Recall

We saw that homogeneous constant coefficient second order linear IVPs

$$ay'' + by' + cy = 0, \quad \text{with } y(0) = y_0,\ y'(0) = v_0$$

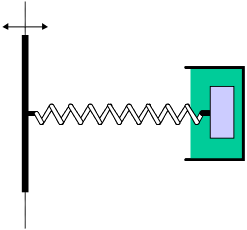
have a general solution

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad \text{with } r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ for } r_1 \neq r_2.$$

Three Questions:

1. What applications motivate these ODEs?
2. How do these solutions behave?
3. What happens when $r_1 = r_2$?

Derivation of spring-dashpot ODE:



$x(t)$ = displacement from rest position

- $x = 0 \Rightarrow$ no elastic restoring force

Newton's 2^{nd} Law:

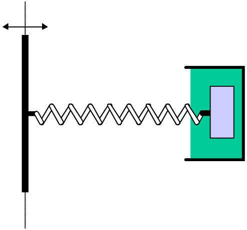
$$F = ma \qquad \text{where } a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

F = sum of forces

$$= \underbrace{\text{elastic restoring force}}_{\substack{\text{Hooke's Law} \\ = -kx}} + \underbrace{\text{drag force}}_{\substack{\text{opposes motion} \\ = -\beta \frac{\mathrm{d}x}{\mathrm{d}t}}} + \underbrace{\text{external forces}}_{f(t)}$$

$$= -kx - \beta \frac{\mathrm{d}x}{\mathrm{d}t} + f(t)$$

Derivation of spring-dashpot ODE:



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Newton's 2^{nd} Law:

$$F = ma \qquad \text{where } a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

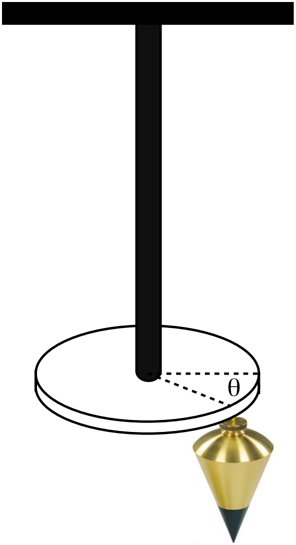
$$F = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t) \qquad \Rightarrow m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t)$$

$$\boxed{mx'' + \beta x' + kx = f(t)}$$

With $f(t) = 0$ (no external forcing) we get an ODE of the form

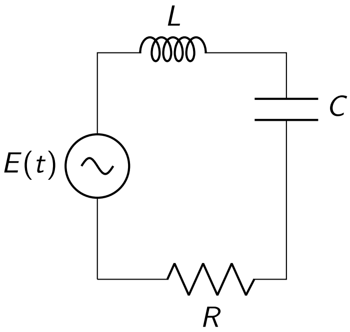
$$ay'' + by' + cy = 0$$

Torsional motion of a weight on a twisted shaft:



$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + k\theta = T(t)$$

L-R-C series circuits:



Q=charge on capacitor

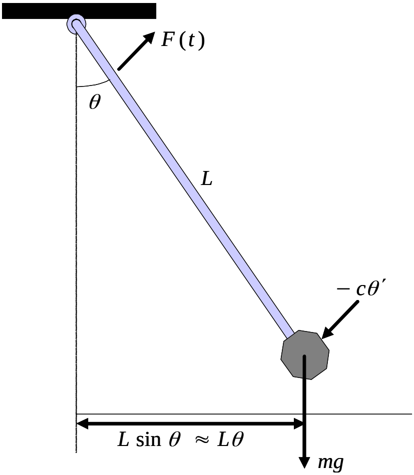
$\frac{dQ}{dt}$=current in circuit

$E(t)$ = applied voltage

Kirchoff's Laws:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Small oscillations of a pendulum:



$$mL^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -cL\frac{\mathrm{d}\theta}{\mathrm{d}t} - mgL\theta + F(t)$$

Equivalence of Problems

These 4 physical systems are modelled identically by:

$$ay'' + by' + cy = f(t)$$

Constants have different physical meaning (& units)

System	a	b	c	f(t)
Spring Dashpot	Mass	Damping Coeff.	Spring Constant	Applied Force
Pendulum	Mass x $(\text{Length})^2$	Damping x Length	Gravitational Moment	Applied Moment
Series Circuit	Inductance	Resistance	Capacitance^{-1}	Imposed Voltage
Twisted Shaft	Moment of Inertia	Damping	Elastic Shaft Constant	Applied Torque

Roots of the characteristic equation (polynomial)

$$ay'' + by' + cy = 0 \qquad \text{guess: } y = e^{rt}$$

$$ar^2 e^{rt} + bre^{rt} + ce^r t = 0 \qquad \Rightarrow \underbrace{ar^2 + br + c}_{\text{char. poly.}} = 0$$

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three main cases:

1. Two distinct real roots: $\underbrace{b^2 - 4ac}_{\text{discriminant}} > 0$

2. Repeated real roots: discriminant $= 0$

3. Complex conjugate roots: discriminant < 0

Case 1: distinct real roots

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}; \qquad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two major subcases:

1. $ac > 0$:

   Both roots have the same sign.

   y_1 and y_2 both grow or decay exponentially.

