Recall: Linearization + Local Behaviour

The non-linear autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \vec{f}(\vec{x}(t))$$

can often be well-approximated near a critical point $\vec{x}^*$ using

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} \approx \mathbf{J}^*\vec{x} - \mathbf{J}^*\vec{x}^* \quad \text{where} \quad \mathbf{J}^* = \mathbf{J}(\vec{x}^*) \quad \text{with} \quad \mathbf{J} = \frac{\mathrm{d}\vec{f}}{\mathrm{d}\vec{x}}.$$

The eigenvalues of the Jacobian matrix, evaluated at a critical point, determine the <u>local</u> behaviour of solutions in some neighbourhood around the critical point.

Classification of 2D Nonlinear Critical Points

Eigenvalues	Stability	Type
$\lambda_1 \leq \lambda_2 < 0$	asymptotically stable	sink (node)
$0 < \lambda_1 \leq \lambda_2$	unstable	source (node)
$\lambda_1 < 0 < \lambda_2$	unstable	saddle
one of $\lambda_1, \lambda_2 = 0$	unclear	unclear
$r \pm i\omega$ with $r < 0$	asymptotically stable	spiral sink
$r \pm i\omega$ with $r > 0$	unstable	spiral source
$r \pm i\omega$ with $r = 0$	unclear	unclear

For the unclear cases, we need higher order approximations...(not covered)

How to find critical points?

Given

$$x' = f(x, y), \quad y' = g(x, y)$$

we want to find the set of points (x^*, y^*) where

$$f(x^*, y^*) = 0 \quad \text{and} \quad g(x^*, y^*) = 0.$$

Consider each constraint separately $\quad \Rightarrow \quad$ nullclines

$$0 = f(x, y) \qquad\qquad 0 = g(x, y)$$

each expression gives some number of curves (i.e. nullclines).

Nullcline = set of points that makes a derivative zero.

Critical points exist at the intersection of the x- and y-nullclines.

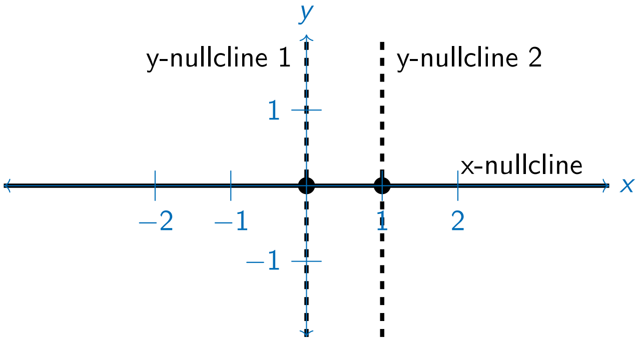
ex: $x' = y$ and $y' = -x + x^2 = x(x-1)$

x-nullclines:

$$0 = y$$
$$y = 0$$

y-nullclines:

$$0 = x(x-1)$$
$$x = 0 \quad \text{and} \quad x = 1$$



critical points: $(0,0)$ and $(1,0)$

ex: Find all critical points for $x' = x - x^2 - 2xy$, $y' = 2y - 2y^2 - 3xy$

x-nullclines:

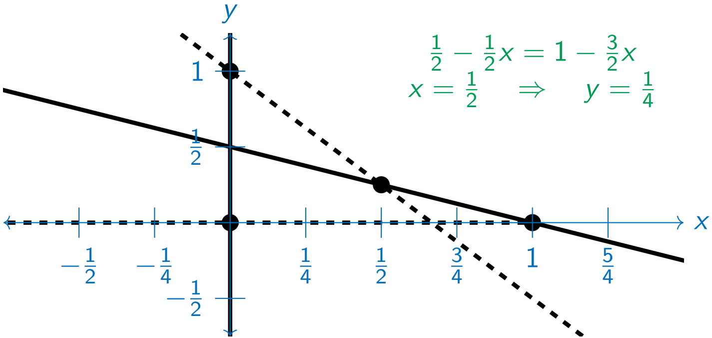
$$0 = x(1 - x - 2y)$$

$x = 0$ and $y = \frac{1}{2} - \frac{1}{2}x$

y-nullclines:

$$0 = y(2 - 2y - 3x)$$

$y = 0$ and $y = 1 - \frac{3}{2}x$



$\frac{1}{2} - \frac{1}{2}x = 1 - \frac{3}{2}x$

$x = \frac{1}{2} \quad \Rightarrow \quad y = \frac{1}{4}$

critical points: $(0,0)$, $(1,0)$, $(0,1)$, and $(1/2 . 1/4)$

ex: Find all critical points for $x' = yx - x - x^3$, $y' = y(x^2 + 1 - y)$

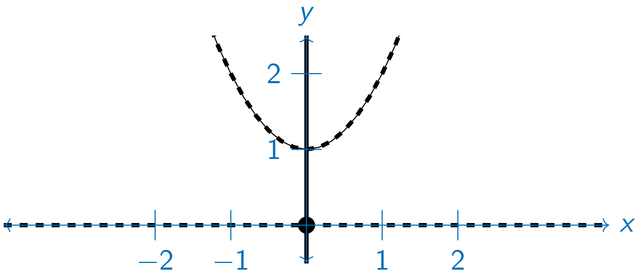
x-nullclines:

$$0 = x(y - 1 - x^2)$$

$x = 0$ and $y = 1 + x^2$

y-nullclines:

$y = 0$ and $y = 1 + x^2$



isolated critical point: $(0, 0)$ non-isolated critical points: $y = 1 + x^2$

A critical point is isolated if it is the only critical point in some small "neighborhood" of the point.

Almost Linear Systems - Conditions for Linearization

A system is called **almost linear** at a critical point $\vec{x}^*$ if

1. the critical point is isolated and
2. the Jacobian matrix at the point is invertible.

In such a case, the higher order terms can safely be ignored and the system behaves like its linearization close to the critical point.

ex: $x' = x^2$, $y' = y^2$ Isolated fixed point at $(0, 0)$.

$$\mathbf{J} = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \quad \Rightarrow \quad \mathbf{J}^* \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This Jacobian matrix cannot be inverted, so we cannot study the fixed point using a linear approximation.