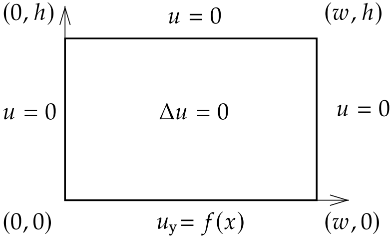
Dirichlet Problem - with a derivative

ex: Consider a rectangular region of width w and height h.

Boundary values are zero at three of the four edges, and the derivative of u obeys some arbitrary function $f(x)$ the fourth edge.

$$\begin{aligned}
\Delta u &= 0 \\
u(0, y) &= 0 && \text{for } 0 < y < h \\
u(x, h) &= 0 && \text{for } 0 < x < w \\
u(w, y) &= 0 && \text{for } 0 < y < h \\
u_y(x, 0) &= f(x) && \text{for } 0 < x < w
\end{aligned}$$



$$u_{xx} + u_{yy} = 0$$

Separation of Variables:

$$u(x,y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const.}$$

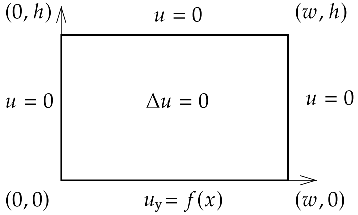
$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

BCs: $X(0) = X(w) = 0$

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

$$Y'' - \lambda Y = 0$$

$$\lambda = \left(\frac{n\pi}{w}\right)^2$$



So far, this is exactly what we had solving the south problem

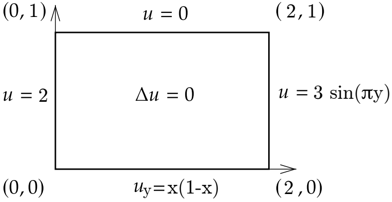
$$u_n(x,y) = a_n \sin\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}(h-y)\right) \qquad u(x,y) = \sum_n u_n$$

$$f(x) = \sum_n b_n \cos\left(\frac{n\pi}{w}x\right) = u_y(x,0)$$

$$u_y(x,0) = \sum_n a_n \frac{n\pi}{w}\cosh\left(\frac{n\pi}{w}h\right)\sin\left(\frac{n\pi}{w}x\right) \qquad \boxed{a_n = \frac{b_n}{\cosh\left(\frac{n\pi}{w}\right)}\frac{w}{n\pi}}$$

ex: Consider the following problem

$$
\begin{aligned}
\Delta u &= 0 \\
u(0,y) &= 2 && \text{for } 0 < y < 1 \\
u(x,h) &= 0 && \text{for } 0 < x < 2 \\
u(2,y) &= 3\sin(\pi y) && \text{for } 0 < y < 1 \\
u_y(x,0) &= x(1-x) && \text{for } 0 < x < 2
\end{aligned}
$$



Strategy: Break the problem down into West, East, and South Problems, then use superposition.

$$u(x,y) = u_W + u_E + u_S$$

West problem

$$u_W(x, y) = \sum_n d_n \sin(n\pi y) \sinh(n\pi(2 - x))$$

$$u_W(0, y) = 2 = \sum_n d_n \sin(n\pi y) \sinh(2n\pi)$$

$$2 = \sum_n b_n \sin(n\pi y) \qquad b_n = \frac{2}{1} \int_0^1 2 \sin(n\pi y) dy = 4 \frac{1 - (-1)^n}{n\pi}$$

$$d_n = \frac{b_n}{\sinh(2n\pi)} = 4 \frac{1 - (-1)^n}{\sinh(2n\pi) n\pi}$$

East problem

$$u_E(x, y) = \sum_n c_n \sin(n\pi y) \sinh(n\pi x)$$

$$u_E(2, y) = 3sin(\pi y) = \sum_n c_n \sin(n\pi y) \sinh(2n\pi)$$

$$3sin(\pi y) = \sum_n b_n \sin(n\pi y) \qquad b_n = \begin{cases} 3 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$c_n = \frac{b_n}{\sinh(2n\pi)} = \begin{cases} \frac{3}{\sinh(2\pi)} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

