

# Laplace Inversion Tactics

1. Function has denominator that can factor.

- factor and use partial fraction decomposition

$$\frac{1}{(s+a)s} = \frac{1}{as} - \frac{1}{a(s+a)} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{a} - \frac{1}{a}e^{-at}$$

2. Denominator has something with  $(s-a)^n$  but numerator has  $s$  appearing in it, not  $s-a$ .

- Split numerator by adding/subtracting so that all appearances of  $s$  are in the form  $s-a$
- use First Shift Theorem to invert

$$\frac{s}{(s-a)^n} = \frac{s-a}{(s-a)^n} + \frac{a}{(s-a)^n} \xrightarrow{\mathcal{L}^{-1}} \frac{e^{at}t^{n-2}}{(n-2)!} + \frac{a}{(n-1)!}e^{at}t^{n-1}$$

3. Numerator has incorrect constant  $A$  for inversion.

- Multiply by  $\frac{\omega}{\omega}$  and swap  $\omega$  with  $A$  :

$$\frac{\omega}{\omega} \frac{A}{\omega^2 + s^2} = \frac{A}{\omega} \frac{\omega}{\omega^2 + s^2} \xrightarrow{\mathcal{L}^{-1}} \frac{A}{\omega} \sin(\omega t)$$

ex:  $y'' + 2y' + 5y = 0, \quad y(0) = y_0, y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 2sY(s) - 2y_0 + 5Y(s) = 0$$

$$Y(s) = \frac{sy_0 + v_0 + 2y_0}{s^2 + 2s + 5} = \frac{sy_0 + v_0 + 2y_0}{\underbrace{s^2 + 2s + 1}_{(s+1)^2} \underbrace{-1 + 5}_{2^2}}$$

$$= \frac{sy_0 + v_0 + 2y_0}{(s+1)^2 + 2^2} = \frac{sy_0}{(s+1)^2 + 2^2} + \frac{v_0 + 2y_0}{(s+1)^2 + 2^2}$$

$$= y_0 \mathcal{L} \left\{ e^{-t} \underbrace{\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2^2} \right\}}_{\cos(2t)} \right\} + (v_0 + y_0) \mathcal{L} \left\{ e^{-t} \underbrace{\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2^2} \right\}}_{\frac{1}{2} \sin(2t)} \right\}$$

$$y(t) = y_0 e^{-t} \cos(2t) + \frac{v_0 + y_0}{2} e^{-t} \sin(2t)$$

ex:  $y'' + 12y' + 36y = 0, \quad y(0) = y_0, y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 12sY(s) - 12y_0 + 36Y(s) = 0$$

$$Y(s) = \frac{sy_0 + v_0 + 12y_0}{s^2 + 12s + 36} = \frac{sy_0 + v_0 + 12y_0}{(s + 6)^2}$$

$$= \frac{\cancel{y_0(s+6)}}{(s+6)^{\cancel{2}}} + \frac{v_0 + 12y_0 - 6y_0}{(s+6)^2}$$

$$= y_0 \underbrace{\frac{1}{s+6}}_{\mathcal{L}\{e^{-6t}\}} + (v_0 + 6y_0) \frac{1}{(s+6)^2}$$

$$= y_0 \mathcal{L}\{e^{-6t}\} + (v_0 + 6y_0) \mathcal{L}\left\{e^{-6t} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right\}$$

$$y(t) = y_0 e^{-6t} + (v_0 + 6y_0) e^{-6t} t$$

$$\text{ex: } y' + 6y = u_1(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$$

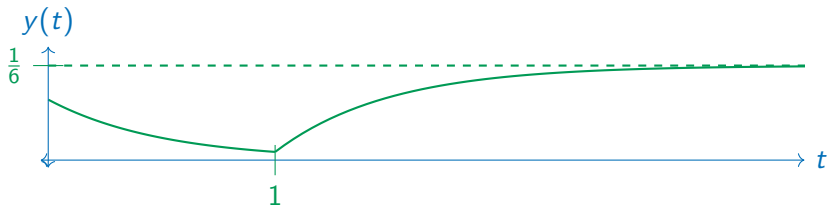
$$sY - y_0 + 6Y = \frac{e^{-s}}{s}$$

$$Y = \frac{\frac{e^{-s}}{s} + y_0}{s + 6} = \frac{e^{-s}}{s(s + 6)} + \frac{y_0}{s + 6}$$

$$Y(s) = \frac{1}{6} \frac{e^{-s}}{s} - \frac{1}{6} \frac{e^{-s}}{s + 6} + \frac{y_0}{s + 6}$$

