Differential Equations

1. What are they and why do we solve them?
2. Terminology
3. Graphical intuition and the direction field

What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. an independent variable.

Given: A DE with an unknown function $y(t)$.

e.x., $$\frac{\mathrm{d}y}{\mathrm{d}t} = -3y(t)$$

<u>or</u>

$$y' = -3y$$

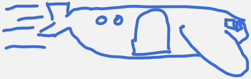
Task: Find the function(s) $y(t)$.

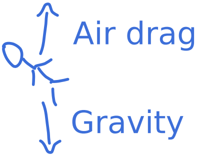
Solution: $y(t) = C_1 e^{-3t}$

Tools:

- Calculus (i.e., integration/differentiation)
- Geuss and check (does some function $f(t)$ make LHS=RHS?)
- Specialized procedures (informed by experience geussing)
- Geometry/Linear Algebra (useful for systems of DEs)

Example: Skydiving





Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v} \qquad \text{DE for } v(t)$$

Example: Epidemiology

Kermack & McKendrick's SIR model

- Susceptible → Infected → Recovered

System of 3 ordinary differential equations:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu(I + R) - \beta SI$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - \gamma I - \mu I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R$$

$t =$ time
$\beta =$ infection rate
$\gamma =$ recovery rate
$\mu =$ birth/death rate

Reproduction number: $R_0 = \frac{\beta}{\mu+\gamma}$

1. $R_0 > 1$: endemic equilibrium
2. $R0 < 1$: disease dies out - basic idea behind "flatten the curve"

Terminology

- Ordinary differential equations (ODE's)
  - A DE with derivatives w.r.t. only one independent variable.
  - If multiple derivatives (e.g., $\partial/\partial t$ and $\partial/\partial x$) we have partial DE (PDE).
- System of differential equations
  - A set (or system) of interdependent DEs.
    - Cannot solve one DE without solving the others.
- Order of a differential equation
  - Order of the highest derivative in the equation.
- Solution of a differential equation
  - A function that satisfies the eq (i.e., makes LHS=RHS) for all values of the independent variable.

Terminology continued

- General Solution
  - A solution with arbitrary constants that can solve all scenarios where a solution exists.

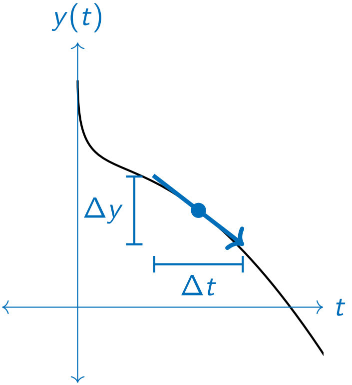
- Arbitrary Constant
  - A constant that arises while solving the ODE (as opposed to a parameter in the equation)

ex: $y' = a$

$$\text{General Solution: } y(t) = at + b$$
$$b = \text{arbitrary constant}$$
$$a = \text{parameter}$$

- Particular solution
  - A solution with NO arbitrary constants, usually because these have been fixed by a constraint.

Graphical intuition, whats does $y' = f(y, t)$ mean?



$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$$

$f(y, t) =$ slope of y(t) in the (t, y)-plane.

$$= \frac{\Delta y}{\Delta t}$$

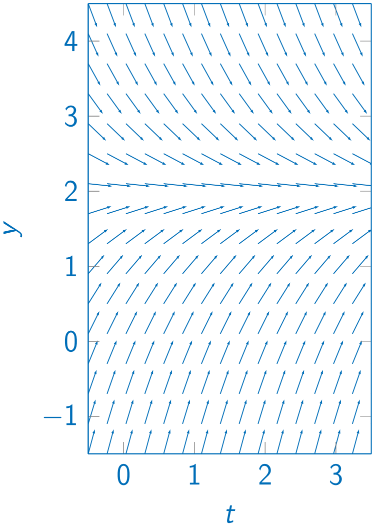
So,

$$y(t + \Delta t) \approx y(t) + \Delta t \cdot f(y(t), t)$$

<u>Direction Field:</u>

1. Draw the slope, $f(y, t)$, as an arrow for every point in the (t, y)-plane.
2. Connect the arrows to get qualitative (approximate) solutions.

Example: $y' = 2 - y$



1. What type of solutions are possible?
   - Monotonically increasing/decreasing
2. What is $y(t)$ as $t \to \infty$?
   - One possibility: $y(t) \to 2$
   - $y = 2$ is a stable steady state.
3. What is the influence of the initial condition?
   - If $y(0) > 2$ decreasing solution.
   - If $y(0) < 2$ increasing solution.
   - If $y(0) = 2$ constant solution.

Doesn't the function $f(y,t)$ tell us everything?

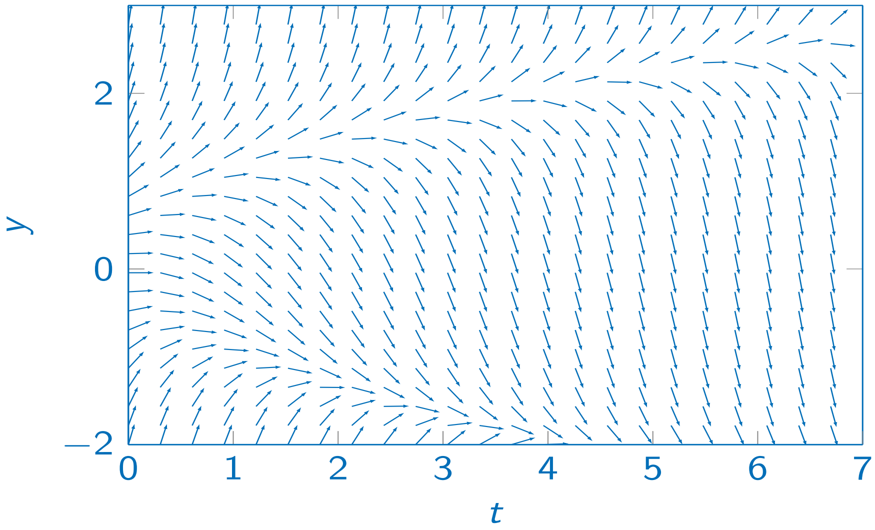
More or less, it gives us all the qualitative properties of solutions.

- Monotonic vs. Transiently Oscillatory vs. Periodic Solutions
- Steady states and their stability
- Influence of initial conditions

So why bother integrating/solving DEs?

- To get quantitative information.
- Impossible to graph direction fields for many systems of ODEs
- Drawing $f(y,t)$ is tedious when there is t-dependence.

Example: $y' = y^2 - t$



Summary

1. What are DEs?
   - Equations involving unknown function(s) and function derivatives.
   - Specify rates of change of certain quantities.
   - Useful for modelling many natural phenomena.

2. Terminology
   - ODEs (& PDEs).
   - Order of DEs, systems of DEs, solutions to DEs, steady states.

3. Graphical Solutions (via Direction Fields)
   - Intuitive way of thinking about DEs
   - Provide qualitative information