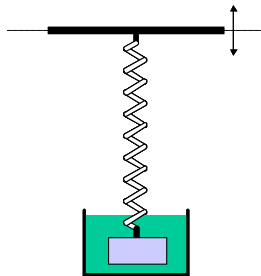


Recall: Spring-dashpot system without forcing



$x(t)$ = displacement from rest position

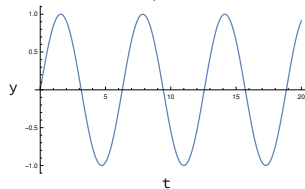
Newton's 2nd Law:

$$F = ma$$

$$x'' + \beta x' + kx = 0$$

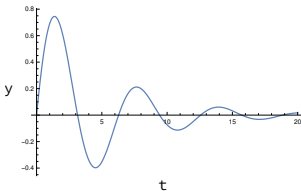
Simple Harmonic Motion

$$\beta = 0$$



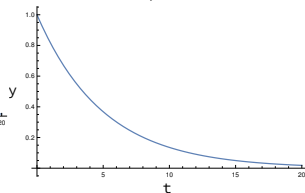
Underdamped Motion

$$\beta > 0$$

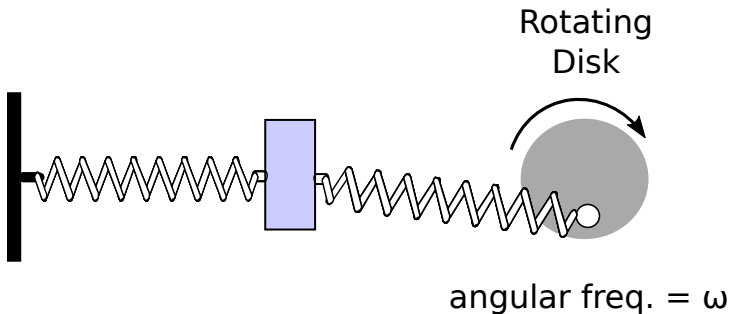


Overdamped Motion

$$\beta \gg 0$$



Spring oscillators with forcing



Harmonic motion with forcing

$$x'' + \omega_0^2 x = f(t) = \frac{F_0}{m} \cos(\omega t) \quad \text{with} \quad \begin{matrix} \omega \neq \omega_0 \\ x(0) = x'(0) = 0 \end{matrix}$$

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \quad x_p = A \cos(\omega t) + B \sin(\omega t)$$

$$x_p'' = -\omega^2 (A \cos(\omega t) + B \sin(\omega t))$$

$$-\omega^2 (A \cos(\omega t) + B \sin(\omega t)) + \omega_0^2 (A \cos(\omega t) + B \sin(\omega t)) = \frac{F_0}{m} \cos(\omega t)$$

$$\underline{\sin(t)} : \quad -\omega^2 B + \omega_0^2 B = 0 \quad \Rightarrow B = 0$$

$$-\omega^2 A + \omega_0^2 A = \frac{F_0}{m} \quad \Rightarrow A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t) \quad \text{with} \quad \omega \neq \omega_0$$

$$x(0) = x'(0) = 0$$

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Initial Conditions:

$$x(0) = 0 = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$c_1 = \frac{F_0}{m(\omega^2 - \omega_0^2)}$$

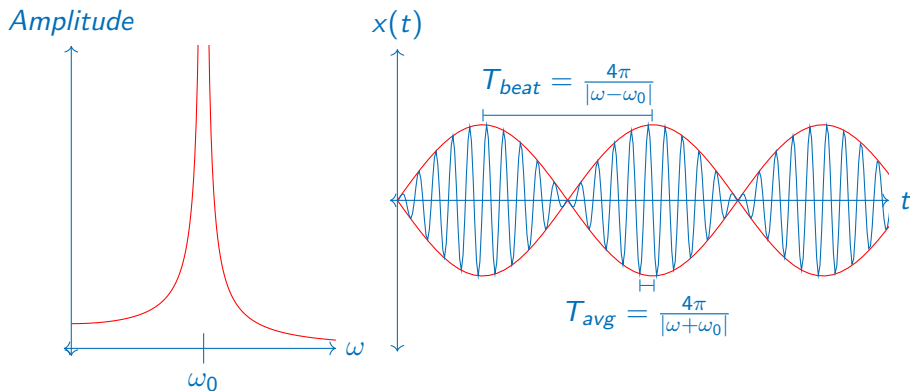
$$x'(0) = 0 = c_2$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \underbrace{(\cos(\omega t) - \cos(\omega_0 t))}_{\sin\left(\frac{\omega_0 + \omega}{2}t\right) \sin\left(\frac{\omega_0 - \omega}{2}t\right)}$$

MATLAB HW3: Find the amplitude of motion numerically.

