Recall:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} \qquad \text{with } \vec{x}(0) = \vec{x}_0$$

Since $\mathbf{A}$ is an $n \times n$ matrix, our solutions live in $\mathbb{R}^n$

Superposition:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n(t)$$

Each $\vec{x}_i(t)$ was found by solving an eigenproblem.

$$\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i \qquad \text{where} \qquad A\vec{v}_i = \lambda_i\vec{v}_i$$

Recall:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n(t)$$

The coefficients c_i come from projecting the initial conditions $\vec{x}_0$ onto the basis

$$\{\vec{x}_1(0), \vec{x}_2(0), \ldots, \vec{x}_n(0)\}$$

$$c_i = \vec{x}_0 \cdot \vec{x}_i(0) \qquad - \qquad \text{change of basis}$$

This an "inner product" defined on the vector space $\mathbb{R}^n$

$$\vec{a} \cdot \vec{b} = \left\langle \vec{a}, \vec{b} \right\rangle = \sum_i a_i \cdot b_i$$

For the rest of the semester, we will be using techniques based on inner products in a function space - effectively an infinite dimensional vector space.

Consider two functions $h(x)$ and $g(x)$ defined on an interval $[a, b]$, then the "standard" inner product is given by

$$\langle h, g\rangle = \int_a^b h(x)g^*(x)dx$$

where g^* is the complex conjugate of g.

The two main techniques we will use are:

1) Laplace Transforms

- ODE does not need to reduce to any specific eigenproblem.
- Project ODE+ICs onto the space of exponential solutions.
    - Use pattern recognition skills to convert projection back into a time-dependent solution.
- Very useful for ODEs with non-smooth forcing and understanding complicated modular systems.

The two main techniques we will use are:

2) Fourier Series

- ODE/PDE must reduce to specific eigenproblem(s)
- Project ICs onto the space of periodic functions (sin/cos basis).
  - Projections can be directly interpreted as constant coefficients in an infinite series solution.
- We may also see some applications with hyperbolic trig. functions.

Solving ODEs: $ay'' + by' + cy = h(t)$ with $\begin{array}{l} y(0) = y_0 \\ y'(0) = v_0 \end{array}$

How do we "normally" solve this?

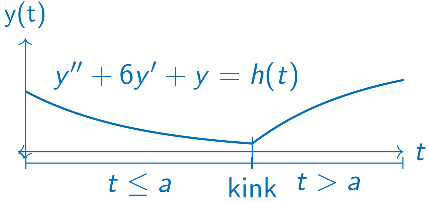
$$y = y_p + y_h$$

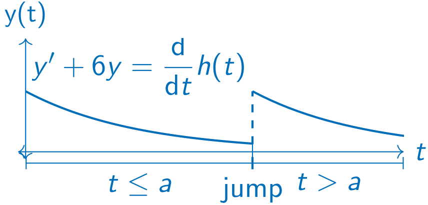
1. Solve homogeneous problem, get $y_h = c_1 y_1 + c_2 y_2$.
2. Apply method of undetermined coefficients, get y_p
   - Requires knowing $h', h'', \ldots$
3. Find c_1 & c_2 by solving: $$\begin{aligned} y_0 &= y_p(t_0) + c_1 y_1(t_0) + c_2 y_2(t_0) \\ v_0 &= y_p'(t_0) + c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{aligned}$$
4. Finally $$y(t) = y_p + c_1 y_1 + c_2 y_2$$

Non-smooth forcing

Suppose some function $h(t)$ jumps abruptly between two values

$$\text{e.g., } h(t) = \begin{cases} 0 & t \leq a \\ 1 & t > a \end{cases}$$





Since the derivative of $h(t)$ is undefined at $t = a$, we cannot use the method of undetermined coefficients...

We must use Laplace Transforms!

Laplace Transforms

Suppose you have a real-valued function $y(t)$ defned for $t \in [0, \infty)$ that solves some ODE of interest.

