Laplace Inversion Tactics

1. Function has denominator that can factor.
   - factor and use partial fraction decomposition
   $$\frac{1}{(s+a)s} = \frac{1}{as} - \frac{1}{a(s+a)} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \frac{1}{a} - \frac{1}{a}e^{-at}$$
2. Denominator has something with $(s-a)^n$ but numerator has s appearing in it, not $s-a$.
   - Split numerator by adding/subtracting so that all appearances of s are in the form $s-a$
   - use First Shift Theorem to invert
   $$\frac{s}{(s-a)^n} = \frac{s-a}{(s-a)^n} + \frac{a}{(s-a)^n} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \frac{e^{at}t^{n-2}}{(n-2)!} + \frac{a}{(n-1)!}e^{at}t^{n-1}$$
3. Numerator has incorrect constant A for inversion.
   - Mulitply by $\frac{\omega}{\omega}$ and swap ω with A :
   $$\frac{\omega}{\omega}\frac{A}{\omega^2+s^2} = \frac{A}{\omega}\frac{\omega}{\omega^2+s^2} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \frac{A}{\omega}\sin(\omega t)$$

ex: $y'' + 2y' + 5y = 0, \quad y(0) = y_0,\ y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 2sY(s) - 2y_0 + 5Y(s) = 0$$

$$Y(s) = \frac{sy_0 + v_0 + 2y_0}{s^2 + 2s + 5} = \frac{sy_0 + v_0 + 2y_0}{\underbrace{s^2 + 2s + 1}_{(s+1)^2} \underbrace{-1 + 5}_{2^2}}$$

$$= \frac{sy_0 + v_0 + 2y_0}{(s+1)^2 + 2^2} = \frac{sy_0}{(s+1)^2 + 2^2} + \frac{v_0 + 2y_0}{(s+1)^2 + 2^2}$$

$$= y_0 \mathcal{L}\left\{ e^{-t} \underbrace{\mathcal{L}^{-1}\left\{ \frac{s}{s + 2^2} \right\}}_{\cos(2t)} \right\} + (v_0 + y_0)\mathcal{L}\left\{ e^{-t} \underbrace{\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 2^2} \right\}}_{\frac{1}{2}\sin(2t)} \right\}$$

$$y(t) = y_0 e^{-t}\cos(2t) + \frac{v_0 + y_0}{2} e^{-t}\sin(2t)$$

ex: $y'' + 12y' + 36y = 0, \quad y(0) = y_0,\ y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 12sY(s) - 12y_0 + 36Y(s) = 0$$

$$\begin{aligned}
Y(s) &= \frac{sy_0 + v_0 + 12y_0}{s^2 + 12s + 36} = \frac{sy_0 + v_0 + 12y_0}{(s+6)^2} \\
&= \frac{y_0\cancel{(s+6)}}{(s+6)^{\cancel{2}}} + \frac{v_0 + 12y_0 - 6y_0}{(s+6)^2} \\
&= y_0 \underbrace{\frac{1}{s+6}}_{\mathcal{L}\{e^{-6t}\}} + (v_0 + 6y_0)\frac{1}{(s+6)^2} \\
&= y_0 \mathcal{L}\left\{e^{-6t}\right\} + (v_0 + 6y_0)\mathcal{L}\left\{e^{-6t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right\}
\end{aligned}$$

$$y(t) = y_0 e^{-6t} + (v_0 + 6y_0)e^{-6t}t$$

ex: $y' + 6y = u_1(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$

$$sY - y_0 + 6Y = \frac{e^{-s}}{s}$$

$$Y = \frac{\frac{e^{-s}}{s} + y_0}{s+6} = \frac{e^{-s}}{s(s+6)} + \frac{y_0}{s+6}$$

$$Y(s) = \frac{1}{6}\frac{e^{-s}}{s} - \frac{1}{6}\frac{e^{-s}}{s+6} + \frac{y_0}{s+6}$$

$$\downarrow \mathcal{L}^{-1}$$

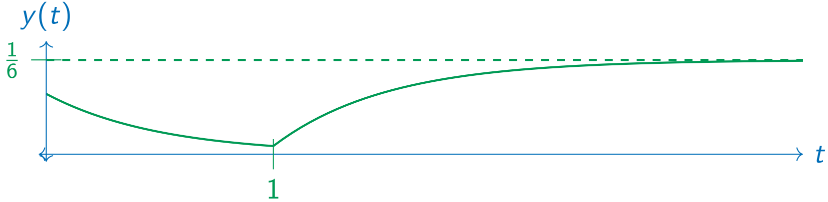
$$y(t) = \frac{1}{6}u(t-1) - \frac{1}{6}e^{-6(t-1)}u(t-1) + y_0 e^{-6t}$$

$$\frac{1}{s(s+6)} = \frac{A}{s} + \frac{B}{s+6}$$

$$\Rightarrow 1 = A(s+6) + Bs$$

$$\Rightarrow \begin{array}{l} 1 = 6A \\ 0 = A + B \end{array}$$

$$\Rightarrow \begin{array}{l} A = \frac{1}{6} \\ B = -\frac{1}{6} \end{array}$$



ex: $y'' + 2y' + 5y = u_5(t) - u_{15}(t), \quad y(0) = y_0,\ y'(0) = v_0$

$$s^2 Y(s) - sy_0 - v_0 + 2sY(s) - 2y_0 + 5Y(s) = \frac{e^{-5s}}{s} - \frac{e^{-15s}}{s}$$

