Fixed points of a scalar ODE

Fixed points of a differential equation make the time derivative zero.

ex:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\,(5 - y)$$

Two Fixed Points y^*:

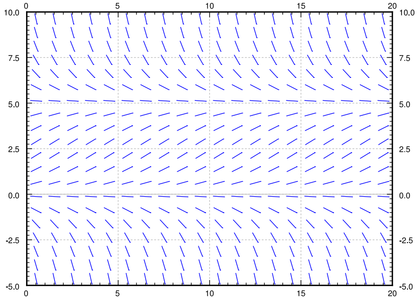
$$y^* = 0$$

and

$$y^* = 5$$

Fixed points are horizontal lines in the slope field

Slope Field: $\frac{\mathrm{d}y}{\mathrm{d}t}$ vs. t



Fixed Points and Stability

Solutions flow towards or away from fixed points:

- Stable fixed points attract nearby solutions.
- Unstable fixed points repel nearby solutions.

ex:

$$\frac{dy}{dt} = y\,(5 - y)$$

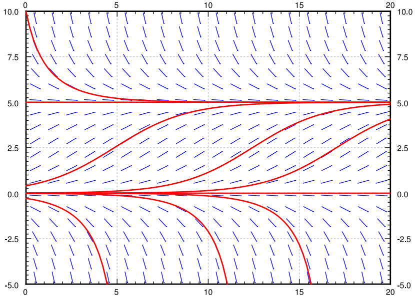
Two Fixed Points y^*:

$y^* = 0$ (unstable)

and

$y^* = 5$ (stable)

Slope Field: $\dfrac{dy}{dt}$ vs. t



2D Vector Fields

The ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

gives us a derivative for each point $\vec{x} \in \mathbb{R}^n$

Restricting ourselves to $\mathbb{R}^2$ with constant $\mathbf{A}$, we can draw an arrow parallel to the derivative at many points in the (x, y)-plane (phase-plan).

Then we can visualize approximate solution flows in the plane.

Note: This system a unique fixed point $\vec{x}^* = \vec{0}$ at the origin.

Qualitative behaviour of solutions: Stability

The origin $\vec{0}$ is always a fixed point for the system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$.

Q: Do solutions approach or move away from this fixed point?
A: It depends on the eigenvalues, λ, of $\mathbf{A}$.

- If all λ's have $\mathrm{Re}(\lambda) < 0$, solutions approach $\vec{0}$ as $t \to \infty$.
  - **Stable fixed point**
- If any λ's have $\mathrm{Re}(\lambda) > 0$, solutions move away from $\vec{0}$ as $t \to \infty$.
  - **Unstable fixed point**
- If a complex conjugate λ pairs exists, solutions exhibit oscillations around $\vec{0}$.
  - **Spiral fixed point**

Classifying Fixed Points (Nodes and Saddles)

The origin $\vec{0}$ is always a fixed point for the system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$.

If all eigevalues are real and ...

- they have the same sign, the fixed point is called a **node**
  - All λ's < 0 $\quad\Rightarrow$ **stable node** (sink)
  - All λ's > 0 $\quad\Rightarrow$ **unstable node** (source)
- some λ's have opposite signs, the fixed point is called a **saddle**

Classifying Fixed Points (Spirals and Centers)

The origin $\vec{0}$ is always a fixed point for the system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$.

If a pair of eigenvalues are complex conjugates, $\lambda_{1,2} = r \pm i\omega$, then the fixed point is called a **spiral** (or focus).

- For a 2×2 matrix, with
  - $r < 0 \quad \Rightarrow$ **stable spiral** (spiral sink)
  - $r > 0 \quad \Rightarrow$ **unstable spiral** (spiral source)
  - $r = 0 \quad \Rightarrow$ **neutral spiral** (spiral center)

