Recall: Initial Value Problems (IVPs).

$$ay'' + by' + cy = h(t), \qquad \begin{array}{l} y(0) = y_0 \\ y'(0) = v_0 \end{array}$$

This is a natural approach for scenarios where we know the initial state of a system (e.g., mass-spring or electrical circuit) and want to predict it behaviour at later times.

In other scenarios, we might know the state of the system at two times $t = 0$ and $t = T$.

$$ay'' + by' + cy = h(t), \qquad \underbrace{\begin{array}{l} y(0) = y_0 \\ y(T) = y_T \end{array} \quad \text{or} \quad \begin{array}{l} y'(0) = v_0 \\ y'(T) = v_T \end{array}}_{\text{boundary conditions}}$$

We call these boundary value problems (BVPs).

Periodic BVPs

In cases where $h(t)$ is a periodic function with period T, i.e.,

$$h(t + T) = h(t) \qquad \forall t,$$

the solution $y(t)$ eventually becomes periodic.

Ignoring any intial transient solutions, we can find the long-term periodic solution by solving an ODE with periodic boundary conditions, given by

$$ay'' + by' + cy = h(t), \qquad \begin{aligned} y(0) &= y(T) \\ y'(0) &= y'(T) \end{aligned}$$

ex: $y'' + y = \cos(2\pi t)$, $\quad \begin{array}{l} y(0) = y(1) \\ y'(0) = y'(1) \end{array}$

$$y(t) = y_h + y_p \qquad \begin{array}{l} y_h = c_1 \cos(t) + c_2 \sin(t) \\ y_p = A\cos(2\pi t) + B\sin(2\pi t) \end{array}$$

M. U. C.: $A = \dfrac{1}{1-4\pi^2}, \ B = 0$

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + \frac{1}{1-4\pi^2}\cos(2\pi t)$$

Boundary Conditions:

$$y(0) = c_1 + \cancel{\frac{1}{1-4\pi^2}} = y(1) = c_1\cos(1) + c_2\sin(1) + \cancel{\frac{1}{1-4\pi^2}}$$

$$c_1 = c_1\cos(1) + c_2\sin(1)$$

$$y'(0) = c_2 = y'(1) = -c_1\sin(1) + c_2\cos(1)$$

$$c_2 = -c_1\sin(1) + c_2\cos(1)$$

$$\begin{aligned} c_1 &= c_1\cos(1) + c_2\sin(1) \\ c_2 &= -c_1\sin(1) + c_2\cos(1) \end{aligned}$$

$$c_2(1-\cos(1)) = -c_1\sin(1)$$

$$c_2 = -\frac{\sin(1)}{1-\cos(1)}c_1$$

$$c_1 = c_1\cos(1) - \frac{\overbrace{\sin^2(1)}^{1-\cos^2(1)}}{1-\cos(1)}c_1$$

$$c_1 = c_1\left(\cos(1) - \frac{(1+\cos(1))\cancel{(1-\cos(1))}}{\cancel{1-\cos(1)}}\right)$$

$$c_1 = c_1\left(\cancel{\cos(1)} - (1+\cancel{\cos(1)})\right)$$

$$c_1 = -c_1 \qquad \Rightarrow c_1 = c_2 = 0$$

$$\boxed{y(t) = \frac{1}{1-4\pi^2}\sin(2\pi t)}$$

Alternatively, we can notice that the homogeneous solution have period 2π whereas the particular solution has period 1. Since the boundary conditions require solutions with period 1, we can only keep the particular solution.

ex: $y'' + y = f(t)$, $\quad \begin{array}{l} y(0) = y(1) \\ y'(0) = y'(1) \end{array}$ with $f(t+1) = f(t)$

Due to its periodicity, we can express $f(t)$ as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + b_n \sin(2n\pi t)$$

This is called the **Fourier Series** of the function $f(t)$, the coefficients a_n and b_n are called **Fourier coefficients**.

