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DAEs arising from traveling wave solutions of PDEs

Stephen L. Campbell*, Wieslaw Marszalek¹

Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205 USA

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Abstract

The study of traveling waves for explicit and implicit PDEs can sometimes result in differential algebraic equations (DAEs) instead of ordinary differential equations. The advantages of using DAEs is discussed as are the implications of DAE theory for the study of traveling waves. A specific type of traveling wave that connects equilibria is examined in more detail. Specific examples are given.

Keywords: Differential algebraic equation; DAE; Traveling waves; Equilibria; Magnetohydrodynamics

AMS classification: 34A09; 34C40

1. Introduction

DAEs are implicitly defined systems of differential equations,

$$F(y', y, t) = 0 \tag{1}$$

with $\partial F/\partial y'$ identically singular. DAEs arise in many different areas and a variety of numerical and analytical tools for working with DAEs have been developed over the last decade [1]. That DAEs arise in the method of lines (MOL) solution of constrained and unconstrained PDEs has been one of the driving forces behind the development of numerical integrators for DAEs. It is less well appreciated that DAEs can arise from PDEs in other ways.

In this paper we show that the search for traveling wave solutions of PDEs can lead naturally to nonlinear DAEs. Even if the original PDE is explicit and without constraints we will show that it is possible for the resulting ordinary differential equations to be a DAE. Following the

* Corresponding author. E-mail: slc@math.ncsu.edu.

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general discussion, we consider a particular example from magnetohydrodynamics; the dissipative magnetohydrodynamic equations with resistivity, viscosity, and thermal conductivity from [24].

To our knowledge this paper is the first discussion of the relationship between the theory of DAEs and traveling waves. The DAE perspective makes it possible to exploit specially designed integrators and other software for the analysis of DAEs. It is shown that by utilizing a DAE perspective we can more easily analyze and explain some of the existing results in the magnetohydrodynamics literature. In this paper we develop the basic theoretical framework that is needed. A more detailed analysis of the magnetohydrodynamics equations using this framework is given in a follow up paper [16] where our approach permits us to show the existence of a special kind of weak shock for certain parameter values.

2. A few comments on DAEs

There has been a considerable amount of work in the last decade on developing analytical and numerical tools for working with DAEs [1, 2, 13, 15]. Many DAEs can be numerically simulated without having to reduce them to an explicit form. As we will see later this can be advantageous for studying traveling wave solutions.

DAEs differ from ODEs in several ways. The DAE is called *solvable* if there is a well defined constant dimensional manifold of solutions and the solutions are uniquely determined by consistent initial conditions. Solvability sometimes also includes the idea of solutions existing for a class of forcing functions.

The solution manifold may be thought of as being defined by a family of constraints. In some applications these constraints are given explicitly but in other problems some or all of the constraints may be defined implicitly. One of our examples will have implicit constraints.

The index is one measure of how singular a DAE is. There are several definitions of the index of a DAE. For our purposes the most important is as follows. If (1) is differentiated k times with respect to t , we get the $(k + 1)n$ derivative array equations [1]

$$\begin{bmatrix} F(y', y, t) \\ F_t(y', y, t) + F_y(y', y, t)y' + F_{y'}(y', y, t)y'' \\ \vdots \\ \frac{d^k}{dt^k}F(y', y, t) \end{bmatrix} = G_k(y', y, t, w) = 0, \quad (2)$$

where $w = [y^{(2)}, \dots, y^{(k+1)}]$. We frequently drop the k subscript on G to simplify our notation. In particular applications, different equations in $F = 0$ are often differentiated a different number of times. This has no effect on our discussion.

For our purposes it suffices to say that the DAE $F = 0$ is index k if $G_k = 0$ uniquely determines y' given consistent t, y and no smaller value of k has this property. A more careful discussion of the index is in [9].

3. General traveling waves

In order to keep our notation reasonably concise we shall consider one dimensional partial differential equations (PDEs). In their most general form they would look something like

$$F(u_t, u_{tt}, u_x, u_{xx}, u_{xt}, u) = 0, \tag{3}$$

where u is a vector variable of the scalar variables t, x . At this point we consider the domain to be the real plane. Of course by letting $v = u_t, w = u_x, p = [u, v, w]$ we can rewrite (3) as a first order system $F(p_t, p_x, p) = 0$. However, it will sometimes be simpler to take our equations in the form of (3).

A *traveling wave solution* of (3) is a solution of the form $u(x, t) = \phi(x - st)$ where $\phi(z)$ is a function of the scalar variable z and s is called the *wave speed*.

Assuming that ϕ is twice differentiable, we get that ϕ must be a solution of the DAE

$$F(-s\phi', s^2\phi'', \phi', \phi'', -s\phi'', \phi) = 0. \tag{4}$$

Traveling waves are important in several different ways. Which part of DAE theory is appropriate depends on what kind of traveling waves are of interest and what the intended application is.

In some areas, such as chemical engineering, the systems of PDEs that arise in the form (3) frequently have second partials of only some components of u occurring. Also the second partials do not appear in all equations. In these situations (4) will always be a DAE.

3.1. Wave families

In the theory for the standard linear time invariant wave equation, traveling waves are used to represent solutions. What is of interest is not the existence of one particular solution which is a traveling wave but rather the existence of an infinite dimensional family of traveling waves all with the same speed.

We shall say that (3) has a *family of traveling waves at wave speed s* if (4) has an infinite dimensional set of solutions. A necessary condition for this to happen is that (4) is not solvable. For nonlinear DAEs that would say that the conditions given in [10] do not hold. For linear time invariant PDEs the situation is simpler.

