$$\text{time-domain} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \text{s-domain}$$

$f(t)$ $\leftrightarrow$	$F(s)$
C $\leftrightarrow$	$\frac{C}{s}$
Ce^{-at} $\leftrightarrow$	$\frac{C}{s+a}$
t^n $\leftrightarrow$	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$ $\leftrightarrow$	$\frac{\omega}{\omega^2+s^2}$
$\cos(\omega t)$ $\leftrightarrow$	$\frac{s}{\omega^2+s^2}$
$\vdots$ $\leftrightarrow$	$\vdots$

Argument Scaling: $t \to \alpha t$ with constant α

$$
\begin{aligned}
\mathcal{L}\{f(\alpha t)\} &= \int_0^\infty e^{-st} f(\alpha t) dt \\
&= \int_0^\infty e^{-\frac{s}{\alpha}u} f(u) \frac{du}{\alpha} \\
&= \frac{1}{\alpha} \underbrace{\int_0^\infty e^{-\frac{s}{\alpha}u} f(u) du}_{F\left(\frac{s}{\alpha}\right)} \\
&= \frac{1}{\alpha} F\left(s/\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
v &= \alpha t \\
du &= \alpha dt
\end{aligned}
$$

First Shift Theorem: Multiplication by $e^{\alpha t}$

$$\begin{aligned}\mathcal{L}\left\{e^{\alpha t}f(t)\right\} &= \int_0^\infty e^{-st}e^{\alpha t}f(t)dt \\ &= \int_0^\infty e^{-(s-\alpha)t}f(t)dt = F(s-\alpha)\end{aligned}$$

ex: Suppose $Y(s) = \frac{1}{s+6}$, find $y(t)$.

$$\begin{aligned}Y(s) &= \underbrace{\frac{1}{s}}_{\mathcal{L}\{1\}} \text{ with } s \to s+6 \\ y(t) &= e^{-6t}\mathcal{L}^{-1}\left\{1/s\right\} \\ y(t) &= e^{-6t}\end{aligned}$$

ex: The LT of $\sin(4t)$ is $G(s) = \frac{4}{s^2+16}$.

What is the inverse of $F(s) = \frac{4}{s^2-6s+25}$?

$$
\begin{aligned}
F(s) &= \frac{4}{\underbrace{s^2 - 6s + 9}_{(s-3)^2} + 16} \\
&= \frac{4}{(s-3)^2 + 16} \\
&= \frac{4}{s^2 + 16} \text{ with } s \to s - 3
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= e^{3t} \mathcal{L}^{-1} \left\{ \frac{4}{s^2 + 16} \right\} \\
&= e^{3t} \sin(4t)
\end{aligned}
$$

Resonance $\Leftrightarrow$ Differentiation in s-domain

$$\mathcal{L}\left\{t^k f(t)\right\} = \int_0^\infty e^{-st} t^k f(t) dt \qquad = \int_0^\infty \underbrace{e^{-st} t}_{-\frac{\mathrm{d}}{\mathrm{d}s} e^{-st}} \; t^{k-1} f(t) dt$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} t^{k-1} f(t) dt \qquad \underset{\text{k-1 more times}}{\overset{\text{repeat same thing}}{\ldots}} \qquad = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} F(s)$$

with k=1

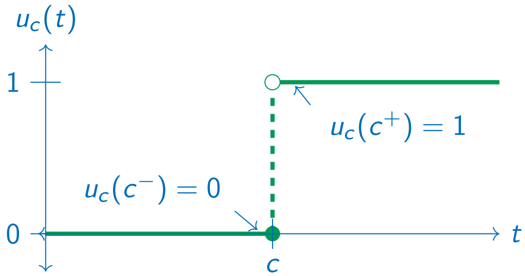
$$\mathcal{L}\left\{tf(t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

$$ex: \mathcal{L}\left\{t \sin(\omega t)\right\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}\left\{t \cos(\omega t)\right\} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

