Introduction

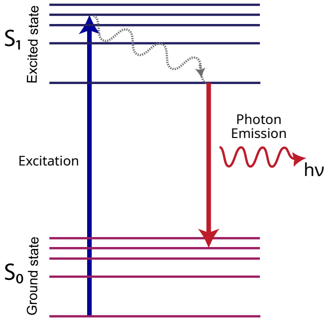
Almost all nonlinear systems cannot be solved analytically. Instead, we can use local stability analysis to gain qualitative infomation about solutions.

1. Classify the stability and type of all fixed points
   - Fixed points can lose stability, collide, and/or dissapear/appear as system parameters are changed.

2. Draw phase plane sketches to get a geometric picture
   - Use the fact that derivatives switch sign when trajectories cross nullclines

Lamps

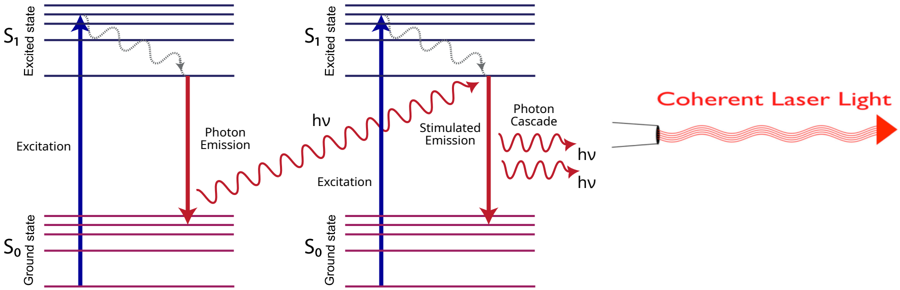
Lamps work at a quantum level by pumping molecules into an excited state, such that they release photons (light particles) as they return back down to the ground state.



The excitation/emission process is random, producing photons with uncorrelated phases and orientation. This is what we call incoherent light.

Lasers (**L**ight **A**mplification by **S**timulated **E**mission of **R**adiation)

Lasers work through a process called **stimulated emission**, photon emission from one molecule triggers the emission of a photon by its neighbour with similar phase and orientation.



If enough stimulated emission happens, we get an emission cascade of photons with highly correlated phase and orientation. This results in coherent light beams that can travel long distances with minimal spread.

A simple model for a laser considers two variables:

1. $N(t) =$ # of "ordinary" excited molecules
2. $n(t) =$ # of "lasing" molecules that will emit photons coherently.

The dynamics of the two variables are given by

$$\begin{aligned} n' &= nN - n \\ N' &= -nN - N + p \end{aligned}$$

where p is a "pumping" parameter that quantifies the rate of excitation.

see Laser Physics by Miloni and Eberly (1988) for more details.

ex: Find all critical points for the system:

$$n' = nN - n, \quad N' = -nN - N + p$$

n-nullclines:

$$0 = n(N-1)$$

$$n = 0 \quad \text{and} \quad N = 1$$

N-nullclines:

$$0 = -N(1+n) + p$$

$$N = \frac{p}{1+n}$$

intersections

$$n = 0 \;\; \& \;\; N = \frac{p}{1+n} \Rightarrow N = p$$

$$N = 1 \;\; \& \;\; N = \frac{p}{1+n} \Rightarrow n = p - 1$$

$$\Rightarrow N = 1$$

critical points: $\underbrace{(0, p)}_{\text{lamp}}$, and $\underbrace{(p-1, 1)}_{\text{laser}}$

ex: Given the simple laser model

$$n' = nN - n, \quad N' = -nN - N + p,$$

find a condition on p that ensures that the laser state is the unique stable steady state.

$$\mathbf{J} = \begin{bmatrix} N-1 & n \\ -N & -n-1 \end{bmatrix}$$

lamp: $(0, p)$

$$\mathbf{J}^* = \begin{bmatrix} p-1 & 0 \\ -p & -1 \end{bmatrix}$$

laser: $(p-1, 1)$

$$\mathbf{J}^* = \begin{bmatrix} 0 & p-1 \\ 1 & -p \end{bmatrix}$$

Find the eigenvalues, i.e., $\det(J^* - \lambda \mathbf{I}) = 0$

$$(p-1-\lambda)(-1-\lambda) = 0$$

$$\lambda = p-1, -1$$

$$-\lambda(-p-\lambda) + 1 - p = 0$$

$$\lambda^2 - p\lambda + 1 - p = 0$$

$$\lambda = \frac{-p}{2} \pm \frac{\sqrt{(p-2)^2}}{2}$$

$$= -1, 1-p$$

ex: Given the simple laser model

$$n' = nN - n, \quad N' = -nN - N + p,$$

find a condition on p that ensures that the laser state is the unique stable steady state.

lamp: $(0, p)$
$\lambda = p - 1, -1$

laser: $(p - 1, 1)$
$\lambda = -1, 1 - p$

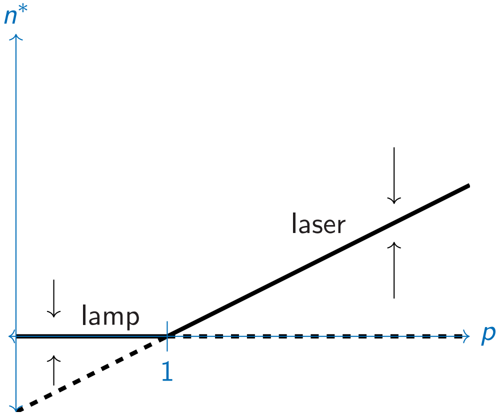
Both steady states have an eigenvalue of -1, independent of p.

