Recall: Inhomogeneous ODEs

1^{st} Order ODEs:

$$y' + p(t)y = g(t)$$

Solve by Method of Integrating Factors

- Requirement: $p(t)$ and $g(t)$ are continuous functions

2^{nd} Order ODEs:

$$ay'' + by' + cy = g(t)$$

Solve by Method of Undetermined Coefficients

- Requirement: $g(t)$ has a finite family of functional forms
    - Pre-requisite: $g(t)$ and all its derivatives are continuous.

How do we handle cases where $g(t)$ is discontinuous?

Laplace Transforms!

What is a Laplace Transform?

An operator:

- Takes as an input a function of one variable, e.g. $y(t)$
- Yields another function of a new variable $Y(s)$
- Mapping between functions in the "time-domain" and the "s domain".

$$\mathcal{L}\left\{y(t)\right\} = Y(s) = \int_0^\infty e^{-st} y(t) dt$$

Convention: Lowercase letter in "time-domain", uppercase in "s-domain"

Fun properties of Laplace Transforms

$$\mathcal{L}\left\{y(t)\right\} = Y(s) = \int_0^\infty e^{-st} y(t) dt$$

1. Laplace transforms "eat" derivatives.
   - Converts ODEs into algebraic expressions:

Solve for $Y(s)$ in the "s-domain" $\Leftrightarrow$ Solve the ODE in the "t-domain"

2. Laplace transforms smooth out discontinuities
   - ODEs with discontinuities become continuous functions of s

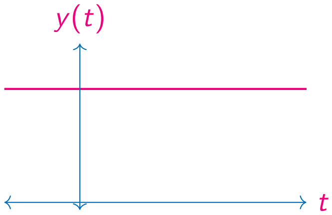
Before solving ODEs, let practice taking LT of simple functions.

ex: $y(t) = \frac{1}{2}$ $\qquad \mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st}\frac{1}{2}\,dt$

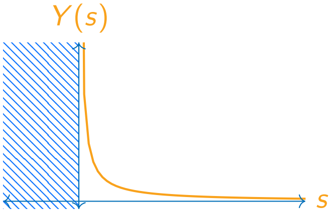
$$= -\frac{1}{2s}e^{-st}\Big|_0^\infty \qquad = -\lim_{A\to\infty}\frac{1}{2s}e^{-st}\Big|_0^A$$

$$= -\frac{1}{2s}\lim_{A\to\infty}\left(e^{-sA}-1\right)$$

$$= \begin{cases} \frac{1}{2s} & \text{if } s>0 \\ DNE & \text{if } s\le 0 \end{cases}$$

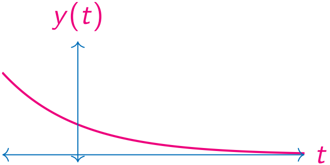


$\xrightarrow{\mathcal{L}}$

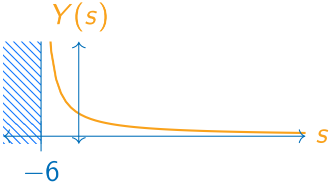


ex: $y(t) = e^{-6t}$

$$\begin{aligned}
\mathcal{L}\left\{y(t)\right\} = Y(s) &= \int_0^\infty e^{-st}e^{-6t}dt \\
&= \int\limits_0^\infty e^{-(s+6)t} = -\frac{1}{s+6}\lim_{A\to\infty}\left(e^{-(s+6)A} - 1\right) \\
&= \begin{cases} \frac{1}{s+6} & \text{if } s > -6 \\ DNE & \text{if } s \leq -6 \end{cases}
\end{aligned}$$



$\xrightarrow{\mathcal{L}}$



General Results

For any constants C and a we have the Laplace Transforms of the following functions $y(t)$:

$y(t) = C$	$\mathcal{L}\left\{C\right\} = \dfrac{C}{s}$	Constant
$y(t) = e^{at}$	$\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	Exponential Function

From now on, we don't worry too much about the domain of definition.

In general, there are always some conditions on s for the integrals to exist.

General Result: Linearity of Laplace Transforms

Given any two function $f(t)$ and $g(t)$ as well as any constant c.

1. $\mathcal{L}\left\{f(t)+g(t)\right\} = \int_0^{\infty} e^{-st}\left(f(t)+g(t)\right) dt$

   $= \mathcal{L}\left\{f(t)\right\} + \mathcal{L}\left\{g(t)\right\} = F(s) + G(s)$

2. $\mathcal{L}\left\{cf(t)\right\} = c\mathcal{L}\left\{f(t)\right\} = cF(s)$

The Laplace tranform is <u>linear</u>.

ex: $y(t) = \cos(at)$ or $y(t) = \sin(at)$

Euler's Identity:

$$e^{iat} = \cos(at) + i\sin(at)$$

$$\begin{aligned}
\mathcal{L}\left\{e^{iat}\right\} &= \mathcal{L}\left\{\cos(at)\right\} + i\mathcal{L}\left\{sin(at)\right\} \\
&= \frac{1}{s-ia} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia} \\
&= \frac{s+ia}{s^2 - ias + ias - i^2a^2} = \frac{s+ia}{s^2+a^2} \\
&= \underbrace{\frac{s}{s^2+a^2}}_{\mathcal{L}\{\cos(at)\}} + i\underbrace{\frac{a}{s^2+a^2}}_{\mathcal{L}\{\sin(at)\}}
\end{aligned}$$

Same result can be found through integration by parts (twice).

ex: $y(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt = \int_0^\infty e^{-st} t dt$$

let $\begin{array}{l} u = t,\ du = dt \\ dv = e^{-st} dt,\ v = -\frac{e^{-st}}{s} \end{array}$

$$\int te^{-st} dt = uv - \int vdu = \frac{te^{-st}}{s} + \int \frac{e^{-st}}{s}$$

$$= \frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} = -\frac{e^{-st}(st+1)}{s^2}$$

$$\mathcal{L}\{t\} = \lim_{A\to\infty} -\frac{e^{-st}(st+1)}{s^2}\Big|_0^A = \lim_{A\to\infty} -\frac{e^{-sA}(sA+1)}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} \qquad (s > 0)$$

For $y(t) = t^k$, integrate by parts k times.

General Results

For any constants C, a, ω, and k we have the Laplace Transforms of the following functions $y(t)$:

$y(t) = C$	$\mathcal{L}\left\{C\right\} = \dfrac{C}{s}$	Constant
$y(t) = e^{at}$	$\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	Exponential Function
$y(t) = \cos(\omega t)$	$\mathcal{L}\left\{\cos \omega t\right\} = \dfrac{s}{s^2+\omega^2}$	Cosine
$y(t) = \sin(\omega t)$	$\mathcal{L}\left\{\sin \omega t\right\} = \dfrac{\omega}{s^2+\omega^2}$	Sine
$y(t) = t^k$	$\mathcal{L}\left\{t^k\right\} = \dfrac{k!}{s^{k+1}}$	Power Function

Summary

- Laplace transform (LT) maps $f(t) \to F(s)$
  - From "t-space" to "s-space" .
  - We will learn to invert the transform in the next lecture.
- LT: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$
  - Evaluation of the integrals is tedious.
  - We use general results to quickly transform functions.
  - Many tables exist online and in textbooks.
- LT is linear because the integral is linear
  1. $\mathcal{L}\{f+g\} = F(s) + G(s)$
  2. $\mathcal{L}\{cf\} = cF(s)$