Laplace Transform of an IVP

Given

$$ay'' + by' + cy = f(t) \quad y(0) = y_0, y'(0) = v_0$$

we want to find $Y(s)$.

$$\begin{aligned} \mathcal{L}\left\{ay'' + by' + cy\right\} &= \mathcal{L}\left\{f(t)\right\} \\ a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + cY(s) &= F(s) \end{aligned}$$

In order to isolate $Y(s)$, we need to be able to compute $\mathcal{L}\left\{y'\right\}$ and $\mathcal{L}\left\{y''\right\}$.

Laplace Transform of Derivatives

Suppose $\mathcal{L}\{y(t)\} = Y(s)$. What is $\mathcal{L}\{y'(t)\}$?

$$\mathcal{L}\{y'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{y'(t)dt}_{dv}$$

$$v = y(t)$$
$$du = -se^{-st}dt$$

$$= e^{-st}y(t)\Big|_0^\infty - (-s)\underbrace{\int_0^\infty e^{-st}y(t)dt}_{\mathcal{L}\{y(t)\}}$$

$$\stackrel{s \geq 0}{=} -y(0) + s\mathcal{L}\{y(t)\}$$

$$= \boxed{sY(s) - y_0}$$

Laplace Transform of Derivatives

Given that $\mathcal{L}\{y'(t)\} = sY(s) - y_0$. What is $\mathcal{L}\{y''(t)\}$?

$$\mathcal{L}\{y''(t)\} = s\mathcal{L}\{y'(t)\} - y'(0) = s[sY(s) - y_0] - \underbrace{y'(0)}_{v_0}$$

$$= \boxed{s^2 Y(s) - sy_0 - v_0}$$

Laplace Transform of an IVP

Given

$$ay'' + by' + cy = f(t) \quad y(0) = y_0, y'(0) = v_0$$

we want to find $Y(s)$.

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + cY(s) = F(s)$$
$$a(s^2Y(s) - sy_0 - v_0) + b(sY(s) - y_0) + cY(s) = F(s)$$

$$\underbrace{(as^2 + bs + c)}_{\text{char. poly.}} Y(s) = F(s) + (as + b)y_0 + av_0$$

$$Y(s) = \frac{F(s)}{as^2 + bs + c} + \frac{(as + b)y_0 + av_0}{as^2 + bs + c}$$

ex: Use Laplace Transforms to solve $y' + 6y = 3$ with $y(0) = 2$.

$$\begin{aligned}
(\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= \mathcal{L}\{3\} \\
\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= \frac{3}{s} \\
sY(s) - \underbrace{y(0)}_{2} + 6Y(s) &= \frac{3}{s} \\
(s+6)Y(s) &= \frac{3}{s} + 2 \\
Y(s) &= \frac{3}{s(s+6)} + \frac{2}{s+6}
\end{aligned}$$

This is the example from pg. 8 of lecture 12.

$$y(t) = \frac{1}{2} + \frac{3}{2}e^{-6t}$$

ex: Solve $y'' + 6y' + 25y = 0$ with $\begin{array}{l} y(0) = 0 \\ y'(0) = 4 \end{array}$ using Laplace Trans.

$$s^2 Y(s) - s\cancel{y(0)} - y'(0) + 6(sY(s) - \cancel{y(0)}) + 25Y(s) = 0$$

$$(s^2 + 6s + 25)Y(s) = \underbrace{y'(0)}_{4}$$

$$Y(s) = \frac{4}{s^2 + 6s + 25}$$

This is the example from pg. 5 of lecture 12.

$$y(t) = e^{-3t}\sin(4t)$$

ex: Solve $y'' + 6y' + 25y = -8e^{-4t}$ with $\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}$ using Lap. Trans.

$$s^2 Y(s) - s\cancel{y(0)} - y'(0) + 6(sY(s) - \cancel{y(0)}) + 25Y(s) = \frac{-8}{s+4}$$

$$\underbrace{(s^2 + 6s + 25)}_{(s+3)^2+16} Y(s) = \frac{-8}{s+4} + 1 = \frac{s-4}{s+4}$$

$$Y(s) = \frac{s-4}{((s+3)^2+16)(s+4)}$$

This is the example from pg. 12 of lecture 12.

$$y(t) = e^{-3t}\left(\frac{8}{17}\cos(4t) + \frac{9}{68}\sin(4t)\right) - \frac{8}{17}e^{-4t}$$

ex: Solve $y'' - 3y' - 4y = 8e^{4t}$ with $\begin{matrix} y(0) = 0 \\ y'(0) = 1 \end{matrix}$ using Laplace Trans.

$$s^2Y(s) - s\cancel{y(0)} - y'(0) - 3(sY(s) - \cancel{y(0)}) - 4Y(s) = \frac{-8}{s+4}$$

$$\underbrace{(s^2 - 3s - 4)}_{(s-4)(s+1)} Y(s) = \frac{8}{s-4} + 1 = \frac{s+4}{s-4}$$

$$Y(s) = \frac{s+4}{(s-4)^2(s+1)}$$

This is the example from pg. 10 of lecture 12.

$$y(t) = \underbrace{\frac{40}{25}te^{4t}}_{y_p} - \underbrace{\frac{3}{25}e^{4t} + \frac{3}{25}e^{-t}}_{y_h}$$

This is a case with mathematical resonance, where we mulitply our guess for y_p by t.

Multiplication by $t \Leftrightarrow$ Differentiation in s-domain

$$\mathcal{L}\left\{t^k f(t)\right\} = \int_0^\infty e^{-st} t^k f(t) dt \qquad = \int_0^\infty \underbrace{e^{-st} t}_{-\frac{\mathrm{d}}{\mathrm{d}s} e^{-st}} \; t^{k-1} f(t) dt$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} t^{k-1} f(t) dt \qquad \underset{\text{k-1 more times}}{\overset{\text{repeat same thing}}{\cdots}} \qquad = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} F(s)$$

with k=1

$$\mathcal{L}\left\{t f(t)\right\} = -\frac{\mathrm{d}}{\mathrm{d}s} F(s)$$

$$ex : \mathcal{L}\left\{t \sin(\omega t)\right\} = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

$$\mathcal{L}\left\{t \cos(\omega t)\right\} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

ex: Solve $y''+4y = 4\cos(2t)$ with $\begin{array}{l} y(0)=0 \\ y'(0)=0 \end{array}$ using Laplace Transforms

$$s^2Y(s) + 4Y(s) = \frac{4s}{s^2+4}$$

$$Y(s) = \frac{4s}{(s^2+4)^2} = \underbrace{\frac{2\omega s}{(s^2+\omega^2)^2}}_{\mathcal{L}\{t\sin(\omega t)\}} \quad \text{with } \omega = 2$$

$$y(t) = t\sin(2t)$$

ex: Find the Laplace Tranform of $f(t) = t^2 e^{3t}$

$$
\begin{aligned}
F(s) &= (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathcal{L}\left\{e^{3t}\right\} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}s^2} \frac{1}{s-3} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \frac{-1}{(s-3)^2} \\
&= \frac{2}{(s-3)^3}
\end{aligned}
$$

ex: Solve $y'' - 6y' + 9y = e^{3t}$ with $\begin{array}{l} y(0) = 0 \\ y'(0) = 0 \end{array}$ using Laplace Transforms

$$\underbrace{(s^2 - 6s + 9)}_{(s-3)^2} Y(s) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-3)^2}$$

$$y(t) = \frac{1}{2} t^2 e^{3t}$$