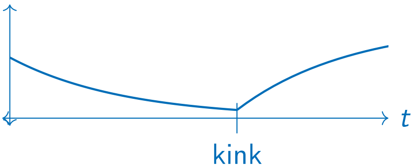
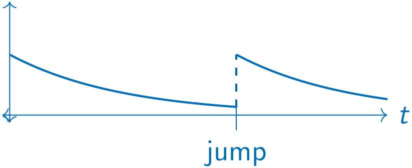
Recall: $ay'' + by' + cy = f(t)$

We have studied this DE extensively, and know how to solve it for a large class of functions $f(t)$ that are continuous.

Note: Discontinuities in $f(t)$ create kinks or jumps in the solution.





How can we deal with a RHS with, a potentially infinite number of, discontinuities?

Idea: Apply a transformation to the RHS to make it "nice".

The Laplace Transform.

Laplace Transform Workflow

Transforms the problem from the time domain to the "solution"-domain.

t − domain

Unknown y(t)
that solves ODE

$\xrightarrow{\text{Solve ODE}}$

Found y(t)

Transform
y(t) and
the ODE

Invert
the
transform

s − domain

Unknown Y(s)
that solves an
algebraic equation

$\xrightarrow{\text{Solve alg. eq.}}$

Isolate Y(s)

Laplace Transform of $y(t)$:

$$\mathcal{L}\left\{y(t)\right\} = Y(s) = \int_0^\infty e^{-st} y(t) dt$$

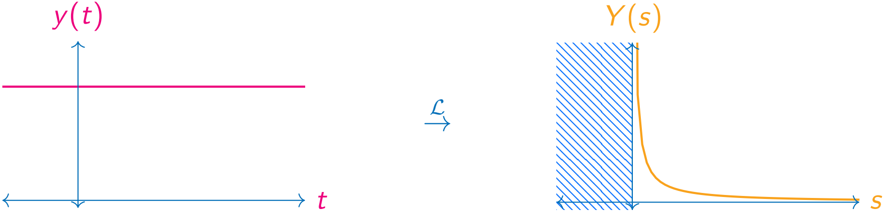
ex: $y(t) = \frac{1}{2}$

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st}\frac{1}{2}\,dt$$

$$= -\frac{1}{2s}e^{-st}\Big|_0^\infty \qquad = -\lim_{A\to\infty}\frac{1}{2s}e^{-st}\Big|_0^A$$

$$= -\frac{1}{2s}\lim_{A\to\infty}\left(e^{-sA} - 1\right)$$

$$= \begin{cases} \frac{1}{2s} & \text{if } s > 0 \\ DNE & \text{if } s \leq 0 \end{cases}$$

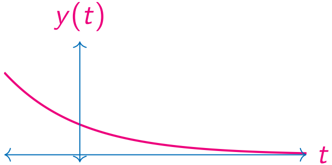


ex: $y(t) = e^{-6t}$

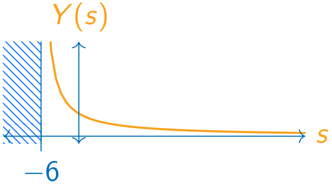
$$\mathcal{L}\left\{y(t)\right\} = Y(s) = \int_0^\infty e^{-st} e^{-6t} dt$$

$$= \int_0^\infty e^{-(s+6)t} = -\frac{1}{s+6} \lim_{A\to\infty} \left(e^{-(s+6)A} - 1\right)$$

$$= \begin{cases} \frac{1}{s+6} & \text{if } s > -6 \\ DNE & \text{if } s \leq -6 \end{cases}$$



$\xrightarrow{\mathcal{L}}$



General Results

For any constants C and a we have the Laplace Transforms of the following functions $y(t)$:

$$y(t) = C \qquad \mathcal{L}\left\{C\right\} = \frac{C}{s} \qquad \text{Constant}$$

$$y(t) = e^{at} \qquad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad \text{Exponential Function}$$

From now on, we don't worry too much about the domain of definition.

In general, there are always some conditions on s for the integrals to exist.

General Result: Linearity of Laplace Transforms

Given any two function $f(t)$ and $g(t)$ as well as any constant c.

1. $\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt$

   $= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} = F(s) + G(s)$

2. $\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\} = cF(s)$

The Laplace tranform is <u>linear</u>.

ex: $y(t) = \cos(at)$ or $y(t) = \sin(at)$

Euler's Identity:

$$e^{iat} = \cos(at) + i\sin(at)$$

$$\begin{aligned}
\mathcal{L}\left\{e^{iat}\right\} &= \mathcal{L}\{\cos(at)\} + i\mathcal{L}\{sin(at)\} \\
&= \frac{1}{s-ia} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia} \\
&= \frac{s+ia}{s^2 - ias + ias - i^2a^2} = \frac{s+ia}{s^2+a^2} \\
&= \underbrace{\frac{s}{s^2+a^2}}_{\mathcal{L}\{\cos(at)\}} + i\underbrace{\frac{a}{s^2+a^2}}_{\mathcal{L}\{\sin(at)\}}
\end{aligned}$$

Same result can be found through integration by parts (twice).

ex: $y(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt = \int_0^\infty e^{-st} t dt$$

let $\begin{array}{l} u = t,\ du = dt \\ dv = e^{-st}dt,\ v = -\frac{e^{-st}}{s} \end{array}$

$$\int te^{-st}dt = uv - \int vdu = \frac{te^{-st}}{s} + \int \frac{e^{-st}}{s}$$

$$= \frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} = -\frac{e^{-st}(st+1)}{s^2}$$

$$\mathcal{L}\{t\} = \lim_{A\to\infty} -\frac{e^{-st}(st+1)}{s^2}\bigg|_0^A = \lim_{A\to\infty} -\frac{e^{-sA}(sA+1)}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} \qquad (s > 0)$$

For $y(t) = t^k$, integrate by parts k times.

General Results

For any constants C, a, ω, and k we have the Laplace Transforms of the following functions $y(t)$:

$y(t) = C$	$\mathcal{L}\left\{C\right\} = \dfrac{C}{s}$	Constant
$y(t) = e^{at}$	$\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	Exponential Function
$y(t) = \cos(\omega t)$	$\mathcal{L}\left\{\cos \omega t\right\} = \dfrac{s}{s^2+\omega^2}$	Cosine
$y(t) = \sin(\omega t)$	$\mathcal{L}\left\{\sin \omega t\right\} = \dfrac{\omega}{s^2+\omega^2}$	Sine
$y(t) = t^k$	$\mathcal{L}\left\{t^k\right\} = \dfrac{k!}{s^{k+1}}$	Power Function

Summary

- Laplace transform (LT) maps $f(t) \to F(s)$
  - From "t-space" to "s-space" .
  - We will learn to invert the transform in the next lecture.
- LT: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$
  - Evaluation of the integrals is tedious.
  - We use general results to quickly transform functions.
  - Many tables exist online and in textbooks.
- LT is linear because the integral is linear
  1. $\mathcal{L}\{f + g\} = F(s) + G(s)$
  2. $\mathcal{L}\{cf\} = cF(s)$