Autonomous DEs

A first order autonomous DE can be written as

$$\frac{dy}{dt} = f(y),$$

i.e., without any explicit time-dependence.

For such an autonomous DE, a point y^* is called a fixed point if

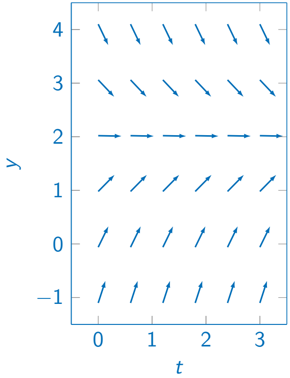
$$f(y^*) = 0$$

Then $y(t) = y^*$ is a constant solution

- an equilibrium solution (steady state)

ex: $y' = 2 - y$

$y = 2$ is a fixed point



Solutions to autonomous DEs flow towards/away from fixed points.

Example: Logistic Equation

Let $r > 0$ be the exponential growth rate of a population with size $y(t)$, and $K > 0$ be carrying capacity of its environment.

Logistic model of population dynamics:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(K - y\right)$$

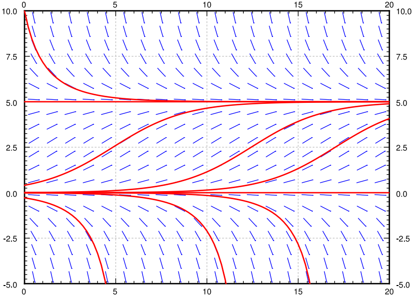
Two Equilibria:

$$ry^*\left(K - y^*\right) = 0$$

$$y^* = 0 \qquad \text{(unstable)}$$

and

$$y^* = K \qquad \text{(stable)}$$



Asymptotic Stability

A fixed point y^* is asymptotically stable, if there exists some initial condition $y_0 \neq y^*$ such that a solution to

$$\begin{aligned} \frac{dy}{dt} &= f(y) \\ \text{with } y(t_0) &= y_0 \end{aligned} \qquad \text{obeys} \qquad \lim_{t \to \infty} |y(t) - y^*| \to 0$$

That is, some solutions eventually converge to y^*.

ex: $\dfrac{dy}{dt} = ry(K - y)$

$y^* = 0$

only $y_0 = 0$ converges to $y^* = 0$
$\Rightarrow$ unstable

$y^* = K$

All $y_0 > 0$ converges to $y = K$
$\Rightarrow$ stable

Phase Lines (review from MATH 100)

For autonomous DEs, there is no horizontal variation in the slopefield.

- We can collapse the (t,y)-plane onto a 1D phase line.

How to draw a (horizontal) phase line diagram:

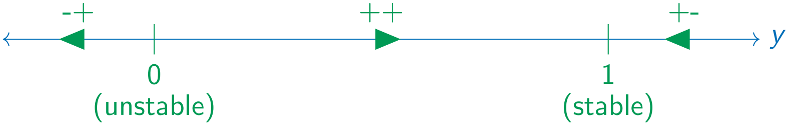
1. Identify all the fixed points y^* for the DE.

2. For each interval between the y^*s as well as $\pm\infty$ evaluate $f(y)$

   - Draw a rightward arrow if $f(y) > 0$

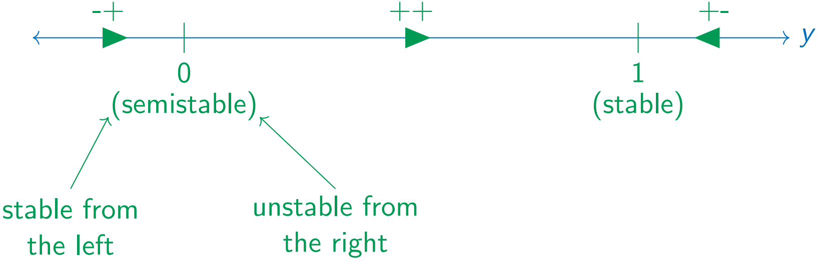
   - Draw a leftward arrow if $f(y) < 0$

ex: $y' = y(1-y)$ (evaluate sign of each term separately)



ex: $y' = y^2(1-y)$

Draw the phase line and classify the stability of the fixed points.



Euler's method (review from MATH 100)

Most DEs cannot be solved analytically.

In this case we can solve them numerically.

The simplest numerical method is called Euler's method.

Consider the first order DE

$$y' = f(y, t).$$

Use the approximation

$$y(t + \Delta t) \approx y(t) + f(y(t), t)\Delta t$$

with some finite Δt.

Euler's Method + Initial Value Problems

We can approximate numerical solutions to the differential equation

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

using the following iteration method:

$$\begin{aligned} y_1 &= y_0 + f(t_0, y_0)\Delta t \\ y_2 &= y_1 + f(t_1, y_1)\Delta t \\ &\vdots \\ y_{i+1} &= y_i + f(t_i, y_i)\Delta t, \end{aligned}$$

where y_i approximates $y(t_i)$ and $\Delta t = t_{i+1} - t_i$.

If you want to approximate $y(T)$ for $T > t_0$ using N steps, then $\Delta t = \frac{T - t_0}{N}$. In this way $y_N \approx y(T)$

Let $y(t)$ be the solution to the initial value problem

$$\frac{dy}{dt} = y, \quad y(0) = 1.$$

Use Euler's method to approximate $y(0.3)$ with step size $\Delta t = 0.1$.

t	$y(t)$	$f(y,t)$	$y(t) + f(y,t)\Delta t$
0	1	1	1+0.1
0.1	1.1	1.1	1.1+0.11
0.2	1.21	1.21	1.21+ 0.121
0.3	1.331		

Approximation Error

The exact solution of $y' = y$ with $y(0) = 1$ is $y = e^t$, so at $t = 0.3$ the solution will be $y(1) = e^{0.3} = 1.34986$.

$$\text{Error} = |y_{\text{approx}} - y_{\text{exact}}|$$

Step size	Numerical solution	Error
$\Delta t = 0.1$	$y(0.3) = y_3 = 1.331$	0.0188588
$\Delta t = 0.05$	$y(0.3) = y_6 = 1.3401$	0.00976317
$\Delta t = 0.025$	$y(0.3) = y_{12} = 1.344891$	0.00496998

Error Bounds: We can prove that for Euler's method

$$\text{Error} \leq c_1 \Delta t \qquad \Rightarrow \qquad \text{first-order method}$$

Higher order numerical schemes:

Improved Euler: Error $\leq c_2 \Delta t^2$ 2nd order method
Runge-Kutta (RK4): Error $\leq c_3 \Delta t^4$ 4th order method