Recall: Constant Coefficients 2^{nd} Order Homogeneous IVP

$$ay'' + by' + cy = h(t) = 0 \qquad \begin{aligned} y(0) &= y_0 \\ y'(0) &= v_0 \end{aligned}$$

ex: Spring-dashpot with no external forcing

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t) \qquad - \qquad \text{three major cases}$$

What about for the inhomogeneous case ($h(t) \neq 0$) ?

General solution:

$$y_g = y_p + y_h$$

Recall, y_p has no arbitrary coefficients.

How can we find y_p?

$ay'' + by' + cy = h(t) \neq 0$

Two major cases:

1. $c = 0$

   Define $v(t) = y'$

   $$av' + bv = h(t)$$

   Solve by method of integrating factors

2. $c \neq 0$

Method of Undetermined Coefficients

$ay'' + by' + cy = h(t) \neq 0$

$c \neq 0$ - Method of Undetermined Coefficients

Basic idea

$$y_g = y_p + y_h$$

We know how to find y_h

- it makes the LHS=0

When we plug y_p into the LHS, it must equal $h(t)$.

- y_p + its derivatives $= h(t)$

Idea: differentiate $h(t)$ a bunch and see what kind of funtion y_p could be.

Find all the functional forms obtained by differentiation

$h(t) = t^2$

$\{t^2, t, \text{constant}\}$ finite set

$h(t) = \cos(3t)$

$\{\sin(3t), \cos(3t)\}$ finite set

$h(t) = \ln(t)$

$\left\{\ln(t), \frac{1}{t}, \frac{1}{t^2}, \frac{1}{t^3}, \ldots\right\}$ infinite set

Find the particular solution to $3y'' + 2y' + y = t^2$

Hint: guess $y_p = A + Bt + Ct^2$

$$y_p' = B + 2Ct \qquad\qquad y_p'' = 2C$$

plug into ODE

$$6C + 2(B + 2Ct) + A + Bt + Ct^2 = t^2$$
$$Ct^2 + (4C + B)t + (A + 2B + 6C) = t^2$$

We need this to be true for all t. Equate the coefficients in front of the different time-dependent functions.

$$\underline{t^2} : C = 1 \qquad \underline{t} : (4C + B) = 0$$
$$\underline{\text{constant}} : A + 2B + 6C = 0 \qquad 4 + B = 0 \qquad B = -4$$
$$A - 8 + 6 = 0 \qquad \Rightarrow A = 2$$

$$y_p = 2 - 4t + t^2$$

Find the particular solution to $y'' + 3y = 24e^{3t}$

Guess $y_p(t) = Ae^{3t}$

$$y_p' = 3Ae^{3t} \qquad y_p'' = 9Ae^{3t}$$

plug into ODE

$$\begin{aligned} 9Ae^{3t} + 3Ae^{3t} &= 24e^{3t} \\ 12Ae^{3t} &= 24e^{3t} \end{aligned}$$

We need this to be true for all t. Equate the coefficients in front of the different time-dependent functions.

$$\underline{e^{3t}} : 12A = 24 \qquad A = 2$$

$$y_p(t) = 2e^{3t}$$

$ay'' + by' + cy = h(t) \neq 0$

Will that approach always work?

Yes, if the function $h(t)$ has a finite family of derivative functional forms.

ex: $e^t, t^8, te^t, \cos(t), \sin(t), \ldots$

No, if it has infinitely many derivatives...

ex: $\ln(t), t^{-1}, t^{-2}, \ldots$

Method of Undetermined Coefficients:

$$ay'' + by' + cy = h(t)$$
$$y(t) = y_p(t) + y_h(t)$$

Form of function $h(t)$	Geuss for $y_p(t)$
$\sum_{j=0}^{N} B_j t^j$	$\sum_{j=0}^{N} A_j t^j$
$e^{\lambda t}$	$Ae^{\lambda t}$
$\sin \omega t$ or $\cos \omega t$	$A \sin \omega t + B \cos \omega t$
$e^{\lambda t} \sin \omega t$ or $e^{\lambda t} \cos \omega t$	$e^{\lambda t} A \sin \omega t + e^{\lambda t} B \cos \omega t$
Additive combinations of above	Additive combinations of above
Multiplicative combinations of above	Multiplicative combinations of above
Part of the homogeneous solution Note[1]	$Ath(t)$ or $At^2 h(t)$ or ...
Anything else	You are out of luck

[1]Note: This corresponds to resonance.

[2]Note: b_j , c_j , b, c, A, and B are all constants in the above table

Find the particular solution to $y'' - 6y' + \frac{25}{4}y = 3te^{2t}$

guess $y_p = Ae^{2t} + Bte^{2t}$

$$y_p' = 2Ae^{2t} + Be^{2t} + 2Bte^{2t} \qquad y_p'' = 2(2A+B)e^{2t} + 2Be^{2t} + 4Bte^{2t}$$

$$= (2A+B)e^{2t} + 2Bte^{2t} \qquad = 4(A+B)e^{2t} + 4Bte^{2t}$$

plug into DE

$$4(A+B)e^{2t} + 4Bte^{2t} - 6(2A+B)e^{2t} - 12Bte^{2t} + \frac{25}{4}Ae^{2t} + \frac{25}{4}Bte^{2t} = 3te^{2t}$$

$$-\frac{7}{4}Bte^{2t} - \frac{1}{4}(7A+8B)\,e^{2t} = 3te^{2t}$$

$$\underline{te^{2t}} : -\frac{7}{4}B = 3 \qquad B = -\frac{12}{7}$$

$$\underline{e^{2t}} : 7A + 8B = 0 \qquad A = \frac{8\cdot 12}{7\cdot 7} = \frac{96}{49}$$

$$y_p = \frac{96}{49}e^{2t} - \frac{12}{7}te^{2t}$$

Solve the IVP $y'' - 3y' - 4y = 3e^{2t}$ with $y(0) = y'(0) = 1$

$$y_g = y_p + c_1 y_1 + c_2 y_2$$

$$y_{1,2} = e^{rt} \quad \Rightarrow \quad (r^2 - 3r - 4) = 0$$

$$r_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1$$

$$y_g = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$$

$$y_g' = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}$$

Initial Conditions:

$$y(0) = 1 = c_1 + c_2 - \frac{1}{2}$$

$$y'(0) = 1 = 4c_1 - c_2 - 1$$

$$y(0) = 1 = c_1 + c_2 - \frac{1}{2}$$
$$y'(0) = 1 = 4c_1 - c_2 - 1$$

Add the two equations

$$2 = 5c_1 - \frac{3}{2} \qquad \frac{7}{2} = 5c_1$$
$$c_1 = \frac{7}{10}$$
$$1 = \frac{7}{10} + c_2 - \frac{5}{10} \qquad 1 = \frac{2}{10} + c_2$$
$$c_2 = \frac{8}{10}$$

$$y(t) = \frac{7}{10}e^{4t} + \frac{8}{10}e^{-t} - \frac{1}{2}e^{2t}$$