

Recall: Linear 1<sup>st</sup> Order ODEs

$$y' + p(t)y = h(t)$$

Operator form:  $L[y] = h(t)$

- Linear 1<sup>st</sup> order operator  $L$
- $h(t)$  is a forcing term (**inhomogeneity**)
- Linear +  $h(t) = 0 \Rightarrow$  **Homogeneous ODE**
- Linear +  $h(t) \neq 0 \Rightarrow$  **Inhomogeneous ODE**

Solved by method of integrating factors:

$$\begin{aligned}\mu(t)y(t) &= \int \mu(t)h(t)dt + C \\ y(t) &= \frac{\int \mu(t)h(t)dt}{\mu(t)} + \underbrace{\frac{C}{\mu(t)}}_{\text{indep. of } h(t)}\end{aligned}$$

# General Solution Structure of Linear ODEs

ex: 1<sup>st</sup> Order Initial Value Problems

$$y' + p(t)y = h(t); \quad y(0) = y_0$$

Associated Homogeneous Problem:    Inhomogeneous DE:

$$\begin{aligned} y_h' + p(t)y_h &= 0 \\ \Rightarrow y_h &= \frac{C}{\mu(t)} \end{aligned}$$

$$y(t) = \frac{\int \mu(t)h(t)dt}{\mu(t)} + y_h(t)$$

$$y(t) = \underbrace{y_p(t)}_{\text{particular part}} + \underbrace{y_h(t)}_{\text{homogeneous part}}$$

All linear DEs have this type of solution structure.

$$y(t) = \text{particular part} + \text{homogeneous part}$$

# Linear 2<sup>nd</sup> order ODEs

General DE:

$$y'' + p(t)y' + q(t)y = h(t)$$

Initial Conditions:

$$y(t_0) = y_0, \quad y'(t_0) = v_0$$

Focus on constant coefficient case first

simplest case: homogeneous

$$ay'' + by' + cy = 0$$

We want intuition for how homogenous solutions work.

For non-constant coefficients: method of reduction of order (DiffyQs §2.1)

# Superposition Principle for Linear Homogeneous ODEs

Suppose the functions  $y_1(t)$  and  $y_2(t)$  both independently solve a linear homogeneous ODE

$$L[y] = 0$$

then

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution to the same ODE.

Proof:

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &\stackrel{\text{Linearity 1}}{=} L[c_1 y_1] + L[c_2 y_2] \\ &\stackrel{\text{Linearity 2}}{=} c_1 L[y_1] + c_2 L[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

# Completeness of solutions

Suppose that  $y_1(t)$  and  $y_2(t)$  both solve a  $2^{nd}$  order linear homogeneous ODE

$$L[y] = 0$$

If  $y_1$  and  $y_2$  are linearly independent functions, then the general solution to the homogeneous problem is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

**Take home message:** If you can find two linearly independent solutions to a homogeneous  $2^{nd}$  order linear DE, then you have found all of them

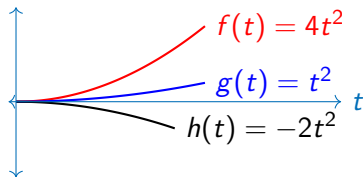
Proof: DiffyQs §2.1 (Theorem 2.1.3)

# Linear dependence of functions

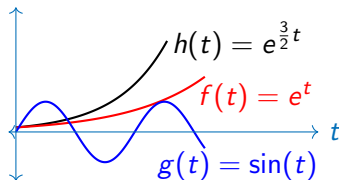
- Two functions  $f(t)$  and  $g(t)$  are **linearly dependent** on the interval  $t \in I = [\alpha, \beta]$  if there exist a non-zero constant  $k$  such that

$$f(t) = kg(t) \quad \forall t \in I$$

## Linearly Dependent



## Not Linearly Dependent



- If functions are not linearly dependent on  $I$ , then we say they are **linearly independent** on  $I$ .

## Solutions to $ay'' + by' + cy = 0$

Lets try an ansatz of  $y(t) = e^{rt}$  ... where  $r$  is unknown (ansatz method)

$$a \frac{d^2}{dt^2} e^{rt} + b \frac{d}{dt} e^{rt} + c e^{rt} = 0$$

$$ar^2 e^{rt} + br e^{rt} + c e^{rt} = 0 \qquad e^{rt} (ar^2 + br + c) = 0$$

Since  $e^{rt} \neq 0$ , the ODE can only have a solution  $y(t) = e^{rt}$  if

$$ar^2 + br + c = 0.$$

This is called the **characteristic equation**.

Possible values of  $r$ :

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

# Roots of the characteristic equation (polynomial)

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three main cases:

1. Two distinct real roots:  $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

2. Repeated real roots: discriminant = 0

3. Complex conjugate roots: discriminant < 0



## Case 1: distinct real roots

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}; \quad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two major subcases:

1.  $ac > 0$  :

Both roots have the same sign.

$y_1$  and  $y_2$  both grow or decay exponentially.

2.  $ac < 0$ :

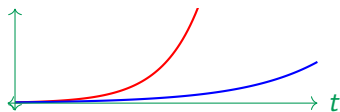
The two roots have opposite sign.

One solution grows exponentially, the other decays exponentially.

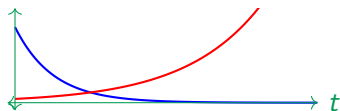
# Qualitative Behaviour: distinct real roots

Sum of real exponential functions, three subcases:

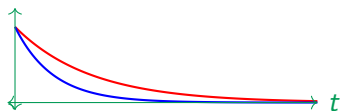
1. All positive roots,  $0 < r_1 < r_2$ .



2. Mixed roots,  $r_1 < 0 < r_2$ .



3. All negative roots,  $r_1 < r_2 < 0$ .



# Summary

- Superposition of Homogeneous solutions

$$y_h = c_1 y_1 + c_2 y_2$$

- Linear independence of homogeneous solutions

$$y_g = c_1 y_1 + c_2 y_2 \quad \Rightarrow \quad y_1 \neq k y_2; \quad k = \text{constant}$$

$y_g$  = general solution, solves ALL scenarios where a solution exists.

- $ay'' + by' + cy = 0$

Try ansatz  $y = e^{rt} \quad \rightarrow \quad e^{rt} \cdot \underbrace{(ar^2 + br + c)}_{\text{characteristic polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$