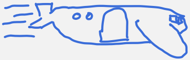
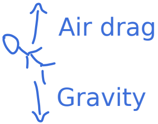
Introduction to Systems: Skydiving





Newton's 2nd Law:

$$ma = F(t) = -mg - \mu v$$

noting that $a = x''$ and $v = x'$, we can rewrite this as

$$x'' + \frac{\mu}{m}x' = -g \quad \Big\} \text{ } 2^{nd} \text{ order ODE, one unknown function}$$

or equivalently,

$$\left.\begin{aligned} x' &= v \\ v' &= -\tfrac{\mu}{m}v - g \end{aligned}\right\} 1^{st} \text{ order ODEs, two unknown functions}$$

How do we find two unknown functions simultaneously?

Systems of DEs: Matrix Notation

$$x \to x_1,\ v \to x_2 \quad \Rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{\mu}{m}x_2 - g \end{aligned}$$

Using matrix notation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\mu}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$= \mathbf{A}\vec{x} + \text{constant vector}$$

General Inhomogeneous IVP:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t), \qquad \vec{x}(t_0) = x_0$$

where $\mathbf{A}$ is an $n \times n$ matrix, and both $\vec{x}$ and $\vec{f}$ are $n \times 1$ column vectors.

Equivalence of problems

For every n^{th} order linear ODE, there is a corresponding system of n 1^{st} order linear ODEs.

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) = h(t)$$

can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t)$$

with

$$\vec{x}(t) = \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}$$

ex: Rewrite $x''' + x' + x = t^2$ as a matrix expression of the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + \vec{b}(t)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= x'' \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} = \mathbf{A}\vec{x} + \vec{b}(t)$$

$$\begin{aligned} x' &= x_2 \\ x'' &= x_3 \\ x''' &= -x' - x + t^2 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ t^2 \end{bmatrix}$$

ex: Rewrite the matrix expression $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\vec{x}$ as a singe second order ODE.

$$x_1' = 2x_1 + 3x_2 \tag{1}$$
$$x_2' = 4x_1 + 5x_2 \tag{2}$$

isolate one of the variables (e.g., x_2)

$$(1) \quad x_2 = \frac{1}{3}\left(x_1' - 2x_1\right) \tag{3}$$

sub back into remaining equation

$$(2) \quad x_2' = 4x_1 + \frac{5}{3}\left(x_1' - 2x_1\right) \overset{(3)}{=} \frac{1}{3}\left(x_1'' - 2x_1'\right)$$

$$4x_1 - \frac{10}{3}x_1 + \frac{5}{3}x_1' = \frac{1}{3}x_1'' - \frac{2}{3}x_1' \qquad \boxed{-\frac{1}{3}x_1'' + \frac{7}{3}x_1' + \frac{2}{3}x_1 = 0}$$

Equivalence of problems

For every system of m coupled n^{th} order linear ODE, there is a corresponding system of $m \cdot n$ 1^{st} order linear ODEs.

ex: m=2, constant coefficients

$$a_n x^{(n)} + c_n y^{(n)} + a_{n-1} x^{(n-1)} + c_{n-1} y^{(n-1)} + \cdots + a_0 x + c_0 y = h(t)$$
$$b_n y^{(n)} + d_n x^{(n)} + b_{n-1} y^{(n-1)} + d_{n-1} x^{(n-1)} + \cdots + b_0 y + d_0 x = g(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t) \quad \text{with} \quad \vec{x}(t) = \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \\ y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

ex: Rewrite the system $\begin{aligned} x'' + \alpha(2x - y) = 0 \\ y'' + \beta(3y - 2x) = 0 \end{aligned}$ as a matrix expression of the form $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= y \\ x_4 &= y' \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} x' \\ x'' \\ y' \\ y'' \end{bmatrix} = \mathbf{A}\vec{x}$$

$$x' = x_2$$

$$x'' = -\alpha(2x - y) = -\alpha(2x_1 - x_3)$$

$$y' = x_4$$

$$y'' = -\beta(3y - 2x) = -\beta(3x_3 - 2x_1) \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\alpha & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 2\beta & 0 & -3\beta & 0 \end{bmatrix}$$

Skydiving: Homogeneous Solutions

Suppose we want to solve $v'' + \frac{\mu}{m} v' = 0$ Guess $v(t) = e^{rt}$

$$r^2 + \frac{\mu}{m} r = 0 \qquad r\left(r + \frac{\mu}{m}\right) = 0$$

$$r = 0, -\frac{\mu}{m} \qquad v(t) = c_1 + c_2 e^{-\frac{\mu}{m}t}$$

$$= c_1 y_1(t) + c_2 y_2(t)$$

What about for the vector expression?

