Recall: Linear 1^{st} Order ODEs $\quad y' + p(t)y = g(t)$

Operator form: $\mathrm{L}\left[y\right] = g(t)$

- Linear 1^{st} order operator L
- $g(t)$ is called the **inhomogeneity**
- Linear $+\ g(t) = 0 \quad \Rightarrow$ **Homogeneous ODE**
- Linear $+\ g(t) \neq 0 \quad \Rightarrow$ **Inhomogeneous ODE**

Solved by method of integrating factors:

$$\mu(t)y(t) = \int \mu(t)g(t)dt + C$$

$$y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} + \underbrace{\frac{C}{\mu(t)}}_{\text{indep. of } g(t)}$$

General Solution Structure of Linear ODEs

ex: 1^{st} Order Initial Value Problems

$$y' + p(t)y = g(t); \qquad y(0) = y_0$$

General Solution:

$$y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)} + \frac{C}{\mu(t)}$$

Associated Homogeneous Problem:

$$y_h' + p(t)y_h = 0 \quad \text{(i.e., } g(t) = 0\text{)}$$

$$\Rightarrow \quad y_h = \frac{C}{\mu(t)}$$

$$y(t) = \underbrace{y_p(t)}_{\text{particular part}} + \underbrace{y_h(t)}_{\text{homogeneous part}}$$

All linear DEs have this type of solution structure.

$$y(t) = \text{particular part} + \text{homogeneous part}$$

Linear 2^{nd} order ODEs

General DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

Initial Conditions:

$$y(t_0) = y_0, \qquad y'(t_0) = v_0$$

Focus on the homogeneous case first

simplest case: constant coefficients

$$ay'' + by' + cy = 0$$

We want intuition for how to obtain and work with homogeneous solutions.

Find two solutions to $y'' - 2y' - 3y = 0$

Based on the constant coefficient first-order DEs, guess $y = e^{rt}$
We call e^{rt} an ansatz or "trial solution" (ansatz method)

$$y' = re^{rt} \qquad y'' = r^2 e^{rt}$$

$$r^2 e^{rt} - 2re^{rt} - 3e^{rt} = 0 \qquad \left(r^2 - 2r - 3\right)\cancel{e^{rt}} = 0$$

$$r^2 - 2r - 3 = 0$$

(characteristic polynomial)

$$r = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 3, -1$$

two solutions:

$$y_1 = e^{3t} \qquad \underline{\text{and}} \qquad y_2 = e^{-t}$$

Find two solutions to $t^2y'' - ty' - 3y = 0$

Use an ansatz of $y = t^k$. (Check the logic by solving $ty' - ay = 0$)

$$y' = kt^{k-1} \qquad y'' = k(k-1)t^{k-2}$$

$$t^2k(k-1)t^{k-2} - tkt^{k-1} - 3t^k = 0 \qquad (k^2 - 2k - 3)\cancel{t^k} = 0$$

$$k^2 - 2k - 3 = 0 \qquad \text{(characteristic polynomial)}$$

$$k = \frac{2 \pm \sqrt{4+12}}{2} = 3, -1$$

two solutions:

$$y_1 = t^3 \qquad \underline{\text{and}} \qquad y_2 = t^{-1}$$

Superposition Principle for Linear Homogeneous ODEs

Suppose the functions $y_1(t)$ and $y_2(t)$ both independently solve a linear homogeneous ODE

$$\mathrm{L}\left[y\right] = 0$$

then

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution to the same ODE.

Proof:

$$\begin{aligned}
\mathrm{L}\left[c_1 y_1 + c_2 y_2\right] &\overset{\text{Linearity 1}}{=} \mathrm{L}\left[c_1 y_1\right] + \mathrm{L}\left[c_2 y_2\right] \\
&\overset{\text{Linearity 2}}{=} c_1 \mathrm{L}\left[y_1\right] + c_2 \mathrm{L}\left[y_2\right] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}$$

Solve $y'' - 2y' - 3y = 0$ with $y(0) = 1,\ y'(0) = 0.$

General Solution

$$y(t) = c_1 e^{3t} + c_2 e^{-t}$$

initial conditions

$$y(0) = c_1 + c_2 = 1 \quad ①$$

$$y'(0) = 3c_1 - c_2 = 0 \quad ② \qquad \Rightarrow c_2 = 3c_1 \quad ③$$

$\underline{③ \to ①:}$ $4c_1 = 1$

$$\boxed{c_1 = \frac{1}{4}}$$

$$\boxed{c_2 = \frac{3}{4}}$$

$$y(t) = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}$$

Solve $t^2y'' - ty' - 3y = 0$ with $y(1) = 2,\ y'(1) = 1.$

General Solution

$$y(t) = c_1 t^3 + c_2 t^{-1}$$

initial conditions

$$y(1) = c_1 + c_2 = 2 \quad ①$$
$$y'(1) = 3c_1 - c_2 = 1 \quad ②$$

$$\underline{① + ②:}\ 4c_1 = 3 \qquad \Rightarrow \boxed{c_1 = \frac{3}{4}} \quad ③$$

