

# MATH 215/255: Elementary Differential Equations

1. What are they and why do we solve them?
2. Terminology

# What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function  $y$  and at least one derivative of  $y$  w.r.t. an independent variable.

Given: A DE with an unknown function  $y(t)$ . e.x.,  $\frac{dy}{dt} = -3y(t)$   
or

$$y' = -3y$$

Task: Find the function(s)  $y(t)$ .

Solution:  $y(t) = C_1 e^{-3t}$

DEs specify the rate of change of one quantity (e.g., the position of an object) with respect to another (e.g., time).

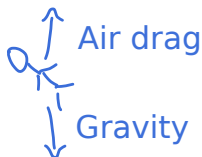
# Why do we solve/study DEs?

DEs provide an intuitive way to describe many types of interactions (e.g., mechanical, biochemical, social, economic, etc.).

Solving and analyzing DEs allows us to:

1. Make predictions about the future (forecasting).
  - Will some quantity grow unboundedly? Oscillate? Decay to zero?
  - With what rate will those things happen?
2. Test possible mechanisms that may explain experimental data.
  - Why does some quantity sometimes oscillate vs reach an equilibrium?

# Example: Skydiving



Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v}$$

DE for  $v(t)$

# Example: Ecology - Lotka-Volterra Model

Predator-Prey Model, 2 variables:

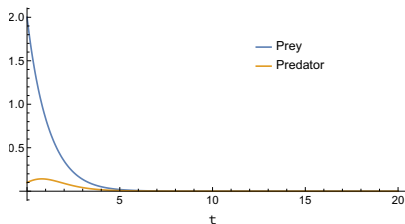
$x$  = prey population and  $y$  = predator population

$$\frac{dx}{dt} = \alpha x - \beta xy, \quad \frac{dy}{dt} = \delta xy - \gamma y$$

We can prove that only these two solutions types are possible

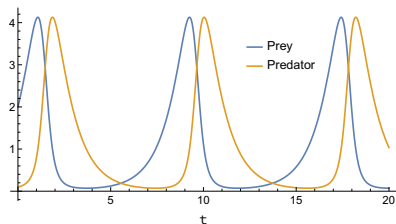
## Mutual Extinction

$$\{\beta = -1, \alpha = -1, \delta = 1, \gamma = 1\}$$



## Predator-Prey Oscillations

$$\{\beta = 1, \alpha = 1, \delta = 1, \gamma = 1\}$$



# Terminology: ODEs vs PDEs

- Ordinary differential equation (ODE) (covered in this course)
  - A DE with derivatives w.r.t. only one independent variable.
  - $\frac{dy}{dt} = y(t) + 3$       or       $\frac{dy}{dt} = \sin(y) + \cos(t)$
- Partial differential equation (PDE) (not covered in this course)
  - A DE with derivatives w.r.t multiple independent variables.
  - Heat/Diffusion eq:  $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$
  - Wave eq:  $\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 u}{\partial x^2}$

Partial derivatives are necessary for solutions to agree when changing coordinate systems (e.g., switch from cartesian to polar coordinates)

# Terminology: Order of a DE

The highest derivative that appears in the DE.

- $y' = y + 3$       first order
- $y' = y^2 + 9$       first order
- $\left(\frac{dy}{dt}\right)^2 = \tan(t)$       first order
- $y'' = -y$       second order
- $\frac{d^4y}{dx^4} = ky$       fourth order

# Terminology: Operator Form $\Rightarrow \mathcal{L}[y(t)] = f(t)$

Everything that depends on the unknown function goes on one side of the equal sign and everything else on the other.

- $\frac{dy}{dt} = y(t) + 3 \quad \rightarrow \quad \frac{dy}{dt} - y(t) = 3$ 
  - $\mathcal{L}[y] = f' - y, \quad f(t) = 3$
- $\frac{dy}{dt} = \sin(y) + \cos(t) \quad \rightarrow \quad \frac{dy}{dt} - \sin(y) = \cos(t)$ 
  - $\mathcal{L}[y] = f' - \sin(y), \quad f(t) = \cos(t)$

The operator  $\mathcal{L}[\cdot]$  encodes the "intrinsic" dynamics that the ODE is modelling.

- Force-displacement relationship of a spring.
- Velocity-drag relationship of a viscous fluid.