Proposition 1. *Let A, B, C, D, E, F be square matrices. The linear time invariant PDE*

$$A \frac{\partial u}{\partial t} + B \frac{\partial^2 u}{\partial t^2} + C \frac{\partial u}{\partial x} + D \frac{\partial^2 u}{\partial x^2} + E \frac{\partial^2 u}{\partial x \partial t} + Fu = 0 \tag{5}$$

has a family of traveling waves with wave speed s if and only if

$$\det (-s\lambda A + s^2\lambda^2 B + \lambda C + \lambda^2 D - s\lambda^2 E + F) = 0 \quad \text{for all } \lambda. \tag{6}$$

Proof. For the system (5) we get that the DAE (4) becomes

$$(s^2 B + D - sE)\phi'' + (-sA + C)\phi' + F\phi = 0. \tag{7}$$

This is a linear time invariant DAE. It is known [3, 4] (see the more recent [11]) that the existence of a finite dimensional family of solutions is equivalent to solvability of (7) which in turn is equivalent to the existence of a λ such that $\lambda^2(s^2B + D - sE) + \lambda(-sA + C) + F$ has nonzero determinate [1]. If a linear time invariant DAE with square coefficients is not solvable, then there always exists an infinite dimensional family of solutions. Thus the proposition holds. This proposition can also be proved directly by rewriting (7) as a first order pencil and using the Kronecker structure of a singular matrix pencil. \square

From (6) we have that if a family of traveling waves exists, then F must be singular.

Example 2. Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E = F = 0.$$

Then $\det(-s\lambda A + s^2\lambda^2 B + \lambda C + \lambda^2 D) = (s^2\lambda^2 + \lambda)\lambda(1 - s)$. Thus the only wave speed for a family of traveling waves is $s = 1$. At $s = 1$, the wave family is $\phi_1(z)$ arbitrary and $\phi_2(z) = \int (c - \phi_1'(z) - \phi_1(z)) dz$.

Note that $s = -1$ is not also a wave speed in Example 2. This contrasts with the usual linear time invariant case where the negative of the wave speed is also a wave speed.

3.2. Traveling waves and equilibria

A very different scenario occurs when one is looking for a traveling wave that connects two equilibria. This occurs in some approaches for proving the existence of shock waves. Classically a researcher would have to reduce the problem to an ordinary differential equation (ODE). Such a reduction might be extremely difficult or even impossible without making simplifying assumptions. Also different reductions might be necessary for different parameter values.

There are advantages to considering a DAE instead. No simplification may be needed. The same DAE model can serve for a variety of parameter values. The time needed for obtaining information can be greatly reduced. In this situation we want to have the DAE to be solvable so that there is a well defined solution manifold. We also want to be able to determine the equilibria of the DAE and their stability properties. Finally we want to be able to use DAE integrators to integrate the DAE in order to help study the system's behavior. We now consider each of these three topics.

3.2.1. The solution manifold

If we are fortunate enough that our DAE takes the form of

$$\phi_1' = f(\phi_1, \phi_2), \quad 0 = g(\phi_1, \phi_2) \tag{8a, b}$$

and $\partial g / \partial \phi_2$ is invertible, then the situation is easy. The DAE is what is called semi-explicit index one and the solution manifold is given by (8b).

In general the situation is more complex if the DAE is not semi-explicit or if it is not index one. General procedures are described in [6, 10]. These papers give ways to determine the index, the

dimension of the solution manifold, and explain how to get a local characterization of the solution manifold.

In order to briefly describe some of these results for later use, let

$$\bar{J}_k = [G_{y'} \ G_w], \quad J_k = [G_{y'} \ G_w \ G_y],$$

where $G = G_k$.

The following assumptions on G_k permit a robust numerical least squares solution of the derivative array equations.

- (A1) Sufficient smoothness of G_k .
- (A2) Consistency of $G_k = 0$ as an algebraic equation.
- (A3) $\bar{J}_k = [G_{y'} \ G_w]$ is 1-full with respect to y' and has constant rank independent of (t, y, y', w) .
- (A4) $J_k = [G_{y'} \ G_w \ G_y]$ has full row rank independent of (t, y, y', w) .

Here the matrix C of the equation $Cx = b$ is said to be 1-full with respect to x_1 if there is a nonsingular matrix Q such that

$$QA = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that (A1)–(A4) are directly in terms of the original equations and their derivatives. Also (A3) and (A4) hold in a full neighborhood since y', y, w are considered to be independent variables in (A1)–(A4). Conditions (A1)–(A4) are numerically verifiable using a combination of symbolic and numeric software [10].

Assumptions (A1)–(A4) can also be used to establish solvability, compute the index, and determine the degrees of freedom. Let $S_k = \{(t, y) : G_k(t, y, \hat{y}) = 0 \text{ is consistent for some } \hat{y}\}$.

Theorem 3 (Campbell and Griepentrog [10]). *Suppose that there is a k such that the derivative arrays G_i of $F(y', y, t)$ satisfy (A1)–(A4) for $i = k$ and $k + 1$ on appropriate neighborhoods. Then the DAE is geometrically (uniformly) solvable on that neighborhood and the solution manifold is S_k . If no smaller value of k satisfies these assumptions, then the index is k .*

Unless stated otherwise, in what follows we shall always assume that we have taken k large enough so that the assumptions of Theorem 3 hold almost everywhere.

In general, one cannot determine the index of a DAE from a linearization of it. There is one important exception.

Theorem 4. *Suppose that $F_{y'}$ has constant rank and $\{F_{y'}, F_y\}$ is an index one pencil independent of y, y', t . Then the DAE $F(y', y, t) = 0$ has index one.*

3.2.2. Equilibria of DAEs

When studying equilibria of DAEs there are several issues. One is finding the equilibrium. The second is determining the stability of the equilibrium point. There is a related issue of whether two equilibria are on the same component of the solution manifold. We now restrict ourselves to time invariant systems.

\bar{y} is an equilibria of $F(y', y) = 0$ if and only if $F(0, \bar{y}) = 0$. If $F_y(0, \bar{y})$ is nonsingular, then \bar{y} is isolated and can in principle be found by solving $F(0, \bar{y}) = 0$ by a numerical or symbolic method.

The invertibility of $F_y(0, \bar{y})$ is important in another way. The following lemma is used in the subsequent theorem.

