Differential Equations

1. What are they and why do we solve them?
2. Basic Terminology
3. Separable Equations

Disclaimer

- Differential equations can be pretty hard - for many different reasons
  - Lots of confusing vocabulary terms
  - Lots of "tricks"
- This course is all about teaching you as many techniques as possible in a very short time.
  - Many things will not fully make sense this term, you may only understand them in a later course.

Disclaimer

- Please ask questions if you are confused! - I may seem annoyed.
  - I may get annoyed with how little time we have to cover a deep topic.
  - I am most likely annoyed at myself for explaining things poorly.
  - I am almost certainly not annoyed with you.
- I will makes lots of helpful algebraic mistakes and clever typographic errors that are carefully designed to assist your learning experience.
  - Please point them out! - Discussion is always good.

What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. one or more independent variables.

Given: A DE with an unknown function $y(t)$.

e.x., $$\frac{dy}{dt} = -3y(t)$$ or $$y' = -3y$$

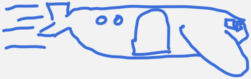
Task: Find the function(s) $y(t)$.

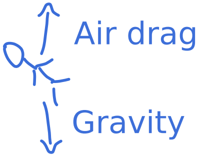
Solution: $y(t) = C_1 e^{-3t}$

Tools:

- Calculus (i.e., integration/differentiation)
- Geuss and check (does some function $f(t)$ make LHS=RHS?)
- Specialized procedures (informed by experience geussing)
- Geometry/Linear Algebra (useful for systems of DEs)

Example: Skydiving





Newton's Second Law:

$$F = ma$$

$$ma = \underbrace{-mg}_{\text{gravitational force}} \quad \underbrace{-\mu v}_{\text{drag force}}$$

$$a = v'$$

$$\boxed{mv' = -mg - \mu v} \qquad \text{DE for } v(t)$$

Terminology

- Ordinary differential equations (ODEs)
  - A DE with derivatives w.r.t. only one independent variable.
  - $\sim$ 80% of this course
- Partial differential equations (PDEs)
  - A DE with multiple derivatives (e.g., $\partial/\partial t$ and $\partial/\partial x$)
  - Partial derivatives are necessary for solutions to match when working in different coordinate systems
  - Polar coordinates
  - Spherical coordinates

Terminology

- Linear DEs
  - Linear combination of the function and its derivatives.
  - Linear: $c_1 + c_2 y + c_3 y' + c_4 y'' = 0$
  - Nonlinear: $c_1(y)^2 + c_2 yy' + c_3(y'')^3 = 0$

- Order of a DE
  - Pick out the highest derivative of $y(x)$ in the DE.
  - If n is the number of derivatives, then the order of the ODE is also n.
  - First order: $y' + 3y^2 = e^x$
  - Fourth order: $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x^2$

Terminology

- Solution of a differential equation
  - Any function that satisfies the eq (i.e., makes LHS=RHS) for <u>all</u> values of the independent variable(s).
  - This seems very silly, but when in doubt it is the most useful thing to remember.

This is not a differential eqaution

Suppose

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2$$

Find $y(t)$ Multiply both sides by dt

$$\frac{\mathrm{d}y}{\mathrm{d}t}dt = t^2 dt$$

Integrate

$$\int dy = \int t^2 dt$$

$$y(t) + C_1 = \frac{t^3}{3} + C_2$$

$$y(t) = \frac{t^3}{3} + C$$

Since $y(t)$ does not appear in the equation - not a differential equation.

A first order linear ODE

Suppose

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 y(t)$$

Find $y(t)$ Divide by y, then mulitply dt

$$\frac{dy}{\cancel{dt}}\frac{1}{y}\cancel{dt} = t^2 dt$$

integrate

$$\int \frac{dy}{y} = \int t^2 dt$$

$$\ln(y) = \frac{t^3}{3} + C$$

exponentiate both sides

$$y(t) = e^{\frac{t^3}{3}+C} = Ce^{\frac{t^3}{3}}$$

A first order nonlinear ODE

Suppose

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos(3t)y^2$$

Find $y(t)$

$$\frac{dy}{\cancel{dt}}\frac{1}{y^2}\cancel{dt} = \cos(3t)dt$$

$$\int \frac{dy}{y^2} = \int \cos(3t)dt$$

$$\frac{-1}{y} = \frac{\sin(3t)}{3} + C_1$$

$$y(t) = \frac{-1}{\frac{\sin(3t)}{3} + C_1} = \frac{-3}{\sin(3t) + C}$$

The generic separable first order ODE

Suppose

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

the dependence on t and y can be divided up into two factors multiplying each other. The functions f and g are known.

$$\frac{dy}{\cancel{dt}}\frac{1}{g(y)}\cancel{dt} = f(t)dt$$

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

$$\Gamma(y) = F(t) + C$$

$$y(t) = \Gamma^{-1}\left(F(t) + C\right)$$

Works as long as $1/g(y)$ and $f(t)$ are integrable functions.

Summary

1. What are DEs?
   - Equations involving unknown function(s) and function derivatives.
   - Specify rates of change of certain quantities.
   - Useful for modelling many natural phenomena.

2. Terminology
   - ODEs (& PDEs).
   - Order of DEs, systems of DEs, solutions to DEs.

3. Separable equations:
   - Move everything related to the unknown function on one side, and everything related to the independent variable on the other side.
   - Integrate, then isolate the unknown function.