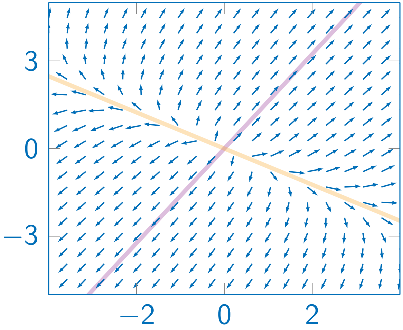
Phase Plane Behaviour of Nodes

$\underline{\lambda_{1,2} \in \mathbb{R}}$ with $|\lambda_2| > |\lambda_1|$

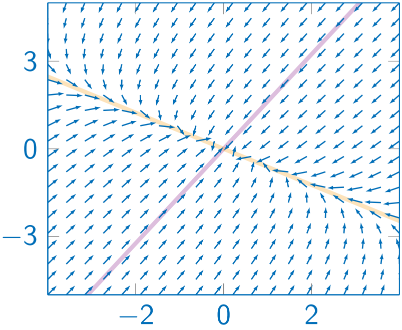
Each eigenvalue/eigenvector is associated with an **eigendirection**

- Solutions flow as straight lines along the eigendirection

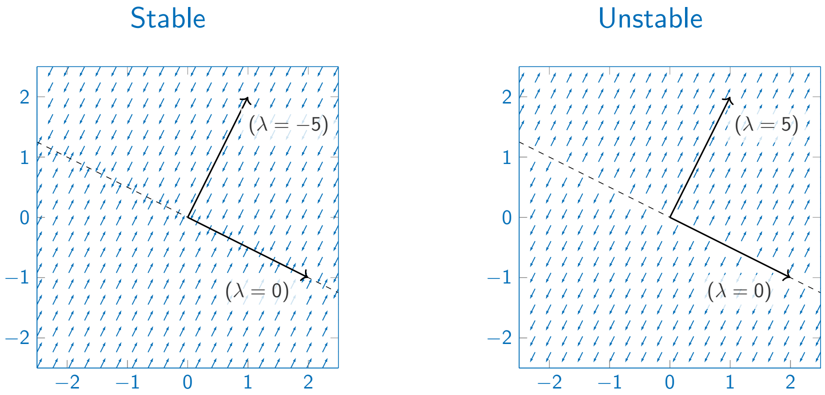
Unstable Node: $0 < \lambda_1 < \lambda_2$



Stable Node: $\lambda_2 < \lambda_1 < 0$



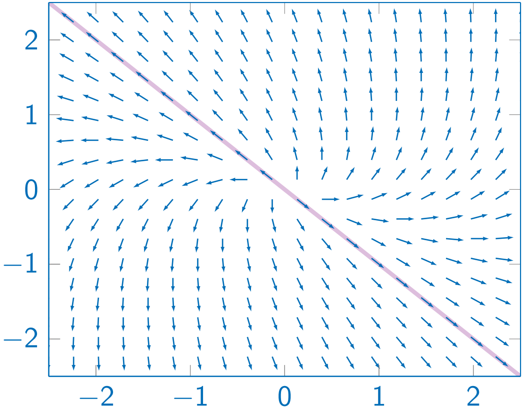
Special Case: Zero Eigenvalue



Line of fixed-points (non-isolated fixed points).

Special Case: Repeated Eigenvalue (**Degenerate Node**) $\lambda_1 = \lambda_2 \in \mathbb{R}$

ex: $\lambda > 0$

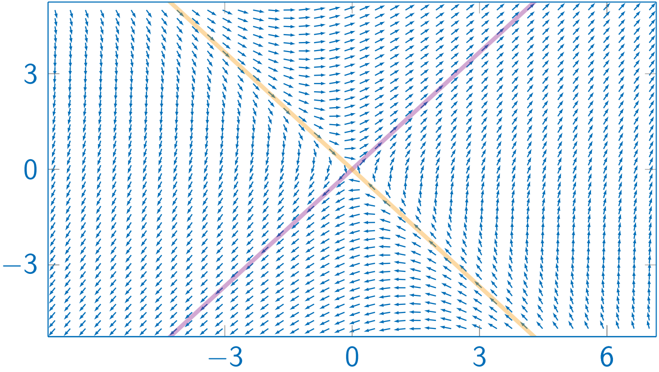


Far from the origin, solutions align with the unique eigendirection. Near the origin, they can rotate.

