So far, we have been dealing with

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x}, \qquad \vec{x}(t_0) = \vec{x}_0 \qquad \text{with } \mathbf{A} \in \mathcal{M}_{n\times n}$$

Homogeneous:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} - \mathbf{A}\vec{x} = \vec{0}$$

General solution:

$$\vec{x}_h(t) = \mathbf{X}(t)\vec{c}, \qquad \vec{c} = \mathbf{X}^{-1}(t_0)\vec{x}_0$$

$\mathbf{X}$ is constructed from n linearly independent eigensolutions

$$\mathbf{X}(t) = \underbrace{\begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix}}_{\begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & \cdots & \cdots \end{bmatrix}} = \begin{bmatrix} x_{1,1}(t) & x_{2,1}(t) & \cdots & x_{n,1}(t) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{1,n}(t) & \cdots & \cdots & x_{n,n}(t) \end{bmatrix}$$

Inhomogeneous Systems

Now we want to solve:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + \vec{f}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} - \mathbf{A}\vec{x} = \vec{f}(t)$$

General Solution:

$$\vec{x}(t) = \vec{x}_h + \vec{x}_p$$

Recall: we already showed this to be true for any linear scalar ODE, nothing changes for a system.

Constant Inhomogeneity

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + \vec{b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} - \mathbf{A}\vec{x} = \vec{b}$$

Method of Undetermined Coefficients:

$$\vec{x}_p = \vec{w}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_p = \vec{0} = \mathbf{A}\vec{w} + \vec{b}$$

$$\vec{w} = -\mathbf{A}^{-1}\vec{b}$$

in practice

$$\left[\begin{array}{c|c} \mathbf{A} & -\vec{b} \end{array}\right]$$

Assuming $\mathbf{A}^{-1}$ exists, i.e., $\det \mathbf{A} \neq 0$.

Non-constant Inhomogeneity

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + f(t)\vec{b}$$

or more generally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + \vec{f}(t)$$

This lecture we cover: method of undetermined coefficients

- Works just like for scalar DEs
- Need to find many unknown vectors, instead of coefficients
- Gets very trick with resonance

Next lecture: variation of parameters (more powerful, no guessing)

Method of Undetermined Coefficients: scalar resonance

$$ay'' + by' + cy = f(t) \qquad \text{gen sol: } y = y_h + y_p$$

Guess y_p based on the form of $f(t)$

If any term in y_p is linearly dependent with y_h, multiply that term by t.

How will this work for a system? $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + f(t)\vec{b}$

$$\vec{x}_h = \sum_i c_i e^{\lambda_i t}(t)\vec{v}_i \quad \text{and} \quad \vec{x}_p \approx \sum_j \frac{\mathrm{d}^j}{\mathrm{d}t^j} f(t)\vec{w}_j$$

Do we check for linear dependence between:

- $e^{\lambda_i t}$ and $\dfrac{\mathrm{d}^j f}{\mathrm{d}t^j}$?
- $\vec{v}_i$ and $\vec{b}$?
- The products of these thing?

Method of Undetermined Coefficients: system resonance

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} \quad = \quad \mathbf{A}\vec{x} \quad + \quad \underbrace{e^{rt}}_{\substack{\text{scalar}\\ \text{function}}} \times \underbrace{\vec{b}}_{\substack{\text{constant}\\ \text{vector}}}$$

With unknown **A** we don't know if we have resonance, lets assume we do.
Try

$$\vec{x}_p = e^{rt}\vec{u} + te^{rt}\vec{w}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_p = e^{rt}\left(r\vec{u} + \vec{w} + rt\vec{w}\right)$$

$$\mathbf{A}\vec{x}_p + \vec{f}(t) = e^{rt}\left(\mathbf{A}\vec{u} + t\mathbf{A}\vec{w} + \vec{b}\right)$$

$$\mathbf{A}\vec{u} + t\mathbf{A}\vec{w} + \vec{b} = r\vec{u} + \vec{w} + rt\vec{w}$$

