

# Fourier Series

Given any periodic function  $f(t)$  with period  $T$ , we can approximate  $f(t)$  as a Fourier series

$$f(t) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)$$

with

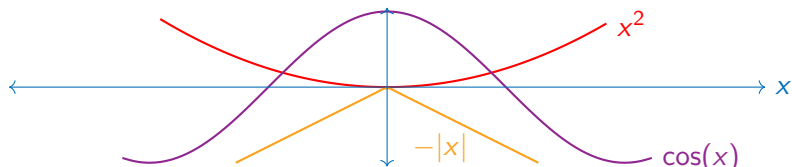
$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T f(t) dt &= \frac{1}{T} \int_{\alpha}^{\alpha+T} f(t) dt \\ a_n &= \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2n\pi t}{T}\right) dt &= \frac{2}{T} \int_{\alpha}^{\alpha+T} f(t) \cos\left(\frac{2n\pi t}{T}\right) dt \\ b_n &= \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2n\pi t}{T}\right) dt &= \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2n\pi t}{T}\right) dt \end{aligned}$$

We integrate over one period, and can choose  $\alpha$  to make the integrals simpler to evaluate.

# Even and Odd Functions:

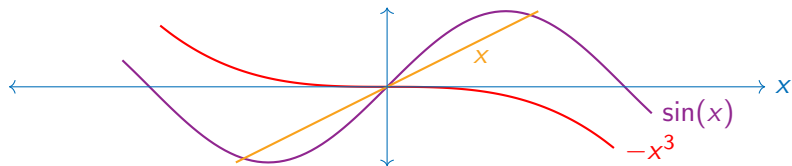
Even Functions:

$$f(x) = f(-x)$$



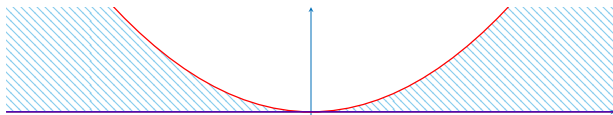
Odd Functions:

$$f(x) = -f(-x)$$

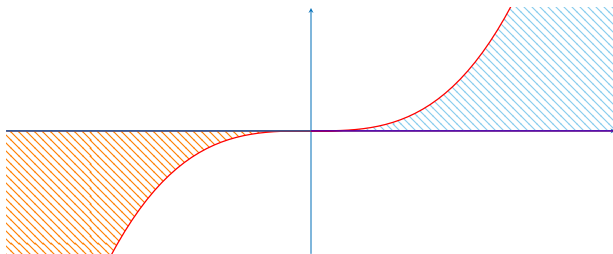


# Even and Odd Functions: Integral Properties

Even Functions: The integral of an even function on the interval  $[-L, L]$  is double its integral on  $[0, L]$



Odd Functions: The integral of an odd function on the interval  $[-L, L]$  is 0.



# Even and Odd Functions: Products of odd/even functions

Works like multiplying real numbers

$$\text{even} \Leftrightarrow +1$$

$$\text{odd} \Leftrightarrow -1$$

$$\text{odd} \cdot \text{odd} = \text{even}$$

$$-1 \cdot -1 = +1$$

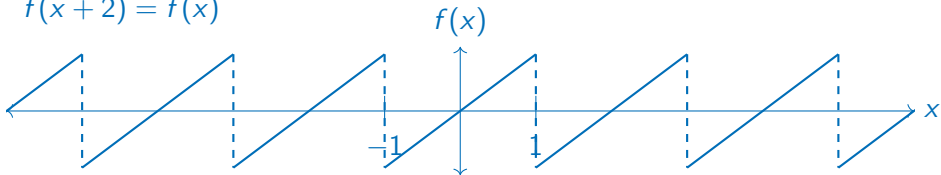
$$\text{even} \cdot \text{even} = \text{even}$$

