

# The Heat Equation and the Laplacian

Recall the 1D heat equation

$$u_t = \alpha u_{xx} = \alpha \frac{\partial^2}{\partial x^2} u(x, t)$$

In higher dimensions  $(x, y, \dots)$  this becomes

$$\begin{aligned} u_t &= \alpha (u_{xx} + u_{yy} + \dots) \\ &= \alpha \Delta u \end{aligned}$$

$\Delta$  is the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \dots = \vec{\nabla} \cdot \vec{\nabla} \quad \text{with } \vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \vdots \end{bmatrix}$$

# Steady State Heat Equation

If a steady state heat distribution is attained, we have  $u_t \rightarrow 0$ .

In 1D this gave us

$$0 = u_{xx} \quad \Rightarrow \quad u_{\infty}(x) = C_1x + C_2$$

Given some boundary conditions, we can find  $u_{\infty}$ .

ex: Suppose our domain is  $x \in [0, L]$  with  $u(0, t) = a$  and  $u(L, t) = b$

$$u_{\infty}(x) = a + \frac{b-a}{L}x$$

# Laplace's Equation

In higher dimensions, we need to solve Laplace's Equation

$$\Delta u = 0$$

Restricting ourselves to 2 cartesian dimensions, we get

$$u_{xx} + u_{yy} = 0$$

No longer as simple as finding linear functions

ex:  $u(x, y) = x^2 - y^2$

$$u_{xx} = 2, \quad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

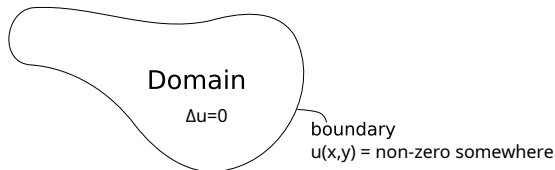
Q: What about BCs?

# Boundary Conditions - Dirichlet Problem

1. Consider a region in the  $x, y$ -plane, the domain for the problem.
2. Specify the value of some unknown function  $u(x, t)$  along the boundaries of the region.
3. Solve Laplace's equation:

$$u_{xx} + u_{yy} = 0$$

4. Impose the boundary conditions.



# A Simple Dirichlet Problem

ex: Consider a rectangular region of width  $w$  and height  $h$ .

Boundary values are zero at three of the four edges, and obey some arbitrary function  $f(x)$  along the fourth edge.

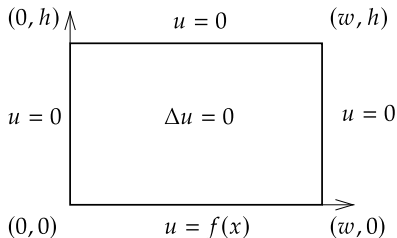
$$\Delta u = 0$$

$$u(0, y) = 0 \quad \text{for } 0 < y < h$$

$$u(x, h) = 0 \quad \text{for } 0 < x < w$$

$$u(w, y) = 0 \quad \text{for } 0 < y < h$$

$$u(x, 0) = f(x) \quad \text{for } 0 < x < w$$



# Solving Dirichlet Problems

$$u_{xx} + u_{yy} = 0$$

Separation of Variables:

$$u(x, y) = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const.}$$

Horizontal Problem:

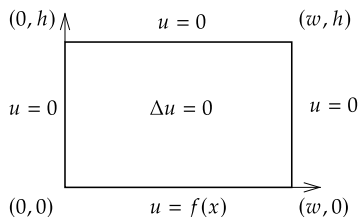
$$X'' + \lambda X = 0$$

Vertical Problem:

$$Y'' - \lambda Y = 0$$

# Solving Dirichlet Problems

$$X'' + \lambda X = 0$$



What are the boundary conditions? What are the allowed solution and values of  $\lambda$ ?

Horizontal Problem:

BCs:  $X(0) = X(w) = 0$

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right) \quad \lambda_n = \left(\frac{n\pi}{w}\right)^2$$

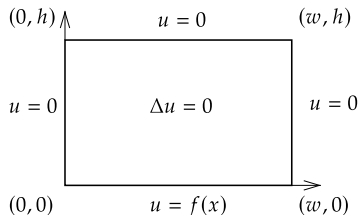
# Solving Dirichlet Problems

$$Y'' - \lambda Y = 0$$

$$\lambda = \left(\frac{n\pi}{w}\right)^2$$

What are the boundary conditions? What types of solutions do we obtain?

Vertical Problem: BCs:  $Y(h) = 0, Y(0) = f(x)$ .



$$\begin{aligned} \text{Try } Y_n = e^{ry} : \quad r^2 e^{ry} - \left(\frac{n\pi}{w}\right)^2 e^{ry} &= 0 \\ r^2 = \left(\frac{n\pi}{w}\right)^2 &= 0 \quad \Rightarrow r = \pm \frac{n\pi}{w} \end{aligned}$$

$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y}$$



$$Y_n(y) = Ae^{\frac{n\pi}{w}y} + Be^{-\frac{n\pi}{w}y} \quad \text{with } Y(h) = 0$$

$$Y(h) = 0 = Ae^{\frac{n\pi}{w}h} + Be^{-\frac{n\pi}{w}h} \quad \Rightarrow A = -Be^{-2\frac{n\pi}{w}h}$$

$$\begin{aligned} Y_n(y) &= B \left[ -e^{-2\frac{n\pi}{w}h} e^{\frac{n\pi}{w}y} + e^{-\frac{n\pi}{w}y} \right] \\ &= B \left[ -e^{-\frac{n\pi}{w}h} e^{\frac{n\pi}{w}(y-h)} + e^{-\frac{n\pi}{w}h} e^{-\frac{n\pi}{w}(y-h)} \right] \\ &= \underbrace{Be^{-\frac{n\pi}{w}h}}_{\text{arbitrary}} \underbrace{\left[ e^{-\frac{n\pi}{w}(y-h)} - e^{\frac{n\pi}{w}(y-h)} \right]}_{\propto \sinh} \\ &= a_n \sinh \left( \frac{n\pi}{w}(h-y) \right) \end{aligned}$$

$$u_n(x, y) = a_n \sin \left( \frac{n\pi}{w}x \right) \sinh \left( \frac{n\pi}{w}(h-y) \right)$$

Eigensolution of the Laplacian that satisfies the zero boundary conditions on the 3 sides.

