Recall

Linear 1^{st} order ODEs: $y' + p(t)y = h(t)$

- $h(t)$ **is called the** inhomogeneity
- $h(t) = 0$: Homogeneous ODE
- $h(t) \neq 0$: Inhomogeneous ODE

Previously, we solved a inhomogeneous linear ODE

$$y' - 3t^2 y = 3t^2$$

where we found the solution

$$y(t) = -1 + Ce^{t^3}$$

What is the solution to the associated homogeneous problem?

Solve $y' - 3t^2y = 0$

Could use integrating factors or convert to a separable equation.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2y$$
$$\int \frac{dy}{y} = \int 3t^2dt$$
$$\ln(y) = t^3 + C$$
$$y(t) = Ce^{t^3}$$

Homogeneous ODEs have solutions defined up to an arbitrary multiplicative constant.

Recap

inhomogeneous problem: $$y' - 3t^2 y = 3t^2$$

$$y(t) = -1 + Ce^{t^3}$$

homogeneous problem: $$y' - 3t^2 y = 0$$

$$y(t) = -Ce^{t^3}$$

The solution to the homogeneous problem is part of the solution of the inhomogeneous problem!

General Solution Structure

General Solutions: Solution with arbitrary constants.

The general solution of all inhomogeneous DEs obeys the following structure

$$y_g = y_p + y_h$$

- y_p: Particular part
  - No arbitrary constants
  - Depends on the inhomogeneity
- y_h: Homogeneous part
  - Solves the associated homogeneous problem
  - Has multiplicative arbitrary constants
  - Independent of the inhomogeneity

More Terminology

Initial Condition: An **initial condition** (IC) is a constraint on a solution that allows for arbitrary constants to be fixed.

ex: $y(0) = 4$

Initial Value Problem: An **initial value problem** (IVP) is an ODE with an IC.

ex: $y' - 3t^2 y = 3t^2$ with $y(0) = 4$

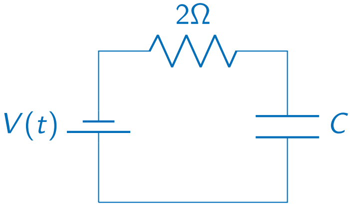
$$y(t) = -1 + Ce^{t^3}$$

$$\underline{t = 0} : y(0) = -1 + C = 4 \qquad C = 5$$

$$y(t) = -1 + 5e^{t^3}$$

Solve the following IVP

An RC circuit comprised of a 2Ω resistor and a capacitor of unknown capacitance C (in farads) is connected to a voltage source. The capacitor has a charge difference Q across its plates, which varies with the applied voltage.



Kirchoff's Law:

$$V(t) = \frac{Q}{C} + 2\frac{\mathrm{d}Q}{\mathrm{d}t}$$

The voltage source is held constant until the capacitor reaches a charge of 5 coulombs, and then at $t = 0$ it is turned off.

Find the charge on the capacitor, $Q(t)$, as a function of time.

IVP:

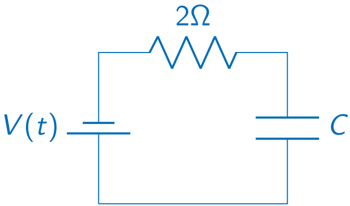
$$2\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = 0; \qquad Q(0) = 5$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{2C} \qquad \int \frac{dQ}{Q} = \int -\frac{1}{2C}dt$$

$$\ln(Q) = -\frac{1}{2C}t + D \qquad Q(t) = De^{-\frac{t}{2C}}$$

Initial Condition:

$$Q(0) = D = 5 \qquad \boxed{Q(t) = 5e^{-\frac{t}{2C}}}$$



Kirchoff's Law:

$$V(t) = \frac{Q}{C} + 2\frac{dQ}{dt}$$

The voltage source is held constant until the capacitor reaches a charge of 5 coulombs, and then at $t = 0$ it is turned off.

At $t = 1$ second, you measure the charge on the capacitor to be 2.5 coulombs. Find the value of C.

$$Q(t) = 5e^{-\frac{t}{2C}}; \qquad Q(1) = 2.5$$

$$\begin{aligned} Q(1) = 5e^{-\frac{1}{2C}} &= 2.5 \\ e^{-\frac{1}{2C}} &= \frac{1}{2} \\ -\frac{1}{2C} &= \ln\left(\frac{1}{2}\right) = -\ln(2) \end{aligned}$$

$$C = \frac{1}{2\ln(2)} \approx 0.7213 \text{ farads}$$

Summary

$$y' + p(t)y = h(t)$$

General solution structure

$$\text{general solution} = \text{particular part} + \text{homogeneous part}$$

- Particular part has no arbitrary constants
  - Determined by the specific inhomogeneity $h(t)$

- Homogeneous part has multiplicative arbitrary constants
  - Solves the associated homogeneous problem $(h(t) = 0)$

Arbitrary constants can be determined by matching initial conditions.

$$y' + p(t)y = h(t); \qquad y(0) = y_0$$