Recall:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} \qquad \text{with } \mathbf{A} \text{ an } n \times n \text{ matrix}$$

We can find n solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ by finding the eigenvalues, λ, and eigenvectors, $\vec{v}$, of the matrix $\mathbf{A}$.

i.e., solving

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \text{and} \quad (\mathbf{A} - \lambda \mathbf{I})\vec{v} = 0$$

Homogeneous Problem and Superposition

Suppose $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\ldots$, $\vec{x}_n(t)$ all solve the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

Then

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

also solves the same homogeneous problem.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} &= \sum_{i=1}^{n} c_i \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_i = \sum_{i=1}^{n} c_i \mathbf{A}(t)\vec{x}_i \\ &= \mathbf{A}(t)\sum_{i=1}^{n} c_i\vec{x}_i \\ &= \mathbf{A}(t)\vec{x}(t)\end{aligned}$$

So, the $\vec{x}(t)$ above is the general solution to the homogeneous problem.

Find the general solution to

$$\frac{dx}{dt} = -3x - 2y$$
$$\frac{dy}{dt} = -2x - 6y$$

$$\begin{aligned}\vec{x}(t) &= c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\\ &= \underbrace{c_1 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{eigenvector}} e^{-2t}}_{\text{eigensolution}} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t}\end{aligned}$$

Notes:

- The two eigenvectors are linearly independent
  - $\Rightarrow$ The two eigensolutions/eigenmodes are linearly independent
- The solution has components along the two eigenvectors/eigendirection
- The component along the second direction decays the fastest.
- c_1 and c_2 are determined from initial conditions

Find the solution to

$$\frac{dx}{dt} = -3x - 2y \qquad x(0) = 5$$
with
$$\frac{dy}{dt} = -2x - 6y \qquad y(0) = 4$$

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t}$$

$$\vec{x}(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2c_1 & c_2 & 5 \\ c_1 & 2c_2 & 4 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|c} 0 & 5c_2 & 13 \\ c_1 & 2c_2 & 4 \end{array}\right]$$

$-\frac{2}{5}R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|c} 0 & 5c_2 & 13 \\ c_1 & 0 & -\frac{6}{5} \end{array}\right]$$

$$c_1 = -\frac{6}{5}$$
$$c_2 = \frac{13}{5}$$

Initial Conditions and Eigenvectors

For first order systems of the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x}$, we have a general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t),$$

where with real and distinct eigenvalue of $\mathbf{A}$, we have: $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$

1. The arbitrary coefficients c_i come from decomposing the initial condition $\vec{x}(0)$ onto the basis of eigenvectors $\{\vec{v}_i\}$.
2. This decomposes the solution into n different "eigenmodes".
   - c_i is the initial "amplitude" of the i^{th} eigenmode
3. Each eigenmode grows $(\lambda_i > 0)$ or shrinks $(\lambda_i < 0)$ over time.

Find the general solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -\lambda & -2 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda(-3-\lambda) - 4 = 0$$

$$\lambda^2 + 3\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda + 4) = 0$$

$$\lambda_{1,2} = 1, -4$$

Find the general solution to

$$\frac{dx}{dt} = -2y$$
$$\frac{dy}{dt} = -2x - 3y$$

$\underline{\lambda_1 = 1 :}$ $\left[\begin{array}{cc|c} 0-1 & -2 & 0 \\ -2 & -3-1 & 0 \end{array}\right]$ $\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 - 2R_1 \to R_2$

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad -1x - 2y = 0$$

$$x = -2y$$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \vec{x}_1(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t$$

Find the general solution to

$$\frac{dx}{dt} = -2y$$
$$\frac{dy}{dt} = -2x - 3y$$

$\underline{\lambda_2 = -4 :}$ $\left[\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 + R_1/2 \to R_2$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 4x - 2y = 0$$

$$2x = y$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-4t}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-4t}$$

Find the eigenvalues for the ODE:

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Characteristic equation

$$(-1-\lambda)^2 + 4 = 0$$
$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$
$$\lambda_{1,2} = -1 \pm 2i$$

Complex conjugate eigenvalues pairs: $\lambda_{1,2} = r \pm i\omega$

Associated eigenvectors are also complex conjugates

$$\vec{v}_{1,2} = \underbrace{\vec{a}}_{\text{real part}} \pm \underbrace{i\vec{b}}_{\text{imaginary part}}$$

Proof:

Suppose $\vec{v}_1 = \vec{a} + i\vec{b}$ with $\lambda_1 = r + i\omega$

$$\mathbf{A}(\vec{a} + i\vec{b}) = (r + i\omega)(\vec{a} + i\vec{b})$$

Take complex conjugate of both sides

$$\mathbf{A}(\vec{a} - i\vec{b}) = (r - i\omega)(\vec{a} - i\vec{b})$$
$$\mathbf{A}\vec{v}_2 = \lambda_2\vec{v}_2$$

Find the eigenvectors for the ODE:

$$\frac{dx}{dt} = -x + 2y$$
$$\frac{dy}{dt} = -2x - y$$

$$\lambda_{1,2} = -1 \pm 2i$$

$\underline{\lambda_1 = -1 + 2i}$

$$\left[\begin{array}{cc|c} -1-(-1+2i) & 2 & 0 \\ -2 & -1-(-1+2i) & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$R_2 - iR_1 \rightarrow R_2$ and $\frac{1}{2}R_1 \rightarrow R_1$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} -ix + y &= 0 \\ y &= ix \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Find the general solution for the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - y$$

$$\lambda_{1,2} = -1 \pm 2i \qquad \vec{v}_{1,2} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-1-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$x(t) = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}$$
$$y(t) = ic_1 e^{(-1+2i)t} - ic_2 e^{(-1-2i)t}$$

Can convert to purely real solution using Euler's identity (not covered).

General solution to the eigenproblem (2x2)

From linear algebra, we know that $\mathbf{A}\vec{v} = \lambda\vec{v}$ has a non-trivial solution $\vec{v}$ only if the determinant of the matrix $\mathbf{A} - \lambda\mathbf{I}$ is identically zero.

That is, the constant λ must satisfy the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = (a-\lambda)\cdot(d-\lambda) - bc = 0$

$$\underbrace{\lambda^2 - (a+d)\lambda + ad - bc}_{\text{characterisitc poly.}} = 0 \quad \Rightarrow \lambda = \frac{a+b \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

General solution to the eigenproblem (2x2)

From linear algebra, we know that $\mathbf{A}\vec{v} = \lambda\vec{v}$ has a non-trivial solution $\vec{v}$ only if the determinant of the matrix $\mathbf{A} - \lambda\mathbf{I}$ is identically zero.

That is, the constant λ must satisfy the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Three possibilites

1. $\mathbf{A}$ has 2 distinct real eigenvalues and eigenvectors✓
2. The eigenvalues/vectors of $\mathbf{A}$ occur as a complex conjugate pair ✓
3. One eigenvalue is repeated (not covered)