Lemma 5. *Suppose that A, B is a regular pencil of matrices, that is, $\det(\lambda A + B)$ is not identically zero. Then $\hat{\lambda} = 0$ is an eigenvalue of the pencil if and only if $\det(B) = 0$.*

Proof. The eigenvalues of the pencil are those λ such that $\det(\lambda A + B) = 0$. If 0 is an eigenvalue, then $\det(B) = 0$. To see the converse, we note that since the pencil is regular there are invertible matrices P, Q such that

$$PAQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PBQ = \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix},$$

where N is a nilpotent matrix of index k . Thus $\det(\lambda A + B)$ is a constant multiple of $\det(\lambda I + D) \det(\lambda N + I) = \det(\lambda I + D)$. If D is singular, then $\det(\lambda A + B) = 0$ for $\lambda = 0$. \square

Given an equilibrium we have found it is possible to analyze its stability directly from the DAE. In general, one has to be very careful with linearizing DAEs. However, the situation is more straightforward around equilibria. The following result uses the above Lemma and follows from [8] which in turn was motivated by [22] which used somewhat different assumptions.

Theorem 6. *Suppose that \bar{y} is an equilibrium of $F(y', y) = 0$. Suppose that the DAE satisfies (A1)–(A4) in a neighborhood of $(0, \bar{y}, 0)$. Let $A = F_{y'}(0, \bar{y})$, $B = F_y(0, \bar{y})$. Suppose that B is nonsingular. Let y be n dimensional and r be the difference in rank of $[G_{y'}, G_w]$ and $[G_{y'}, G_y, G_w]$ for this system at $(0, \bar{y}, 0)$. Then*

- (i) *The local linearization $A\tilde{y}' + B\tilde{y} = B\bar{y}$ and the original DAE $F(y', y) = 0$ have the same dimensional solution manifold in a neighborhood of \bar{y} .*
- (ii) *$y' = \tilde{y}' + O(\|y - \bar{y}\|^2)$.*
- (iii) *The dimension of the solution manifold is $n - r$.*

Thus if the pencil $\lambda A + B$ has $n - r$ finite eigenvalues with nonzero real part, then they will determine the stability properties of \bar{y} on the solution manifold of $F(y', y) = 0$.

Proof. Item 1 and 2 are from [8]. They imply that the eigenvalues of the pencil will determine the stability of the equilibrium provided the number of nonzero eigenvalues with nonzero real part is the same as the dimension of the solution manifold. Item 3 is from [5]. The final conclusion now follows. \square

3.2.3. Simulation

One advantage of an implicit formulation is that simulations can be performed prior to an extensive analysis and problem reduction. BDF based codes such as DASSL are available for index one DAEs [1]. Index two and three DAEs may be integrated by Implicit Runge Kutta (IRK) codes such as RADAU5 [15] if they have a Heisenberg structure. The integration of higher index fully implicit DAEs is more difficult but techniques are under development [12].

4. Conservative form PDEs

In this section we shall examine traveling waves solutions for PDEs of the form

$$[h(u)]_t + [p(u)]_x = \mu[D(u)u_x]_x. \tag{9}$$

The existence of traveling wave solutions to (9) connecting equilibria with $\mu > 0$ is often related to the existence of shock waves to the PDE with $\mu = 0$ [23].

Keeping the same variable u for the traveling wave function we get that (4) for (9) is

$$-sh'(u)u' + [p(u)]' = \mu[D(u)u']'. \tag{10}$$

We want our solution $u(z)$ to go from a left equilibrium u_L to a right equilibrium u_R . That is we want $\lim_{z \rightarrow -\infty} u(z) = u_L$, $\lim_{z \rightarrow \infty} u(z) = u_R$, and $\lim_{|z| \rightarrow \infty} u'(z) = 0$. Integrating (10) from $-\infty$ to t and using that u_L is the left endpoint and that u' has zero limit as $t \rightarrow -\infty$ gives the DAE

$$-s(h(u) - h(u_L)) + p(u) - p(u_L) = \mu D(u)u'. \tag{11}$$

The DAE (11) is a linearly implicit DAE. However, as we shall see it need not be index one. An equilibrium for (11) must satisfy

$$-s(h(u) - h(u_L)) + p(u) - p(u_L) = 0. \tag{12}$$

At a given equilibrium \bar{u} the Jacobian for (12) is $R(\bar{u}) = -sh'(\bar{u}) + p'(\bar{u})$. If there is a traveling wave solution connecting the equilibria, then the left and right equilibria determine the wave speed via (12). A type of converse holds.

Theorem 7. Fix u_L . Suppose that there exists a solution \hat{u} of the DAE (11) which connects the equilibrium u_L with the equilibrium u_R with wave speed \hat{s} . Suppose that the assumptions (A1)–(A4) hold for (11) in a neighborhood of \hat{u} and u_L, u_R and for s near \hat{s} .

- (i) If $-\hat{s}h'(u_R) + p'(u_R)$ is nonsingular, then for s near \hat{s} there will be a right equilibrium $u_R(s)$ and a solution of (11) connecting u_L to $u_R(s)$.
- (ii) Suppose $[-\hat{s}h'(u_R) + p'(u_R), h(u_R) - h(u_L)]$ is invertible when its i th column is deleted. Let δ be the i th component of u_R . Then there will exist a new right equilibrium $\tilde{u}_R(\delta)$ with this same i th component and a solution connecting u_L and $\tilde{u}_R(\delta)$ for a wave speed $s(\delta)$ near \hat{s} .

Proof. The proof is straightforward application of the implicit function theorem and standard ODE theory, once we see that the assumptions (A1)–(A4) holding as s varies ensures that the solution manifold has fixed dimension and also varies smoothly with the chosen parameters [5, 10]. \square

At an equilibrium \bar{u} , the linearization of (11) is

$$A = \mu D(\bar{u}), \quad B = sh'(\bar{u}) - p'(\bar{u}). \tag{13}$$

Note that Eq. (9) includes the form

$$[h(u)]_t + [p(u)]_x = \mu[d(u)]_{xx} \tag{14}$$

by letting $D(u) = d'(u)$. However, (9) is more general since not every $D(u)$ can be written as $d'(u)$ for some $d(u)$. The DAE (11) could also be written as a semiexplicit DAE by letting $v = u'$ but that would increase the index by one.

