Recall: Linear IVPs

Given the initial state of a system (e.g., mass-spring or electrical circuit), predict its behaviour at later times.

1. Scalar ODEs:

$$\mathrm{L}\left[y(t)\right] = f(t) \quad \text{with} \quad y(0) = y_0 \text{ (and maybe } y'(0) = v_0\text{)}$$

- 1^{st} order $\Rightarrow$ solve by integrating factor
- 2^{nd} order $\Rightarrow$ solve by undetermined coefficients
  - Equivalent to a system of 2 1^{st} order ODEs

2. Systems of 1^{st} order ODEs:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t) \quad \text{with} \quad \vec{x}(0) = \vec{x}_0$$

General solution: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}\int \mathbf{X}^{-1}\vec{f}(t)dt \;$ with $\; \vec{c} = \mathbf{X}^{-1}(0)\vec{x}_0$

Fundamental Matrices and Solution Bases

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t) \quad \text{with} \quad \vec{x}(0) = \vec{x}_0$$

General solution: $\vec{x}(t) = \mathbf{X}(t)\vec{c} + \mathbf{X}\int \mathbf{X}^{-1}\vec{f}(t)dt \quad$ with $\quad \vec{c} = \mathbf{X}^{-1}(0)\vec{x}_0$

The n columns of the fundamental matrix $\mathbf{X}$ are the L.I. solutions to

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_i = \mathbf{A}(t)\vec{x}_i \quad \text{for } i = 1, \dots, n$$

These **fundamental solutions** form a basis for all homogeneous solutions.

$$\vec{x}_h = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n \quad \text{with } c_i = \frac{\vec{x}_i(0)\cdot\vec{x}_0}{||\vec{x}_i(0)||}$$

Other solution bases can be found through linear combinations...

The Flow Matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} \quad \text{with} \quad \vec{x}(0) = \vec{x}_0$$

$\vec{x}(t) = \mathbf{X}(t)\vec{c} \quad \text{with} \quad \vec{c} = \mathbf{X}^{-1}(0)\vec{x}_0 \quad \Rightarrow \quad \vec{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\vec{x}_0.$

Let $\mathbf{\Psi}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)$ be a fundamental matrix in a new basis set

$$\{\vec{\psi}_1(t), \ldots, \vec{\psi}_n(t)\}$$

Observe that at $t = 0$ we have

$$\mathbf{\Psi}(0) = \mathbf{X}(0)\mathbf{X}^{-1}(0) = \mathbf{I}$$

So, a homogeneous solution with an initial condition $\vec{x}_h(0) = \vec{x}_0$ can be written as

$$\vec{x}_h(t) = \mathbf{\Psi}(t)\vec{x}_0$$

Orthonormal Solution Bases

Let $\mathbf{\Psi}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0)$ be a fundamental matrix in a new basis set

$$\{\vec{\psi}_1(t), \ldots, \vec{\psi}_n(t)\} \qquad \text{with} \qquad \mathbf{\Psi}(0) = \mathbf{I}$$

Note: $\vec{\psi}_i(0) = \underbrace{[0, \ldots, 1, \ldots, 0]^T}_{i^{th} \text{ component is non-zero}}$ $\Rightarrow \{\vec{\psi}_i(0)\}$ is an orthonormal basis, i.e.,

$$\vec{\psi}_i(0) \cdot \vec{\psi}_j(0) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

This means that if we write

$$\vec{x}(t) = c_1\vec{\psi}_1(t) + \cdots + c_n\vec{\psi}_n(t) \qquad \text{with} \qquad \vec{x}(0) = \vec{x}_0$$

then $c_i = \vec{\psi}_i(0) \cdot \vec{x}_0 = i^{\text{th}}$ component of $\vec{x}_0$.

Q: How can we do this when $n \to \infty$?

ex: PDEs: $\mathrm{L}\left[u(x,t)\right] = 0$ with $u(x,0) = u_0(x)$

BVPs

Before we solve PDEs, we need to discuss ODE boundary value problems.

