# Recall: we have always rearranged products as sums

ex: $Y(s) = \frac{1}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs + D}{s^2 + 1}$

After partial fraction decomposition...

$$
\begin{aligned}
A = 0, B = 1, C = 0, D = -1 \\
Y(s) &= \frac{1}{s^2} - \frac{1}{s^2 + 1} \\
&= \mathcal{L}\{t\} - \mathcal{L}\{\sin(t)\}
\end{aligned}
$$

$$y(t) = t - \sin(t)$$

It is possible to deal with the product directly!

# Convolutions

We denote the convolution of two functions $f$ and $g$ by the symbol $f * g$, with

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

**Convolutions are:**

1. Commutative/Symmetric
   - $f * g = g * f$
2. Linear
   - $f * (g + h) = f * g + f * h$, where $h$ is a function
   - $f * (cg) = c(f * g) = (cf) * g$, where $c$ is a constant
3. Associative
   - $f * (g * h) = (f * g) * h$

Convolutions are useful for inverting products of Laplace Transforms

# Convolution Theorem

If $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$ are known functions, then

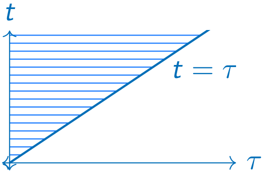
$$\boxed{\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g} = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$$

or conversely

$$\boxed{\mathcal{L}\{f * g\} = F(s) \cdot G(s)}$$

# Proof of the convolution theorem with $h(t) = f(t) * g(t)$

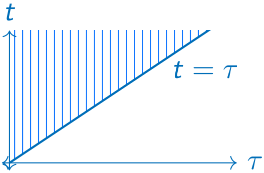
$$\mathcal{L}\left\{h(t)\right\} = \int_0^\infty e^{-st} h(t) dt = \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau) g(t-\tau) e^{-st} d\tau \, dt$$




equivalent areas

$\Leftrightarrow$

switch integration order




$$= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau) g(t-\tau) e^{-st} dt \, d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau) e^{-s\tau} \int_{t=\tau}^{\infty} g(t-\tau) e^{-s(t-\tau)} d\tau \, dt \qquad \text{let } u = t - \tau$$

$$= \underbrace{\int_{\tau=0}^{\infty} f(\tau) e^{-s\tau} d\tau}_{F(s)} \underbrace{\int_{u=0}^{\infty} g(u) e^{-su} du}_{G(s)} \qquad t = \tau \Rightarrow u = 0$$

$$= F(s) G(s)$$

ex: $y'' + y = t$ with $y(0) = y'(0) = 0.$

Use the convolution theorem to find $y(t)$.

$$s^2 Y(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \mathcal{L}\left\{t^2\right\} \cdot \mathcal{L}\left\{\sin(t)\right\}$$

$$y(t) = t * \sin(t) = \int_0^t (t-\tau)\sin(\tau)d\tau$$

$$\overset{\text{by parts}}{=} \left[-t\cos(\tau) - \sin(\tau) + \tau\cos(\tau)\right]_{\tau=0}^{t}$$

$$= \cancel{-t\cos(t)} - \sin(t) + \cancel{t\cos(t)} + t\cos(0) + \cancel{\sin(0)} + \cancel{0\cos(0)}$$

$$= t - sin(t)$$

ex: $y'' + y = g(t)$ with $y(0) = 3,\ y'(0) = 5.$

Use the convolution theorem to find an general expression for $y(t)$.

$$\begin{aligned} s^2 Y(s) - 3s - 5 + Y(s) &= G(s) \\ (s^2+1)Y(s) &= G(s) + 3s + 5 \\ Y(s) &= \frac{G(s)}{s^2+1} + 3\frac{s}{s^2+1} + 5\frac{1}{s^2+1} \end{aligned}$$

$$y(t) = \sin(t) * g(t) + 3\cos(t) + 5\sin(t)$$

we call $\sin(t)$ the impulse response function.

$$y(t) = \underbrace{\int_0^t \sin(t-\tau)g(\tau)d\tau}_{\text{particular part}} + \underbrace{3\cos(t) + 5\sin(t)}_{\text{homogeneous part}}$$

This approach allows us to solve whole classes of ODEs at once.

