

# Homogeneous Heat Equation

$$u_t = \alpha u_{xx} \quad \text{with} \quad \begin{array}{l} u(0, t) = u(L, t) = 0 \\ \text{or} \\ u_x(0, t) = u_x(L, t) = 0 \end{array} \quad \text{and} \quad u(x, 0) = u_0(x)$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} \left( a_n \cos \left( \frac{n\pi}{L} x \right) + b_n \sin \left( \frac{n\pi}{L} x \right) \right)$$

$$\underline{u(0, t) = u(L, t) = 0:}$$

$$\underline{u_x(0, t) = u_x(L, t) = 0:}$$

Fourier sine series

$$a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L u_0(x) \sin \left( \frac{n\pi}{L} x \right) dx$$

Fourier cosine series

$$a_n = \frac{2}{L} \int_0^L u_0(x) \cos \left( \frac{n\pi}{L} x \right) dx$$

$$b_n = 0$$

# Inhomogeneous Heat Equation

2 types of inhomogeneities:

1. Inhomogeneous BCs (e.g.,  $u(0, t) \neq 0$  or  $u_x(0, t) \neq 0 \dots$  )
2. Source/Sink Inhomogeneity (Inhomogeneous PDE)

$$u_t = \alpha u_{xx} + \sigma(x)$$

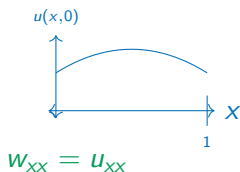
$\sigma(x)$  accounts for local heat production/removal.

Overall approach to solving both is the same, but each example can have its own quirks

ex:  $u_t = \alpha u_{xx}$ ,  $u(0, t) = u(1, t) = 1$   
 $u(x, 0) = 1 + x(1 - x)$  on  $[0, 1]$

Trick: define  $w(x, t) = u(x, t) - 1$

$$w_t = u_t$$



let's write down a PDE for  $w(x, t)$

$$w_t = \alpha w_{xx}$$

$$w(0, t) = w(1, t) = 0$$

$$w(x, 0) = x(1 - x)$$

we've solved this one before

$$w(x, t) = - \sum_{n=1}^{\infty} \frac{4}{\pi^3} \frac{(-1)^n - 1}{n^3} \sin(n\pi x) e^{-\alpha n^2 \pi^2 t}$$

$$u(x, t) = 1 + w(x, t)$$

# Inhomogeneous Heat Equation: General Approach

$$u(x, t) = \underbrace{w(x, t)}_{\substack{\text{Transient} \\ \text{Solution} \\ \text{—} \\ \text{Homogeneous Part}}} + \underbrace{u_{\infty}(x)}_{\substack{\text{Steady State} \\ \text{Solution} \\ \text{—} \\ \text{Inhomogeneous Part}}}$$

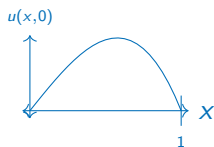
Four steps:

1. Find  $u_{\infty}(x)$  (if it exists)
2. Write down a homogeneous IBVP for  $w(x, t) = u(x, t) - u_{\infty}(x)$
3. Solve for  $w(x, t)$
4. Final solution:  $u(x, t) = w(x, t) + u_{\infty}(x)$

ex:  $u_t = \alpha u_{xx}$ ,

$$u_x(0, t) = u_x(1, t) = 1$$

$$u(x, 0) = x(1 - x^2) \text{ on } [0, 1]$$



1. Find  $u_\infty$

$$u_t = \alpha u_{xx} = 0$$

$$u_\infty(x) = \cancel{C_0}x + C_1 \quad \text{1 from the BCs}$$

The heat flux ( $-\alpha u_x$ ) is the same at both ends is the same. There is no net change in the total amount of heat in the rod.

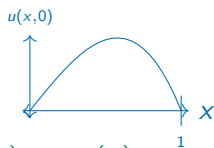
$$\begin{aligned} \int_0^1 u(x, 0) dx &= \int_0^1 u_\infty(x) dx \\ \int_0^1 x(1 - x^2) dx &= \int_0^1 x + C_1 dx = \frac{1}{2} + C_1 \\ \frac{1}{2} - \frac{1}{4} &= \frac{1}{2} + C_1 \quad \Rightarrow C_1 = -\frac{1}{4} \end{aligned}$$

$$u_\infty = x - \frac{1}{4}$$

ex:  $u_t = \alpha u_{xx}$ ,

$$u_x(0, t) = u_x(1, t) = 1$$

$$u(x, 0) = x(1 - x^2) \text{ on } [0, 1]$$



2. Write down a homogeneous IBVP for  $w(x, t) = u(x, t) - u_\infty(x)$

$$w_t = \alpha w_{xx}$$

$$w_x(0, t) = w_x(L, t) = 0$$

$$w(x, 0) = u(x, 0) - u_\infty(x)$$

$$= \frac{1}{4} - x^3$$

3. Solve for  $w(x, t)$

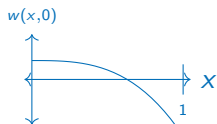
Since we have zero flux boundary conditions for  $w$ , we know it should be represented as a sum of cos terms

