Review: Evaluating Integrals

Indefinite integrals OR antiderivatives

$$F(t) = \int f(t)dt$$

Only defined up to an (arbitrary) additive constant

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[F(t) + C\right] = f(t)$$

Definite integrals

$$\int_{t_0}^{t} f(s)ds = F(t) + \cancel{C} - \left(F(t_0) + \cancel{C}\right)$$
$$= F(t) - F(t_0)$$

Intuitive approach to integrating a DE

Suppose we have $y' = f(t)$ with an initial condition $y(0) = y_0$, then we can find the general solution with

$$y(t) = \int f(t)\mathrm{d}t = F(t) + C,$$

and then use the initial condition to find the particular solution.

ex: Solve $y' = e^{-3t}$ with $y(0) = 2$.

$$y(t) = \int e^{-3t}dt = -\frac{1}{3}e^{-3t} + C \qquad \text{general solution}$$

$$y(0) = 2 = -\frac{1}{3} + C \implies C = \frac{7}{3}$$

$$y(t) = \frac{7}{3} - \frac{1}{3}e^{-3t} \qquad \text{particular solution}$$

Particular solutions have NO arbitrary constants.

Integration becomes harder once we have $y' = f(y,t)$.

ex: Solve $y' = 2y^2 t$ with y(0)=1/4.

hint: $\frac{dy}{dt} = g(y)h(t) \Leftrightarrow \frac{dy}{g(y)} = h(t)dt.$ (seperable equation)

$$\frac{dy}{y^2} = 2t\,dt \qquad \int \frac{dy}{y^2} = 2\int t\,dt$$

$$\Rightarrow -\frac{1}{y} = t^2 + C \qquad \frac{1}{y} = -t^2 + C$$

$$y = \frac{1}{C - t^2}$$

initial condition $y(0) = 1/4$

$$y(0) = \frac{1}{C} = \frac{1}{4} \Rightarrow C = 4 \qquad \boxed{y(t) = \frac{1}{4 - t^2}}$$

Solution blows up to $+\infty$ at $t = 2$.

Integration becomes harder once we have $f = f(y, t)$.

ex: Try to solve $y' = -ay + t^3$ where a is some constant.

Lets try a similar strategy

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = t^3$$
$$\mathrm{d}y + ay\mathrm{d}t = t^3 dt$$
$$y(t) + a\int y(t)\mathrm{d}t = \frac{t^4}{4} + C$$

We dont know $y(t)$, so can't integrate it. $\Rightarrow$ Need more tricks

Can we always integrate $y' = f(y, t)$?

Potential issues:

- Does the integral exist?
  - Does not mean: Can you solve the integral?
  - Means: Does the derivative have a well-defined value for all time?
  - e.g., $y' = \frac{y}{t-1}$ is undefined at $t = 1$.
    - Solutions may not be defined for all values of the indep. variable.

- Is there only one solution $y(t)$?
  - Yes if and only if:
    1. $f(y, t)$ is well-defined and
    2. differentiable in y everywhere along the solution $y(t)$.
  - Need to specify initial conditions to get a unique particular solution.

Classifying differential equations

- Linear first-order: $y' + p(t)y = g(t); \qquad y(t_0) = y_0$
  - Linear with respect to y, y', y'' ..., $p(t)$doesn't matter for linearity.
  - Homogeneous: $g(t) = 0$
  - Inhomogeneous: $g(t)$ is not zero everywhere.
  - Constant coefficient $p(t) = a$, with constant a.
    - Solvable if $g(t)$ is "nice", and has unique solutions.

Solution Structure of Linear DEs

The linear ODE

$$\mathrm{L}[y] = g(t)$$

ex: $y' + ay = t^3$ is a 1^{st} order linear ODE

always has the general solution structure

$$y(t) = Cy_h(t) + y_p(t)$$

where y_h is the general solution to $\mathrm{L}[y_h] = 0$ (i.e., the **associated homogeneous problem**),

y_p is a particular solution to $\mathrm{L}[y_p] = g(t)$,

and C is any constant.

Proof:

$$\mathrm{L}[y(t)] = \mathrm{L}[Cy_h + y_p] \overset{\text{Linearity 1}}{=} \mathrm{L}[Cy_h] + \mathrm{L}[y_p]$$

$$\overset{\text{Linearity 2}}{=} C\underbrace{\mathrm{L}[y_h]}_{=0} + \underbrace{\mathrm{L}[y_p]}_{=g(t)} = g(t)$$

First-Order Homogeneous Constant Coefficient Equations

$$y' + ay = 0$$

where $a \neq 0$ is a constant.
General solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = 0 \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -ay$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = -a \quad \Rightarrow \int \frac{\mathrm{d}y}{y} = -\int a\,\mathrm{d}t$$

$$\ln|y| = -at + C_1 \quad \Rightarrow e^{\ln|y|} = e^{-at+C_1}$$

$$|y| = e^{-at}\underbrace{e^{C_1}}_{C_2} \quad \Rightarrow \quad y = \underbrace{\pm C_2}_{C} e^{-at}$$

$$\boxed{y(t) = Ce^{-at}}$$

The ODE

$$y' + 2y = t$$

has a particular solution given by $y_p = \frac{1}{2}t - \frac{1}{4}$.
Find the general solution, and then solve the following IVP:

$$y' + 2y = t, \quad \text{with } y(0) = -1.$$

General Solution:

$$y(t) = Ce^{-2t} + \frac{1}{2}t - \frac{1}{4}$$

Impose the initial condtion constraint $y(0) = -1$

$$y(0) = C - \frac{1}{4} = -1 \qquad \Rightarrow C = -\frac{3}{4}$$

$$\boxed{y(t) = -\frac{3}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}}$$

Note: y_p alone can only solve an IVP with $y(0) = -1/4$

The ODE

$$\frac{1}{3}y' = y + \frac{1}{3}e^{3t}$$

has a particular solution given by $y_p = te^{3t}$.

Find the general solution, and then solve the following IVP:

$$\frac{1}{3}y' = y + \frac{1}{3}e^{3t}, \quad \text{with } y(0) = -1.$$

Re-arrange DE:

$$y' - 3y = e^{3t} \qquad \Rightarrow \qquad \text{Gen. Soln. } y(t) = Ce^{3t} + te^{3t}$$

initial condtion $y(0) = -1$

$$y(0) = C = -1 \qquad \Rightarrow \qquad \boxed{y(t) = -e^{3t} + te^{3t}}$$

Note: y_p alone can only solve an IVP with $y(0) = 0$

Summary of Lecture 2

- Integrating $y' = f(y,t)$:
  - Indefinite integral (antiderivative) gives general solution.
  - Can we always integrate a 1st order DE?
    - No, but we avoid those cases.
- Linear constant coefficient DEs: $y' + ay = g(t)$
  - Solving the homogeneous problem: $y_h' + ay_h = 0 \Rightarrow y_h = Ce^{-at}$
    - General form of inhomogeneous solution: $y = y_h + y_p$
    - y_p alone can only solve a unique IVP, but not the general case.
  - How do we find $y_p(t)$?
    - Next Lecture: Method of Undetermined Coefficients