Recall:

$$y' + ay = g(t) \qquad \text{with } g(t) \neq 0$$

has the general solution

$$y = y_h + y_p \qquad \text{where } y_h = Ce^{-at}$$

$y_h' + ay_h = 0$, so we need y_p and its derivative to be related to $g(t)$.

Idea:

Make y_p a linear combination of all the functional forms contained in $g(t)$ and all the functions obtained from differentiating $g(t)$

Method of Undetermined Coefficients

$$y' + ay = g(t)$$

ex: $y' - y = t^3 \implies y_h = Ce^t$

To find y_p:

1. Find the functional forms obtained from differentiating $g(t)$.
   ex: $g(t) = t^3$

$$g' = 3\underline{t^2}$$
$$g'' = 6\underline{t}$$
$$g''' = \underline{6} \qquad \text{(constant)}$$

$$\text{family of functional forms } = \{t^3, t^2, t, \text{constant}\}$$

2. Take a linear combintation of all the functional forms you find.

$$y_p = At^3 + Bt^2 + Ct + D$$

Method of Undetermined Coefficients

To find y_p:

1. Find the functional forms obtained from differentiating $g(t)$.
2. Take a linear combintation of all the functional forms you find.
3. Plug the guess back into the ODE, solve for coefficients.
   ex: $y' - y = t^3$

$$y_p = At^3 + Bt^2 + Ct + D \qquad y_p' = 3At^2 + 2Bt + C$$

plug into DE

$$y_p' - y_p = 3At^2 + 2Bt + C - At^3 - Bt^2 - Ct - D = t^3$$

$$-At^3 + (3A - B)t^2 + (2B - C)t + (C - D) = t^3$$

Method of Undetermined Coefficients

To find y_p:

1. Find the functional forms obtained from differentiating $g(t)$.
2. Take a linear combintation of all the functional forms you find.
3. Plug the guess back into the ODE, solve for coefficients.

   ex: $y' - y = t^3$

$$-At^3 + (3A - B)t^2 + (2B - C)t + (C - D) = t^3$$

match coeffs. for each function of t

$$\underline{t^3:}\quad -A = 1 \implies \boxed{A = -1}$$

$$\underline{t^2:}\quad 3A - B = 0 \implies \boxed{B = -3}$$

$$\underline{t^1:}\quad 2B - C = 0 \implies \boxed{C = -6}$$

$$\underline{t^0:}\quad C - D = 0 \implies \boxed{D = -6}$$

$$\boxed{y_p = -t^3 - 3t^2 - 6t - 6}$$

Find the general solution to $y' - 5y = e^t$

$$y_h = Ce^{5t}$$

Family of functional forms: $\{e^t\}$

Guess: $y_p = Ae^t$

Substitute: $Ae^t - 5Ae^t = e^t$

Solve for A:

$$-4Ae^t = e^t \Rightarrow A = -\frac{1}{4}$$

$$y_p = -\frac{1}{4}e^t$$

$$\boxed{y = Ce^{5t} - \frac{1}{4}e^t}$$

Find the general solution to $y' + y = \sin(t)$

$$y_h = Ce^{-t}$$

Family of functional forms: $\{\sin(t), \cos(t)\}$

Guess: $y_p = A\sin(t) + B\cos(t)$

Sub into DE:

$$A\cos(t) - B\sin(t) + A\sin(t) + B\cos(t) = \sin(t)$$
$$(A - B)\sin(t) + (A + B)\cos(t) = \sin(t)$$

Group by funcs of t:

$\underline{\sin(t):}$ $A - B = 1$

$$-2B = 1 \implies B = -\frac{1}{2}$$

$\underline{\cos(t):}$ $A + B = 0 \Rightarrow A = -B$

$$\implies A = \frac{1}{2}$$

$$y_p = \frac{1}{2}(\sin(t) - \cos(t))$$

$$\boxed{y(t) = Ce^{-t} + \frac{1}{2}(\sin(t) - \cos(t))}$$

Find the general solution to $y' - 5y = e^{5t}$

$$y_h = Ce^{5t}$$

Family of functional forms: $\{e^{5t}\}$

Guess: $y_p = Ae^{5t}$

Substitute:

$$A5e^{5t} - 5Ae^{5t} = e^{5t}$$

$$0 = e^{5t}$$

We are stuck :(

This is called <u>mathematical resonance</u>, one of the functional forms is proportional to y_h.

Guess: $y_p = tAe^{5t}$

Compute Derivative':

$$y_p' = Ae^{5t} + 5tAe^{5t}$$

Substitute:

$$Ae^{5t} + 5tAe^{5t} - 5Ae^{t} = e^{5t}$$

Solve for A:

$$Ae^{5t} = e^{5t} \Rightarrow A = 1$$

$$y_p = Ae^{5t}$$

$$\boxed{y = Ce^{5t} + te^{5t}}$$

Summary of Lecture 3

Method of Undetermined Coefficients:

$$y' + ay = g(t)$$

General Solution:

$$y = y_h + y_p \qquad \text{where } y_h = Ce^{-at}$$

1. Find the functional forms obtained from differentiating $g(t)$.
2. Take a linear combintation of all the functional forms you find.
   - If any of the functional forms are e^{-at} swap for te^{at}
3. Plug the guess back into the ODE, solve for coefficients.

Note: this will not work if there infinitely many functional forms.

- e.g., $g(t) = t^{-1}$ family $= \{t^{-1}, t^{-2}, t^{-3}, \ldots\}$