Phase Plane Behaviour of Saddles

$\lambda_{1,2} \in \mathbb{R}$ with $|\lambda_1| < |\lambda_2|$

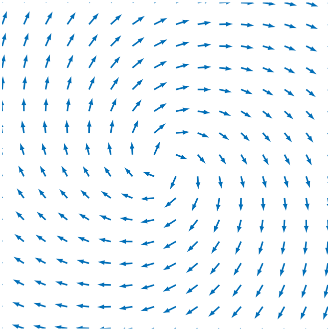
Saddle: $\lambda_1 < 0 < \lambda_2$

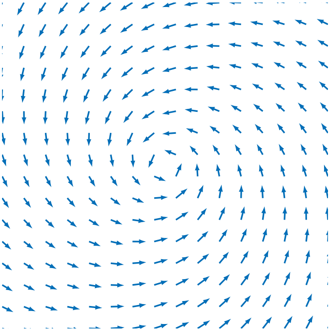


- Eigendirections with $\mathrm{Re}(\lambda) > 0$ are repelling
- Eigendirections with $\mathrm{Re}(\lambda) < 0$ are attracting

Spiral Fixed-Points ($\lambda_{1,2} = r \pm i\omega$)

$r > 0$ $r < 0$

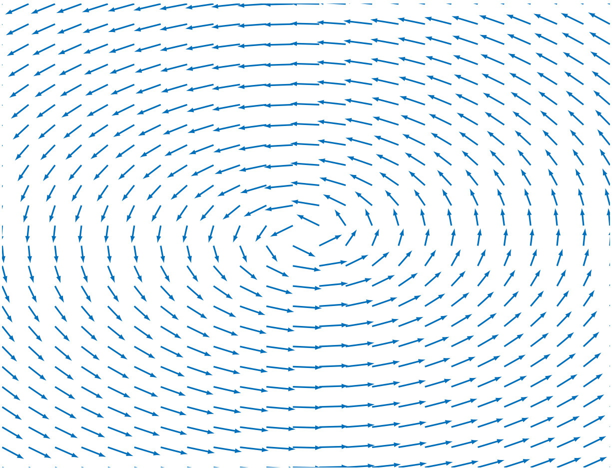




Unstable spiral Stable spiral

We will see how the eigenvectors dictate the direction of rotation shortly.

Center Fixed-Point ($\lambda_{1,2} = \pm i\omega$)



The origin has neutral stability.
Solutions travel around the origin indefinitely.

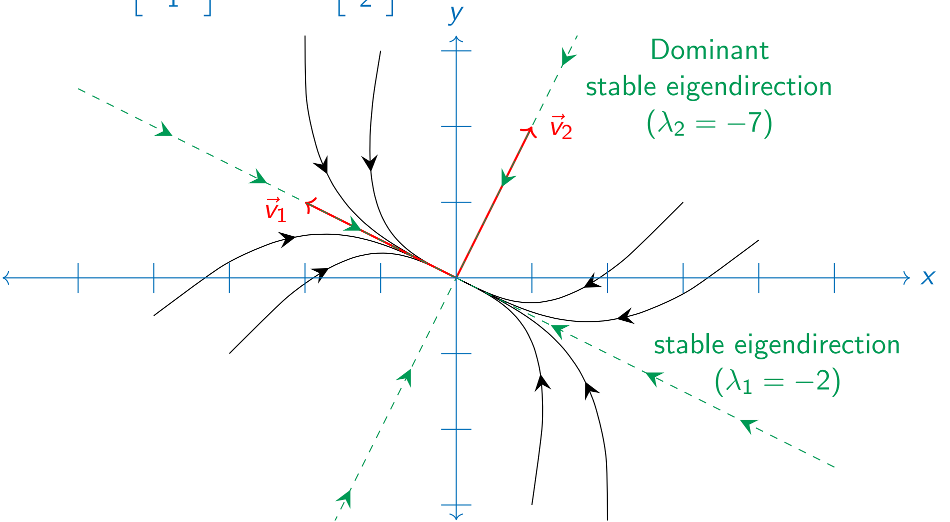
Sketching Solutions near Node/Saddle Fixed Points in the Phase-Plane

For a 2×2 system $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x}$ with real distinct eigenvalues

1. Find the eigenvalues/vectors of the matrix $\mathbf{A}$.
2. Draw the eigendirections (eigenvectors) of the system.
   - Eigendirections with $\lambda > 0$ are repelling
   - Eigendirections with $\lambda > 0$ are attracting
3. Draw a few sample solution flows.
   - As solutions get closer to an eigendirection, they align themselves with that direction.

