The Heat Equation and the Laplacian

Recall the 1D heat equation

$$u_t = \alpha u_{xx} = \alpha \frac{\partial^2}{\partial x^2} u(x,t)$$

In higher dimentions $(x, y, \dots)$ this becomes

$$\begin{aligned} u_t &= \alpha \left(u_{xx} + u_{yy} + \dots \right) \\ &= \alpha \Delta u \end{aligned}$$

Δ is the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \dots \quad = \vec{\nabla} \cdot \vec{\nabla} \qquad \text{with } \vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \vdots \end{bmatrix}$$

Steady State Heat Equation

If a steady state heat distribution is attained, we have $u_t \to 0$.

In 1D this gave us

$$0 = u_{xx} \qquad \Rightarrow u_\infty(x) = C_1 x + C_2$$

Given some boundary conditions, we can find u_∞.

ex: Suppose our domain is $x \in [0, L]$ with $u(0,t) = a$ and $u(L,t) = b$

$$u_\infty(x) = a + \frac{b-a}{L} x$$

Laplace's Equation

In higher dimensions, we need to solve Laplace's Equation

$$\Delta u = 0$$

Restrciting ourselves to 2 dimensions, we get

$$u_{xx} + u_{yy} = 0$$

No longer as simple as finding linear functions

<u>ex</u>: $u(x, y) = x^2 - y^2$

$$u_{xx} = 2, \qquad u_{yy} = -2 \qquad \Rightarrow u_{xx} + u_{yy} = 0$$

Solutions to Laplace's Equation are called **harmonic functions**

Boundary Conditions - Dirichlet Problem

Consider a region in the x, y-plane.

We specify certain values of some unknown function $u(x, t)$ along the boundaries of the region.

We then try to find a solution to the Laplace equation

$$u_{xx} + u_{yy} = 0$$

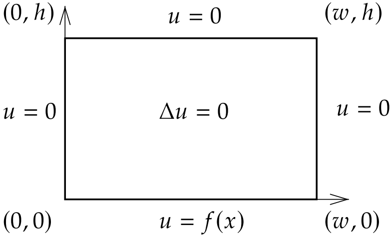
defined on this region such that agrees with the values we specified on the boundary.

Boundary Conditions - Dirichlet Problem

ex: Consider a rectangular region of width w and height h.

Boundary values are zero at three of the four edges, and obey some arbitrary function $f(x)$ along the fourth edge.

$$
\begin{aligned}
\Delta u &= 0 & \\
u(0, y) &= 0 & \text{for } 0 < y < h \\
u(x, h) &= 0 & \text{for } 0 < x < w \\
u(w, y) &= 0 & \text{for } 0 < y < h \\
u(x, 0) &= f(x) & \text{for } 0 < x < w
\end{aligned}
$$



Solving Dirichlet Problems

$$u_{xx} + u_{yy} = 0$$

Separation of Variables:

$$u(x, y) = X(x)Y(y)$$

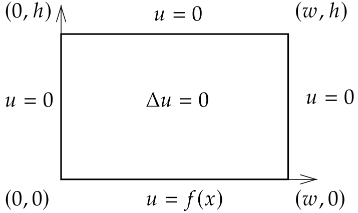
$$X''Y + XY'' = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const.}$$

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

Solving Dirichlet Problems

$$X'' + \lambda X = 0$$



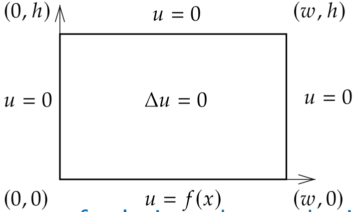
What are the boundary conditions? What are the allowed solution and values of λ?

BCs: $X(0) = X(w) = 0$

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

Solving Dirichlet Problems

$$Y'' - \lambda Y = 0$$



$$\lambda = \left(\frac{n\pi}{w}\right)^2$$

What are the boundary conditions? What types of solutions do we obtain?

BCs: $Y(h) = 0$, less clear what is going on for $y = 0$.