# Impulse Response Function

Suppose we want to solve

$$ay'' + by' + cy = g(t), \quad \text{with } y(0) = y_0, y'(0) = v_0,$$

then we can define the impulse response function $f(t)$ as the solution to

$$af'' + bf' + cf = \delta(t), \quad \text{with } y(0) = 0, y'(0) = 0$$

$$F(s) = \frac{1}{as^2 + bs + c} \qquad\qquad f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$$

then

$$Y(s) = G(s)\frac{1}{as^2 + bs + c} + \frac{(as + b)y_0 + av_0}{as^2 + bs + c}$$

$$y(t) = g(t) * f(t) + \mathcal{L}^{-1}\left\{\frac{ay_0 s + (by_0 + av_0)}{as^2 + bs + c}\right\}$$

Use the convolution theorem to find $c > 1$ such that the solution to

$$y'' + y = \delta(t-1) - \delta(t-c) \quad \text{with } y(0) = 0, y'(0) = 0$$

is zero for $t \geq c$. Assume $t > c$, then

$$y(t) = \int_0^t f(t-\tau)(\delta(\tau-1) - \delta(\tau-c))dt = f(t-1) - f(t-c)$$

where $f(t)$ is the impulse response function, i.e., it solves

$$f'' + f = \delta(t)$$

$$s^2F(s) + F(s) = 1 \quad \Rightarrow F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin(t)$$

$$\sin(t-1) - \sin(t-c) = 0 \quad \Rightarrow t - c + 2m\pi = t - 1 \quad m \in \mathbb{Z}^+$$

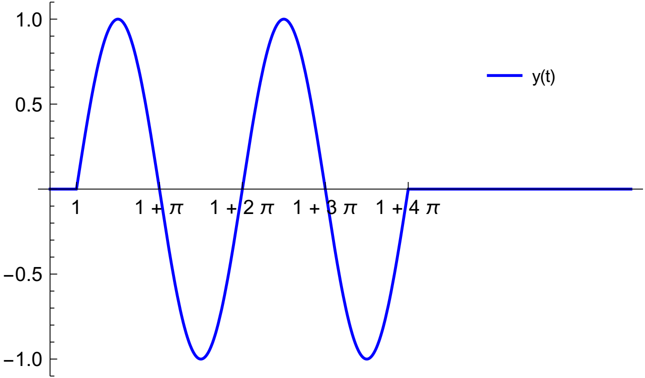
$$c = 1 + 2m\pi$$

ODE:

$$y'' + y = \delta(t-1) - \delta(t-c) \quad \text{with } y(0) = 0, y'(0) = 0$$

Solution:

$$y(t) = u(t-1)\sin(t-1) - u(t-c)\sin(t-c), \quad \text{with } c = 1 + 2m\pi$$



The convolution theorem also allows us to solve integro-differential equations. ex: $y'(t) = e^t + 2\int_0^t y(t-\tau)e^\tau\,d\tau$ with $y(0) = 0$

$$sY(s) = \frac{1}{s-1} + 2\mathcal{L}\left\{\int_0^t y(t-\tau)e^\tau\,d\tau\right\} = \frac{1}{s-1} + 2Y(s)\frac{1}{s-1}$$

$$\left(s - \frac{2}{s-1}\right)Y(s) = \frac{1}{s-1} \qquad \Rightarrow \qquad \frac{\overbrace{s^2 - s - 2}^{(s-2)(s+1)}}{\cancel{s-1}}Y(s) = \frac{1}{\cancel{s-1}}$$

$$Y(s) = \frac{1}{(s-2)(s+1)}$$

$$y(t) = \int_0^t e^{2\tau}e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t e^{3\tau}\,d\tau = \frac{e^{-t}}{3}\left[e^{3\tau}\right]_{\tau=0}^{t}$$

$$= \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

# Convolution Theorem: Special Case

Let $f(t)$ be an integrable function and $g(t) = 1$.

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = F(s)\mathcal{L}\left\{1\right\}$$

$$= \frac{F(s)}{s} \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

ex: Find the inverse Laplace tranforms of $H(s) = \frac{1}{s(s+7)}$

$$h(t) = \int_0^t e^{-7\tau}d\tau = -\frac{1}{7}\left[e^{-7\tau}\right]_{\tau=0}^{t} = -\frac{1}{7}\left(e^{-7t} - 1\right)$$