MATH 215/255: Elementary Differential Equations

Lecture 1: Differential Equations

1. What are they and why do we solve them?
2. Terminology

What is a Differential Equation?

A differential equation (DE) is an equation involving an unknown function y and atleast one derivative of y w.r.t. an independent variable.

Given: A DE with an unknown function $y(t)$.

e.x., $$\frac{\mathrm{d}y}{\mathrm{d}t} = -3y(t)$$

or

$$y' = -3y$$

Task: Find the function(s) $y(t)$.

Solution: $y(t) = Ce^{-3t}$

DEs specify the rate of change of one variable (e.g., the position of an object) with respect to another (e.g., time).

Why do we solve/study DEs?

DEs provide an intuitive way to describe many types of interactions (e.g., mechanical, biochemical, social, economic, etc.).

Solving and analyzing DEs allows us to:

1. Make predictions about the future (forecasting).
   - Will some variable grow unboundedly? Oscillate? Decay to zero?
   - With what rate will those things happen?
2. Test mechanisms that may explain experimental data.
   - e.g., determine why a variable sometimes oscillates vs. equilibrates?

Example: Skydiving

Newton's Second Law:

$$F = ma$$

$$ma(t) = \underbrace{-mg}_{\text{gravitational force}} \qquad \underbrace{-\mu v(t)}_{\text{drag force}}$$

Rewrite as:

1^{st} order linear ODE	2^{nd} order linear ODE	1^{st} order linear system
sub. $a = v'$	sub. $a = x''$, $v = x'$	$\boxed{\begin{aligned} x' &= v \\ mv' &= -mg - \mu v \end{aligned}}$
$\boxed{mv' = -mg - \mu v}$	$\boxed{mx'' = -mg - \mu x'}$	

We will learn different methods to solve these three types of DEs.

Example: Ecology - Lotka-Volterra Model

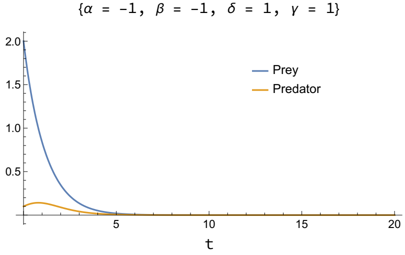
Predator-Prey Model, 2 variables:

x = prey population and y = predator population

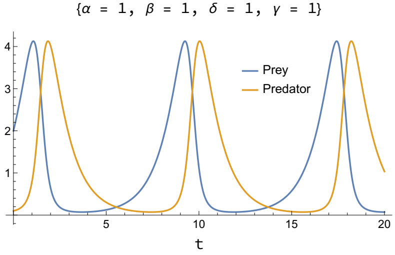
$$\frac{dx}{dt} = \alpha x - \beta xy, \qquad \frac{dy}{dt} = \delta xy - \gamma y$$ 1st order nonlinear system

We can prove that only these two solutions types are possible

Mutual Extinction



Predator-Prey Oscillations



Terminology: ODEs vs PDEs

- Ordinary differential equation (ODE) weeks 1-10
  - A DE with derivatives w.r.t. only one independent variable.
  - ex: $\underbrace{\frac{dy}{dt} = y(t) + 3}_{1^{st}\text{ order linear ODE}}$ or $\underbrace{\frac{dy}{dt} = \sin(y) + \cos(t)}_{1^{st}\text{ order nonlinear ODE}}$
- Partial differential equation (PDE) weeks 11-13
  - A DE with derivatives w.r.t multiple independent variables.
  - ex: Temperature of a metal rod, given by $u(x, t)$.

  Heat/Diffusion eq: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$ - 2^{nd} order linear PDE

Partial derivatives are necessary for solutions to agree when changing coordinate systems (e.g., switch from cartesian to polar coordinates)

Terminology: Order of a DE

The highest derivative that appears in the DE.