$$+1 \cdot +1 = +1$$

$$\text{even} \cdot \text{odd} = \text{odd}$$

$$+1 \cdot -1 = -1$$

Find the Fourier Series representation of  $f(x) = x$  for  $x \in [-1, 1]$  with  $f(x+2) = f(x)$



$$T = 2$$

$$a_n = \frac{1}{1} \int_{-1}^1 x \cos(n\pi x) dx \quad \text{let} \quad \begin{array}{ll} u = x & du = dx \\ dv = \cos(n\pi x) dx & v = \frac{\sin(n\pi x)}{n\pi} \end{array}$$

$$\int_{-L}^L x \cos(n\pi x) dx = \underbrace{\left( \underbrace{x}_{\text{odd func.}} \underbrace{\frac{\sin(n\pi x)}{n\pi}}_{\text{odd func.}} \right)}_{\text{even func.}} \bigg|_{-1}^1 - \int_{-1}^1 \underbrace{\frac{\sin(n\pi x)}{n\pi}}_{\text{odd func.}} dx$$

$$= 0 \quad \Rightarrow \text{No cos terms in the Fourier Series}$$

Find the Fourier Series representation of  $f(x) = x$  for  $x \in [-1, 1]$  with  $f(x+2) = f(x)$

$$b_n = \frac{1}{1} \int_{-1}^1 x \sin(n\pi x) dx$$

$$\text{let } \begin{array}{ll} u = x & du = dx \\ dv = \sin(n\pi x) dx & v = -\frac{\cos(n\pi x)}{n\pi} \end{array}$$

$$\begin{aligned} \int_{-1}^1 x \sin(n\pi x) dx &= -\left( x \frac{\cos(n\pi x)}{n\pi} \right) \Big|_{-1}^1 + \int_{-1}^1 \frac{\cos(n\pi x)}{n\pi} dx \\ &= \frac{-1}{n\pi} (\cos(n\pi) + \cos(n\pi)) + \cancel{\frac{\sin(n\pi x)}{n^2\pi^2} \Big|_{-1}^1} \end{aligned}$$

use the identity  $\cos(n\pi) = (-1)^n$

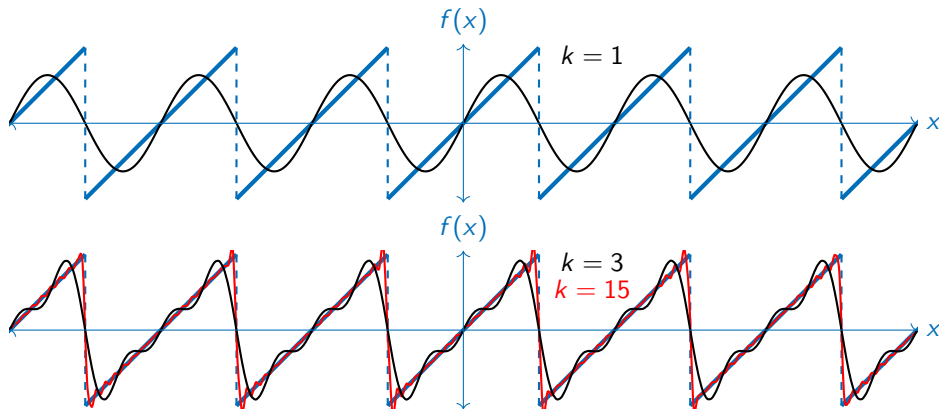
$$= -2 \frac{(-1)^n}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} -2 \frac{(-1)^n}{n\pi} \sin(n\pi x)$$

