Recall: Superposition for Linear Homogeneous ODEs

Suppose the linearly independent functions $y_1(t)$ and $y_2(t)$ both independently solve a 2^{nd} order linear homogeneous ODE

$$\mathsf{L}[y] = 0$$

then

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution to the ODE.

Proof:

$$\begin{aligned} \mathsf{L}[c_1 y_1 + c_2 y_2] &\overset{\text{Linearity 1}}{=} \mathsf{L}[c_1 y_1] + \mathsf{L}[c_2 y_2] \\ &\overset{\text{Linearity 2}}{=} c_1 \mathsf{L}[y_1] + c_2 \mathsf{L}[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

Superposition for Linear Inhomogeneous ODEs

Suppose the linearly independent functions $y_1(t)$ and $y_2(t)$ both solve a 2nd order linear homogeneous ODE

$$\mathrm{L}[y] = 0 \tag{1}$$

and a particular solution $y_p(t)$ solves the linear inhomogeneous ODE

$$\mathrm{L}[y_p] = f(t) \neq 0 \tag{2}$$

then $y_g = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homogeneous or complementary solution}} + y_p$ is the general solution to Eq. (2).

Proof:

$$\mathrm{L}[c_1 y_1 + c_2 y_2 + y_p] \overset{\text{Linearity 1}}{=} \mathrm{L}[c_1 y_1] + \mathrm{L}[c_2 y_2] + \mathrm{L}[y_p]$$

$$\overset{\text{Linearity 2}}{=} \underbrace{c_1 \mathrm{L}[y_1] + c_2 \mathrm{L}[y_2]}_{0} + \underbrace{\mathrm{L}[y_p]}_{f(t)} = f(t)$$

For proof of uniqueness of y_p, see DiffQs §2.5.1

Solution Structure for Linear Inhomogeneous Problems

The general solution to

$$\mathrm{L}\left[y(t)\right] = f(t) \qquad \text{with } f(t) \neq 0$$

where $\mathrm{L}\left[\cdot\right]$ is linear is always of the form

$$y_g(t) = y_h(t) + y_p(t)$$

where y_h solves $\mathrm{L}\left[y\right] = 0$ the associated homogeneous problem.

ex: Second order $\mathrm{L}\left[y\right] \neq 0$ with $y(0) = y_0$, $y'(0) = v_0$

$$\begin{aligned} y_h &= c_1 y_1(t) + c_2 y_2(t) \\ y(0) = y_0 &= c_1 y_1(0) + c_2 y_2(0) + y_p(0) \\ y'(0) = v_0 &= c_1 y_1'(0) + c_2 y_2'(0) + y_p'(0) \end{aligned}$$

Solve for c_1 and c_2. Don't forget to include y_p.

How to find y_p? Method of Undetermined Coefficients

Suppose we get:

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad \text{guess } y_h = e^{rt}$$

Suppose we get:

$$ay'' + by' + cy = f(t)$$

Guess $y_p =$ similar to $f(t)$

This approach works for:

- $f(t) =$ polynomial func.
- $f(t) =$ exponential func.
- $f(t) =$ sin or cos
- additive combinations of above
- multiplicative combinations of above

How to find y_p? Method of Undetermined Coefficients

Suppose we get:

$$ay'' + by' + cy = Dt + E \quad \Rightarrow \quad \text{guess } y_p = Ft + H$$

Plug in guess to find F and H:

$$y_p' = F, \qquad y_p'' = 0$$

$$bF + cFt + cH = Dt + E \qquad \text{(one eq. two unknowns)}$$

group by powers of t to obtain two eqs.

$$\underline{t^1}: \quad cFt = Dt \qquad \Rightarrow F = D/c$$

$$\underline{t^0}: \quad bF + cH = D$$

$$b\frac{D}{c} + cH = E \qquad \Rightarrow H = \frac{E}{c} - b\frac{D}{c^2}$$

Works as long as $c \neq 0$...otherwise the algebra is impossible.

