Differential Equations

1. What are they and why do we solve them?
2. Terminology

What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. an independent variable.

Given: A DE with an unknown function $y(t)$.

e.x.,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3y(t)$$

<u>or</u>

$$y' = -3y$$

Task: Find the function(s) $y(t)$.

Solution: $y(t) = C_1 e^{-3t}$

DEs specify the rate of change of one quantity (e.g., the position of an object) with respect to another (e.g., time).

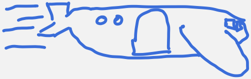
Why do we solve/study DEs?

DEs provide an intuitive way to describe many types of interactions (e.g., mechanical, biochemical, social, economic, etc.).

Solving and analyzing DEs allows us to:

1. Make predictions about the future (forecasting).
   - Will some quantity grow unboundedly? Oscillate? Decay to zero?
   - With what rate will those things happen?
2. Test possible mechanisms that may explain experimental data.
   - Why does some quantity sometimes oscillate vs reach an equilibrium?

Example: Skydiving



Air drag

Gravity

Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v} \qquad \text{DE for } v(t)$$

Example: Ecology - Lotka-Volterra Model

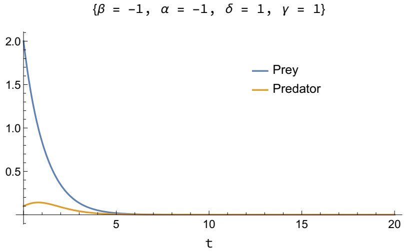
Predator-Prey Model, 2 variables:

x = prey population and y = predator population

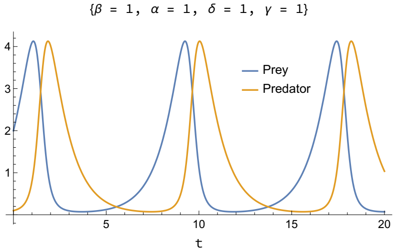
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \delta xy - \gamma y$$

We can prove that only two possible solutions exist

Mutual Extinction



Predator-Prey Oscillations



Terminology: ODEs vs PDEs

- Ordinary differential equation (ODE) (covered in this course)
  - A DE with derivatives w.r.t. only one independent variable.
  - $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t) + 3$ or $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \sin(y) + \cos(t)$
- Partial differential equation (PDE) (not covered in this course)
  - A DE with derivatives w.r.t multiple independent variables.
  - Heat/Diffusion eq: $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$
  - Wave eq: $\dfrac{\partial^2 u}{\partial t^2} = c\dfrac{\partial^2 u}{\partial x^2}$

  Partial derivatives are necessary for solutions to agree when changing coordinate systems (e.g., switch from cartesian to polar coordinates)

Terminology: Order of a DE

The highest derivative that appears in the DE.

- $y' = y + 3$ first order
- $y' = y^2 + 9$ first order
- $\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tan(t)$ first order
- $y'' = -y$ second order
- $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = ky$ fourth order

Terminology: Operator Form $\Rightarrow$ $\mathrm{L}[y(t)] = f(t)$

Everything that depends on the unknown function goes on one side of the equal sign and everything else on the other.

- $\dfrac{dy}{dt} = y(t) + 3 \quad \rightarrow \quad \dfrac{dy}{dt} - y(t) = 3$
  - $\mathrm{L}[y] = f' - y, \quad f(t) = 3$
- $\dfrac{dy}{dt} = \sin(y) + \cos(t) \quad \rightarrow \quad \dfrac{dy}{dt} - \sin(y) = \cos(t)$
  - $\mathrm{L}[y] = f' - \sin(y), \quad f(t) = \cos(t)$

The operator $\mathrm{L}[\cdot]$ encodes the "intrinsic" dynamics that the ODE is modelling.

- Force-displacement relationship of a spring.
- Velocity-drag relationship of a viscous fluid.

Terminology: Linearity of DEs - $\mathsf{L}[y(t)] = f(t)$

If the operator $\mathsf{L}[\cdot]$ is linear, then the DE is linear.

