Recall:

So far, we have always been rearranging s-domain functions to be sums of easy to invert terms.

ex: $Y(s) = \frac{1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}$

After partial fraction decomposition (try it for practice)

$$A = 0, B = 1, C = 0, D = -1$$

$$\begin{aligned} Y(s) &= \frac{1}{s^2} - \frac{1}{s^2+1} \\ &= \mathcal{L}\{t\} - \mathcal{L}\{\sin(t)\} \end{aligned}$$

$$y(t) = t - \sin(t)$$

Suppose we wish to solve
$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$
for general $g(t)$.

$$as^2Y(s) + bsY(s) + cY(s) = G(s)$$

$$Y(s) = \frac{G(s)}{as^2 + bs + c}$$

$$Y(s) = \underbrace{F(s)G(s)}_{\text{How do we invert a product?}} \qquad \text{with } F(s) = \frac{1}{as^2 + bs + c}$$

We need to use the convolution theorem!

Convolutions

We denote the convolution of two functions f and g by the symbol $f * g$, with

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Note: $f * g = g * f$ (convolutions are symmetric)

Convolutions are useful for inverting products of Laplace Transforms ex: Find $h(t) = t * t^2$

$$
\begin{aligned}
h(t) &= \int_0^t (t-\tau)\tau^2 d\tau \\
&= \int_0^t t\tau^2 d\tau - \int_{\tau=0}^t \tau^3 d\tau \\
&= t\left[\frac{\tau^3}{3}\right]_{\tau=0^t} - \left[\frac{\tau^4}{4}\right]_0^t \\
&= \frac{t^4}{3} - \frac{t^4}{4} \\
&= \frac{t^4}{12}
\end{aligned}
$$

ex: Find $h(t) = t * \sin(t)$

$$
\begin{aligned}
h(t) &= \int_0^t (t-\tau)\sin(\tau)d\tau \\
&= \int_0^t t\sin(\tau)d\tau - \int_0^t \tau\sin(\tau)d\tau \\
&= t\left[-\cos(\tau)\right]_{\tau=0}^t \\
&\quad - \left(-[\tau\cos(\tau)]_{\tau=0}^t + \int_0^t \cos(\tau)d\tau\right) \\
&= t - t\cos(t) - \left(-t\cos(t) + [\sin(\tau)]_{\tau=0}^t\right) \\
&= t - \sin(t)
\end{aligned}
$$

let

$u = \tau, \quad du = d\tau$

$dv = \sin(\tau), v = -\cos(\tau)$

Convolution Theorem

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$ are known functions, then

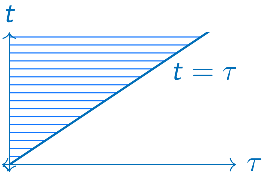
$$\boxed{\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g} = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

or conversely

$$\boxed{\mathcal{L}\{f * g\} = F(s) \cdot G(s)}$$

Proof of the convolution theorem

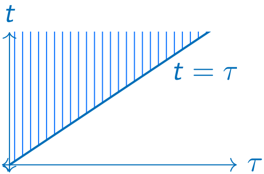
$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st} h(t) dt = \int_{t=0}^\infty \int_{\tau=0}^t f(\tau) g(t-\tau) e^{-st} d\tau \, dt$$



equivalent areas

$\Leftrightarrow$

switch integration order



$$= \int_{\tau=0}^\infty \int_{t=\tau}^\infty f(\tau) g(t-\tau) e^{-st} dt \, d\tau$$

$$= \int_{\tau=0}^\infty f(\tau) e^{-s\tau} \int_{t=\tau}^\infty g(t-\tau) e^{-s(t-\tau)} d\tau \, dt \qquad \text{let } u = t - \tau$$

$$= \underbrace{\int_{\tau=0}^\infty f(\tau) e^{-s\tau} d\tau}_{F(s)} \underbrace{\int_{u=0}^\infty g(u) e^{-su} du}_{G(s)} \qquad t = \tau \Rightarrow u = 0$$

$$= F(s) G(s)$$

Suppose $y(t)$ solves $y'' + y = t, \quad y(0) = y'(0) = 0.$

Show that $y(t) = \frac{1}{2}t^2 * \cos(t)$ and use the convolution theorem to find an explicit representation of y(t).

$$s^2 Y(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2\left(s^2+1\right)}$$

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{2}t^2 * \cos(t)\right\} &= \frac{1}{2}\mathcal{L}\left\{t^2\right\} \cdot \mathcal{L}\left\{\cos(t)\right\} \\ &= \frac{1}{2}\frac{2}{s^3} \cdot \frac{s}{s^2+1} \\ &= \frac{1}{s^2\left(s^2+1\right)} \checkmark\end{aligned}$$

$$\begin{aligned}Y(s) &= \frac{1}{s^2} \cdot \frac{1}{s^2+1} \\ &= \mathcal{L}\left\{t\right\} \cdot \mathcal{L}\left\{\sin(t)\right\} \\ y(t) &= t * \sin(t) = t - \sin(t)\end{aligned}$$

Inhomogeneous IVPs via Laplace transforms

Consider the constant coefficient 2^{nd} order DE:

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= v_0 \end{aligned}$$

Take LT

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - av_0 = G(s)$$

Solve for $Y(s)$:

$$Y(s) = \underbrace{\frac{(as + b)y_0 + av_0}{as^2 + bs + c}}_{Y_h} + \underbrace{\frac{G(s)}{as^2 + bs + c}}_{Y_p}$$

effects of initial conditions (Homogeneous Part)

effects of forcing function (Particular Part)

Inhomogeneous IVPs via Laplace transforms

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned} \quad \rightarrow \quad Y(s) = \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{Y_h(s)} + \underbrace{\frac{G(s)}{as^2+bs+c}}_{Y_p(s)}$$

1. Break up $Y_h(s)$ using partial frac. decomp. & invert $Y_h(s) \to y_h(t)$.
2. Define the **Transfer Function**:

$$F(s) = \frac{1}{as^2+bs+c}$$

3. Invert $F(s) \to f(t)$. The function $f(t)$ is called the **impulse response** function.
4. From the convolution theorem with $Y_p(s) = F(s)G(s)$

$$y_p(t) = f * g$$

5. Finally

$$y(t) = y_h(t) + y_p(t)$$

ex: $y'' + 4y = t^3, \quad y(0) = y'(0) = 0.$

Find an appropriate impulse response function and express the ODE's solution as a convolution integral.

$$\text{Transfer Function: } F(s) = \frac{1}{s^2+4} = \frac{1}{2}\mathcal{L}\{\sin(2t)\}$$

$$\text{Impulse Response: } f(t) = \frac{1}{2}\sin(2t)$$

$$\begin{aligned} y(t) &= \sin(2t) * t^3 \\ &= \frac{1}{2}\int_0^t \sin(2(t-\tau))\tau^3 d\tau \end{aligned}$$