Homogeneous Heat Equation

$u_t = \alpha u_{xx}$
for $x \in [0, L]$ with $u(0,t) = u(L,t) = 0$ or $u_x(0,t) = u_x(L,t) = 0$ and $u(x,0) = u_0(x)$

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)}$$

$\underline{u(0,t) = u(L,t) = 0}$:

Fourier sine series

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$\underline{u_x(0,t) = u_x(L,t) = 0}$:

Fourier cosine series

$$a_n = \frac{2}{L}\int_0^L u_0(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$b_n = 0$$

Inhomogeneous Heat Equation

2 types of inhomogeneities:

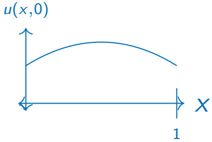
1. Inhomogeneous BCs (e.g., $u(0, t) \neq 0$ or $u_x(0, t) \neq 0 \ldots$)
2. Source/Sink Inhomogeneity (Inhomogeneous PDE)

$$u_t = \alpha u_{xx} + \sigma(x)$$

$\sigma(x)$ accounts for local heat production/removal.

Overall approach to solving both is the same, but each example can have its own quirks ex: $u_t = \alpha u_{xx}$, $u(0,t) = u(1,t) = 1$

$u(x,0) = 1 + x(1-x)$ on $[0,1]$



Trick: define $w(x,t) = u(x,t) - 1$

$$w_t = u_t \qquad w_{xx} = u_{xx}$$

let's write down a PDE for $w(x,t)$

$$w_t = \alpha w_{xx} \qquad w(0,t) = w(L,t) = 0$$

$$w(x,0) = x(1-x)$$

we've solved this one before

$$w(x,t) = -\sum_{n=1}^{\infty} \frac{4}{\pi^3} \frac{(-1)^n - 1}{n^3} \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

$$u(x,t) = 1 + w(x,t)$$

Inhomogeneous Heat Equation: General Approach

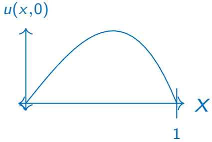
$$u(x,t) = \underbrace{w(x,t)}_{\substack{\text{Transient}\\ \text{Solution}\\ \text{—}\\ \text{Homgeneous Part}}} + \underbrace{u_\infty(x)}_{\substack{\text{Steady State}\\ \text{Solution}\\ \text{—}\\ \text{Inhomgeneous Part}}}$$

Four steps:

1. Find $u_\infty(x)$ (if it exists)
2. Write down a homogeneous IBVP for $w(x,t) = u(x,t) - u_\infty(x)$
3. Solve for $w(x,t)$
4. Final solution: $u(x,t) = w(x,t) + u_\infty(x)$

ex: $u_t = \alpha u_{xx}$, $\quad u_x(0,t) = u_x(1,t) = 1$

$u(x,0) = x(1-x^2)$ on $[0,1]$



1. Find u_∞

1 from the BCs

$$u_t = \alpha u_{xx} = 0 \qquad\qquad u_\infty(x) = \cancel{C_0} x + C_1$$

The heat flux $(-\alpha u_x)$ is the same at both ends is the same. There is no net change in the total amount of heat in the rod.

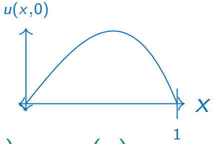
$$\int_0^1 u(x,0)dx = \int_0^1 u_\infty(x)dx$$

$$\int_0^1 x(1-x^2)dx = \int_0^1 x + C_1 dx = \frac{1}{2} + C_1$$

$$\frac{1}{2} - \frac{1}{4} = \frac{1}{2} + C_1 \quad \Rightarrow C_1 = -\frac{1}{4}$$

$$u_\infty = x - \frac{1}{4}$$

ex: $u_t = \alpha u_{xx}$, $\quad u_x(0,t) = u_x(1,t) = 1$
$\quad u(x,0) = x(1-x^2)$ on $[0,1]$



2. Write down a homogeneous IBVP for $w(x,t) = u(x,t) - u_\infty(x)$

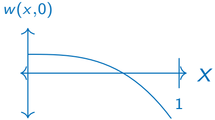
$$w_t = \alpha w_{xx} \qquad \begin{aligned} w_x(0,t) &= w_x(L,t) = 0 \\ w(x,0) &= u(x,0) - u_\infty(x) \\ &= \frac{1}{4} - x^3 \end{aligned}$$

