

## Recall:

We saw how to use integration to solve the following first order ODEs

$$y' = f(t), \quad y' = f(y), \quad \text{and} \quad y' = g(y)f(t).$$

What happens if

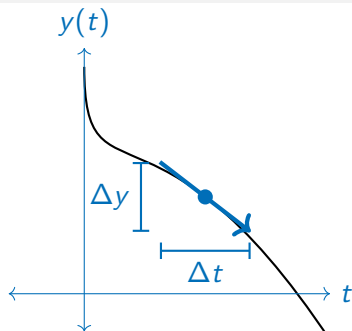
1. we cannot solve one of the integrals, or
2. we get a more general ODE of the form  $y' = f(y, t)$ ?

Exact analytical solutions are not always possible.

- 99.99% of all the possible ODEs involve impossible integrals.

Today we will build intuition around how to find approximate solutions.

# Graphical intuition, whats does $y' = f(y, t)$ mean?



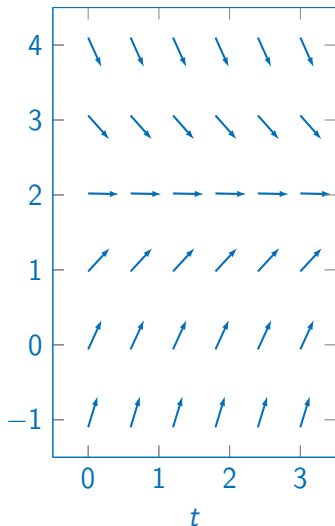
$$\frac{dy}{dt} = f(y, t)$$

$$f(y, t) = \text{slope of } y(t) \text{ in the } (t, y)\text{-plane.}$$
$$\approx \frac{\Delta y}{\Delta t}$$

## Slope Field Diagram:

1. Draw an arrow with slope  $f(y, t)$  for some points in the  $(t, y)$ -plane.
2. Starting from some initial condition(s), trace along with the arrows to get approximate solutions.

## Example: $y' = 2 - y$ , try different initial conditions



1. What type of solutions are possible?
  - Monotonically increasing/decreasing
2. What is  $y(t)$  as  $t \rightarrow \infty$ ?
  - Unique possibility:  $y(t) \rightarrow 2$
  - $y = 2$  is a stable steady state.
3. What is the influence of the initial condition?
  - If  $y(0) > 2$  decreasing solution.
  - If  $y(0) < 2$  increasing solution.
  - If  $y(0) = 2$  constant solution.

## slopefield.m

A convenient MATLAB function that will plot slopefields for you!

```
function slopefield(f,t,y)

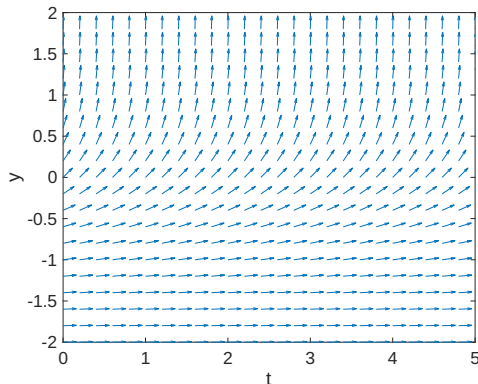
[T,Y] = meshgrid(t,y);
dydt = f(T,Y);
theta = atan(dydt);
L = min(min(diff(t)),min(diff(y)))*0.8;
dy = L*sin(theta);
dt = L*cos(theta);
quiver(t,y,dt,dy,0)
axis equal; axis([t(1),t(end),y(1),y(end)]);

end
```

Copy-paste into a file called **slopefield.m** in your MATLAB development environment

ex:  $y' = e^{2y}$

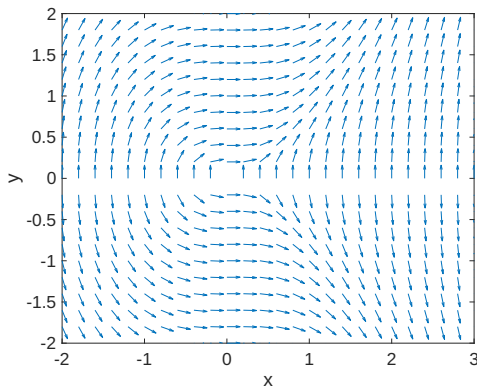
```
f = @(t,y) exp(2*y);  
t = 0:0.2:5;  
y = -2:0.2:2;  
slopefield(f,t,y)
```



Note: The slope is positive for all values of  $y$ . Blowup happens for all initial conditions.

ex:  $y' = \frac{x^2}{y}$

```
f = @(t,y) t.^2./y;  
t = -2:0.2:3;  
y = -2:0.2:2;  
slopefield(f,t,y)
```



Trace a solution backwards from  $y(0) = 1$ .

Why does the solution not exist for  $x < -\sqrt[3]{3/2}$ ?

$f(x, y)$  is discontinuous  $y = 0$

# Picard's Theorem

A unique solution to

$$y' = f(t, y), \quad \text{with } y(t_0) = y_0$$

exists for  $t$  near  $t_0$  if

1.  $f(t, y)$  is continuous (as a function of two variables) and
2.  $\partial f / \partial y$  exists and is continuous near  $(t_0, y_0)$ .

ex:  $y' = 2\sqrt{|y|}$ , with  $y(0)=0$

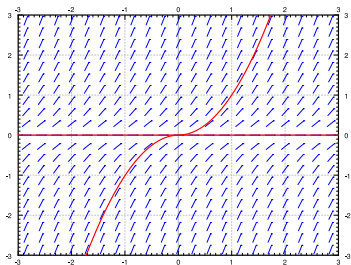
Analytically, we find 2 solutions

1. By inspection:

$$y(t) = 0$$

2. Integrate for positive and negative  $y$  separately:

$$y(t) = \begin{cases} t^2 & t \geq 0 \\ -t^2 & t < 0 \end{cases}$$



Picard's uniqueness theorem does not apply, because  $\frac{\partial}{\partial y} 2\sqrt{|y|}$  does not exist at  $y = 0$ .