

Recall: Constant Coefficient Homogeneous 2nd Order ODE

$ay'' + by' + cy = 0$ we need 2 linearly independent solutions

Try ansatz $y = e^{rt}$ \rightarrow $e^{rt} \cdot \underbrace{(ar^2 + br + c)}_{\text{characteristic polynomial}} = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2. Repeated real roots: discriminant = 0

3. Complex conjugate roots: discriminant < 0

Repeated real root ($r_1 = r_2 = r$)

Straighforward solution

$$y_1 = e^{rt}$$

with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

We need another solution that is linearly independent of y_1

Lets try

$$y_2 = ty_1$$

Checking $y_2 = te^{rt}$

$$ay'' + by' + cy = 0 \quad \text{with } b^2 - 4ac = 0$$

$$\text{Try: } y_2 = te^{rt}$$

$$\Rightarrow r_{1,2} = r = \frac{-b}{2a}$$

$$y_2' = e^{rt} + rte^{rt}$$

$$y_2'' = 2re^{rt} + r^2te^{rt}$$

plug these into the ODE

$$a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + ce^{rt} = 0$$

$$(ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = 0$$

$$\text{sub in } r = \frac{-b}{2a}$$

$$\underbrace{\left(\cancel{a} \frac{b^2}{4\cancel{a}^2} - \frac{b^2}{2a} + c \right)}_{\frac{1}{4a}(4ac - b^2) = 0} te^{rt} + \underbrace{\left(\cancel{2a} \frac{\cancel{-b}}{\cancel{2a}} + b \right)}_0 e^{rt} = 0$$

$$0 = 0 \checkmark$$

Constant Coefficient Homogeneous 2nd Order ODE

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$$y_h = c_1 e^{rt} + c_2 t e^{rt}$$

3. Complex conjugate roots: discriminant < 0

Solve the IVP: $y'' + 4y' + 4y = 0$

$$\begin{aligned} y(0) &= 2 \\ y'(0) &= 0 \end{aligned}$$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 4}}{2} = \frac{-4 \pm 0}{2} = -2$$

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

initial conditions:

$$y(0) = 2 = c_1$$

$$y'(0) = 0 = -2c_1 + c_2(e^{-2t} - 2te^{-2t}) \Big|_{t=0}$$

$$= -4 + c_2 \quad \Rightarrow \quad c_2 = 4$$

$$\boxed{y(t) = 2e^{-2t} + 4te^{-2t}}$$

Complex roots ($b^2 - 4ac < 0$)

Roots are given by:

$$r_1 = \lambda + i\mu$$

$$\text{where } i = \sqrt{-1}$$

$$r_2 = \lambda - i\mu$$

$$\lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

The two functions $e^{(\lambda+i\mu)t}$ & $e^{(\lambda-i\mu)t}$ are solutions to $ay'' + by' + cy = 0$.

What is the exponential of a complex number?

Euler's formula:

$$e^{\pm i\alpha} = \cos \alpha \pm i \sin \alpha$$

$$\begin{aligned}
 e^{(\lambda \pm i\mu)t} &= e^{\lambda t} e^{\pm i\mu t} \\
 &= \underbrace{e^{\lambda t}}_{\text{Real}} \underbrace{[\cos(\mu t) \pm i \sin(\mu t)]}_{\text{Complex}}
 \end{aligned}$$

We don't want complex solutions, try a linear combination of the two.

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = \frac{e^{\lambda t + i\mu t} + e^{\lambda t - i\mu t}}{2}$$

$$\begin{aligned}
 \tilde{y}_1 &= \frac{e^{\lambda t}}{2} [\cos(\mu t) + \cancel{i \sin(\mu t)} + \cos(\mu t) - \cancel{i \sin(\mu t)}] = \frac{e^{\lambda t}}{2} 2 \cos(\mu t) \\
 &= e^{\lambda t} \cos(\mu t)
 \end{aligned}$$

$$\text{Similarly, } \tilde{y}_2 = \frac{y_1 - y_2}{2i} = \frac{e^{\lambda t + i\mu t} - ie^{\lambda t - i\mu t}}{2i} \rightarrow \tilde{y}_2 = e^{\lambda t} \sin(\mu t)$$

Complex roots ($r_{1,2} = \lambda \pm i\mu$)

The functions $y_1 = e^{\lambda t} \cos(\mu t)$ and $y_2 = e^{\lambda t} \sin(\mu t)$ are linearly independent real solutions.

Sketch the two functions if you are not convinced.

General solution: $y_h = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$

Find the general solution to: $y'' + 6y = 0$

$$r_{1,2} = \frac{\pm\sqrt{-4 \cdot 6}}{2} = \pm\sqrt{-6} = \pm i\sqrt{6}$$

$$y_h = c_1 \cos(\sqrt{6}t) + c_2 \sin(\sqrt{6}t)$$

Solve the IVP: $y'' + 2y' + 5y = 0$

$$y(0) = 1$$
$$y'(0) = -1$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}}{2}i = -1 \pm 2i$$

$$y_h = e^{-t} (c_1 \cos(2t) + c_2 \sin(2t))$$

initial conditions:

$$y(0) = 1 = c_1$$

$$y'(0) = -1 = -c_1 + (-2c_1 \sin(0) + 2c_2 \cos(0)) = -c_1 + 2c_2$$

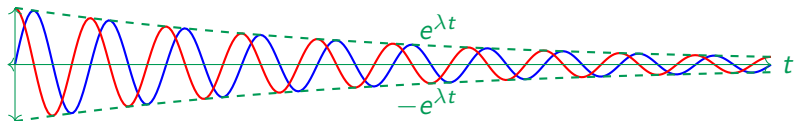
$$-1 = -1 + 2c_2 \quad \Rightarrow \quad c_2 = 0$$

$$y(t) = e^{-t} \cos(2t)$$

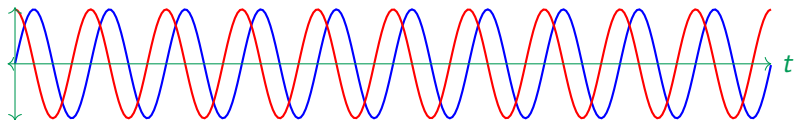
Qualitative Behaviour: complex roots

Three subcases:

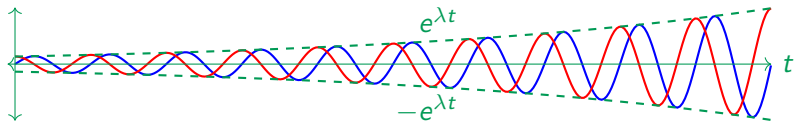
1. $\lambda < 0 \Rightarrow$ Exponentially decaying oscillations.



2. $\lambda = 0 \Rightarrow$ Sustained periodic oscillations.



3. $\lambda > 0 \Rightarrow$ Exponentially growing oscillations.



Summary

- For linear ODEs:
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 - Pick an ansatz (e.g., e^{rt})
 - Write down the characteristic equation
 - Find the roots
 - If you don't have enough functions, make a new one by multiplying by t
- Write down the general solution according to the roots
 - Real and distinct $\Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
 - Real and repeated $\Rightarrow y_h = c_1 e^{rt} + c_2 t e^{rt}$
 - Complex $\Rightarrow y_h = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$
- Fit the constants c_1 and c_2 to the initial conditions