Recall: Solutions flows and fixed points (critical points)

The origin $\vec{0}$ is the unique fixed point for the linear system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$.

How about the (constant) non-homogeneous linear system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + \vec{f}$?

$$\vec{0} = \mathbf{A}\vec{x}^* + \vec{f} \quad \Rightarrow \quad \vec{x}^* = -\mathbf{A}^{-1}\vec{f}$$

is the fixed point
or critical point

A constant inhomogeneity just shifts everything from the origin.

The global behaviour of solutions is fully determined by the eigenvalues and eigenvectors of $\mathbf{A}$:

1. Eigendirections with real eigenvalues and:
   - $\lambda < 0$ attract solutions towards the fixed point.
   - $\lambda > 0$ repel solutions away from the fixed point.
2. Complex conjugate eigenvalues produce oscillations.

2-Dimensional Autnomous Nonlinear Systems

Consider the autonomous nonlinear system

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(x,y), \qquad \frac{\mathrm{d}}{\mathrm{d}t}y = g(x,y)$$

where f and g are nonlinear function with no explicit time-dependence.

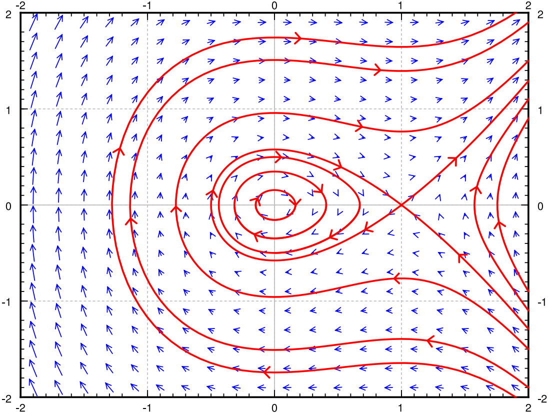
This can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \vec{f}(\vec{x})$$

We can plot the vector field $\begin{bmatrix} x' \\ y' \end{bmatrix}$ over the (x,y)-plane.

This vector field gives us qualitative information about solution flows.

ex: The vector field for the the system $x' = y$ and $y' = -x + x^2$



This system has two critical points:

$$x = y = 0 \qquad x = 1, \quad y = 0$$
(center) (saddle)

Linearization & Local Solution Flow

To classifiy critical points of a nonlinear system, we linearize it around its critical points $\vec{x}^*$ (e.g., for a 2D system $\vec{x}^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$).

Any solution that starts at $\vec{x}^*$ stays there forever.

So we perturb the fixed point by setting $\vec{x}(t) = \vec{x}^* + \vec{u}(t)$ where $\vec{u}(0)$ is infinitesimally small.

Thus we are interested in

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \vec{f}(x), \quad \text{with } \vec{x}(0) = \vec{x}^* + \vec{u}(0) \quad \text{where} \quad \vec{u}(0) \approx \vec{0}$$

expanding $\vec{x}(t)$ we get

$$= \vec{f}(\vec{x}^* + \vec{u})$$

Linearization & Local Solution Flow

To classifiy critical points of a nonlinear system, we linearize it around its critical points $\vec{x}^*$ (e.g., for a 2D system $\vec{x}^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{u} = \vec{f}(\vec{x}^* + \vec{u}) \quad \text{where} \quad \vec{u}(0) \approx \vec{0}$$

make a first-order Taylor expansion in $\vec{u}$ around $\vec{u} = \vec{0}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} \approx \cancelto{\vec{0}}{\vec{f}(\vec{x}^*)} + \left.\frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}^*} \vec{u} + \cdots + h.o.t.$$

where is $\dfrac{\partial \vec{f}}{\partial \vec{x}}$ called the Jacobian matrix. For our 2D system, it is given by

$$\mathbf{J} = \frac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial}{\partial x} f(x,y) & \dfrac{\partial}{\partial y} f(x,y) \\ \dfrac{\partial}{\partial x} g(x,y) & \dfrac{\partial}{\partial y} g(x,y) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

Linearization & Local Solution Flow

To classifiy critical points of a nonlinear system, we linearize it around its critical points $\vec{x}^*$ (e.g., for a 2D system $\vec{x}^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$).

After the subsititution $\vec{u} = \vec{x} - \vec{x}^*$, near the critical point $\vec{x}^*$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} \approx \mathbf{J}^* \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x^* \\ y^* \end{bmatrix} \right) \quad \text{where } \mathbf{J}^* = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} \approx \mathbf{J}^*\vec{x} - \underbrace{\mathbf{J}^*\vec{x}^*}_{\text{constant}} = \text{a (constant) non-homogeneous linear system}$$

This linear approximation holds locally.

Thus the local behaviour of solutions near critical points is determined by the eigenvalues and eigenvectors of $\mathbf{J}$.

ex: Find the Jaobian matrix for the system

$$x' = f(x,y), \quad y' = g(x,y) \quad \text{with} \quad f(x,y) = y, \quad g(x,y) = -x + x^2$$

$$\mathbf{J} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2x-1 & 0 \end{bmatrix}$$

ex: Find a linear approximation to the above system at the two following critical points

$\underline{x = y = 0}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\underline{x = 1, \quad y = 0}$

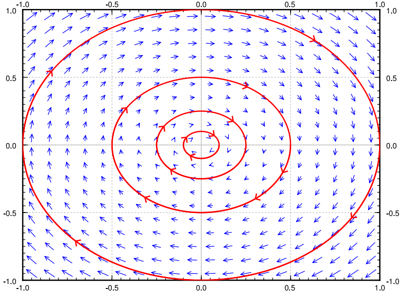
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y \end{bmatrix}$$

Approximate Local Solution Flows

$\underline{x = y = 0}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues: $\pm i$ (center)



$\underline{x = 1, \quad y = 0}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y \end{bmatrix}$$

Eigenvalues: ± 1 (saddle)