Recall:

Operator form of a DE:

$$\mathrm{L}\left[y(t)\right] = f(t)$$

For the next few weeks, we will focus on first order DEs

$$y' = f(t, y).$$

We start with the two simple cases

$$y' = f(t) \quad \text{and} \quad y' = f(y)$$

Integrals as solutions

$$y' = f(t) \quad \text{and} \quad y' = f(y)$$

Strategy:

1. Move everything that depends on y to one side of the equal sign.
2. Move everything that depends on t to the other side.
3. Integrate and isolate y.

ex: $y' = \cos(t)$

$$\frac{dy}{dt} = \cos(t)$$

$$\int dy = \int \cos(t)dt$$

$$y(t) = \sin(t) + C \qquad \text{(General Solution)}$$

Incorporating Initial Conditions

ex: $y' = e^{-2t}$, with $y(0) = 1$

$$\frac{dy}{dt} = e^{-2t}$$
$$\int dy = \int e^{-2t} dt$$
$$y(t) = -\frac{1}{2}e^{-2t} + C \qquad \text{(General Solution)}$$

impose the initial condition

$$y(0) = 1 = -\frac{1}{2} + C \qquad \Rightarrow \qquad C = 1 + \frac{1}{2}$$

$$y(t) = \frac{3}{2} - \frac{1}{2}e^{-2t} \qquad \text{(Particular Solution)}$$

ex: $y' = e^{2y}$, with y(1)=0

$$\frac{dy}{dt} = e^{2y} \qquad \int e^{-2y}\,dy = \int dt$$

$$-\frac{1}{2}e^{-2y} = t + C$$

$$e^{-2y} = -2t - 2C$$

$$y(t) = -\frac{1}{2}\ln(-2t - 2C)$$

impose the initial condition

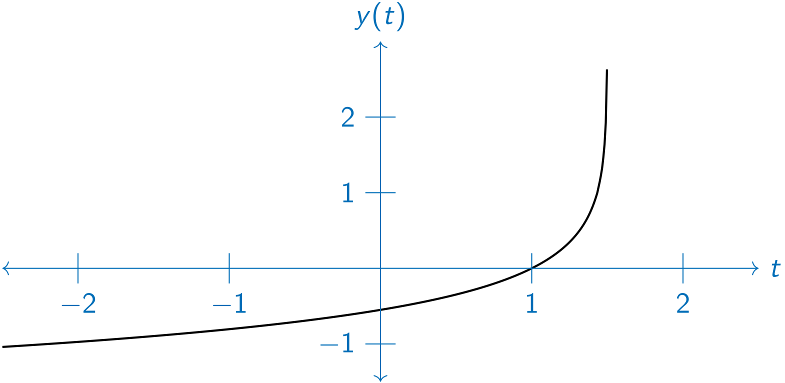
$$y(1) = 0 = -\frac{1}{2}\ln(-2 - 2C) \qquad \Rightarrow -2 - 2C = 1$$

$$C = -\frac{3}{2}$$

$$y(t) = -\frac{1}{2}\ln(-2t + 3)$$

ex: $y' = e^{2y}$, with y(1)=0 $\quad\Rightarrow\quad y(t) = -\frac{1}{2}\ln(-2t+3)$

Solution blows-up in finite time!



Domain of definition: $t \in \left(-\infty, \frac{3}{2}\right)$

Outside this domain, the solution does not exist.

Separable Equations

Suppose you are given

$$\frac{dy}{dx} = f(x)g(y)$$

where the functions f and g are known. Proceeding as before...

$$\frac{\mathrm{d}y}{g(y)} = f(t)dt$$

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

$$\Gamma(y) = F(t) + C$$

$$y(t) = \Gamma^{-1}\left(F(t) + C\right)$$

Works as long as $1/g(y)$ and $f(t)$ are integrable functions.

ex: $y' = -ty, \quad y(0) = 5$

$$\int \frac{\mathrm{d}y}{y} = \int -t\mathrm{d}t$$

$$\ln(y) = -\frac{t^2}{2} + C_1$$

exponentiate both sides

$$y = e^{-\frac{t^2}{2}+C_1} = e^{C_1}e^{-\frac{t^2}{2}} = C_2e^{-\frac{t^2}{2}}$$

impose the initial condition

$$y(0) = 5 = C_2$$

$$\boxed{y(t) = 5e^{-\frac{t^2}{2}}}$$

ex: $\dfrac{dy}{dx} = \frac{x^2}{y}, \quad y(0) = 1$

$$\int y\,dy = \int x^2 dx$$

$$\frac{1}{2}y^2(x) = \frac{1}{3}x^3 + C$$

$$y = \pm\sqrt{\frac{2}{3}x^3 + C}$$

impose the initial condition for both cases

$$y(0) = 1 = \sqrt{C} \qquad\qquad y(0) = 1 = -\sqrt{C}$$

$$C = 1 \qquad\qquad C \text{ does not exist}$$

$$y(x) = \sqrt{\frac{2}{3}x^3 + 1} \qquad\qquad \text{for } x > -\sqrt[3]{3/2}$$

ex: $y' = y^2 \quad y(0) = 1$

$$\int \frac{\mathrm{d}y}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C$$

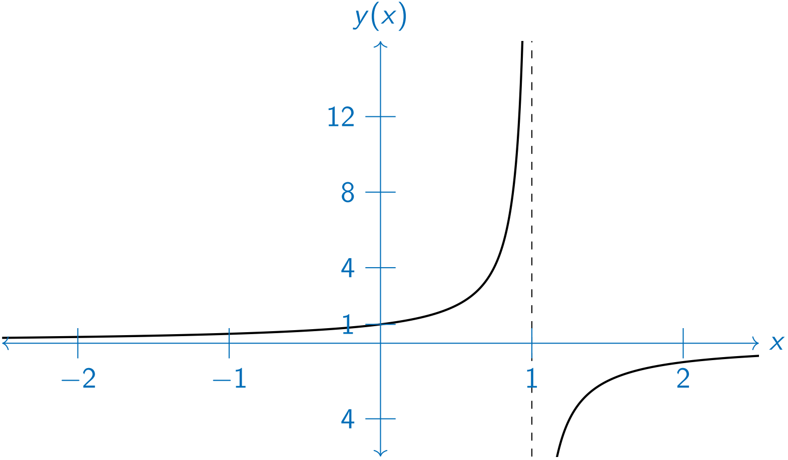
$$y(x) = -\frac{1}{x+C}$$

impose the initial conditions

$$y(0) = 1 = -\frac{1}{C} \qquad C = -1$$

$$y(x) = -\frac{1}{x-1}$$

$y(x) = -\frac{1}{x-1}$



Domain of definition: $t \in (-\infty, 1) \cup (1, \infty)$