18.06.2021

Universitatea din București
Facultatea de Matematică și Informatică
Calculatoare și Tehnologia Informației
Grupa 161
Crețu Laurențiu-Vasile

Examen
Calcul numeric

1) $\begin{cases} x + y + 2z = 7 \\ x + 2y + 3z = 1 \\ x + y + z = 10 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 1 \\ 10 \end{pmatrix}$$

$$\bar{A} = [A \mid b] = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 10 \end{array}\right)$$

$$\max_{i,j=\overline{1,3}} |a_{ij}| = |a_{23}| = 3$$

$$\bar{A} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \\ 1 & 1 & 1 & 10 \end{array}\right) \\ \begin{array}{cccc} x & y & z & \end{array} \end{array} \overset{C_1 \leftrightarrow C_3}{\sim} \begin{array}{c} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 1 & 1 & 7 \\ 1 & 1 & 1 & 10 \end{array}\right) \\ \begin{array}{cccc} z & y & x & \end{array} \end{array}$$

$$\bar{A} \underset{L_3 \leftarrow L_3 - \frac{1}{3}L_1}{\overset{L_2 \leftarrow L_2 - \frac{2}{3}L_1}{\sim}} \begin{array}{c} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \frac{19}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{29}{3} \end{array}\right) \\ \begin{array}{cccc} z & y & x & \end{array} \end{array}$$

$$\max_{i,j=\overline{2,3}} |a_{ij}| = \frac{2}{3} = |a_{33}|$$

$$\overline{A} \overset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{29}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & \frac{19}{3} \end{array}\right) \overset{c_2 \leftrightarrow c_3}{\sim} \left(\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{29}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{19}{3} \end{array}\right)$$

$$\begin{array}{ccc} z & y & x \end{array} \qquad \begin{array}{ccc} z & x & y \end{array}$$

$$\overline{A} \overset{L_3 \leftarrow L_3 - \frac{1}{2}L_2}{\sim} \left(\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{29}{3} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{array}\right)$$

$$\begin{array}{ccc} z & x & y \end{array}$$

$$\begin{cases} 3z + x + 2y = 1 \Rightarrow 3z = 1 - 16 + 6 = -9 \Rightarrow \boxed{z = -3} \\ \frac{2}{3}x + \frac{1}{3}y = \frac{29}{3} \Rightarrow \frac{2}{3}x = \frac{29}{3} + 1 \Rightarrow \frac{2}{3}x = \frac{32}{3} \Rightarrow \boxed{x = 16} \\ -\frac{1}{2}y = \frac{3}{2} \Rightarrow \boxed{y = -3} \end{cases}$$

2)

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -4 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

Inițializăm matricele L și w:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$k = 1$:

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{-4}{3} = -\frac{4}{3}$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{3}$$

$$A \underset{L_3 \leftarrow L_3 - \frac{1}{3} L_1}{\overset{L_2 \leftarrow L_2 + \frac{4}{3} L_1}{\sim}} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & \frac{10}{3} \\ 0 & 4 & \frac{2}{3} \end{pmatrix}$$

Matricea L devine: $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$

$k = 2$:

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{4}{1} = 4$$

$$A \overset{L_3 \leftarrow L_3 - 4L_2}{\sim} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & \frac{10}{3} \\ 0 & 0 & -\frac{38}{3} \end{pmatrix}$$

Matricea L devine: $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & 1 & 0 \\ \frac{1}{3} & 4 & 1 \end{pmatrix}$

În concluzie, factorizarea LU a matricei A este:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{3} & 1 & 0 \\ \frac{1}{3} & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & \frac{10}{3} \\ 0 & 0 & -\frac{38}{3} \end{pmatrix}$$

3) $f : [0,5] \to \mathbb{R}, \quad f(x) = 3^x + 3\cdot 4^x - 13x^2$

$(0, 3, 5)$

$x_1 = 0$

$x_2 = 3$

$x_3 = 5$

$$P_2(x) = f[x_1] + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2)$$

$$= f[0] + f[0,3](x-0) + f[0,3,5](x-0)(x-3)$$

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_1, x_2, \dots x_{n+1}]$
0	$f[0] = 2$		
3	$f[3] = 102$	$f[0,3] = \frac{97}{3}$	
5	$f[5] = 2990$	$f[3,5] = 1444$	$f[0,3,5] = \frac{847}{5}$

