

# Mini-Projet : Traveling waves and spiral waves in excitable media : Belousov-Zhabotinsky

Students :

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## 1 Introduction

In this mini-projet, we focus on the analysis and simulation of traveling waves and spiral waves for the BZ reaction in excitable media. The objectives of the project are threefold : 1- to conduct an analysis similar to what we have done for the Nagumo equation in PC7, but for a stiffer problem, 2- to analyze to what extent a splitting method can reproduce the dynamics of the wave when we want to use large time steps compared to the fastest scales in the system, 3- to study, depending on time, the dynamics in 2D of spiral waves along the line of [1].

## 2 Traveling waves for the BZ reaction in 1D

We are concerned with the numerical approximation of a model for the Belousov-Zhabotinski reaction, a catalyzed oxidation of an organic species by acid bromated ion (see [2] for more details and illustrations). We thus consider the model detailed in [4] and coming from the classic work of [3] which takes into account three species: hypobromous acid  $\text{HBrO}_2$ , bromide ions  $\text{Br}^-$ , and cerium (IV). Denoting by  $a = [\text{Ce(IV)}]$ ,  $b = [\text{HBrO}_2]$ , and  $c = [\text{Br}^-]$ , we obtain a very stiff system of three PDEs:

$$\begin{cases} \partial_t a - D_a \partial_x^2 a = \frac{1}{\mu}(-qa - ab + fc), \\ \partial_t b - D_b \partial_x^2 b = \frac{1}{\epsilon}(qa - ab + b(1 - b)), \\ \partial_t c - D_c \partial_x^2 c = b - c, \end{cases} \quad (1)$$

$x \in \mathbb{R}$ , with diffusion coefficients  $D_a$ ,  $D_b$  and  $D_c$ , and the real positive parameters:  $f$ , small  $q$ , and small  $\epsilon$  and  $\mu$ , such that  $\mu \ll \epsilon \ll 1$ .

The dynamical system associated with this system models reactive excitable media with a large time scale spectrum (see [4] for more details). The spatial configuration with the addition of diffusion develops propagating wavefronts with steep spatial gradients. Hence, this model presents all the difficulties associated with a stiff multi-scale configuration. The advantages of applying a splitting strategy to this problem have already been studied and presented in [1]. In what follows, we will briefly consider a 1D case of (1) in order to illustrate the errors of splitting schemes for stiff problems.

We consider then a 1D configuration of problem (1) with homogeneous Neumann boundary conditions and the following parameters, taken from [4]:

$$\epsilon = 10^{-2}, \quad \mu = 10^{-5}, \quad f = 3, \quad q = 2 \times 10^{-4}, \quad (2)$$

with diffusion coefficients:

$$D_a = 1/400, \quad D_b = 1/400, \quad D_c = 0.6/400, \quad (3)$$

for a space region of  $[0, 4]$ . Several level of sufficiently refined uniform mesh are considered while the exact solution  $T^s$  is approximated by a reference or *quasi-exact* solution  $\mathcal{T}^s$  of the semi-discretized coupled reaction-diffusion problem (1), performed by Radau5 with very fine tolerances. The splitting schemes consider Radau5 and ROCK4 as integration methods for the reaction and diffusion problems.

**2.1** Simulate the propagation of the wave in time. Show that there is a limit splitting time step for which the velocity of the wave is not predicted accurately any more.

**2.2** Characterize the error for several splitting time steps and plot the error as a function of splitting time step.

**2.3** Explain why this strategy can be considered as optimal in order to conduct a simulation of a traveling wave where stiffness is present at several levels.

### 3 Traveling waves for the BZ reaction in 2D

Following, what has been done in [1, 5], propose a study of the 2D case in terms of simulation and splitting methods.

### References

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