Fourier Series of T-periodic function

Expand $f(x)$ in the Fourier basis

$$\mathrm{FS}[f(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right)$$

$$a_n = \left\langle f(x), \cos\left(\frac{2n\pi}{T}x\right)\right\rangle = \int_{\alpha}^{\alpha+T} f(x)\cos\left(\frac{2n\pi}{T}x\right) dx$$

$$b_n = \left\langle f(x), \sin\left(\frac{2n\pi}{T}x\right)\right\rangle = \int_{\alpha}^{\alpha+T} f(x)\sin\left(\frac{2n\pi}{T}x\right) dx$$

The Fourier basis is orthonormalized

$$\left\langle \sin\left(\frac{2n\pi}{T}x\right), \sin\left(\frac{2m\pi}{T}x\right)\right\rangle = \begin{cases} 1 & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Homogeneous Heat Equation

Let $u(x,t)$ be a function of space $0 < x < L$ and time t.

$u_t = \alpha u_{xx}$ with $u(x,0) = u_0(x)$ for $x \in (0,L)$ and

1) $u(0,t) = u(L,t) = 0$

or

2) $u_x(0,t) = u_x(L,t) = 0$

Fundamental Solution (Separation of Variables):

$$u_n = e^{-\alpha \frac{n^2\pi^2}{L^2} t}\left(A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)\right) \qquad n = 0, 1, \ldots, \infty$$

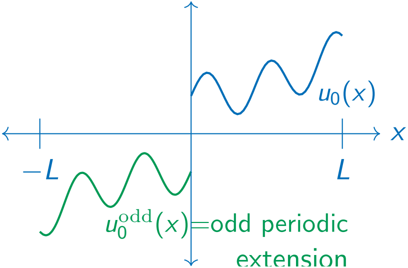
Initial Condition:

$$u(x,0) = u_0(x) = \sum_{n=0}^{\infty} u_n(x,0) = \underbrace{\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)}_{\text{Fourier Series of a } 2L\text{-periodic func.}}$$

Case 1: $u_t = \alpha u_{xx}$ with $u(x,0) = u_0(x)$ for $x \in (0,L)$ and $u(0,t) = u(L,t) = 0$

Only sin terms satisfy the BCs $\Rightarrow A_n = 0$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$



FS is 2L periodic, but u_0 is only defined for $x \in (0,L)$

⋮

Extend u_0 as an odd func. for $x \in [-L, L]$

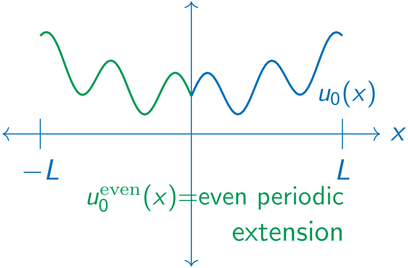
Then

$$B_n = \frac{2}{2L}\int_{-L}^{L} \underbrace{\overbrace{u_0^{\mathrm{odd}}(x)}^{\text{odd func.}}\,\overbrace{\sin\left(\frac{n\pi}{L}x\right)}^{\text{odd func.}}}_{\text{even func.}}\,dx = \frac{2}{L}\int_{0}^{L} u_0(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Case 2: $u_t = \alpha u_{xx}$ with $u(x,0) = u_0(x)$ for $x \in (0, L)$ and $u_x(0,t) = u_x(L,t) = 0$

Only cos terms satisfy the BCs $\Rightarrow B_n = 0$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$



FS is 2L periodic, but u_0 is only defined for $x \in (0, L)$

$\vdots$

Extend u_0 as even func. for $x \in [-L, L]$

Then

$$A_n = \frac{2}{2L}\int_{-L}^{L} = \underbrace{\overbrace{u_0^{\text{even}}(x)}^{\text{even func.}}\overbrace{\cos\left(\frac{n\pi}{L}x\right)}^{\text{even func.}}}_{\text{even func.}} dx = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{n\pi}{L}x\right) dx$$