$$f[0] = f(0) = 3^0 + 3\cdot 4^0 - 13\cdot 0^2 = 1 + 4 = 5$$

$$f[3] = f(3) = 3^3 + 3\cdot 4^3 - 13\cdot 3^2 = 27 + 192 - 117 = 102$$

$$f[5] = f(5] = 3^5 + 3\cdot 4^5 - 13\cdot 5^2 = 243 + 3072 - 325$$
$$= 2990$$

$$f[0,3] = \frac{f(3) - f(0)}{3-0} = \frac{102-5}{3} = \frac{97}{3}$$

$$f[3,5] = \frac{f(5) - f(3)}{5-3} = \frac{2990-102}{2} = 1444$$

$$f[0,3,5] = \frac{f[3,5] - f[0,3]}{5-0} = \frac{1444 - \frac{97}{3}}{5} = \frac{847}{5}$$

$$P_2(x) = 2 + \frac{97}{3}x + \frac{847}{5}x(x-3) = 2 + \frac{97}{3}x + \frac{847}{5}(x^2-3x)$$

$$= 2 + \frac{97}{3}x + \frac{847}{5}x^2 - \frac{2541}{5}x = 2 - x\frac{7138}{15} + \frac{847}{5}x^2.$$

4) $$I = \int_{-4}^{-1} \left(2\cdot x^4 + 4x^3 + \frac{2}{2}x^1 + 6\cdot x^0\right) dx$$

$$I = \int_{-4}^{-1} \left(2x^4 + 4x^3 + x + 6\right) dx.$$

a) $m = 3$

$$I^m_{trapez} = \frac{h}{2}\left(f(a) + 2\sum_{k=2}^{m} f(x_k) + f(b)\right)$$

$$h = \frac{b-a}{m}$$

$$x_{k+1} = a + k\cdot h, \quad k = \overline{0,m}$$

$m = 3$: $\quad h = \frac{-1+4}{3} = 1$

$$x_1 = -4 + 0\cdot 1 = -4$$

$$x_2 = -4 + 1\cdot 1 = -3$$

$$x_3 = -4 + 2\cdot 1 = -2$$

$$I^3_{trapez} = \frac{1}{2}\left(f(-4) + 2\left(f(x_2) + f(x_3)\right) + f(-1)\right) =$$

$$f(-4) = 2\cdot(-4)^4 + 4\cdot(-4)^3 - 4 + 6 = 512 - 256 + 2 = 258$$

$$f(-1) = 2\cdot(-1)^4 + 4\cdot(-1)^3 - 1 + 6 = 2 - 4 + 5 = 3$$

$$f(-3) = 2\cdot(-3)^4 + 4\cdot(-3)^3 - 3 + 6 = 162 - 108 + 3 = 57$$

$$f(-2) = 2\cdot(-2)^4 + 4\cdot(-2)^3 - 2 + 6 = 32 - 32 + 4 = 4$$

$$I^3_{trapez} = \frac{1}{2}\left(258 + 2(57+4) + 3\right) = \frac{1}{2}\left(261 + 122\right) =$$

$I^{3}_{trapez} = \frac{383}{2}$

b) $er = \frac{b-a}{12} \cdot f''(\xi) \cdot h^2, \quad \xi \in (a,b)$

$a = -4, \quad b = -1, \quad h = \frac{-1+4}{m} = 1$

$f(x) = 2x^4 + 4x^3 + x + 6$

$f'(x) = 8x^3 + 12x^2 + 1$

$f''(x) = 24x^2 + 24x = 24x(x+1)$

$er = \frac{-1+4}{12} \cdot f''(\xi) \cdot 1^2 = \frac{1}{\cancel{4}_{1}} \cdot \overset{6}{\cancel{24}}\, \xi(\xi+1) =$

$= 6\xi^2 + 6, \quad \xi \in (-4,-1)$

$eroarea = \left| I(f) - I^{3}_{trapez} \right|$

$I(f) = \int_{-4}^{-1} (2x^4 + 4x^3 + x + 6)\,dx =$

$= 2\frac{x^5}{5}\Big|_{-4}^{-1} + 4\frac{x^4}{4}\Big|_{-4}^{-1} + \frac{x^2}{2}\Big|_{-4}^{-1} + 6x\Big|_{-4}^{-1} =$

$= 2\left(-\frac{1}{5} + \frac{1024}{5}\right) + 4\left(\frac{1}{4} - 4^3\right) + \left(\frac{1}{2} - \frac{16}{2}\right) + (-6+24)$