Its Laplace Transform is given by

$$Y(s) = \mathcal{L}\{y(t)\} = \langle e^{-st}, y(t) \rangle$$

$$= \int_0^{\infty} e^{-st} y(t) dt$$

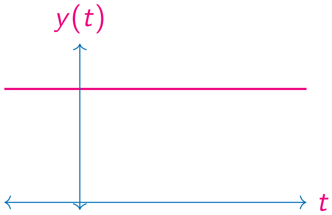
ex: $y(t) = \frac{1}{2}$

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st}\frac{1}{2}dt$$

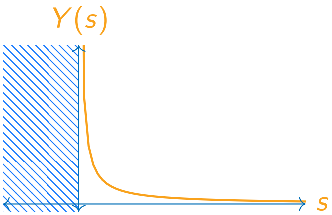
$$= -\frac{1}{2s}e^{-st}\Big|_0^\infty \qquad = -\lim_{A\to\infty}\frac{1}{2s}e^{-st}\Big|_0^A$$

$$= -\frac{1}{2s}\lim_{A\to\infty}\left(e^{-sA}-1\right)$$

$$= \begin{cases} \frac{1}{2s} & \text{if } s > 0 \\ DNE & \text{if } s \leq 0 \end{cases}$$



$\xrightarrow{\mathcal{L}}$

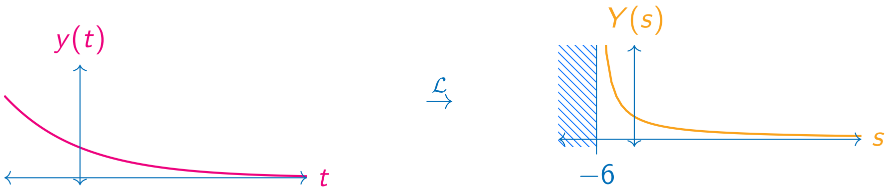


ex: $y(t) = e^{-6t}$

$$\mathcal{L}\left\{y(t)\right\} = Y(s) = \int_0^\infty e^{-st}e^{-6t}dt$$

$$\cdots$$

$$= \begin{cases} \frac{1}{s+6} & \text{if } s > -6 \\ DNE & \text{if } s \leq -6 \end{cases}$$



Laplace Transform of Derivatives

Suppose $\mathcal{L}\{y(t)\} = Y(s)$. What is $\mathcal{L}\{y'(t)\}$?

$$v = y(t)$$
$$du = -se^{-st}dt$$

$$\mathcal{L}\{y'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u}\underbrace{y'(t)dt}_{dv}$$

$$= e^{-st}y(t)\Big|_0^\infty - (-s)\underbrace{\int_0^\infty e^{-st}y(t)dt}_{\mathcal{L}\{y(t)\}}$$

$$\stackrel{s\geq 0}{=} -y(0) + s\mathcal{L}\{y(t)\}$$

$$= \boxed{sY(s) - y_0}$$

Laplace Transform of Derivatives

Given that $\mathcal{L}\{y'(t)\} = sY(s) - y_0$. What is $\mathcal{L}\{y''(t)\}$?

$$\mathcal{L}\{y''(t)\} = s\mathcal{L}\{y'(t)\} - y'(0) = s\left[sY(s) - y_0\right] - \underbrace{y'(0)}_{v_0}$$

$$= \boxed{s^2 Y(s) - sy_0 - v_0}$$

Linearity of Laplace Transforms:

1. $\mathcal{L}\left\{f(t)+g(t)\right\} = \int_0^\infty e^{-st}\left(f(t)+g(t)\right)dt$

   $= \mathcal{L}\left\{f(t)\right\} + \mathcal{L}\left\{g(t)\right\} = F(s) + G(s)$

2. $\mathcal{L}\left\{cf(t)\right\} = c\mathcal{L}\left\{f(t)\right\} = cF(s)$

ex: What is the Laplace transform of $y' + 6y$ with $y(0) = y_0$?

$$\begin{aligned}\mathcal{L}\left\{y'+6y\right\} &= \mathcal{L}\left\{y'\right\} + \mathcal{L}\left\{6y\right\}\\ &= s\mathcal{L}\left\{y\right\} - y_0 + 6\mathcal{L}\left\{y\right\}\\ &= (s+6)Y(s) - y_0\end{aligned}$$

ex: Find $Y(s)$ for $y' + 6y = 3$ with $y(0) = y_0$.

$$(s+6)Y(s) - y_0 = \mathcal{L}\{3\}$$

$$\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \frac{3}{s}$$

$$sY(s) - y_0 + 6Y(s) = \frac{3}{s}$$

$$Y(s) = \underbrace{\frac{3}{s(s+6)}}_{\mathcal{L}\{???\}} + \underbrace{\frac{y_0}{s+6}}_{y_0\mathcal{L}\{e^{-6t}\}}$$

ex: Solve $y' + 6y = 3$ with $y(0) = y_0$ using Laplace Transforms.