$$w(x, t) = \frac{a_0}{2} + \sum e^{-\alpha n^2 \pi^2 t} a_n \cos(n\pi x)$$

ex:  $w_t = \alpha w_{xx}$ ,

$$w_x(0, t) = w_x(1, t) = 0$$

$$w(x, 0) = \frac{1}{4} - x^3 \text{ on } [0, 1]$$



$$a_n = 2 \int_0^1 \left( \frac{1}{4} - x^3 \right) \cos(n\pi x) dx$$

wolfram 
$$-\frac{3(-1)^n (\pi^2 n^2 - 2) + 6}{\pi^4 n^4}$$

$$a_0 = \frac{2}{1} \int_0^1 \frac{1}{4} - x^3 dx$$

$$= 2 \left[ \frac{x}{4} - \frac{x^4}{4} \right] \Big|_0^1$$

$$= 0$$

$$u(x, t) = \underbrace{x - \frac{1}{4}}_{u_\infty} + \underbrace{0}_{a_0/2}$$

$$+ \sum_{n=1}^{\infty} e^{-\alpha n^2 \pi^2 t} a_n \cos(n\pi x)$$

Suppose your cheap landlord has used an insulated wire of length  $L$  as a basic fuse. It is made of a metal that readily converts electrical current into heat when your electrical system is near its limit. Under these conditions, its internal heat production is well-described by a source function

$$\sigma(x) = 80 \frac{^{\circ}\text{C}}{\text{s}} \sin\left(\pi \frac{x}{L}\right),$$

and its internal temperature follows the inhomogeneous heat equation

$$u_t = 0.01 u_{xx} + \sigma(x).$$

Find the solution  $u(x, t)$  assuming that the end of the rods are connected to ice baths (i.e., its is very cold in your apartment).

$$u_t = 0.01 u_{xx} + \sigma(x) \qquad \text{as } t \rightarrow \infty, u(x, t) \rightarrow u_{\infty}(x)$$

$$0 = 0.01 u_{\infty}''(x) + \sigma(x) \qquad \Rightarrow u_{\infty}''(x) = -100\sigma(x)$$



$$\frac{d^2}{dx^2} u_\infty(x) = -100 \times 80 \sin\left(\pi \frac{x}{L}\right)$$

$$\frac{d}{dx} u_\infty(x) = -8,000 \int \sin\left(\pi \frac{x}{L}\right) dx$$

$$\frac{d}{dx} u_\infty(x) = 8,000 \frac{L}{\pi} \cos\left(\pi \frac{x}{L}\right) + C_1$$

$$u_\infty(x) = \int 8,000 \frac{L}{\pi} \cos\left(\pi \frac{x}{L}\right) + C_1 dx$$

$$u_\infty(x) = 8,000 \frac{L^2}{\pi^2} \sin\left(\pi \frac{x}{L}\right) + C_1 x + C_2$$

Find  $C_1$  and  $C_2$  by matching the boundary conditions  $u(0) = u(L) = 0$

$$u_\infty(0) = 0 = C_2 \quad \Rightarrow C_2 = 0$$

$$u_\infty(L) = 0 = C_1 L \quad \Rightarrow C_1 = 0$$

$$u_\infty(x) = 8,000 \frac{L^2}{\pi^2} \sin\left(\pi \frac{x}{L}\right)$$

Define a new PDE for  $w(x, t) = u(x, t) - u_\infty$

Assume that the wire is initially at thermal equilibrium with the environment, i.e.,

$$u(x, 0) = 0$$

$$w_t = 0.01 w_{xx}$$

$$w(0, t) = w(L, t) = 0$$

$$w(x, 0) = \underbrace{-8,000 \frac{L^2}{\pi^2} \sin\left(\pi \frac{x}{L}\right)}_{\text{Fourier Sine Series}}$$

only  $n = 1$  term is non-zero

$$w(x, t) = -8,000 \frac{L^2}{\pi^2} e^{-0.01 \frac{\pi^2}{L^2} t} \sin\left(\pi \frac{x}{L}\right)$$

$$u(x, t) = 8,000 \frac{L^2}{\pi^2} \left(1 - e^{-0.01 \frac{\pi^2}{L^2} t}\right) \sin\left(\pi \frac{x}{L}\right)$$

## Formulas for finding $u_\infty(x)$ : Inhomogeneous BCs

$$u_t = \alpha u_{xx} \quad u(x, t) = u_0(x) \quad \text{with } x \in [0, L]$$

At steady state  $u_t = 0 \implies u = u_\infty(x)$

$$u_\infty''(x) = 0 \implies u_\infty(x) = C_1 + C_2x$$

- $u(0, t) = a, \quad u(L, t) = b$

$$u_\infty(x) = a + \frac{b-a}{L}x$$

- $u_x(0, t) = u_x(L, t) = a$

$$u_\infty(x) = \left( \frac{\int_0^L u_0(x) dx}{L} - \frac{aL}{2} \right) + ax$$

## Formulas for finding $u_\infty(x)$ : Inhomogeneous PDE

$$u_t = \alpha u_{xx} + \sigma(x) \quad u(x, t) = u_0(x) \quad \text{with } x \in [0, L]$$

$$\alpha u_\infty''(x) = -\sigma(x) \implies u_\infty(x) = \underbrace{-\frac{1}{\alpha} \iint \sigma(x) dx^2}_{S(x)} + C_1 x + C_2$$

- $u(0, t) = a, \quad u(L, t) = b$

$$C_2 = a - S(0)$$

$$C_1 = \frac{b - S(L) - C_2}{L}$$