5. The p -system

Our first example is the p -system. It is reasonably simple and can be used to illustrate several of the ideas given above. The PDE is

$$u_t - v_x = 0, \quad v_t + [p(u)]_x = \mu u_{xx}. \tag{15a,b}$$

The corresponding DAE (11) is

$$-s(u - u_L) - (v - v_L) = 0, \quad -s(v - v_L) + p(u) - p(u_L) = \mu u'. \tag{16a,b}$$

This is a semi-explicit DAE which is always index one and (16a) gives the solution manifold. An equilibrium (u_R, v_R) different from (u_L, v_L) must satisfy

$$-s(u_R - u_L) - (v_R - v_L) = 0, \quad -s(v_R - v_L) + p(u_R) - p(u_L) = 0. \tag{17a,b}$$

The matrix pencil (13) in (u, v) variables is given by

$$A = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}, \quad B = \begin{bmatrix} s & 1 \\ -p'(u) & s \end{bmatrix}. \tag{18}$$

B is nonsingular if $p'(u) + s^2 \neq 0$. Also $\det(\lambda A + B) = (-\lambda\mu + p'(u) + s^2)$ so that the eigenvalue at an equilibrium is

$$\lambda = \frac{p'(u) + s^2}{\mu}.$$

Note that $\lambda \neq 0$ precisely when $\det(B) \neq 0$ as promised by Lemma 5.

What is required in order to get the required traveling wave with two equilibria whose stability is determined by the linearization? Since the solution manifold is one dimensional, in order to get a trajectory from u_L to u_R we must have u_L is unstable and u_R is stable. This can only happen if $p'(u_L) + s^2 > 0$ and $p'(u_R) + s^2 < 0$. From (17a) we see that s and the right equilibrium are carefully linked for this problem. We could discuss this example in more detail, but instead turn to a more complex equation.

6. Example from magnetohydrodynamics

The dissipative magnetohydrodynamics equations with resistivity, viscosity, and thermal conductivity, are $\nabla \cdot \mathbf{B} = 0$ and [24]

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{19a}$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} = -\nabla \cdot \left(\rho \mathbf{v} \mathbf{v} + \mathbf{I} \left(p + \frac{B^2}{2} \right) - \mathbf{B} \mathbf{B} \right) + \nu \nabla^2 \mathbf{v} + \left(\mu + \frac{\nu}{3} \right) \nabla(\nabla \cdot \mathbf{v}), \quad (19b)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (19c)$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla \cdot \left(\left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + p \right) \mathbf{v} + \mathbf{E} \times \mathbf{B} \right) + \nabla \cdot \sigma \cdot \mathbf{v} + \kappa \nabla^2 \left(\frac{p}{\rho} \right). \quad (19d)$$

The specific meaning of these variables is discussed in [24] and [16]. If we consider one dimensional flow in the x direction only, then the x component B^x of the magnetic induction is constant and the y and z components, B^y, B^z can be taken as functions of x, t .

For notational convenience let μ replace $\mu + \frac{4}{3}\nu$. Letting $\rho = u_1$, the three components of \mathbf{v} be u_2, u_3, u_4 , and $[B^y, B^z, E] = [u_5, u_6, u_7]$, we obtain the traveling wave DAE (10) as

$$-su_1' + (u_1 u_2)' = 0, \quad (20a)$$

$$-s(u_1 u_2)' + (u_1 u_2^2 + P^*)' = \mu u_2'', \quad (20b)$$

$$-s(u_1 u_3)' + (u_1 u_2 u_3 - B^x u_5)' = \nu u_3'', \quad (20c)$$

$$-s(u_1 u_4)' + (u_1 u_2 u_4 - B^x u_6)' = \nu u_4'', \quad (20d)$$

$$-su_5' + (u_2 u_5 - B^x u_3)' = \eta u_5'', \quad (20e)$$

$$-su_6' + (u_2 u_6 - B^x u_4)' = \eta u_6'', \quad (20f)$$

$$\begin{aligned} & -su_7' + [(u_7 + P^*)u_2 - B^x(B^x u_2 + u_3 u_5 + u_4 u_6)] \\ & = \frac{\mu}{2}(u_2'')^2 + \frac{\nu}{2}(u_3^2 + u_4^2)'' + \frac{\eta}{2}(u_5^2 + u_6^2)'' + \kappa \left(\frac{p}{u_1} \right)'' \end{aligned} \quad (20g)$$

The parameters η, κ, μ, ν are resistivity, thermal conductivity, and the two viscosity coefficients. P^* and p are given by

$$P^* = p + \frac{1}{2}((B^x)^2 + u_5^2 + u_6^2), \quad (21)$$

$$p = \frac{1}{2}(\gamma - 1)[2u_7 - u_1(u_2^2 + u_3^2 + u_4^2) - (u_5^2 + u_6^2 + (B^x)^2)], \quad (22)$$

where B^x and γ are constants.

The DAE that results from (20) will be linearly implicit. However, if $\kappa \neq 0$, then the nullspace of the coefficient of u' will not be constant. If $\kappa = 0$, then the nullspace will be constant but the range will not be. Several algorithms have been developed, especially in the chemical engineering literature to estimate the index and determine the dimension of the solution manifold [17, 18]. These algorithms rely on the graph describing which variables are in which equations. For $\kappa = 0$, these algorithms will underestimate the number of constraints. (They miss the implicitly defined \hat{f}_7 equation in Proposition 8).

The DAE can be made semi-explicit by introducing another variable but then it will be index two.

A detailed analysis of the system (20) is performed in [16]. Here we are illustrating the general ideas. To be precise we consider a special case.