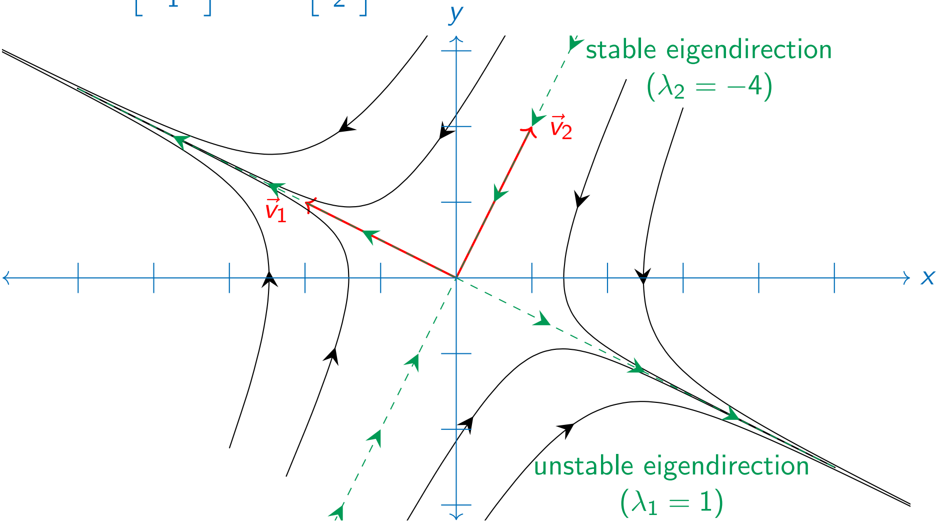
Sketch solutions to $\dfrac{dx}{dt} = -3x - 2y$, $\dfrac{dy}{dt} = -2x - 6y$ in the phase-plane

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t}$$



Sketch solutions to $\dfrac{dx}{dt} = -2y$, $\dfrac{dy}{dt} = -2x - 3y$ in the phase-plane

$$\vec{x}(t) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-4t}$$



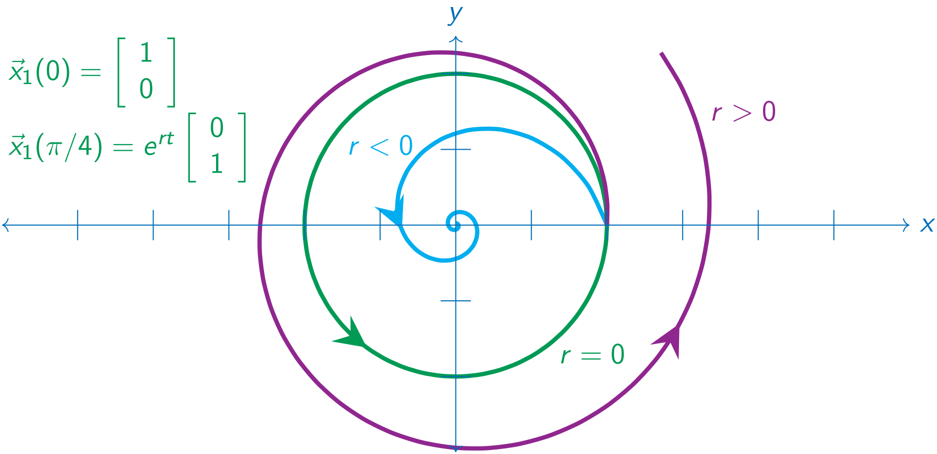
Sketch the solution behaviours for

$$\frac{dx}{dt} = rx + -2y$$
$$\frac{dy}{dt} = 2x + ry$$

$$\vec{x}_1 = e^{rt}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)\right) \qquad \vec{x}_2 = e^{rt}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t)\right)$$

$$\vec{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_1(\pi/4) = e^{rt}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



