

Dirichlet Problems on Rectangular Domains

$$u_N = \sum_n a_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(\frac{n\pi}{w}y\right)$$

$$u_S = \sum_n b_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(\frac{n\pi}{w}(h-y)\right)$$

$$u_E = \sum_n c_n \sin\left(\frac{n\pi}{h}y\right) \sinh\left(\frac{n\pi}{h}x\right)$$

$$u_W = \sum_n d_n \sin\left(\frac{n\pi}{h}y\right) \sinh\left(\frac{n\pi}{h}(w-x)\right)$$

We can rewrite u_S and u_W using the identity

$$\sinh(a-b) = \sinh(a)\cosh(b) - \cosh(a)\sinh(b)$$

From

$$\sinh(a - b) = \sinh(a) \cosh(b) - \cosh(a) \sinh(b)$$

we obtain

$$\begin{aligned} \sinh\left(\frac{n\pi}{w}(h - y)\right) &= \sinh\left(\frac{n\pi}{w}h\right) \cosh\left(\frac{n\pi}{w}y\right) \\ &\quad - \cosh\left(\frac{n\pi}{w}h\right) \sinh\left(\frac{n\pi}{w}y\right) \end{aligned}$$

\vdots

$$u_S = \sum_n b_n \sin\left(\frac{n\pi}{w}x\right) \left[\sinh\left(\frac{n\pi}{w}h\right) \cosh\left(\frac{n\pi}{w}y\right) - \cosh\left(\frac{n\pi}{w}h\right) \sinh\left(\frac{n\pi}{w}y\right) \right]$$

$$u_W = \sum_n d_n \sin\left(\frac{n\pi}{h}y\right) \left[\sinh\left(\frac{n\pi}{h}w\right) \cosh\left(\frac{n\pi}{h}x\right) - \cosh\left(\frac{n\pi}{h}w\right) \sinh\left(\frac{n\pi}{h}x\right) \right]$$