6.1. η only

Suppose that $\eta \neq 0$ but $\mu = \nu = \kappa = 0$. Then the DAE for the waves is

$$-su_1 + u_1u_2 - c_1 = 0, \tag{23a}$$

$$-su_1u_2 + u_1u_2^2 + P^* - c_2 = 0, \tag{23b}$$

$$-su_1u_3 + u_1u_2u_3 - B^xu_5 - c_3 = 0, \tag{23c}$$

$$-su_1u_4 + u_1u_2u_4 - B^xu_6 - c_4 = 0, \tag{23d}$$

$$-su_5 + u_2u_5 - B^xu_3 - c_5 = \eta u'_5, \tag{23e}$$

$$-su_6 + u_2u_6 - B^xu_4 - c_6 = \eta u'_6, \tag{23f}$$

$$-su_7 + (u_7 + P^*)u_2 - B^x(B^xu_2 + u_3u_5 + u_4u_6) - c_7 = \eta(u_5u'_5 + u_6u'_6), \tag{23g}$$

where the c_i are constants determined by u_L . That is, c_i is the i th entry of the vector $h(u_L) - p(u_L)$ in (11). The Jacobians of the DAE $F(u', u) = 0$ are

$$F_{u'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 & \eta u_5 & \eta u_6 & 0 \end{bmatrix},$$

$$F_u = - \begin{bmatrix} u_2 - s & u_1 & 0 & 0 & 0 & 0 & 0 \\ (u_2 - s)u_2 + P_1^* & (2u_2 - s)u_1 + P_2^* & P_3^* & P_4^* & P_5^* & P_6^* & P_7^* \\ (u_2 - s)u_3 & u_1u_3 & (u_2 - s)u_1 & 0 & -B^x & 0 & 0 \\ (u_2 - s)u_4 & u_1u_4 & 0 & (u_2 - s)u_1 & 0 & -B^x & 0 \\ 0 & u_5 & -B^x & 0 & u_2 - s & 0 & 0 \\ 0 & u_6 & 0 & -B^x & 0 & u_2 - s & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \end{bmatrix},$$

where $P_i^* = \partial P^* / \partial u_i$ and the α_i are nonzero entries. Let

$$\Theta = - \begin{bmatrix} -s + u_2 & u_1 & 0 & 0 & 0 \\ -su_2 + u_2^2 + P_1^* & -su_1 + 2u_1u_2 + P_2^* & P_3^* & P_4^* & P_7^* \\ -su_3 + u_2u_3 & u_1u_3 & -su_1 + u_1u_2 & 0 & 0 \\ -su_4 + u_2u_4 & u_1u_4 & 0 & -su_1 + u_1u_2 & 0 \\ \alpha_1 & \alpha_2 - u_5^2 - u_6^2 & \alpha_3 + B^xu_5 & \alpha_4 + B^xu_6 & \alpha_7 \end{bmatrix}, \tag{24}$$

To simplify the remaining discussion we rewrite (23) as

$$f_1(u_1, u_2) = 0, \tag{25a}$$

$$f_2(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = 0, \tag{25b}$$

$$f_3(u_1, u_2, u_3, u_5) = 0, \tag{25c}$$

$$f_4(u_1, u_2, u_4, u_6) = 0, \tag{25d}$$

$$f_5(u_2, u_3, u_5) = \eta u'_5, \tag{25e}$$

$$f_6(u_2, u_4, u_6) = \eta u'_6, \tag{25f}$$

$$f_7(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = \eta(u_5 u'_5 + u_6 u'_6). \tag{25g}$$

Proposition 8. For the system (23) and with Θ defined by (24) we have that

- (i) If $\det(\Theta) \neq 0$ and $u_1 \neq 0$, then the DAE (23) is an index one DAE with a 2 dimensional solution manifold.
- (ii) If the pencil $\{F_{u'}, F_u\}$ has 2 nonzero eigenvalues with nonzero real parts at an equilibrium, then these eigenvalues determine the stability properties of that equilibrium on the solution manifold.
- (iii) The solution manifold is given by (23a)–(23d) and $\widehat{f}_7 = -u_5 f_5 - u_6 f_6 + f_7 = 0$.

Proof. Adding $-u_5$ times row 5 to row 7 and $-u_6$ times row 6 to row 7 converts the pencil $\{F_{u'}, F_u\}$ to the pencil

$$\left\{ Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta I & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} \Theta_{11} & * & \Theta_{12} \\ * & * & * \\ \Theta_{21} & * & \Theta_{22} \end{bmatrix} \right\} \text{ where } \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}. \tag{26}$$

The pencil $\{F_{u'}, F_u\}$ is index one if and only if Θ is nonsingular. If $F_{u'}$ has constant rank, then the DAE is index one if and only if the pencil of its linearization is also index one. This direct relationship between the index of the pencil and the index of the DAE is not true for higher index DAEs. \square

The solution manifold given by $f_1=0, \dots, f_4=0, \widehat{f}_7=0$ contains all the equilibrium points. Provided its Jacobian is full row rank we get a well defined manifold.

There are several problems which can now arise in order for there to exist a traveling wave that connects two equilibria.

First the solution manifold may consist of more than one component. In order to connect them the equilibria need to be on the same component.

Secondly, even if the equilibria both lie on the same component of the manifold, there can be difficulties. The equation $\det(\Theta(u)) = 0$ defines another, possibly empty, manifold which we will call the *singularity manifold*. In some problems the singularity manifold may intersect the solution manifold. If the equilibria are on opposite sides of the singularity surface, it may not be possible to connect them with a solution.

Table 1
Four equilibrium points

u_L	u_{R1}	u_{R2}	u_{R3}
15	13.1798	10.0611	27.5113
0.5	0.4309	0.2546	0.7274
0.2	-0.3155	-0.1086	-0.0326
0.2	-0.3155	-0.1086	-0.0326
-1	0.9332	0.1572	-0.1279
-1	0.9332	0.1572	-0.1279
10	8.9599	4.8305	20.5711

Table 2
Nonzero eigenvalues and $\det(\Theta)$ at each of the four equilibria

	u_L	u_{R1}	u_{R2}	u_{R3}
Eigenvalue 1	0.0333	-0.0357	-0.2035	0.2586
Eigenvalue 2	0.8557	0.3993	-0.2121	0.2607
$\det(\Theta)$	-34.200	-72.922	-178.985	65.234

6.1.1. Specific MHD example

We will now give a specific illustration of the above ideas. Let

$$u_L = [15, 0.5, 0.2, 0.2, -1, -1, 10], \quad B^x = 2, \quad \gamma = 1.4, \quad s = 1.$$

Solving the equilibrium equations we get that there are 3 other equilibrium points. They are given in Table 1.

