Homogeneous Heat Equation

$u_t = \alpha u_{xx}$ for $x \in [0, L]$ with $u(0,t) = u(L,t) = 0$ <u>or</u> $u_x(0,t) = u_x(L,t) = 0$ and $u(x,0) = u_0(x)$

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2\pi^2}{L^2} t} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)}$$

$\underline{u(0,t) = u(L,t) = 0}$:

Fourier sine series

$$a_n = 0$$
$$b_n = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$\underline{u_x(0,t) = u_x(L,t) = 0}$:

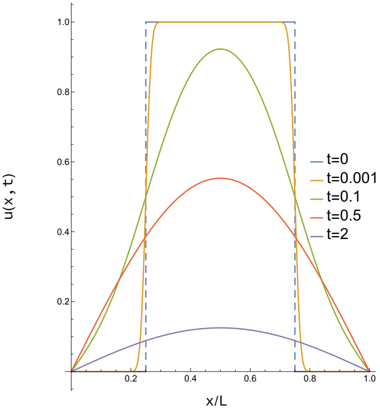
Fourier cosine series

$$a_n = \frac{2}{L}\int_0^L u_0(x) \cos\left(\frac{n\pi}{L}x\right) dx$$
$$b_n = 0$$

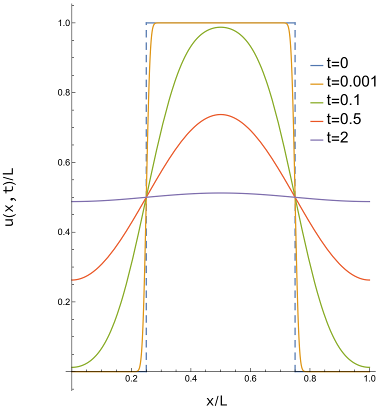
The heat equation smooths out initial conditions.

ex: $u_0(x) = u\left(x - \frac{L}{4}\right) - u\left(x - \frac{3L}{4}\right)$

$u(0,t) = u(L,t) = 0$:

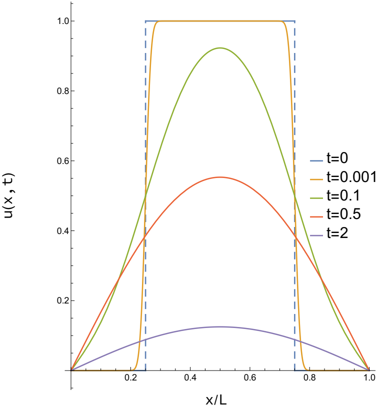


$u_x(0,t) = u_x(L,t) = 0$:



Short-term Behaviour

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{e^{-\alpha \frac{n^2\pi^2}{L^2} t}}_{\text{Exp. Decay}} \cdot \underbrace{\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)}_{n^{th} \text{ Fourier Mode}}$$



- Sharp edges in the initial condition require high frequncy modes ($n \gg 1$).
- The exp. decay rate grows with n^2.
- High frequency modes decay the fastest.
- Sharp features get smoothed out very quickly.

Long-term Behaviour

As $t \to \infty$,

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha\frac{n^2\pi^2}{L^2}t}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)\\ &= \frac{a_0}{2}\end{aligned}$$

For finite $t \gg 0$, we can just keep the first non-zero term in the infinite sum. All other terms decay much much faster.

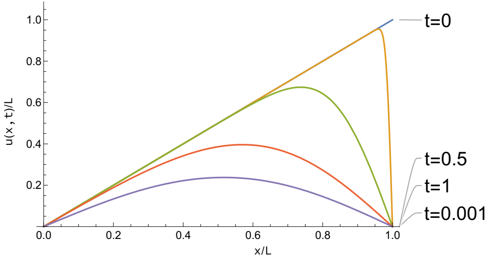
$\underline{u(0,t) = u(L,t) = 0}$:

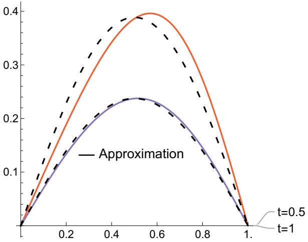
$$u(x,t) \approx e^{-\alpha\frac{\pi^2}{L^2}t} b_1 \sin\left(\frac{\pi}{L}x\right)$$

$\underline{u_x(0,t) = u_x(L,t) = 0}$:

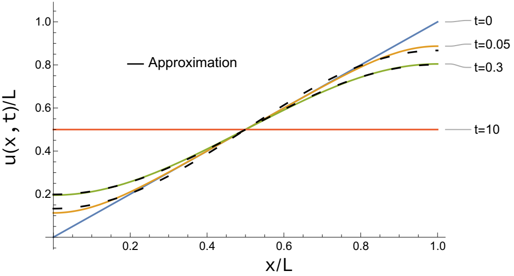
$$u(x,t) \approx \frac{a_0}{2} + e^{-\alpha\frac{\pi^2}{L^2}t} a_1 \cos\left(\frac{\pi}{L}x\right)$$

