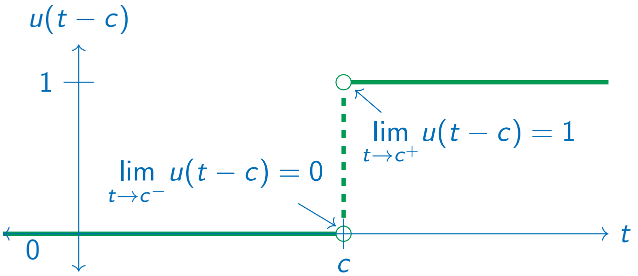
The Heaviside Step Function: $u(t-c)$ or $H(t-c)$ or $u_c(t)$

Used to model effects that "turn-on" at some time c.

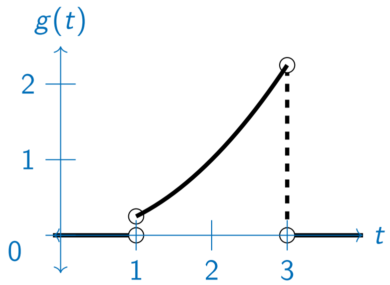


$$u(t-c) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t > c \end{cases}$$

Write the piecewise function

$$g(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{4}t^2 & 1 < t < 3 \\ 0 & t > 3 \end{cases}$$

in terms of the heaviside function.

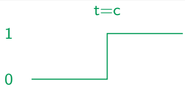


$$g(t) = \frac{1}{4}t^2 \underbrace{u(t-1)}_{\substack{\text{turn on} \\ \text{at } t=1}} \times \underbrace{(1-u(t-3))}_{\substack{\text{turn off} \\ \text{at } t=3}}$$

or

$$g(t) = \frac{1}{4}t^2 \underbrace{u(t-1)}_{\substack{\text{turn on} \\ \text{at } t=1}} - \frac{1}{4}t^2 \underbrace{u(t-3)}_{\substack{\text{turn off} \\ \text{at } t=3}}$$

Laplace Tranform of the Heaviside Function



$$\mathcal{L}\left\{u(t-c)\right\} = \int_0^\infty e^{-st}u(t-c)dt = \lim_{A\to c^+}\int_A^\infty e^{-st}dt$$

$$\overset{s\geq 0}{=} \lim_{A\to c^+} \frac{1}{s}e^{-sA}$$

$$= \boxed{e^{-sc}\frac{1}{s}} = e^{-sc}\mathcal{L}\left\{1\right\}$$

How about the more general pattern $e^{-sc}F(s)$?

Second Shift Theorem

$$\boxed{\mathcal{L}\left\{f(t-c)u(t-c)\right\} = e^{-sc}F(s)}$$

Proof of Second Shift Theorem

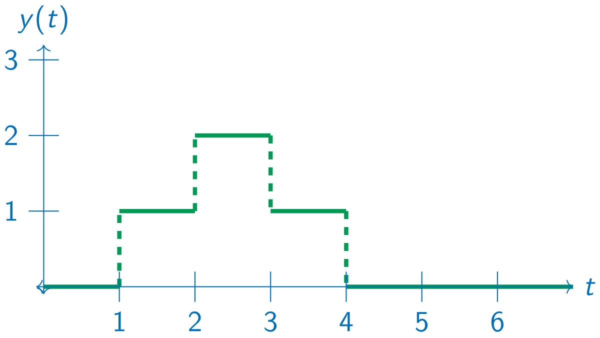
$$\begin{aligned}
\mathcal{L}\{f(t-c)u(t-c)\} &= \int_0^\infty e^{-st}\underbrace{f(t-c)u(t-c)}_{0 \text{ for } t<c}dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&= \int_0^\infty e^{-s(v+c)}f(v)dv \\
&= e^{-sc}\int_0^\infty e^{-sv}f(v)dv = e^{-sc}\mathcal{L}\{f(t)\}
\end{aligned}$$

$$v = t - c$$
$$dv = dt$$

$$\boxed{\mathcal{L}\{f(t-c)u(t-c)\} = e^{-sc}F(s)}$$

ex: Suppose $Y(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$, find and sketch $y(t)$.

$$\begin{aligned}
Y(s) &= \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} \\
&= e^{-s}\mathcal{L}\{1\} + e^{-2s}\mathcal{L}\{1\} - e^{-3s}\mathcal{L}\{1\} - e^{-4s}\mathcal{L}\{1\} \\
&= [u(t)\cdot 1]\Big|_{t=t-1} \cdots \\
&= u(t-1) + u(t-2) - u(t-3) - u(t-4)
\end{aligned}$$



ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+s^2}$, find $y(t)$.

$$\begin{aligned} y(t) &= \left[u(t)\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\ &= u(t-4)\left[\sin(3t)\right]_{t=t-4} \\ &= u(t-4)\sin(3(t-4)) \end{aligned}$$

ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+(s+11)^2}$, find $y(t)$.

$$\begin{aligned} y(t) &= \left[u(t)\mathcal{L}^{-1}\left\{\frac{3}{9+(s+11)^2}\right\}\right]_{t=t-4} \\ &= u(t-4)\left[e^{-11t}\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\ &= u(t-4)e^{-11(t-4)}\left[\sin(3t)\right]_{t=t-4} \end{aligned}$$

Find the Laplace transform of

$$g(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{4}t^2 & 1 < t < 3 \\ 0 & t > 3 \end{cases}$$