The nonzero finite eigenvalues for the pencil of the linearization at each equilibrium and the value of $\det(\Theta)$ are given in Table 2.

The information in Table 2 combined with the theoretical results given earlier, make it possible to draw several conclusions. The equilibria u_L, u_{R3} are unstable, u_{R2} is stable, and u_{R1} is a saddle. The fact that $\det(\Theta)$ changes sign shows that the singularity manifold intersects the solution manifold and cuts it into submanifolds. The sign of $\det(\Theta)$ will be constant on each connected submanifold. The equilibrium u_{R3} must lie on a different submanifold than u_L, u_{R1}, u_{R2} since $\det(\Theta)$ has a different sign at u_{R3} . Thus a traveling wave solution of the type that we seek must originate at u_L and proceed to either u_{R1} or u_{R2} .

In general it is not possible to easily visualize a high dimensional DAE. However, for the system (25) it is possible to do so. Using a symbolic language, such as MAPLE, one can solve $f_1 = 0, f_3 = 0, f_4 = 0, \hat{f}_7 = 0$ for u_1, u_3, u_4, u_7 and get a DAE in u_2, u_5, u_6 .

$$u'_5 = g_1(u_5, u_2, \beta), \quad u'_6 = g_2(u_6, u_2, \beta), \quad 0 = h(u_2, u_5, u_6, \beta), \tag{27a,b,c}$$

where β includes parameters such as $\eta, u_L, s, \gamma, B^x$.

h is a quadratic in u_2 . One could try to solve for u_2 . This has been done elsewhere in the literature [24]. However, this breaks the problem into two subproblems. There are advantages to leaving the

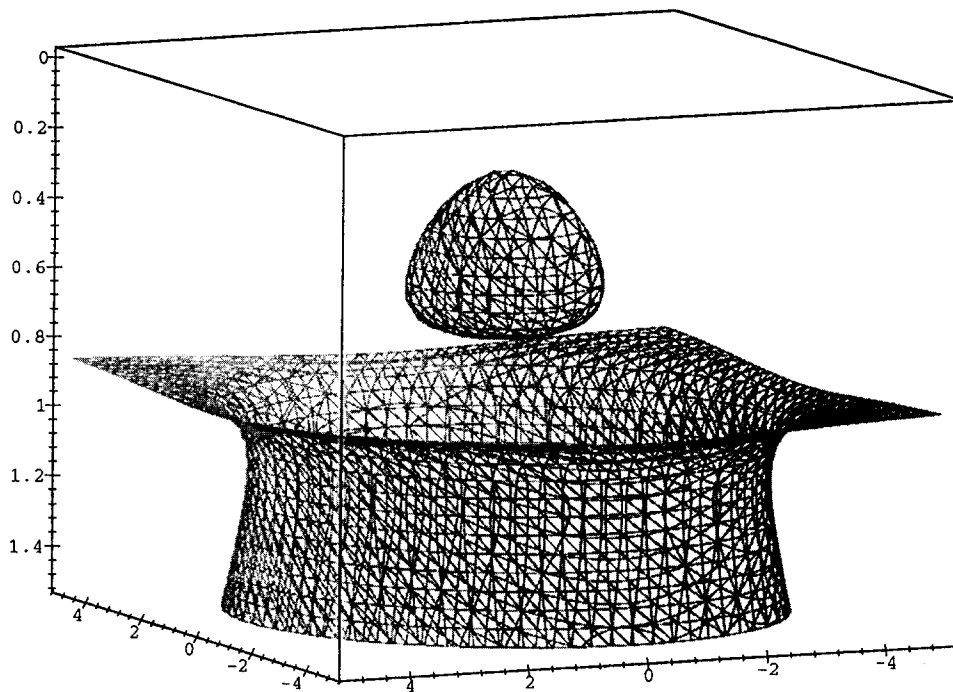


Fig. 1. Solution manifold: $0 < u_2 < 1.4$, $-5 < u_5, u_6 < 5$.

problem in the form of (27). We shall merely allude to them here. In [16] this analysis is carried out in detail.

Returning to our specific example, we see in Fig. 1 that the solution manifold given by the graph of (27c) consists of two surfaces which resemble an ‘egg floating over a volcano’. The vertical direction is the u_2 direction. For this particular problem, all 4 equilibria lie on the egg which we denote \mathcal{E} .

The singularity manifold $\det(\Theta) = 0$ in the reduced set of coordinates is now given by $h_{u_2} = 0$. This surface is graphed in Fig. 2.

We know that the singularity manifold must intersect \mathcal{E} since the sign of $\det(\Theta)$ was not constant on the equilibria. The curve of intersection is at the equator of \mathcal{E} as shown in Fig. 3. The equilibrium u_{R3} lies on the bottom half of \mathcal{E} (larger u_2). This is called the supersonic region in the MHD literature [24]. The other three equilibria lie on the top half of \mathcal{E} (smaller u_2). This is known as the subsonic region.

By looking at the DAE (27) we can examine what happens as parameters are varied and, in particular, we can place an equilibrium on the singularity curve. The consequences of this are also examined in [16].

