D'Alembert's Method: $y_{tt} = c^2 y + xx$

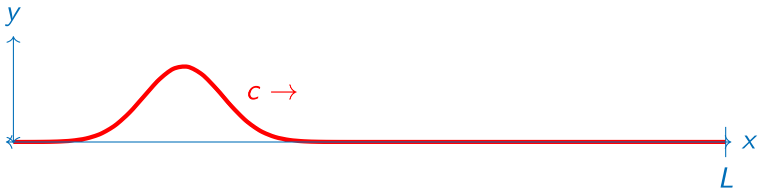
The Separation of Variables/Fourier method can be used to analyze the spectrum of frequencies present in the wave.

Useful to understand music - not useful to visualize a wave.

D'Alembert's Method: an alternative approach based on change of coordinates:

$$\xi = x - ct, \qquad \eta = x + ct$$

Basic idea, solutions are waves travelling with speed $\pm c$.



D'Alembert's Method

$$\xi = x - ct, \qquad \eta = x + ct$$

Let's compute $\dfrac{\partial^2}{\partial x^2}$ in this new coordinate system.

Apply the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cancelto{1}{\frac{\partial \xi}{\partial x}} \frac{\partial}{\partial \eta} + \cancelto{1}{\frac{\partial \eta}{\partial x}} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\
\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\
&= \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}
\end{aligned}$$

D'Alembert's Method

$$\xi = x - ct, \qquad \eta = x + ct$$

Let's compute $\dfrac{\partial^2}{\partial t^2}$ in this new coordinate system.

Apply the chain rule:

$$\frac{\partial}{\partial t} = \cancelto{-c}{\frac{\partial \xi}{\partial t}} \frac{\partial}{\partial \eta} + \cancelto{+c}{\frac{\partial \eta}{\partial x}} \frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial}{\partial t} = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)$$

$$= c^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)$$

D'Alembert's Method

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \qquad \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2}$$

Write down the wave equation in the new coordinate system.

$$y_{tt} = c^2 y_{xx}$$

$$c^2\frac{\partial^2 y}{\partial \xi^2} - 2c^2\frac{\partial^2 y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 y}{\partial \eta^2} = c^2\frac{\partial^2 y}{\partial \xi^2} + c^2 2\frac{\partial^2 y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 y}{\partial \eta^2}$$

$$0 = 4c^2\frac{\partial^2 y}{\partial \xi \partial \eta}$$

D'Alembert's Method

$$0 = \frac{\partial^2 y}{\partial \xi \partial \eta}$$

Find the general solution to the wave equation:

Drop the multiplicative constants, integrate w.r.t. ξ

$$y_\eta(\xi, \eta) = \int \frac{\partial^2 y}{\partial \xi \partial \eta} d\xi = \int 0 d\xi = C(\eta)$$

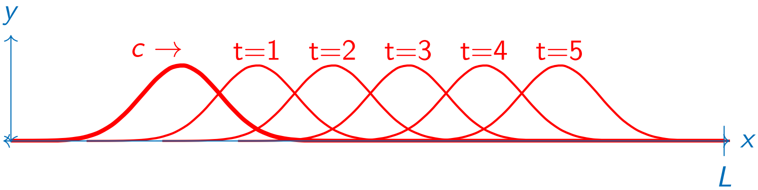
$$y(\xi, \eta) = \int C(\eta) d\eta = A(\eta) + B(\xi)$$

where A and B are single variable functions.

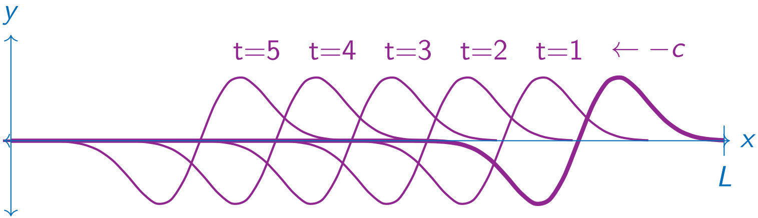
$$y(x, t) = A(x - ct) + B(x + ct)$$

$$y(x,t) = A(x-ct) + B(x+ct)$$

$A(x-ct)$: moves rightwards



$B(x+ct)$: moves leftwards



These are just two unrelated examples, the wave profiles A and B are not completely different like in this slide.

D'Alembert's Formula

General Solution to:

$$y_{tt} = c^2 y_{xx} \quad 0 < x < L \qquad\qquad y(0) = y(L) = 0,$$
$$y(x,0) = f(x) \qquad\qquad y_t(x,0) = g(x)$$

$$\boxed{y(x,t) = A(x-ct) + B(x+ct)}$$

Let F be the odd periodic extension of $f(x)$, and G be the odd periodic extension of $g(x)$. Then we have

$$A(z) = \frac{1}{2}\left[F(z) - \frac{1}{c}\int_0^z G(x)dx\right] \quad \text{and} \quad B(z) = \frac{1}{2}\left[F(z) + \frac{1}{c}\int_0^z G(x)dx\right]$$

Proof: DiffyQs §3.8.2

Intepreting D'Alembert's Formula

$$y_{tt} = c^2 y_{xx} \quad 0 < x < L \qquad\qquad y(0) = y(L) = 0,$$
$$y(x,0) = f(x) \qquad\qquad y_t(x,0) = g(x)$$

$$y(x,t) = A(x-ct) + B(x+ct) \qquad \text{with}$$
$$A(z) = \frac{1}{2}\left[F(z) - \frac{1}{c}\int_0^z G(x)dx\right] \quad \text{and} \quad B(z) = \frac{1}{2}\left[F(z) + \frac{1}{c}\int_0^z G(x)dx\right]$$

F and G are and odd periodic extensions of $f(x)$ and $g(x)$, respectively.

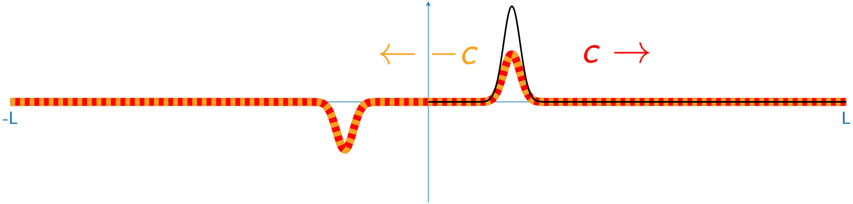
Suppose $g(x) = 0$, then $G(x) = 0$ so

$$A(x) = B(x) = \frac{1}{2}F(z)$$

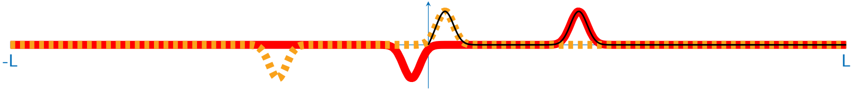
We have odd extensions of the initial condition moving left and right.

This leads to reflection of the wave at the domain boundaries.

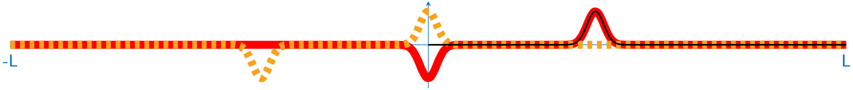
$t = 0$: A and B sum to give the initial condition [Desmos Version]



$t = 0.8$: B hits the boundary at $x = 0$



$t = 1$: primary wave from B annihilates with the odd extension of A



$t = 1.8$: wave is reflected after hitting boundary