Fourier Series

Given any periodic function $f(t)$ with period T, we can approximate $f(t)$ as a Fourier series

$$f(t) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)$$

with

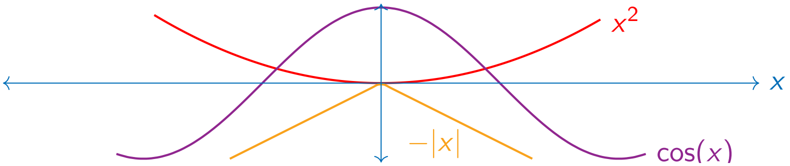
$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)dt & &= \frac{2}{T}\int_\alpha^{\alpha+T} f(t)dt \\
a_n &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt & &= \frac{2}{T}\int_\alpha^{\alpha+T} f(t)\cos\left(\frac{2n\pi t}{T}\right)dt \\
b_n &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt & &= \frac{2}{T}\int_\alpha^{\alpha+T} f(t)\sin\left(\frac{2n\pi t}{T}\right)dt
\end{aligned}$$

We always integrate over one period, and can choose α to make the integrals simpler to evaluate.

Even and Odd Functions:

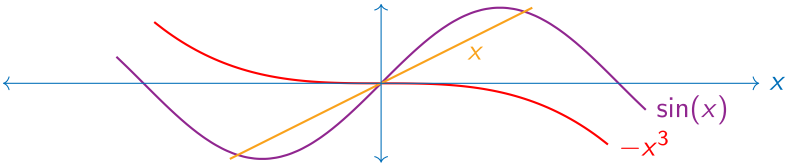
Even Functions:

$$\boxed{f(-x) = f(x)}$$



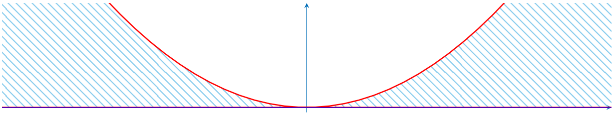
Odd Functions:

$$\boxed{f(-x) = -f(x)}$$

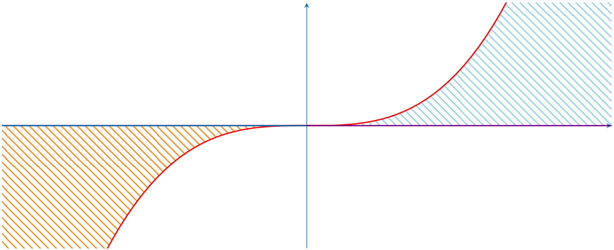


Even and Odd Functions: Integral Properties

Even Functions: The integral of an even function on the interval $[-L, L]$ is double its integral on $[0, L]$



Odd Functions: The integral of an odd function on the interval $[-L, L]$ is 0.



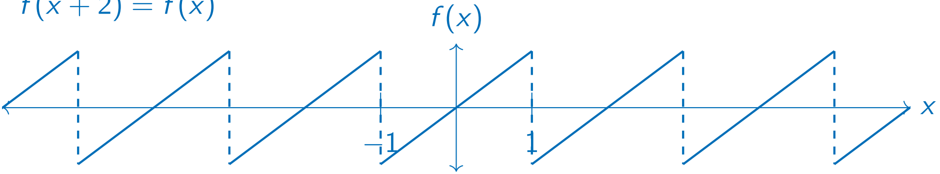
Even and Odd Functions: Products of odd/even functions

Works like multiplying real numbers

$$\text{even} \Leftrightarrow +1$$
$$\text{odd} \Leftrightarrow -1$$

$$\text{odd} \cdot \text{odd} = \text{even} \qquad \text{even} \cdot \text{even} = \text{even} \qquad \text{even} \cdot \text{odd} = \text{odd}$$
$$-1 \cdot -1 = +1 \qquad +1 \cdot +1 = +1 \qquad +1 \cdot -1 = -1$$

Find the Fourier Series representation of $f(x) = x$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$



