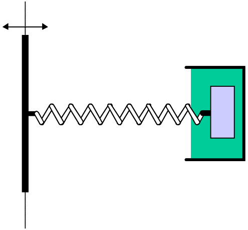
Derivation of spring-dashpot ODE:



$x(t)$ = displacement from rest position

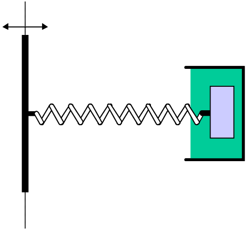
- $x = 0 \Rightarrow$ no elastic restoring force

Newton's 2^{nd} Law:

$$F = ma \qquad \text{where } a = \frac{d^2x}{dt^2}$$

$$
\begin{aligned}
F &= \text{sum of forces} \\
&= \underbrace{\text{elastic restoring force}}_{\substack{\text{Hooke's Law} \\ = -kx}} + \underbrace{\text{drag force}}_{\substack{\text{opposes motion} \\ = -\beta \frac{dx}{dt}}} + \underbrace{\text{external forces}}_{f(t)} \\
&= -kx - \beta\frac{dx}{dt} + f(t)
\end{aligned}
$$

Derivation of spring-dashpot ODE:



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Newton's 2^{nd} Law:

$$F = ma \qquad \text{where } a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

$$F = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t)$$

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t)$$

$$\boxed{mx'' + \beta x' + kx = f(t)}$$

Simple Harmonic Motion

Spring with no damping and no forcing, i.e.

$$mx'' + kx = 0$$

- Similar: frictionless pendulum, circuit with no resistance. . . .
- General solution: Homogeneous problem

$$\text{Guess}: x(t) = e^{rt} \qquad mr^2e^{rt} + ke^{rt} = 0$$

$$mr^2 + k = 0$$

$$r = \pm\frac{\sqrt{-4km}}{2m} \quad = \pm i\sqrt{\frac{k}{m}}$$

$$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t); \qquad \omega_0 = \sqrt{k/m}$$

Simple Harmonic Motion: $mx'' + kx = 0$

- General solution:

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad \omega_0 = \sqrt{k/m}$$

- All solutions have periodic motion with period: $T = \frac{2\pi}{\omega_0}$

- The quantity ω_0 is called the **natural frequency**.

- Obtaining a unique solution requires two initial conditions:
  - $x(0) =$ initial displacement away from rest position.
  - $x'(0) =$ initial velocity of the mass

Amplitude-phase form of solution:

Solution can also be expressed as

$$x(t) = R\cos(\omega_0 t - \varphi)$$

- R = amplitude of motion
    - max displacement
- φ = phase angle (initial phase)
    - by convention $-\pi < \varphi < \pi$

Gen Solution: $R\cos(\omega t-\varphi) = \quad c_1\cos(\omega t) \quad + \quad c_2\sin(\omega t)$

Trig ident: $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$\Rightarrow \quad R\cos(\omega t-\varphi) = R\left[\cos(\varphi)\cos(\omega t) + \sin(\omega t)\sin(\varphi)\right]$

by visual inspection

$$R\cos\varphi = c_1 \tag{1}$$

$$R\sin\varphi = c_2 \tag{2}$$

(2)/(1):

$$\frac{R\sin\varphi}{R\cos\varphi} = \tan\varphi = \frac{c_2}{c_1}$$

$$\Rightarrow \boxed{\varphi = \arctan\left(\frac{c_2}{c_1}\right) \text{ if } c_1 > 0}$$

Lookup "atan2" for details

$(1)^2 + (2)^2$:

$$R^2\underbrace{\left(\cos^2\varphi + \sin^2\varphi\right)}_{1} = c_1^2 + c_2^2$$

$$R^2 = c_1^2 + c_2^2$$

$$\boxed{R = \pm\sqrt{c_1^2 + c_2^2}}$$

Check at $t = 0$ to determine + or -

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 1cm and released with a velocity of 14 cm/s at $t = 0$.