Similar procedure works for cos or sin as the scalar function.

$$\vec{x}_p = e^{rt}\vec{u} + te^{rt}\vec{w} \qquad \Rightarrow \qquad \mathbf{A}\vec{u} + t\mathbf{A}\vec{w} + \vec{b} = r\vec{u} + \vec{w} + rt\vec{w}$$

True for all t, need the coefficients to match

<u>t:</u>

$$\mathbf{A}\vec{w} = r\vec{w}$$

$\vec{w}$ is an eigenvector with eigenvalue r.

But if r is not an eignevalue of $\mathbf{A}$, then $\vec{w} = \vec{0}$. (i.e., no resonance)

<u>constant:</u>

$$\mathbf{A}\vec{u} + \vec{b} = r\vec{u} + \vec{w}$$

$$(\mathbf{A} - r\mathbf{I})\vec{u} = \vec{w} - \vec{b}$$

Easy to solve if r is not an eigenvalue (i.e., $\vec{w} = \vec{0}$)

$$\vec{u} = -(\mathbf{A} - r\mathbf{I})^{-1}\vec{b} \qquad \text{or} \qquad \left[\begin{array}{c|c} \mathbf{A} - r\mathbf{I} & -\vec{b} \end{array}\right]$$

Conclusion: sufficient to check the scalar function for resonance

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x} + e^{rt}\vec{b}, \qquad \vec{x} = \vec{x}_h + \vec{x}_p, \qquad \text{where } \vec{x}_p = e^{rt}\vec{u} + te^{rt}\vec{w}$$

$$\text{with} \quad \mathbf{A}\vec{w} = r\vec{w} \quad \text{and} \quad (\mathbf{A} - r\mathbf{I})\vec{u} = \vec{w} - \vec{b}$$

If r is an eigenvalue, we cannot invert $(\mathbf{A} - r\mathbf{I})$...

Recall: $\vec{w}$ is only defined up to a mulitplicative constant (α)

$$(\mathbf{A} - r\mathbf{I})\vec{u} = \alpha\vec{w} - \vec{b}$$

Choose α such that we can make

$$\text{LHS} = \text{RHS}$$

That is quite vague, lets do an example!

Find a particular solution to $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x} + e^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{aligned}
\det\left(\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}\right) &= 0 \\
(2-\lambda)(-2-\lambda)+3 &= 0 \\
\lambda^2 + 2\lambda - 2\lambda - 4 + 3 &= 0 \\
\lambda^2 - 1 &= 0 \qquad\qquad \lambda_{1,2} = \pm 1
\end{aligned}$$

One eigensolution has a prefactor e^{-t} and so does the inhomogeneity, we have resonance.

Need to find the $\lambda = -1$ eigenvector.

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \qquad \begin{array}{l} 3x - y = 0 \\ y = 3x \end{array} \qquad\qquad \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find a particular solution to $\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x} + e^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\vec{x}_p = e^{-t}\left(\vec{u} + t\vec{w}\right) \qquad (\mathbf{A}+\mathbf{I})\vec{u} = \alpha\vec{w} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha - 2 \\ 3\alpha + 1 \end{bmatrix}$$

$$3u_1 - u_2 = \alpha - 2 = 3\alpha + 1 \qquad \alpha - 2 = 3\alpha + 1 \qquad \Rightarrow \alpha = -\frac{3}{2}$$

$$3u_1 - u_2 = -\frac{7}{2} \qquad \text{or } u_2 = 3u_1 + \frac{7}{2} \qquad \vec{u} = \begin{bmatrix} 1 \\ \frac{13}{2} \end{bmatrix}$$

$$\vec{x}_p = e^{-t}\left(\begin{bmatrix} 1 \\ \frac{13}{2} \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