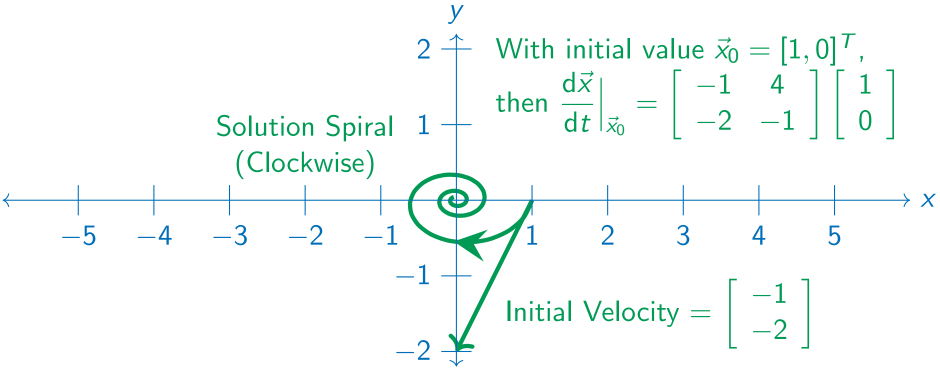
Quickly determining clockwise/counterclowise rotations

Given $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x}$ with complex-conjugate eigenvalues/eigenvectors.

To determine rotation direction:

1. pick a point (e.g., $\vec{x}_0 = [1, 0]^T$)
2. evaluate the derivative $\mathbf{A}\vec{x}_0$

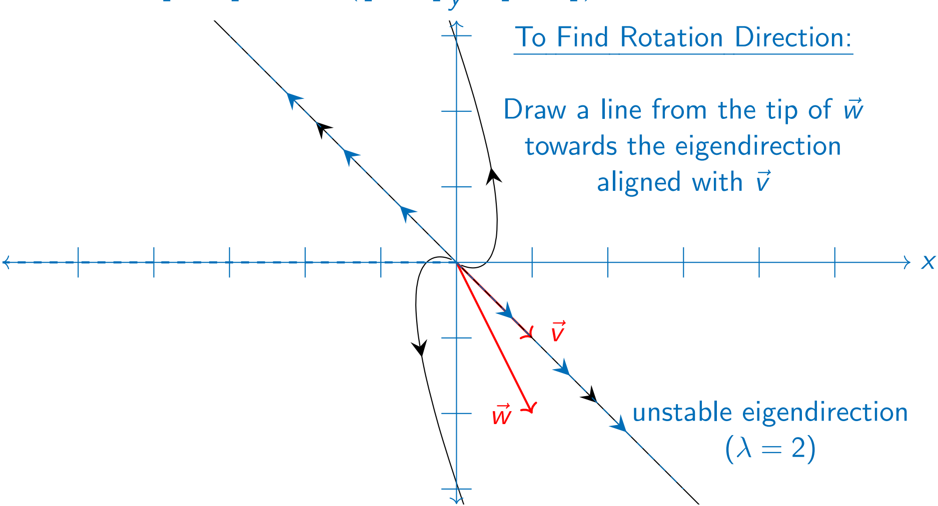
ex: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} -1 & 4 \\ -2 & -1 \end{bmatrix}\vec{x}$ has eigenvalues $\lambda_{1,2} = -1 \pm 2\sqrt{2}i$



With initial value $\vec{x}_0 = [1, 0]^T$, then $\left.\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right|_{\vec{x}_0} = \begin{bmatrix} -1 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Sketch solutions to $\dfrac{dx}{dt} = x - y$, $\dfrac{dy}{dt} = x + 3y$ in the phase-plane

$$\vec{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$



To Find Rotation Direction:

Draw a line from the tip of $\vec{w}$ towards the eigendirection aligned with $\vec{v}$

unstable eigendirection ($\lambda = 2$)