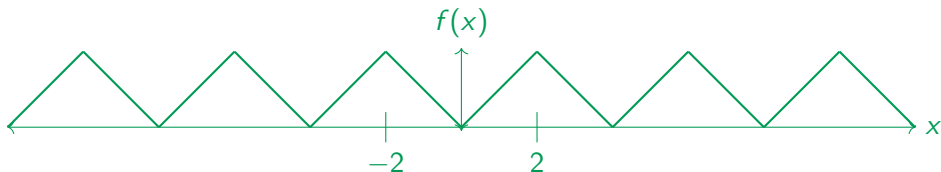
# Finite Fourier Series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^k a_n \cos(\omega_n x) + \sum_{n=1}^k b_n \sin(\omega_n x) \quad \omega_n = n \frac{2\pi}{T}$$

$$a_n = 0 \quad b_n = -2 \frac{(-1)^n}{n\pi}$$



Compute the Fourier Series for  $f(x) = |x|$  for  $x \in [-2, 2]$  with  $f(x+4) = f(x)$



$$a_n = \frac{1}{2} \int_{-2}^2 \underbrace{|x|}_{\text{even func.}} \underbrace{\cos\left(n\frac{\pi}{2}x\right)}_{\text{even func.}} dx$$

even func.

The integral of an even function on  $[0, L]$  is half its integral from  $[-L, L]$

$$a_n = \int_0^2 x \cos\left(n\frac{\pi}{2}x\right) dx$$



Compute the Fourier Series for  $f(x) = |x|$  for  $x \in [-2, 2]$  with  $f(x+4) = f(x)$

$$a_n = \int_0^2 x \cos\left(n\frac{\pi}{2}x\right) dx$$

$$\begin{aligned} \text{let } u &= x & du &= dx \\ dv &= \cos\left(n\frac{\pi}{2}x\right) & v &= 2\frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} \end{aligned}$$

$$\begin{aligned} &= 2 \left( x \frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} \right) \Big|_0^2 - 2 \int_0^2 \frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} dx \\ &= \frac{4}{n^2\pi^2} \cos\left(n\frac{\pi}{2}x\right) \Big|_0^2 = \frac{4}{n^2\pi^2} [\cos(n\pi) - 1] \\ &= \frac{4}{n^2\pi^2} [(-1)^n - 1] = \begin{cases} -\frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

Compute the Fourier Series for  $f(x) = |x|$  for  $x \in [-2, 2]$  with  $f(x+4) = f(x)$

$\frac{a_0}{2}$  is the average value of the function (DC component)

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 |x| dx = \int_0^2 x dx \\ &= \left. \frac{x^2}{2} \right|_0^2 \\ &= \frac{4}{2} - 0 \\ &= 2 \end{aligned}$$

Compute the Fourier Series for  $f(x) = |x|$  for  $x \in [-2, 2]$  with  $f(x+4) = f(x)$

$$b_n = \frac{1}{2} \int_{-2}^2 \underbrace{|x|}_{\text{even func.}} \underbrace{\sin\left(n\frac{\pi}{2}x\right)}_{\text{odd func.}} dx$$

odd func.

Any integral that is symmetric about  $x = 0$  of an odd function is zero

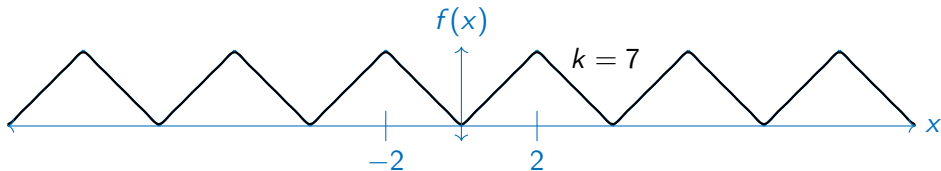
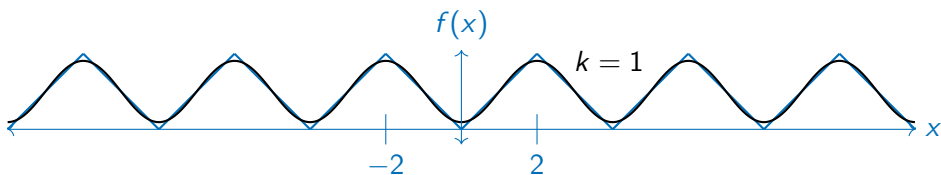
...opposite AUC on both sides...

