

Recall: 2^{nd} order linear homogeneous constant coeff. ODEs

$$ay'' + by' + cy = 0, \quad \text{with } y(0) = y_0, \quad y'(0) = v_0$$

has a characteristic polynomial

$$ar^2 + br + c = 0 \quad \text{with root(s) } r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solution is determined by the type of root(s):

1. Real and distinct $\Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
2. Real and repeated $\Rightarrow y_h = c_1 e^{rt} + c_2 t e^{rt}$
3. Complex $\Rightarrow y_h = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \quad \text{with } r_{1,2} = \alpha \pm i\beta$

ex: Spring-Dashpot System

$$mx'' + \beta x' + kx = f(t)$$

m = mass, β = damping coeff., k = spring const., $f(t)$ = external forcing

Simple Harmonic Motion

Spring with no damping and no forcing, i.e.

$$mx'' + kx = 0$$

- Similar: frictionless pendulum, circuit with no resistance. . . .
- General solution: Homogeneous problem

Guess : $x(t) = e^{rt}$

$$mr^2 e^{rt} + ke^{rt} = 0$$

$$mr^2 + k = 0$$

$$r = \pm \frac{\sqrt{-4km}}{2m} = \pm i \sqrt{\frac{k}{m}}$$

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t); \quad \omega_0 = \sqrt{k/m}$$

Simple Harmonic Motion: $mx'' + kx = 0$

- General solution:

$$\text{Guess: } x = e^{rt}, \quad mr^2 - k = 0 \Rightarrow r = \pm i\sqrt{k/m}$$

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad \text{with } \omega_0 = \sqrt{k/m}$$

- All solutions have periodic motion with period: $T = \frac{2\pi}{\omega_0}$
- The quantity ω_0 is called the **natural frequency**.
- Obtaining a unique solution requires two initial conditions:
 - $x(0)$ = initial displacement away from rest position.
 - $x'(0)$ = initial velocity of the mass

Amplitude-phase form of solution:

Solution can also be expressed as

$$x(t) = R \cos(\omega_0 t - \varphi)$$

- R = amplitude of motion
 - max displacement
- φ = phase angle (initial phase)
 - by convention $-\pi < \varphi < \pi$

Gen Solution: $R \cos(\omega t - \varphi) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

Trig ident: $\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$

$$\Rightarrow R \cos(\omega t - \varphi) = R [\cos(\varphi) \cos(\omega t) + \sin(\omega t) \sin(\varphi)]$$

by visual inspection

$$R \cos \varphi = c_1 \quad (1)$$

$$R \sin \varphi = c_2 \quad (2)$$

(2)/(1):

$$\frac{R \sin \varphi}{R \cos \varphi} = \tan \varphi = \frac{c_2}{c_1}$$

$$\Rightarrow \varphi = \arctan \left(\frac{c_2}{c_1} \right) \text{ if } c_1 > 0$$

(1)² + (2)²:

$$R^2 \underbrace{(\cos^2 \varphi + \sin^2 \varphi)}_1 = c_1^2 + c_2^2$$

$$R^2 = c_1^2 + c_2^2$$

$$R = \pm \sqrt{c_1^2 + c_2^2}$$

Lookup “atan2” for details

Check at $t = 0$ to determine + or -

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 1cm and released with a velocity of 14 cm/s at $t = 0$.

1. Find the displacement for $t > 0$.
2. Find the natural frequency, period, amplitude, and phase angle of motion.

$$3x'' + 15x = 0$$

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$\omega_0 = \sqrt{k/m} = \sqrt{15/3} = \sqrt{5}$$

Initial Conditions:

$$x(0) = 0.01\text{m} = c_1$$

$$x'(0) = 0.14\text{m/s} = \omega_0 c_2 = \sqrt{5}c_2$$

$$c_2 = \frac{0.14}{\sqrt{5}}$$

$$x(t) = 0.01 \cos \sqrt{5}t + \frac{0.14}{\sqrt{5}} \sin \sqrt{5}t$$

Amplitude and phase angle

$$x(t) = R \cos(\omega_0 t - \phi)$$

$$R = \sqrt{c_1^2 + c_2^2}$$

$$= \sqrt{0.01^2 + \frac{0.14^2}{5}}$$

$$\approx 0.0626$$

$$\varphi = \arctan \left(\frac{c_2}{c_1} \right)$$

$$= \arctan \left(\frac{14}{\sqrt{5}} \right)$$

$$\approx 1.412$$

$$x(t) \approx 0.0626 \cos \left(\frac{t}{\sqrt{5}} - 1.412 \right)$$

Free oscillations with damping

$$x'' + \beta x' + kx = 0$$

What happens for small and large β ?

