Linear Second Order Non-Constant Cofficient IVPs

$$p(t)y'' + q(t)y' + r(t)y = g(t), \quad \text{with } y(t_0) = y_0, y'(t_0) = v_0$$

Homogeneous Problems ($g(t) = 0$):

Find two linearly independent solutions y_1 and y_2.

$$\text{Superposition:} \quad y = c_1 y_1 + c_2 y_2$$

Methods:

Reduction of Order or Ansatz Method

Inhomogeneous Problems ($g(t) \neq 0$): $y = c_1 y_1 + c_2 y_2 + y_p$

Reduction of Order or Method of Undetermined Coefficients
(only works if you are lucky)

Euler Equations: $at^2y'' + bty' + cy = 0$

Guess: $y = t^k$ $\qquad y' = kt^{k-1} \qquad y'' = k(k-1)t^{k-2}$

$$ak(k-1)\underbrace{t^2t^{k-2}}_{t^k} + bk\underbrace{tt^{k-1}}_{t^k} + ct^k = 0$$

$$\underbrace{ak^2 + (b-a)k + c}_{\text{char. poly.}} = 0 \qquad k = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

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Three Cases:

1. Real Distinct Roots: $y = c_1 t^{k_1} + c_2 t^{k_2}$

2. Complex Conjugate Roots: $k = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$

$$y_{1,2} = t^{\alpha \pm i\beta} = t^{\alpha} t^{\pm i\beta} = t^{\alpha} e^{\ln\left(t^{\pm i\beta}\right)} = t^{\alpha} e^{\pm i\beta \ln(t)}$$

$$= t^{\alpha}\left(\cos(\beta \ln(t)) \pm i \sin(\beta \ln(t))\right) \quad \Rightarrow \quad \begin{array}{l} y_1 = t^{\alpha}\cos(\beta \ln(t)) \\ y_2 = t^{\alpha}\sin(\beta \ln(t)) \end{array}$$

3. Repeated real: $y = c_1 t^k + c_2 y_2$ where $y_2(t) = ??? \neq t^{k+1}$

Find one solution to $t^2y'' + 3ty' + y = 0$

Guess: $y = t^k \quad \Rightarrow \quad y' = kt^{k-1}$ and $y'' = k(k-1)t^{k-2}$

$$k(k-1)t^k + 3t^k + t^k = 0$$

$$k^2 + (3-1)k + 1 = 0 \qquad k = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

So we find one solution

$$y_1 = t^{-1}$$

How can we find y_2?

The trick of multiplying by t does not work :(

Find the second solution to $t^2y'' + 3ty' + y = 0$

Guess: $y_2 = u(t)y_1$ where $u(t)$ is unknown and $y_1 = t^{-1}$

$$y_2' = u't^{-1} - ut^{-2}, \qquad y_2'' = u''t^{-1} - u't^{-2} - u't^{-2} + 2ut^{-3}$$

Plug into DE:

$$\begin{aligned} t^2(u''t - 2u't^{-2} + 2ut^{-3}) + 3t(u't^{-1} - ut^{-2}) + ut^{-1} &= 0 \\ u''t + (-2+3)u' + (2-3+1)ut^{-1} &= 0 \\ u''t + u' &= 0 \end{aligned}$$

let $w = u'$

$tw' + w = 0$ first order linear ODE $\Rightarrow$ integrating factor

$$w' + \frac{1}{t}w = 0 \quad \Rightarrow \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|} = t$$

$$tw(t) = C \quad \Rightarrow w = C/t$$

$$u' = w = C/t \Rightarrow u(t) = \int C/t dt = C\ln(t) \Rightarrow \quad \boxed{y_2 = t^{-1}\ln(t)}$$

Reduction of Order: $p(t)y'' + q(t)y' + r(t)y = 0$

1. Given one solution $y_1(t)$, guess $y_2 = u(t)y_1$ where $u(t)$ is unknown

2. Plug y_2 into ODE. If you are lucky all the terms with $u(t)$ will cancel.

   - New ODE for $u(t)$: $s(t)u'' + v(t)u' = 0$ (2^{nd} order $\to$ 1^{st} order)
   - Let $w = u'$ $\Rightarrow$ $w' + \frac{v(t)}{s(t)}w = 0$
   - Integrating factor: $\mu(t) = e^{\int \frac{v(t)}{s(t)}dt}$ $\Rightarrow$ $w = C/\mu(t)$
   - Then $u(t) = \int 1/\mu(t)dt$ (without loss of generality, set $C = 1$)

3. General Solution:

$$y_h = c_1 y_1(t) + c_2 y_1(t) \int 1/\mu(t)dt$$

Note: This method only works if you already have $y_1(t)$.

Suppose that $y_1 = \frac{1}{1+t}$ solves $-\frac{1}{2}(1+t)y'' + ty' + y = 0$,

find y_2. Guess: $y_2 = \frac{u(y)}{1+t}$, $\quad y_2' = \frac{u'}{1+t} - \frac{u}{(1+t)^2}$,

$$y_2'' = \frac{u''}{1+t} - \frac{u'}{(1+t)^2} - \frac{u'}{(1+t)^2} + 2\frac{u}{(1+t)^3}$$

$$-\frac{1}{2}(1+t)\left(\frac{u''}{1+t} - 2\frac{u'}{(1+t)^2} + 2\frac{u}{(1+t)^3}\right) + t\left(\frac{u'}{1+t} - \frac{u}{(1+t)^2}\right) + \frac{u(y)}{1+t} = 0$$

$$-\frac{1}{2}u'' + \cancel{(1+t)}\frac{u'}{\cancel{1+t}} + \cancel{(-t-1)}\overset{-1}{\frac{u}{(1+t)^{\cancel{2}}}} + \frac{u}{1+t} = 0$$

$$-\frac{1}{2}u'' + u' = 0$$

let $w = u'$, $\qquad \Rightarrow -\frac{1}{2}w' + w = 0$

$$-\frac{1}{2}w' + w = 0$$

$$w' - 2w = 0 \qquad \Rightarrow \mu = e^{\int -2dt} = e^{-2t}$$

$$\mu w = e^{-2t} = C \qquad \Rightarrow w = Ce^{2t}$$

$$u = \int w dt = \int Ce^{2t} dt = \frac{C}{2}e^{2t}$$

$$y_2 = \frac{u}{1+t} \qquad \boxed{y_2 = \frac{e^{2t}}{1+t}}$$

Given that $y_1 = t^2$ solves $y'' - 2t^{-2}y = 0$ find y_2.

Guess: $y_2 = u(t)t^2, \quad y_2' = u't^2 + 2tu, \quad y_2'' = u''t^2 + 2tu' + 2u + 2tu'$

$$u''t^2 + 4tu' + 2u - 2t^{-2}ut^2 = 0$$
$$u''t^2 + 4tu' = 0$$

let $w = u'$

$$t^2w' + 4tu' = 0$$
$$w' + \frac{4}{t}w = 0$$
$$w = 1/\mu(t) = t^{-4}$$

<u>integrating factor</u>

$$\mu = e^{4\int \frac{1}{t}dt} = e^{4\ln(t)} = t^4$$

$$u(t) = \int wdt = \int t^{-4}dt$$
$$= -\frac{1}{3}t^{-3}$$

$$\boxed{y_2 = t^{-3}y_1(t) = t^{-1}}$$