$\underline{u(0,t) = u(L,t) = 0}$: $u(x,t) \approx e^{-\alpha \frac{\pi^2}{L^2} t} b_1 \sin\left(\frac{\pi}{L} x\right)$

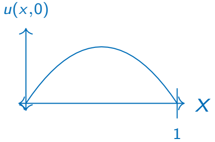




$\underline{u_x(0,t) = u_x(L,t) = 0}$: $u(x,t) \approx \frac{a_0}{2} + e^{-\alpha \frac{\pi^2}{L^2} t} a_1 \cos\left(\frac{\pi}{L} x\right)$



ex: $u_t = \alpha u_{xx}$, $\quad u(0,t) = u(1,t) = 0$
$u(x,0) = x(1-x)$ on $[0,1]$



BCs $\Rightarrow$ only sin terms survive.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = x(1-x)$$

$$b_n = \frac{2}{1}\int_0^1 u(x,0)\sin(n\pi x)dx = 2\int_0^1 x(1-x)\sin(n\pi x)dx$$

$$\overset{\text{wolfram}}{=} 2\left(\frac{-2((-1)^n - 1)}{n^3\pi^3}\right) \qquad \Rightarrow \qquad \boxed{b_n = -\frac{4}{\pi^3}\frac{(-1)^n - 1}{n^3}}$$

https://www.wolframalpha.com/input?i=integral+of+x%281-x%29sin%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer

Given

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{4}{\pi^3} \frac{(-1)^n - 1}{n^3} \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

suppose that after 100 time units the hottest point in the domain is at half of its initial temperature, find an approximation for α.

Initial Condition: $u_0(x) = x(1-x)$

$$\Rightarrow x_{\max} = \frac{1}{2} \qquad\qquad u_0(x_{\max}) = \frac{1}{4}$$

Long-term: $u(x,t) \approx \dfrac{8}{\pi^3} \sin(\pi x) e^{-\alpha \pi^2 t}$

approximation also has its max at $x = 1/2$

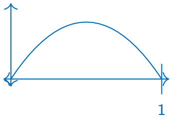
$$\frac{8}{\pi^3} e^{-\alpha \pi^2 100} \approx \frac{1}{8}$$

$$\alpha \approx -\frac{1}{100\pi^2} \log\left(\frac{\pi^3}{64}\right)$$

$$\alpha \approx 7.34 \times 10^{-4}$$

ex: $u_t = 0.1 u_{xx}$, $\quad u_x(0,t) = u_x(1,t) = 0$

$u(x,0) = x(1-x)$ on $[0,1]$



BCs $\Rightarrow$ only cos terms survive.

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-0.1 n^2 \pi^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = x(1-x)$$

$$a_n = \frac{2}{1}\int_0^1 u(x,0)\cos(n\pi x)dx = 2\int_0^1 x(1-x)\sin(n\pi x)dx$$

$$\overset{\text{wolfram}}{=} -2\frac{(-1)^n + 1}{\pi^2 n^2}$$

$$a_0 = \frac{2}{1}\int_0^1 x(1-x)dx = \frac{2}{6}$$

https://www.wolframalpha.com/input?i=integral+of+x%281-x%29cos%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer

https://www.wolframalpha.com/input?i=integral+of+x%281-x%29+from+0+to+1

Given

$$u(x,t) = \frac{1}{6} - 2\sum_{n=1}^{\infty} \frac{(-1)^n + 1}{\pi^2 n^2} \cos(n\pi x) e^{-0.1n^2\pi^2 t}$$

approximately how long does it take for the endpoints of the domain to be within 1% of their steady state value?

$n = 1$ has a coefficient of zero, use $n = 2$ instead for approximation.

$$\text{Long-term: } u(x,t) \approx \frac{1}{6} - 2\frac{2}{4\pi^2}\ \cos(2\pi x)e^{-0.4\pi^2 t}$$

$$\text{Steady state: } u_\infty(x) = \frac{1}{6}$$

$$u_\infty(x) - u(L,t) \approx \frac{1}{\pi^2}\cos(2\pi x)e^{-0.4\pi^2 t}$$

$$x = 0 \text{ or } x = 1\text{: } \frac{1}{\pi^2}e^{-0.4\pi^2 t} \approx 1\% \times \frac{1}{6}$$

$$t \approx -\frac{1}{0.4\pi^2}\log\left(\frac{0.01\pi^2}{6}\right)$$

$$\approx 1.04 \text{ time units}$$