two LI homogeneous solutions $\rightarrow$ two LI vectors

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-\frac{\mu}{m}t} \begin{bmatrix} 1 \\ -\frac{\mu}{m} \end{bmatrix}$$

$$= c_1 \vec{x}_1(t) \quad + \quad c_2 \vec{x}_2(t)$$

Homogeneous Problem and Superposition

Suppose $\vec{x}_1(t)$, $\vec{x}_2(t)$, ..., $\vec{x}_k(t)$ all solve the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

Then

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_k\vec{x}_k(t)$$

also solves the same homogeneous problem.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \sum_{i=1}^{n} c_i \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_i &= \sum_{i=1}^{n} c_i \mathbf{A}(t)\vec{x}_i \\ &= \mathbf{A}(t)\sum_{i=1}^{n} c_i\vec{x}_i \\ &= \mathbf{A}(t)\vec{x}(t)\end{aligned}$$

Finding the solution vectors $\vec{x}_i$

Given

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x},$$

we want to find one of its solution vectors.

For a scalar equation, if we have $y' = ay$ then we know the solution is $y = ce^{at}$.

Lets guess $\vec{x}(t) = e^{\lambda t}\vec{v}$, and plug it into the ODE

$$\lambda e^{\lambda t}\vec{v} = \mathbf{A}e^{\lambda t}\vec{v}$$
$$\lambda\vec{v} = \mathbf{A}\vec{v}$$

$\vec{v}$ is an eigenvector of $\mathbf{A}$, and λ is its associated eigenvalue.

Eigenvectors/Eigenvalues

- A $n \times n$ matrix has n eigenvalues and eigenvectors (except in some special situations that we will not consider)
- The eigenvalues are obtained from the roots of the characteristic polynomial obtained from

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0,$$

  where "det" is short for "determinant" and $\mathbf{I}$ is the $n \times n$ identity matrix
- The eigenvectors are computed by solving the linear system

$$(\mathbf{A} - \lambda \mathbf{I})v = 0$$

  once each of the eigenvalues λ is found.
- the eigenvectors are not unique, defined up to arbitrary mulitplicative constant

Find the eigenvalues/vectors associated with $\begin{aligned}\frac{dx}{dt} &= -3x - 2y \\ \frac{dy}{dt} &= -2x - 6y\end{aligned}$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -3-\lambda & -2 \\ -2 & -6-\lambda \end{bmatrix}\right) = 0$$

$$(3+\lambda)(6+\lambda) - 4 = 0$$

$$\lambda^2 + 9\lambda + 18 - 4 = 0$$

$$\lambda^2 + 9\lambda + 14 = 0$$

$$\lambda = \frac{-9 \pm \sqrt{81 - 4 \cdot 14}}{2} = \frac{-9 \pm \sqrt{81 - 56}}{2}$$

$$= \frac{-9 \pm \sqrt{25}}{2} = \frac{-9 \pm 5}{2}$$

$$\lambda_{1,2} = -2, -7$$

Find the eigenvalues/vectors associated with $\frac{dx}{dt} = -3x - 2y$, $\frac{dy}{dt} = -2x - 6y$

$\underline{\lambda_1 = -2:}$ $\quad \mathbf{A}\vec{v}_1 = -2\vec{v}_1 \qquad (\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \vec{0}$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \vec{v}_1 = \vec{0}$$

<u>Augmented matrix:</u> $\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 - 2R_1 \to R_2$

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad -1x - 2y = 0$$

$$x = -2y$$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \vec{x}_1(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t}$$

Find the eigenvalues/vectors associated with $\dfrac{dx}{dt} = -3x - 2y$, $\dfrac{dy}{dt} = -2x - 6y$

$\underline{\lambda_2 = -7 :}$ $\quad \mathbf{A}\vec{v}_2 = -7\vec{v}_2$ $\qquad (\mathbf{A} + 7\mathbf{I})\vec{v}_2 = \vec{0}$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \vec{v}_2 = \vec{0}$$

<u>Augmented matrix:</u> $\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 1 & -4 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 + 2R_1 \to R_2$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$4x - 2y = 0$$

$$x = y/2$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t}$$

Summary

- Linear n^{th} order DEs can be converted to a system of n 1^{st} order DEs
- Homogeneous system: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)x$
  - Need to find n linearly independent solutions $\{\vec{x}_1(t),\ \vec{x}_2(t),\ \ldots,\ \vec{x}_n(t)\}$
  - These solution vectors are based on the eigenvectors/eigenvalues of $\mathbf{A}$
  - $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$
- Finding eigenvalues/eignevectors, we need to solve

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})v = 0$$