Review

Linear First Order DE:

$$y' + p(t)y = g(t)$$

$$\underbrace{y_h' + p(t)y_h = 0}_{\text{Homogeneous Problem}}$$

Linearity $\Rightarrow$ General Solution Structure:

$$y_g = \underbrace{\underbrace{y_p}_{\textbf{Particular Part}} \text{ - Method of Undetermined Coefficients} + \underbrace{y_h}_{\textbf{Homogeneous Part}} \text{ - General Solution}}_{\text{Method of Integrating Factors}}$$

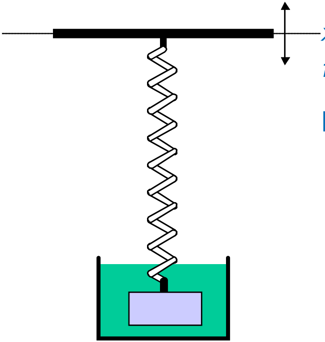
What happens when we add in y''

Overview

Second order linear differential equations

- Where do these DEs arise?
- Existence & uniqueness of solutions
- Superposition again:
  - General solution = homogeneous solution + particular solution

Spring-dashpot system:

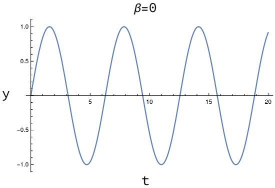


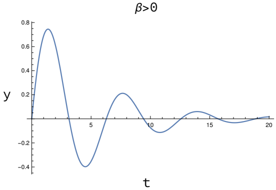
$x(t) =$ displacement from rest position
$f(t) =$ applied force

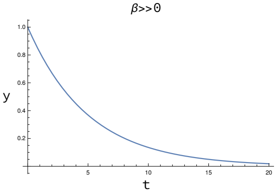
Newton's 2^{nd} Law:

$$F = ma$$

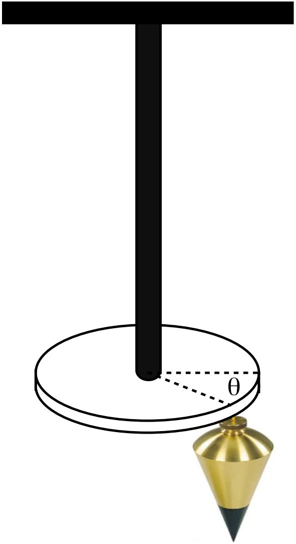
$$-kx - \beta\frac{dx}{dt} + f(t) = m\frac{d^2x}{dt^2}$$





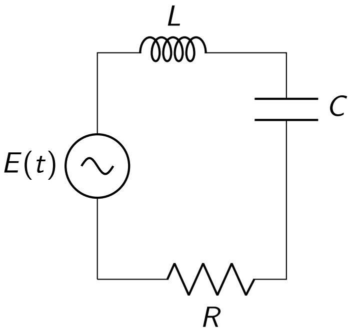


Torsional motion of a weight on a twisted shaft:



$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + k\theta = T(t)$$

L-R-C series circuits:



Q=charge on capacitor

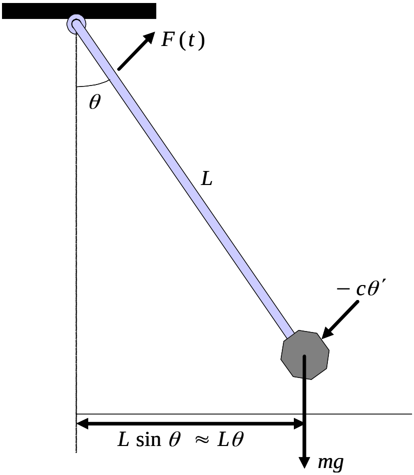
$\frac{dQ}{dt}$=current in circuit

$E(t)$ = applied voltage

Kirchoff's Laws:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Small oscillations of a pendulum:



$$mL^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -cL\frac{\mathrm{d}\theta}{\mathrm{d}t} - mgL\theta + F(t)$$

Equivalence of Problems

These 4 physical systems are modelled identically by:

$$Ay'' + By' + Cy = D(t)$$

Constants have different physical meaning (& units)

