Recall

We saw how to solve ODEs of the form

$$y' = f(t) \cdot g(y) \qquad - \qquad \text{Separable Equations}$$

$$\frac{dy}{g(y)} = f(t)dt \ldots \text{integrate and isolate } y$$

Now we want to solve something a little more general...

$$y' = f(t) \cdot g(y) + h(t)$$

This is not separable :(

To keep things simple, we focus on linear equations with $g(y) = y$

General form of a 1^{st} order linear ODE

$$y' + p(t)y = h(t)$$

Operator form:

- Stuff with y goes on the left
- stuff without y goes on the right.

Two cases:

- $h(t) = 0$: call this a Homogeneous equation
  - Separable - we know how to solve
  - Homogeneous $\Leftrightarrow$ separable for linear equations only
- $h(t) \neq 0$: call this a Non-Homogenous or Inhomogeneous equation
  - $h(t) =$ the inhomogeneity

ex: Solve $y' + 2ty = t^2 e^t$

First we do something very odd, multiply everything by e^{t^2}

$$e^{t^2} y' + \underbrace{2te^{t^2}}_{\frac{d}{dt} e^{t^2}} y = t^2 e^{t^3} \quad \rightarrow \quad \underbrace{e^{t^2} \frac{d}{dt} y + \frac{d}{dt} e^{t^2} y}_{\text{product rule applied to } e^{t^2} y} = t^2 e^{t^3}$$

$$\frac{d}{dt}\left(e^{t^2} y\right) = t^2 e^{t^3} \qquad \int d\left(e^{t^2} y\right) = \int t^2 e^{t^3} dt$$

$$e^{t^2} y = \frac{1}{3} \int e^u du \qquad \begin{aligned} u &= t^3 \\ du &= 3t^2 \end{aligned}$$

$$\begin{aligned} e^{t^2} y &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{t^3} + C \end{aligned}$$

$$y(t) = \frac{1}{3} e^t + Ce^{-t^2}$$

Method of integrating factors: $y' + p(t)y = h(t)$

1. Multiply by the integrating factor $\mu(t) = e^{\int p(t)\mathrm{d}t}$

$$\mu(t)y' + \underbrace{p(t)\mu(t)}_{\mu'(t)}y = h(t)\mu(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot y) = h(t)\mu(t)$$

$$\mathrm{d}(\mu \cdot y) = h(t)\mu(t)\mathrm{d}t$$

2. Integrate:

$$\mu \cdot y = \int h(t)\mu(t)\mathrm{d}t + C$$

3. Isolate $y(t)$:

$$y(t) = \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$

Existence of solutions

$$y(t) = \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

For solutions to exist, we need the derivative of y to be defined at all points in time.

That is, y must be differentiable.

- What properties does $p(t)$ need to ensure $\mu(t)$ is differentiable?
  - Continuous and integrable.
- What properties does $h(t)$ need to ensure $y(t)$ is differentiable?
  - Continuous and $\mu(t) \cdot h(t)$ is integrable.

These conditions are pretty lax, but evaluating the integrals can be tough.

ex: Find the general solution of $y' - 3y = e^t$.

$$\boxed{\begin{aligned} y(t) &= \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)} \\ \mu(t) &= e^{\int p(t)\mathrm{d}t} \end{aligned}}$$

$$\begin{aligned}
\mu(t) &= e^{\int(-3)dt} = e^{-3t} \\
e^{-3t}y' - 3e^{-3t}y &= e^t e^{-3t} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-3t}y\right) &= e^{-2t} \\
\mathrm{d}\left(e^{-3t}y\right) &= e^{-2t}\mathrm{d}t \\
e^{-3t}y(t) &= \int e^{-2t}\mathrm{d}t + C \\
y(t) &= \frac{-\frac{1}{2}e^{-2t} + C}{e^{-3t}} \\
&= \underbrace{-\frac{1}{2}e^t}_{y_p} + \underbrace{Ce^{3t}}_{y_h}
\end{aligned}$$

ex: Find the general solution of $y' - 3t^2 y = 3t^2$.

$$\boxed{\begin{aligned} y(t) &= \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)} \\ \mu(t) &= e^{\int p(t)\mathrm{d}t} \end{aligned}}$$

$$\mu(t) = e^{-\int 3t^2 dt} = e^{-t^3}$$

$$e^{-t^3} y(t) = \int 3t^2 e^{-t^3} \mathrm{d}t$$

$$\text{let } u = t^3$$
$$du = 3t^2$$

$$\int 3t^2 e^{-t^3}\mathrm{d}t = \int e^{-u} du$$
$$= -e^{-u} + C$$

$$e^{-t^3} y(t) = -e^{-t^3} + C$$
$$y(t) = -1 + Ce^{t^3}$$

Summary

- General linear 1st order ODE: $y' + p(t)y = h(t)$
  - $h(t) = 0$: Homogeneous ODE
  - $h(t) \neq 0$: Inhomogeneous ODE
  - To solve, turn the LHS into an exact derivative:

$$y' + p(t)y \quad \rightarrow \quad \mathrm{d}(\mu \cdot y) = \mu' y + \mu y'$$

    - Multiply by an integrating factor $\mu(t) = e^{\int p(t)\mathrm{d}t}$
    - General solution: $y(t) = \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$

- This method solves all linear 1st order ODEs, provided that $p(t)$ and $g(t)$ are continuous.