 $\downarrow \mathcal{L}^{-1}$ 

$$y(t) = \frac{1}{6} u(t - 1) - \frac{1}{6} e^{-6(t-1)} u(t - 1) + y_0 e^{-6t}$$



$$\frac{1}{s(s+6)} = \frac{A}{s} + \frac{B}{s+6}$$

$$\Rightarrow 1 = A(s + 6) + Bs$$

$$\Rightarrow \begin{aligned} 1 &= 6A \\ 0 &= A + B \end{aligned}$$

$$\Rightarrow \begin{aligned} A &= \frac{1}{6} \\ B &= -\frac{1}{6} \end{aligned}$$

ex:  $y'' + 2y' + 5y = u_5(t) - u_{15}(t), \quad y(0) = y_0, y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 2sY(s) - 2y_0 + 5Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-15s}}{s}$$

$$Y(s) = \frac{\left(\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right) + sy_0 + v_0 + 2y_0}{s^2 + 2s + 5}$$

$$= (e^{-5s} - e^{-15s}) \underbrace{\frac{1}{s(s^2 + 2s + 5)}}_{F(s)} + \underbrace{\frac{1}{s^2 + 2s + 5}}_{\text{homogeneous part}}$$

$$y(t) = u_5(t) [\mathcal{L}^{-1}\{F(s)\}]_{t=t-5}$$

$$- u_{15}(t) [\mathcal{L}^{-1}\{F(s)\}]_{t=t-15}$$

$$+ y_0 e^{-t} \cos(2t) + \frac{v_0 + y_0}{2} e^{-t} \sin(2t)$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+2s+5)} \right\} = ???$$

$$F(s) = \frac{1}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$1 = As^2 + 2As + 5A + Bs^2 + Cs$$

$$\text{constant terms: } 1 = 5A \quad \Rightarrow A = \frac{1}{5}$$

$$s \text{ terms: } 0 = 2A + C \quad \Rightarrow C = -2A = -\frac{2}{5}$$

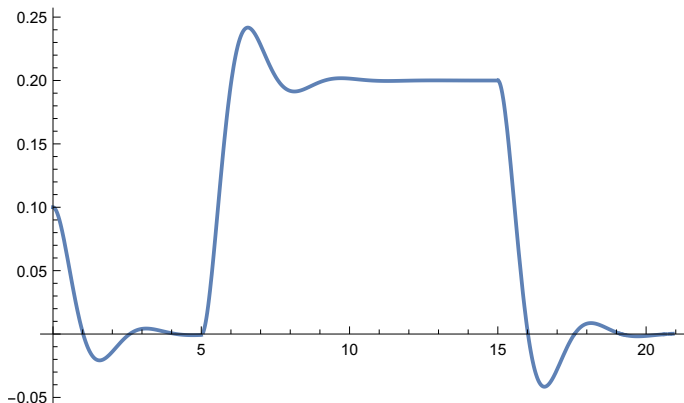
$$s^2 \text{ terms: } 0 = A + B \quad \Rightarrow B = -A = -\frac{1}{5}$$

$$F(s) = \frac{1}{5s} - \frac{1}{5} \frac{s+2}{(s+1)^2 + 2^2}$$

$$= \frac{1}{5s} - \frac{1}{5} \left( \frac{s+1}{(s+1)^2 + 2^2} - \frac{1}{(s+1)^2 + 2^2} \right)$$

$$f(t) = \frac{1}{5} \left( 1 - e^{-t} \left( \cos(2t) - \frac{1}{2} \sin(2t) \right) \right)$$

$$\begin{aligned} y(t) = & u_5(t) \frac{1}{5} \left( 1 - e^{-(t-5)} \left( \cos(2(t-5)) - \frac{1}{2} \sin(2(t-5)) \right) \right) \\ & - u_{15}(t) u_5(t) \frac{1}{5} \left( 1 - e^{-(t-15)} \left( \cos(2(t-15)) - \frac{1}{2} \sin(2(t-15)) \right) \right) \\ & + y_0 e^{-t} \cos(2t) + \frac{v_0 + y_0}{2} e^{-t} \sin(2t) \end{aligned}$$



ex:  $y'' + y' = 1$ ,  $y(0) = y_0$ ,  $y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + sY(s) - y_0 = \frac{1}{s}$$

$$(s^2 + s)Y = sy_0 + y_0 + v_0 + \frac{1}{s}$$

$$Y(s) = \frac{\cancel{sy_0}}{\cancel{s}(s+1)} + \frac{y_0 + v_0}{s(s+1)} + \frac{1}{s^2(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

$$A = 1, B = -1$$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$1 = A(s+1) + Bs(s+1) + Cs^2$$

$$\underline{s=0}: A=1, \quad \underline{s=-1}: C=1$$

$$= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$\underline{s=1}: 2 + 2B + \cancel{1} = \cancel{1}$$

$$Y(s) = \frac{\cancel{y_0}}{\cancel{s+1}} + \frac{y_0 + v_0}{s} - \frac{\cancel{y_0} + v_0}{s+1} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \quad B = -1$$



continuing ...  $Y(s) = \frac{y_0 + v_0 - 1}{s} + \frac{1 - v_0}{s+1} + \frac{1}{s^2}$

$$\begin{aligned} y(t) &= y_0 + v_0 + 1 + (1 - v_0)e^{-t} + t \\ &= \underbrace{c_1 + c_2 e^{-t}}_{y_h} + \underbrace{t}_{y_p} \end{aligned}$$

M.U.C.

$$y'' + y' = 1$$

homogeneous problem

$$r^2 + r = 0 \Rightarrow r(r+1) = 0 \quad \Rightarrow y_h = c_1 + c_2 e^{-t}$$

RHS in nullspace of operator

$$y_p = Bt \dots$$

put it all together and then solve for  $c_1$  and  $c_2 \dots$