The coefficients are obtained by taking the inner product of the function $f(t)$ and the Fourier basis

$$\{\cos(2n\pi t),\ \sin(2n\pi t)\} \qquad n = 0, \ldots, \infty$$

$y'' + y = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + b_n \sin(2n\pi t)$ Let's guess

$$y(t) = \sum_n y_n(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi t) + B_n \sin(2n\pi t)$$

Apply M.U.C. term-by-term for the different values of n.

For $n \neq 0$, the n^{th} particular solution is

$$\begin{aligned} y_n &= A_n \cos(2n\pi t) + B_n \sin(2n\pi t) \\ y_n'' &= -4n^2\pi^2 A_n \cos(n\pi t) - 4n^2\pi^2 B_n \sin(n\pi t) \\ \underline{\text{ODE}}: \quad y_n'' + y_n &= a_n \cos(2n\pi t) + b_n \sin(2n\pi t) \end{aligned}$$

$$\begin{aligned} A_n(1 - 4n^2\pi^2)\cos(n\pi t) + B_n(1 - 4n^2\pi^2)\sin(n\pi t) & \qquad A = \frac{a_n}{1 - 4n\pi^2} \\ = a_n \cos(n\pi t) + b_n \sin(n\pi t) & \qquad B = \frac{b_n}{1 - 4n\pi^2} \end{aligned}$$

$y'' + y = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$

For $n = 0$, we have

$$\begin{aligned} y_0 &= A, y_0'' = 0 \\ A &= \frac{1}{2}a_0 \end{aligned}$$

$$y(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{a_n}{1-4n\pi^2}\cos(2n\pi t) + \frac{b_n}{1-4n\pi^2}\sin(2n\pi t)$$

Given a specific periodic $f(t)$, we can find its Fourier coefficients a_n and b_n and use the solution above.

Fourier Series

Given any periodic function $f(t)$ with period T, we can approximate $f(t)$ as a Fourier series

$$f(t) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)$$

with

$$\begin{aligned}
a_0 &= \langle f(t), 1 \rangle && = \frac{2}{T}\int_0^T f(t)dt \\
a_n &= \left\langle f(t), \cos\left(\frac{2n\pi t}{T}\right)\right\rangle && = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt \\
b_n &= \left\langle f(t), \sin\left(\frac{2n\pi t}{T}\right)\right\rangle && = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt
\end{aligned}$$

If $f(t)$ is a continuous function, then the approximation becomes an equality.

The Fourier Basis

Computing the Fourier series of a function $f(t)$ is effectively projecting $f(t)$ onto the space of T-periodic functions, decomposing the function into infinitely many components.

Vector

$$\vec{x} = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}^T$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c_1 = \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$c_3 = \vec{x} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2π Periodic Function

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

$$a_n = \langle f(t), \cos(n\pi t) \rangle$$
$$b_n = \langle f(t), \sin(n\pi t) \rangle$$

The Fourier Basis is Orthonormalized

Consider m and n to be any two positive integers or zero, then we have

$$\left\langle \cos\left({}^{2n\pi t}/_{T}\right), \sin\left({}^{2m\pi t}/_{T}\right)\right\rangle = 0 \qquad \forall m, n \qquad \text{(Orthogonality)}$$

$$\begin{aligned}\left\langle \sin\left({}^{2n\pi t}/_{T}\right), \sin\left({}^{2m\pi t}/_{T}\right)\right\rangle &= \left\langle \cos\left({}^{2n\pi t}/_{T}\right), \cos\left({}^{2m\pi t}/_{T}\right)\right\rangle \\ &= \begin{cases} 1 & \text{if } m = n \qquad \text{(Normalization)} \\ 0 & \text{otherwise} \qquad \text{(Orthogonality)} \end{cases}\end{aligned}$$

The normalization condition is the reason for the factors of $\frac{2}{T}$ (or $\frac{1}{T}$) in front of the Fourier coefficient integrals.