Conditions for linearity:

Given any two functions f and g and a constant c, $\mathsf{L}[\cdot]$ is linear if

1. $\mathsf{L}[f+g] = \mathsf{L}[f] + \mathsf{L}[g]$
2. $\mathsf{L}[cf] = c\mathsf{L}[f]$

Terminology: Linearity of DEs - $\mathrm{L}\left[y(t)\right] = f(t)$

If the operator $\mathrm{L}\left[\cdot\right]$ is linear, then the DE is linear.

In practice:

Does the operator have either of the following:

1. any nonlinear functions of y (or its derivatives) or
2. any products of y and its derivatives

ex: $\mathrm{L}\left[y\right] = y'' + y$

Linear

ex: $\mathrm{L}\left[y\right] = y' + \sin(y'')$

Nonlinear

ex: $\mathrm{L}\left[y\right] = y' + y'y$

Nonlinear

Terminology: Autonomous DEs - $\mathrm{L}\left[y(t)\right] = f(t)$

If both $\mathrm{L}\left[\cdot\right]$ and $f(t)$ do not explicitly depend on the independent variable, then the DE is autonomous.

- $y' = y \quad \rightarrow \quad y' - y = 0$ Autonomous
- $y' = y^2 + 3 \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y^2 = 3$ Autonomous
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y + \tan(t) \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y = \tan(t)$ Non-autonomous
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -3ty \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} + 3ty = 0$ Non-autonomous

$f(t)$ is often called the (external) forcing term.

constant or zero-forcing $\Rightarrow$ Autonomous DE

Classifying ODEs

- $x'' + x^2 = t$
  - Order: 2
  - Linear: No
  - Autnomous: No

- $\dfrac{\mathrm{d}^4 x}{\mathrm{d}t^4} = 0$
  - Order: 4
  - Linear: Yes
  - Autnomous: Yes

Terminology: Solution to an ODE

A solution of an ODE is a function that satisifes the ODE.

ex: Is $y = Ce^{-t} + t - 1$ a solution to $y' + y = t$?

$$\begin{aligned} y' &= -Ce^{-t} + 1 \\ y' + y &= \cancel{-Ce^{-t}} + \cancel{1} + \cancel{Ce^{-t}} + t - \cancel{1} \\ &= t \quad \checkmark \end{aligned}$$

Here C is an arbitrary constant that can have any value.

Any solution with an arbitrary constant is called a general solution

We can obtain a unique solution by imposing some constraint, a solution with no arbitrary constants is called a particular solution

Initial Value Problems

ODEs of the form

$$\mathrm{L}\left[y\right] = f(t), \text{ with } y(t_0) = y_0,$$

where t_0 and y_0 are numerical values (usually real-valued).

ex: Find the particular solution to $y' + y = t$ with $y(0) = 0$?

Start with the general solution

$$y(t) = Ce^{-t} + t - 1$$

evaluate at $t = t_0 = 0$, make that equal to $y_0 = 0$

$$y(0) = C - 1 = 0 \quad \Rightarrow C = 1$$

$$\boxed{y(t) = e^{-t} + t - 1}$$

Summary

1. What are DEs?
   - Equations involving unknown function(s) and function derivatives.
   - Specify rates of change of certain quantities.
   - Useful for modelling many natural phenomena.
2. Terminology
   - ODEs (& PDEs).
   - Order of DEs, Linear DEs, Autonomous DEs, Solutions to DEs
3. Initial Value Problems
   - The most "standard" way to obtain a unique solution
   - Specify solution value at some initial time

For next class...you will need access to MATLAB

1. Create a MathWorks account
   - Go to matlab.mathworks.com
   - Click "No account? Create one!"
   - Enter your UBC email address and follow the instructions
     - You can obtain one from here using "Activate Student Email"
   - Note it may take a few hours to activate your MathWorks account
2. Use MATLAB Online
   - Go to matlab.mathworks.com
   - Sign in with your UBC email address and MathWorks password