Recall

Linear 1^{st} Order ODEs: $y' + p(t)y = h(t)$

- $h(t)$ **is called the inhomogeneity**
- $h(t) = 0$**: Homogeneous ODE**
- $h(t) \neq 0$**: Inhomogeneous ODE**

General Solution:

$$y_{\text{general}} = y_{\text{particular}} + y_{\text{homogeneous}}$$
$$y_g(t) = y_p(t) + Cy_h(t)$$

- Particular part: depends on $h(t)$ - no arbitrary constants
  - $h(t) = 0 \Rightarrow y_p = 0$
- Homogeneous part: independent of $h(t)$ - multiplicative arbitrary constant

Recall

Linear 1^{st} Order Initial Value Problems (IVPs):

$$y' + p(t)y = h(t); \qquad y(0) = y_0$$

General Solution + Initial Condition:

$$\begin{aligned} y(0) &= y_p(0) + Cy_h(0) \\ &= y_0 \\ C &= \frac{y_0 - y_p(0)}{y_h(0)} \end{aligned}$$

1 Integration Constant + 1 Initial condition = Unique Solution

Linear 2^{nd} order ODEs

General: $y'' + p(t)y' + q(t)y = h(t)$

ICs: $y(0) = y_0, \qquad y'(0) = v_0$

For this week (and beyond):

1. Constant coefficient, homog.

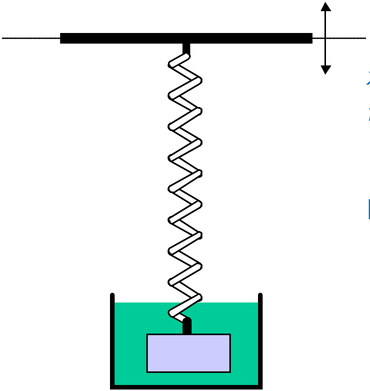
$$ay'' + by' + cy = 0$$

2. Constant coefficient, inhomog.

$$ay'' + by' + cy = h(t)$$

Where do 2^{nd} order linear ODEs come from?

Spring-dashpot system:



$x(t) =$ displacement from rest position
$f(t) =$ applied force

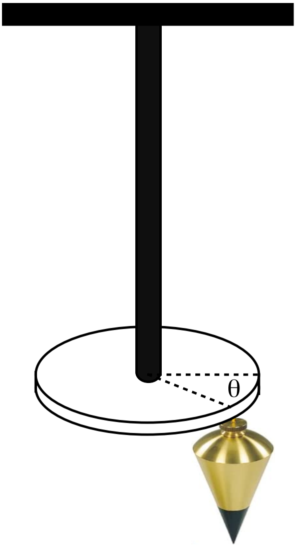
Newton's 2^{nd} Law:

$$F = ma$$

$$-kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t) = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

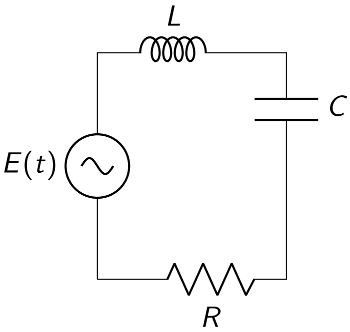
$$mx'' + \beta x' + kx = f(t)$$

Torsional motion of a weight on a twisted shaft:



$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + k\theta = T(t)$$

L-R-C series circuits:



Q=charge on capacitor

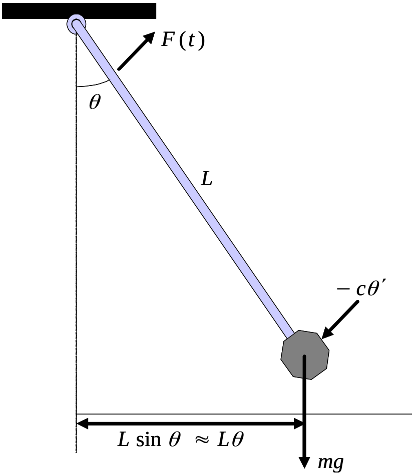
$\frac{dQ}{dt}$=current in circuit

$E(t)$ = applied voltage

Kirchoff's Laws:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Small oscillations of a pendulum:



$$mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -cL\frac{\mathrm{d}\theta}{\mathrm{d}t} - mgL\theta + F(t)$$

