Limited Immunity in Epidemics

Purpose. The purpose of this lab is to give you practice using calculus to analyze the solutions of systems of differential equations.

Preview. This laboratory assumes that you are familiar with the SIR model for the spread of infectious diseases. We shall develop a more sophisticated model which allows for the possibility that not all people who recover from a disease keep their immunity to it. We shall see that under this assumption the disease does not die out but remains in a fixed percentage of the population.

Part I: The SIRS Model. The SIR model predicts some interesting behavior of actual epidemics, but it makes some unrealistic predictions as well. For example, a glaring one is that it predicts that after the epidemic passes the number of infected people falls off exponentially to zero. In other words, the disease essentially vanishes. (After the number of infected people falls below, say 10^{-6}, it's hard to justify saying that anyone is actually infected.) However, there are many reasons that a disease doesn't actually disappear after an epidemic. In this lab, we want to concentrate on just one: The effect of *limited immunity*, i.e., a certain fraction of the 'recovered' population loses immunity and becomes susceptible again. This is actually not an uncommon phenomenon, though, thankfully, few of the really virulent diseases seem to display it.

We shall keep the notation that we used in the SIR model; that is $S(t), I(t)$, and $R(t)$ represent, respectively, the susceptible, infected, and recovered populations at time t. We keep the assumptions made in deriving the SIR model, but suppose now that recovered individuals have some tendency to lose their immunity. Specifically, we suppose that every day a certain fraction, $\mu > 0$, of the recovered population loses immunity and becomes susceptible again.

(a) Explain why this situation can be modeled by the differential equations

$$\begin{aligned}\frac{dS}{dt} &= -\alpha S(t)I(t) + \mu R(t)\\ \frac{dI}{dt} &= \alpha S(t)I(t) - \lambda I(t)\\ \frac{dR}{dt} &= \lambda I(t) - \mu R(t)\end{aligned}$$

where α, λ, and μ are positive constants. Prove that $M = S(t)+I(t)+R(t)$ remains constant in time.

(b) Let $s = S/M$, $i = I/M$, and $r = R/M$ denote the fractions of the total population that are susceptible, infected, and recovered, respectively. Show that s, i, and r satisfy the differential equations

$$\begin{aligned} \frac{ds}{dt} &= -\beta s(t)i(t) + \mu r(t) \\ \frac{di}{dt} &= \beta s(t)i(t) - \lambda i(t) \\ \frac{dr}{dt} &= \lambda i(t) - \mu r(t) \end{aligned}$$

where $\beta = \alpha M$. Prove that $s(t) + i(t) + r(t) = 1$ for all time. Reduce the system above to a system of two differential equations

$$\frac{ds}{dt} = -\beta s(t)i(t) + \mu(1 - s(t) - i(t)) \tag{1}$$

$$\frac{di}{dt} = \beta s(t)i(t) - \lambda i(t) \tag{2}$$

by substituting $r = 1-i-s$ in the first equation. Explain why $i(t)+s(t) \leq 1$ for all t.

(c) Let the triangle T consist of the set of points in the $s - i$ plane such that $i \geq 0$, $s \geq 0$, and $s + i \leq 1$. Note that at each time t, the point $(s(t), i(t))$ is in this triangle. We will do a phase plane analysis of the equati ons (1) and (2), assuming that the parameter values are $\beta = 0.6$, $\lambda = 0.34$ and $\mu = 0.1$.

Nullclines. Sketch the nullclines for $\frac{di}{dt}$. These are curves consisting of points (s, i) for which

$$\frac{di}{dt} = 0.$$

At these points, the number of infecteds is neither increasing nor decreasing. One of these nullclines, which we will call L, divides the triangle into two parts. In one part, i is increasing and in the other part i is decreasing. Figure out which is which.

In the same triangular region, sketch the nullcline for $\frac{ds}{dt}$. This nullcline is the set of points (s, i) at which the number of susceptibles is neither increasing nor decreasing; i.e., the points at which

$$\frac{ds}{dt} = 0.$$

Let H denote this nullcline. Note that H divides T into two parts. In one part, s is increasing and in the other part s is decreasing. Figure out which is which.

Together, the curve H and the line L divide T into four regions, and in each region we know whether s and i are increasing or decreasing. This information tells us the general direction of a solution trajectory in each of the four regions. Indicate these directions on your phase plane diagram.

(d) *Equilibria.* At the equilibria, neither the number of infecteds nor the number of susceptibles is changing. Using the given values of β, λ and μ, find the equilibria (i.e., the constant solutions) for the system of dif ferential equations given by (1) and (2). Describe what these two equilibrium states represent in terms of an epidemic. How are the equilibria related to the nullclines for $\frac{di}{dt}$ and $\frac{ds}{dt}$?

(e) Choose an initial point within the triangle T and use your phase plane analysis to help draw a rough sketch of the solution curve $(s(t), i(t))$. Now choose several other initial points and sketch the solution curves. Describe, in words, what you expect to see in the long term for an epidemic modeled by this equation. Do the equilibria appear to be stable or unstable?

(f) Write down the recursion formulas you would need in order to use Euler's method to study solutions of the SIRS model given by equations (1) and (2).

(g) Assume the initial conditions are $s(0) = 0.9$ and $i(0) = 0.1$. Using a time step of $\Delta t = 1$ day, compute the points $(s(n), i(n))$ for the first 5 days of the epidemic.

(h) A plot obtained by using Euler's method for 100 days appears on the next page. After day 20, only every other point is shown. Describe the behavior of the solution.

(i) Show that, in general, the nullclines for $\dfrac{di}{dt} = \beta si - \lambda i$ are the lines $i = 0$ and $s = \lambda/\beta$. Confirm that for the values of β, λ and μ given above, one of the equilibria from part (d) occurs on the line $s = \lambda/\beta$. What does the condition that λ is less than β mean in terms of the epidemic, and what effect does this relationship have on the significance of the equilibrium? If λ were greater than β, what would this say about the equilibria? What would it say about the epidemic?

