Recall: Laplace Transform Workflow

Transforms the problem from the time domain to the "solution"-domain.

$$\begin{array}{cccc}
t-domain & \text{Unknown y(t) that solves ODE} & \xrightarrow{\text{Solve ODE}} & \text{Found y(t)} \\
 & \Big\downarrow \text{Transform y(t) and the ODE} & & \Big\uparrow \text{Invert the transform} \\
s-domain & \text{Unknown Y(s) that solves an algebraic equation} & \xrightarrow{\text{Solve alg. eq.}} & \text{Isolate Y(s)}
\end{array}$$

Laplace Transform as a mapping: $\mathcal{L} : f(t) \mapsto F(s)$

This mapping is bijective for most practical cases

Bijectivity of the Laplace Transform (not true for all functions)

$$\mathcal{L} : F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\text{time-domain} \;\rightleftarrows\; \text{s-domain}$$

$$\mathcal{L}^{-1} : y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$f(t) \longleftrightarrow F(s)$$

$$C \longleftrightarrow \frac{C}{s}$$

$$Ce^{at} \longleftrightarrow \frac{C}{s-a}$$

$$t^n \longleftrightarrow \frac{n!}{s^{n+1}}$$

$$\sin(\omega t) \longleftrightarrow \frac{\omega}{\omega^2+s^2}$$

$$\cos(\omega t) \longleftrightarrow \frac{s}{\omega^2+s^2}$$

$$\vdots \qquad\qquad \vdots$$

ex: Suppose $Y(s) = \frac{1}{s+6}$, find $y(t)$.

$$Y(s) = \frac{C}{s-a} \qquad C = 1, \quad a = -6$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} = e^{-6t}$$

ex: Suppose $G(s) = \frac{12}{s^2+16}$, find g(t).

$$G(s) = C\frac{\omega}{s+\omega^2} \qquad C = 3, \quad \omega = 4$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{12}{s^2+16}\right\} = 3\sin(4t)$$

ex: Suppose $Y(s) = \frac{s-1}{9+s^2}$, find $y(t)$

$$\begin{aligned} Y(s) &= \frac{-1}{9+s^2} + \frac{s}{9+s^2} \\ &= A\underbrace{\frac{\omega}{s^2+\omega^2}}_{\mathcal{L}\{\sin\omega t\}} + \underbrace{\frac{s}{s^2+\omega^2}}_{\mathcal{L}\{\cos\omega t\}} \end{aligned}$$

$$\omega = 3$$

$$= A\frac{3}{s^2+9} + \frac{s}{s^2+9}$$

$$A \cdot 3 = -1$$

$$Y(s) = \frac{-1}{3}\frac{3}{s^2+9} + \frac{s}{s^2+9}$$

$$A = -\frac{1}{3}$$

$$y(t) = -\frac{1}{3}\sin(3t) + \cos(3t)$$

ex: Suppose $H(s) = \frac{4}{s^2+6s+25}$, find h(t).

Kind of looks like previous example, but denominator is not quite right.

Try completing the square.

$$as^2 + bs + c = a(s+d)^2 + e \qquad \begin{array}{l} d = \frac{b}{2a} \\ e = c - \frac{b^2}{4a} \end{array}$$

$$\begin{aligned} H(s) &= \frac{4}{s^2 + 6s + 25} \\ &= \frac{4}{(s+3)^2 + 16} \end{aligned} \qquad \begin{array}{l} d = \frac{6}{2} = 3 \\ e = 25 - \frac{36}{4} = 16 \end{array}$$

Looks similar to LT of $sin(4t)$, but s is shifted.

First Shift Theorem: Multiplication by e^{at}

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)dt \qquad \text{let } \tilde{s} = s-a$$

$$\int_0^\infty e^{\tilde{s}t}f(t)dt = F(\tilde{s})$$

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)}$$

Take home:

If you recognize something in the s-domain but with $s \to s-a$, multiply by e^{at} in the time domain.

ex: Suppose $H(s) = \frac{4}{(s+3)^2+16}$, find h(t).

$$H(s) = \left.\frac{4}{s^2+16}\right|_{s=s+3} \quad = \quad \mathcal{L}\left\{\sin(4t)\right\}\Big|_{s=s+3}$$

$$h(t) = e^{-3t}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = e^{-3t}\sin(4t)$$

ex: Suppose $Y(s) = \frac{3}{s(s+6)} + \frac{2}{s+6}$, find $y(t)$.

$$Y(s) = \underbrace{\frac{3}{s(s+6)}}_{\mathcal{L}\{???\}} + 2\mathcal{L}\left\{e^{-6t}\right\}$$