$$\underline{③ \to ①:}\ \frac{3}{4} + c_2 = 2 \qquad \Rightarrow \boxed{c_2 = \frac{5}{4}}$$

$$y(t) = \frac{3}{4}t^3 + \frac{5}{4}t^{-1}$$

Satisfying Initial Conditions: The General Case

Given two functions y_1 and y_2 that both solve

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{with } y(t_0) = y_0, \quad y'(t_0) = v_0,$$

then the general solution is

$$y = c_1 y_1 + c_2 y_2.$$

Try to satisfy the initial condition

@$t = t_0$

$$\begin{aligned} c_1 y_1 + c_2 y_2 &= y_0 \\ c_1 y_1' + c_2 y_2' &= v_0 \end{aligned} \qquad \longrightarrow \underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{c}} \underset{=}{=} \underbrace{\begin{bmatrix} y_0 \\ v_0 \end{bmatrix}}_{\vec{b}}$$

Solution: $\quad \vec{c} = A^{-1}\vec{b} \quad$ if A^{-1} exists $\quad$ (i.e., if $\det A \neq 0$)

Note: $\det A$ is called the Wronskian determinant.

Wronskian Determinant

For two differentiable functions, $y_1(t)$ and $y_2(t)$, the Wronskian determinant is denoted $W\left(y_1, y_2\right)(t)$, and is given by

$$W\left(y_1, y_2\right)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

- $W\left(y_1, y_2\right)(t_0) \neq 0 \implies y = c_1 y_1 + c_2 y_2$ can satisfy ANY initial conditions at t_0.

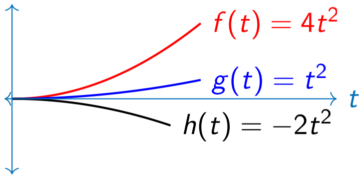
- $W\left(y_1, y_2\right)(t) \neq 0$ for some t is equivalent to stating that y_1 and y_2 are linearly independent functions.

Linear dependence of functions

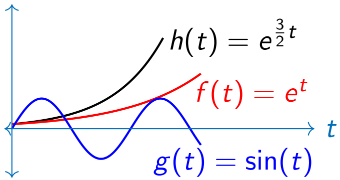
- Two functions $f(t)$ and $g(t)$ are **linearly dependent** (or L.D.) if there exist a non-zero constant k such that

$$f(t) = kg(t) \quad \forall t$$

Linearly Dependent



Not Linearly Dependent



- If functions are not linearly dependent, then we say they are **linearly independent** (or L.I.).

Fundamental sets of solutions

Suppose that $y_1(t)$ and $y_2(t)$ both solve a 2^{nd} order linear homogeneous ODE

$$\mathrm{L}\,[y] = 0$$

If y_1 and y_2 are **linearly independent** functions, then the we call the set

$$\{y_1, y_2\}$$

a **fundamental set of solutions**, and the general solution for all homogeneous problem IVPs is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

Take home message: If you can find two linearly independent solutions to a homogeneous 2^{nd} order linear DE, then you have found all of them

Proof: DiffyQs §2.1 (Theorem 2.1.3)

Constant Coefficient IVP: The General Case

$$ay'' + by' + cy = 0 \qquad \text{with } y(t_0) = y_0, \quad y'(t_0) = v_0.$$

Guess $y = e^{rt}$.

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \qquad ar^2 + br + c = 0 \quad \text{(char. poly.)}$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \Rightarrow \begin{array}{l} y_1 = e^{r_1 t} \\ y_2 = e^{r_2 t} \end{array}$$

check for linear independence

$$\begin{aligned} W &= e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t} \\ &= r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} \\ &= (r_2 - r_1) e^{(r_1+r_2)t} \\ &\neq 0 \qquad \text{if } r_1 \neq r_2 \qquad \Rightarrow y_1 \text{ and } y_2 \text{ are L.I.} \end{aligned}$$

Fundamental solution: $\quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Summary: 2^{nd} Order Homogeneous ODEs

$$y'' + p(t)y' + q(t)y = 0$$

- Always has two linearly independent solutions y_1 and y_2.
  - Ansatz method allows us to guess them.
  - Constant coefficients: $ay'' + by' + cy = 0 \quad \Rightarrow y = e^{rt}$
- Superposition Principle:
  - The linear combination $y = c_1 y_1 + c_2 y_2$ also solves the homogeneous ODE.
- Fundamental Solutions:
  - Linear combinations of y_1 and y_2 can solve all IVPs.
  - Use the Wronskian to determine linear independence and establish a fundamental set of solutions.