6.1.2. A cautionary example

One of the advantages of modern symbolic and numeric software is that they make it possible to rapidly experiment with a variety of parameter values and physical configurations. However, care must still be exercised to make sure the relevant theoretical assumptions hold. Our last example

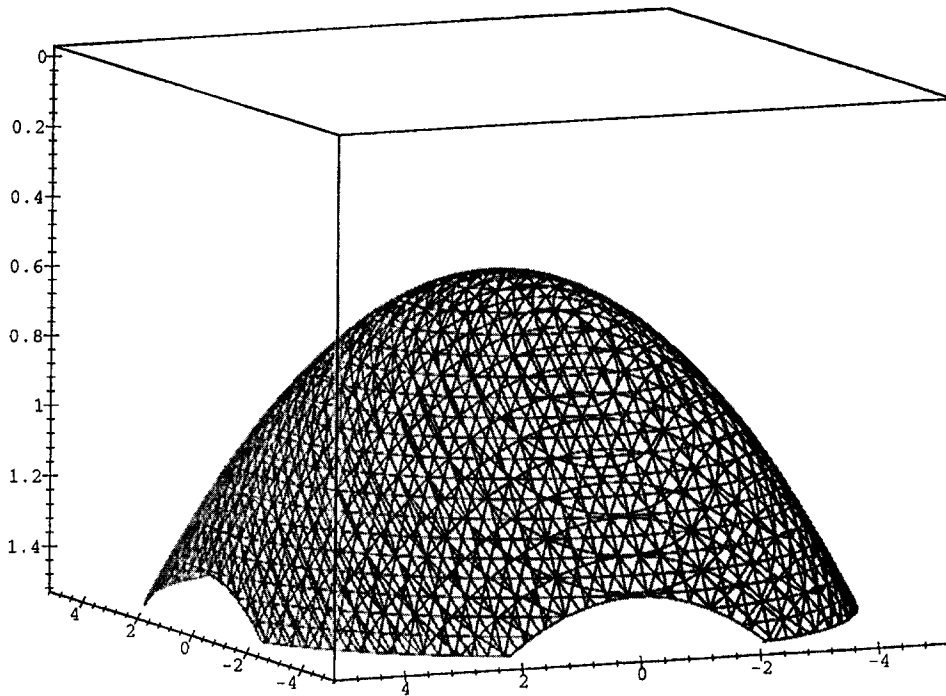


Fig. 2. Singularity manifold.

illustrates this point. Suppose that $B^x = 0$, $s = u_{2L}$ and that the remaining components of u_L are chosen to satisfy

$$u_{7L} + \frac{3}{4}(u_{5L}^2 + u_{6L}^2) - \frac{1}{2}su_{1L} - \frac{1}{2}u_{1L}(u_{3L}^2 + u_{4L}^2) = 0.$$

Then the DAE (27) becomes

$$u'_5 = (u_2 - s)u_5, \quad u'_6 = (u_2 - s)u_6, \quad 0 = (u_2 - s)(u_5 + u_6 + \frac{3}{4}(u_5^2 + u_6^2)). \tag{28a, b, c}$$

The linearization around an equilibrium (u_5, u_6, u_2) is given by the pencil

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} u_2 - s & 0 & u_5 \\ 0 & u_2 - s & u_6 \\ (u_2 - s)(1 + \frac{3}{2}u_5) & (u_2 - s)(1 + \frac{3}{2}u_6) & u_5 + u_6 + \frac{3}{4}(u_5^2 + u_6^2) \end{bmatrix} \tag{29}$$

The surface given by the constraint (28c) consists of a cylinder \mathcal{C} and a plane \mathcal{P} ($u_2 = s$) which is perpendicular to the cylinder. See Fig. 4. The equilibrium points are all of \mathcal{P} and the line $u_5=0, u_6=0$ and u_2 arbitrary on \mathcal{C} . We denote this line by \mathcal{L} and the points on \mathcal{P} but not on \mathcal{C} by $\mathcal{P} - \mathcal{C}$.

However, this information does not correctly capture the full solution behavior of this example. To see this we examine the system more carefully. Let $(u_2 - s)Q$ be the right hand side of (28c). If $Q \neq 0$, that is $u \notin \mathcal{C}$, then the DAE is semi-explicit index one and as noted earlier, (A1)–(A4) hold.

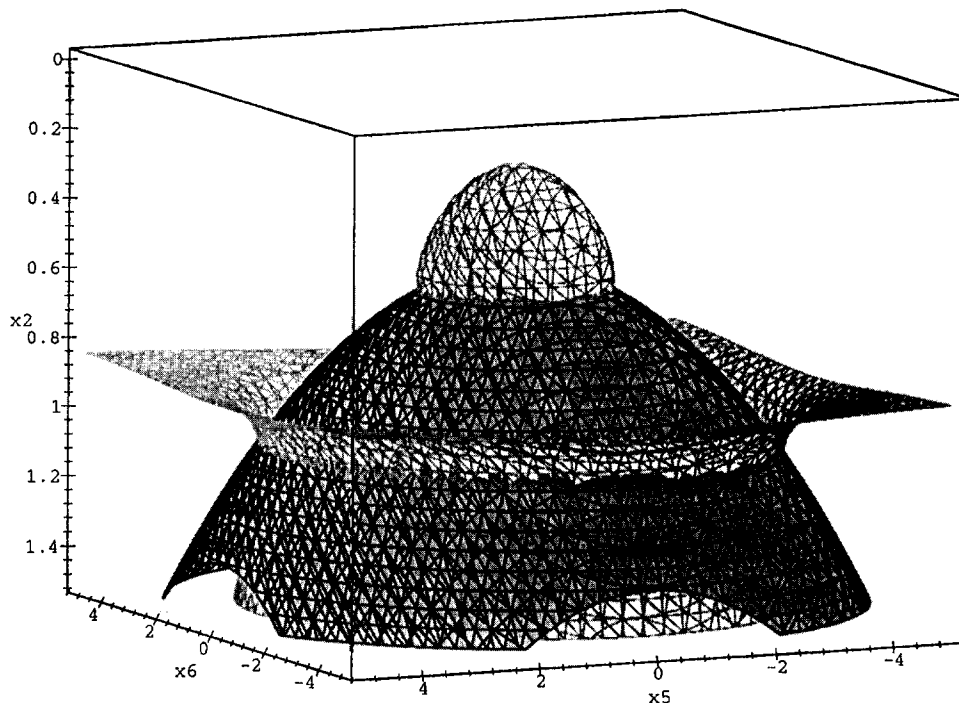


Fig. 3. Solution and singularity manifolds.