## Terminology: Linearity of DEs - $L[y(t)] = f(t)$

If the operator  $L[\cdot]$  is linear, then the DE is linear.

### Conditions for linearity:

Given any two functions  $f$  and  $g$  and a constant  $c$ ,  $L[\cdot]$  is linear if

$$1. \quad L[f + g] = L[f] + L[g]$$

$$2. \quad L[cf] = cL[f]$$

# Terminology: Linearity of DEs - $L[y(t)] = f(t)$

If the operator  $L[\cdot]$  is linear, then the DE is linear.

In practice:

Does the operator have either of the following:

1. any nonlinear functions of  $y$  (or its derivatives) or
2. any products of  $y$  and its derivatives

ex:  $L[y] = y'' + y$

Linear

ex:  $L[y] = y' + \sin(y'')$

Nonlinear

ex:  $L[y] = y' + y'y$

Nonlinear

# Terminology: Autonomous DEs - $L[y(t)] = f(t)$

If both  $L[\cdot]$  and  $f(t)$  do not explicitly depend on the independent variable, then the DE is autonomous.

$$\bullet \quad y' = y \quad \rightarrow \quad y' - y = 0 \quad \text{Autonomous}$$

$$\bullet \quad y' = y^2 + 3 \quad \rightarrow \quad \frac{dy}{dt} - y^2 = 3 \quad \text{Autonomous}$$

$$\bullet \quad \frac{dy}{dt} = y + \tan(t) \quad \rightarrow \quad \frac{dy}{dt} - y = \tan(t) \quad \text{Non-autonomous}$$

$$\bullet \quad \frac{dy}{dt} = -3ty \quad \rightarrow \quad \frac{dy}{dt} + 3ty = 0 \quad \text{Non-autonomous}$$

$f(t)$  is often called the (external) forcing term.

constant or zero-forcing  $\Rightarrow$  Autonomous DE

# Classifying ODEs

- $x'' + x^2 = t$ 
  - Order: 2
  - Linear: No
  - Autonomous: No
  
- $\frac{d^4x}{dt^4} = 0$ 
  - Order: 4
  - Linear: Yes
  - Autonomous: Yes

# Terminology: Solution to an ODE

A solution of an ODE is a function that satisfies the ODE.

ex: Is  $y = Ce^{-t} + t - 1$  a solution to  $y' + y = t$ ?

compute derivative(s):  $y' = -Ce^{-t} + 1$

evaluate ODE:  $y' + y = \cancel{-Ce^{-t}} + \cancel{1} + \cancel{Ce^{-t}} + t - \cancel{1}$   
 $= t \quad \checkmark$

Here  $C$  is an arbitrary constant that can have any value.

Any solution with an arbitrary constant is called a general solution

A solution with no arbitrary constants is called a particular solution

We eliminate arbitrary constant by using constraints

# Initial Value Problems

Add a constraint at  $t = t_0$ , e.g.

$$L[y] = f(t), \text{ with } y(t_0) = y_0,$$

where  $t_0$  and  $y_0$  are numerical values (usually real-valued).

ex: Find the particular solution to  $y' + y = t$  with  $y(0) = 4$ ?

Start with the general solution

$$y(t) = Ce^{-t} + t - 1$$

evaluate at  $t = t_0 = 0$ , make that equal to  $y_0 = 4$

$$y(0) = C - 1 = 4 \quad \Rightarrow \quad C = 5$$

$$y(t) = 5e^{-t} + t - 1$$

# Summary

## 1. What are DEs?

- Equations involving unknown function(s) and function derivatives.
- Specify rates of change of certain quantities.
- Useful for modelling many natural phenomena.

## 2. Terminology

- ODEs (& PDEs).
- Order of DEs, Linear DEs, Autonomous DEs, Solutions to DEs

## 3. Initial Value Problems

- The most "standard" way to obtain a unique solution
- Specify solution value at some initial time

# For next class...you will need access to MATLAB

## 1. Create a MathWorks account

- Go to [matlab.mathworks.com](https://matlab.mathworks.com)
- Click "No account? Create one!"
- Enter your UBC email address and follow the instructions
  - You can obtain one from [here](#) using "Activate Student Email"
- Note it may take a few hours to activate your MathWorks account

## 2. Use MATLAB Online

- Go to [matlab.mathworks.com](https://matlab.mathworks.com)
- Sign in with your UBC email address and MathWorks password