$$\Rightarrow b_n = 0$$

# Finite Fourier Series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^k a_n \cos(\omega_n x) + \sum_{n=1}^k b_n \sin(\omega_n x) \quad \omega_n = n \frac{2\pi}{T}$$

$$a_0 = 2 \quad a_n = \begin{cases} -\frac{8}{n^2 \pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad b_n = 0$$



# Even and Odd Functions: Fourier Series

Even Function:  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x)$

Proof:

$$b_n = \int_{-L}^L \underbrace{\text{even func.} \times \sin(\omega_n)}_{\text{odd func.}} = 0$$

Odd Function:  $f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\omega_n x)$

Proof:

$$a_n = \int_{-L}^L \underbrace{\text{odd func.} \times \cos(\omega_n)}_{\text{odd func.}} = 0$$

Even function, only cos terms

-

Odd function, only sin terms

Compute the Fourier Series for  $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$  for  $x \in [-1, 1]$

with  $f(x+2) = f(x)$

$$a_n = \frac{1}{1} \int_0^1 x \cos(n\pi x) dx$$

$$\text{let } \begin{array}{ll} u = x & du = dx \\ dv = \cos(n\pi x) dx & v = \frac{\sin(n\pi x)}{n\pi} \end{array}$$

$$\int_0^L x \cos(n\pi x) dx = \left( x \frac{\sin(n\pi x)}{n\pi} \right) \Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi} dx$$

$$= \frac{\cos(n\pi x)}{n^2 \pi^2} \Big|_0^1 = \frac{(-1)^n - 1}{n^2 \pi^2} = \begin{cases} -\frac{2}{n^2 \pi^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

This expression breaks down for  $n = 0 \dots$

Compute the Fourier Series for  $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$  for  $x \in [-1, 1]$   
with  $f(x+2) = f(x)$

$n = 0$ :

$\frac{a_0}{2}$  is the average value of the function (DC component)

$$a_0 = \frac{1}{1} \int_0^1 x dx$$

$$= \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

Compute the Fourier Series for  $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$  for  $x \in [-1, 1]$

with  $f(x+2) = f(x)$

$$b_n = \frac{1}{1} \int_0^1 \underbrace{\underbrace{x}_{\text{odd func.}} \underbrace{\sin(n\pi x)}_{\text{odd func.}}}_{\text{even func.}} dx$$

The integral of an even function on  $[0, L]$  is half its integral from  $[-L, L]$

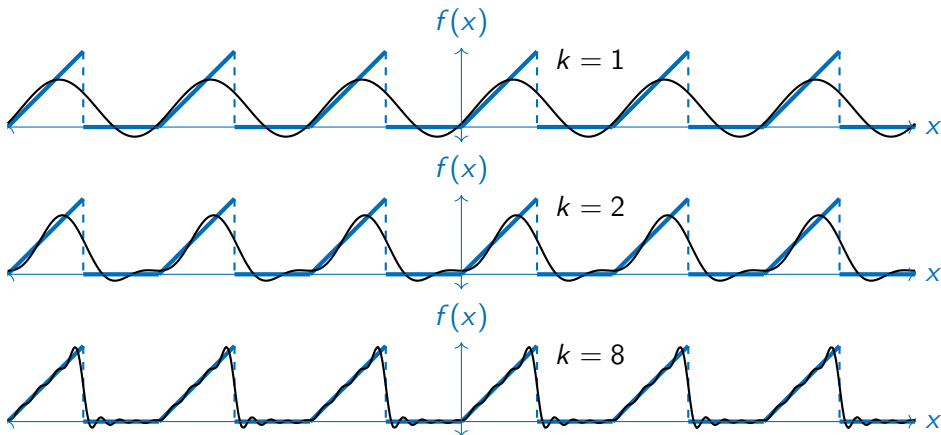
$$b_n = \frac{1}{2} \times -2 \frac{(-1)^n}{n\pi} = -\frac{(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi^2} \cos(n\pi x) + \sum_{n=1}^{\infty} -\frac{(-1)^n}{n\pi} \sin(n\pi x)$$



