Recall: We previsouly saw...

1. General linear 1st order ODE: $y' + p(t)y = h(t)$

   - To solve, multiply by $\mu(t)$ turn the LHS into a total derivative:

$$y' + p(t)y \quad \to \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot y) = \mu y' + \mu' y$$

2. Total derivative of the function of two variables $\Phi(x, y)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(x, y) = \Phi_x + \Phi_y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

   or equivalently

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{with } M_y = N_x$$

What can we do if $M_y \neq N_x$?

Idea: multiply by an integrating factor u(x,y)

General Idea

Suppose we have

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{with } M_y \neq N_x,$$

we want to multiply the whole equation by $u(x, y)$.

Goal:

$$(uM)_y = (uN)_x$$

$$u_y M + u M_y = u_x N + u N_x \qquad \text{This is a PDE for } u(x, y)$$

PDEs are outside the scope of this course.

We restrict ourselves to two cases:

1. $u = u(x)$
2. $u = u(y)$

ex: Solve $(1-x-y^2)+2y\dfrac{dy}{dx}=0$ Not separable, not linear

$$M_y=-2y, \quad N_x=0\neq M_y \qquad \text{not exact}$$

Try $u=u(y)$

$$(uM)_y=(uN)_x$$

$$u_yM+uM_y=\cancel{u_x}^{0}N+u\cancel{N_x}^{0}$$

$u'(y)(1-2x-y^2)-u(y)2y=0$ Not solvable (x-dependence)

Try $u=u(x)$

$$(uM)_y=(uN)_x$$

$$\cancel{u_y}^{0}M+uM_y=u_xN+u\cancel{N_x}^{0}$$

$$-u(x)\cancel{2y}=u'(x)\cancel{2y} \qquad u'=-u$$

$$u(x)=Ce^{-x} \qquad \text{arbitrarily choose } C=1$$

ex: Solve $(1-x-y^2)e^{-x} + 2ye^{-x}\frac{dy}{dx} = 0$ We know this DE is exact!

$$\Phi(x,y) = \int M(x,y)dx + h(y)$$

$$= \int (1-x-y^2)e^{-x}dx + h(y)$$

$$= \underbrace{(1-y^2)\int e^{-x}dx}_{(y^2-1)e^{-x}} - \underbrace{\int xe^{-x}dx}_{I(x)} + h(y)$$

$$I(x) = \int xe^{-x}dx \qquad \text{let} \quad \begin{array}{l} u = x,\ du = dx \\ dv = e^{-x}dx,\ v = -e^{-x} \end{array}$$

$$= uv - \int vdu = xe^{-x} + \int e^{-x}dx = (x-1)e^{-x}$$

$$\Phi(x,y) = (y^2+x)e^{-x} + h(y)$$

$$\Phi_y = 2ye^{-x} + h'(y) = 2ye^{-x} \qquad \Rightarrow h' = 0 \Rightarrow h = C$$

$$\Phi(x,y) = (y^2+x)e^{-x} + C \qquad \text{Solution:} \quad \boxed{(y^2+x)e^{-x} = C}$$

ex: Find the integrating factor for $2x + (x^2 + 2e^y)\frac{\mathrm{d}y}{\mathrm{d}x}$

$$M = 2x, \quad N = x^2 + 2e^y, \qquad \text{with } M_y = 0 \neq N_x = 2x$$

Not exact, try $u = u(x)$

$$(uM)_y = (uN)_x$$

$$\cancelto{0}{u_y} M + u\cancelto{0}{M_y} = u_x N + uN_x$$

$$0 = u'(x^2 + 2e^y) + 2xu(y) \qquad \text{Not separable, not linear}$$

Try $u = u(y)$

$$(uM)_y = (uN)_x$$

$$u_y M + u\cancelto{0}{M_y} = \cancelto{0}{u_x} N + uN_x$$

$$u'(y)\cancel{2x} = u(y)\cancel{2x} \qquad u' = u$$

$$\boxed{u(y) = e^y}$$

ex: Consider $\Phi(x, y) = x^2y^3 + y^2x^3 = 2$.

1. Find a non-exact DE whose solution is given by expression above.
2. Find a solution value for x when $y = 1$.

1. Start by making an exact DE: $\Phi_x + \Phi_y \dfrac{dy}{dx} = 0$

$$(2xy^3 + 3y^2x^2) + (3x^2y^2 + 2yx^3)\frac{dy}{dx} = 0$$

divide by the common factor xy

$$(2y^2 + 3yx) + (3xy + 2x^2)\frac{dy}{dx} = 0$$

$$M_y = 4y + 3x \qquad N_x = 3y + 4x \neq M_y$$

2. Plug $y = 1$ into the expression

$$x^2 + x^3 = 2 \qquad \text{by inspection, we find } x = 1$$