Try $Y_n = e^{ry}$

$$r^2 e^{ry} - \left(\frac{n\pi}{w}\right)^2 e^{ry} = 0$$

$$r^2 = \left(\frac{n\pi}{w}\right)^2 = 0 \qquad \Rightarrow r = \pm\frac{n\pi}{w}$$

$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y}$$

$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y} \qquad \text{with } Y(h) = 0$$

$$Y(h) = 0 = Ae^{\frac{n\pi}{w}h} + Be^{-\frac{n\pi}{w}h} \qquad \Rightarrow A = -Be^{-2\frac{n\pi}{w}h}$$

$$\begin{aligned}
Y_n(y) &= B\left[-e^{-2\frac{n\pi}{w}h}e^{\frac{n\pi}{w}y} + e^{-\frac{n\pi}{w}y}\right] \\
&= B\left[-e^{-\frac{n\pi}{w}h}e^{\frac{n\pi}{w}(y-h)} + e^{-\frac{n\pi}{w}h}e^{-\frac{n\pi}{w}(y-h)}\right] \\
&= \underbrace{Be^{-\frac{n\pi}{w}h}}_{\text{arbitrary}}\underbrace{\left[e^{-\frac{n\pi}{w}(y-h)} - e^{\frac{n\pi}{w}(y-h)}\right]}_{\propto\text{sinh}} \\
&= a_n \sinh\left(\frac{n\pi}{w}(h-y)\right)
\end{aligned}$$

$$u_n(x,y) = a_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}(h-y)\right)$$

Eigensolution of the Laplacian that satifies the zero boundary conditions on the 3 sides.

The non-zero boundary condition

$$u_n(x,y) = a_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}(h-y)\right), \quad u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

$u(x,0) = f(x)$ - Express the boundary condition as Fourier Series
Given the appearance of our u_n, we clearly need a Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{w}x\right) \qquad \text{with } b_n = \frac{2}{w}\int_0^w f(x)\sin\left(\frac{n\pi}{w}x\right)dx$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}h\right)$$

need equality between the two series

$$\Rightarrow a_n = \frac{b_n}{\sinh\left(\frac{n\pi}{w}h\right)}$$

The Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

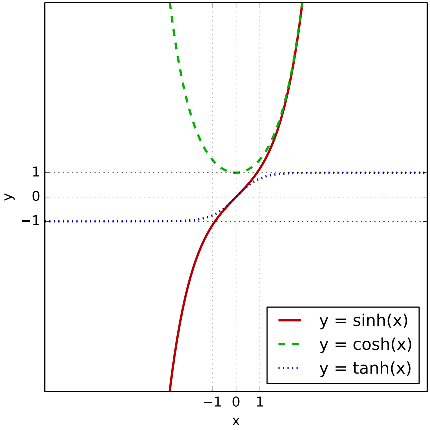
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$

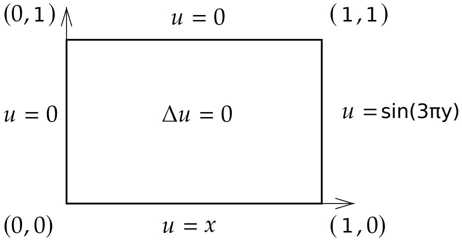


source: Wikipedia

Boundary conditions on two sides

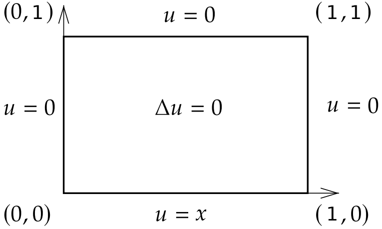
ex: Consider a rectangular region of width 1 and height 1.

$$\begin{aligned} \Delta u &= 0 & & \\ u(0,y) &= 0 & &\text{for } 0 < y < h \\ u(x,h) &= 0 & &\text{for } 0 < x < w \\ u(w,y) &= \sin(3\pi x) & &\text{for } 0 < y < h \\ u(x,0) &= x & &\text{for } 0 < x < w \end{aligned}$$

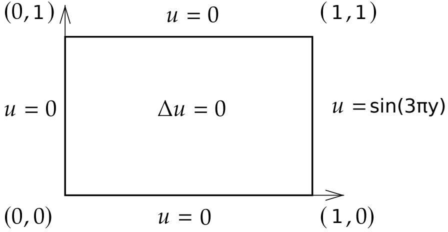


Breaking it down by directions

South Problem



East Problem



$$u(x, y) = u_N + u_S + u_E + u_W$$

The west and north problems have zero solutions, no need to solve them.

