Generalizing D'Alembert's Solution

The solution method is agnostic to boundary conditions:

$$y_{tt} = c^2 y_{xx} \qquad a < x < b$$

$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

$$\boxed{y(x,t) = A(x-ct) + B(x+ct)}$$

$$A(z) = \frac{1}{2}\left[F(z) - \frac{1}{c}\int_a^z G(x)dx\right] \quad B(z) = \frac{1}{2}\left[F(z) + \frac{1}{c}\int_a^z G(x)dx\right]$$

$$\begin{aligned} y(x,t) &= \frac{F(x-ct)+F(x+ct)}{2} + \frac{\int_a^{x+ct} G(s)ds - \int_a^{x-ct} G(s)ds}{2c} \\ &= \boxed{\frac{F(x-ct)+F(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds} \end{aligned}$$

Generalizing D'Alembert's Solution

$$y_{tt} = c^2 y_{xx} \qquad a < x < b$$

$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

$$\boxed{y(x,t) = \frac{F(x-ct) + F(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds}$$

Pick $F(z)$ and $G(z)$ to match the boundary conditions.

- $y(a,t) = y(b,t) = 0$
  $\Rightarrow$ F & G are odd periodic extensions of f and g
- $y_x(a,t) = y_x(b,t) = 0$
  $\Rightarrow$ F & G are even periodic extensions of f and g

Separation of variables with zero-derivative boundaries

$$y_{tt} = c^2 y_{xx} \quad 0 < x < L \qquad y_x(0,t) = y_x(L,t) = 0$$
$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

Perform even periodic extensions of the initial conditions

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$
$$g(x) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi}{L}x\right)$$

$$y(x,t) = \frac{d_0}{2}t + \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[d_n \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right) + a_n \cos\left(\frac{n\pi c}{L}t\right)\right]$$

Suppose you are given

$$y_{tt} = c^2 y_{xx} \quad 0 < x < L \qquad y_x(0,t) = y_x(L,t) = 0$$
$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

Compare the two methods for computing the solution to this problem

Separation of variables

1. Find the even periodic extension of $f(x)$
   - $a_n =$ an integral
   - $a_0 =$ another integral
2. $$y(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right)$$

Infinite sums are too costly, in practice we only get approximate solutions.

https://www.desmos.com/calculator/auxhik8lsh

Suppose you are given

$$y_{tt} = c^2 y_{xx} \quad 0 < x < L \qquad y_x(0,t) = y_x(L,t) = 0$$

$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

Compare the two methods for computing the solution to this problem

D'Alembert

1. Define an even extension of $f(x)$ onto [-L,L]

$$f^{\text{even}}(x) = \begin{cases} f(x) & 0 \leq x < L \\ -f(-x) & -L \leq x < 0 \end{cases}$$

2.

$$y(x,t) = \frac{1}{2}\left[f^{\text{even}}(mod(x - ct + L, 2L) - L)\right.$$
$$\left. + f^{\text{even}}(mod(x + ct + L, 2L) - L)\right]$$

Exact solution, no infinite sums or integrals required.

`https://www.desmos.com/calculator/auxhik8lsh`

Other boundary conditions?

Suppose we have $y(0,t) = y_0 \neq 0$ and $y(L,t) = y_L \neq 0$

Proceed like with the heat equation

$$y_{\mathrm{p}}(x) = y_0 + \frac{y_L - y_0}{L}x$$

define $w(x,t) = y(x,t) - y_p(x)$

$$\begin{aligned} w_{tt} &= c^2 w_{xx} \quad 0 < x < L & \qquad w(0,t) &= w(L,t) = 0 \\ w(x,0) &= f(x) - y_p(x) & y_t(x,0) &= g(x) \end{aligned}$$

Same strategy applies for non-zero derivatives.

Periodic extensions of functions defined for $a < x < b$

We have seen how to make odd/even periodic extensions of functions defined for $0 < x < L$

- Graphically
- Fourier Series with period $2L$
  - even=Fourier Cosine Series
  - odd=Fourier Sine Series

How does this work for a function defined for $a < x \leq b$?

1. Define a new coordinate $\tilde{x} = x - a$
2. Take a Fourier series in $\tilde{x}$ with $L = b - a$

Hint: You need to do this for Assignment Q5c

What happens as $a \to -\infty$ and $b \to \infty$

$$y_{tt} = c^2 y_{xx} \qquad -\infty < x < \infty$$

$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

$$\boxed{y(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds}$$

The wave extends over all space... eventually.

However, the wave travels at finite speed.

Information about the initial condition at a point x_0 is always contained in the interval $[x_0 - ct, x_0 + ct]$.

What type of solutions do we get for $-\infty < x < \infty$?

For $0 < x < L$, we got

$$y(x,t) \approx \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi t}{T_n}\right) + b_n \sin\left(\frac{2\pi t}{T_n}\right) \right] \left[\sin\left(\frac{2\pi x}{\lambda_n}\right) + \cos\left(\frac{2\pi x}{\lambda_n}\right) \right]$$

a_n and b_n are found from periodic extensions of the initial conditions

$$T_n = \frac{2L}{cn} \qquad \lambda_n = \frac{2L}{n} \qquad \text{(periods of the vibrational modes)}$$

alterntively, we can rewrite this as

$$y(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T_n}t} e^{i\frac{2\pi}{\lambda_n}x} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i\left(\frac{x}{\lambda_n} + \frac{t}{T_n}\right)}$$

$$c_n = \begin{cases} a_0/2 & n = 0 \\ \frac{1}{2}(a_n + b_n) & n > 0 \\ \frac{1}{2}(a_{|n|} - b_{|n|}) & n < 0 \end{cases}$$

What type of solutions do we get for $-\infty < x < \infty$?

For $-\infty < x < \infty$, we get

Inspired by the finite case, we try separation of variables with $y(x,t) = e^{\alpha x} e^{\beta t}$

$$
\begin{aligned}
y_{tt} &= c^2 y_{xx} \\
\beta^2 e^{\alpha x} e^{\beta t} &= c^2 \alpha^2 e^{\alpha x} e^{\beta t} \\
\beta^2 &= c^2 \alpha^2 \qquad\qquad \beta = \pm c\alpha
\end{aligned}
$$

For general initial conditions, β and α are purely imaginary. Let $\alpha = ik$

$$y(x,t) = \int_{-\infty}^{\infty} w(k) e^{2\pi i k(x+ct)} dk$$

The function $w(k)$ is found from a Fourier Transform of the initial conditions
NOT ON EXAM

What type of solutions do we get for $-\infty < x < \infty$?

For $-\infty < x < \infty$, we get

For some very special initial conditions, Fourier Transforms are not necessary.

ex: $y_{tt} = c^2 y_{xx}$ with $y(x,0) = e^{-2x}$

Try $y_{1,2}(x,t) = e^{\alpha x \pm c\alpha t}$

$$\text{Initial condition: } y(x,0) = e^{-2x} = e^{\alpha x} \Rightarrow \alpha = -2$$
$$y_{1,2}(x,t) = e^{-2x \mp 2ct}$$

Summary

- D'Alembert solution to the wave equation is complementary to the Fourier series solution:
  - Same solution, different perspective
- Two waves one moving left and another moving right
  - The two waves are fully determined by the initial conditions.
  - The initial conditions propagate through the domain at finite speed
- Finite Domain: use appropriate periodic extension + wave interference to match boundary conditions.
- Infinite Domain: No need to use periodic extensions, there are no boundary conditions.