

# Homogeneous Heat Equation

$$u_t = \alpha u_{xx} \quad \text{with} \quad \begin{array}{l} u(0, t) = u(L, t) = 0 \\ \text{or} \\ u_x(0, t) = u_x(L, t) = 0 \end{array} \quad \text{and} \quad u(x, 0) = u_0(x)$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} \left( a_n \cos \left( \frac{n\pi}{L} x \right) + b_n \sin \left( \frac{n\pi}{L} x \right) \right)$$

$$\underline{u(0, t) = u(L, t) = 0:}$$

$$\underline{u_x(0, t) = u_x(L, t) = 0:}$$

Fourier sine series

$$a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L u_0(x) \sin \left( \frac{n\pi}{L} x \right) dx$$

Fourier cosine series

$$a_n = \frac{2}{L} \int_0^L u_0(x) \cos \left( \frac{n\pi}{L} x \right) dx$$

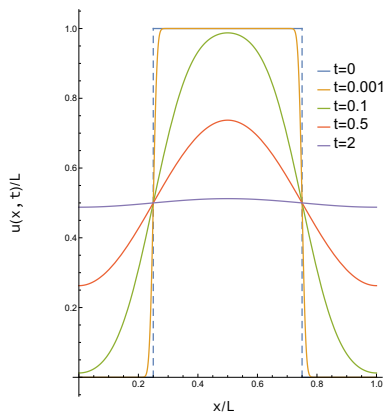
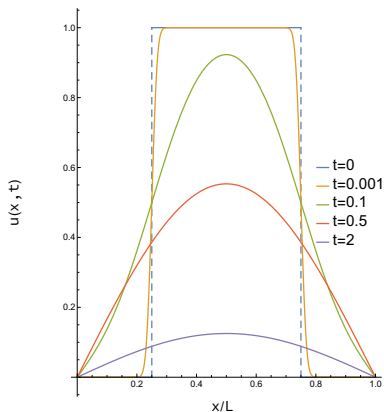
$$b_n = 0$$

The heat equation smooths out initial conditions.

ex:  $u_0(x) = u_{\frac{L}{4}}(x) - u_{\frac{3L}{4}}(x) = H\left(x - \frac{L}{4}\right) - H\left(x - \frac{3L}{4}\right)$

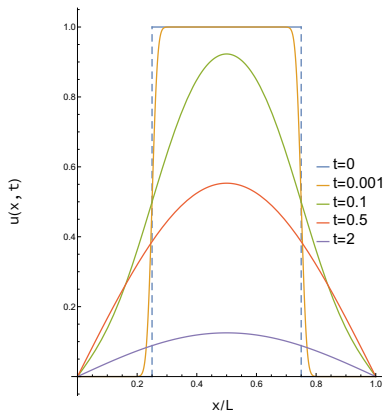
$u(0, t) = u(L, t) = 0$ :

$u_x(0, t) = u_x(L, t) = 0$ :



# Short-term Behaviour

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{e^{-\alpha \frac{n^2 \pi^2}{L^2} t}}_{\text{Exp. Decay}} \cdot \underbrace{\left( a_n \cos \left( \frac{n\pi}{L} x \right) + b_n \sin \left( \frac{n\pi}{L} x \right) \right)}_{n^{\text{th}} \text{ Fourier Mode}}$$



- Sharp edges in the initial condition require high frequency modes ( $n \gg 1$ ).
- The exp. decay rate grows with  $n^2$ .
- High frequency modes decay the fastest.
- Sharp features get smoothed out very quickly.

# Long-term Behaviour

As  $t \rightarrow \infty$ ,

$$\begin{aligned}\lim_{t \rightarrow \infty} u(x, t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} \left( a_n \cos \left( \frac{n\pi}{L} x \right) + b_n \sin \left( \frac{n\pi}{L} x \right) \right) \\ &= \frac{a_0}{2}\end{aligned}$$

For finite  $t \gg 0$ , we can just keep the first non-zero term in the infinite sum. All other terms decay much much faster.

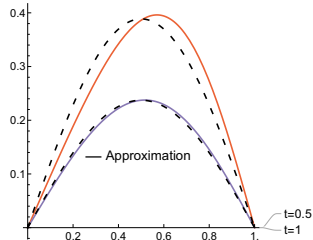
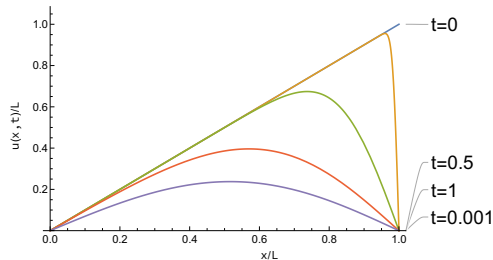
$$\underline{u(0, t) = u(L, t) = 0:}$$

$$\underline{u_x(0, t) = u_x(L, t) = 0:}$$

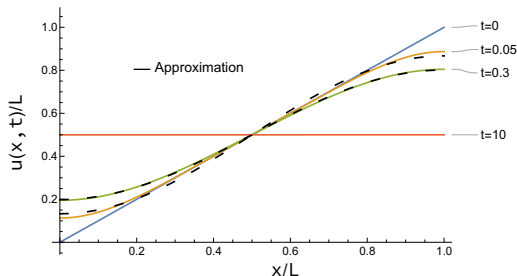
$$u(x, t) \approx e^{-\alpha \frac{\pi^2}{L^2} t} b_1 \sin \left( \frac{\pi}{L} x \right)$$

$$u(x, t) \approx \frac{a_0}{2} + e^{-\alpha \frac{\pi^2}{L^2} t} a_1 \cos \left( \frac{\pi}{L} x \right)$$

$$u(0, t) = u(L, t) = 0: u(x, t) \approx e^{-\alpha \frac{\pi^2}{L^2} t} b_1 \sin\left(\frac{\pi}{L} x\right)$$