1. Find the displacement for $t > 0$.
2. Find the natural frequency, period, amplitude, and phase angle of motion.

$$3x'' + 15x = 0$$

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$w_0 = \sqrt{k/m} = \sqrt{15/3} = \sqrt{5}$$

Initial Conditions:

$$x(0) = 0.01\text{m} = c_1$$

$$x'(0) = 0.14\text{m/s} = \omega_0 c_2 = \sqrt{5} c_2$$

$$c_2 = \frac{0.14}{\sqrt{5}}$$

$$\boxed{x(t) = 0.01\cos\sqrt{5}t + \frac{0.14}{\sqrt{5}}\sin\sqrt{5}t}$$

Amplitude and phase angle

$$x(t) = R\cos(\omega_0 t - \phi)$$

$$\begin{aligned} R &= \sqrt{c_1^2 + c_2^2} \\ &= \sqrt{0.01^2 + \frac{0.14^2}{5}} \\ &\approx 0.0626 \end{aligned}$$

$$\begin{aligned} \varphi &= \arctan\left(\frac{c_2}{c_1}\right) \\ &= \arctan\left(\frac{14}{\sqrt{5}}\right) \\ &\approx 1.412 \end{aligned}$$

$$x(t) \approx 0.0626\cos\left(\frac{t}{\sqrt{5}} - 1.412\right)$$

Free oscillations with damping

$$x'' + \beta x' + kx = 0$$

What happens for small and large β?

- $\beta \to 0$, no damping $\Rightarrow$ recover simple harmonic oscillations
- $\beta \to \infty$, infinite damping $\Rightarrow$ no oscillations

Characterisitc equation: $mr^2 + \beta r + k = 0$ has two roots

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Note: Since $\beta > 0$, the real part of $r_{1,2}$ is always negative
$\Rightarrow$ exponentially decaying solutions.

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Three cases:

1. $\beta^2 < 4km$: roots are complex conjugates
   - exp * (sin + cos) solutions - **underdamped** motion

2. $\beta^2 > 4km$: two distinct real (negative) roots
   - (bi-exponential solutions) - **overdamped** motion

3. $\beta^2 = 4km$: repeated real root
   - ($e^{rt} + te^{rt}$ solutions) - **critically damped** motion

Underdamped Motion

General solution:

$$x(t) = e^{-\frac{\beta}{2m}t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right)$$

where $\omega_1 = \sqrt{k/m - (\beta/2m)^2} \leq \sqrt{k/m}$ (damping slows the oscillations)

We can write this in amplitude-phase form

$$x(t) = e^{-\frac{\beta}{2m}t} R \cos(\omega_1 t - \varphi)$$

- $e^{-\frac{\beta}{2m}t} R$ is the time-varying amplitude (or amplitude)
- ω_1 is called the quasi-frequency of motion
  - $T = 2\pi/\omega_1$ is the quasi-period of motion
- φ is the phase angle of motion (or phase shift)

Again, we have $\varphi = \arctan\left(\dfrac{c_2}{c_1}\right)$, $R = \sqrt{c_1^2 + c_2^2}$

Overdamped Motion:

General Solution:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

with $r_1 < r_2 < 0$.

Critically-Damped: $\beta = 2\sqrt{k \cdot m}$

General Solution:

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

with $r = \frac{-\beta}{2m} = -\sqrt{k/m} < 0$.

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

1. Determine the value of the damping consant β for which the system is critically damped.
2. Find the displacement $x(t)$ of the mass if the system is critically damped, assuming an inital velocity of 0 cm/s.

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

1. Determine the value of the damping consant β for which the system is critically damped.

$$\begin{aligned} \text{Critical damping:} \quad \beta^2 &= 4km \\ &= 4 \cdot 3 \cdot 15 = 180 \\ \beta &= \sqrt{180} \approx 13.416 \end{aligned}$$

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

2. Find the displacement $x(t)$ of the mass if the system is critically damped, assuming an inital velocity of 0 cm/s.

$$x(t) = c_1 e^{-\sqrt{\frac{k}{m}}t} + c_2 t e^{-\sqrt{\frac{k}{m}}t} = c_1 e^{-\sqrt{5}t} + c_2 t e^{-\sqrt{5}t}$$

Initial conditions:

$$\begin{aligned}
x(0) &= 0.03\text{m} = c_1 \\
x'(0) &= 0 = -\sqrt{5}c_1 + c_2\left(1 - \sqrt{5} \cdot 0 \cdot 1\right) = -\sqrt{5}c_1 + c_2 \\
c_2 &= \sqrt{5}c_1 = \sqrt{5} \cdot 0.03 \approx 0.067 \\
x(t) &= 0.03e^{-\sqrt{5}t} + 0.067te^{-\sqrt{5}t}
\end{aligned}$$