System	A	B	C	D
Spring Dashpot	Mass	Damping Coeff.	Spring Constant	Applied Force
Pendulum	Mass x $(\text{Length})^2$	Damping x Length	Gravitational Moment	Applied Moment
Series Circuit	Inductance	Resistance	Capacitance^{-1}	Imposed Voltage
Twisted Shaft	Moment of Inertia	Damping	Elastic Shaft Constant	Applied Torque

General Linear 2nd Order DE's

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{1}$$

- $g(t)$ represents the "forcing" term, an external influence.
  - Homogeneous if $g(t) = 0$ for all $t \in I$
  - Inhomogeneous if $g(t) \neq 0$ for all $t \in I$
- $p(t)$ and $q(t)$ represent the intrinsic properties of the physical system.
  - Often consant, but not always
  - e.g., an aging spring could be modelled by $q = q(t)$.
- Solution to (1) is defined on an interval $t \in I = (\alpha, \beta)$

Initial Conditions

IVP = Initial Value Problem

- Consists of Eq. (1) plus a set of initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = v_0 \tag{2}$$

Why are there two initial conditions?

- 2nd order $\Rightarrow$ integrate twice to solve $\Rightarrow$ two constants of integration

$$\Downarrow$$

Need two initial conditions for a well-defined solution

Existence and uniqueness of solutions

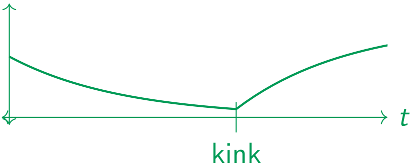
Theorem 1: Consider the IVP

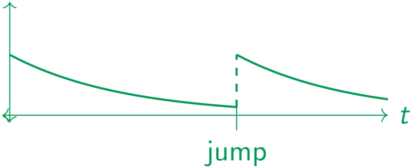
$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \qquad t \in I$$
$$y(t_0) = y_0, \quad y'(t_0) = v_0, \qquad t_0 \in I$$

where $p(t)$, $q(t)$, and $g(t)$ are **continuous** for $t \in I = (\alpha, \beta)$

There is a single solution to this IVP, and the solution is defined throughout the time interval I

Notes: Discontinuities create kinks or jumps in the solution.





Superposition Principle

Theorem 2: Suppose y_1 and y_2 are solutions to

$$\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1 = g_1(t), \qquad t \in I$$
$$\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2 = g_2(t), \qquad t \in I.$$

Then, for any values of the constants c_1 and c_2, the linear combination $y = c_1y_1 + c_2y_2$ satisifes

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = c_1g_1(t) + c_2g_2(t), \qquad t \in I.$$

Solution Structure

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = g(t) \tag{1}$$

$$y(t_0) = y_0, \quad y'(t_0) = v_0 \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}y}{\mathrm{d}t} + q(t)y = 0 \tag{3}$$

$$\underbrace{y_g(t)}_{\substack{\textbf{General}\\ \text{Solution}\\ \text{of (1)}}} = \underbrace{y_p(t)}_{\substack{\textbf{Particular}\\ \text{Solution}\\ \text{of (1)}}} + \underbrace{y_c(t)}_{\substack{\text{Particular solution of (3) + (2)}\\ =\\ \textbf{Complementary}\text{ part}}}$$

Homogeneous Solution Structure

1^{st} Order DE:

$$y' + q(t)y = 0$$
$$\Rightarrow y_h = c_1 y_1$$
$$ex. \quad p(t) = a \Rightarrow y_h = Ce^{-at}$$

2^{nd} Order DE:

$$y'' + p(t)y' + q(t)y = 0$$
$$\Rightarrow y_h = c_1 y_1 + c_2 y_2$$

Both y_1 and y_2 make the LHS=0, independently.

How to construct y_c

We need y_h to satisfy (3) and the two initial values (2)

$$y(t_0) = y_0, \quad y'(t_0) = v_0 \tag{2}$$

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{3}$$

$$y_h = c_1 y_1 + c_2 y_2$$

Do some algebra to solve for c_1 and c_2.

Is the algebra always possible?

Can we always satisfy (2) with $c_1 y_1 + c_2 y_2$?

