Differential Equations

1. What are they and why do we solve them?
2. Terminology

What is a Differential Equation?

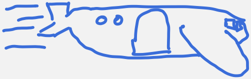
A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. an independent variable.

Given: A DE with an unknown function $y(t)$. e.x., $\frac{dy}{dt} = -3y(t)$ or $y' = -3y$

Task: Find the function(s) $y(t)$. Solution: $y(t) = C_1 e^{-3t}$

Tools:
- Calculus (i.e., integration/differentiation)
- Guess and check (does some function $f(t)$ make LHS=RHS?)
- Specialized procedures (informed by experience geussing)
- Geometry/Linear Algebra (useful for systems of DEs)

Example: Skydiving



Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v} \qquad \text{DE for } v(t)$$

Example: Epidemiology

Kermack & McKendrick's SIR model

- Susceptible $\rightarrow$ Infected $\rightarrow$ Recovered

System of 3 ordinary differential equations:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu(I + R) - \beta SI$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - \gamma I - \mu I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R$$

$t =$ time
$\beta =$ infection rate
$\gamma =$ recovery rate
$\mu =$ birth/death rate

Reproduction number: $R_0 = \frac{\beta}{\mu+\gamma}$

1. $R_0 > 1$: endemic equilibrium
2. $R_0 < 1$: disease dies out - basic idea behind "flatten the curve"

Terminology: ODEs vs PDEs

- Ordinary differential equation (ODE) (covered in this course)
    - A DE with derivatives w.r.t. only one independent variable.
    - $\frac{\mathrm{d}y}{\mathrm{d}t} = y(t) + 3$ or $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin(y) + \cos(t)$
- Partial differential equation (PDE) (not covered in this course)
    - A DE with derivatives w.r.t multiple independent variables.
    - Heat/Diffusion eq: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$
    - Wave eq: $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$

  Partial derivatives are necessary for solutions to agree when changing coordinate systems (e.g., switch from cartesian to polar coordinates)

Terminology: Order of a DE

The highest derivative that appears in the DE.

- $y' = y + 3$ first order
- $y' = y^2 + 9$ first order
- $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tan(t)$ first order
- $y'' = -y$ second order
- $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = ky$ fourth order

Terminology: Operator Form $\Rightarrow$ $\mathrm{L}\left[y(t)\right] = f(t)$

Everything that depends on the unknown function goes on one side of the equal sign and everything else on the other.

- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t) + 3 \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y(t) = 3$
  - $\mathrm{L}\left[y\right] = f' - y, \quad f(t) = 3$
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \sin(y) + \cos(t) \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - \sin(y) = \cos(t)$
  - $\mathrm{L}\left[y\right] = f' - \sin(y), \quad f(t) = \cos(t)$

The operator $\mathrm{L}\left[\cdot\right]$ encodes the "intrinsic" dynamics that the ODE is modelling.

- Force-displacement relationship of a spring.
- Velocity-drag relationship of a viscous fluid.

Terminology: Linearity of DEs - $\mathsf{L}[y(t)] = f(t)$

If the operator $\mathsf{L}[\cdot]$ is linear, then the DE is linear.

Conditions for linearity:

Given any two functions f and g and a constant c, $\mathsf{L}[\cdot]$ is linear if

1. $\mathsf{L}[f+g] = \mathsf{L}[f] + \mathsf{L}[g]$
2. $\mathsf{L}[cf] = c\mathsf{L}[f]$

Terminology: Linearity of DEs - $\mathrm{L}[y(t)] = f(t)$

If the operator $\mathrm{L}[\cdot]$ is linear, then the DE is linear.

In practice:

Does the operator have either of the following:

1. any nonlinear functions of y (or its derivatives) or
2. any products of y and its derivatives

ex: $\mathrm{L}[y] = y'' + y$

Linear

ex: $\mathrm{L}[y] = y' + \sin(y'')$

Nonlinear

ex: $\mathrm{L}[y] = y' + y'y$

Nonlinear

Terminology: Autonomous DEs - $\mathrm{L}\left[y(t)\right] = f(t)$

If both $\mathrm{L}\left[\cdot\right]$ and $f(t)$ do not explicitly depend on the independent variable, then the DE is autonomous.

- $y' = y \quad \rightarrow \quad y' - y = 0$ Autonomous
- $y' = y^2 + 3 \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y^2 = 3$ Autonomous
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y + \tan(t) \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y = \tan(t)$ Non-autonomous
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -3ty \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} + 3ty = 0$ Non-autonomous

$f(t)$ is often called the (external) forcing term.

constant or zero-forcing $\Rightarrow$ Autonomous DE

Classifying ODEs

- $x'' + x^2 = t$
  - Order: 2
  - Linear: No
  - Autnomous: No

- $\dfrac{\mathrm{d}^4 x}{\mathrm{d}t^4} = 0$
  - Order: 4
  - Linear: Yes
  - Autnomous: Yes

Terminology: Solution to an ODE

A solution of an ODE is a function that satisifes the ODE.

ex: Is $y = Ce^{-t} + t - 1$ a solution to $y' + y = t$?

$$\begin{aligned} y' &= -Ce^{-t} + 1 \\ y' + y &= \cancel{-Ce^{-t}} + \cancel{1} + \cancel{Ce^{-t}} + t - \cancel{1} \\ &= t \quad \checkmark \end{aligned}$$

Here C is an arbitrary constant that can have any value.

Any solution with an arbitrary constant is called a general solution

We can obtain a unique solution by imposing some constraint, a solution with no arbitrary constants is called a particular solution

Initial Value Problems

ODEs of the form

$$\mathrm{L}\left[y\right] = f(t), \text{ with } y(t_0) = y_0,$$

where t_0 and y_0 are numerical values (usually real-valued).

ex: Find the particular solution to $y' + y = t$ with $y(0) = 0$?

Start with the general solution

$$y(t) = Ce^{-t} + t - 1$$

evaluate at $t = t_0 = 0$, make that equal to $y_0 = 0$

$$y(0) = C - 1 = 0 \quad \Rightarrow C = 1$$

$$\boxed{y(t) = e^{-t} + t - 1}$$

Summary

1. What are DEs?
    - Equations involving unknown function(s) and function derivatives.
    - Specify rates of change of certain quantities.
    - Useful for modelling many natural phenomena.
2. Terminology
    - ODEs (& PDEs).
    - Order of DEs, Linear DEs, Autonomous DEs, Solutions to DEs
3. Initial Value Problems
    - The most ”standard” way to obtain a unique solution
    - Specify solution value at some initial time