For $p < 1$, the lamp state is a stable node and the laser state is a saddle (unstable).

For $p > 1$, the lamp state is a saddle and the laser state is stable node.

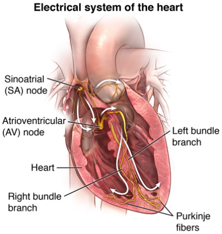
Laser Threshold

The laser and lamp steady states collide at $p = 1$ and exchange their stability.

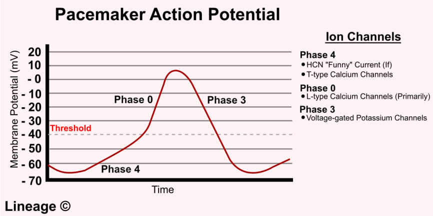


Your heart beats cyclically

The Sinoatrial (SA) node is the main clock.



source: https://www.hopkinsmedicine.org/health/conditions-and-diseases/anatomy-and-function-of-the-hearts-electrical-system



source: https://step1.medbullets.com/cardiovascular/108016/sa-node-action-potential

These oscillations must be robust to perturbations, otherwise you would die :(

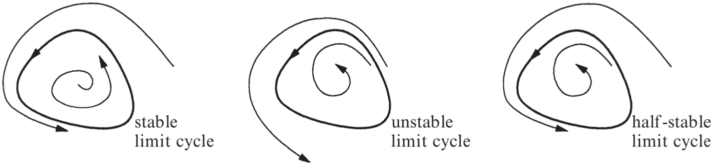
Such periodic solutions are called stable limit cycles

- i.e., a cycle (or closed loop) that the nonlinear system approaches in the limit $t \to \infty$

Limit Cycles

A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.

Spiral trajectories $\Rightarrow$ complex conjugate eignevalues.



Stable limit cycles form around unstable spiral fixed points

Van der Pol Oscillator

The Van der Pol Oscillator is a simple model that exhibits limit cycles. Its dynamics are given by

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = a - x$$

where a is a forcing parameter.

Find the nullclines for this model.

x-nullcline:

$$y = \frac{1}{3}x^3 - x$$

y-nullcline:

$$x = a$$

Van der Pol Oscillator

The Van der Pol Oscillator has a unique fixed point at

$$x = a, \quad y = \frac{1}{3}a(a^2 - 3)$$

with eigenvalues given by

$$\lambda = \frac{1}{2}\left(1 - a^2 \pm \sqrt{a^4 - 2a^2 - 3}\right).$$

These are complex conjugate eigenvalues that become stable for $|a| > 1$.

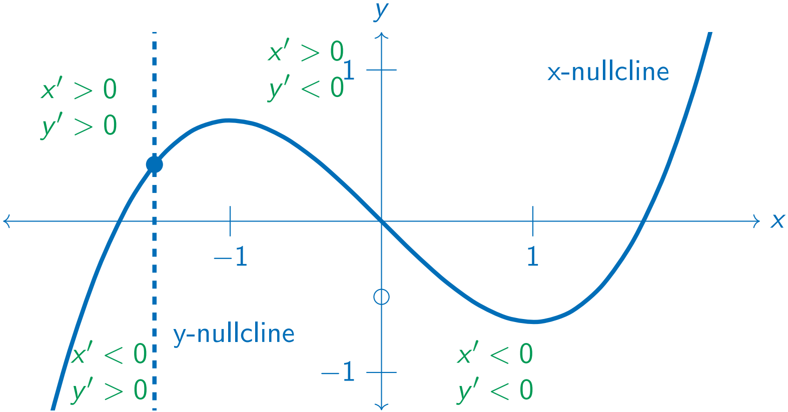
Unstable spiral for $|a| < 1$.

Stable spiral for $|a| > 1$.

Use the nullclines to sketch a solution trajectory for

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = a - x$$

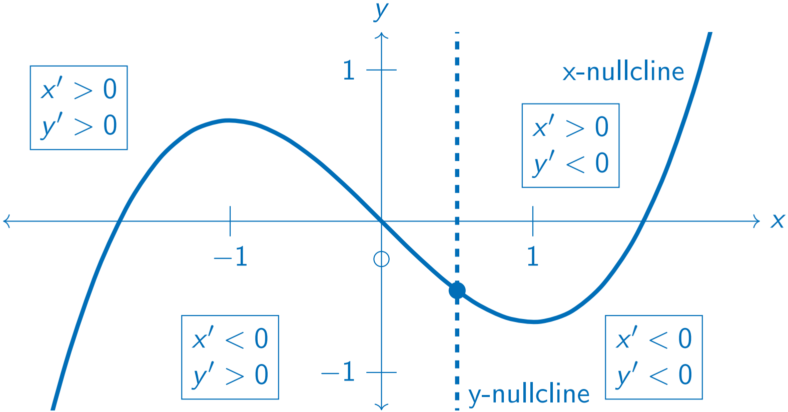
with a=-1.5 starting at (0,-0.5)



Use the nullclines to sketch a solution trajectory for

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = a - x$$

with a=0.5 starting at (0,-0.25)



Note: the solution trajectory should approach a stable limit cycle