- $y' = y + 3$ first order
- $y' = y^2 + 9$ first order
- $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tan(t)$ first order
- $y'' = -y$ second order
- $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = ky$ fourth order

Terminology: Operator Form $\Rightarrow$ $\mathrm{L}\left[y(t)\right] = f(t)$

Everything that depends on the unknown function goes on one side of the equal sign and everything else on the other.

- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t) + 3 \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - y(t) = 3$
  - $\mathrm{L}\left[y\right] = y' - y, \quad f(t) = 3$
- $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \sin(y) + \cos(t) \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} - \sin(y) = \cos(t)$
  - $\mathrm{L}\left[y\right] = y' - \sin(y), \quad f(t) = \cos(t)$

Terminology: Operator Form $\Rightarrow$ $\mathrm{L}[y(t)] = f(t)$

The operator $\mathrm{L}[\cdot]$ encodes the "intrinsic" dynamics that the ODE is modelling.

- Force-displacement relationship of a spring.
- Velocity-drag relationship of a viscous fluid.

In many physics-based applications, $\mathrm{L}[\cdot]$ does not depend explicitly on the independent variable t.

$f(t)$ is often called the (external) forcing term.

It typically accounts for external influences that could be varied or turned off.

Terminology: Linearity of DEs - $\mathrm{L}[y(t)] = f(t)$

If the operator $\mathrm{L}[\cdot]$ is linear, then the DE is linear.

Conditions for linearity:

Given any two functions f and g and a constant c, $\mathrm{L}[\cdot]$ is linear if

1. $\mathrm{L}[f+g] = \mathrm{L}[f] + \mathrm{L}[g]$

2. $\mathrm{L}[cf] = c\mathrm{L}[f]$

Terminology: Linearity of DEs - $\mathrm{L}[y(t)] = f(t)$

If the operator $\mathrm{L}[\cdot]$ is linear, then the DE is linear.

In practice:

Does the operator have either of the following:

1. any nonlinear functions of y (or its derivatives) or
2. any products of y and its derivatives

ex: $\mathrm{L}[y] = y'' + y$ ex: $\mathrm{L}[y] = y' + \sin(y'')$ ex: $\mathrm{L}[y] = y' + y'y$

Linear Nonlinear Nonlinear

Terminology: Solution to an ODE

A solution of an ODE is a function that satisifes the ODE.

ex: Is $y = Ce^{-t} + t - 1$ a solution to $y' + y = t$?

compute derivative(s): $y' = -Ce^{-t} + 1$

evaluate ODE: $y' + y = \cancel{-Ce^{-t}} + \cancel{1} + \cancel{Ce^{-t}} + t - \cancel{1}$

$= t \quad \checkmark$

Here C is an arbitrary constant that can have any value.

Any solution with an arbitrary constant is called a general solution

A solution with no arbitrary constants is called a particular solution

We eliminate arbitrary constant by using constraints

Initial Value Problems

Add a constraint at $t = t_0$, e.g.

$$\mathrm{L}[y] = f(t), \text{ with } y(t_0) = y_0,$$

where t_0 and y_0 are numerical values (usually real-valued).

ex: Find the particular solution to $y' + y = t$ with $y(0) = 4$?

Start with the general solution

$$y(t) = Ce^{-t} + t - 1$$

evaluate at $t = t_0 = 0$, make that equal to $y_0 = 4$

$$y(0) = C - 1 = 4 \quad \Rightarrow C = 5$$
$$\boxed{y(t) = 5e^{-t} + t - 1}$$

Summary

1. What are DEs?
   - Equations involving unknown function(s) and function derivatives.
   - Specify rates of change of certain quantities.
   - Useful for modelling many natural phenomena.

2. Terminology
   - ODEs (& PDEs).
   - Order of DEs, Linear DEs, Solutions to DEs.

3. Initial Value Problems
   - A straightforward way to obtain a unique solution.
   - Specify solution value at some initial time t_0.