

Differential Equations

1. What are they and why do we solve them?
2. Terminology
3. Graphical intuition and the direction field

What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. an independent variable.

Given: A DE with an unknown function $y(t)$.

$$\text{e.x., } \frac{dy}{dt} = -3y(t) \quad \text{or}$$

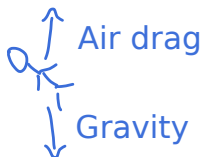
$$y' = -3y$$

Task: Find the function(s) $y(t)$.

$$\text{Solution: } y(t) = C_1 e^{-3t}$$

- Tools:
- Calculus (i.e., integration/differentiation)
 - Guess and check (does some function $f(t)$ make LHS=RHS?)
 - Specialized procedures (informed by experience geussing)
 - Geometry/Linear Algebra (useful for systems of DEs)

Example: Skydiving



Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v} \quad \text{DE for } v(t)$$

Example: Epidemiology

Kermack & McKendrick's SIR model

- Susceptible \rightarrow Infected \rightarrow Recovered

System of 3 ordinary differential equations:

$$\frac{dS}{dt} = \mu(I + R) - \beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

t = time

β = infection rate

γ = recovery rate

μ = birth/death rate

Reproduction number: $R_0 = \frac{\beta}{\mu + \gamma}$

1. $R_0 > 1$: endemic equilibrium

2. $R_0 < 1$: disease dies out - basic idea behind "flatten the curve"

Terminology: ODEs vs PDEs

- Ordinary differential equation (ODE) (covered in this course)
 - A DE with derivatives w.r.t. only one independent variable.
 - $\frac{dy}{dt} = y(t) + 3$ or $\frac{dy}{dt} = \sin(y) + \cos(t)$
- Partial differential equation (PDE) (not covered in this course)
 - A DE with derivatives w.r.t multiple independent variable.
 - Heat/Diffusion eq: $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$
 - Wave eq: $\frac{\partial^2 u}{\partial t^2} = c \frac{\partial^2 u}{\partial x^2}$
 - Partial derivatives are necessary for solutions to agree when changing coordinate systems (e.g., switch from cartesian to polar coordinates)

Terminology: Order of a DE

The highest derivative that appears in the DE.

- $y' = y + 3$ first order
- $y' = y^2 + 9$ first order
- $\left(\frac{dy}{dt}\right)^2 = \tan(t)$ first order
- $y'' = -y$ second order
- $\frac{d^4y}{dx^4} = ky$ fourth order

Terminology: Operator Form $\Rightarrow \mathcal{L}[y(t)] = f(t)$

Everything that depends on the unknown function goes on one side of the equal sign and everything else on the other.

- $\frac{dy}{dt} = y(t) + 3 \quad \rightarrow \quad \frac{dy}{dt} - y(t) = 3$
 - $\mathcal{L}[y] = f' - y, \quad f(t) = 3$
- $\frac{dy}{dt} = \sin(y) + \cos(t) \quad \rightarrow \quad \frac{dy}{dt} - \sin(y) = \cos(t)$
 - $\mathcal{L}[y] = f' - \sin(y), \quad f(t) = \cos(t)$

The operator $\mathcal{L}[\cdot]$, encodes the "intrinsic" dynamics that the ODE is modelling.

- Force-displacement relationship of a spring.
- Velocity-drag relationship of a viscous fluid.

Terminology: Linearity of DEs - $L[y(t)] = f(t)$

If the operator $L[\cdot]$ is linear, then the DE is linear.

Conditions for linearity:

Given any two functions f and g and a constant c , a linear operator satisfies

$$1. \quad L[f + g] = L[f] + L[g]$$

$$2. \quad L[cf] = cL[f]$$

In practice: does the operator have any nonlinear functions?

ex: $L[y] = y'' + y$

Linear

ex: $L[y] = y' + \sin(y)$

Nonlinear

Terminology: Autonomous DEs - $L[y(t)] = f(t)$

If both $L[\cdot]$ and $f(t)$ do not explicitly depend on the independent variable, then the DE is autonomous.

• $y' = y \rightarrow y' - y = 0$ Autonomous

• $y' = y^2 + 3 \rightarrow \frac{dy}{dt} - y^2 = 3$ Autonomous

• $\frac{dy}{dt} = y + \tan(t) \rightarrow \frac{dy}{dt} - y = \tan(t)$ Non-autonomous

• $\frac{dy}{dt} = -3ty \rightarrow \frac{dy}{dt} + 3ty = 0$ Non-autonomous

$f(t)$ is often called the (external) forcing term.

constant or zero-forcing \Rightarrow Autonomous DE

Classifying ODEs

- $x'' + x^2 = t$
 - Order: 2
 - Linear: No
 - Autonomous: No

- $\frac{d^4x}{dt^4} = 0$
 - Order: 4
 - Linear: Yes
 - Autonomous: Yes

Terminology: Solution to an ODE

A solution of an ODE is a function that satisfies the ODE.

ex: Is $y = Ce^{-t} + t - 1$ a solution to $y' + y = t$?

$$\begin{aligned}y' &= -Ce^{-t} + 1 \\y' + y &= \cancel{-Ce^{-t}} + \cancel{1} + \cancel{Ce^{-t}} + t - \cancel{1} \\&= t \quad \checkmark\end{aligned}$$

Here C is an arbitrary constant that can have any value.

Any solution with an arbitrary constant is called a general solution

We can obtain a unique solution by imposing some constraint, a solution with no arbitrary constants is called a particular solution

Initial Value Problems

ODEs of the form

$$L[y] = f(t), \text{ with } y(t_0) = y_0,$$

where t_0 and y_0 are numerical values (usually real-valued).

ex: Find the particular solution to $y' + y = t$ with $y(0) = 0$?

Start with the general solution

$$y(t) = Ce^{-t} + t - 1$$

evaluate at $t = t_0 = 0$, make that equal to $y_0 = 0$

$$y(0) = C - 1 = 0 \quad \Rightarrow \quad C = 1$$

$$\boxed{y(t) = e^{-t} + t - 1}$$

Summary

1. What are DEs?

- Equations involving unknown function(s) and function derivatives.
- Specify rates of change of certain quantities.
- Useful for modelling many natural phenomena.

2. Terminology

- ODEs (& PDEs).
- Order of DEs, Linear DEs, Autonomous DEs, Solutions to DEs

3. Initial Value Problems

- The most "standard" way to obtain a unique solution
- Specify solution value at some initial time