Suppose you have data about an ODE solution at $t = 0$ and $t = T$.

$$ay'' + by' + cy = f(t), \qquad \underbrace{\begin{matrix} y(0) = y_0 \\ y(T) = y_T \end{matrix} \quad \underline{\text{or}} \quad \begin{matrix} y'(0) = v_0 \\ y'(T) = v_T \end{matrix}}_{\text{boundary conditions (BCs)}}$$

We call these boundary value problems (BVPs).

Periodic BVPs

In cases where $f(t)$ is a periodic function with period T, i.e.,

$$f(t+T) = f(t) \qquad \forall t,$$

the solution $y(t)$ eventually becomes T-periodic.

We can find the long-term periodic solution by solving an ODE with periodic boundary conditions, given by

$$ay'' + by' + cy = f(t), \qquad \underbrace{\begin{array}{l} y(0) = y(T) \\ y'(0) = y'(T) \end{array}}_{2^{nd}\text{ order}\Rightarrow 2\text{ BCs}}$$

ex: $y'' + y = \cos(2\pi t)$, $\quad \begin{aligned} y(0) &= y(1) \\ y'(0) &= y'(1) \end{aligned}$

$$y(t) = y_h + y_p \qquad \begin{aligned} y_h &= c_1 \cos(t) + c_2 \sin(t) \\ \text{geuss: } y_p &= A\cos(2\pi t) + B\sin(2\pi t) \end{aligned}$$

$$\underline{\text{M. U. C.:}}\ A = \frac{1}{1 - 4\pi^2},\ B = 0$$

$$y(t) = c_1 \cos(t) + c_2 \sin(t) + \frac{1}{1 - 4\pi^2}\cos(2\pi t)$$

Boundary Conditions:

$$y(0) = c_1 + \cancel{\frac{1}{1 - 4\pi^2}} = y(1) = c_1 \cos(1) + c_2 \sin(1) + \cancel{\frac{1}{1 - 4\pi^2}}$$

$$c_1 = c_1 \cos(1) + c_2 \sin(1)$$

$$y'(0) = c_2 = y'(1) = -c_1 \sin(1) + c_2 \cos(1)$$

$$c_2 = -c_1 \sin(1) + c_2 \cos(1)$$

$$\begin{aligned} c_1 &= c_1\cos(1) + c_2\sin(1) \\ c_2 &= -c_1\sin(1) + c_2\cos(1) \end{aligned}$$

$$c_2(1-\cos(1)) = -c_1\sin(1)$$

$$c_2 = -\frac{\sin(1)}{1-\cos(1)}c_1$$

$$c_1 = c_1\cos(1) - \frac{\overbrace{\sin^2(1)}^{1-\cos^2(1)}}{1-\cos(1)}c_1$$

$$c_1 = c_1\left(\cos(1) - \frac{(1+\cos(1))\cancel{(1-\cos(1))}}{\cancel{1-\cos(1)}}\right)$$

$$c_1 = c_1\left(\cancel{\cos(1)} - (1+\cancel{\cos(1)})\right)$$

$$c_1 = -c_1 \qquad \Rightarrow c_1 = c_2 = 0$$

$$\boxed{y(t) = \frac{1}{1-4\pi^2}\sin(2\pi t)}$$

Alternative approach:

Notice that $y_h(t) = y_h(t+2\pi)$ and $y_p(t) = y_p(t+1)$.

Since the BCs require solutions with period 1, we know the homogeneous part of the solution is zero.

ex: $y'' + y = f(t)$, $\begin{array}{l} y(0) = y(1) \\ y'(0) = y'(1) \end{array}$ with $f(t+1) = f(t)$

Due to its periodicity, we can express $f(t)$ as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + b_n \sin(2n\pi t)$$

This is called the **Fourier Series** of the function $f(t)$, the coefficients a_n and b_n are called **Fourier coefficients**.

The coefficients are obtained by taking the inner product of the function $f(t)$ and the Fourier basis

$$\{\cos(2n\pi t),\ \sin(2n\pi t)\} \qquad n = 0, \ldots, \infty$$

$$y'' + y = \tfrac{1}{2}a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos(2n\pi t) + b_n \sin(2n\pi t), \qquad \begin{array}{l} y(0) = y(1) \\ y'(0) = y'(1) \end{array}.$$

$$\underline{\text{Guess:}}\; y(t) = \sum_n y_n(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi t) + B_n \sin(2n\pi t)$$

Apply M.U.C. term-by-term for the different values of n.

