Recall: D'Alembert's Solution

$$y_{tt} = c^2 y_{xx} \qquad\qquad 0 < x < L$$
$$y(x,0) = f(x) \qquad\qquad y_t(x,0) = g(x)$$

$$y(x,t) = A(x-ct) + B(x+ct) \qquad \text{with}$$

$$A(z) = \frac{1}{2}\left[F(z) - \frac{1}{c}\int_0^z G(x)dx\right] \quad \text{and} \quad B(z) = \frac{1}{2}\left[F(z) + \frac{1}{c}\int_0^z G(x)dx\right]$$

F and G are and odd periodic extensions of $f(x)$ and $g(x)$, respectively.

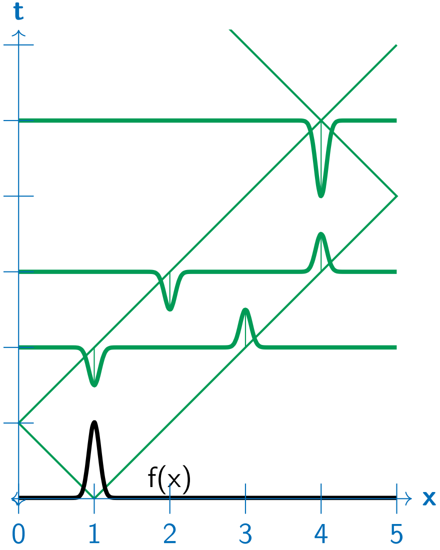
Simple case: g(x)=0

The initial condition $f(x)$ splits into two waves that move in opposite direction with speed c and reflect off the boundaries.

$$y_{tt} = y_{xx} \qquad 0 < x < 5$$
$$y(x,0) = f(x) \qquad y_t(x,0) = 0$$

Suppose $f(x)$ is zero everywhere except very close to x=1 where it has a local maxmimum.

Find the first time and location where the solution has a single local extremum for $t > 0$.



Given $\quad y_{tt} = 25 y_{xx} \qquad\qquad y(0,t) = y(3,t) = 0,$

$y(x,0) = y_0(x) \quad$ for $0 < x < 3 \qquad\qquad y_t(x,0) = 0$

Find the solution $y(1,1000)$. We have $c = 5$ and $L = 3$.

D'Alembert's solution:

$$y(1,1000) = \frac{F(1 - c\cdot 1000) + F(1 + c\cdot 1000)}{2}$$

where F is an <u>odd</u> periodic extension of the initial condition

$$F(z) = \begin{cases} y_0(z) & 0 \le z \le 3 \\ -y_0(-z) & -3 < z < 0 \end{cases} \qquad \text{with } F(z + n6) = F(z) \quad \forall n \in \mathbb{N}$$

Unfortunately $x \pm ct = \underbrace{1 \pm 5000}_{-4999,\ 5001}$ is not in the interval $(-3,3]$

We exploit the periodicity property $F(z + n6) = F(z)$ to map -4999 and 5001 onto $(-3, 3]$.

Need to find two appropriate integers n_1 and n_2

$$
\begin{aligned}
-3 &< -4999 + n_1 6 \leq 3 \\
4996 &< n_1 6 \leq 5002 \\
\frac{4996}{6} &< n_1 \leq \frac{5002}{6} \\
832.66 &< n_1 \leq 833.6 \\
n_1 &= 833
\end{aligned}
$$

$$
\begin{aligned}
-3 &< 5001 + n_2 6 \leq 3 \\
-5004 &< n_2 6 \leq -4998 \\
\frac{-5004}{6} &< n_2 \leq \frac{-4998}{6} \\
-833 &< n_2 \leq -834 \\
n_2 &= -834
\end{aligned}
$$

So,

$$-4999 \to -4999 + 833 \cdot 6 = -1 \qquad 5001 \to 5001 - 834 \cdot 6 = -3$$

$$-4999 \to -1 \qquad 5001 \to -3$$

Finally

$$y(1,1000) = \frac{F(-1) + F(-3)}{2}$$

where

$$F(z) = \begin{cases} y_0(z) & 0 \le z \le 3 \\ -y_0(-z) & -3 < z < 0 \end{cases}$$

so

$$y(1,1000) = -\frac{y_0(1) + y_0(3)}{2}$$

<u>ex</u>: $y_0(x) = x^3 - \frac{9}{2}x^2 + 7$

$$y(1,1000) = -\frac{1 - \frac{9}{2} + 7 + 27 - \frac{9}{2} \cdot 9 + 7}{2} = \frac{3}{2}$$

D'Alembert's Solution with other Boundary Conditions

The solution method is agnostic to boundary conditions

$$y_{tt} = c^2 y_{xx} \qquad a < x < b$$

$$y(x,0) = f(x) \qquad y_t(x,0) = g(x)$$

$$\boxed{\begin{gathered} y(x,t) = A(x-ct) + B(x+ct) \qquad \text{with} \\ A(z) = \frac{1}{2}\left[F(z) - \frac{1}{c}\int_a^z G(x)dx\right] \quad \text{and} \quad B(z) = \frac{1}{2}\left[F(z) + \frac{1}{c}\int_a^z G(x)dx\right] \end{gathered}}$$

ex: $0 < x < L \quad y_x(0,t) = y_x(L,t) = 0$

F and G are even periodic extensions of $f(x)$ and $g(x)$, respectively.

ex: $-\infty < x < \infty$

$F = f$

$G = g$