2+\sigma_2^2)} [\Omega_{12} + 2s^2(d\sigma_1 + \omega_{12}\sigma_2) \wedge (d\sigma_2 - \omega_{12}\sigma_1)]$

goes to 0 as $s \to \infty$, except nbhd $V_1, \cdots, V_t$ near $p_1, \cdots, p_t$.

Choose frame s.t. ω_{12} vanishes at each p_i.

Then $\lim\limits_{s\to\infty} \int_M \sigma_s^* \Phi = \sum\limits_i \lim\limits_{s\to\infty} \int_{V_i} \frac{1}{\pi} e^{-(s\sigma_1)^2-(s\sigma_2)^2} d(s\sigma_1) \wedge d(s\sigma_2)$

approaches ± 1 on each V_i

E.g. $L = T\Sigma$, $f: \Sigma \to \mathbb{R}$ Morse f.

Then $\sum\limits_{i=1}^{k} \omega(\nabla f, p_i) = \#\max - \#\text{saddle} + \#\min$

Gauss-Bonnet formula.

$\langle c_1(T\Sigma), \Sigma \rangle = \frac{1}{2\pi}\int_\Sigma F_A = \frac{1}{2\pi}\int_\Sigma K\, dA = \chi(\Sigma)$

Universal bundle

$z = (z_1, \cdots z_i \cdots) \quad w = (w_1, \cdots w_i \cdots) \in \mathbb{C}^\infty \setminus 0$

$z \sim w$ if $z = \lambda w \quad \lambda \in \mathbb{C}^*$

$\underline{\mathbb{C}P^\infty}$ $\quad U_i = \{ [z_j] \mid z_i \neq 0 \}$ open cover

$\downarrow$

$\mathbb{C}^\infty \quad \left(\frac{z_1}{z_i} \cdots\cdots \frac{z_{i-1}}{z_i} \quad \frac{z_{i+1}}{z_i} \cdots \right)$

Universal bundle $\quad \begin{array}{c} \mathbb{C} \hookrightarrow E_\infty \\ \downarrow \\ \mathbb{C}P^\infty \end{array} = \{ (V, z) \in \mathbb{C}P^\infty \times \mathbb{C}^\infty \mid z \in V \}$

$g_{ij} : U_i \cap U_j \to GL(1, \mathbb{C}) \quad = \frac{z_i}{z_j}$

Infinite Grassmannian $\quad G_m(\mathbb{C}^\infty) = \{ V_m \subset \mathbb{C}^\infty \}$

M: smooth mnfd. $\Gamma : [M, G_m(\mathbb{C}^\infty)] \leftrightarrow \mathrm{Vect}^{\mathbb{C}}_m(M)$

$\Gamma(F) = F^* E_\infty$

① Surjectivity. Given $\begin{array}{c} \mathbb{C}^m \to E \\ \pi \downarrow \\ M \end{array} \sim \underset{\text{cuntr}}{\{U_0, \cdots, U_n\}}$

S_i^k : k-simplices in M.

$\Downarrow$

$b_i^k \quad$ barycentric subdivision $\longrightarrow U_k$.

$\begin{tikzcd} \pi^{-1}(U_k) \arrow[r, "\psi_k"] \arrow[rrd, bend left, "\eta_k"] & U_k \times \mathbb{C}^m \arrow[d, "\pi_2"] \\ & \mathbb{C}^m \end{tikzcd}$

$\{ \zeta_0, \cdots, \zeta_n \}$ partition of unity $\sim$ open cover $\{U_\alpha\}$

$\tilde{F} : E \to \mathbb{C}^{m(n+1)}$

$\vdots \quad e \to (\zeta_i(\pi(e)) \eta_i(e))$

$\Downarrow$

$F : M \to G_m(\mathbb{C}^\infty)$

$p \to \tilde{F}(E_p)$

② Inj.

$\mathbb{C}_e^\infty$ $\{(z_1, \cdots) \in \mathbb{C}^\infty \mid z_1 = z_3 = \cdots = 0\}$

$\mathbb{C}_o^\infty$ $z_2 = z_4 = \cdots = 0$

$\tilde{T}_e$ $u_{2j} = z_j$ $\tilde{T}_o$ $u_{2j-1} = z_j$ $\mathbb{C}^\infty \to \mathbb{C}_{o/e}^\infty$

Induce maps $T_e : G_m(\mathbb{C}^\infty) \to G_m(\mathbb{C}_e^\infty)$

$T_o : G_m(\mathbb{C}^\infty) \to G_m(\mathbb{C}_o^\infty)$

$\tilde{H}_e : \mathbb{C}^\infty \times I \to \mathbb{C}^\infty$ induces map

$(z, t) \to tz + (1-t)\tilde{T}_e(z)$ $H_e : G_m(\mathbb{C}^\infty) \times I \to G_m(\mathbb{C}^\infty)$

T_e to id

Similarly define H_o.

$F, G : M \to G_m(\mathbb{C}^\infty)$ $F^* E_\infty = G^* E_\infty$

claim $T_e \circ F \sim T_o \circ G$

$$\begin{array}{ccc} T_e \circ F & \sim & T_o \circ G \\ \uparrow & & \uparrow \\ \tilde{T}_e \circ \tilde{F} & & \tilde{T}_o \circ \tilde{G} \end{array}$$

$\tilde{H} : E \times I \to \mathbb{C}^\infty$, $(e,t) \to t\,\tilde{T}_e \circ \tilde{F}(e,t) + (1-t)\,\tilde{T}_o \circ \tilde{G}(e,t)$

$\downarrow$

$H : M \times I \to G_m(\mathbb{C}^\infty)$, F to G.

Homo - group of $\mathbb{CP}^\infty$

$$\begin{array}{ccc} \mathbb{C}-\{0\} & \hookrightarrow & \mathbb{CP}^\infty \\ & & \downarrow \\ & & \mathbb{C}^\infty - \{0\} \end{array}$$

long exact sequence ~ homotopy

$\mathbb{C}^\infty - \{0\} \simeq S^\infty$, $\pi_k = 0$, $\forall k \in \mathbb{N}^+$.

$\pi_k(\mathbb{CP}^\infty) = \mathbb{Z}$ k even

$K(\overset{s}{\mathbb{Z}}, 2)$ 0 otherwise

$\mathrm{Vect}_1^{\mathbb{C}}(M) \cong [M, K(\mathbb{Z}, 2)] \cong H^2(M; \mathbb{Z})$

Δ Instead of $\mathbb{CP}^\infty$, we can work on $\mathbb{CP}^N$, $N \geq \dim M + 1$.

On $\begin{array}{c} \mathbb{C} \hookrightarrow E_\infty \\ \downarrow \\ \mathbb{CP}^\infty \end{array}$, define a connection and $c_1(E_\infty)$ s.t. $F^*(c_1(E_\infty)) = c_1(E)$ when $\Gamma(F) = E$

Similarly, we have $\Gamma: [M, G_m(\mathbb{R}^\infty)] \xrightarrow{\sim} \mathrm{Vect}_m^{\mathbb{R}}(M)$

$\sim \quad (\mathbb{H}^\infty)] \xrightarrow{\sim} \mathrm{Vect}_m^{\mathbb{H}}(M)$