Partial fraction decomposition

$$\frac{3}{s(s+6)} = \frac{A}{s} + \frac{B}{s+6}$$

multiply by denominator

$$\begin{aligned} 3 &= A(s+6) + B \cdot s \\ &= 6A + (A+B)s \end{aligned}$$

True for all $s \Rightarrow$ coefficients must match

constant terms: $3 = 6A$ $\qquad A = \frac{1}{2}$

s terms: $0 = A + B$ $\qquad B = -A = -\frac{1}{2}$

$$
\begin{aligned}
Y(s) &= \frac{1}{2s} - \frac{1}{2}\frac{1}{s+6} + \frac{2}{s+6} \\
&= \frac{1}{2s} + \frac{3}{2}\frac{1}{s+6} \\
&= \mathcal{L}\left\{\frac{1}{2}\right\} + \frac{3}{2}\mathcal{L}\left\{e^{-6t}\right\}
\end{aligned}
$$

$$
y(t) = \frac{1}{2} + \frac{3}{2}e^{-6t}
$$

ex: Suppose $Y(s) = \frac{s+4}{(s-4)^2(s+1)}$, find $y(t)$

$$\frac{s+4}{(s-4)^2(s+1)} = \frac{A}{(s-4)^2} + \frac{B}{(s-4)} + \frac{C}{(s+1)}$$

$$s+4 = A(s+1) + B\underbrace{(s-4)(s+1)}_{s^2-3s-4} + C\underbrace{(s-4)^2}_{s^2-8s+16}$$

$$s+4 = (B+C)s^2 + (A-3B-8C)s + A - 4B + 16C$$

match coeffiecients

$$\underline{s^2:} \quad B + C = 0 \quad \Rightarrow B = -C \qquad \boxed{B = -\frac{3}{25}}$$

$$\underline{s:} \quad A - 3B - 8C = 1$$

$$A - 5C = 1 \quad \Rightarrow A = 1 + 5C \qquad \boxed{A = \frac{40}{25}}$$

$$\underline{s^0:} A - 4B + 16C = 4$$

$$1 + 25C = 4 \Rightarrow C = \frac{3}{25}$$

$$Y(s) = \frac{40}{25}\underbrace{\frac{1}{(s-4)^2}}_{\mathcal{L}\{t\}\Big|_{s=s-4}} - \frac{3}{25}\underbrace{\frac{1}{(s-4)}}_{\mathcal{L}\{e^{4t}\}} + \frac{3}{25}\underbrace{\frac{1}{(s+1)}}_{\mathcal{L}\{e^{-t}\}}$$

$$\begin{aligned} y(t) &= \frac{40}{25}e^{4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{25}e^{4t} + \frac{3}{25}e^{-t} \\ &= \underbrace{\frac{40}{25}te^{4t}}_{y_p} \underbrace{- \frac{3}{25}e^{4t} + \frac{3}{25}e^{-t}}_{y_h} \end{aligned}$$

Next lecture, we learn how to find Y(s) from an ODE.

ex: Suppose $Y(s) = \frac{s-4}{((s+3)^2+16)(s+4)}$, find $y(t)$

$$\frac{s-4}{((s+3)^2+16)(s+4)} = \frac{As+B}{(s+3)^2+16} + \frac{C}{s+4}$$

$$s-4 = (As+B)(s+4) + C(s^2+6s+25)$$

$$s-4 = (A+C)s^2 + (4A+B+6C)s + 4B + 25C$$

match coeffiecients

$$\underline{s^2:} \quad A+C=0 \quad \Rightarrow A=-C \qquad \boxed{A=\frac{8}{17}}$$

$$\underline{s:} \quad 4A+B+6C=1$$

$$B+2C=1 \quad \Rightarrow B=1-2C \qquad \boxed{B=\frac{33}{17}}$$

$$\underline{s^0:} \quad 4B+25C=-4$$

$$4+17C=-4 \Rightarrow \boxed{C=\frac{-8}{17}}$$

$$Y(s) = \frac{8}{17}\underbrace{\frac{s}{(s+3)^2+16}}_{\mathcal{L}\{e^{-3t}\cos(4t)\}} + \frac{33}{17}\underbrace{\frac{1}{(s+3)^2+16}}_{\frac{1}{4}\mathcal{L}\{e^{-3t}\sin(4t)\}} - \frac{8}{17}\underbrace{\frac{1}{s+4}}_{\mathcal{L}\{e^{-4t}\}}$$

$$y(t) = \underbrace{e^{-3t}\left(\frac{8}{17}\cos(4t) + \frac{33}{68}\sin(4t)\right)}_{y_h} - \underbrace{\frac{8}{17}e^{-4t}}_{y_p}$$

Next lecture, we learn how to find Y(s) from an ODE.

Summary: Inverting the Laplace transform of a function

ex: $Y(s) = \frac{s+4}{(s-4)^2(s+1)}$

1. Do some algebra to get a sum of "easy" terms
   - partial fraction decomposition
   - completing the square

   ex: $Y(s) = \frac{8}{5}\frac{1}{(s-4)^2} - \frac{1}{5}\frac{1}{(s-4)} + \frac{1}{5}\frac{1}{(s+1)}$

2. Transform back from $Y(s)$ to $y(t)$ using Laplace transform tables
   - Tackle each term in the sum individually.
   - Go slowly when applying shift theorems

ex:

$$\begin{aligned} y(t) &= \frac{8}{5}e^{4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{5}e^{4t} + \frac{1}{5}e^{-t} \\ &= \frac{8}{5}te^{4t} - \frac{1}{5}e^{4t} + \frac{1}{5}e^{-t} \end{aligned}$$