# Recall: Constant coefficient $2^{nd}$ order homogeneous ODEs

$$ay'' + by' + cy = 0$$

Guess $y = e^{rt}$ char. poly.: $ar^2 + br + c = 0$

These have three possible homogeneous solutions:

1. $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. $y_h = c_1 e^{rt} + c_2 t e^{rt}$, $r = \frac{-b}{2a}$
3. $y_h = e^{-\frac{b}{2a} t} \left(\cos(\omega t) + \sin(\omega t)\right)$, $\omega = \sqrt{4ac - b^2}$

What about when we have non-constant coefficient ODEs?

ex:

$$p(t)y'' + q(t)y' + r(t)y = 0$$

# Special Case: Euler Equations

$$at^2y'' + bty' + cy = 0$$

Guess: $y = t^k$

$$y' = kt^{k-1} \qquad y'' = (k-1)kt^{k-2}$$

$$at^2(k-1)kt^{k-2} + btkt^{k-1} + ct^k = 0$$

factor out the $t^k$

$$a(k-1)k + bk + c = 0$$
$$ak^2 + (b-a)k + c = 0$$
$$k_{1,2} = \frac{a-b \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

# Special Case: Euler Equations

$$at^2y'' + bty' + cy = 0$$

Guess: $y = t^k$

$$k_{1,2} = \frac{a - b \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

Three cases:

1. Real and distinct values of k

$$y = c_1 t^{k_1} + c_2 t^{k_2}$$

2. Repeated real values of k

$$y = c_1 t^k + c_2?? \qquad k = \frac{a-b}{2a}$$

3. Complex conjugate values $\qquad y =????????$

(unclear how to make it real)

# Case 2: Repeated real values of $k$

$at^2y'' + bty' + cy = 0$

$y_1 = t^k$ $\qquad k = \dfrac{a-b}{2a}$ $\qquad (b-a)^2 - 4ac = 0$

Guess $y_2 = r(t)y_1(t) = rt^k$

$$y' = r't^k + kr't^{k-1} \qquad y'' = r''t^k + 2kr't^{k-1} + (k-1)krt^{k-2}$$

$$ar''t^{k+2} + a2kr't^{k+1} + a(k-1)krt^{k+1} + br't^{k+1} + bkrt^k + crt^k = 0$$

factor out $t^k$, refactor as an ODE for $r$

$$at^2r'' + (2ak+b)tr' + (a(k-1)k + bk + c)\,r = 0$$

$$at^2r'' + (2ak + b)tr' + (a(k-1)k + bk + c)\, r = 0$$

$$k = \frac{a-b}{2a} \qquad \& \qquad (b-a)^2 - 4ac = 0$$

$$\begin{aligned} a(k-1)k + bk + c &= ak^2 - ak + bk + c \\ &= \frac{(a-b)^2}{4a} + (b-a)\frac{a-b}{2a} + c \\ &= \frac{(a-b)^2}{4a} - \frac{(a-b)^2}{2a} + c \\ &= -\frac{(a-b)^2}{4a} + c \\ &= -\frac{1}{4a}\left((a-b)^2 - 4ac\right) = 0 \end{aligned}$$

$$at^2 r'' + (2ak + b)tr' + = 0$$

$$k = \frac{a-b}{2a} \qquad \& \qquad (b-a)^2 - 4ac = 0$$

$$(2ak + b)t = (a - b + b)t = at$$

$$at^2 r'' + atr' = 0$$

$$r'' + \frac{1}{t} r' = 0$$

define a new function $v(t) = r'$

$$v' + \frac{1}{t} v = 0$$

solve by integrating factors

$at^2y'' + bty' + cy = 0$

$y_1 = t^k,$ $y_2 = r(t)t^k,$ $r' = v(t)$

$$v' + \frac{1}{t}v = 0$$

$$\mu(t) = e^{\int \frac{dt}{t}} = e^{\ln(t)} = t$$

$$tv(t) = \int 0 \cdot dt = C \qquad v(t) = \frac{C}{t}$$

$$r(t) = \int v(t)dt = C\ln(t)$$

$$\boxed{y_h = c_1t^k + c_2t^k\ln(t)}$$

# Reduction of order

Given

$$p(t)y'' + q(t)y' + r(t)y = 0$$

and one solution $y_1(t)$, we want to find the second solution $y_2(t)$.

1. Guess $y_2(t) = a(t)y_1(t)$
2. Substitute into ODE, rearrange to get an ODE for $a'(t) = v(t)$
3. Solve for $v(t)$ using integrating factors
4. Integrate $v(t)$ to find $a(t)$ and $y_2(t)$

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

# Inhomogeneous ODEs

Given

$$p(t)y'' + q(t)y' + r(t)y = h(t)$$

and one solution $y_1(t)$, we want to find the general solution.

1. Find $y_2(t)$ by reduction of order
2. Use the method of undetermined coefficients to find the particular part of the solution
   - Watch out for mathematical resonance

$$y_g = y_p + c_1 y_1(t) + c_2 y_2(t)$$

# Case 3: Complex conjugate values of $k$

$$
\begin{aligned}
y &= t^k \\
&= e^{\ln\left(t^k\right)} \\
&= e^{k ln(t)} \\
y_{1,2} &= e^{\lambda ln(t)} e^{\pm i\mu ln(t)} \\
&= t^{\lambda} e^{\pm i\mu ln(t)}
\end{aligned}
$$

$$
\begin{aligned}
k_{1,2} &= \frac{a-b \pm \sqrt{(b-a)^2 - 4ac}}{2a} \\
&= \frac{a-b}{2a} \pm i\frac{\sqrt{4ac-(b-a)^2}}{2a} \\
&= \lambda \pm i\mu
\end{aligned}
$$

$$y_h = t^{\lambda}\left(c_1 \cos(\mu \ln(t)) + c_2 \sin(\mu \ln(t))\right)$$