Recall: Homogeneous Systems of n ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

has n linearly independent **fundamental solutions**

$$\{\vec{x}_1(t),\ \vec{x}_2(t),\ \dots,\ \vec{x}_n(t)\}$$

and its general solution is given by

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

or

$$\vec{x} = \mathbf{X}(t)\vec{c} \quad \text{where} \quad \mathbf{X}(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X} = \mathbf{A}\mathbf{X}$$

Q: Non-homogeneous systems?

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t)$$

1. Integrating Factors
2. Undetermined Coefficients
3. Variation of Parameters

Variation of Parameters: $\dfrac{d}{dt}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t)$

Due to the linearity of the DE, we know the solution structure is

$$\vec{x}(t) = \underbrace{\vec{x}_h(t)}_{\text{homog. part}} + \underbrace{\vec{x}_p(t)}_{\text{particular part}} \quad \text{where } \vec{x}_h = \mathbf{X}(t)\vec{c} \quad \text{and} \quad \frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X}$$

LHS: $\dfrac{d}{dt}\vec{x}_p = \mathbf{X}'\vec{u} + \mathbf{X}\vec{u}'$

$= \mathbf{A}\mathbf{X}\vec{u} + \mathbf{X}\vec{u}'$

Guess: $\vec{x}_p = \mathbf{X}(t)\vec{u}(t)$

$$\cancel{\mathbf{A}\mathbf{X}\vec{u}} + \mathbf{X}\vec{u}' = \cancel{\mathbf{A}\mathbf{X}\vec{u}} + \vec{f}(t)$$

RHS: $\mathbf{A}\mathbf{X}\vec{u} + \vec{f}(t)$

$$\vec{u}' = \mathbf{X}^{-1}\vec{f}(t) = \vec{g}(t)$$

where $\vec{g}$ solves $\mathbf{X}\vec{g} = \vec{f}(t)$

$$\vec{u}(t) = \int \underbrace{\mathbf{X}^{-1}(t)\vec{f}(t)}_{\vec{g}(t)}\,dt$$

$$\boxed{\vec{x}_p = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\vec{f}(t)dt}$$

Suppose $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ and $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ solve $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$.

Find the particular solution to $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^t \\ e^{-2t} \end{bmatrix}$.

Fundamental Matrix: $\mathbf{X} = \begin{bmatrix} e^{-2t} & e^{4t} \\ -e^{-2t} & e^{4t} \end{bmatrix}$

$\vec{x}_p = \mathbf{X} \int \vec{g}(t)dt$ where $\mathbf{X}\vec{g} = \begin{bmatrix} 2e^t \\ e^{-2t} \end{bmatrix}$

$$\left[\begin{array}{cc|c} e^{-2t} & e^{4t} & 2e^t \\ -e^{-2t} & e^{4t} & e^{-2t} \end{array}\right] \xrightarrow{R_1+R_2 \to R_1} \left[\begin{array}{cc|c} 0 & 2e^{4t} & 2e^t + e^{-2t} \\ -e^{-2t} & e^{4t} & e^{-2t} \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 2e^{4t} & 2e^t + e^{-2t} \\ -e^{-2t} & 0 & \frac{1}{2}e^{-2t} - e^t \end{array}\right]$$

$$\begin{aligned} 2e^{4t}g_2 &= 2^e t + e^{-2t} \\ -e^{-2t}g_1 &= \tfrac{1}{2}e^{-2t} - e^t \end{aligned}$$

$$\vec{g} = \begin{bmatrix} -\frac{1}{2} + e^{3t} \\ e^{-3t} + \frac{1}{2}e^{-6t} \end{bmatrix}$$

$$\vec{x}_p = \underbrace{\begin{bmatrix} e^{-2t} & e^{4t} \\ -e^{-2t} & e^{4t} \end{bmatrix}}_{\text{fund. matrix } \mathbf{X}} \underbrace{\int g(t)dt}_{\vec{u}'dt} \quad \text{with} \quad \vec{g} = \begin{bmatrix} -\frac{1}{2} + e^{3t} \\ e^{-3t} + \frac{1}{2}e^{-6t} \end{bmatrix}$$