South problem

$$u_S(x,y) = \sum_n a_n \sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}(1-y)\right)$$

$$\frac{\partial}{\partial y}u_S(x,0) = x(1-x) = \sum_n a_n \sin\left(\frac{n\pi}{2}x\right)\frac{n\pi}{2}\cosh\left(\frac{n\pi}{2}\right)$$

$$x(1-x) = \sum_n b_n \sin\left(\frac{n\pi}{2}x\right) \qquad \boxed{a_n = -\frac{2}{n\pi}\frac{b_n}{\cosh\left(\frac{n\pi}{2}\right)}}$$

$$b_n = \frac{2}{2}\int_0^2 x(1-x)\sin\left(\frac{n\pi}{2}x\right)dx = \frac{4(-1)^n\left(\pi^2 n^2-4\right)+16}{\pi^3 n^3}$$

Finally,

$$u(x,y) = u_W + u_E + u_S$$

Derivatives in higher dimensions

For $u = u(x, y)$, we can compute two derivatives along the coordinate axes

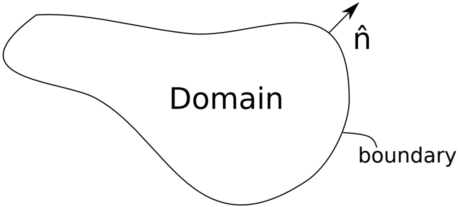
$$u_x = \frac{\partial}{\partial x} u(x, y)$$
$$u_y = \frac{\partial}{\partial y} u(x, y)$$

The generalization of this is a directional derivative (or gradient) $\vec{\nabla} u$

$$\vec{\nabla} u = \underbrace{\begin{bmatrix} u_x \\ u_y \end{bmatrix}}_{\text{vector}}$$

Neumann Boundary Conditions

Consider $\vec{x} \in \mathbb{R}^d$ restricted to a closed domain, where the boundary of the domain has a outer unit normal vector $\hat{n}$.



The Neumann boundary condition is given by

$$\frac{\partial u}{\partial \hat{n}} = f(\vec{x}) \qquad \text{for } \vec{x} \text{ along the boundary}$$

$$\frac{\partial u}{\partial \hat{n}} = \vec{\nabla} u \cdot \hat{n}$$

Neumann Boundary Condition Example - Electrostatics

Consider Maxwell's 1st equation

$$\vec{\nabla} \cdot \vec{E}(x, y) = \frac{\rho}{\varepsilon_0} = \frac{\text{charge density}}{\text{permittivity of free space}}$$

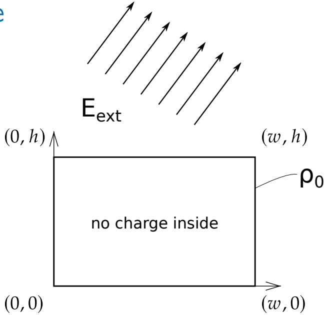
applied to a charged 2D box with with charge density ρ_0 that sits in an external electric field $\vec{E}_{ext} = \vec{E}_{ext}(x, y)$.

The electric field is the negative gradient of the electric potential $u(x, y)$

$$\vec{E} = -\vec{\nabla} u$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = -\nabla^2 u = -\Delta u$$

$$\Rightarrow -\Delta u = \frac{\rho}{\varepsilon_0}$$



Neumann Boundary Condition Example - Electrostatics

$$\Delta u = 0 \qquad \text{for } (x, y) \text{ inside the box}$$

Boundary Conditions:

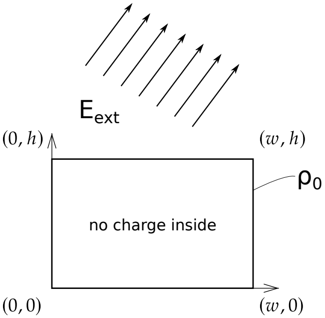
$$-\frac{\partial u}{\partial \hat{n}} - \vec{E}_{ext}(x,t)\cdot\hat{n} = \frac{\rho_0}{\varepsilon_0}$$

