Recall: we have always rearranged products as sums

ex: $Y(s) = \frac{1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs + D}{s^2 + 1}$

After partial fraction decomposition...

$$A = 0, B = 1, C = 0, D = -1$$

$$\begin{aligned} Y(s) &= \frac{1}{s^2} - \frac{1}{s^2 + 1} \\ &= \mathcal{L}\{t\} - \mathcal{L}\{\sin(t)\} \end{aligned}$$

$$y(t) = t - \sin(t)$$

It is possible to deal with the product directly!

Convolutions

We denote the convolution of two functions f and g by the symbol $f * g$, with

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Convolutions are:

1. Commutative/Symmetric
   - $f * g = g * f$
2. Linear
   - $f * (g + h) = f * g + f * h$, where h is a function
   - $f * (cg) = c(f * g) = (cf) * g$, where c is a constant
3. Associative
   - $f * (g * h) = (f * g) * h$

Convolutions are useful for inverting products of Laplace Transforms

Convolution Theorem

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$ are known functions, then

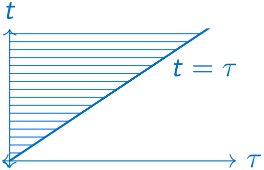
$$\boxed{\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g} = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

or conversely

$$\boxed{\mathcal{L}\{f * g\} = F(s) \cdot G(s)}$$

Proof of the convolution theorem with $h(t) = f(t) * g(t)$

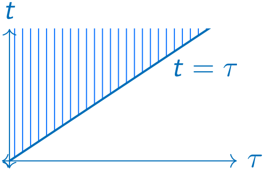
$$\mathcal{L}\{h(t)\} = \int_0^{\infty} e^{-st} h(t) dt = \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau) g(t-\tau) e^{-st} d\tau \, dt$$



equivalent areas

$\Leftrightarrow$

switch integration order



$$= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau) g(t-\tau) e^{-st} dt \, d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau) e^{-s\tau} \int_{t=\tau}^{\infty} g(t-\tau) e^{-s(t-\tau)} d\tau \, dt \qquad \text{let } u = t - \tau$$

$$= \underbrace{\int_{\tau=0}^{\infty} f(\tau) e^{-s\tau} d\tau}_{F(s)} \underbrace{\int_{u=0}^{\infty} g(u) e^{-su} du}_{G(s)} \qquad t = \tau \Rightarrow u = 0$$

$$= F(s) G(s)$$

ex: $y'' + y = t$ with $y(0) = y'(0) = 0$.

Use the convolution theorem to find $y(t)$.

$$s^2 Y(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \mathcal{L}\left\{t^2\right\} \cdot \mathcal{L}\left\{\sin(t)\right\}$$

$$y(t) = t * \sin(t) = \int_0^t (t-\tau)\sin(\tau)d\tau$$

$$\overset{\text{by parts}}{=} \left[-t\cos(\tau) - \sin(\tau) + \tau\cos(\tau)\right]_{\tau=0}^{t}$$

$$= \cancel{-t\cos(t)} - \sin(t) + \cancel{t\cos(t)} + t\cos(0) + \cancel{\sin(0)} + \cancel{0\cos(0)}$$

$$= t - sin(t)$$

ex: $y'' + y = \sin(t)$ with $y(0) = y'(0) = 0.$

Use the convolution theorem to find $y(t)$.

$$\begin{aligned}
s^s Y(s) + Y(s) &= \frac{1}{s^2+1} \\
Y(s) &= \frac{1}{s^2+1} \cdot \frac{1}{s^2+1} \\
y(t) &= \int_0^t \sin(t-\tau)\sin(\tau)d\tau \\
&= \int_0^t (\sin(t)\cos(\tau) - \cos(t)\sin(\tau))\sin(\tau)d\tau \\
&= \sin(t)\int_0^t \cos(\tau)\sin(\tau)d\tau - \cos(t)\int_0^t \sin^2(\tau)d\tau
\end{aligned}$$