$T = 2$ cosine terms:

$$a_n = \frac{2}{2}\int_{-1}^{1} \underbrace{\underbrace{x}_{\text{odd func.}} \; \underbrace{\cos(n\pi x)}_{\text{even func.}}}_{\text{odd func.}} \, dx$$

$$= 0 \qquad \Rightarrow \qquad \text{No cos terms in the Fourier Series}$$

Find the Fourier Series representation of $f(x) = x$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$ sine terms:

$$b_n = \frac{1}{1}\int_{-1}^{1} x\sin(n\pi x)dx \quad \text{let} \quad \begin{array}{ll} u = x & du = dx \\ dv = \sin(n\pi x)dx & v = -\frac{\cos(n\pi x)}{n\pi} \end{array}$$

$$\int_{-1}^{1} x\sin(n\pi x)dx = -\left(x\frac{\cos(n\pi x)}{n\pi}\right)\Big|_{-1}^{1} + \int_{-1}^{1}\frac{\cos(n\pi x)}{n\pi}dx$$

$$= \frac{-1}{n\pi}(\cos(n\pi) + cos(n\pi)) + \cancel{\frac{\sin(n\pi x)}{n^2\pi^2}\Big|_{-1}^{1}}$$

use the identity $\cos(n\pi) = (-1)^n$

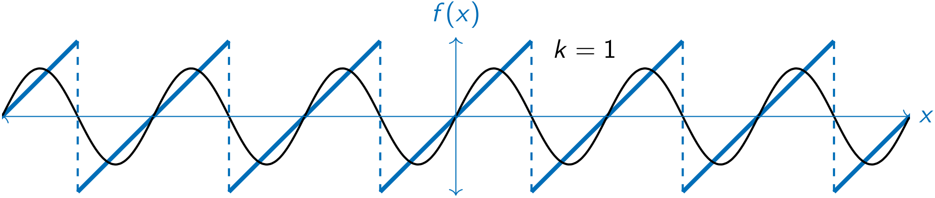
$$= -2\frac{(-1)^n}{n\pi}$$

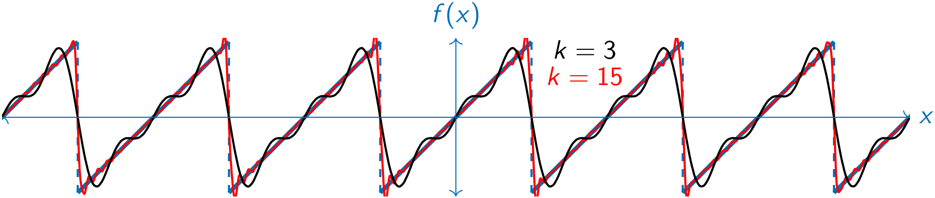
$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin(n\pi x)$$

Finite Fourier Series

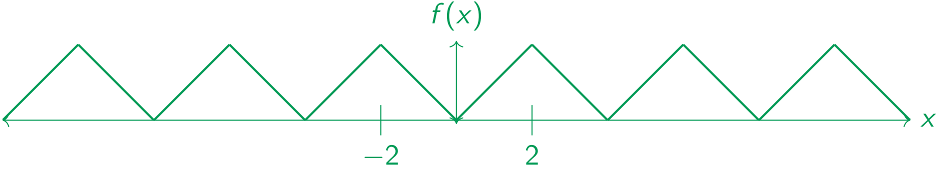
$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos(\omega_n x) + \sum_{n=1}^{k} b_n \sin(\omega_n x) \qquad \omega_n = n\frac{2\pi}{T}$$

$a_n = 0 \qquad b_n = -2\frac{(-1)^n}{n\pi}$





Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$



$$a_n = \frac{1}{2} \int_{-2}^{2} \underbrace{\underbrace{|x|}_{\text{even func.}} \underbrace{\cos\left(n\frac{\pi}{2}x\right)}_{\text{even func.}}}_{\text{even func.}} dx$$

The integral of an even function on $[0, L]$ is half its integral from $[-L, L]$

$$a_n = \int_{0}^{2} x \cos\left(n\frac{\pi}{2}x\right) dx$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$

$$a_n = \int_0^2 x \cos\left(n\frac{\pi}{2}x\right) dx$$

$$\text{let} \quad \begin{matrix} u = x & du = dx \\ dv = \cos\left(n\frac{\pi}{2}x\right) & v = 2\frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} \end{matrix}$$

$$= \cancelto{0}{2\left(x\frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi}\right)\Big|_0^2} - 2\int_0^2 \frac{\sin\left(n\frac{\pi}{2}x\right)}{n\pi} dx$$

$$= \frac{4}{n^2\pi^2}\cos\left(n\frac{\pi}{2}x\right)\Big|_0^2 = \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} -\frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$

$\frac{a_0}{2}$ is the average value of the function (DC component)

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} |x| dx = \int_0^2 x dx \\ &= \frac{x^2}{2}\bigg|_0^2 \\ &= \frac{4}{2} - 0 \\ &= 2 \end{aligned}$$

Compute the Fourier Series for $f(x) = |x|$ for $x \in [-2, 2]$ with $f(x+4) = f(x)$