- $\beta \rightarrow 0$, no damping \Rightarrow recover simple harmonic oscillations
- $\beta \rightarrow \infty$, infinite damping \Rightarrow no oscillations

Characteristic equation: $mr^2 + \beta r + k = 0$ has two roots

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Note: Since $\beta > 0$, the real part of $r_{1,2}$ is always negative
 \Rightarrow exponentially decaying solutions.

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Three cases:

1. $\beta^2 < 4km$: roots are complex conjugates
 - $\exp * (\sin + \cos)$ solutions - **underdamped** motion
2. $\beta^2 > 4km$: two distinct real (negative) roots
 - (bi-exponential solutions) - **overdamped** motion
3. $\beta^2 = 4km$: repeated real root
 - ($e^{rt} + te^{rt}$ solutions) - **critically damped** motion

Underdamped Motion

General solution:

$$x(t) = e^{-\frac{\beta}{2m}t} (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$$

where $\omega_1 = \sqrt{k/m - (\beta/2m)^2} \leq \omega_0$ (damping slows the oscillations)

We can write this in amplitude-phase form

$$x(t) = e^{-\frac{\beta}{2m}t} R \cos(\omega_1 t - \varphi)$$

- $e^{-\frac{\beta}{2m}t} R$ is the time-varying amplitude (or amplitude)
- ω_1 is called the quasi-frequency of motion
 - $T = 2\pi/\omega_1$ is the quasi-period of motion
- φ is the phase angle of motion (or phase shift)

Again, we have $\varphi = \arctan\left(\frac{c_2}{c_1}\right), \quad R = \sqrt{c_1^2 + c_2^2}$

Overdamped Motion: $\beta^2 - 4km > 0$

General Solution:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

with $r_1 < r_2 < 0$.

Critically-Damped: $\beta^2 - 4km = 0 \Rightarrow \beta = 2\sqrt{k \cdot m}$

General Solution:

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}$$

with $r = \frac{-\beta}{2m} = -\sqrt{k/m} < 0$.

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

1. Determine the value of the damping constant β for which the system is critically damped.
2. Find the displacement $x(t)$ of the mass if the system is critically damped, assuming an initial velocity of 0 cm/s.

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, and is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

1. Determine the value of the damping constant β for which the system is critically damped.

$$\begin{aligned}\text{Critical damping: } \beta^2 &= 4km \\ &= 4 \cdot 3 \cdot 15 = 180 \\ \beta &= \sqrt{180} \approx 13.416\end{aligned}$$

A spring has spring constant of 15 N/m. A mass of 3 kg is attached to the spring, is then displaced from the equilibrium position by 3cm and released with an initial velocity of 0 cm/s at $t = 0$.

2. Find the displacement $x(t)$ of the mass if the system is critically damped, assuming an initial velocity of 0 cm/s.

$$x(t) = c_1 e^{-\sqrt{\frac{k}{m}}t} + c_2 t e^{-\sqrt{\frac{k}{m}}t} = c_1 e^{-\sqrt{5}t} + c_2 t e^{-\sqrt{5}t}$$

Initial conditions:

$$x(0) = 0.03\text{m} = c_1$$

$$x'(0) = 0 = -\sqrt{5}c_1 + c_2 (1 - \sqrt{5} \cdot 0 \cdot 1) = -\sqrt{5}c_1 + c_2$$

$$c_2 = \sqrt{5}c_1 = \sqrt{5} \cdot 0.03 \approx 0.067$$

$$x(t) = 0.03e^{-\sqrt{5}t} + 0.067te^{-\sqrt{5}t}$$