- If so, then $y = c_1 y_1 + c_2 y_2$ is the general solution of (3)

$$\begin{aligned} c_1 y_1 + c_2 y_2 &= y_0 \\ c_1 y_1' + c_2 y_2' &= v_0 \end{aligned}$$

Express in matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$

$$A \quad x \quad = \quad b$$

Solution: $\quad x = A^{-1} b \qquad\qquad$ if A^{-1} exists

Under what conditions does the matrix A^{-1} exist?

$$\det A \neq 0$$

Wronskian Determinant

For two differentiable functions, $y_1(t)$ and $y_2(t)$, the Wronskian determinant is denoted $W(y_1, y_2)(t)$, and is defined by

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

- $W(y_1, y_2)(t_0) \neq 0$ is the condition that allows $c_1 y_1 + c_2 y_2$ to satisfy ANY initial conditions at t_0.

A set of 2 solutions, $y_1(t)$ and $y_2(t)$, to Eq. (3) on the time interval I, is called a **fundamental set of solutions** if $W(y_1, y_2)(t) \neq 0$ for some $t \in I$:

$$y_c = c_1 y_1 + c_2 y_2$$

Generality of Fundamental Solutions

For 2 solutions $y_1(t)$ and $y_2(t)$ of Eq. (3) on I, show that if $W(y_1, y_2)(t) \neq 0$ for for some $t_0 \in I$, then in fact $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

$$y'' + p(t)y' + q(t)y = 0 \qquad \Rightarrow \qquad y'' = -p(t)y' - q(t)y$$

$$\begin{aligned}
W &= y_1 y_2' - y_1' y_2 \\
\frac{\mathrm{d}W}{\mathrm{d}t} &= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} \\
&= y_1 \left[-p(t)y_2' - q(t)y_2\right] - \left[-p(t)y_1' - q(t)y_1\right] y_2 \\
&= -p(t)y_1 y_2' - \cancel{q(t)y_1 y_2} + p(t)y_1' y_2 + \cancel{q(t)y_1 y_2} \\
&= -p(t)y_1 y_2' + p(t)y_1' y_2 = -p(t)\left[y_1 y_2' - y_1' y_2\right] \\
W' &= -p(t)W
\end{aligned}$$

We can solve $W' + p(t)W = 0$ using the method of integrating factors!

$$W' + p(t)W = 0 \qquad \Rightarrow \qquad \mu = e^{\int p(t)dt}$$

$$\mu(t)W(t) = \int 0dt + C$$
$$W(t) = \frac{C}{\mu(t)} \qquad \Rightarrow W(t) = Ce^{-\int p(t)dt}$$

Assuming the integral of $p(t)$ is finite, we have two cases:

1. $C = 0 \qquad \Rightarrow W(t) = 0$ for all t
2. $C \neq 0 \qquad \Rightarrow W(t) \neq 0$ for all t

Suppose that at some t_0 we find that $W(t_0) \neq 0$

$$\Rightarrow C \neq 0 \Rightarrow W(t) \neq 0 \quad \text{for all t}$$

Linear dependence of functions

- Two functions f and g are **linearly dependent** on $I = (\alpha, \beta)$ if there exist constants $k_1$and k_2 , not both zero, such that

$$k_1 f + k_2 g = 0 \quad \forall t \in I$$

- The functions are **linearly independent** on I if they are not linearly dependent on I.

- Note that if f and g are differentiable functions:
  - $W(f, g)(t_0) \neq 0 \quad \Leftrightarrow \quad f$ & g are <u>linearly independent</u> on I.
  - f & g are linearly dependent $\quad \Rightarrow \quad \underline{W(f, g)(t) = 0}$.
  - but $W(f, g)(t) = 0 \quad \cancel{\Rightarrow} \quad f$ & g are linearly dependent

Summary for linear homogeneous equations

- For linear equations with continuous coefficients functions $p(t)$ & $q(t)$, there exists a unique solution to an IVP.
  - Linear combinations of solutions are solutions
  - The Wronskian of two solutions is a determinant
- If the Wronskian of 2 solutions is non-zero at a point, then any initial values can be satisfied at that point. The general solution of the DE is a linear combination of those 2 solutions
- Such a set of solutions is called a **fundamental set of solutions**
- If y_1 & y_2 are 2 solutions to (3), the following 4 statements are equivalent:
  - The functions y_1 & y_2 are a fundamental set of solutions on I.
  - The functions y_1 & y_2 are linearly independent on I.
  - $W(f,g)(t) \neq 0$ for all $t \in I$.
  - $W(f,g)(t_0) \neq 0$ for some $t_0 \in I$.