For $n \neq 0$, the n^{th} particular solution is

$$\begin{aligned} y_n &= A_n \cos(2n\pi t) + B_n \sin(2n\pi t) \\ y_n'' &= -4n^2\pi^2 A_n \cos(n\pi t) - 4n^2\pi^2 B_n \sin(n\pi t) \\ \underline{\text{ODE}}: \quad y_n'' + y_n &= a_n \cos(2n\pi t) + b_n \sin(2n\pi t) \end{aligned}$$

$$\begin{aligned} A_n(1-4n^2\pi^2)\cos(n\pi t) + B_n(1-4n^2\pi^2)\sin(n\pi t) & \qquad A_n = \frac{a_n}{1-4n\pi^2} \\ = a_n \cos(n\pi t) + b_n \sin(n\pi t) & \qquad B_n = \frac{b_n}{1-4n\pi^2} \end{aligned}$$

$y'' + y = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + b_n \sin(2n\pi t),$ $\qquad \begin{matrix} y(0) = y(1) \\ y'(0) = y'(1) \end{matrix}$.

$$\underline{\text{Guess:}}\ y(t) = \sum_n y_n(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi t) + B_n \sin(2n\pi t)$$

Apply M.U.C. term-by-term for the different values of n.

For $n = 0$, we have

$$\begin{aligned} y_0 &= A, y_0'' = 0 \\ A &= \frac{1}{2}a_0 \\ y(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{a_n}{1-4n\pi^2}\cos(2n\pi t) + \frac{b_n}{1-4n\pi^2}\sin(2n\pi t) \end{aligned}$$

Given a specific periodic function $f(t)$, we can find its Fourier coefficients a_n and b_n and use the BVP solution above.

Inner Products

Dot products are an example of an inner product for Euclidean vector spaces.

$$\langle \vec{x}, \vec{y} \rangle = \sum_i x_i y_i$$

5 basic properties define an inner product: wikipedia

To define inner products for function spaces, sums are replaced by integrals.

For T-periodic functions f and g we define the following inner product:

$$\langle f, g \rangle = \frac{2}{T} \int_0^T f(t) g(t) dt$$

Fourier Series

Given any periodic function $f(t)$ with period T, we can approximate $f(t)$ as a Fourier series

$$f(t) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)$$

with

$$\begin{aligned}
a_0 &= \langle f(t), 1 \rangle & &= \frac{2}{T}\int_0^T f(t)dt \\
a_n &= \left\langle f(t), \cos\left(\frac{2n\pi t}{T}\right) \right\rangle & &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right) dt \\
b_n &= \left\langle f(t), \sin\left(\frac{2n\pi t}{T}\right) \right\rangle & &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right) dt
\end{aligned}$$

If $f(t)$ is a continuous function, then the approximation becomes an equality.

The Fourier Basis is Orthonormalized

Consider m and n to be any two positive integers or zero, then we have

$$\left\langle \cos\left({}^{2n\pi t}/_{T}\right), \sin\left({}^{2m\pi t}/_{T}\right) \right\rangle = 0 \qquad \forall m, n \qquad \text{(Orthogonality)}$$

$$\begin{aligned} \left\langle \sin\left({}^{2n\pi t}/_{T}\right), \sin\left({}^{2m\pi t}/_{T}\right) \right\rangle &= \left\langle \cos\left({}^{2n\pi t}/_{T}\right), \cos\left({}^{2m\pi t}/_{T}\right) \right\rangle \\ &= \begin{cases} 1 & \text{if } m = n \neq 0 \qquad \text{(Normalization)} \\ 0 & \text{otherwise} \qquad \text{(Orthogonality)} \end{cases} \end{aligned}$$

The normalization condition is the reason for the factors of $\frac{2}{T}$ in front of the Fourier coefficient integrals.