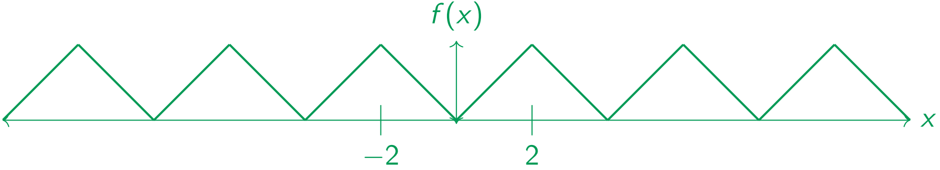
Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$



$$a_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{|x|}_{\text{even func.}} \underbrace{\cos\left(n\frac{\pi}{2}x\right)}_{\text{even func.}}}_{\text{even func.}} dx$$

The integral of an even function on $[0, L]$ is half its integral from $[-L, L]$

$$a_n = \int_{0}^{2} x\cos\left(n\frac{\pi}{2}x\right) dx$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$

$$a_n = \int_0^2 x \cos\left(n\frac{\pi}{2}x\right) dx$$

$$\text{let} \begin{array}{ll} u = x & du = dx \\ dv = \cos\left(n\frac{\pi}{2}x\right) & v = 2\frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} \end{array}$$

$$= \cancelto{0}{2\left(x\frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi}\right)\Big|_0^2} - 2\int_0^2 \frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} dx$$

$$= \frac{4}{n^2\pi^2}\cos\left(n\frac{\pi}{2}x\right)\Big|_0^2 = \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} -\frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x + 4) = f(x)$

$\frac{a_0}{2}$ is the average value of the function (DC component)

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} |x| dx = \int_0^2 x dx \\ &= \frac{x^2}{2}\bigg|_0^2 \\ &= \frac{4}{2} - 0 \\ &= 2 \end{aligned}$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$

$$b_n = \frac{1}{2} \int_{-2}^{2} \underbrace{\underbrace{|x|}_{\text{even func.}} \underbrace{\sin\left(n\frac{\pi}{2}x\right)}_{\text{odd func.}}}_{\text{odd func.}} dx$$

Any integral that is symmetric about $x = 0$ of an odd function is zero

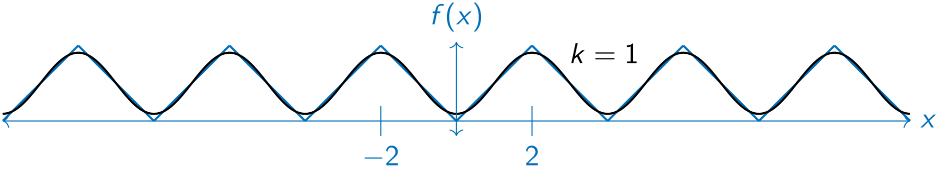
...opposite AUC on both sides...

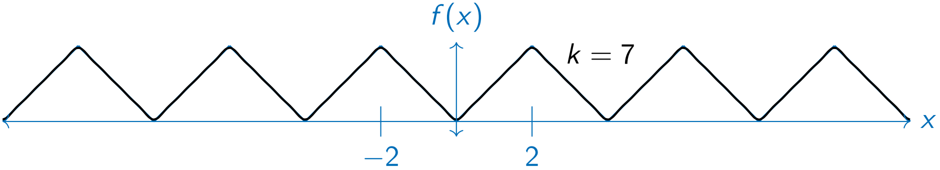
$$\Rightarrow b_n = 0$$

Finite Fourier Series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos(\omega_n x) + \sum_{n=1}^{k} b_n \sin(\omega_n x) \qquad \omega_n = n\frac{\pi}{L}$$

$$a_0 = 2 \qquad a_n = \begin{cases} -\frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad b_n = 0$$

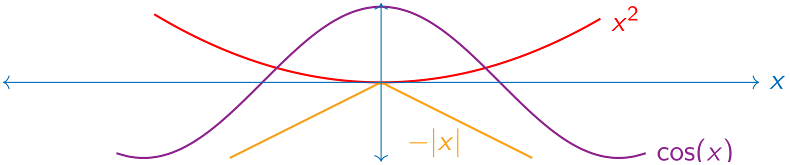




Even and Odd Functions:

