Recall:

We saw how to solve ODEs of the following forms:

1. $y' = p(t) \cdot g(y)$ — Separable Equations
2. $y' + ay = g(t)$ — Constant Coefficient

Now we want to solve something a little more general...

$$y' + p(t)y = g(t)$$

We use a trick involving the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}(y \cdot \mu) = y'\mu + y\mu' = h(t) \qquad \text{where } \mu = \mu(t)$$

solving for the product $y \cdot \mu$

$$\mathrm{d}(y \cdot \mu) = h(t)\mathrm{d}t \qquad \text{integrate} \Rightarrow (y \cdot \mu) = H(t) + C$$

$$\Rightarrow y = \frac{H(t) + C}{\mu(t)}$$

ex: Solve $y' + 2ty = t$

First we do something very odd, multiply everything by e^{t^2}

$$e^{t^2} y' + \underbrace{2te^{t^2}}_{\frac{\mathrm{d}}{\mathrm{d}t} e^{t^2}} y = te^{t^2} \quad \rightarrow \quad \underbrace{e^{t^2} \frac{\mathrm{d}}{\mathrm{d}t} y + \frac{\mathrm{d}}{\mathrm{d}t} e^{t^2} y}_{\text{product rule applied to } (y \cdot e^{t^2})} = te^{t^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t^2} y\right) = t^2 e^{t^3} \qquad \int d\left(e^{t^2} y\right) = \int te^{t^2} dt$$

$$e^{t^2} y = \frac{1}{3} \int e^u du \qquad \begin{aligned} u &= t^2 \\ du &= 2t \end{aligned}$$

$$\begin{aligned} e^{t^2} y &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{t^2} + C \end{aligned}$$

$$y(t) = \underbrace{\frac{1}{2} e^t}_{y_p} + \underbrace{Ce^{-t^2}}_{y_h}$$

Method of integrating factors: $y' + p(t)y = h(t)$

1. Multiply by the integrating factor $\mu(t) = e^{\int p(t)\mathrm{d}t}$

$$\mu(t)y' + \underbrace{p(t)\mu(t)}_{\mu'(t)}y = h(t)\mu(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot y) = h(t)\mu(t)$$

$$\mathrm{d}(\mu \cdot y) = h(t)\mu(t)\mathrm{d}t$$

2. Integrate:

$$\mu \cdot y = \int h(t)\mu(t)\mathrm{d}t + C$$

3. Isolate $y(t)$:

$$y(t) = \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$

ex: Find the general solution of $y' - 3y = e^t$.

$$\boxed{\begin{aligned} y(t) &= \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)} \\ \mu(t) &= e^{\int p(t)\mathrm{d}t} \end{aligned}}$$

$$\begin{aligned}
\mu(t) &= e^{\int (-3)dt} = e^{-3t} \\
e^{-3t}y' - 3e^{-3t}y &= e^t e^{-3t} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-3t}y\right) &= e^{-2t} \\
\mathrm{d}\left(e^{-3t}y\right) &= e^{-2t}\mathrm{d}t \\
e^{-3t}y(t) &= \int e^{-2t}\mathrm{d}t + C \\
y(t) &= \frac{-\frac{1}{2}e^{-2t} + C}{e^{-3t}} \\
&= \underbrace{-\frac{1}{2}e^t}_{y_p} + \underbrace{Ce^{3t}}_{y_h}
\end{aligned}$$

ex: Find the general solution of $y' - 3t^2 y = 3t^2$.

$$y(t) = \frac{\int h(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t)\mathrm{d}t}$$

$$\mu(t) = e^{-\int 3t^2 dt} = e^{-t^3}$$

$$e^{-t^3} y(t) = \int 3t^2 e^{-t^3} \mathrm{d}t$$

let $u = t^3$
$du = 3t^2$

$$\int 3t^2 e^{-t^3} \mathrm{d}t = \int e^{-u} du$$
$$= -e^{-u} + C$$

$$e^{-t^3} y(t) = -e^{-t^3} + C$$

$$\boxed{y(t) = -1 + Ce^{t^3}}$$

ex: $ty' + 5y = 24t^3$, with $y(1) = 2$

$$y' + \frac{5y}{t} = 24t^2$$

$$\mu(t) = e^{\int \frac{5}{t}\,dt} = e^{5\ln(t)} = \left(e^{\ln(t)}\right)^5 = t^5$$

$$t^5 y(t) = \int 24t^7\,dt = 3t^8 + C$$

$$y(t) = 3t^3 + Ct^{-5}$$

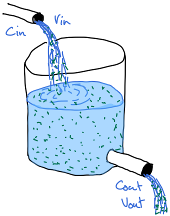
$$y(1) = 2 = 3 + C \qquad C = -1$$

$$\boxed{y(t) = 3t^3 - t^{-5}}$$

Application: Mixing Tank

At time $t = 0$ a tank contains 10 kg of salt dissolved in 200 liters of water. Water containing $\frac{1}{2}$kg of salt per liter enters the tank at a rate of 2 L/min and the well mixed solution leaves the tank at the same rate.

(a) Let $Q(t)$ be the amount of salt in kilograms in the solution after t minutes have passed. Find a formula for the rate of change in the amount of salt, $\frac{dQ}{dt}$, in terms of the amount of salt in the solution, $Q(t)$.



$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \text{in} - \text{out} = 0.5\frac{kg}{L} \times 2\frac{L}{min} - \frac{Q\ kg}{200L} \times 2\frac{L}{min}$$

$$= 1 - \frac{Q}{100}$$

(b) Find $Q(t)$.

$$\begin{gathered}\mu(t) = e^{t/100} \\ Q(0) = 10\end{gathered} \quad \Rightarrow \quad Q(t) = 100 - 90e^{-t/100}$$

Summary

- General linear 1st order ODE: $y' + p(t)y = h(t)$
  - To solve, turn the LHS into an total derivative:

$$y' + p(t)y \quad \to \quad \mathrm{d}(\mu \cdot y) = \mu y' + \mu' y$$

    - Multiply by an integrating factor $\mu(t) = e^{\int p(t)\mathrm{d}t}$
    - General solution: $y(t) = \frac{\int g(t)\mu(t)\mathrm{d}t + C}{\mu(t)}$

- This method solves all linear 1st order ODEs, provided that $p(t)$ and $h(t)$ are continuous.