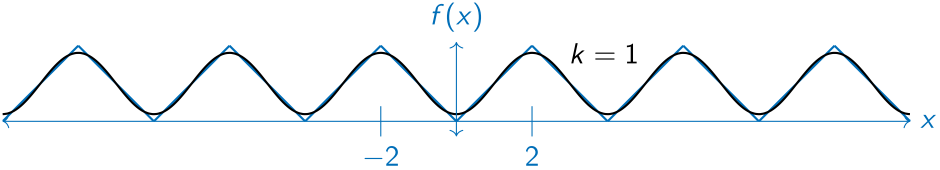
$$b_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\underbrace{|x|}_{\text{even func.}} \underbrace{\sin\left(n\frac{\pi}{2}x\right)}_{\text{odd func.}}}_{\text{odd func.}} dx$$

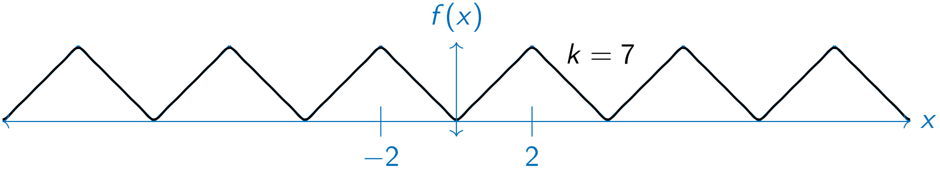
$$\Rightarrow b_n = 0$$

Finite Fourier Series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos(\omega_n x) + \sum_{n=1}^{k} b_n \sin(\omega_n x) \qquad \omega_n = n\frac{2\pi}{T}$$

$$a_0 = 2 \qquad a_n = \begin{cases} -\frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad b_n = 0$$





Even and Odd Functions: Fourier Series

Even Function: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x)$ $\omega_n = n\frac{2\pi}{T}$

Proof:

$$b_n = \int_{-L}^{L} \underbrace{\text{even func.} \times \sin(\omega_n)}_{\text{odd func.}} = 0$$

$\omega_n = n\frac{2\pi}{T}$

Odd Function: $f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\omega_n x)$ $\omega_n = n\frac{2\pi}{T}$

Proof:

$$a_n = \int_{-L}^{L} \underbrace{\text{odd func.} \times \cos(\omega_n)}_{\text{odd func.}} = 0$$

Even function, only cos terms - Odd function, only sin terms

Compute the Fourier Series for $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$

$$a_n = \frac{1}{1}\int_0^1 x\cos(n\pi x)dx$$

$$\text{let} \quad \begin{array}{ll} u = x & du = dx \\ dv = \cos(n\pi x)dx & v = \frac{\sin(n\pi x)}{n\pi} \end{array}$$

$$\int_0^L x\cos(n\pi x)dx = \left(x\frac{\sin(n\pi x)}{n\pi}\right)\Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}dx$$

$$= \frac{\cos(n\pi x)}{n^2\pi^2}\Big|_0^1 = \frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} -\frac{2}{n^2\pi^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

This expression breaks down for $n = 0$...

Compute the Fourier Series for $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$

$\underline{n = 0}$:

$\frac{a_0}{2}$ is the average value of the function (DC component)

$$a_0 = \frac{1}{1} \int_0^1 x dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Compute the Fourier Series for $f(x) = \begin{cases} x & 0 \leq x < 1 \\ 0 & -1 < x < 0 \end{cases}$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$

$$b_n = \frac{1}{1}\int_0^1 \underbrace{\underbrace{x}_{\text{odd func.}} \underbrace{\sin(n\pi x)}_{\text{odd func.}}}_{\text{even func.}} dx$$

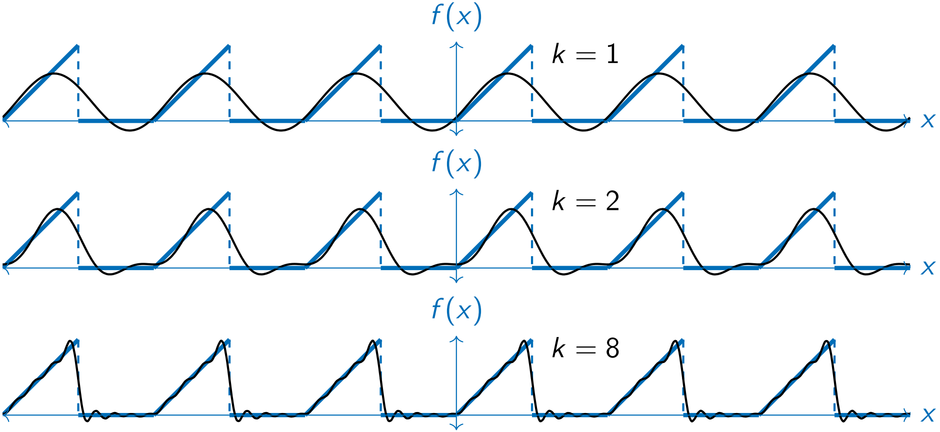
The integral of an even function on [0, L] is half its integral from [-L,L]

