Recall: Homogeneous Linear Systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} \qquad \text{with } \mathbf{A} \text{ an } n \times n \text{ constant matrix}$$

We can (usually) find n solutions $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$ by determining

1. the eigenvalues λ_i, and
2. the eigenvectors, $\vec{v}_i$

of matrix $\mathbf{A}$.

i.e., solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0$$

then we have the general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

The fundamental set of solutions

Suppose you find n linearly independent n-dimensional vector functions

$$\{\vec{x}_1(t),\ \vec{x}_2(t),\ \ldots,\ \vec{x}_n(t)\}$$

that each solves $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}x$.

This is called the **fundamental set of solutions**.

- All solution lie within the span of this set

Sometimes we can only find find $n-1$ eigenvalues

- We'll need some trick to find the "missing" fundamental solution

The fundamental matrix

Construct a matrix $\mathbf{X}(t)$ with each vector as a column

$$\mathbf{X}(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) & \cdots & x_{n,1}(t) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{1,n}(t) & \cdots & \cdots & x_{n,n}(t) \end{bmatrix}$$

Recall that:

$$\text{L.I. columns of } \mathbf{X} \quad \Leftrightarrow \quad \det(\mathbf{X}) \neq 0 \quad \Leftrightarrow \quad \mathbf{X}^{-1} \text{ exists}$$

General solution to the homogeneous IVP

The general solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)x, \quad \vec{x}(t_0) = \vec{x}_0$$

is given by

$$\vec{x} = \mathbf{X}(t)\vec{c} \qquad \text{with} \;\; \vec{c} = \mathbf{X}^{-1}(t_0)\vec{x}_0$$

Proof:
We know

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t) \quad \Leftrightarrow \quad \vec{x}(t) = \mathbf{X}(t)\vec{c}$$

where $\vec{c}$ is a column vector, to find it we match the initial condition

$$\mathbf{X}(t_0)\vec{c} = \vec{x}_0 \quad \Rightarrow \quad \vec{c} = \mathbf{X}^{-1}(t_0)\vec{x}_0$$

Note on inverting matrices

Given the linear system of equations

$$\mathbf{M}\vec{c} = \vec{b}$$

we have the formal solution

$$\vec{c} = \mathbf{M}^{-1}\vec{b}.$$

Simple formulas for inverting 2×2 matrices exist, but I do not recommend using them.

My recommendation, use the augmented matrix

$$\left[M \middle| \vec{b}\right]$$

to solve for the entries in $\vec{c}$. This way no memorization required.

Find the solution to $\dfrac{dx}{dt} = -3x - 2y$, $\dfrac{dy}{dt} = -2x - 6y$ with $x(0) = 5$, $y(0) = 4$

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t} = \begin{bmatrix} -2e^{-2t} & e^{-7t} \\ e^{-2t} & e^{-7t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x}(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2c_1 & c_2 & 5 \\ c_1 & 2c_2 & 4 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|c} 0 & 5c_2 & 13 \\ c_1 & 2c_2 & 4 \end{array}\right]$$

$-\frac{2}{5}R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 0 & 5c_2 & 13 \\ c_1 & 0 & -\frac{6}{5} \end{array}\right]$$

$$c_1 = -\frac{6}{5}$$
$$c_2 = \frac{13}{5}$$

General solution to the eigenproblem (2x2 constant matrix)

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

$$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0 \quad \Leftrightarrow \quad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Three possibilites:

1. 2 distinct real eigenvalues/vectors ✓
2. A complex conjugate pair of eigenvalues/vectors
3. One eigenvalue is repeated, only one eigenvector

Find the eigenvalues for the ODE:

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Characteristic equation

$$(-1-\lambda)^2 + 4 = 0$$
$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$
$$\lambda_{1,2} = -1 \pm 2i$$

Complex conjugate eigenvalues pairs: $\lambda_{1,2} = r \pm i\omega$

Associated eigenvectors are also complex conjugates

$$\vec{v}_{1,2} = \vec{a} \pm i\vec{b} \qquad \text{where} \qquad \begin{array}{l} \mathrm{Re}(\vec{v}_1) = \vec{a} \\ \mathrm{Im}(\vec{v}_1) = \vec{b} \end{array}$$

