Partial Differential Equations (PDEs)

A PDE is a DE for a function $u(x_1, x_2, \ldots)$ that depends on 2 or more independent variables and contains partial derivatives taken with respect to at least 2 variables.

ex: 1D Heat Equation

$$\frac{\partial}{\partial t}u(x,t) = \alpha \frac{\partial^2}{\partial x^2}u(x,t)$$

or

$$u_t = \alpha u_{xx}$$

Partial derivatives are used so we get the same results in non-euclidean coordinates (e.g., speherical, polar, etc)

Trick: convert PDE into an (infinite) system of ODEs and solve those ODEs instead.

1D Heat Equation - An Initial Boundary Value Problem

Let $u(x, t)$ be the temperature at the position $x \in \underbrace{[0, L]}_{\text{domain}}$ and the time t

The heat equation tells us the evolution of the distribution of heat throughout the domain (e.g., a 1D rod) from some initial condition $u_0(x)$.

(Homogeneous)

Boundary Conditions

Initial Condition

$u_t = \alpha u_{xx}$ for $x \in [0, L]$ with $\overbrace{u(0, t) = u(L, t) = 0}^{\text{rod connected to ice baths}}$ or $\underbrace{u_x(0, t) = u_x(L, t) = 0}_{\text{insulated rod}}$ and $u(x, 0) = u_0(x)$

$\alpha =$ thermal diffusivity

Separation of Variables

$$\frac{\partial}{\partial t}u(x,t) = \alpha\frac{\partial^2}{\partial x^2}u(x,t)$$

Guess: $u(x,t) = X(x)T(t)$

$$X(x)T'(t) = \alpha X''(x)T(t)$$

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}$$

This can only be true if each fraction is a constant

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda \qquad \Leftrightarrow \qquad \begin{gathered} X''(x) + \lambda X(x) = 0 \\ \text{and} \\ T'(t) = -\alpha\lambda T(t) \end{gathered}$$

$$X'' + \lambda X = 0 \quad \text{with BCs:} \quad X(0) = X(L) = 0 \quad \text{or} \quad X'(0) = X'(L) = 0$$

Non-zero solutions exist only for $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $n \in \mathbb{N}^+$ or $n = 0$.

$\underline{\lambda = 0}$

$$X_0'' = 0$$

$$X_0(x) = A_0 + \cancel{B_0 x}$$

does not match BCs

$\underline{\lambda > 0}$

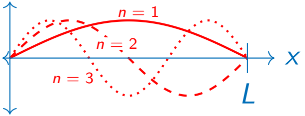
$$X_n'' + \lambda_n X_n = 0 \qquad \text{try} \qquad X_n(x) = c_n e^{rx}$$

$$r^2 + \lambda_n = 0 \qquad \Rightarrow r = \pm i\sqrt{\lambda_n}$$

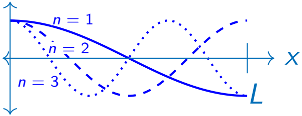
$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)$$

Choose sin or cos based on the BCs of the problem.

$\sin(n\pi x/L)$



$\cos(n\pi x/L)$



Time-Dependence + Superposition

$$u_n(x,t) = T_n(t)X_n(x) \quad \text{with} \quad T_n' = -\alpha\lambda_n T \quad \Leftrightarrow \quad T_n(t) = c_n e^{-\alpha\lambda_n t}$$

$$u_n(x,t) = c_n e^{-\alpha\omega_n^2 t}\left(A_n \cos(\omega_n x) + B_n \sin(\omega_n x)\right)$$

We have infinitely many solutions to the linear PDE, any linear combinations of those solutions is also a solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

Initial condtion: $u(x,0) = u_0(x)$ Pick $c_n = 1$ (without loss of generality)

$$u_0(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos(\omega_i x) + B_n \sin(\omega_n x)\right) \qquad \omega_n = \frac{n\pi}{L}$$

Find A_n and B_n by taking a Fourier Series expansion of $u_0(x)$

ICs + BCs $\Rightarrow$ Periodic extension

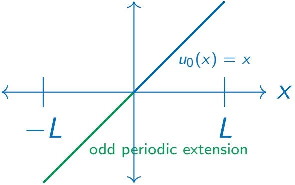
$$u_0(x) = A_0 + \sum_{i=n}^{\infty} \left(A_n \cos(\omega_i n x) + B_n \sin(\omega_n x)\right) \qquad \omega_n = \frac{n\pi}{L}$$

$u_0(x)$ is defined for $x \in [0, L]$ only.

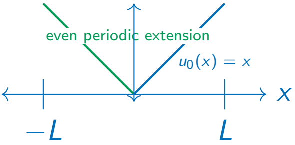
Our spatial solution is the Fourier Series of a 2L periodic function!

BCs: $X(0) = X(L) = 0$ or $X'(0) = X'(L) = 0$

- sin terms satisfy $X(0) = X(L) = 0$
  1. Extend $u_0(x)$ as an <u>odd</u> function
  2. Take its FS: <u>Fourier Sine Series</u>



- cos terms satisfy $X'(0) = X'(L) = 0$
  1. Extend $u_0(x)$ as an <u>even</u> function
  2. Take its FS: <u>Fourier Cosine Series</u>



Solution to $u_t = \alpha u_{xx}$ with $0 < x < L$ and $u(x,0) = u_0(x)$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

If $u(0,t) = u(L,t) = 0$: Fourier sine series

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

If $u_x(0,t) = u_x(L,t) = 0$: Fourier cosine series

$$a_n = \frac{2}{L}\int_0^L u_0(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$b_n = 0$$

ex: $u_t = \alpha u_{xx}$ for $0 < x < L$ with $u(x,0) = x$

Two possible homogeneous BCs:

$u(0,t) = u(L,t) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx = -L\frac{2(-1)^n}{\pi n}$$

$u_x(0,t) = u_x(L,t) = 0$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2L\left((-1)^n - 1\right)}{\pi^2 n^2}$$

$$a_0 = \frac{2}{L}\int_0^L x dx = L$$

ex: $u_t = \alpha u_{xx}$ for $0 < x < L$ with $\begin{array}{l} u(x,0) = \sin(5x/L) \\ u(0,t) = u(L,t) = 0 \end{array}$

From the BCs,

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \sin(5x/L) \sin(\frac{n\pi}{L} x) dx \\ &= \begin{cases} 1 & \text{if } n = 5 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$u(x,t) = e^{-\alpha \frac{25\pi^2}{L^2} t} \sin\left(\frac{5\pi}{L} x\right)$$