Review

- Linear DEs: $y' + p(t)y = g(t)$; $\quad y(t_0) = y_0$
  - **Homogeneous**: $g(t) = 0$
  - **Inhomogeneous**: $g(t)$ is not zero everywhere.
  - **Constant coefficient** $p(t) = a$, with constant a.
    - General solution of form: $y_g = y_h + y_p$
    - $y_h = Ce^{-at}$
    - y_p – Method of undetermined coefficients
      -Skipped, will revisit

What to do if $p(t)$ is not constant, or if undetermined coefficients doesn't work for a specific $g(t)$?

- Use the method of integrating factors.

Want to solve $y' + p(t)y = g(t)$

Instead, lets do something easier. Solve

$$\frac{\mathrm{d}}{\mathrm{d}t}(q \cdot y) = h(t)$$

with some known $q(t)$ and $\int h(t)dt = H(t) + C$.

$$\begin{array}{ccc} \frac{\mathrm{d}}{\mathrm{d}t}(q(t) \cdot y(t)) = h(t) & \to & \mathrm{d}(q \cdot y) = h(t)\mathrm{d}t \\ q(t)y(t) = H(t) + C & \leftarrow & \int \mathrm{d}(q(t)y(t)) = \int h(t)\mathrm{d}t \end{array}$$

$$\Rightarrow y(t) = \frac{H(t) + C}{q(t)}$$

Idea

It would be very nice if we could multiply

$$y' + p(t)y = g(t)$$

by some magical function $\mu(t)$ such that

$$\mu(t) \cdot \left[y' + p(t)y\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[q(t) \cdot y(t)\right]$$

Unique choice of $\mu(t)$:

$$\mu(t) = e^{\int p(t)dt}$$

See full notes on Canvas for details.

Method of integrating factors: $y' + p(t)y = g(t)$

1. Multiply by the integrating factor $\mu(t) = e^{\int p(t)\mathrm{d}t}$

$$\mu(t)y' + \underbrace{p(t)\mu(t)}_{\mu'(t)}y = g(t)\mu(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot y) = g(t)\mu(t)$$

$$\mathrm{d}(\mu \cdot y) = g(t)\mu(t)\mathrm{d}t$$

2. Integrate:

$$\mu \cdot y = \int g(t)\mu(t)\mathrm{d}t + C$$

3. Isolate $y(t)$:

$$y(t) = \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$

Existence and Uniqueness

$$y(t) = \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

- What properties does $p(t)$ need to ensure $\mu(t)$ is differentiable?
  - Continuous and integrable.

- What properties does $g(t)$ need to ensure $y(t)$ is differentiable?
  - Continuous and $\mu(t) \cdot g(t)$ is integrable.

These conditions are pretty lax:

- almost all linear 1st Order DEs can be solved.

Evaluating the integrals can be tough.

ex: Find the general solution of $y' - 3y = e^t$.

$$\boxed{\begin{aligned} y(t) &= \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)} \\ \mu(t) &= e^{\int p(t)\mathrm{d}t} \end{aligned}}$$

$$\begin{aligned}
\mu(t) &= e^{\int(-3)dt} = e^{-3t} \\
e^{-3t}y' - 3e^{-3t}y &= e^t e^{-3t} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-3t}y\right) &= e^{-2t} \\
\mathrm{d}\left(e^{-3t}y\right) &= e^{-2t}\mathrm{d}t \\
e^{-3t}y(t) &= \int e^{-2t}\mathrm{d}t + C \\
y(t) &= \frac{-\frac{1}{2}e^{-2t} + C}{e^{-3t}} \\
&= \underbrace{-\frac{1}{2}e^t}_{y_p} + \underbrace{Ce^{3t}}_{y_h}
\end{aligned}$$

ex: Find the general solution of $y' - 3t^2 y = 3t^2$.

$$\boxed{\begin{aligned} y(t) &= \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)} \\ \mu(t) &= e^{\int p(t)\mathrm{d}t} \end{aligned}}$$

$$\mu(t) = e^{-\int 3t^2 dt} = e^{-t^3}$$

$$e^{-t^3} y(t) = \int 3t^2 e^{-t^3} \mathrm{d}t$$

let $u = t^3$

$du = 3t^2$

$$\int 3t^2 e^{-t^3} \mathrm{d}t = \int e^{-u} du$$

$$= -e^{-u} + C$$

$$e^{-t^3} y(t) = -e^{-t^3} + C$$

$$y(t) = \underbrace{-1}_{y_p} + \underbrace{Ce^{t^3}}_{y_h}$$

Solution Structure for $y' + ay = g(t)$

Do we always have a general solution of the form $y_g = y_p + y_h$?

Yes, always.

For all linear DEs.

Solution Structure for $y' + p(t)y = g(t)$

Do we always have a general solution of the form $y_g = y_p + y_h$?
Suppose y_p and y_h solve the following DEs

$$y_p' + p(t)y_p = g(t) \tag{1}$$
$$y_h' + p(t)y_h = 0 \tag{2}$$

What DE does y_g solve?

$$\begin{aligned} y_g' + p(t)y_g &= y_p' + y_g' + p(t)(y_p + y_h) \\ &= \left(y_p' + p(t)y_p\right) + \left(y_h' + p(t)y_h\right) \\ &= g(t) + 0 = g(t) \end{aligned}$$

y_g also solves Eq. (1)! Give me some y_p that solves (1), I can always add y_h and still have a solution to (1).

Solution Structure for $y' + p(t)y = g(t)$

Do we always have a general solution of the form $y_g = y_p + y_h$?
Suppose y_p solves the following DEs

$$y_p' + p(t)y_p = g(t) \tag{1}$$

Is y_p necessarily a particular solution? Proof by contradiction
Assume y_p has some arbitrary constant.

Additive Constant: $y_p \to y_p + C$

$$\begin{aligned} y_p' + a(y_p + C) &= y_p' + ay_p + aC \\ &= g(t) + aC \\ &\neq g(t) \end{aligned}$$

Multiplicative Constant: $y_p \to Cy_p$

$$\begin{aligned} Cy_p' + aCy_p &= C\left(y_p' + ay_p\right) \\ &= Cg(t) \\ &\neq g(t) \end{aligned}$$

y_p cannot have any arbitrary constants!

Summary

- Looked at general linear 1st order DEs: $y' + p(t)y = g(t)$
  - Tried to turn the LHS into an exact derivative:

$$y' + p(t)y \quad \rightarrow \quad \mathrm{d}(q \cdot y) = q'y + qy'$$

  - Multiplied by an integrating factor $\mu(t)$
  - After some algebra find that $\mu = q$
  - General solution: $y(t) = \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$
  - Integrating factor: $\mu(t) = e^{\int p(t)\mathrm{d}t}$
- Found that we could solve all linear 1st order DEs by this method, provided e.g. that $p(t)$and $g(t)$ are continuous.
- General solution to a linear DE: $y_g = y_p + y_h$