Partial fraction decomposition

$$Y(s) = \frac{3}{s(s+6)} + \frac{y_0}{s+6} = \frac{A}{s} + \frac{B}{s+6} + \frac{y_0}{s+6}$$

$$3 + \cancel{y_0 \cdot s} = A(s+6) + B \cdot s + \cancel{y_0 \cdot s}$$

$$3 = 6A + (A+B)s$$

True for all $s \Rightarrow$ coefficients must match

constant terms: $3 = 6A$ $\qquad A = \frac{1}{2}$

s terms: $0 = A + B$ $\qquad B = -A = -\frac{1}{2}$

$$Y(s) = \frac{1}{2s} - \frac{1}{2(s+6)} + \frac{y_0}{s+6} = \frac{1}{2s} + \left(y_0 - \frac{1}{2}\right)\frac{1}{s+6}$$

$$
\begin{aligned}
Y(s) &= \frac{1}{2s} + \left(y_0 - \frac{1}{2}\right)\frac{1}{s+6} \\
&= F(s) + \left(y_0 - \frac{1}{2}\right) G(s)
\end{aligned}
$$

$$F(s) = \frac{1}{2s} \Rightarrow f(t) = \frac{1}{2}$$

$$G(s) = \frac{1}{s+6} \Rightarrow g(t) = e^{-6t}$$

$$
\begin{aligned}
&= \mathcal{L}\left\{\frac{1}{2}\right\} + \left(y_0 - \frac{1}{2}\right)\mathcal{L}\left\{e^{-6t}\right\} \\
y(t) &= \frac{1}{2} + \left(y_0 - \frac{1}{2}\right) e^{-6t}
\end{aligned}
$$

General Laplace Transform Method for IVPs

1. Take Laplace transform of the entire DE (ex. $y' + 6y = 3$)
   - Use linearity and rules for transforming derivatives.

$$sY(s) - y_0 + 6Y(s) = \frac{3}{s}$$

2. Solve the resulting equation for $Y(s)$

$$Y(s) = \frac{3}{s(s+6)} + \frac{y_0}{s+6}$$

3. Do some algebra to get a sum of "easy" terms

$$Y(s) = \frac{1}{2s} + \left(y_0 - \frac{1}{2}\right)\frac{1}{s+6}$$

4. Transform back from $Y(s)$ to $y(t)$
   - Tackle each term in the sum individually.

$$y(t) = \frac{1}{2} + \left(y_0 - \frac{1}{2}\right)e^{-6t}$$

What are "easy" terms?

Spot a term in s-space → you now know what it corresponds to in t-space

$$\mathcal{L}\{C\} = \frac{C}{s} \qquad \text{Constant}$$

$$\mathcal{L}\left\{Ce^{-at}\right\} = \frac{C}{s+a} \qquad \text{Exponential func.}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{Power func.}$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{\omega^2 + s^2}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{\omega^2 + s^2}$$

Tables exist with even more terms...

ex: Use Laplace transforms to solve $y'' + 9y = 0$ with $y(0) = 1$ and $y'(0) = -1$.

$$s^2Y(s) - sy_0 - v_0 + 9Y(s) = 0$$

$$(9+s^2)Y(s) - s\cancelto{1}{y_0} - \cancelto{-1}{v_0} = 0$$

$$(9+s^2)Y(s) = -1 + s$$

$$\begin{aligned} Y(s) &= \frac{-1}{9+s^2} + \frac{s}{9+s^2} \\ &= A\frac{3}{9+s^2} + \frac{s}{9+s^2} \qquad\qquad A\cdot 3 = -1 \\ Y(s) &= \frac{-1}{3}\frac{3}{9+s^2} + \frac{s}{9+s^2} \qquad\qquad A = -\frac{1}{3} \end{aligned}$$

Invert the transform

$$y(t) = -\frac{1}{3}\sin(3t) + \cos(3t)$$

Summary

- Laplace transform (LT): $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
- Maps $f(t) \to F(s)$, from "t-space" to "s-space"
- LT is linear because the integral is linear
- LT of derivatives (from integration by parts)

$$\mathcal{L}\left\{y'(t)\right\} = sY(s) - y_0$$
$$\mathcal{L}\left\{y''(t)\right\} = s^2 Y(s) - sy_0 - v_0$$

- Using these we can take LT of constant coefficient DE's
- Solve the algebraic equation and invert the transform
    - Use tables of LT's to do this