Proof:

Suppose $\vec{v}_1 = \vec{a} + i\vec{b}$ with $\lambda_1 = r + i\omega$

$$\mathbf{A}(\vec{a} + i\vec{b}) = (r + i\omega)(\vec{a} + i\vec{b})$$

Take complex conjugate of both sides

$$\mathbf{A}(\vec{a} - i\vec{b}) = (r - i\omega)(\vec{a} - i\vec{b})$$
$$\mathbf{A}\vec{v}_2 = \lambda_2\vec{v}_2$$

Find the eigenvectors for the ODE:

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

$$\lambda_{1,2} = -1 \pm 2i$$

$\underline{\lambda_1 = -1 + 2i}$

$$\left[\begin{array}{cc|c} -1-(-1+2i) & 2 & 0 \\ -2 & -1-(-1+2i) & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$R_2 - iR_1 \rightarrow R_2$ and $\frac{1}{2}R_1 \rightarrow R_1$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} -ix + y &= 0 \\ y &= ix \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Find the general solution for the ODE:

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

$$\lambda_{1,2} = -1 \pm 2i \qquad \vec{v}_{1,2} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

$$\vec{x}(t) = c_1 \underbrace{e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{e^{(-1-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}}_{\vec{x}_2(t)}$$

Here both eigensolutions are complex-valued.

Convert to purely real solution using Euler's identity.

Conversion to real-valued vector solutions: $\lambda_{1,2} = r \pm i\omega$

Suppose you have

$$\vec{x}(t) = c_1 \overbrace{e^{(r+i\omega)t}(\vec{a} + i\vec{b})}^{\vec{x}_1(t)} + c_2 \overbrace{e^{(r-i\omega)t}(\vec{a} - i\vec{b})}^{\vec{x}_2(t)}$$

Then the first eigensolution can be written as

$$\begin{aligned}
\vec{x}_1(t) &= e^{rt}(\cos(\omega t) + i\sin(\omega t))(\vec{a} + i\vec{b}) \\
&= e^{rt}\left[(\cos(\omega t)\vec{a} - \sin(\omega t)\vec{b}) + i(\sin(\omega t)\vec{a} + \cos(\omega t)\vec{b})\right] \\
&= \mathrm{Re}\left(\vec{x}_1\right) + i\mathrm{Im}\left(\vec{x}_1\right) \quad \text{and also } \vec{x}_2 = \mathrm{Re}\left(\vec{x}_1\right) - i\mathrm{Im}\left(\vec{x}_1\right)
\end{aligned}$$

Choose c_1 and c_2 as complex conjugate, then

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 = (c_1 + c_2)\mathrm{Re}\left(\vec{x}_1\right) + i(c_1 - c_2)\mathrm{Im}\left(\vec{x}_1\right)$$

with $c_1 + c_2 = d_1 \in \mathbb{R}$ and $c_1 - c_2 = -id_2$ where $d_2 \in \mathbb{R}$

$$\vec{x}(t) = \underbrace{d_1\mathrm{Re}(\vec{x}_1(t))}_{\text{Real}} + \underbrace{d_2\mathrm{Im}(\vec{x}_1(t))}_{\text{Real}}$$

Find the real-valued solution for the ODE: $\frac{dx}{dt} = -x + 2y$, $\frac{dy}{dt} = -2x - y$

$$\vec{x}_1(t) = e^{(-1+2i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{-t}(\cos(2t) + i\sin(2t))\left(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{a}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}}\right)$$

$$\mathrm{Re}(\vec{x}_1(t)) = e^{-t}\left(\cos(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

$$\mathrm{Im}(\vec{x}_1(t)) = e^{-t}\left(\sin(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ -c_1\sin(2t) + c_2\cos(2t) \end{bmatrix}$$

Solve the IVP:

$$\frac{dx}{dt} = -x + 2y \qquad \text{with} \qquad x(0) = 2$$
$$\frac{dy}{dt} = -2x - y \qquad\qquad\qquad y(0) = 7$$

General Solution:

$$\vec{x}(t) = e^{-t}\begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ -c_1\sin(2t) + c_2\cos(2t) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} c_1 = 2 \\ c_2 = 7 \end{matrix}$$