Find a particular solution to $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$

$$\det\left(\begin{bmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

$$(1+\lambda)(1+\lambda) + 4 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\vec{x}_{1,2} \approx e^{-t}\left[\sin(2t) \pm \cos(2t)\right]\vec{v}_{1,2}$$

Two scalar functions $\sin(t)$ and e^{-3t}...No resonance!

family of functional forms $= \{\sin(t), \cos(t), e^{-3t}\}$

guess: $\vec{x}_p = \sin(t)\vec{a} + \cos(t)\vec{b} + e^{-3t}\vec{c}$

Find a particular solution to $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$

$\vec{x}_p = \sin(t)\vec{a} + \cos(t)\vec{b} + e^{-3t}\vec{c}$

LHS: $\frac{d}{dt}\vec{x}_p = \cos(t)\vec{a} - \sin(t)\vec{b} - 3e^{-3t}\vec{c}$

RHS: $\mathbf{A}\left(\sin(t)\vec{a} + \cos(t)\vec{b} + e^{-3t}\vec{c}\right) + \sin(t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^{-3t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\underline{\cos(t):}\quad \vec{a} = \mathbf{A}\vec{b}$ ①

$\underline{\sin(t):}\quad -\vec{b} = \mathbf{A}\vec{a} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{A}\vec{a} + \vec{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

① $\Rightarrow \quad \mathbf{A}\mathbf{A}\vec{b} + \vec{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad (\mathbf{A}^2 + \mathbf{I})\vec{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ ②

$\underline{e^{-3t}:}\quad -3\vec{c} = \mathbf{A}\vec{c} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad\qquad (\mathbf{A} + 3\mathbf{I})\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ③

Solve ② for $\vec{b}$ and ③ for $\vec{c}$, then use ① to find $\vec{a}$...

Find a particular solution to $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$

$$\vec{x}_p = \sin(t)\vec{a} + \cos(t)\vec{b} + e^{-3t}\vec{c}$$

$$(\mathbf{A}^2 + \mathbf{I})\vec{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} -2 & 8 \\ -2 & -2 \end{bmatrix}\vec{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 8 & -2 \\ -2 & -2 & 0 \end{array}\right] \qquad R_1 - R_2 \to R_1 \left[\begin{array}{cc|c} 0 & 10 & -2 \\ -2 & -2 & 0 \end{array}\right]$$

$$R_2 + \tfrac{2}{10}R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 0 & 10 & -2 \\ -2 & 0 & -\frac{4}{10} \end{array}\right] \qquad \begin{array}{ll} 10b_2 = -2 & \Rightarrow b_2 = -\frac{2}{10} \\ -2b_1 = -\frac{4}{10} & \Rightarrow b_1 = \frac{2}{10} \end{array}$$

$$\vec{b} = \frac{1}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{a} = \mathbf{A}\vec{b} = \frac{1}{5}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find a particular solution to $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 2\sin t \\ -e^{-3t} \end{bmatrix}$

$$\vec{x}_p = \sin(t)\vec{a} + \cos(t)\vec{b} + e^{-3t}\vec{c}$$

$$(\mathbf{A} + 3\mathbf{I})\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}\vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & 2 & 1 \end{array}\right] \qquad R_1 + 2R_2 \to R_1 \left[\begin{array}{cc|c} 4 & 0 & 2 \\ 1 & 2 & 1 \end{array}\right]$$

$R_2 - \frac{1}{4}R_1 \to R_2$

$$\left[\begin{array}{cc|c} 4 & 0 & 2 \\ 0 & 2 & \frac{1}{2} \end{array}\right] \qquad \begin{array}{ll} 4c_1 = 2 & \Rightarrow c_1 = \frac{1}{2} \\ 2c_2 = \frac{1}{2} & \Rightarrow c_2 = \frac{1}{4} \end{array}$$

$$\vec{c} = \frac{1}{4}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{x}_p = \frac{1}{5}\left(\sin(t)\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \cos(t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \frac{1}{4}e^{-3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$