2. $ac < 0$:

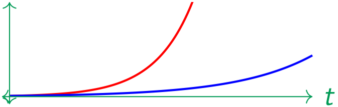
   The two roots have opposite sign.

   One solution grows exponentially, the other decays exponentially.

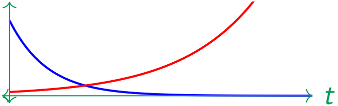
Qualitative Behaviour: distinct real roots

Sum of real exponential functions, three subcases:

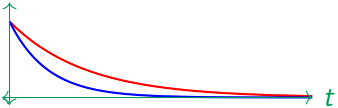
1. All positive roots, $0 < r_1 < r_2$.



2. Mixed roots, $r_1 < 0 < r_2$.



3. All negative roots, $r_1 < r_2 < 0$.



Case 2: Repeated real root $(r_1 = r_2 = r)$ $y_h = c_1 y_1 + c_2 y_2$

Straighforward solution

$$y_1 = e^{rt}$$

with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$$

We need another solution that is linearly independent of y_1

Lets try

$$y_2 = q(t) y_1(t)$$

Unique choice

$$q(t) = Ct \qquad \Rightarrow \qquad y_2(t) = te^{rt}$$

Proof that $y_2 = te^{rt}$

$$ay'' + by' + cy = 0 \qquad \text{with } b^2 - 4ac = 0 \quad \Rightarrow \quad r_{1,2} = r = \frac{-b}{2a}$$

Try: $y_2 = q(t)e^{rt}, \quad y_2' = q'e^{rt} + rqe^{rt}$

$$y_2'' = q''e^{rt} + 2rq'e^{rt} + r^2qe^{rt}$$

plug these into the ODE

$$a\left(q''e^{rt} + 2rq'e^{rt} + r^2qe^{rt}\right) + b\left(q'e^{rt} + rqe^{rt}\right) + cqe^{rt} = 0$$

$$aq''e^{rt} + (2ar + b)\,q'e^{rt} + \underbrace{(ar^2 + br + c)}_{\text{char. poly.}=0}qe^{rt} = 0$$

sub in $r = \frac{-b}{2a}$

$$aq''e^{rt} + \underbrace{\left(\cancel{2a}\frac{-b}{\cancel{2a}} + b\right)}_{0}e^{rt} = 0$$

$$aq''e^{rt} = 0 \quad \Rightarrow \quad q'' = 0 \quad \Rightarrow \quad q(t) = Ct + D$$

$D = 0$ due to linear independence between y_2 and y_1

Checking for linear independence when $r_1 = r_2 = r$

Both $y_1 = e^{rt}$ and $y_2 = te^{rt}$ are solutions.

$W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$?

$$\begin{aligned} W &= e^{rt}\left(rte^{rt} + e^{rt}\right) - re^{rt}te^{rt} \\ &= \cancel{rte^{2rt}} + e^{2rt} - \cancel{rte^{2rt}} \\ &= e^{2rt} \neq 0 \end{aligned}$$

y_1 and y_2 are linearly independent!

General solution: $y_h = c_1 e^{rt} + c_2 te^{rt}$

Solve the IVP: $y'' + 4y' + 4y = 0$ $\quad$ $y(0) = 2$, $y'(0) = 0$

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 4}}{2} = \frac{-4 \pm 0}{2} =, -2$$

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

initial conditions:

$$y(0) = 2 = c_1$$

$$y'(0) = 0 = -2c_1 + c_2(e^{-2t} - 2te^{-2t})\Big|_{t=0}$$

$$= -4 + c_2 \qquad \Rightarrow \qquad c_2 = 4$$

$$\boxed{y(t) = 2e^{-2t} + 4te^{-2t}}$$

Review: Complex Numbers

Square root of a negative number:

Suppose $w > 0$

$$\sqrt{-w} = i\sqrt{w}$$

$$i = \text{imaginary unit}$$

$$i \times i = -1$$

Suppose $a, b \in \mathbb{R}$

Complex Number: $z = a + ib$
Complex Conjugate: $\bar{z} = a - ib$

$$\frac{z + \bar{z}}{2} = \frac{2a}{2} = a = \mathrm{Re}(z) \in \mathbb{R}$$

$$\frac{z - \bar{z}}{2i} = \frac{2ib}{2i} = b = \mathrm{Im}(z) \in \mathbb{R}$$

Case 3: Complex roots ($b^2 - 4ac < 0$) $y_h = c_1 y_1 + c_2 y_2$

Roots are given by:

$$r_1 = \alpha + i\beta \qquad \text{where } i = \sqrt{-1}$$
$$r_2 = \alpha - i\beta$$

$$\alpha = \frac{-b}{2a}, \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The two functions $y_1 = e^{(\alpha + i\beta)t}$ & $y_2 = e^{(\alpha - i\beta)t} = \bar{y}_1$ are solutions.