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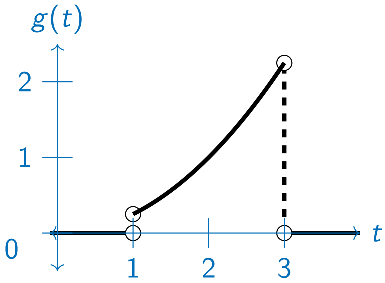


$$\begin{aligned} g(t) &= \frac{1}{4}t^2 u(t-1) - \frac{1}{4}t^2 u(t-3) \\ &= f_1(t-1)u(t-1) + f_2(t-3)u(t-3) \\ G(s) &= e^{-s}F_1(s) - e^{-3s}F_2(s) \end{aligned}$$

Need to find f_1 and f_2. Let $z_1 = t - 1 \quad \Rightarrow \quad t = z_1 + 1$

$$f_1(t) = \frac{1}{4}t^2 = \frac{1}{4}(z_1+1)^2$$

$$f_1(z_1) = \frac{1}{4}z_1^2 + \frac{1}{2}z_1 + \frac{1}{4} \quad \Rightarrow F_1(s) = \frac{1}{2s^3} + \frac{1}{2s^2} + \frac{1}{4s}$$

Let Let $z_2 = t - 3 \quad \Rightarrow \quad t = z_2 + 3$

$$\begin{aligned} f_2(t-3) &= \frac{1}{4}t^2 = \frac{1}{4}(z_2+3)^2 \\ &= \frac{1}{4}(z_2^2 + 6z_2 + 9) \Rightarrow F_2(s) = \frac{1}{2s^3} + \frac{3}{2s^2} + \frac{9}{4s} \end{aligned}$$

$$G(s) = \left(\frac{1}{2s^3} + \frac{1}{2s^2} + \frac{1}{4s}\right)e^{-s} - \left(\frac{1}{2s^3} + \frac{3}{2s^2} + \frac{9}{4s}\right)e^{-3s}$$

ex: $y' + 6y = \begin{cases} 0 & t < 1 \\ \frac{1}{4}t^2 & t > 1 \end{cases}$ with $y(0) = 1$

$$sY(s) - 1 + 6Y(s) = \left(\frac{1}{2s^3} + \frac{1}{2s^2} + \frac{1}{4s}\right)e^{-s}$$

$$Y(s) = \left(\frac{1}{2(s+6)s^3} + \frac{1}{2(s+6)s^2} + \frac{1}{4s(s+6)}\right)e^{-s} + \frac{1}{s+6}$$

Apply partial fraction decomp. to each term

$$\frac{1}{2s^3(s+6)} = -\frac{1}{72s^2} + \frac{1}{12s^3} + \frac{1}{432s} - \frac{1}{432(s+6)}$$

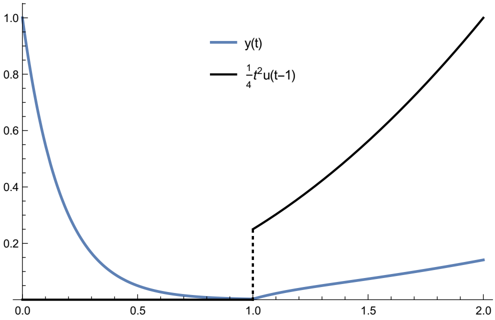
$$\frac{1}{2s^2(s+6)} = \frac{1}{12s^2} - \frac{1}{72s} + \frac{1}{72(s+6)}$$

$$\frac{1}{4s(s+6)} = \frac{1}{24s} - \frac{1}{24(s+6)}$$

$$Y(s) = \frac{1}{432}\left(\frac{36}{s^3} + \frac{30}{s^2} + \frac{13}{s} - \frac{13}{s+6}\right)e^{-s} + \frac{1}{s+6}$$

$$Y(s) = \frac{1}{432}\left(\frac{36}{s^3} + \frac{30}{s^2} + \frac{13}{s} - \frac{13}{(s+6)}\right)e^{-s} + \frac{1}{s+6}$$

$$\begin{aligned} y(t) &= \frac{1}{432}u(t-1)\left[18t^2 + 30t + 13 - 13e^{6t}\right]_{t=t-1} + e^{-6t} \\ &= \frac{1}{432}u(t-1)\left(18(t-1)^2 + 30(t-1) + 13 - 13e^{6(t-1)}\right) + e^{-6t} \end{aligned}$$



Products in Laplace Space

The second shift theorem is a special case of a product in Laplace space

$$\mathcal{L}\left\{f(t-c)u(t-c)\right\} = \underbrace{e^{-sc}}_{G(s)}F(s)$$

Inversion of the general case requires the use of a convolution

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

ex: $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\frac{2}{s^2}\right\}$ Note: $\mathcal{L}^{-1}\left\{e^{-3s}\frac{2}{s^3}\right\} = u(t-3)(t-3)^2$

$$= \int_0^t u(\tau-3)2(t-\tau)d\tau = \begin{cases} 0 & t < 3 \\ \int_3^t 2(t-\tau)d\tau & t > 3 \end{cases}$$

$$= \begin{cases} 0 & t < 3 \\ \left[2t\tau - \tau^2\right]_{\tau=3}^{t} & t > 3 \end{cases} = u(t-3)(t-3)^2$$