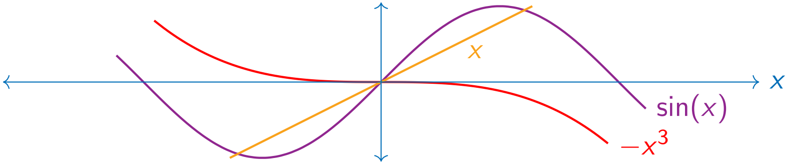
Even Functions:

$$f(x) = f(-x)$$



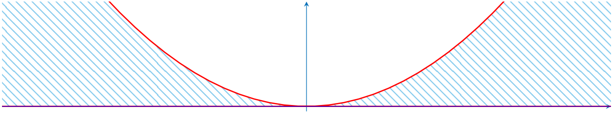
Odd Functions:

$$f(x) = -f(x)$$

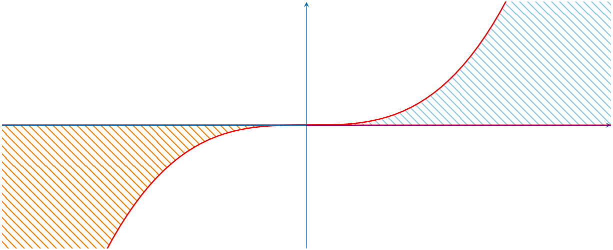


Even and Odd Functions: Integral Properties

Even Functions: The integral of an even function on the interval $[-L, L]$ is double its integral on $[0, L]$



Odd Functions: The integral of an odd function on the interval $[-L, L]$ is 0.



Even and Odd Functions: Products of odd/even functions

Works like multiplying real numbers

$$\text{even} \Leftrightarrow +1$$
$$\text{odd} \Leftrightarrow -1$$

$$\text{odd} \cdot \text{odd} = \text{even} \qquad \text{even} \cdot \text{even} = \text{even} \qquad \text{even} \cdot \text{odd} = \text{odd}$$
$$-1 \cdot -1 = +1 \qquad +1 \cdot +1 = +1 \qquad +1 \cdot -1 = -1$$

Even and Odd Functions: Fourier Series

Even Function: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x)$

Proof:

$$b_n = \int_{-L}^{L} \underbrace{\text{even func.} \times \sin(\omega_n)}_{\text{odd func.}} = 0$$

Odd Function: $f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\omega_n x)$

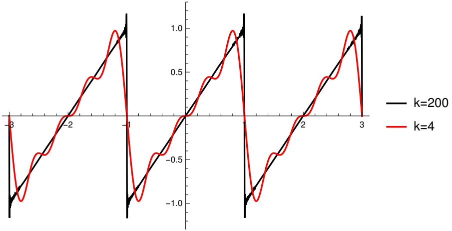
Proof:

$$a_n = \int_{-L}^{L} \underbrace{\text{odd func.} \times \cos(\omega_n)}_{\text{odd func.}} = 0$$

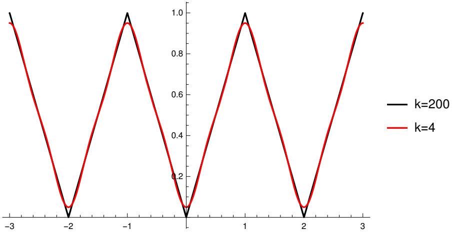
Even function, only cos terms - Odd function, only sin terms

Fourier Series Convergence

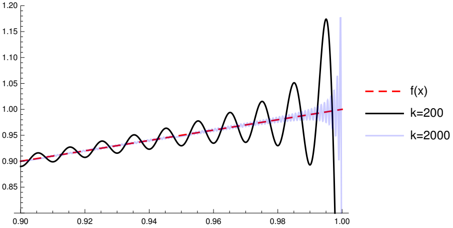
$f(x) = x$ for $x \in [-1, 1]$ with $f(t+2) = f(t)$



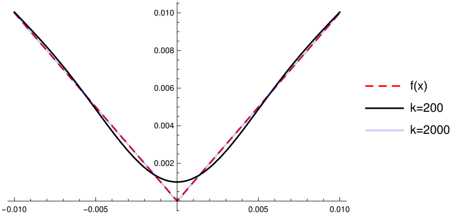
$f(x) = |x|$ for $x \in [-1, 1]$ with $f(t+2) = f(t)$



Zoom in on discontinuity

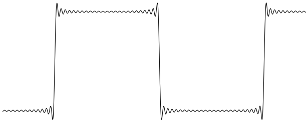


Zoom in on discontinuity



Fourier Series Convergence

- The Fourier Series of any continuous function converges (as $k \to \infty$) to the function value at every point. $\Rightarrow f(x) = \mathrm{FS}(f(x))$
- The Fourier Series of a function with jump discontinuities exhibits **Gibb's phenomena**
  - High frequency over/undershooting of the function



- The Fourier Series converges to the midpoint between the two function values at any point of discontinuity. $\Rightarrow f(x) \approx \mathrm{FS}(f(x))$
- The rate of convergence of smooth functions is faster than for functions with discontinuities.