(Previously) A first order linear ODE

Suppose

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 y(t)$$

Find $y(t)$ Divide by y, then mulitply dt

$$\frac{dy}{\cancel{dt}}\frac{1}{y}\cancel{dt} = t^2 dt$$

integrate

$$\int \frac{dy}{y} = \int t^2 dt$$

$$\ln(y) = \frac{t^3}{3} + C$$

exponentiate both sides

$$y(t) = e^{\frac{t^3}{3}+C} = Ce^{\frac{t^3}{3}}$$

Formally

$$\int \frac{dy}{y} = \ln(|y|) + C.$$

Why did I ignore the absolute value? Can I always do that?

Lets consider

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)y$$

$$\frac{dy}{\cancel{dt}}\frac{1}{y}\cancel{dt} = f(t)dt$$

integrate

$$\int \frac{dy}{y} = \int f(t)dt$$

$$\ln(|y|) = F(t) + C_1$$

C_1 is an arbitrary constant (of integration)

$$\ln(|y(t)|) = F(t) + C_1$$

Two cases:

$y < 0$

$$|y| = -y$$

$$\ln(-y) = F(t) + C_1$$
$$-y = e^{C_1} e^{F(t)}$$

$$y = -e^{C_1} e^{F(t)}$$
$$= Ce^{F(t)}$$

$y > 0$

$$|y| = y$$

$$\ln(y) = F(t) + C_1$$
$$y = e^{C_1} e^{F(t)}$$

$$= Ce^{F(t)}$$

Since $e^{F(t)} > 0$, $y(t)$ cannot change sign - Always ok to ignore $|y|$