Review: Method of Undetermined Coefficients

$ay'' + by' + cy = h(t) \neq 0 \qquad \Rightarrow \qquad y = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$

How to find y_p:

1. Differentiate $h(t)$ to get functional forms

$$\{f_1(t), f_2(t), f_3(t), \dots\}$$

2. If there are a finite number of functions, guess

$$y_p = Af_1(t) + Bf_2(t) + Cf_3(t) + \dots$$

3. Plug y_p into ODE
4. Solve for undetermined coefficients $(A, B, C \dots)$

Solving for undetermined coefficients $(A, B, C \ldots)$

$ay'' + by' + cy = h(t) \neq 0 \qquad \Rightarrow \quad y = y_p + c_1 y_1(t) + c_2 y_2(t)$

$$y_p = Af_1(t) + Bf_2(t) + Cf_3(t) + \ldots$$

We obtain a linear system of equations, n equations n unknowns

$$\mathbf{M} \begin{bmatrix} A \\ B \\ C \\ \vdots \end{bmatrix} = \mathbf{v}.$$

If any of the functions f_i are linearly dependent with $y_1(t)$ or $y_2(t)$, we have an underdetermined system of equations...we won't find a unique solution.

This situation is called mathematical resonance.

Dealing with mathematical resonance

$ay'' + by' + cy = h(t) \neq 0 \qquad \Rightarrow \qquad y = y_p + c_1 y_1(t) + c_2 y_2(t)$

Suppose the family of functional for $h(t)$ is

$$\{f_1(t), f_2(t), f_3(t)\}$$

where $f_3(t)$ and $y_1(t)$ are linearly dependent.

Recall: when we had repeated roots for the homogeneous case, we obtained a new linearly dependent function by multiplying by t.

Then we should guess

$$y_p = Af_1(t) + Bf_2(t) + Ctf_3(t)$$

If $tf_3(t)$ is linearly dependent with $y_2(t)$, then use $t^2 f_3(t)$ instead.

A simple example of mathematical resonance

$$y'' + y = h(t) = \sin \omega t$$
$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

Family of functional forms for $h(t)$

$$\{\cos(\omega t), \sin(\omega t)\}$$

Naive guess:

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$

Notice that if $\omega = 1$, then $y_p = y_h$ (not good!)

$\underline{\omega \neq 1}$

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$

$\underline{\omega = 1}$

$$y_p = At\cos(\omega t) + Bt\sin(\omega t)$$

Practice

(1) $y' + 6y = \cos t + t^2$

$$y_h = c_1 e^{-6t}$$

$$\text{family} = \left\{\cos t, \sin t, t^2, t, 1\right\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + E$$

(2) $y'' = t^2$ $\qquad y_h = c_1 + c_2 t$

$$\text{family} = \left\{t^2, \underline{t}, \underline{1}\right\}$$

$$y_p = At^2 + Bt^4 + Ct^4$$

(3) $y'' + 3y' + 2y = 5e^{-t}$

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

$$\text{family} = \left\{\underline{e^{-t}}\right\}$$

$$y_p = Ate^{-t}$$

(4) $y'' + 2y' + y = 12e^{-t}$

$$y_h = c_1 e^{-t} + c_2 te^{-t}$$

$$\text{family} = \left\{\underline{e^{-t}}\right\}$$

$$y_p = At^2 e^{-t}$$

(5) $y'' + 6y' = \cos t + t^2$

$$y_h = c_1 e^{-6t} + c_2$$

$$\text{family} = \left\{\cos t, \sin t, t^2, t, \underline{1}\right\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + Et^3$$

Find the general solution of $y'' + 5y' + 4y = e^{-4t}$

$$r_{1,2} = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

$$y_h = c_1 e^{-t} + \underbrace{c_2 e^{-4t}}_{\propto h(t)}$$

Try:

$$y_p = Ate^{-4t}$$

$$y_p' = A\left(e^{-4t} - 4te^{-4t}\right)$$

$$y_p'' = -Ae^{-4t} - 4A\left(e^{-4t} - 4te^{-4t}\right)$$

$$= -8Ae^{-4t} + 16Ate^{-4t}$$

plug into DE:

$$-8Ae^{-4t} + 16Ate^{-4t} + 5Ae^{-4t} - 20Ate^{-4t} + 4Ate^{-4t} = e^{-4t}$$

$$(-8+5)Ae^{-4t} + (20-20)te^{-4t} = e^{-4t}$$

$$-3Ae^{-4t} = e^{-4t}$$

$$A = -\frac{1}{3}$$

$$y = c_1 e^{-4t} + c_2 e^{-t} - \frac{1}{3} te^{-4t}$$

Find the general solution of $y'' + 4y' + 4y = e^{-2t}$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 16}}{2} = -2$$

$$y_h = \underbrace{c_1 e^{-2t}}_{\propto h(t)} + c_2 t e^{-2t}$$

Try:

$$y_p = At^2 e^{-2t}$$

$$y_p' = A\left(2te^{-2t} - 2t^2 e^{-2t}\right)$$

$$y_p'' = 2A\left(e^{-2t} - 2te^{-2t}\right) - 2A\left(2te^{-2t} - 2t^2 e^{-2t}\right)$$

$$= 4At^2 e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t}$$

plug into DE:

$$4At^2e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t} + 8Ate^{-2t} - 8At^2e^{-2t} + 4At^2e^{-2t} = e^{-2t}$$

$$(-8+8)At^2e^{-2t} + (-8+8)te^{-2t} + 2Ae^{-2t} = e^{-2t}$$

$$2Ae^{-2t} = e^{-2t}$$

$$2A = 1 \qquad \Rightarrow A = \frac{1}{2}$$

$$y = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{2t}$$

Method of Undetermined Coefficients:

$$ay'' + by' + cy = h(t)$$
$$y(t) = y_p(t) + y_h(t)$$

Form of function $h(t)$	Geuss for $y_p(t)$
$\sum_{j=0}^{N} B_j t^j$	$\sum_{j=0}^{N} A_j t^j$
$e^{\lambda t}$	$Ae^{\lambda t}$
$\sin \omega t$ or $\cos \omega t$	$A \sin \omega t + B \cos \omega t$
$e^{\lambda t} \sin \omega t$ or $e^{\lambda t} \cos \omega t$	$e^{\lambda t} A \sin \omega t + e^{\lambda t} B \cos \omega t$
Additive combinations of above	Additive combinations of above
Multiplicative combinations of above	Multiplicative combinations of above
Part of the homogeneous solution Note[1]	$Ath(t)$ or $At^2 h(t)$ or ...
Anything else	You are out of luck

[1]Note: This corresponds to resonance.

[2]Note: b_j , c_j , b, c, A, and B are all constants in the above table