Recall:

We saw how to use integration to solve the following first order ODEs

$$y' = f(t), \quad y' = f(y), \text{ and } \quad y' = g(y)f(t).$$

What happens if

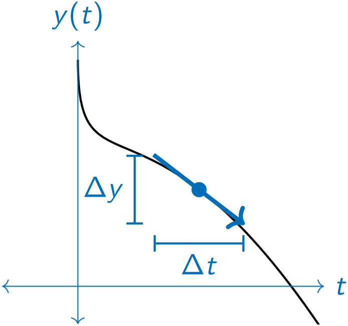
1. we cannot solve one of the integrals, or
2. we get a more general ODE of the form $y' = f(y, t)$?

Exact analytical solutions are not always possible.

- 99.99% of all the possible ODEs involve impossible integrals.

Today we will build intuition around how to find approximate solutions.

Graphical intuition, whats does $y' = f(y, t)$ mean?



$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$$

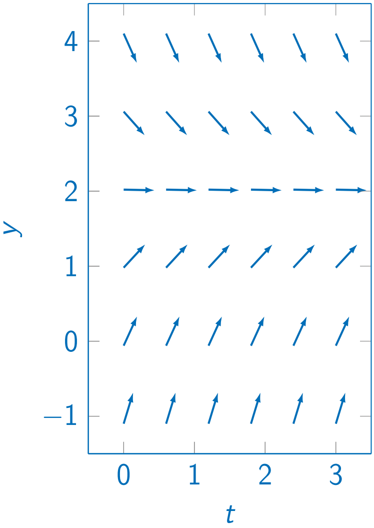
$$f(y,t) = \text{slope of y(t) in the } (t, y)\text{-plane.}$$
$$\approx \frac{\Delta y}{\Delta t}$$

Slope Field Diagram:

1. Draw an arrow with slope $f(y, t)$ for some points in the (t, y)-plane.

2. Stating from some initial condition(s), trace along with the arrows to get approximate solutions.

Example: $y' = 2 - y$, try different initial conditions



1. What type of solutions are possible?
   - Monotonically increasing/decreasing
2. What is $y(t)$ as $t \to \infty$?
   - Unique possibility: $y(t) \to 2$
   - $y = 2$ is a stable steady state.
3. What is the influence of the initial condition?
   - If $y(0) > 2$ decreasing solution.
   - If $y(0) < 2$ increasing solution.
   - If $y(0) = 2$ constant solution.

slopefield.m

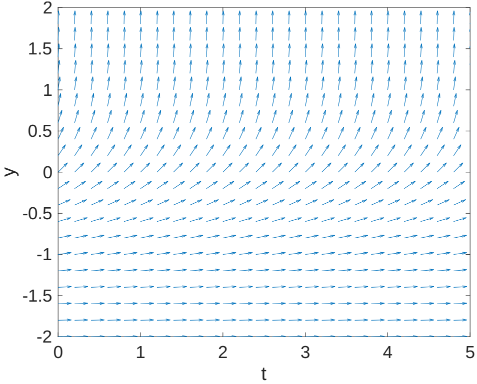
A convenient MATLAB function that will plot slopefields for you!

```
function slopefield(f,t,y)

[T,Y] = meshgrid(t,y);
dydt = f(T,Y);
theta = atan(dydt);
L = min(min(diff(t)),min(diff(y)))*0.8;
dy = L*sin(theta);
dt = L*cos(theta);
quiver(t,y,dt,dy,0)
axis equal; axis([t(1),t(end),y(1),y(end)]);

end
```

Copy-paste into a file called **slopefield.m** in your MATLAB development environment

ex: $y' = e^{2y}$

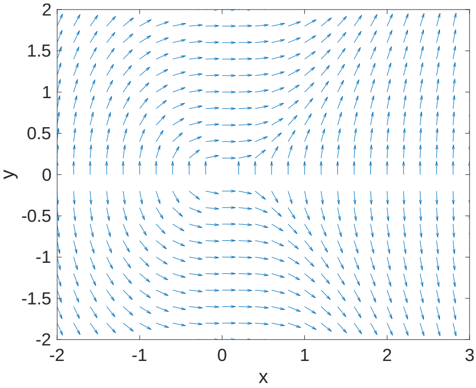
```
f = @(t,y) exp(2*y);
t = 0:0.2:5;
y = -2:0.2:2;
slopefield(f,t,y)
```



Note: The slope is positive for all values of y. Blowup happens for all initial conditions.

ex: $y' = \frac{x^2}{y}$

```
f = @(t,y) t.^2./y;
t = -2:0.2:3;
y = -2:0.2:2;
slopefield(f,t,y)
```



Trace a solution backwards from $y(0) = 1$.

Why does the solution not exist for $x < -\sqrt[3]{3/2}$?

$f(x, y)$ is discontinous $y = 0$

Picard's Theorem

A unique solution to

$$y' = f(t, y), \quad \text{with } y(t_0) = y_0$$

exists for t near t_0 if

1. $f(t, y)$ is continuous (as a function of two variables) and
2. $\partial f / \partial y$ exists and is continuous near (t_0, y_0).

ex: $y' = 2\sqrt{|y|}$, with y(0)=0

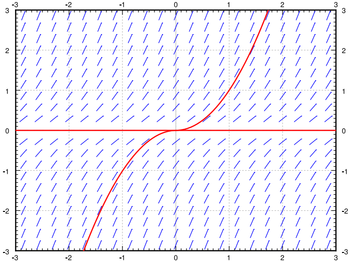
Analytically, we find 2 solutions

1. By inspection:

$$y(t) = 0$$

2. Integrate for positive and negative y separately:

$$y(t) = \begin{cases} t^2 & t \geq 0 \\ -t^2 & t < 0 \end{cases}$$



Picard's uniqueness theorem does not apply, because $\dfrac{\partial}{\partial y} 2\sqrt{|y|}$ does not exist at $y = 0$.