Introduction

There are many scenarios where catastrophic behaviour arises:

- Plastic deformation of solids
- Financial collapse
- Climate change tipping points

Two typical properties:

1. Small change in a parameter produces drastic change in system equilibrium.
2. Hard to reverse, cannot just reverse the small parameter change (hysteresis)

Simple model of a catastrophe

We can model a generic catastrophe by modifying the dynamics of the Van der Pol oscillator:

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = \frac{x}{2} - b - y$$

where b is some parameter

Find the nullclines for this model.

x-nullcline:

$$y = \frac{1}{3}x^3 - x$$

y-nullcline:

$$y = \frac{x}{2} - b$$

The system

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = \frac{x}{2} - b - y$$

has a critical point at (0,0) when $b = 0$. Classify it.

$$\mathbf{J} = \begin{bmatrix} -x^2 + 1 & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \qquad \mathbf{J}^* = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\det(\mathbf{J}^* - \lambda I) = (1-\lambda)(-1-\lambda) - \frac{1}{2} = 0$$

$$\lambda^2 - 1 - \frac{1}{2} = 0$$

$$\lambda^2 - \frac{3}{2} = 0$$

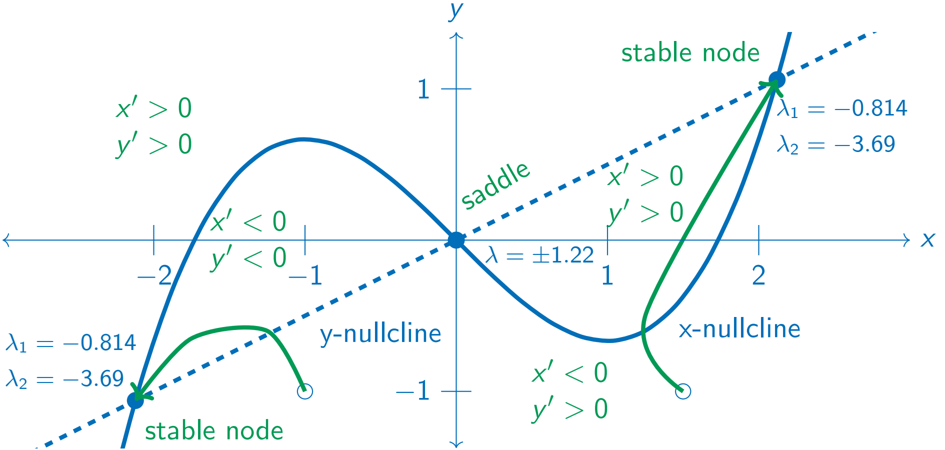
$$\lambda = \pm\frac{\sqrt{4\frac{3}{2}}}{2}$$
$$= \pm\sqrt{3/2}$$
$$\approx \pm 1.22$$

(saddle)

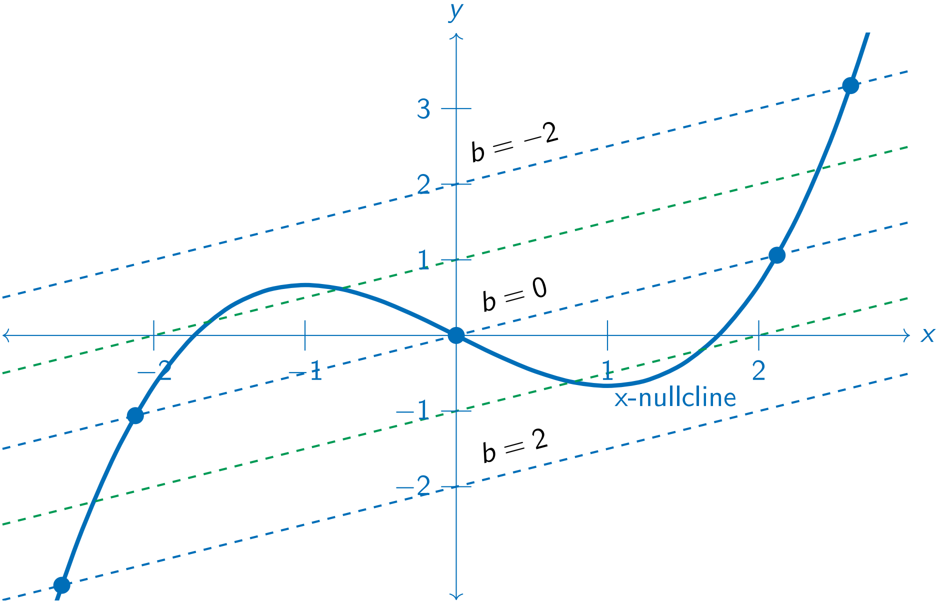
Sketch solution trajectories for the system

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = \frac{x}{2} - b - y$$

with $b = 0$ and initial conditions at (-1,-1) and (1.5,-1).



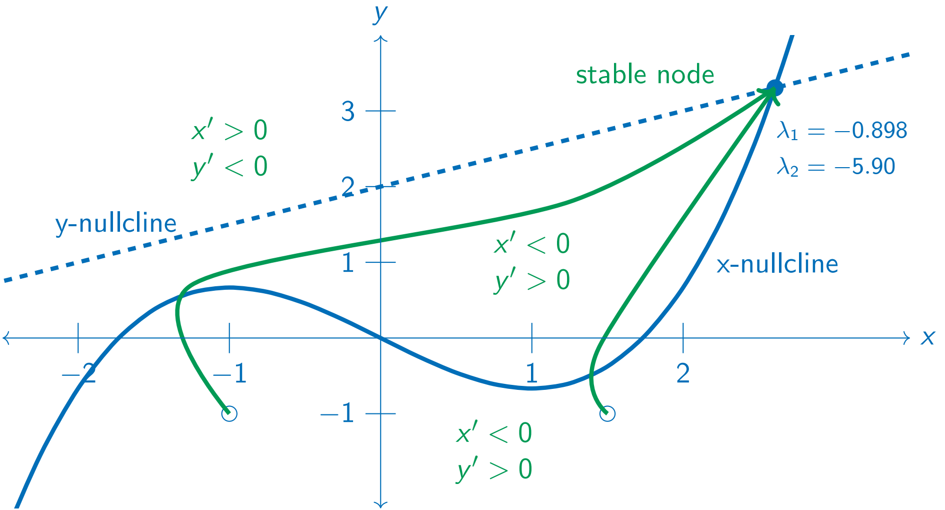
Varying b changes the number of nullcline intersections



Sketch solution trajectories for the system

$$x' = y - \frac{1}{3}x^3 + x, \quad y' = \frac{x}{2} + 2 - y$$

with initial conditions at (-1,-1) and (1.5,-1).



Catastrophe and Hysteresis