hint: $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$$y(t) = \sin(t) \int_0^t \cos(\tau)\sin(\tau)d\tau - \cos(t) \int_0^t \sin^2(\tau)d\tau$$

$$y(t) = \frac{\sin(t)}{2} \int_0^t \sin(2\tau)d\tau - \frac{\cos(t)}{2} \int_0^t (1-\cos(2\tau))d\tau$$

$$= -\frac{\sin(t)}{4}\left[\cos(2\tau)\right]_{\tau=0}^{t} - \frac{\cos(t)}{2}\left[\tau - \frac{\sin(2\tau)}{2}\right]_{\tau=0}^{t}$$

$$= -\frac{\sin(t)(\cos(2t)-1)}{4} - \frac{t\cos(t)}{2} + \frac{\cos(t)\sin(2t)}{4}$$

$$= \frac{1}{4}\sin(t) - \frac{t}{2}\cos(t) - \frac{1}{4}\underbrace{\cos(2t)}_{1-2\sin(t)}\sin(t) + \frac{1}{4}\underbrace{\sin(2t)}_{2\cos(t)\sin(t)}\cdot\cos(t)$$

$$= \cancel{\frac{1}{4}\sin(t)} - \frac{t}{2}\cos(t) - \cancel{\frac{1}{4}\sin(t)} + \frac{2}{4}\cancelto{1}{\left(\sin^2(t)+\cos^2(t)\right)}\sin(t)$$

$$= \frac{1}{2}\sin(t) - \frac{t}{2}\cos(t)$$

hint: $\cos(\alpha)\sin(\alpha) = \frac{1}{2}\sin(2\alpha)$ and $\sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$

ex: $y'' + y = g(t)$ with $y(0) = 3,\ y'(0) = 5.$

Use the convolution theorem to find an general expression for $y(t)$.

$$\begin{aligned} s^2 Y(s) - 3s - 5 + Y(s) &= G(s) \\ (s^2+1)Y(s) &= G(s) + 3s + 5 \\ Y(s) &= \frac{G(s)}{s^2+1} + 3\frac{s}{s^2+1} + 5\frac{1}{s^2+1} \end{aligned}$$

$$y(t) = \sin(t) * g(t) + 3\cos(t) + 5\sin(t)$$

$$y(t) = \underbrace{\int_0^t \sin(t-\tau)g(\tau)d\tau}_{\text{particular part}} + \underbrace{3\cos(t) + 5\sin(t)}_{\text{homogeneous part}}$$

We can solve whole classes of ODEs at once.

The convolution theorem also allows us to solve integro-differential equations. ex: $y'(t) = e^t + 2\int_0^t y(t-\tau)e^\tau\,d\tau$ with $y(0) = 0$

$$sY(s) = \frac{1}{s-1} + 2\mathcal{L}\left\{\int_0^t y(t-\tau)e^\tau\,d\tau\right\} = \frac{1}{s-1} + 2Y(s)\frac{1}{s-1}$$

$$\left(s - \frac{2}{s-1}\right)Y(s) = \frac{1}{s-1} \qquad \Rightarrow \qquad \frac{\overbrace{s^2 - s - 2}^{(s-2)(s+1)}}{\cancel{s-1}}Y(s) = \frac{1}{\cancel{s-1}}$$

$$Y(s) = \frac{1}{(s-2)(s+1)}$$

$$y(t) = \int_0^t e^{2\tau}e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t e^{3\tau}\,d\tau = \frac{e^{-t}}{3}\left[e^{3\tau}\right]_{\tau=0}^{t}$$

$$= \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

Convolution Theorem: Special Case

Let $f(t)$ be an integrable function and $g(t) = 1$.

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = F(s)\mathcal{L}\left\{1\right\}$$

$$= \frac{F(s)}{s} \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

ex: Find the inverse Laplace tranforms of $H(s) = \frac{1}{s(s+7)}$

$$h(t) = \int_0^t e^{-7\tau}d\tau = -\frac{1}{7}\left[e^{-7\tau}\right]_{\tau=0}^{t} = -\frac{1}{7}\left(e^{-7t} - 1\right)$$