Recall: Linear 1^{st} Order ODEs $\quad y' + p(t)y = h(t)$

Operator form: $\mathrm{L}[y] = h(t)$

- Linear 1^{st} order operator L
- $h(t)$ is a forcing term **(inhomogeneity)**
- Linear + $h(t) = 0 \quad \Rightarrow$ **Homogeneous ODE**
- Linear + $h(t) \neq 0 \quad \Rightarrow$ **Inhomogeneous ODE**

Solved by method of integrating factors:

$$\mu(t)y(t) = \int \mu(t)h(t)dt + C$$

$$y(t) = \frac{\int \mu(t)h(t)dt}{\mu(t)} + \underbrace{\frac{C}{\mu(t)}}_{\text{indep. of } h(t)}$$

General Solution Structure of Linear ODEs

ex: 1^{st} Order Initial Value Problems

$$y' + p(t)y = h(t); \qquad y(0) = y_0$$

Associated Homogeneous Problem:

$$y_h' + p(t)y_h = 0$$

$$\Rightarrow \quad y_h = \frac{C}{\mu(t)}$$

Inhomogeneous DE:

$$y(t) = \frac{\int \mu(t)h(t)dt}{\mu(t)} + y_h(t)$$

$$y(t) = \underbrace{y_p(t)}_{\text{particular part}} + \underbrace{y_h(t)}_{\text{homogeneous part}}$$

All linear DEs have this type of solution structure.

$$y(t) = \text{particular part} + \text{homogeneous part}$$

Linear 2^{nd} order ODEs

General DE:

$$y'' + p(t)y' + q(t)y = h(t)$$

Initial Conditions:

$$y(t_0) = y_0, \qquad y'(t_0) = v_0$$

Focus on constant coefficient case first

simplest case: homogeneous

$$ay'' + by' + cy = 0$$

We want intuition for how homogenoues solutions work.

For non-constant coefficients: method of reduction of order (DiffyQs §2.1)

Superposition Principle for Linear Homogeneous ODEs

Suppose the functions $y_1(t)$ and $y_2(t)$ both independently solve a linear homogeneous ODE

$$\mathrm{L}[y] = 0$$

then

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution to the same ODE.

Proof:

$$\begin{aligned}
\mathrm{L}[c_1 y_1 + c_2 y_2] &\overset{\text{Linearity 1}}{=} \mathrm{L}[c_1 y_1] + \mathrm{L}[c_2 y_2] \\
&\overset{\text{Linearity 2}}{=} c_1 \mathrm{L}[y_1] + c_2 \mathrm{L}[y_2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}$$

Completeness of solutions

Suppose that $y_1(t)$ and $y_2(t)$ both solve a 2^{nd} order linear homogeneous ODE

$$\mathrm{L}[y] = 0$$

If y_1 and y_2 are linearly independent functions, then the general solution to the homogeneous problem is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

Take home message: If you can find two linearly independent solutions to a homogeneous 2^{nd} order linear DE, then you have found all of them

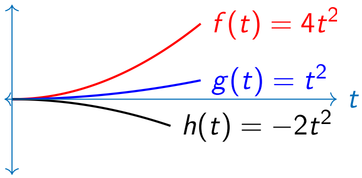
Proof: DiffyQs §2.1 (Theorem 2.1.3)

Linear dependence of functions

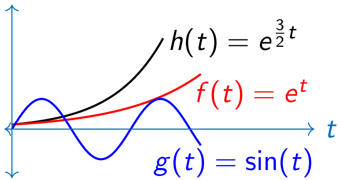
- Two functions $f(t)$ and $g(t)$ are **linearly dependent** on the interval $t \in I = [\alpha, \beta]$ if there exist a non-zero constant k such that

$$f(t) = kg(t) \quad \forall t \in I$$

Linearly Dependent



Not Linearly Dependent



- If functions are not linearly dependent on I, then we say they are **linearly independent** on I.

Solutions to $ay'' + by' + cy = 0$

Lets try an ansatz of $y(t) = e^{rt}$ ... where r is unknown (ansatz method)

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{rt} + b\frac{\mathrm{d}}{\mathrm{d}t}e^{rt} + ce^{rt} = 0$$

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \qquad e^{rt}\left(ar^2 + br + c\right) = 0$$

Since $e^{rt} \neq 0$, the ODE can only have a solution $y(t) = e^{rt}$ if

$$ar^2 + br + c = 0.$$

This is called the **characteristic equation**.

Possible values of r:

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots of the characteristic equation (polynomial)

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three main cases:

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

2. Repeated real roots: discriminant $= 0$

3. Complex conjugate roots: discriminant < 0

Case 1: distinct real roots

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}; \qquad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two major subcases:

1. $ac > 0$:

   Both roots have the same sign.

   y_1 and y_2 both grow or decay exponentially.

2. $ac < 0$:

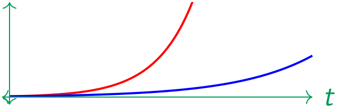
   The two roots have opposite sign.

   One solution grows exponentially, the other decays exponentially.

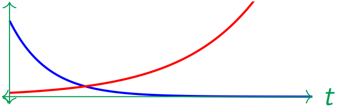
Qualitative Behaviour: distinct real roots

Sum of real exponential functions, three subcases:

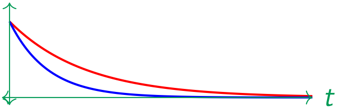
1. All positive roots, $0 < r_1 < r_2$.



2. Mixed roots, $r_1 < 0 < r_2$.



3. All negative roots, $r_1 < r_2 < 0$.



Summary

- Superposition of Homogeneous solutions

$$y_h = c_1 y_1 + c_2 y_2$$

- Linear independence of homoegeneous solutions

$$y_g = c_1 y_1 + c_2 y_2 \quad \Rightarrow \quad y_1 \neq k y_2; \quad k = \text{constant}$$

  y_g = general solution, solves ALL scenarios where a solution exists.

- $ay'' + by' + cy = 0$

  Try ansatz $y = e^{rt}$ $\quad \rightarrow \quad e^{rt} \cdot \underbrace{(ar^2 + br + c)}_{\text{characterisitc polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$