ex: Find the particular solution to $y'' + 2y' + 2y = 2t^2 - 2$.
Guess: $y_p = At^2 + Bt + C$

$$y_p' = 2At + B, \quad y_p'' = 2A$$

$$2A + 2(2At + B) + 2(At^2 + Bt + C) = 2t^2 - 2$$

group by powers of t:

$\underline{t^2:}$ $\quad 2At^2 = 2t^2 \quad \Rightarrow A = 1$

$\underline{t^1:}$ $\quad 4\cancel{A}^{1}t + 2Bt = 0$

$4 + 2B = 0 \quad \Rightarrow B = -2$

$\underline{t^0:}$ $\quad 4\cancel{A}^{1}t + 2Bt = -2$

$2\cancel{A}^{1} + 2\cancel{B}^{-2} + 2C = -2$

$2 - 4 + 2C = -2 \quad \Rightarrow C = 0$

$$\boxed{y_p = t^2 - 2t}$$

ex: Solve $y'' + 2y' + 2y = 2t^2 - 2$ with $y(0) = 1, y'(0) = -3$.

$$y = c_1 y_1 + c_2 y_2 + \underbrace{y_p}_{t^2 - 2t}$$

$$y_{1,2} = e^{rt}$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y(t) = e^{-t}\left(c_1 \cos(t) + c_2 \sin(t)\right) + t^2 - 2t$$

initial conditions:

$$y(0) = 1 = c_1 \quad \Rightarrow c_1 = 1$$

$$y'(t) = -e^{t}\left(c_1 \cos(t) + c_2 \sin(t)\right) + e^{t}\left(c_2 \cos(t) - c_1 \sin(t)\right) + 2t - 2$$

$$y'(0) = -3 = -c_1 + c_2 - 2 \Rightarrow c_2 = 0$$

$$\boxed{y(t) = e^{-t}\cos(t) + t^2 - 2t}$$

Mathematical Resonance & Undetermined Coefficients

Given an inhomogeneous linear DE $\mathrm{L}[y] = f(t) \neq 0$, we say that mathematical resonance occurs when $f(t)$ (or its derivatives) has the same form as y_h.

$$\text{result} = \text{impossible algebra}$$

$$\underline{ex:} \quad x'' + x = \sin \omega t$$

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

$\underline{\omega \neq 1}$ (no problem)

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$

$\underline{\omega = 1}$ (resonance)

$$y_p = At\cos(t) + Bt\sin(t)$$

Trick: Multiply the naïve guess for $y_p(t)$ by t^k where k is large enough to ensure that y_p is not of the same form as y_h.

Method of Undetermined Coefficients - Resonance

$ay'' + by' + cy = f(t)$

1. Solve the associated homogeneous eqn. to get $y_h(t)$:

$$y_h = c_1 y_1 + c_2 y_2$$

2. Determine the "family of functional forms" for $f(t)$ by differentiating:

   ex: $\cos(3t)$ $\quad \{\cos(3t), \sin(3t)\} \Rightarrow y_p = A\cos(3t) + B\sin(3t)$

   ex: $t^2 e^{-t}$ $\quad \left\{t^2 e^{-t}, te^{-t}, e^{-t}\right\} \Rightarrow y_p = At^2 e^{-t} + Bte^{-t} + Ce^{-t}$

   ex: $t\sin 2t$ $\quad \{t\sin 2t, t\cos 2t, \sin 2t, \cos 2t\} \Rightarrow y_p = \cdots$

   If the family for $f(t)$ has N members, $y_p(t)$ must have N L.I. terms.

3. Check for that none of the family members look like $y_1(t)$ or $y_2(t)$.

   Multiply family members by t until there is no more resonance.