Beat phenomena

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2}t\right) \sin\left(\frac{\omega_0 - \omega}{2}t\right)$$



What happens to $x(t)$ as $\omega \rightarrow \omega_0$?

Resonance ($\omega \rightarrow \omega_0$)

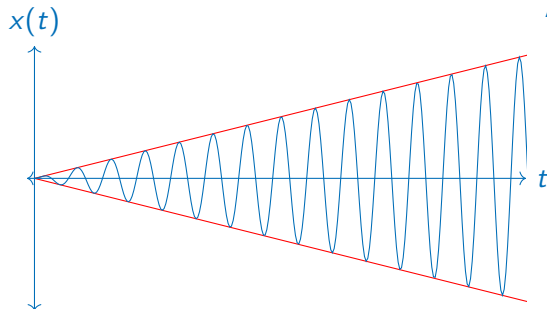
$$x'' + \omega_0^2 x = f(t) = \frac{F_0}{m} \cos(\omega_0 t)$$

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \quad x_p = ?$$

$$y_p \neq A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

$$y_p = \cancel{A}t \cos(\omega_0 t) + Bt \sin(\omega_0 t)$$

$$B = \frac{F_0}{2m\omega_0}$$



As $t \rightarrow \infty$

Amplitude $\rightarrow \infty$

Finite input power

Very large response

Physical Resonance

Damped oscillator with forcing

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

$$x(t) = x_h(t) + x_p(t)$$

Three cases:

$$x_h(t) = \begin{cases} e^{-\frac{\beta}{2m}t} (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)) & \beta^2 < 4km \\ c_1 e^{r_1 t} + c_2 e^{r_2 t} & \beta^2 > 4km \\ c_1 e^{rt} + c_2 t e^{rt} & \beta^2 = 4km \end{cases}$$

All exponents are negative

$$\Rightarrow x_h \rightarrow 0 \text{ as } t \rightarrow \infty$$

Damped oscillator with forcing

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

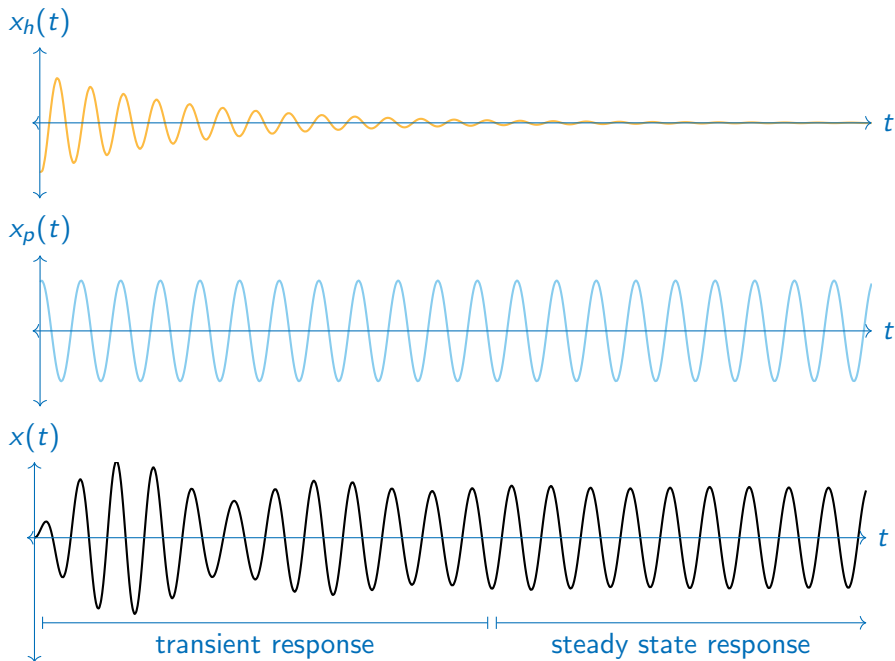
$$x(t) = x_h(t) + x_p(t)$$

From the Method of Undetermined Coefficients:

$$x_p = A \cos(\omega t) + B \sin(\omega t)$$

As $t \rightarrow \infty$

$$x(t) \rightarrow x_p(t)$$



$$mx'' + \beta x' + kx = \cos(\omega t)$$

How does the amplitude of the steady state response vary with ω ?