3. Solve for $w(x,t)$

Since we have zero flux boundary conditions for w, we know it should be represented as a sum of cos terms

$$w(x,t) = \frac{a_0}{2} + \sum e^{-\alpha n^2 \pi^2 t} a_n \cos(n\pi x)$$

<u>ex</u>: $w_t = \alpha w_{xx}$, $\quad w_x(0,t) = w_x(1,t) = 0$
$\quad w(x,0) = \frac{1}{4} - x^3$ on $[0,1]$



$$a_n = 2\int_0^1 \left(\frac{1}{4} - x^3\right)\cos(n\pi x)dx$$

$$\overset{\text{wolfram}}{=} -\frac{3(-1)^n\left(\pi^2 n^2 - 2\right) + 6}{\pi^4 n^4}$$

$$a_0 = \frac{2}{1}\int_0^1 \frac{1}{4} - x^3 dx$$
$$= 2\left[\frac{x}{4} - \frac{x^4}{4}\right]\Big|_0^1$$
$$= 0$$

$$u(x,t) = \underbrace{x - \frac{1}{4}}_{u_\infty} + \underbrace{0}_{a_0/2}$$
$$+ \sum_{n=1}^{\infty} e^{-\alpha n^2\pi^2 t} a_n \cos(n\pi x)$$

https://www.wolframalpha.com/input?key=&i=integral+of+%281%2F4-x%5E3%29cos%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer

Suppose your cheap landlord has used an insulated wire of length L as a basic fuse. It is made of a metal that readily converts electrical current into heat when your electrical system is near its limit. Under these conditions, its internal heat production is well-described by a source function

$$\sigma(x) = 80\frac{^\circ\mathrm{C}}{s}\sin\left(\pi\frac{x}{L}\right),$$

and its internal temperature follows the inhomogeneous heat equation

$$u_t = 0.01u_{xx} + \sigma(x).$$

Find the solution $u(x, t)$ assuming that the end of the rods are connected to ice baths (i.e., its is very cold in your apartment).

$$u_t = 0.01u_{xx} + \sigma(x) \qquad \text{as } t \to \infty,\ u(x,t) \to u_\infty(x)$$

$$0 = 0.01u_\infty''(x) + \sigma(x) \qquad \Rightarrow u_\infty''(x) = -100\sigma(x)$$

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_\infty(x) &= -100 \times 80 \sin\left(\pi \frac{x}{L}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x} u_\infty(x) &= -8,000 \int \sin\left(\pi \frac{x}{L}\right) dx \\
\frac{\mathrm{d}}{\mathrm{d}x} u_\infty(x) &= 8,000 \frac{L}{\pi} \cos\left(\pi \frac{x}{L}\right) + C_1 \\
u_\infty(x) &= \int 8,000 \frac{L}{\pi} \cos\left(\pi \frac{x}{L}\right) + C_1 dx \\
u_\infty(x) &= 8,000 \frac{L^2}{\pi^2} \sin\left(\pi \frac{x}{L}\right) + C_1 x + C_2
\end{aligned}$$

Find C_1 and C_2 by matching the boundary conditions $u(0) = u(L) = 0$

$$\begin{aligned}
u_\infty(0) &= 0 = C_2 & \Rightarrow C_2 = 0 \\
u_\infty(L) &= 0 = C_1 L & \Rightarrow C_1 = 0
\end{aligned}$$

$$u_\infty(x) = 8,000 \frac{L^2}{\pi^2} \sin\left(\pi \frac{x}{L}\right)$$

Define a new PDE for $w(x,t) = u(x,t) - u_\infty$

Assume that the wire is initially at thermal equilbrium with the environment, i.e.,

$$u(x,0) = 0$$

$$w_t = 0.01 w_{xx} \qquad w(0,t) = w(L,t) = 0$$

$$w(x,0) = \underbrace{-8,000\frac{L^2}{\pi^2}\sin\left(\pi\frac{x}{L}\right)}_{\text{Fourier Sine Series}}$$

only $n = 1$ term is non-zero

$$w(x,t) = -8,000\frac{L^2}{\pi^2}e^{-0.01\frac{\pi^2}{L^2}t}\sin\left(\pi\frac{x}{L}\right)$$

$$u(x,t) = 8,000\frac{L^2}{\pi^2}\left(1 - e^{-0.01\frac{\pi^2}{L^2}t}\right)\sin\left(\pi\frac{x}{L}\right)$$

Formulas for finding $u_\infty(x)$: Inhomogeneous BCs

$$u_t = \alpha u_{xx} \qquad u(x,t) = u_0(x) \quad \text{with } x \in [0, L]$$

At steady state $u_t = 0 \implies u = u_\infty(x)$

$$u_\infty''(x) = 0 \implies u_\infty(x) = C_1 + C_2 x$$

- $u(0,t) = a, \quad u(L,t) = b$

$$u_\infty(x) = a + \frac{b-a}{L}x$$

- $u_x(0,t) = u_x(L,t) = a$

$$u_\infty(x) = \left(\frac{\int_0^L u_0(x)dx}{L} - \frac{aL}{2} \right) + ax$$

Formulas for finding $u_\infty(x)$: Inhomogeneous PDE

$$u_t = \alpha u_{xx} + \sigma(x) \qquad u(x,t) = u_0(x) \quad \text{with } x \in [0, L]$$

$$\alpha u_\infty''(x) = -\sigma(x) \implies u_\infty(x) = \underbrace{-\frac{1}{\alpha}\iint \sigma(x)\,dx^2}_{S(x)} + C_1 x + C_2$$

- $u(0,t) = a, \quad u(L,t) = b$

$$C_2 = a - S(0) \qquad\qquad C_1 = \frac{b - S(L) - C_2}{L}$$