$$\vec{u}(t) = \int \vec{g}(t)dt = \begin{bmatrix} -\frac{t}{2} + \frac{1}{3}e^{3t} \\ -\frac{1}{3}e^{-3t} - \frac{1}{12}e^{-6t} \end{bmatrix}$$

$$\begin{aligned} \vec{x}_p &= \mathbf{X}\vec{u}(t) \\ &= \begin{bmatrix} -\frac{t}{2}e^{-2t} + \cancel{\frac{1}{3}e^{t}} - \cancel{\frac{1}{3}e^{t}} - \frac{1}{2}e^{-2t} \\ \frac{t}{2}e^{-2t} - \frac{1}{3}e^{t} - \frac{1}{3}e^{t} - \frac{1}{2}e^{-2t} \end{bmatrix} \end{aligned}$$

$$= \frac{t}{2}e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{3}e^{t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} - \frac{1}{12}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solve $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 0 & 1 \\ -4/t^2 & 4/t \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}$, where two linearly independent solutions of the homogeneous equation are

$$\vec{x}_1 = t^3 \begin{bmatrix} t \\ 4 \end{bmatrix}, \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

$$\text{Fundamental Matrix:} \qquad \mathbf{X} = \begin{bmatrix} t^4 & t \\ 4t^3 & 1 \end{bmatrix}$$

$$\vec{x}_p = \mathbf{X} \int \vec{g}(t) dt \qquad \text{where } \mathbf{X}\vec{g} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$\left[\begin{array}{cc|c} t^4 & t & 0 \\ 4t^3 & 1 & t \end{array}\right] \xrightarrow{R_1 - tR_2 \to R_1} \left[\begin{array}{cc|c} -3t^4 & 0 & -t^2 \\ 4t^3 & 1 & t \end{array}\right]$$

$$\xrightarrow{R_1 - \frac{4}{3t} R_2 \to R_2} \left[\begin{array}{cc|c} -3t^4 & 0 & -t \\ 0 & 1 & -\frac{t}{3} \end{array}\right]$$

$$-3t^4 g_1 = -t^2$$
$$g_2 = -\frac{t}{3}$$

$$\vec{g} = \begin{bmatrix} \frac{1}{3} t^{-2} \\ -\frac{t}{3} \end{bmatrix}$$

$$\vec{x}_p = \underbrace{\begin{bmatrix} t^4 & t \\ 4t^3 & 1 \end{bmatrix}}_{\text{fund. matrix } \mathbf{X}} \underbrace{\int g(t)dt}_{\vec{u}'dt} \quad \text{with } \vec{g} = \begin{bmatrix} \frac{1}{3}t^{-2} \\ -\frac{t}{3} \end{bmatrix}$$

$$\vec{u}(t) = \int \vec{g}(t)dt = \begin{bmatrix} -\frac{1}{3}t^{-1} \\ -\frac{t^2}{6} \end{bmatrix}$$

$$\begin{aligned} \vec{x}_p &= \mathbf{X}\vec{u}(t) \\ &= \begin{bmatrix} -\frac{1}{3}t^3 - \frac{t^3}{6} \\ -\frac{4}{3}t^2 - \frac{1}{6}t^2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t^3 \\ -\frac{3}{2}t^2 \end{bmatrix} \\ &= -\frac{t^2}{2}\begin{bmatrix} t \\ 3 \end{bmatrix} \end{aligned}$$

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$$\vec{x} = c_1 t^3 \begin{bmatrix} t \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix} - \frac{t^2}{2} \begin{bmatrix} t \\ 3 \end{bmatrix}$$

Solve the IVP $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 0 & 1 \\ -4/t^2 & 4/t \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}$ with $\vec{x}(1) = \begin{bmatrix} 3 \\ -\frac{3}{2} \end{bmatrix}$

$$\vec{x}(1) = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$$

$\underline{\text{bottom row:}}$ $4c_1 + c_2 = \dfrac{3}{2} - \dfrac{3}{2} = 0 \quad \Rightarrow c_2 = -4c_1$

$\underline{\text{top row:}}$ $c_1 + c_2 = 3 + \dfrac{1}{2} = \dfrac{7}{2}$

$$-3c_1 = \frac{5}{2} \quad \Rightarrow \boxed{c_1 = -\frac{7}{6}, \quad c_2 = \frac{28}{6}}$$