Recall:

- General linear 1st order ODE: $y' + p(t)y = h(t)$
  - To solve, turn the LHS into an total derivative:

$$y' + p(t)y \quad \to \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot y) = \mu y' + \mu' y$$

We will now explore the idea of total derivatives in more depth:

Consider a function of two variables $\Phi(x, y)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\Phi(x,y) &= \frac{\partial}{\partial x}\Phi(x,y) + \frac{\partial}{\partial y}\Phi(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \underbrace{\underbrace{\Phi_x}_{\text{partial}} + \underbrace{\Phi_y}_{\text{partial}}\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{total}} \end{aligned}$$

Solve $(y+2x)+(x-3y^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$

Idea: write as a total derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}\Phi(x,y)=0 \quad \Rightarrow \Phi(x,y)=\text{const.}$

$$\text{LHS}=\frac{\mathrm{d}}{\mathrm{d}x}\Phi(x,y)=\Phi_x+\Phi_y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\Phi_x = y+2x$$
$$\Phi_y = x-3y^2$$

$$\Phi(x,y)=\int \Phi_x dx + h(y) = \int y+2xdx+h(y)$$
$$= xy+x^2+h(y)$$
$$\Phi_y = x-3y^2 = x+h'(y) \qquad \Rightarrow \qquad h'(y)=-3y^2$$
$$\Rightarrow h(y)=-y^3+C$$

$$\Rightarrow \Phi(x,y)=xy+x^2-y^3+C$$

Implicit solution: $\quad xy+x^2-y^3=C$

Exact Equations

A DE $M(,x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ is called exact if there exist a $\Phi(x,y)$ such that

$$M = \Phi_x, \qquad N = \Phi_y.$$

The function $\Phi(x,y)$ is called a potential.

For exact eqs., the implicit function $y(x)$ given by

$$\Phi(x,y) = C$$

is the solution.

Example: Spring Potential

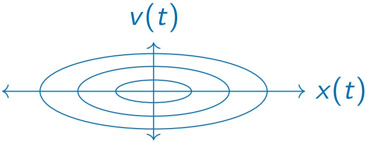
Consider a mass m at position $x(t)$ moving with speed $v(t)$ while attached to a spring with zero rest length and stiffness k:

$$\Phi(x, v) = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

Assuming no forcing or friction, energy is conserved

$$\Phi(x, v) = E_0, \qquad E_0 = \text{initial energy}$$

producing motion tracing out an ellipse in $(x, v)-$space:



All these ellipses satisfy a DE:

$$kx + mv\frac{\mathrm{d}v}{\mathrm{d}x} = 0$$

When is a DE $M(,x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ exact?

Since $\Phi_{xy} = \Phi_{yx}$, a necessary and sufficient that

$$M_y = N_x$$

Theorem: If $M_y = N_x$, near any (x_0, y_0) there is locally a function $\Phi(x, y)$ such that $\Phi_x = M$ & $\Phi_y = N$.

N.B.: $\Phi(x, y)$ exists locally, maybe not globally (e.g., if M or N are piecewise functions).

General Solution Method for an Exact DE.

$$M(,x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad \text{with } M_y = N_x$$

1. Since $\Phi_x = M$, initially fix y

$$\Phi(x,y) = \int M(x,y)dx = Q(x,y) + \underbrace{h(y)}_{\text{const.}}$$

2. Then note that $N = \Phi_y = \frac{\partial}{\partial y}Q + h'$
   - $h' = N - Q_y$ (sanity check: must be independent of x)
   - Integrate to find $h(y)$
3. Implicit solution $\Phi(x,y) = C$

ex: $(2x + \sin(y)) + (1 + x)\cos(y)\frac{dy}{dx} = 0$

Decide if the DE is exact. If yes, find the solution.

$$M = 2x + \sin(y) \qquad N = (1 + x)\cos(y)$$

$$M_y = \cos(y) \qquad N_x = \cos(y) \quad \text{exact}\checkmark$$

$$\begin{aligned}\Phi(x, y) &= \int 2x + \sin(y)dx \\ &= x^2 + x\sin(y) + h(y) \\ \Phi_y &= N \\ x\cos(y) + h'(y) &= (1 + x)\cos(y)\end{aligned}$$

$$\begin{aligned}h'(y) &= \cos(y) \\ h(y) &= \sin(y) + C\end{aligned}$$

$$\Phi(x, y) = x^2 + (x + 1)\sin(y) + C$$

Implicit solution: $x^2 + (x + 1)\sin(y) = C$