North side:

$$-\left(u_x\hat{i} + u_y\hat{j}\right)\cdot\hat{j} = \frac{\rho_0}{\varepsilon_0} + \vec{E}_{ext}(x,h)\cdot\hat{j}$$
$$u_y(x,h) = f(x)$$

West Side:

$$-\left(u_x\hat{i} + u_y\hat{j}\right)\cdot -\hat{i} = \frac{\rho_0}{\varepsilon_0} + \vec{E}_{ext}(0,y)\cdot -\hat{i}$$
$$u_x(0,y) = g(x)$$

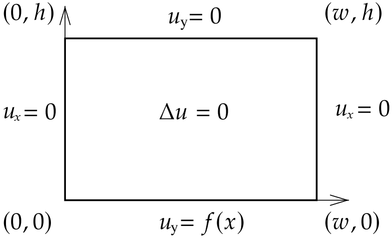


Neumann Problem

ex: Consider a rectangular region of width w and height h.

Normal derivatives are zero at three of the four edges, and the derivative of u obeys some arbitrary function $f(x)$ the fourth edge.

$$
\begin{aligned}
\Delta u &= 0 \\
u_x(0,y) &= 0 \qquad \text{for } 0 < y < h \\
u_y(x,h) &= 0 \qquad \text{for } 0 < x < w \\
u_x(w,y) &= 0 \qquad \text{for } 0 < y < h \\
u_y(x,0) &= f(x) \quad \text{for } 0 < x < w
\end{aligned}
$$



Solution procedure is almost identical to the Dirichlet problem but with sin $\to$ cos and sinh $\to$ cosh

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

BCs: $\dfrac{\partial}{\partial x} X(0) = \dfrac{\partial}{\partial x} X(w) = 0$

$$X_n(x) = \cos\left(\frac{n\pi}{w} x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

$$\Rightarrow Y_n'' + \left(\frac{n\pi}{w}\right)^2 Y_n = 0 \qquad \Rightarrow Y_n = Ae^{\frac{n\pi}{w} y} + Be^{-\frac{n\pi}{w} y}$$

BC: $\dfrac{\partial}{\partial y} Y_n(h) = 0$

$$\Rightarrow B = Ae^{2\frac{n\pi}{w}} \qquad \Rightarrow Y_n(y) = a_n \cosh\left(\frac{n\pi}{w}(y-h)\right)$$

Finally,

$$\boxed{u_n(x,y) = a_n \cosh\left(\frac{n\pi}{w}(y-h)\right) \cos\left(\frac{n\pi}{w} x\right)} \qquad u(x,y) = \sum_n u_n(x,y)$$

Major difference with Dirichlet problem: $n = 0$ gives a non-zero solution

$$X_0(x) = \cos(0) = 1$$

$$Y_0'' = 0 \Rightarrow Y_0 = my + a_0$$

$$\frac{\partial}{\partial y} Y_0(h) = 0 = m \qquad \Rightarrow m = 0$$

$$Y_0 = a_0$$

$$u(x, y) = a_0 + \sum_n a_n \cosh\left(\frac{n\pi}{w}(y - h)\right) \cos\left(\frac{n\pi}{w} x\right)$$

Impossible to actually determine a_0.

- e.g., electrical potentials can be shifted arbitrarily

Pure Neumann problems do not have unique solutions.

The non-zero boundary condition

$$u_n(x,y) = a_n \cos\left(\frac{n\pi}{w}x\right)\cosh\left(\frac{n\pi}{w}(y-h)\right), \quad u(x,y) = a_0 + \sum_{n=1}^{\infty} u_n(x,y)$$

$u_y(x,0) = f(x)$ - Express the boundary condition as a Fourier Series

$$u_y(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{n\pi}{w}\sinh\left(\frac{n\pi}{w}h\right)\cos\left(\frac{n\pi}{w}x\right)$$

Given the appearance of our $u_y(x,h)$, we clearly need a Cosine series

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{w}x\right) \qquad \text{with } b_n = \frac{2}{w}\int_0^w f(x)\cos\left(\frac{n\pi}{w}x\right)dx$$

need equality between the two series

$$\Rightarrow b_0 = 0 = \int_0^w f(x)dx \qquad a_n = \frac{b_n}{\sinh\left(\frac{n\pi}{w}h\right)}\frac{w}{n\pi}$$