The South Problem

Special case of the previous problem we did with $f(x) = x$

$$u_s(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sinh(n\pi(1 - y))$$

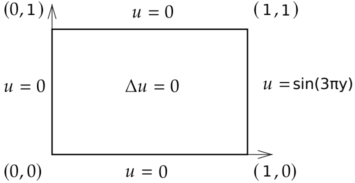
$$a_n = \frac{b_n}{\sinh(n\pi)}$$

$$b_n = 2\int_0^1 x \sin(n\pi x) dx$$
$$= -2\frac{(-1)^n}{n\pi}$$

https://www.wolframalpha.com/input?i=integral+of+x*sin%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer

The East Problem

$$
\begin{aligned}
\Delta u_E &= 0 & & \\
u_E(0, y) &= 0 & & \text{for } 0 < y < h \\
u_E(x, h) &= 0 & & \text{for } 0 < x < w \\
u_E(w, y) &= \sin(3\pi y) & & \text{for } 0 < y < h \\
u_E(x, 0) &= 0 & & \text{for } 0 < x < w
\end{aligned}
$$



$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$

Now we have two zero BCs for $Y(y)$

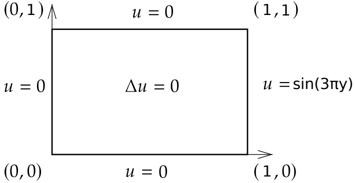
$$Y(0) = Y(1) = 0 \qquad \Rightarrow Y_n(y) = \sin(n\pi y)$$

with

$$\lambda_n = -(n\pi)^2$$

The East Problem

$$X'' + \lambda X = 0 \qquad \text{with } \lambda = -(n\pi)^2$$



$$X'' - (n\pi)^2 X = 0$$

$$X(x) = Ae^{n\pi x} + Be^{-n\pi x}$$

BC @ x=0: $\quad 0 = A + B \qquad\qquad B = -A$

$$X(x) = A(e^{n\pi x} - e^{-n\pi x})$$

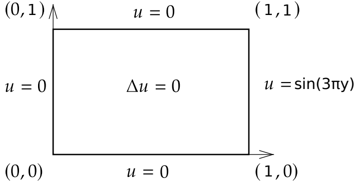
$$= a_n \sinh(n\pi x)$$

$$u_n(x, y) = a_n \sin(n\pi y) \sinh(n\pi x)$$

The East Problem

$$u_n(x,y) = a_n \sin(n\pi y) \sinh(n\pi x)$$

$$u_E = \sum_{n=1}^{\infty} u_n(x,y)$$

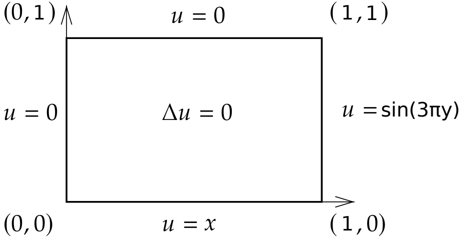


BC @ x=1

$$u(1,y) = \sin(3\pi y) = \sum_n a_n \sin(n\pi y) \sinh(n\pi)$$

$$a_n = \begin{cases} \frac{1}{\sinh(3\pi)} & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u_E = u_3(x,y) = \frac{\sinh(3\pi x)}{\sinh(3\pi)} \sin(3\pi y)$$



$$u(x,t) = u_S + u_E + \cancelto{0}{u_N} + \cancelto{0}{u_W}$$

$$= \sum_{n=1}^{\infty} -2\frac{(-1)^n}{n\pi} \sin\left(n\pi x\right) \sinh\left(n\pi(1-y)\right)$$

$$+ \frac{\sinh(3\pi x)}{\sinh(3\pi)} \sin(3\pi y)$$

Separation of Variables on Rectangular Domains

- General Approach: $u(x,y) = u_N + u_S + u_E + u_W$
- Each u_I solves a Dirichlet problem with <u>one</u> non-zero BC
  - If a problem has all zero BCs, then u_I is zero

$$u_N = \sum_n a_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}y\right)$$
$$u_S = \sum_n b_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}(h-y)\right)$$
$$u_E = \sum_n c_n \sin\left(\frac{n\pi}{h}y\right)\sinh\left(\frac{n\pi}{h}x\right)$$
$$u_W = \sum_n d_n \sin\left(\frac{n\pi}{h}y\right)\sinh\left(\frac{n\pi}{h}(w-x)\right)$$

- Find the arbitrary coefficients by comparing each series solution to a Fourier series of the non-zero BC.