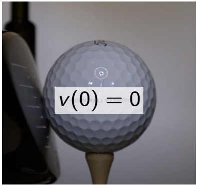
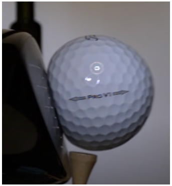
Imagine hitting a gowf ball





source: https://www.youtube.com/watch?v=6TA1s1oNpbk&t=80s

The ball is initially at rest, then a large amount of energy/momentum is rapidly transferred from the gowf club to the ball.

Neglecting air drag and the deformation of the ball, Newton's 2^{nd} law tells us that

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{club}(t), \qquad v(t) = v(0) + \int_0^t \frac{F_{club}(t)}{m} dt$$

Imagine hitting a gowf ball

Suppose that $v_0 = 0$ and that the club is in contact with the ball for $t \in [0, \tau]$ with $0 < \tau \ll 1$.

Then a simple model is $$F_{club}(t) = \begin{cases} \frac{C}{\tau} & 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}$$ with $C = \text{const.}$

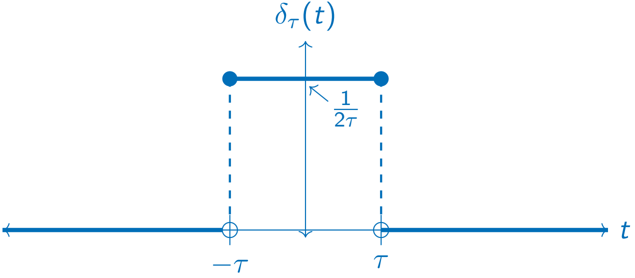
$$v(\tau) = \int_0^\tau \frac{C}{\tau} d\tau = C$$

To approximate a fast event, we take the limit as $\tau \to 0^+$

$$v(0^+) = C$$

For fast events, we do not care about the dynamics of the forcing function just its integral.

General Approximation: normalized step pulse (δ_τ)



Function:

$$\delta_\tau = \begin{cases} \frac{1}{2\tau} & |t| \le \tau \\ 0 & |t| > \tau \end{cases}$$

$$= \frac{u_{-\tau}(t) - u_\tau(t)}{2\tau}$$

Integral:

$$I(\tau) = \int_{-\infty}^{\infty} \delta_\tau(t)dt = \int_{-\tau}^{\tau} \frac{dt}{2\tau}$$

$$= 1$$

Taking the limit $\tau \to 0$

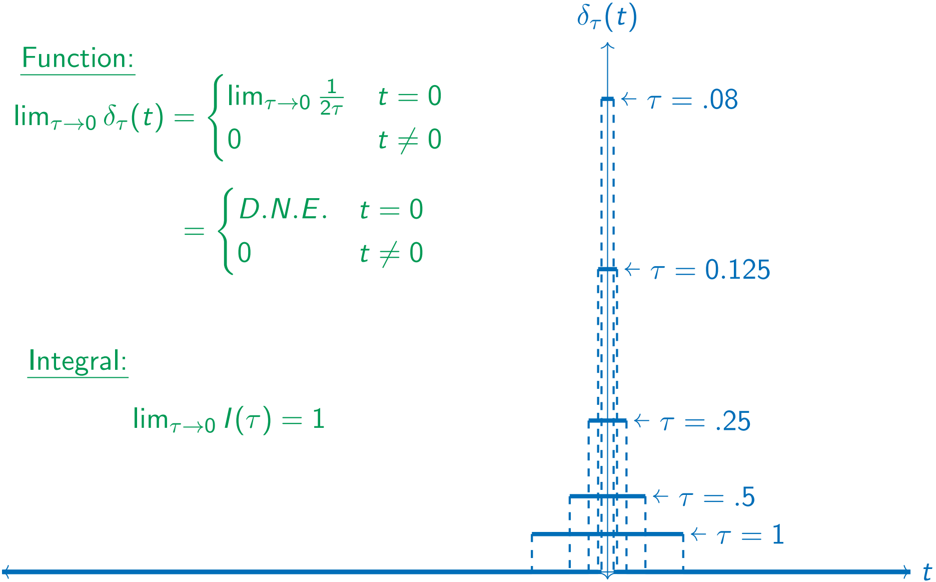
Function:

$$\lim_{\tau\to 0} \delta_\tau(t) = \begin{cases} \lim_{\tau\to 0} \frac{1}{2\tau} & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$= \begin{cases} D.N.E. & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Integral:

$$\lim_{\tau\to 0} I(\tau) = 1$$



Delta Dirac Function: $\delta(t) = \lim_{\tau \to 0} \delta_\tau(t)$

- **Note:** $\delta(t)$ is not really well-defined in the conventional sense
  - It is a "generalized" function with the two following properties:

    1. $\delta(t) = 0$ for $t \neq 0$

       $\delta_\tau(0) \to \infty$ as $\tau \to 0$, but functions cannot have a value of ∞

    2. $\int_{-\infty}^{\infty} \delta(t) dt = 1$

- The delta function acts like an intense pulse, of unit strength.
  - This is also called an **impulse**:
    - An action that happens arbitrarily fast but with finite magnitude.

  - ex: accelerating a golf ball with a golf club

Delta Dirac Function: $\delta(t) = \lim_{\tau \to 0} \delta_\tau(t)$

Theorem: For differentiable functions $f(t)$

$$\int_{-\infty}^{\infty} \delta(t-c) f(t) dt = f(c)$$

Integrating with $\delta(t-c)$ essentially "selects" the value of the integrand at $t = c$.

More generally,

$$\int_a^b \delta(t-c) f(t) dt = \begin{cases} f(c) & a \le c \le b \\ 0 & \text{otherwise} \end{cases}$$

Sketch of proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \delta(t-c)f(t)dt &= \int_{-\infty}^{\infty} \left(\lim_{\tau\to 0} \delta_\tau(t-c)\right) f(t)dt \\
&= \lim_{\tau\to 0} \int_{-\infty}^{\infty} \delta_\tau(t-c)f(t)dt \\
&= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{c-\tau}^{c+\tau} f(t)dt \\
&= \lim_{\tau\to 0} \frac{F(c+\tau)-F(c+\tau)}{2\tau} = f(c)
\end{aligned}
$$

Laplace transform of $\delta(t)$

Integrating with $\delta(t-c)$ essentially "selects" the value of the integrand at $t = c$

Assuming $c \geq 0$

$$\begin{aligned}\mathcal{L}\left\{\delta(t-c)\right\} &= \int_0^{\infty} e^{-st}\delta(t-c)dt \\ &= \int_{-\infty}^{\infty} e^{-st}\delta(t-c)dt = e^{-sc}\end{aligned}$$

Special case: $c = 0$

$$\mathcal{L}\left\{\delta(t)\right\} = \lim_{c\to 0} \mathcal{L}\left\{\delta(t-c)\right\} = \lim_{c\to 0} e^{-sc} = 1$$

Solve: $$2y'' + y' + 2y = \delta(t) \qquad \text{with} \quad \begin{matrix} y(0) = 0 \\ y'(0) = 0 \end{matrix}$$

$$2s^2 Y(s) + sY(s) + 2Y(s) = 1 \qquad\qquad Y(s) = \frac{1}{2s^2 + s + 2}$$

complete the square: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$

$$Y(s) = \frac{1}{2(s+\frac{1}{4})^2 + \frac{15}{8}} \times \frac{\frac{1}{2}}{\frac{1}{2}} \qquad = \frac{1}{2}\frac{1}{(s+\frac{1}{4})^2 + \underbrace{\frac{15}{16}}_{\omega^2}} \times \frac{\omega}{\omega}$$

$$= \frac{1}{2}\frac{\frac{\sqrt{15}}{4}}{\frac{\sqrt{15}}{4}}\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}} \qquad = \frac{2}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

$$y(t) = \frac{2}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t\right)$$

This is the **impulse response** of the system. That is, how the system responds to an impulse of forcing.

Solve: $$2y'' + y' + 2y = \delta(t-5) \quad \text{with} \quad \begin{array}{l} y(0) = 0 \\ y'(0) = 0 \end{array}$$

$$Y(s) = e^{-5s} \cdot \frac{1}{2s^2 + s + 2}$$

by the convolution theorem

$$\begin{aligned} y(t) &= \delta(t-5) * \underbrace{\frac{2}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t\right)}_{f(t)} \\ &= \int_0^t f(t-\tau)\delta(\tau-5)d\tau \\ &= \begin{cases} 0 & t < 5 \\ f(t-5) & t \geq 5 \end{cases} \\ &= u_5(t) f(t-5) \end{aligned}$$

Solve: $2y'' + y' + 2y = g(t)$ with $\begin{matrix} y(0) = 0 \\ y'(0) = 0 \end{matrix}$

$$Y(s) = G(s) \cdot \frac{1}{2s^2 + s + 2}$$

by the convolution theorem

$$y(t) = g(t) * \underbrace{\frac{2}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t\right)}_{f(t)}$$

$$= \int_0^t f(t-\tau) g(\tau) d\tau$$

Take-home message: we can get the solution for any forcing function $g(t)$ by convolving with $f(t)$ the **impulse response function**.