$$Y(s) = \frac{\left(\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\right) + sy_0 + v_0 + 2y_0}{s^2 + 2s + 5}$$

$$= \left(e^{-5s} - e^{-15s}\right) \underbrace{\frac{1}{s(s^2+2s+5)}}_{F(s)} + \underbrace{\frac{sy_0 + v_0 + 2y_0}{s^2+2s+5}}_{\text{homogeneous part}}$$

$$y(t) = u_5(t)\left[\mathcal{L}^{-1}\{F(s)\}\right]_{t=t-5}$$
$$- u_{15}(t)\left[\mathcal{L}^{-1}\{F(s)\}\right]_{t=t-15}$$
$$+ y_0 e^{-t}\cos(2t) + \frac{v_0 + y_0}{2} e^{-t}\sin(2t)$$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\} = ???$

$$F(s) = \frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

$$1 = As^2 + 2As + 5A + Bs^2 + Cs$$

$$\text{constant terms: } 1 = 5A \qquad \Rightarrow A = \frac{1}{5}$$

$$s \text{ terms: } 0 = 2A + C \qquad \Rightarrow C = -2A = -\frac{2}{5}$$

$$s^2 \text{ terms: } 0 = A + B \qquad \Rightarrow B = -A = -\frac{1}{5}$$

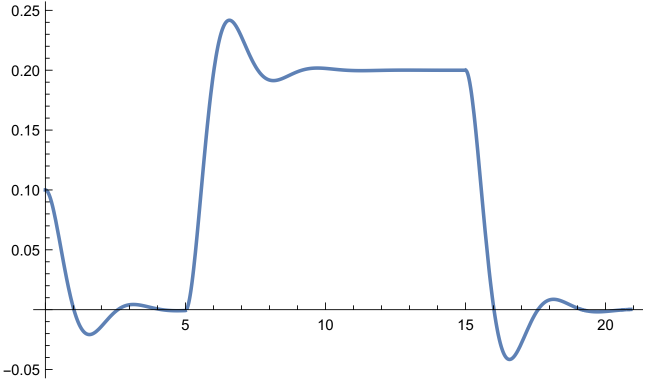
$$F(s) = \frac{1}{5s} - \frac{1}{5}\frac{s+2}{(s+1)^2+2^2}$$

$$= \frac{1}{5s} - \frac{1}{5}\left(\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right)$$

$$f(t) = \frac{1}{5}\left(1 - e^{-t}\left(cos(2t) - \frac{1}{2}\sin(2t)\right)\right)$$

$$y(t) = u_5(t)\frac{1}{5}\left(1 - e^{-(t-5)}\left(cos(2(t-5)) - \frac{1}{2}\sin(2(t-5))\right)\right)$$
$$- u_{15}(t)\frac{1}{5}\left(1 - e^{-(t-15)}\left(cos(2(t-15)) - \frac{1}{2}\sin(2(t-15))\right)\right)$$
$$+ y_0 e^{-t}\cos(2t) + \frac{v_0 + y_0}{2}e^{-t}\sin(2t)$$

$y_0 = 0.1, v_0 = 0$



ex: $y'' + y' = 1,\ y(0) = y_0,\ y'(0) = v_0$

$$s^2 Y(s) - s y_0 - v_0 + sY(s) - y_0 = \frac{1}{s}$$

$$(s^2 + s)Y = s y_0 + y_0 + v_0 + \frac{1}{s}$$

$$Y(s) = \frac{\cancel{s} y_0}{\cancel{s}(s+1)} + \frac{y_0 + v_0}{s(s+1)} + \frac{1}{s^2(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \qquad 1 = A(s+1) + Bs$$

$$A = 1, B = -1$$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} \qquad 1 = A(s+1) + Bs(s+1) + Cs^2$$

$\underline{s = 0:}\ A = 1,\quad \underline{s = -1:}\ C = 1$

$$= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$\underline{s = 1:}\ 2 + 2B + \cancel{1} = \cancel{1}$

$$Y(s) = \cancel{\frac{y_0}{s+1}} + \frac{y_0 + v_0}{s} - \frac{\cancel{y_0} + v_0}{s+1} + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \qquad B = -1$$

continuing … $Y(s) = \frac{y_0+v_0-1}{s} + \frac{1-v_0}{s+1} + \frac{1}{s^2}$

$$
\begin{aligned}
y(t) &= y_0 + v_0 + 1 + (1 - v_0)e^{-t} + t \\
&= \underbrace{c_1 + c_2 e^{-t}}_{y_h} + \underbrace{t}_{y_p}
\end{aligned}
$$

M.U.C.

$$y'' + y' = 1$$

homogeneous problem

$$r^2 + r = 0 \Rightarrow r(r+1) = 0 \qquad \Rightarrow y_h = c_1 + c_2 e^{-t}$$

RHS in nullspace of operator

$$y_p = Bt \ldots$$

put it all together and then solve for c_1 and c_2 …