However, it will be more convenient to consider (A1)–(A4) directly since we need the calculations later.

One differentiation of the DAE would suffice for considering solutions on \mathcal{P} but two are needed for \mathcal{C} . Differentiating the equations in (28) twice and forming the Jacobian we see that

$$[G_{y'} \ G_w] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & -1 & 0 & 0 & 0 & 0 \\ * & * & Q & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & 0 & * & * & 0 & * & -1 & 0 & 0 \\ 0 & * & * & 0 & * & * & 0 & -1 & 0 \\ * & * & * & * & * & Q & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

where $*$ is a possibly nonzero entry whose value is not important for this discussion. If $Q \neq 0$, then the rank of (30) is 8 (has corank 1). If only the first 6 rows are considered the matrix has rank 5 (has corank 1) and the matrix is still one full. One can easily show that (28a) satisfies assumptions (A1)–(A4) and the requirements of Theorem 3 and (28) is a solvable index one DAE on $\mathcal{P} - \mathcal{C}$.

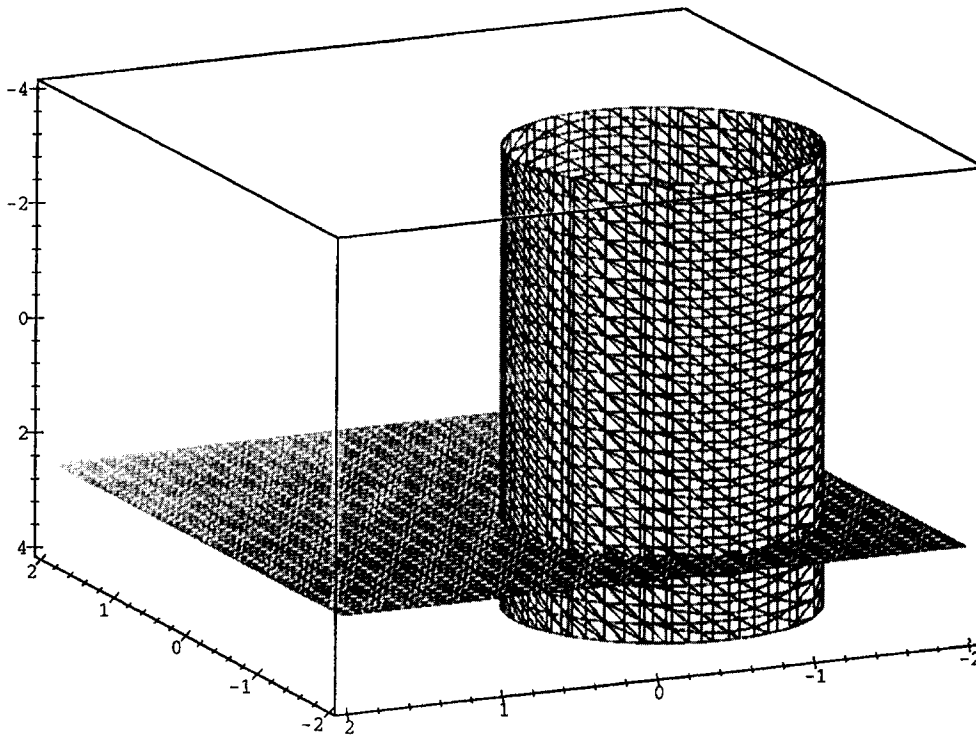


Fig. 4. Solution manifold: $-4 < u_2 < 4$, $-2 < u_5$, $u_6 < 2$.

At an equilibrium on $\mathcal{P} - \mathcal{C}$ we have the pencil

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & u_5 \\ 0 & 0 & u_6 \\ 0 & 0 & Q \end{bmatrix}, \tag{31}$$

where Q is nonzero. The pencil is regular and does correctly capture the index and solution behavior. Suppose, however, that we are on the \mathcal{C} so that $Q = 0$. Then the matrix (30) is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & -1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & -1 & 0 & 0 \\ 0 & * & * & 0 & * & * & 0 & -1 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

One-fullness and constant rank are invariant under invertible row operations and column operations that do not originate with the first three columns. Using the first two rows to zero below them and

the 7th and 8th columns to zero to the left, we get

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

which has rank no more than 7. But as noted earlier, the rank is 8 if $Q \neq 0$. A similar calculation, which we omit, works for 3 differentiations which is the maximum needed since the system has three variables. Thus (A3) is violated since the rank is not constant in a full neighborhood of any points on \mathcal{C} and the DAE is not uniformly solvable on \mathcal{C} or any submanifold of \mathcal{C} .

We now look at the solutions on \mathcal{C} . Since the problem appears to be index 2 we expect another constraint. Differentiating the constraint (28c) we get, after some simplification, that we have the additional constraint

$$u_5^2 + u_6^2 = 0 \tag{34}$$

on \mathcal{C} . Thus $u_5 = 0, u_6 = 0$ which is the line \mathcal{L} . We already know that \mathcal{L} consists of a line of equilibria stretching up the side of the cylinder. However, this does not fully describe the solutions. If $u_5 = 0$ and $u_6 = 0$, then the DAE (28) on \mathcal{C} puts no restrictions on u_2 . Thus the DAE on \mathcal{C} has solutions $u_5 = 0, u_6 = 0$ and u_2 arbitrary. The DAE is not solvable. DAEs with an infinite number of solutions have been discussed elsewhere [14]. However, some sort of regularization, motivated by physical considerations, is needed to pick out a particular solution. We will not discuss this further here.

We note that the pencil (29) on \mathcal{C} is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & u_5 \\ 0 & 0 & u_6 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

which is not regular.

7. Conclusions

We have seen that the traveling waves for PDEs can satisfy a differential algebraic equation which may not be in the simple semi-explicit index one form. The existing DAE theory and algorithms permit analysis and simulation to be performed based on the original DAE model. An example from magnetohydrodynamics has been used as an illustration.

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