Inhomogeneous IVPs via Laplace transforms

Consider the constant coefficient 2^{nd} order DE:

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= v_0 \end{aligned}$$

Take LT

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - av_0 = G(s)$$

Solve for $Y(s)$:

$$Y(s) = \underbrace{\frac{(as + b)y_0 + av_0}{as^2 + bs + c}}_{Y_h} + \underbrace{\frac{G(s)}{as^2 + bs + c}}_{Y_p}$$

effects of initial conditions (Homogeneous Part) — effects of forcing function (Particular Part)

Inhomogeneous IVPs via Laplace transforms

$$\begin{aligned} ay'' + by' + cy &= g(t) \\ y(0) &= y_0 \\ y'(0) &= y_0' \end{aligned} \quad \to \quad Y(s) = \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{Y_h(s)} + \underbrace{\frac{G(s)}{as^2+bs+c}}_{Y_p(s)}$$

1. Break up $Y_h(s)$ using partial frac. decomp. & invert $Y_h(s) \to y_h(t)$.
2. Define the **Transfer Function**:

$$F(s) = \frac{1}{as^2+bs+c}$$

3. Invert $F(s) \to f(t)$. The function $f(t)$ is called the **impulse response** function.
4. From the convolution theorem with $Y_p(s) = F(s)G(s)$

$$y_p(t) = f * g$$

5. Finally

$$y(t) = y_h(t) + y_p(t)$$

ex: $y'' + 4y = t^3, \quad y(0) = y'(0) = 0.$

Find an appropriate impulse response function and express the ODE's solution as a convolution integral.

$$s^2 Y(s) - s + 4Y(s) = \mathcal{L}\left\{t^3\right\}$$

$$Y(s) = \mathcal{L}\left\{t^3\right\} \cdot \frac{1}{s^2+4}$$

$$\text{Transfer Function: } F(s) = \frac{1}{s^2+4} = \frac{1}{2}\mathcal{L}\left\{\sin(2t)\right\}$$

$$\text{Impulse Response: } f(t) = \frac{1}{2}\sin(2t)$$

$$\begin{aligned} y(t) &= \frac{1}{2}\sin(2t) * t^3 \\ &= \frac{1}{2}\int_0^t \sin(2(t-\tau))\tau^3 d\tau \end{aligned}$$

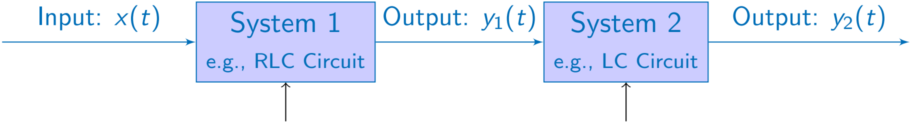
ex: $y'' + 4y = t^4, \quad y(0) = 1, y'(0) = 0.$

Express the ODE's solution in terms of a convolution integral.

$$\begin{aligned} s^2 Y(s) - s + 4Y(s) &= \mathcal{L}\left\{t^4\right\} \\ Y(s) &= \left(s + \mathcal{L}\left\{t^4\right\}\right)\frac{1}{s^2+4} \\ &= \frac{s}{s^2+4} + \mathcal{L}\left\{t^4\right\} \cdot \frac{1}{s^2+4} \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\sin(2t) * t^4 \\ &= \cos(2t) + \frac{1}{2}\int_0^t \sin(2(t-\tau))\tau^4 d\tau \end{aligned}$$

Application: Linear Time Invariant Systems



$$F_1(s) = \frac{1}{a_1 s^2 + b_1 s + c_1} \qquad F_2(s) = \frac{1}{a_2 s^2 + b_2 s + c_2}$$

$$\begin{aligned} y_1(t) &= x(t) * f_1(t) \\ y_2(t) &= y_1(t) * f_2(t) \\ &= x(t) * f_1(t) * f_2(t) \\ &= x(t) * (f_1(t) * f_2(t)) \end{aligned}$$

$f_1(t) * f_2(t) =$ Impulse response for total system