Recall: Constant Coefficient Homogeneous 2^{nd} Order ODE

$ay'' + by' + cy = 0$ we need 2 linearly independent solutions

Try ansatz $y = e^{rt}$ $\rightarrow$ $e^{rt} \times \underbrace{(ar^2 + br + c)}_{\text{characterisitc polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

   $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2. Repeated real roots: discriminant $= 0$ $r = \frac{-b}{2a}$

3. Complex conjugate roots: discriminant < 0 $\sqrt{\text{negative } \#}$

Repeated real root $(r_1 = r_2 = r)$ $\qquad y_h = c_1 y_1 + c_2 y_2$

Straighforward solution

$$y_1 = e^{rt}$$

with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

We need another solution that is linearly independent of y_1

Lets try

$$y_2 = q(t) y_1(t)$$

Unique choice

$$q(t) = Ct \qquad \Rightarrow \qquad y_2(t) = te^{rt}$$

Proof that $y_2 = te^{rt}$

$$ay'' + by' + cy = 0 \qquad \text{with } b^2 - 4ac = 0 \quad \Rightarrow \quad r_{1,2} = r = \frac{-b}{2a}$$

Try: $y_2 = q(t)e^{rt}, \quad y_2' = q'e^{rt} + rte^{rt}$

$$y_2'' = q''e^{rt} + 2rq'e^{rt} + r^2te^{rt}$$

plug these into the ODE

$$a\left(q''e^{rt} + 2rq'e^{rt} + r^2te^{rt}\right) + b\left(q'e^{rt} + rte^{rt}\right) + cqe^{rt} = 0$$

$$aq''e^{rt} + (2ar + b)\,q'e^{rt} + \underbrace{\left(ar^2 + br + c\right)}_{\text{char. poly.}=0} qe^{rt} = 0$$

sub in $r = \frac{-b}{2a}$

$$aq''e^{rt} + \underbrace{\left(\cancel{2a}\frac{-b}{\cancel{2a}} + b\right)}_{0} e^{rt} = 0$$

$$aq''e^{rt} = 0 \quad \Rightarrow \quad q'' = 0 \quad \Rightarrow \quad q(t) = Ct + D$$

$D = 0$ due to linear independence between y_2 and y_1

Constant Coefficient Homogeneous 2^{nd} Order ODE

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$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

   $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2. Repeated real roots: discriminant $= 0$
   $y_h = c_1 e^{rt} + c_2 t e^{rt}$

3. Complex conjugate roots: discriminant < 0

Solve the IVP: $y'' + 4y' + 4y = 0$ $\quad y(0) = 2$, $y'(0) = 0$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 4}}{2} = \frac{-4 \pm 0}{2} =, -2$$

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

initial conditions:

$$y(0) = 2 = c_1$$

$$y'(0) = 0 = -2c_1 + c_2(e^{-2t} - 2te^{-2t})\Big|_{t=0}$$

$$= -4 + c_2 \qquad \Rightarrow \qquad c_2 = 4$$

$$\boxed{y(t) = 2e^{-2t} + 4te^{-2t}}$$

Review: Complex Numbers

Square root of a negative number:

Suppose $w > 0$

$$\sqrt{-w} = i\sqrt{w}$$

i = imaginary unit

$$i \times i = -1$$

Suppose $a, b \in \mathbb{R}$

Complex Number: $z = a + ib$
Complex Conjugate: $\bar{z} = a - ib$

$$\frac{z + \bar{z}}{2} = \frac{2a}{2} = a = \mathrm{Re}(z)$$

$$\frac{z - \bar{z}}{2i} = \frac{2ib}{2i} = b = \mathrm{Im}(z)$$

Complex roots ($b^2 - 4ac < 0$) $\quad y_h = c_1 y_1 + c_2 y_2$

Roots are given by:

$$r_1 = \lambda + i\mu \qquad \text{where } i = \sqrt{-1}$$
$$r_2 = \lambda - i\mu$$

$$\lambda = \frac{-b}{2a}, \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

The two functions $y_1 = e^{(\lambda + i\mu)t}$ & $y_2 = e^{(\lambda - i\mu)t} = \bar{y}_1$ are solutions.

What is the exponential of a complex number?