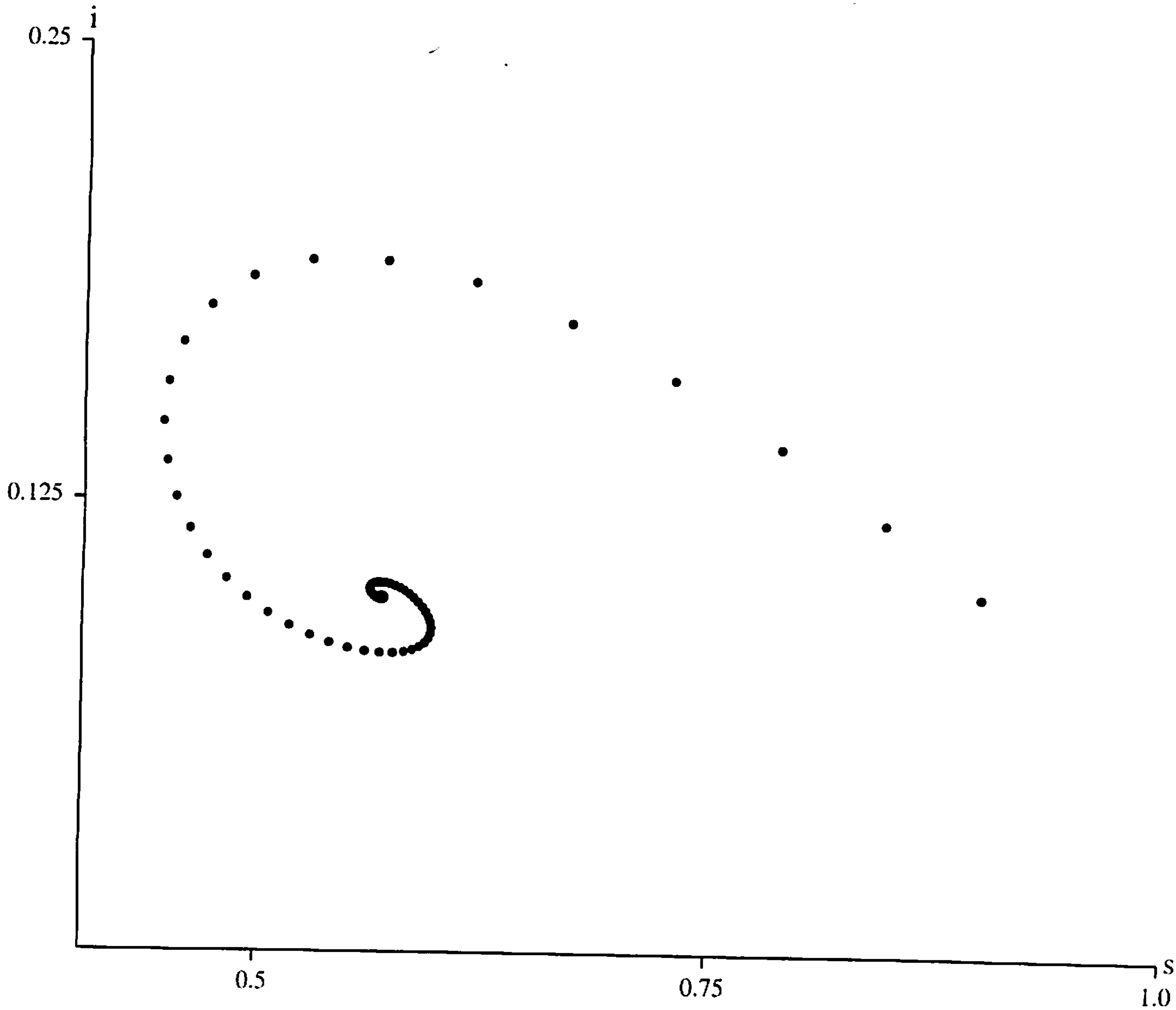

Figure 1

Part II: Cure, But No Immunity. Some diseases (such as most sexually transmitted diseases) confer no immunity, though they can be cured by antibiotics. At any given time, a certain percentage of the infected population is treated and cured, but these individuals go right back into the 'susceptible' population. Thus, there is no population of 'removed' individuals.

(a) Explain why a reasonable model for this situation is

$$\begin{aligned}\frac{dS}{dt} &= -\alpha S(t)I(t) + \gamma I(t)\\ \frac{dI}{dt} &= \alpha S(t)I(t) - \gamma I(t)\end{aligned}$$

where α and γ are positive constants. Explain the meaning of α and γ. Prove that $M = S(t) + I(t)$ is constant.

(b) Show that the fractional populations $s(t) = S(t)/M$ and $i(t) = I(t)/M$ satisfy the differential equations

$$\frac{ds}{dt} = (\gamma - \beta s(t))i(t) \tag{3}$$

$$\frac{di}{dt} = -(\gamma - \beta s(t))i(t) \tag{4}$$

where $\beta = \alpha M$. Prove that $s(t) + i(t) = 1$ for all t.

(c) Do a complete phase plane analysis of the system of equations (3) and (4). What are the nullclines? What are the equilibrium points? What is the behavior of the solution as $t \to \infty$? Be sure to distinguish between the two cases $\gamma > \beta$ and $\gamma < \beta$.

(d) Why would it be an important social goal to make the ratio $\frac{\gamma}{\beta} > 1$? Discuss ways in which this could be achieved.

Report. Your report should include complete answers to all questions in Part I and Part II.

(rb, 97; mcr, 97; sf 98)