$$x_p = A \cos(\omega t) + B \sin(\omega t)$$

$$x_p' = -\omega A \sin(\omega t) + \omega B \cos(\omega t)$$

$$x_p'' = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)$$

$$\begin{aligned} m(-\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)) + \beta(-\omega A \sin(\omega t) + \omega B \cos(\omega t)) \\ + k(A \cos(\omega t) + B \sin(\omega t)) = \cos(\omega t) \end{aligned}$$

$$\begin{aligned} (Ak + B\beta\omega - mA\omega^2) \cos(\omega t) + (Bk - A\beta\omega - mB\omega^2) \sin(\omega t) \\ = \cos(\omega t) \end{aligned}$$

Match coefficients

$$\underline{\cos}: \quad (Ak + B\beta\omega - mA\omega^2) = 1$$

$$\underline{\sin}: \quad (Bk - A\beta\omega - mB\omega^2) = 0$$

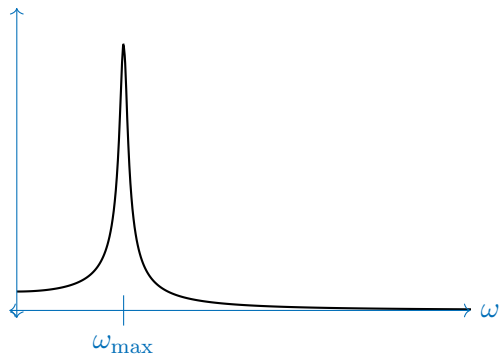
$$B = A \frac{\beta\omega}{k - m\omega^2}$$

$$\left(Ak + A \frac{(\beta\omega)^2}{k - m\omega^2} - mA\omega^2 \right) = 1$$

$$A = \frac{k - m\omega^2}{(\beta\omega)^2 - (k - m\omega^2)^2} \quad B = \frac{\beta\omega}{(\beta\omega)^2 - (k - m\omega^2)^2}$$

Amplitude:

$$\begin{aligned}
 R &= \sqrt{A^2 + B^2} \\
 &= \frac{\sqrt{(k - m\omega^2)^2 + (\beta\omega^2)^2}}{(\beta\omega)^2 - (k - m\omega^2)^2} \\
 &= \frac{1}{\sqrt{k^2 - 2km\omega^2 + m^2\omega^4 + \omega^2\beta^2}}
 \end{aligned}$$



MATLAB HW3:

Find ω_{\max} numerically.

Amplitude vs Forcing Frequency

Steady-state response: $x_p = R \cos(\omega t - \phi)$ with

$$\begin{aligned} R(\omega) &= \frac{1}{\sqrt{k^2 - 2km\omega^2 + m^2\omega^4 + \omega^2\beta^2}} \\ &= \frac{1}{\sqrt{\beta^2\omega^2 + (k - m\omega^2)^2}} \end{aligned}$$

What value of ω creates the largest amplitude response?

$$\frac{d}{d\omega} R = -\frac{\omega (\beta^2 - 2km + 2m^2\omega^2)}{(\beta^2\omega^2 + (k - m\omega^2)^2)^{3/2}} = 0$$

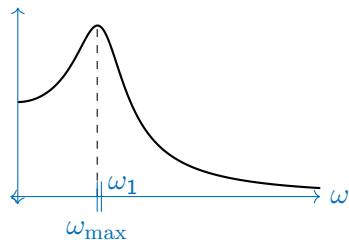
Critical points: $\omega^* = 0, \quad \pm \sqrt{\frac{k}{m} - \frac{1}{2} \left(\frac{\beta}{m} \right)^2}$

Amplitude vs Forcing Frequency

Underdamped:

$$x_h = e^{-\frac{\beta}{2m}t} (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$$

$$\omega_1 = \sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}$$

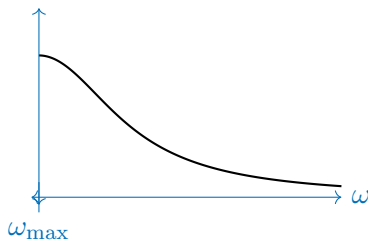


$$\omega_{\max} = \sqrt{\frac{k}{m} - \frac{1}{2} \left(\frac{\beta}{m}\right)^2} \approx \omega_1$$

Overdamped:

$$x_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$



$$\omega_{\max} = 0$$

Resonance with damped oscillators?

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

What happens as $\omega \rightarrow \omega_{\max}$?

- Transient response
 - amplitude and duration both grow larger
- Steady state response
 - amplitude grows larger

This phenomenon is called quasi-resonance.

ω_{\max} is called the practical resonance frequency