$= \frac{2046}{5} + 1 - 256 - \frac{15}{2} + 18 = 164,7$

5) $f:[a,b]\to\mathbb{R},\ m\in\mathbb{N},\ m\geq 3$

$a=x_1<x_2<\ldots<x_{m+1}=b$

$P_m(x)=f[x_1]+f[x_1,x_2](x-x_1)+a_3(x-x_1)(x-x_2)+$

$+a_4(x-x_1)(x-x_2)(x-x_3)+\ldots+a_{m+1}(x-x_1)\cdots(x-x_m)$

$P_m(x_3)=f(x_3)$

a) $P_m(x_3)=f[x_1]+f[x_1,x_2](x_3-x_1)+a_3(x_3-x_1)(x_3-x_2)$

$+a_4(x_3-x_1)(x_3-x_2)\underbrace{(x_3-x_3)}_{0}+\ldots+a_{m+1}(x_3-x_1)\ldots(x_3-x_m)=$

$=f[x_1]+f[x_1,x_2](x_3-x_1)+a_3(x_3-x_1)(x_3-x_2)$

$$f[x_1,x_2,x_3]=\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}=$$

$$f[x_2,x_3]=\frac{f[x_3]-f[x_2]}{x_3-x_2}$$

$$f[x_1,x_2]=\frac{f[x_2]-f[x_1]}{x_2-x_1}\quad\Rightarrow$$

$$\Rightarrow f[x_1,x_2,x_3]=\frac{\dfrac{f[x_3]-f[x_2]}{x_3-x_2}-\dfrac{f[x_2]-f[x_1]}{x_2-x_1}}{x_3-x_1}=$$

$$=\frac{f[x_3]-f[x_2]}{x_3-x_2}\cdot\frac{1}{x_3-x_1}-\frac{f[x_2]-f[x_1]}{x_2-x_1}\cdot\frac{1}{x_3-x_1}=$$

$$=\frac{f[x_3]}{(x_3-x_2)(x_3-x_1)}-\frac{f[x_2]}{(x_3-x_2)(x_3-x_1)}-\frac{f[x_2]}{(x_2-x_1)(x_3-x_1)}+\frac{f[x_1]}{(x_2-x_1)(x_3-x_1)}$$

$$=\frac{f[x_3]}{(x_3-x_2)(x_3-x_1)}-\frac{f[x_1]}{(x_3-x_2)(x_3-x_1)}+\frac{f[x_1]}{(x_3-x_2)(x_3-x_1)}-$$

$$-\frac{f[x_2]}{(x_3-x_2)(x_3-x_1)} - \frac{f[x_2]}{(x_2-x_1)(x_3-x_1)} + \frac{f[x_1]}{(x_2-x_1)(x_3-x_1)} =$$

$$= \underbrace{\frac{f[x_3]-f[x_1]}{(x_3-x_2)(x_3-x_1)}}_{=u} - \frac{f[x_2]-f[x_1]}{(x_3-x_2)(x_3-x_1)} - \frac{f[x_2]-f[x_1]}{(x_3-x_1)(x_2-x_1)} =$$

$$= u - \frac{f[x_2]-f[x_1]}{x_2-x_1} \cdot \frac{x_2-x_1}{(x_3-x_2)(x_3-x_1)} - \frac{f[x_1,x_2]}{x_3-x_1} =$$

$$= u - f[x_1,x_2] \cdot \frac{x_2-x_1}{(x_3-x_2)(x_3-x_1)} - \frac{f[x_1,x_2]}{x_3-x_1} =$$

$$= u - f[x_1,x_2] \left(\frac{x_2-x_1}{(x_3-x_2)(x_3-x_1)} + \frac{1}{x_3-x_1} \right) =$$

$$= u - f[x_1,x_2] \cdot \frac{\cancel{x_2-x_1+x_3-x_2}}{(x_3-x_2)\cancel{(x_3-x_1)}} =$$

$$= u - \frac{f[x_1,x_2]}{x_3-x_2} = \frac{f[x_3]-f[x_1]}{(x_3-x_2)(x_3-x_1)} - \frac{f[x_1,x_2]}{x_3-x_2}$$

//

5) $P_n(x_3) = f(x_3) \Rightarrow f[x_1] + f[x_1,x_2](x_3-x_1) +$

$+ a_3(x_3-x_1)(x_3-x_2) = f(x_3) \Leftrightarrow$

$\Leftrightarrow f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x_3-x_1) + a_3(x_3-x_1)(x_3-x_2) =$

$= a_1 + a_2 x_3 + a_3 x_3^2 \Leftrightarrow$

$\cancel{f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x_3-x_1)}$