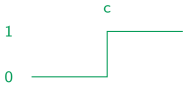
The Heaviside Step Function: $u_c(t)$ or $u(t-c)$ or $H(t-c)$

Used to model effects that "turn-on" at some time c.



$$u_c(t) = \begin{cases} 0 & \text{if } t \leq c \\ 1 & \text{if } t > c \end{cases}$$

Laplace Transform of Heaviside



$$\mathcal{L}\left\{u_c(t)\right\} = \int_0^\infty e^{-st}u_c(t)dt = \int_c^\infty e^{-st}dt = \frac{1}{s}e^{-sc}$$
$$= \boxed{e^{-sc}\frac{1}{s}} = e^{-sc}\mathcal{L}\left\{1\right\}$$

Q: In general, how can we invert $e^{-sc}\mathcal{L}\left\{f(t)\right\}$?

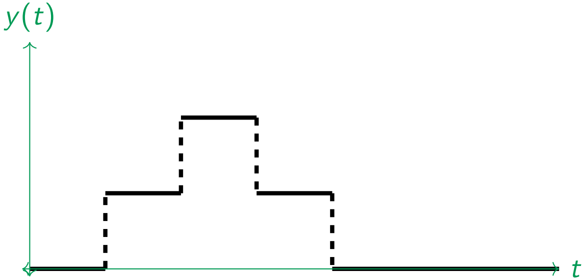
Second Shift Theorem

$$\begin{aligned}
\mathcal{L}\left\{f(t-c)u_c(t)\right\} &= \int_0^\infty e^{-st} \underbrace{f(t-c)u_c(t)}_{0 \text{ for } t<c} dt \\
&= \int_c^\infty e^{-st} f(t-c) dt \\
&= \int_0^\infty e^{-s(u+c)} f(u) du \\
&= e^{-sc} \int_0^\infty e^{-su} f(u) du = e^{-sc} \mathcal{L}\left\{f(t)\right\}
\end{aligned}$$

$$u = t - c$$
$$du = dt$$

ex: Suppose $Y(s) = \frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$, find and sketch $y(t)$.

$$\begin{aligned} Y(s) &= \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} \\ &= e^{-s}\mathcal{L}\left\{1\right\} + e^{-2s}\mathcal{L}\left\{1\right\} - e^{-3s}\mathcal{L}\left\{1\right\} - e^{-4s}\mathcal{L}\left\{1\right\} \\ &= u_1(t) \cdot 1\Big|_{t \to t-c} \dots \\ &= u_1(t) + u_2(t) - u_3(t) - u_4(t) \end{aligned}$$



ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+s^2}$, find $y(t)$.

$$
\begin{aligned}
y(t) &= u_4(t)\left[\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\
&= u_4(t)\left[\sin(3t)\right]_{t=t-4} \\
&= u_4(t)\sin(3(t-4))
\end{aligned}
$$

ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+(s+11)^2}$, find $y(t)$.

$$
\begin{aligned}
y(t) &= u_4(t)\left[e^{-11t}\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\
&= u_4(t)\left[e^{-11t}\sin(3t)\right]_{t=t-4} \\
&= u_4(t)e^{-11(t-4)}\sin(3(t-4))
\end{aligned}
$$

Common Laplace Transforms

$$\mathcal{L}\{y'(t)\} = sY(s) - y_0 \qquad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy_0 - v_0$$

$\mathcal{L}\{C\} = \dfrac{C}{s}$	Constant
$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$	Power Func.
$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$	First Shift Theorem
$\mathcal{L}\{u_c(t-c)\} = e^{-sc}\dfrac{1}{s}$	Heaviside Transfer
$\mathcal{L}\{f(t-c)u(t-c)\} = e^{-sc}F(s)$	Second Shift Theorem
$\mathcal{L}\{t^nf(t)\} = (-1)^n\dfrac{\mathrm{d}^n}{\mathrm{d}s^n}F(s)$	Resonance
$\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{\omega^2+s^2}$	
$\mathcal{L}\{\cos\omega t\} = \dfrac{s}{\omega^2+s^2}$	