What is the exponential of a complex number?

Euler's formula:

$$e^{\pm i\alpha} = \cos\alpha \pm i \sin\alpha$$

$$y_{1,2} = e^{(\alpha \pm i\beta)t} = e^{\alpha t} e^{\pm i\beta t}$$
$$= \underbrace{e^{\alpha t}}_{\text{Real}} \underbrace{[\underbrace{\cos(\beta t)}_{\text{Real}} \pm \underbrace{i \sin(\beta t)}_{\text{Imaginary}}]}_{\text{Complex Conjugates}} \qquad y_1 = \bar{y}_2$$

We want purely real solutions

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = \mathrm{Re}(y_1) = e^{\alpha t}\cos(\beta t) \in \mathbb{R}$$
$$\tilde{y}_2 = \frac{y_1 - y_2}{2i} = \mathrm{Im}(y_1) = e^{\alpha t}\sin(\beta t) \in \mathbb{R}$$

Complex roots ($r_{1,2} = \alpha \pm i\beta$)

The functions $y_1 = e^{\alpha t}\cos(\beta t)$ and $y_2 = e^{\alpha t}\sin(\beta t)$ are linearly independent real solutions.

Sketch the two functions if you are not convinced.

General solution: $y_h = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$

Find the general solution to: $y'' + 6y = 0$

$$r_{1,2} = \frac{\pm\sqrt{-4 \cdot 6}}{2} = \pm\sqrt{-6} = \pm i\sqrt{6}$$

$$y_h = c_1 \cos\left(\sqrt{6}t\right) + c_2 \sin\left(\sqrt{6}t\right)$$

Solve the IVP: $y'' + 2y' + 5y = 0$ $\quad y(0) = 1$, $y'(0) = -1$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}}{2} i = -1 \pm 2i$$

$$y_h = e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$

initial conditions:

$$y(0) = 1 = c_1$$

$$y'(0) = -1 = -c_1 + \left(-2c_1 \sin(0) + 2c_2 \cos(0)\right) = -c_1 + 2c_2$$

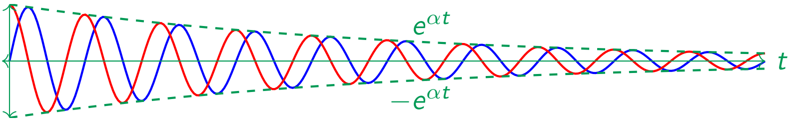
$$-1 = -1 + 2c_2 \qquad \Rightarrow \qquad c_2 = 0$$

$$\boxed{y(t) = e^{-t} \cos(2t)}$$

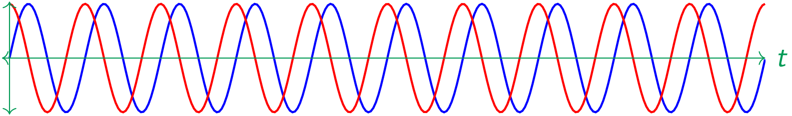
Qualitative Behaviour: complex roots

Three subcases:

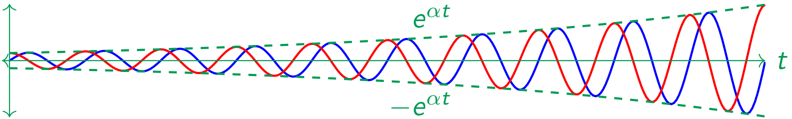
1. $\alpha < 0 \quad \Rightarrow \quad$ Exponentially decaying oscillations.



2. $\alpha = 0 \quad \Rightarrow \quad$ Sustained periodic oscillations.



3. $\alpha > 0 \quad \Rightarrow \quad$ Exponentially growing oscillations.



Summary

- For homogeneous linear constant coeffiecient ODEs:
  - Pick an ansatz (e.g., e^{rt})
  - Write down the characteristic equation
  - Find the roots
    - If you don't have enough functions, make a new one by multiplying by t
- Write down the general solution according to the roots
  - Real and distinct $\Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
  - Real and repeated $\Rightarrow y_h = c_1 e^{rt} + c_2 t e^{rt}$
  - Complex $\Rightarrow y_h = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$ with $r_{1,2} = \alpha \pm i\beta$
- Fit the constants c_1 and c_2 to the initial conditions