Recall: Eigenproblem for 2×2 Linear Systems of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

$$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0 \quad \Leftrightarrow \quad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Three possibilites:

1. 2 distinct real eigenvalues/vectors ✓
2. A complex conjugate pair of eigenvalues/vectors ✓
3. One eigenvalue is repeated, only one eigenvector (To Do)

ex: Find the general solution to $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{x}$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2 \qquad \text{repeated eigenvalue}$$

We can find one fundamental solution from the eigenvector $\vec{v}$

$$(\mathbf{A} - 2\mathbf{I})\vec{v} = \vec{0} \qquad \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$$v_1 + v_2 = 0 \qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so $\vec{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Finding the "extra" fundamental solution with repeated λ

Guess $\vec{x}_2(t) = (\vec{w} + t\vec{u})\, e^{\lambda t}$, where $\vec{w}$ and $\vec{u}$ are constant vectors

Plug guess into ODE:

$$\underline{\text{ODE:}} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_2 = \mathbf{A}\vec{x}_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_2 = \lambda\vec{w}e^{\lambda t} + \vec{u}e^{\lambda t} + \lambda\vec{u}te^{\lambda}t$$

$$\underline{\text{ODE:}} \quad \lambda\vec{w} + \vec{u} + \lambda\vec{u}t = \mathbf{A}\vec{w} + t\mathbf{A}\vec{u}$$

group by powers of t

$$\underline{t^1}: \quad \mathbf{A}\vec{u} = \lambda\vec{u} \qquad \Rightarrow \qquad \vec{u} = \text{the eigenvector}$$

$$\underline{t^0}: \quad \mathbf{A}\vec{w} = 2\vec{w} + \vec{u}$$

$$(\mathbf{A} - \lambda\mathbf{I})\,\vec{w} = \vec{u} \qquad \Rightarrow \qquad \vec{w} = \text{a generalized eigenvector}$$

ex: Find the general solution to $\dfrac{d}{dt}\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\vec{x}$

We have $\vec{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{x}_2(t) = \left(\vec{w} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

We know

$$(\mathbf{A} - 2\mathbf{I})\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{Aug. Matrix: } \left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} -1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \Rightarrow -w_1 - w_2 = 1$$

$$w_2 = -1 - w_1$$

$$\vec{x}_2(t) = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \qquad \vec{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

Note: Repeated Eigenvalues

The diagonal matrix

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

has a repeated eigenvalue $\lambda = a$, with two eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Generally, if the char. poly has a factor

$$(\lambda - a)^m = 0$$

then the eigenvalue has algebraic multiplicity m.

If we can find k eignevectors, the geometric multiplicity of λ is k.

Note: Defective Eigenvalues

Generally, if the char. poly has a factor

$$(\lambda - a)^m = 0$$

then the eigenvalue has algebraic multiplicity m.

If we can find k eignevectors, the geometric multiplicity of λ is k.

We say an eigenvalue λ is defective if $k < m$. ex: $\lambda = 2$ for $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

We can find $m - k$ **generalized eigenvectors** recursively by solving

$$(\mathbf{A} - \lambda I)\vec{w}_n = \vec{w}_{n-1} \quad \text{for } n = 1, \ldots, m - k$$

where $\vec{w}_0$ is the ordinary eigenvector for λ