$$b_n = \frac{1}{2} \times -2\frac{(-1)^n}{n\pi} = -\frac{(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) + \sum_{n=1}^{\infty} -\frac{(-1)^n}{n\pi}\sin(n\pi x)$$

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos(\omega_n x) + \sum_{n=1}^{k} b_n \sin(\omega_n x) \qquad \omega_n = n\frac{2\pi}{T}$$

$a_0 = \frac{1}{2}$ $\quad a_n = \frac{(-1)^n - 1}{n^2\pi^2}$ $\quad b_n = -\frac{(-1)^n}{n\pi}$



General Fourier Series of a T-periodic Function

Even Function: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x)$

Fourier Cosine Series

Odd Function: $f(x) \approx \sum_{n=1}^{\infty} b_n \sin(\omega_n x)$

Fourier Sine Series

Neither even or odd: $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x)$

Fourier Series

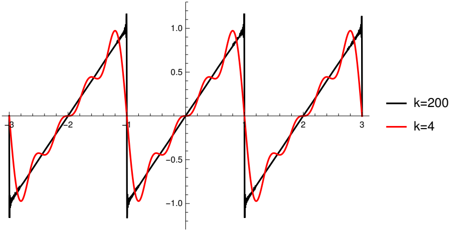
In all cases: $\omega_n = n\frac{2\pi}{T}$

Observations:

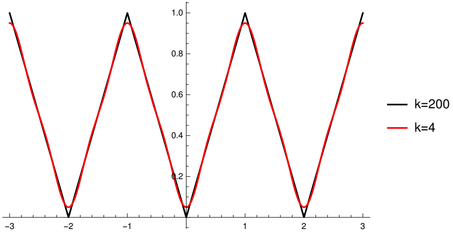
- Take more terms in the Fourier Series and it appears to converge to $f(x)$, (even if $f(x)$ has discontinuities!)
- The coefficients a_n & b_n that we calculate decrease to zero as $n \to \infty$
- The "DC" component $\frac{a_0}{2}$ is simply the mean value of $f(x)$
- Fourier Series overshoot/undershoot the function $f(x)$ at points of discontinuity.
- The above are common features of Fourier series expansions with arbitrary functions.

Fourier Series Convergence

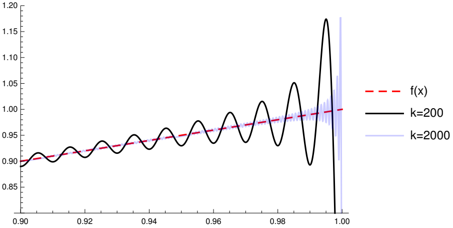
$f(x) = x$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$



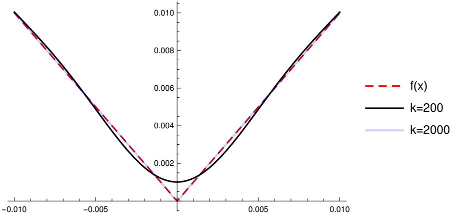
$f(x) = |x|$ for $x \in [-1, 1]$ with $f(x+2) = f(x)$



Zoom in on discontinuity

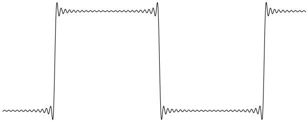


Zoom in on discontinuity



Fourier Series Convergence

- The Fourier Series of any continuous function converges (as $\#$ of terms $\to \infty$) to the function value at every point. $\Rightarrow f(x) = \text{FS}[f]$
- The Fourier Series of a function with jump discontinuities exhibits **Gibb's phenomena**
    - High frequency over/undershooting of the function



- The Fourier Series converges to the midpoint between the two function values at any point of discontinuity x_*. $\Rightarrow f(x) \approx \text{FS}[f]$

$$FS[f](x_*) = \frac{f(x_*^+) + f(x_*^-)}{2}$$

- The rate of convergence of smooth functions is faster than for functions with discontinuities.