Equivalence of Problems

These 4 physical systems are modelled identically by:

$$Ay'' + By' + Cy = D(t)$$

Constants have different physical meaning (& units)

System	A	B	C	D
Spring Dashpot	Mass	Damping Coeff.	Spring Constant	Applied Force
Pendulum	Mass x $(\text{Length})^2$	Damping x Length	Gravitational Moment	Applied Moment
Series Circuit	Inductance	Resistance	Capacitance^{-1}	Imposed Voltage
Twisted Shaft	Moment of Inertia	Damping	Elastic Shaft Constant	Applied Torque

General Linear 2nd Order DE's

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = h(t)$$

- $h(t)$ represents the "forcing" term, an external influence.
  - $h(t) = 0$: solutions tell you the intrinsic behaviour of the system
    - e.g., pull on a spring-mass system, and then let it go
  - $h(t) \neq 0$: solutions tell you the response of the system to forcing
    - e.g., periodically hit a spring-mass system
- $p(t)$ and $q(t)$ represent the intrinsic properties of the physical system.
  - Often consant, but not always
    - e.g., an aging spring could be modelled by $q = q(t)$.

ex: $y'' + 3y' = 0$ We can reduce the order by integrating!

$$y' + 3y = C \qquad \left(e^{3t}y\right)' = ce^{3t}$$

$$e^{3t}y = \underbrace{\frac{C}{3}}_{c_1} e^{3t} + c_2 \qquad y = c_1 + c_2 e^{-3t}$$

This is a special ODE where we can simplify by integrating.

In general, solve using the ansatz $y = e^{rt}$:

$$y' = re^{rt} \qquad y'' = r^2 e^{rt}$$

$$r^2 e^{rt} + 3re^{rt} = 0 \qquad r(r+3) = 0$$

$$\Rightarrow r = 0, -3$$

$$y = c_1 e^0 + c_2 e^{-3t} \qquad = c_1 + c_2 e^{-3t}$$

ex: Solve $y'' + 3y' = 0$ with $y(0) = 2$ and $y'(0) = -2$

$$y_g(t) = c_1 + c_2 e^{-3t}$$
$$y_g'(t) = -3c_2 e^{3t}$$

$$y(0) = c_1 + c_2 = 2$$
$$y'(0) = -3c_2 = -2$$

$$c_2 = \frac{2}{3}$$
$$c_1 + \frac{2}{3} = 2 = \frac{6}{3}$$

$$c_1 = \frac{4}{3}$$

$$y(t) = \frac{4}{3} + \frac{2}{3}e^{-3t}$$

ex: Find the general solution to $-2y'' + 5y' + 3y = 0$

Hint: guess an ansatz $y = e^{rt}$

$$y' = re^{rt} \qquad y'' = r^2 e^{rt}$$

$$-2r^2e^{rt} + 5re^{rt} + 3e^{rt} = 0 \qquad -2r^2 + 5r + 3 = 0$$

$$\begin{aligned} r_{1,2} &= \frac{-5 \pm \sqrt{25+24}}{-4} = \frac{-5 \pm \sqrt{49}}{-4} \\ &= \frac{-5 \pm 7}{-4} = \frac{2}{-4}, \frac{-12}{-4} \\ &= -\frac{1}{2}, 3 \end{aligned}$$

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{3t}$$

Summary

- Linear 2^{nd} order ODEs are useful for modelling many physical systems
- 2^{nd} order $\Rightarrow$ 2 Initial Conditions
- Linear 2^{nd} order homogeneous ODEs with constant coefficients

$$ay'' + by' + cy = 0$$

Ansatz method:

1. Guess $y(t) = e^{rt}$
2. Plug guess into the ODE
3. Find (up to) 2 values of r

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$