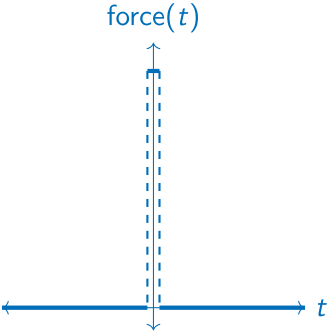
Imagine Hitting a Golf Ball

source: `https://www.youtube.com/watch?v=6TA1s1oNpbk&t=80s`

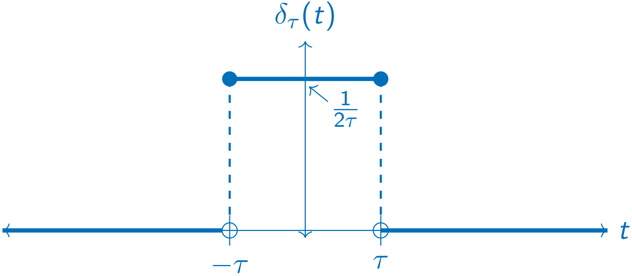
The ball is initially at rest, and suddenly a large force is applied to the ball for a very brief period of time.

The details of what happens at $t = 0$ are too complicated.



Idea: Approximate the force as an infinitesimally short impulse.

Normalized Step Pulse (δ_τ)



Function:

$$\delta_\tau = \begin{cases} \frac{1}{2\tau} & |t| \le \tau \\ 0 & |t| > \tau \end{cases}$$

$$= \frac{u(t+\tau) - u(t-\tau)}{2\tau}$$

Integral:

$$I(\tau) = \int_{-\infty}^{\infty} \delta_\tau(t)dt = \int_{-\tau}^{\tau} \frac{dt}{2\tau}$$

$$= 1$$

Taking the limit $\tau \to 0$

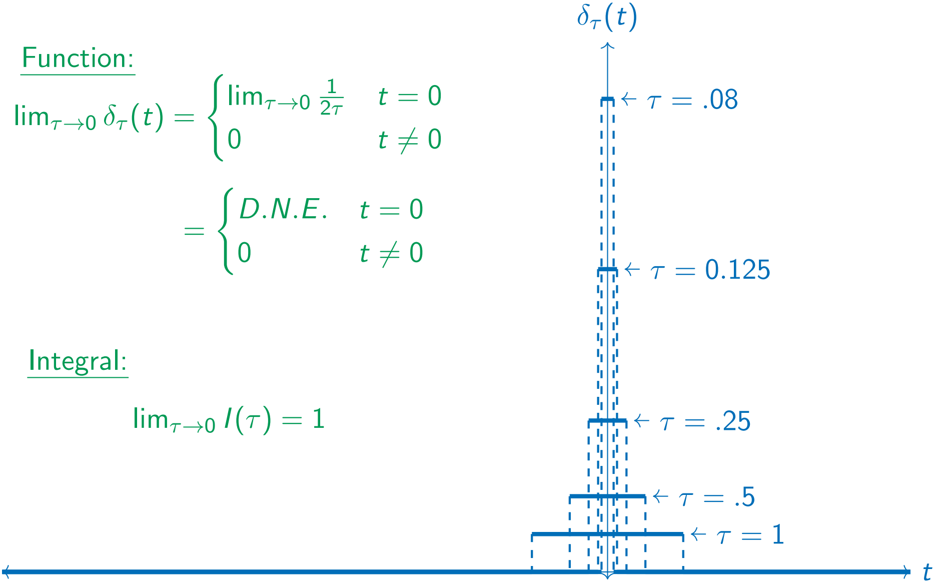
Function:

$$\lim_{\tau\to 0} \delta_\tau(t) = \begin{cases} \lim_{\tau\to 0} \frac{1}{2\tau} & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$= \begin{cases} D.N.E. & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Integral:

$$\lim_{\tau\to 0} I(\tau) = 1$$



Delta Dirac Function: $\delta(t) \approx \lim_{\tau \to 0} \delta_\tau(t)$

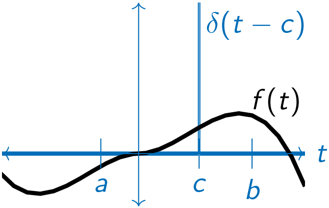
Theorem: For any function $f(t)$ that is integrable in some neighbourhood around c

$$\int_{-\infty}^{\infty} \delta(t-c) f(t)dt = f(c)$$

Integrating against $\delta(t-c)$ essentially "selects" the value of the integrand at $t = c$.

More generally

$$\int_a^b \delta(t-c) f(t)dt = \begin{cases} f(c) & a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$$



Sketch of "proof":

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(t-c)f(t)dt &= \int_{-\infty}^{\infty} \left(\lim_{\tau\to 0} \delta_\tau(t-c)\right) f(t)dt \\
&= \lim_{\tau\to 0} \int_{-\infty}^{\infty} \delta_\tau(t-c)f(t)dt \\
&= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{c-\tau}^{c+\tau} f(t)dt \\
&= \lim_{\tau\to 0} \frac{F(c+\tau)-F(c-\tau)}{2\tau} = f(c)
\end{aligned}$$

Laplace transform of $\delta(t)$

Integrating against $\delta(t-c)$ essentially "selects" the value of the integrand at $t = c$

Assuming $c > 0$

$$\begin{aligned}\mathcal{L}\left\{\delta(t-c)\right\} &= \int_0^\infty e^{-st}\delta(t-c)dt \\ &= \int_{-\infty}^\infty e^{-st}\delta(t-c)dt = e^{-sc}\end{aligned}$$

Special case: $c = 0$

$$\mathcal{L}\left\{\delta(t)\right\} = \lim_{c\to 0^+} \mathcal{L}\left\{\delta(t-c)\right\} = \lim_{c\to 0^+} e^{-sc} = 1$$

Solve: $y'' + 6y' + 45y = 6\delta(t-5)$ with $\begin{array}{l} y(0) = 0 \\ y'(0) = 0 \end{array}$

$$(s^2 + 6s + 45)Y(s) = 6e^{-5s} \quad \Rightarrow \quad Y(s) = e^{-5s} \cdot \underbrace{\frac{6}{\underbrace{s^2 + 6s + 45}_{(s+3)^2+36}}}_{\mathcal{L}\{e^{-3t}\sin(6t)\}}$$

by the convolution theorem

$$\begin{aligned} y(t) &= \delta(t-5) * \underbrace{e^{-3t}\sin(6t)}_{f(t)} \\ &= \int_0^t f(t-\tau)\delta(\tau-5)d\tau \\ &= \begin{cases} 0 & t < 5 \\ f(t-5) & t \geq 5 \end{cases} \\ &= u(t-5)f(t-5) \end{aligned}$$

Delta Dirac "Function": $\delta(t) \approx \lim_{\tau \to 0} \delta_\tau(t)$

- **Note:** $\delta(t)$ is not really well-defined in the conventional sense
  - It is a "generalized" function with the three following properties:

    1. $\delta(t) = 0$ for $t \neq 0$

       - $\delta_\tau(0)$ D.N.E. in the limit $\tau \to 0$, which is problematic.

       - Rather than its value, this "function" is defined via integral properties

    2. $\int_{-\infty}^{\infty} \delta(t)dt = 1$

    3. $\int_a^b \delta(t-c)f(t)dt = \begin{cases} f(c) & a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$

- The delta function acts like an intense pulse of unit strength.
  - This is also called an **impulse**:
    - An action that happens arbitrarily fast but with finite magnitude.
    - ex: accelerating a golf ball with a golf club