Lecture 4 Summary

Linear 2^{nd} order homogeneous ODEs with constant coefficients

$$ay'' + by' + cy = 0$$

Ansatz method:

1. Guess $y(t) = e^{rt}$
2. Plug guess into the ODE
3. Find (up to) 2 values of r

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Some questions:

1. Why do we add up the two solutions? Are there more solutions?
2. Does this guessing method always work?

Differential Operators

$\mathsf{L}\left[y(t)\right] := \dfrac{\mathrm{d}}{\mathrm{d}t}y(t)$ is a differential operator applied to the function $y(t)$.

Takes a function $y(t)$ and returns it derivative

$\mathsf{L}\left[y(t)\right] := ay'' + by' + cy$ is also a differential operator...

Operator form of a generic inhomogeneous ODE:

$$\mathsf{L}\left[y(t)\right] = h(t)$$

Linear Operators

An operator is called **linear** if it has the following propertes for any two function $f(t)$ and $g(t)$ and any constant c.

1. $\mathsf{L}\left[f(t)+g(t)\right]=\mathsf{L}\left[f(t)\right]+\mathsf{L}\left[g(t)\right]$
2. $\mathsf{L}\left[cf(t)\right]=c\mathsf{L}\left[f(t)\right]$

Note: Linear ODEs have linear differential operators

Superposition Principle

Suppose the functions $y_1(t)$ and $y_2(t)$ both independently solve a linear homogeneous ODE

$$\mathrm{L}\left[y\right] = 0$$

then

$$y_h = c_1 y_1 + c_2 y_2$$

is also a solution to the same ODE.

Proof:

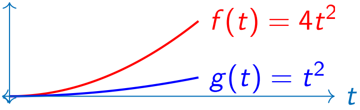
$$\begin{aligned}
\mathrm{L}\left[c_1 y_1 + c_2 y_2\right] &\overset{\text{Linearity 1}}{=} \mathrm{L}\left[c_1 y_1\right] + \mathrm{L}\left[c_2 y_2\right] \\
&\overset{\text{Linearity 2}}{=} c_1 \mathrm{L}\left[y_1\right] + c_2 \mathrm{L}\left[y_2\right] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}$$

Linear dependence of functions

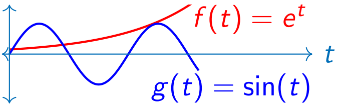
- Two functions $f(t)$ and $g(t)$ are **linearly dependent** on the interval $t \in I = [\alpha, \beta]$ if there exist a non-zero constant k such that

$$f(t) = kg(t) \quad \forall t \in I$$

Linearly Dependent



Not Linearly Dependent



- If functions are not linearly dependent on I, then we say they are **linearly independent** on I.

Completeness of solutions

Suppose that $y_1(t)$ and $y_2(t)$ both solve a 2^{nd} order linear homogeneous ODE

$$\mathrm{L}[y] = 0$$

If y_1 and y_2 are linearly independent functions, then the general solution to the homogeneous problem is

$$y_g(t) = c_1 y_1(t) + c_2 y_2(t)$$

General Solution: Solution with arbitrary constants that solves ALL possible scenarios where solutions exist.

Proof = long and tedious (linear algebra + method of integrating factors)

Take home message:

$$2^{nd} \text{ order linear homogeneous} \Rightarrow \exists\ 2 \text{ linearly independent solutions}$$

Solutions to $ay'' + by' + cy = 0$

Lets try an ansatz of $y(t) = e^{rt}$ ... where r is unknown (ansatz method)

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{rt} + b\frac{\mathrm{d}}{\mathrm{d}t}e^{rt} + ce^{rt} = 0$$

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \qquad e^{rt}\left(ar^2 + br + c\right) = 0$$

Since $e^{rt} \neq 0$, the ODE can only have a solution $y(t) = e^{rt}$ if

$$ar^2 + br + c = 0$$

called the **characteristic equation**.

Possible values of r:

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Roots of the characteristic equation (polynomial)

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three main cases:

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

2. Repeated real roots: discriminant $= 0$

3. Complex conjugate roots: discriminant < 0

Case 1: distinct real roots

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}; \qquad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two major subcases:

1. $ac \geq 0$:

   Both roots have the same sign.

   y_1 and y_2 both grow or decay exponentially.

2. $ac < 0$:

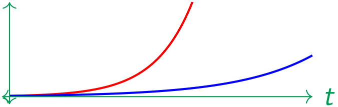
   The two roots have opposite sign.

   One solution grows exponentially, the other decays exponentially.

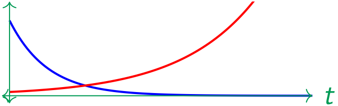
Qualitative Behaviour: distinct real roots

Sum of real exponential functions, three subcases:

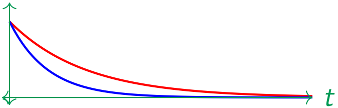
1. All positive roots, $0 < r_1 < r_2$.



2. Mixed roots, $r_1 < 0 < r_2$.



3. All negative roots, $r_1 < r_2 < 0$.



Summary

- Superposition of Homogeneous solutions

$$y_h = c_1 y_1 + c_2 y_2$$

- Linear independence of homoegeneous solutions

$$y_g = c_1 y_1 + c_2 y_2 \quad \Rightarrow \quad y_1 \neq k y_2; \quad k = \text{constant}$$

  $y_g =$ general solution, solves ALL scenarios where a solution exists.

- $ay'' + by' + cy = 0$

  Try ansatz $y = e^{rt}$ $\quad \rightarrow \quad e^{rt} \cdot \underbrace{\left(ar^2 + br + c\right)}_{\text{characterisitc polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$