Recall: Constant Coefficient Homogeneous 2^{nd} Order ODE

$ay'' + by' + cy = 0$ we need 2 linearly independent solutions

Try ansatz $y = e^{rt}$ $\rightarrow$ $e^{rt} \cdot \underbrace{(ar^2 + br + c)}_{\text{characterisitc polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

   $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2. Repeated real roots: discriminant $= 0$

3. Complex conjugate roots: discriminant < 0

Repeated real root ($r_1 = r_2 = r$)

Straighforward solution

$$y_1 = e^{rt}$$

with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

We need another solution that is linearly independent of y_1

Lets try

$$y_2 = ty_1$$

Checking $y_2 = te^{rt}$

$$ay'' + by' + cy = 0 \qquad \text{with } b^2 - 4ac = 0$$

Try: $y_2 = te^{rt}$ $\quad \Rightarrow r_{1,2} = r = \frac{-b}{2a}$

$$y_2' = e^{rt} + rte^{rt}$$

$$y_2'' = 2re^{rt} + r^2te^{rt}$$

plug these into the ODE

$$a\left(2re^{rt} + r^2te^{rt}\right) + b\left(e^{rt} + rte^{rt}\right) + ce^{rt} = 0$$

$$\left(ar^2 + br + c\right)te^{rt} + (2ar + b)e^{rt} = 0$$

sub in $r = \frac{-b}{2a}$

$$\underbrace{\left(\cancel{a}\frac{b^2}{4a^{\cancel{2}}} - \frac{b^2}{2a} + c\right)}_{\frac{1}{4a}(4ac-b^2)=0} te^{rt} + \underbrace{\left(\cancel{2a}\frac{-b}{\cancel{2a}} + b\right)}_{0} e^{rt} = 0$$

$$0 = 0 \checkmark$$

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1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

   $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2. Repeated real roots: discriminant $= 0$
   $y_h = c_1 e^{rt} + c_2 t e^{rt}$

3. Complex conjugate roots: discriminant < 0

Solve the IVP: $y'' + 4y' + 4y = 0$ $\quad y(0) = 2$, $y'(0) = 0$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 4}}{2} = \frac{-4 \pm 0}{2} =, -2$$

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

initial conditions:

$$y(0) = 2 = c_1$$

$$y'(0) = 0 = -2c_1 + c_2(e^{-2t} - 2te^{-2t})\Big|_{t=0}$$

$$= -4 + c_2 \qquad \Rightarrow \qquad c_2 = 4$$

$$\boxed{y(t) = 2e^{-2t} + 4te^{-2t}}$$

Complex roots ($b^2 - 4ac < 0$)

Roots are given by:

$$r_1 = \lambda + i\mu \qquad \text{where } i = \sqrt{-1}$$
$$r_2 = \lambda - i\mu$$

$$\lambda = \frac{-b}{2a}, \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

The two functions $e^{(\lambda+i\mu)t}$ & $e^{(\lambda-i\mu)t}$ are solutions to $ay'' + by' + cy = 0$.

What is the exponential of a complex number?

Euler's formula:

$$e^{\pm i\alpha} = \cos\alpha \pm i \sin\alpha$$

$$e^{(\lambda \pm i\mu)t} = e^{\lambda t} e^{\pm i\mu t}$$
$$= \underbrace{e^{\lambda t}}_{\text{Real}} \underbrace{[\cos(\mu t) \pm i \sin(\mu t)]}_{\text{Complex}}$$

We don't want complex solutions, try a linear combination of the two.

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = \frac{e^{\lambda t + i\mu t} + e^{\lambda t - i\mu t}}{2}$$
$$\tilde{y}_1 = \frac{e^{\lambda t}}{2} [\cos(\mu t) + \cancel{i \sin(\mu t)} + \cos(\mu t) - \cancel{i \sin(\mu t)}] = \frac{e^{\lambda t}}{2} 2 \cos(\mu t)$$
$$= e^{\lambda t} \cos(\mu t)$$

Similarly, $\tilde{y}_2 = \dfrac{y_1 - y_2}{2i} = \dfrac{e^{\lambda t + i\mu t} - ie^{\lambda t - i\mu t}}{2i} \quad \rightarrow \quad \tilde{y}_2 = e^{\lambda t} \sin(\mu t)$

Complex roots ($r_{1,2} = \lambda \pm i\mu$)

The functions $y_1 = e^{\lambda t}\cos(\mu t)$ and $y_2 = e^{\lambda t}\sin(\mu t)$ are linearly independent real solutions.

Sketch the two functions if you are not convinced.

General solution: $y_h = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$

Find the general solution to: $y'' + 6y = 0$

$$r_{1,2} = \frac{\pm\sqrt{-4 \cdot 6}}{2} = \pm\sqrt{-6} = \pm i\sqrt{6}$$

$$y_h = c_1 \cos\left(\sqrt{6}t\right) + c_2 \sin\left(\sqrt{6}t\right)$$

Solve the IVP: $y'' + 2y' + 5y = 0$ $\qquad y(0) = 1$, $y'(0) = -1$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}}{2} i = -1 \pm 2i$$

$$y_h = e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$

initial conditions:

$$y(0) = 1 = c_1$$

$$y'(0) = -1 = -c_1 + \left(-2c_1 \sin(0) + 2c_2 \cos(0)\right) = -c_1 + 2c_2$$

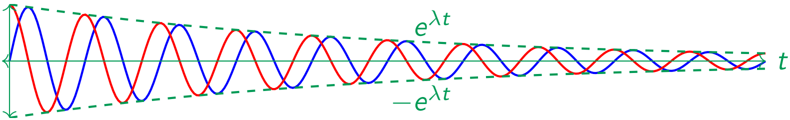
$$-1 = -1 + 2c_2 \qquad \Rightarrow \qquad c_2 = 0$$

$$\boxed{y(t) = e^{-t} \cos(2t)}$$

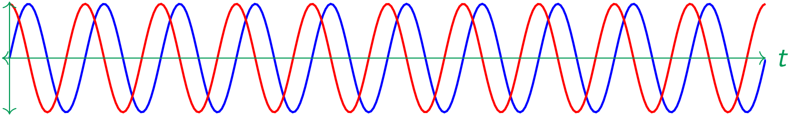
Qualitative Behaviour: complex roots

Three subcases:

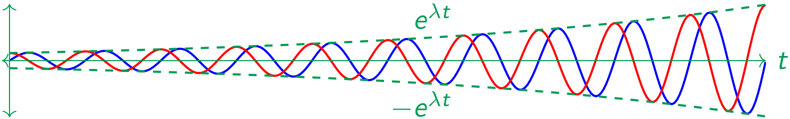
1. $\lambda < 0 \quad \Rightarrow \quad$ Exponentially decaying oscillations.



2. $\lambda = 0 \quad \Rightarrow \quad$ Sustained periodic oscillations.



3. $\lambda > 0 \quad \Rightarrow \quad$ Exponentially growing oscillations.



Summary

- For linear ODEs:
  - Pick an ansatz (e.g., e^{rt})
  - Write down the characteristic equation
  - Find the roots
    - If you don't have enough functions, make a new one by multiplying by t
- Write down the general solution according to the roots
  - Real and distinct $\Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - Real and repeated $\Rightarrow y_h = c_1 e^{rt} + c_2 t e^{rt}$
  - Complex $\Rightarrow y_h = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$
- Fit the constants c_1 and c_2 to the initial conditions