$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^k a_n \cos(\omega_n x) + \sum_{n=1}^k b_n \sin(\omega_n x) \quad \omega_n = n \frac{2\pi}{T}$$

$$a_0 = \frac{1}{2} \quad a_n = \frac{(-1)^n - 1}{n^2 \pi^2} \quad b_n = -\frac{(-1)^n}{n\pi}$$

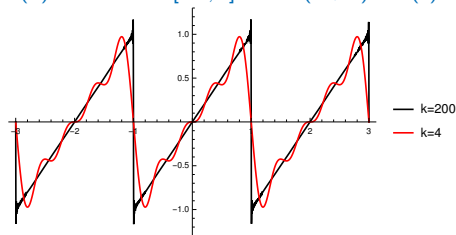


## Observations:

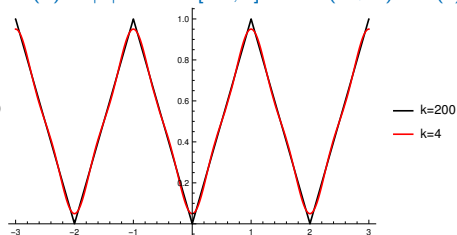
- Take more terms in the Fourier Series and it appears to converge to  $f(x)$  , (even if  $f(x)$  has discontinuities!)
- The coefficients  $a_n$  &  $b_n$  that we calculate decrease to zero as  $n \rightarrow \infty$
- Initial coefficient  $a_0$  is simply the mean value of  $f(x)$
- Fourier Series appears over/undershoot the function  $f(x)$  at points of discontinuity.
- The above are common features of Fourier series expansions with arbitrary functions.

# Fourier Series Convergence

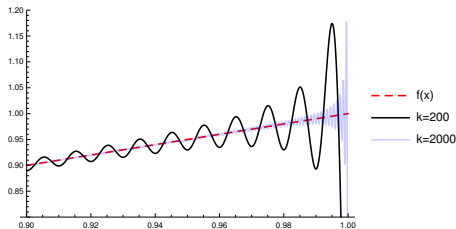
$f(x) = x$  for  $x \in [-1, 1]$  with  $f(t+2) = f(t)$



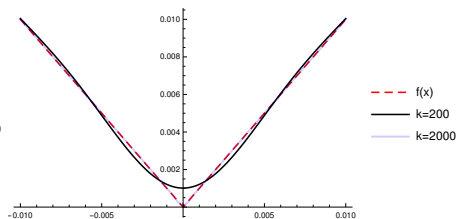
$f(x) = |x|$  for  $x \in [-1, 1]$  with  $f(t+2) = f(t)$



Zoom in on discontinuity

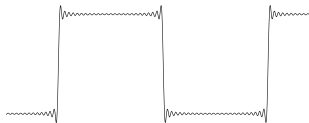


Zoom in on discontinuity



# Fourier Series Convergence

- The Fourier Series of any continuous function converges (as # of terms  $\rightarrow \infty$ ) to the function value at every point.  $\Rightarrow f(x) = \text{FS}[f]$
- The Fourier Series of a function with jump discontinuities exhibits **Gibb's phenomena**
  - High frequency over/undershooting of the function



- The Fourier Series converges to the midpoint between the two function values at any point of discontinuity  $x_*$ .  $\Rightarrow f(x) \approx \text{FS}[f]$

$$\text{FS}[f](x_*) = \frac{f(x_*^+) + f(x_*^-)}{2}$$

- The rate of convergence of smooth functions is faster than for functions with discontinuities.