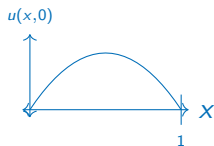
$$u_x(0, t) = u_x(L, t) = 0: u(x, t) \approx \frac{a_0}{2} + e^{-\alpha \frac{\pi^2}{L^2} t} a_1 \cos\left(\frac{\pi}{L} x\right)$$



ex:  $u_t = \alpha u_{xx}$ ,

$$u(0, t) = u(1, t) = 0$$

$$u(x, 0) = x(1 - x) \text{ on } [0, 1]$$



BCs  $\Rightarrow$  only sin terms survive.

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = x(1 - x)$$

$$b_n = \frac{2}{1} \int_0^1 u(x, 0) \sin(n\pi x) dx = 2 \int_0^1 x(1 - x) \sin(n\pi x) dx$$

$$\stackrel{\text{wolfram}}{=} 2 \left( \frac{-2((-1)^n - 1)}{n^3 \pi^3} \right) \Rightarrow$$

$$b_n = -\frac{4}{\pi^3} \frac{(-1)^n - 1}{n^3}$$

[https://www.wolframalpha.com/input?i=integral+of+x%281-x%29sin%28n\\*pi\\*x%29+from+0+to+1+assuming+n+is+an+integer](https://www.wolframalpha.com/input?i=integral+of+x%281-x%29sin%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer)

Given

$$u(x, t) = - \sum_{n=1}^{\infty} \frac{4}{\pi^3} \frac{(-1)^n - 1}{n^3} \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

suppose that after 100 time units the hottest point in the domain is at half of its initial temperature, find an approximation for  $\alpha$ .

Initial Condition:  $u_0(x) = x(1 - x)$

$$\Rightarrow x_{\max} = \frac{1}{2} \qquad u_0(x_{\max}) = \frac{1}{4}$$

$$\text{Long-term: } u(x, t) \approx \frac{8}{\pi^3} \sin(\pi x) e^{-\alpha \pi^2 t}$$

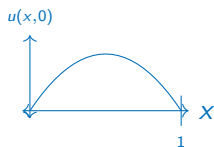
approximation also has its max at  $x = 1/2$

$$\frac{8}{\pi^3} e^{-\alpha \pi^2 100} \approx \frac{1}{8} \qquad \alpha \approx -\frac{1}{100\pi^2} \log\left(\frac{\pi^3}{64}\right)$$

$$\alpha \approx 7.34 \times 10^{-4}$$

ex:  $u_t = 0.1u_{xx}$ ,  $u_x(0, t) = u_x(1, t) = 0$   
 $u(x, 0) = x(1 - x)$  on  $[0, 1]$

BCs  $\Rightarrow$  only cos terms survive.



$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-0.1n^2\pi^2 t}$$

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = x(1 - x)$$

$$a_n = \frac{2}{1} \int_0^1 u(x, 0) \cos(n\pi x) dx = 2 \int_0^1 x(1 - x) \sin(n\pi x) dx$$

$$\stackrel{\text{wolfram}}{=} -2 \frac{(-1)^n + 1}{\pi^2 n^2}$$

$$a_0 = \frac{2}{1} \int_0^1 x(1 - x) dx = \frac{2}{6}$$

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<https://www.wolframalpha.com/input?i=integral+of+x%281-x%29+from+0+to+1>



Given

$$u(x, t) = \frac{1}{6} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{\pi^2 n^2} \cos(n\pi x) e^{-0.1n^2\pi^2 t}$$

approximately how long does it take for the endpoints of the domain to be within 1% of their steady state value?

$n = 1$  has a coefficient of zero, use  $n = 2$  instead for approximation.

$$\text{Long-term: } u(x, t) \approx \frac{1}{6} - 2 \frac{2}{4\pi^2} \cos(2\pi x) e^{-0.4\pi^2 t}$$

$$\text{Steady state: } u_{\infty}(x) = \frac{1}{6}$$

$$u_{\infty}(x) - u(L, t) \approx \frac{1}{\pi^2} \cos(2\pi x) e^{-0.4\pi^2 t}$$

$$x = 0 \text{ or } x = 1: \frac{1}{\pi^2} e^{-0.4\pi^2 t} \approx 1\% \times \frac{1}{6}$$

$$t \approx -\frac{1}{0.4\pi^2} \log \left( \frac{0.01\pi^2}{6} \right)$$

$$\approx 1.04 \text{ time units}$$