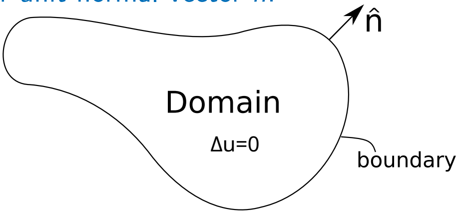
Laplace's Equation - Neumann Problem

Consider $\vec{x} \in \mathbb{R}^d$ restricted to a closed domain, where the boundary of the domain has a outer unit normal vector $\hat{n}$.



Solve Laplace's Equation

$$\Delta u = 0$$

with boundary condition is given by

$$\frac{\partial u}{\partial \hat{n}} = f(\vec{x}) \qquad \text{for } \vec{x} \text{ along the boundary}$$

$$\frac{\partial u}{\partial \hat{n}} = \vec{\nabla} u \cdot \hat{n}$$

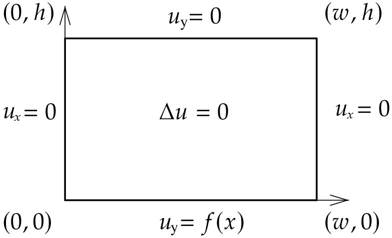
ex: Maxwell's first law with external electric field

Neumann South Problem on a Rectangle

ex: Consider a rectangular region of width w and height h.

Normal derivatives are zero at three of the four edges, and the derivative of u obeys some arbitrary function $f(x)$ the fourth edge.

$$
\begin{aligned}
\Delta u &= 0 & & \\
u_x(0, y) &= 0 & &\text{for } 0 < y < h \\
u_y(x, h) &= 0 & &\text{for } 0 < x < w \\
u_x(w, y) &= 0 & &\text{for } 0 < y < h \\
u_y(x, 0) &= f(x) & &\text{for } 0 < x < w
\end{aligned}
$$



Solution procedure is almost identical to the Dirichlet problem but with $\sin \to \cos$ and $\sinh \to \cosh$

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

BCs: $\dfrac{\partial}{\partial x} X(0) = \dfrac{\partial}{\partial x} X(w) = 0$

$$X_n(x) = \cos\left(\frac{n\pi}{w}x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

$$\Rightarrow Y_n'' + \left(\frac{n\pi}{w}\right)^2 Y_n = 0 \qquad \Rightarrow Y_n = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y}$$

BC: $\dfrac{\partial}{\partial y} Y_n(h) = 0$

$$\Rightarrow B = Ae^{2\frac{n\pi}{w}} \qquad \Rightarrow Y_n(y) = a_n \cosh\left(\frac{n\pi}{w}(y-h)\right)$$

Finally,

$$\boxed{u_n(x,y) = a_n \cosh\left(\frac{n\pi}{w}(y-h)\right)\cos\left(\frac{n\pi}{w}x\right)} \qquad u(x,y) = \sum_n u_n(x,y)$$

We can repeat the same process for all the sub-problems.

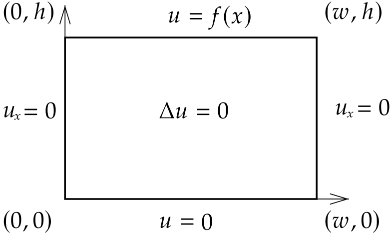
$$u_N = a_0 + \sum_n a_n \cos\left(\frac{n\pi}{w}x\right)\cosh\left(\frac{n\pi}{w}y\right)$$
$$u_S = b_0 + \sum_n b_n \cos\left(\frac{n\pi}{w}x\right)\cosh\left(\frac{n\pi}{w}(h-y)\right)$$
$$u_E = c_0 + \sum_n c_n \cos\left(\frac{n\pi}{h}y\right)\cosh\left(\frac{n\pi}{h}x\right)$$
$$u_W = d_0 + \sum_n d_n \cos\left(\frac{n\pi}{h}y\right)\cosh\left(\frac{n\pi}{h}(w-x)\right)$$

To find the unknown coefficients, match the series solution derivative with a Cosine series of the boundary conditions.

The boundary condition (normal derivative) must integrate to zero over the boundary.

Mixed Neumann/Dirichlet Problem

$$
\begin{aligned}
\Delta u &= 0 \\
u_x(0,y) &= 0 \qquad \text{for } 0 < y < h \\
u(x,h) &= f(x) \quad \text{for } 0 < x < w \\
u_x(w,y) &= 0 \qquad \text{for } 0 < y < h \\
u(x,0) &= 0 \qquad \text{for } 0 < x < w
\end{aligned}
$$



$$u_{xx} + u_{yy} = 0$$

Separation of Variables:

$$u(x, y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const.}$$

Horizontal Problem:

$$X'' + \lambda X = 0$$

Vertical Problem:

$$Y'' - \lambda Y = 0$$

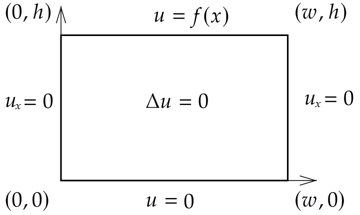
BCs: $X'(0) = X'(w) = 0$

$$X_n(x) = \cos\left(\frac{n\pi}{w}x\right) \qquad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

with the special case $n = 0$

$$X_0(x) = 1$$

$$Y'' - \lambda Y = 0$$



$$\lambda = \left(\frac{n\pi}{w}\right)^2$$

<u>$n \neq 0$</u>

$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y}$$

BC @ x=0: $0 = A + B$ $\Rightarrow B = -A$

$$Y_n(y) = A\left(e^{\frac{n\pi}{w}y} - e^{-\frac{n\pi}{w}y}\right)$$

$$= a_n \sinh\left(\frac{n\pi}{w}y\right)$$

$$u_n(x, y) = a_n \sinh\left(\frac{n\pi}{w}y\right)\cos\left(\frac{n\pi}{w}x\right)$$

<u>n=0</u>

$$Y_0'' = 0 \Rightarrow Y_0(y) = a_0 y + b \qquad Y_0(0) = 0 \Rightarrow b = 0$$

The non-zero boundary condition

$$u_n(x,y) = a_n \cos\left(\frac{n\pi}{w}x\right)\sinh\left(\frac{n\pi}{w}y\right), \quad u(x,y) = a_0 y + \sum_{n=1}^{\infty} u_n(x,y)$$

$u(x,h) = f(x)$ - Express the boundary condition as a Fourier Series

$$u(x,h) = f(x) = a_0 h + \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{w}h\right)\cos\left(\frac{n\pi}{w}x\right)$$

Given the appearance of our $u(x,h)$, we clearly need a Cosine series

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{w}x\right) \qquad \text{with } b_n = \frac{2}{w}\int_0^w f(x)\cos\left(\frac{n\pi}{w}x\right)dx$$

need equality between the two series

$$\Rightarrow a_0 = \frac{b_0}{2h} \qquad \text{and} \qquad a_n = \frac{b_n}{\sinh\left(\frac{n\pi}{w}h\right)}$$