We can repeat the same process for all the sub-problems.

$$u_N = a_0 + \sum_n a_n \cos\left(\frac{n\pi}{w}x\right)\cosh\left(\frac{n\pi}{w}y\right)$$

$$u_S = b_0 + \sum_n b_n \cos\left(\frac{n\pi}{w}x\right)\cosh\left(\frac{n\pi}{w}(h-y)\right)$$

$$u_E = c_0 + \sum_n c_n \cos\left(\frac{n\pi}{h}y\right)\cosh\left(\frac{n\pi}{h}x\right)$$

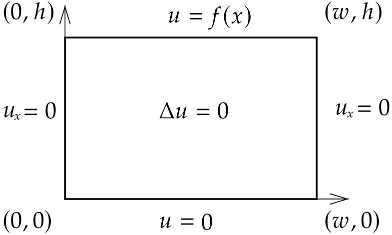
$$u_W = d_0 + \sum_n d_n \cos\left(\frac{n\pi}{h}y\right)\cosh\left(\frac{n\pi}{h}(w-x)\right)$$

To find the unknow coefficients, match the series solution derivative with a Cosine series of the boundary condition.

The boundary condition (normal derivative) must integrate to zero over the boundary.

Mixed Neumann/Dirichlet Problem

$$
\begin{aligned}
\Delta u &= 0 & & \\
u_x(0, y) &= 0 & & \text{for } 0 < y < h \\
u(x, h) &= f(x) & & \text{for } 0 < x < w \\
u_x(w, y) &= 0 & & \text{for } 0 < y < h \\
u(x, 0) &= 0 & & \text{for } 0 < x < w
\end{aligned}
$$



$$u_{xx} + u_{yy} = 0$$

Separation of Variables:

$$u(x, y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const.}$$

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

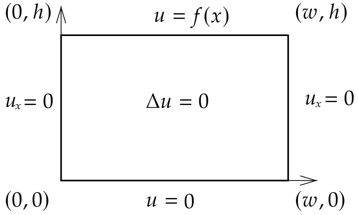
BCs: $X_x(0) = X_x(w) = 0$

$$X_n(x) = \cos\left(\frac{n\pi}{w}x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

with the special case $n = 0$

$$X_0(x) = 1$$

$$Y'' - \lambda Y = 0$$



$$\lambda = \left(\frac{n\pi}{w}\right)^2$$

$\underline{n \neq 0}$

$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y}$$

BC @ x=0: $0 = A + B$ $\Rightarrow B = -A$

$$Y_n(y) = A\left(e^{\frac{n\pi}{w}y} - e^{-\frac{n\pi}{w}y}\right)$$

$$= a_n \sinh\left(\frac{n\pi}{w}y\right)$$

$$u_n(x, y) = a_n \sinh\left(\frac{n\pi}{w}y\right)\cos\left(\frac{n\pi}{w}x\right)$$

$\underline{n=0}$

$$Y_0'' = 0 \Rightarrow Y_0 = a_0 y + b \qquad Y_0(0) = 0 \Rightarrow b = 0$$

The non-zero boundary condition

$$u_n(x,y) = a_n \cos\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}y\right), \quad u(x,y) = a_0 y + \sum_{n=1}^{\infty} u_n(x,y)$$

$u(x,h) = f(x)$ - Express the boundary condition as a Fourier Series

$$u(x,h) = f(x) = a_0 h + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{w}h\right)\cos\left(\frac{n\pi}{w}x\right)$$

Given the appearance of our $u(x,h)$, we clearly need a Cosine series

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{w}x\right) \qquad \text{with } b_n = \frac{2}{w}\int_0^w f(x)\cos\left(\frac{n\pi}{w}x\right)dx$$

need equality between the two series

$$\Rightarrow a_0 = \frac{b_0}{2h} \qquad \text{and} \qquad a_n = \frac{b_n}{\sinh\left(\frac{n\pi}{w}h\right)}$$