Euler's formula:

$$e^{\pm i\alpha} = \cos\alpha \pm i \sin\alpha$$

$$
\begin{aligned}
y_{1,2} = e^{(\lambda \pm i\mu)t} &= e^{\lambda t} e^{\pm i\mu t} \\
&= \underbrace{e^{\lambda t}}_{\text{Real}} \underbrace{\left[\underbrace{\cos(\mu t)}_{\text{Real}} \pm \underbrace{i \sin(\mu t)}_{\text{Imaginary}}\right]}_{\text{Complex Conjugates}} \qquad y_1 = \bar{y}_2
\end{aligned}
$$

We don't want imaginary solutions

$$
\begin{aligned}
\tilde{y}_1 &= \frac{y_1 + y_2}{2} = \mathrm{Re}(y_1) = e^{\lambda t} \cos(\mu t) \\
\tilde{y}_2 &= \frac{y_1 - y_2}{2i} = \mathrm{Im}(y_1) = e^{\lambda t} \sin(\mu t)
\end{aligned}
$$

Complex roots ($r_{1,2} = \lambda \pm i\mu$)

The functions $y_1 = e^{\lambda t} \cos(\mu t)$ and $y_2 = e^{\lambda t} \sin(\mu t)$ are linearly independent real solutions.

Sketch the two functions if you are not convinced.

General solution: $y_h = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$

Find the general solution to: $y'' + 6y = 0$

$$r_{1,2} = \frac{\pm\sqrt{-4\cdot 6}}{2} = \pm\sqrt{-6} = \pm i\sqrt{6}$$

$$y_h = c_1 \cos\left(\sqrt{6}t\right) + c_2 \sin\left(\sqrt{6}t\right)$$

Solve the IVP: $y'' + 2y' + 5y = 0$ $\quad y(0) = 1$, $\quad y'(0) = -1$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}}{2} i = -1 \pm 2i$$

$$y_h = e^{-t} \left(c_1 \cos(2t) + c_2 \sin(2t) \right)$$

initial conditions:

$$y(0) = 1 = c_1$$

$$y'(0) = -1 = -c_1 + (-2c_1 \sin(0) + 2c_2 \cos(0)) = -c_1 + 2c_2$$

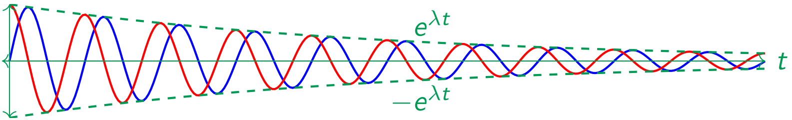
$$-1 = -1 + 2c_2 \qquad \Rightarrow \qquad c_2 = 0$$

$$\boxed{y(t) = e^{-t} \cos(2t)}$$

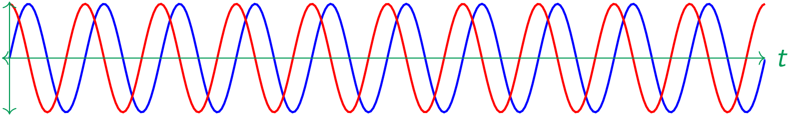
Qualitative Behaviour: complex roots

Three subcases:

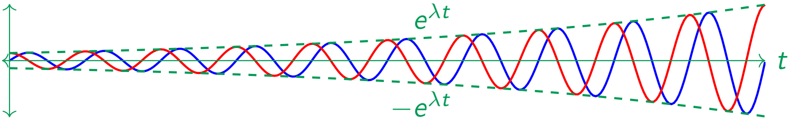
1. $\lambda < 0 \quad \Rightarrow \quad$ Exponentially decaying oscillations.



2. $\lambda = 0 \quad \Rightarrow \quad$ Sustained periodic oscillations.



3. $\lambda > 0 \quad \Rightarrow \quad$ Exponentially growing oscillations.



Summary

- For linear ODEs:
  - Pick an ansatz (e.g., e^{rt})
  - Write down the characteristic equation
  - Find the roots
    - If you don't have enough functions, make a new one by multiplying by t
- Write down the general solution according to the roots
  - Real and distinct $\Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - Real and repeated $\Rightarrow y_h = c_1 e^{rt} + c_2 t e^{rt}$
  - Complex $\Rightarrow y_h = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$
- Fit the constants c_1 and c_2 to the initial conditions