Practice spotting resonance

(1) $y' + 6y = \cos t + t^2$

$$y_h = c_1 e^{-6t}$$

$$\text{family} = \{\cos t, \sin t, t^2, t, 1\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + E$$

(2) $y'' = t^2$ $\qquad y_h = c_1 + c_2 t$

$$\text{family} = \{t^2, \underline{t}, \underline{1}\}$$

$$y_p = At^2 + Bt^3 + Ct^4$$

(3) $y'' + 3y' + 2y = 5e^{-t}$

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

$$\text{family} = \{\underline{e^{-t}}\}$$

$$y_p = Ate^{-t}$$

(4) $y'' + 2y' + y = 12e^{-t}$

$$y_h = c_1 e^{-t} + c_2 te^{-t}$$

$$\text{family} = \{\underline{e^{-t}}\}$$

$$y_p = At^2 e^{-t}$$

(5) $y'' + 6y' = \cos t + t^2$

$$y_h = c_1 e^{-6t} + c_2$$

$$\text{family} = \{\cos t, \sin t, t^2, t, \underline{1}\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + Et^3$$

Find the general solution of $y'' + 5y' + 4y = e^{-4t}$

$$r_{1,2} = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

$$y_h = c_1 e^{-t} + \underbrace{c_2 e^{-4t}}_{\propto f(t)}$$

Try:

$$y_p = Ate^{-4t}$$

$$y_p' = A\left(e^{-4t} - 4te^{-4t}\right)$$

$$y_p'' = -Ae^{-4t} - 4A\left(e^{-4t} - 4te^{-4t}\right)$$

$$= -8Ae^{-4t} + 16Ate^{-4t}$$

plug into DE:

$$-8Ae^{-4t} + 16Ate^{-4t} + 5Ae^{-4t} - 20Ate^{-4t} + 4Ate^{-4t} = e^{-4t}$$

$$(-8+5)Ae^{-4t} + (20-20)te^{-4t} = e^{-4t}$$

$$-3Ae^{-4t} = e^{-4t}$$

$$A = -\frac{1}{3}$$

$$y = c_1 e^{-4t} + c_2 e^{-t} - \frac{1}{3} te^{-4t}$$

Find the general solution of $y'' + 4y' + 4y = e^{-2t}$

$$r_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = -2$$

$$y_h = \underbrace{c_1 e^{-2t}}_{\propto f(t)} + c_2 t e^{-2t}$$

Try:

$$\begin{aligned}
y_p &= At^2 e^{-2t} \\
y_p' &= A\left(2te^{-2t} - 2t^2 e^{-2t}\right) \\
y_p'' &= 2A\left(e^{-2t} - 2te^{-2t}\right) - 2A\left(2te^{-2t} - 2t^2 e^{-2t}\right) \\
&= 4At^2 e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t}
\end{aligned}$$

plug into DE:

$$4At^2e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t} + 8Ate^{-2t} - 8At^2e^{-2t} + 4At^2e^{-2t} = e^{-2t}$$

$$(-8+8)At^2e^{-2t} + (-8+8)te^{-2t} + 2Ae^{-2t} = e^{-2t}$$

$$2Ae^{-2t} = e^{-2t}$$

$$2A = 1 \qquad \Rightarrow A = \frac{1}{2}$$

$$y = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{2t}$$

Method of Undetermined Coefficients:

$$ay'' + by' + cy = f(t)$$
$$y_g(t) = y_p(t) + y_h(t)$$

Form of function $f(t)$	Guess for $y_p(t)$
$\sum_{j=0}^{N} d_j t^j$	$\sum_{j=0}^{N} A_j t^j$
$e^{\lambda t}$	$Ae^{\lambda t}$
$\sin(\omega t)$ or $\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$e^{\lambda t}\sin\omega t$ or $e^{\lambda t}\cos\omega t$	$e^{\lambda t}A\sin\omega t + e^{\lambda t}B\cos\omega t$
Additive combinations of above	Additive combinations of above
Multiplicative combinations of above	Multiplicative combinations of above
Part of the homogeneous solution Note[1]	$Atf(t)$ or $At^2f(t)$
Anything else	You are out of luck

[1]Note: This corresponds to resonance.

[2]Note: a, b, c, d_j , A_j , A, and B are all constants in the above table