

Recall

We saw how to solve ODEs of the form

$$y' = f(t) \cdot g(y) \quad - \quad \text{Separable Equations}$$

$$\frac{dy}{g(y)} = f(t)dt \dots \text{integrate and isolate } y$$

Now we want to solve something a little more general...

$$y' = f(t) \cdot g(y) + h(t)$$

This is not separable :(

Simple Case:

- linear equations with $g(y) = y$

ex: Solve $y' + 2ty = t^2 e^t$

First we do something very odd, multiply everything by e^{t^2}

$$e^{t^2} y' + \underbrace{2te^{t^2}}_{\frac{d}{dt} e^{t^2}} y = t^2 e^{t^3} \quad \rightarrow \quad \underbrace{e^{t^2} \frac{d}{dt} y + \frac{d}{dt} e^{t^2} y}_{\text{product rule applied to } e^{t^2} y} = t^2 e^{t^3}$$

$$\frac{d}{dt} (e^{t^2} y) = t^2 e^{t^3}$$

$$\int d(e^{t^2} y) = \int t^2 e^{t^3} dt$$

$$e^{t^2} y = \frac{1}{3} \int e^u du$$

$$\begin{aligned} u &= t^3 \\ du &= 3t^2 \end{aligned}$$

$$\begin{aligned} e^{t^2} y &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{t^3} + C \end{aligned}$$

$$y(t) = \frac{1}{3} e^t + C e^{-t^2}$$

Method of integrating factors: $y' + p(t)y = h(t)$

1. Multiply by the integrating factor $\mu(t) = e^{\int p(t)dt}$

$$\mu(t)y' + \underbrace{p(t)\mu(t)}_{\mu'(t)}y = h(t)\mu(t)$$

$$\frac{d}{dt}(\mu \cdot y) = h(t)\mu(t)$$

$$d(\mu \cdot y) = h(t)\mu(t)dt$$

2. Integrate:

$$\mu \cdot y = \int h(t)\mu(t)dt + C$$

3. Isolate $y(t)$:

$$y(t) = \frac{\int h(t)\mu(t)dt + C}{\mu(t)}$$

ex: Find the general solution of $y' - 3y = e^t$.

$$y(t) = \frac{\int h(t)\mu(t)dt + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t)dt}$$

$$\mu(t) = e^{\int (-3)dt} = e^{-3t}$$

$$e^{-3t}y' - 3e^{-3t}y = e^te^{-3t}$$

$$\frac{d}{dt}(e^{-3t}y) = e^{-2t}$$

$$d(e^{-3t}y) = e^{-2t}dt$$

$$e^{-3t}y(t) = \int e^{-2t}dt + C$$

$$y(t) = \frac{-\frac{1}{2}e^{-2t} + C}{e^{-3t}}$$

$$= -\frac{1}{2}e^t + Ce^{3t}$$

ex: Find the general solution of $y' - 3t^2y = 3t^2$.

$$y(t) = \frac{\int h(t)\mu(t)dt + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t)dt}$$

$$\mu(t) = e^{-\int 3t^2 dt} = e^{-t^3}$$

$$e^{-t^3}y(t) = \int 3t^2 e^{-t^3} dt$$

$$e^{-t^3}y(t) = -e^{-t^3} + C$$

$$y(t) = -1 + Ce^{t^3}$$

$$\text{let } u = t^3$$
$$du = 3t^2$$

$$\int 3t^2 e^{-t^3} dt = \int e^{-u} du$$
$$= -e^{-u} + C$$

ex: $ty' + 5y = 24t^3$, with $y(1) = 2$

$$y' + \frac{5y}{t} = 24t^2$$

$$\begin{aligned}\mu(t) &= e^{\int \frac{5}{t} dt} = e^{5 \ln(t)} \\ &= \left(e^{\ln(t)}\right)^5 = t^5\end{aligned}$$

$$t^5 y(t) = \int 24t^7 dt = 3t^8 + C$$

$$y(t) = 3t^3 + Ct^{-5}$$

$$y(1) = 2 = 3 + C$$

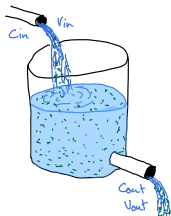
$$C = -1$$

$$\boxed{y(t) = 3t^3 - t^{-5}}$$

Application: Mixing Tank

At time $t = 0$ a tank contains 10 kg of salt dissolved in 200 liters of water. Water containing $\frac{1}{2}$ kg of salt per liter enters the tank at a rate of 2 L/min and the well mixed solution leaves the tank at the same rate.

- (a) Let $Q(t)$ be the amount of salt in kilograms in the solution after t minutes have passed. Find a formula for the rate of change in the amount of salt, $\frac{dQ}{dt}$, in terms of the amount of salt in the solution, $Q(t)$.



$$\begin{aligned}\frac{dQ}{dt} &= \text{in} - \text{out} = 0.5 \frac{\text{kg}}{\text{L}} \times 2 \frac{\text{L}}{\text{min}} - \frac{Q \text{ kg}}{200 \text{ L}} \times 2 \frac{\text{L}}{\text{min}} \\ &= 1 - \frac{Q}{100}\end{aligned}$$

- (b) Find $Q(t)$.

$$\begin{aligned}\mu(t) &= e^{t/100} \\ Q(0) &= 10\end{aligned} \quad \Rightarrow \quad Q(t) = 100 - 90e^{-t/100}$$

Summary

- General linear 1st order ODE: $y' + p(t)y = h(t)$

- To solve, turn the LHS into an exact derivative:

$$y' + p(t)y \rightarrow d(\mu \cdot y) = \mu' y + \mu y'$$

- Multiply by an integrating factor $\mu(t) = e^{\int p(t)dt}$

- General solution: $y(t) = \frac{\int g(t)\mu(t)dt + C}{\mu(t)}$

- This method solves all linear 1st order ODEs, provided that $p(t)$ and $h(t)$ are continuous.