## The non-zero boundary condition

$$u_n(x, y) = a_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(\frac{n\pi}{w}(h-y)\right), \quad u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$$

$u(x, 0) = f(x)$  - Express the boundary condition as Fourier Series

Given the appearance of our  $u_n$ , we clearly need a Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{w}x\right) \quad \text{with } b_n = \frac{2}{w} \int_0^w f(x) \sin\left(\frac{n\pi}{w}x\right) dx$$

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(n\pi \frac{h}{w}\right)$$

need equality between the two series

$$\Rightarrow a_n = \frac{b_n}{\sinh\left(n\pi \frac{h}{w}\right)}$$

# The Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

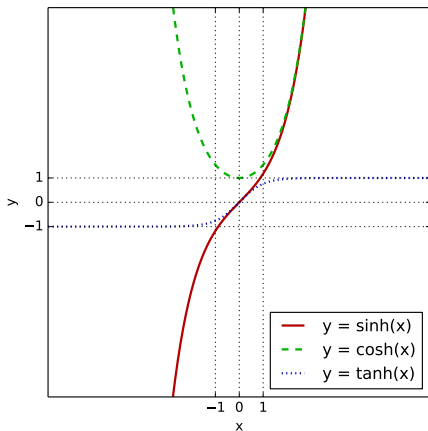
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$



# Boundary conditions on two sides

ex: Consider a rectangular region of width 1 and height 1.

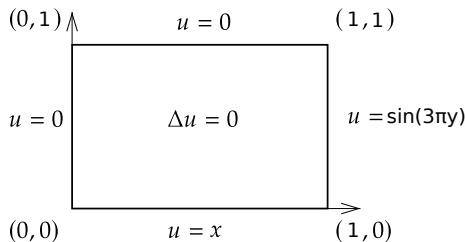
$$\Delta u = 0$$

$$u(0, y) = 0 \quad \text{for } 0 < y < h$$

$$u(x, h) = 0 \quad \text{for } 0 < x < w$$

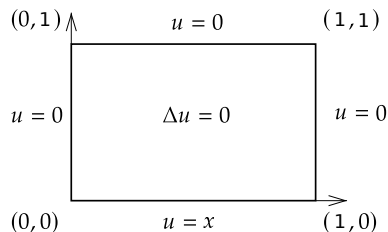
$$u(w, y) = \sin(3\pi y) \quad \text{for } 0 < y < h$$

$$u(x, 0) = x \quad \text{for } 0 < x < w$$

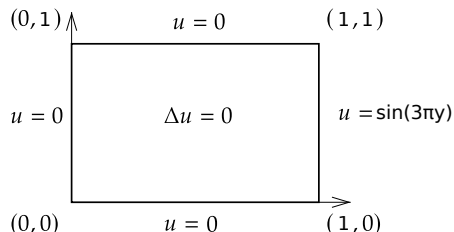


# Breaking it down by directions

## South Problem



## East Problem



$$u(x, y) = u_N + u_S + u_E + u_W$$

The west and north problems have zero solutions, no need to solve them.

# The South Problem

Special case of the previous problem we did with  $f(x) = x$

$$u_s(x, y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \sinh(n\pi(1-y))$$

$$a_n = \frac{b_n}{\sinh(n\pi)}$$

$$\begin{aligned} b_n &= 2 \int_0^1 x \sin(n\pi x) dx \\ &= -2 \frac{(-1)^n}{n\pi} \end{aligned}$$

[https://www.wolframalpha.com/input?i=integral+of+x\\*sin%28n\\*pi\\*x%29+from+0+to+1+assuming+n+is+an+integer](https://www.wolframalpha.com/input?i=integral+of+x*sin%28n*pi*x%29+from+0+to+1+assuming+n+is+an+integer)

# The East Problem

$$\Delta u_E = 0$$

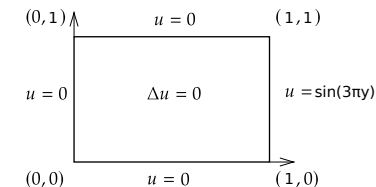
$$u_E(0, y) = 0 \quad \text{for } 0 < y < h$$

$$u_E(x, h) = 0 \quad \text{for } 0 < x < w$$

$$u_E(w, y) = \sin(3\pi y) \quad \text{for } 0 < y < h$$

$$u_E(x, 0) = 0 \quad \text{for } 0 < x < w$$

Horizontal Prob:  $X'' + \lambda X = 0$



Vertical Prob:  $Y'' - \lambda Y = 0$

Now we have two zero BCs for  $Y(y)$

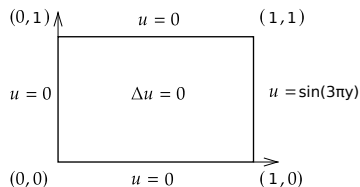
$$Y(0) = Y(1) = 0 \quad \Rightarrow \quad Y_n(y) = \sin(n\pi y)$$

with

$$\lambda_n = -(n\pi)^2$$

# The East Problem

$$X'' + \lambda X = 0 \quad \text{with } \lambda = -(n\pi)^2$$



$$X'' - (n\pi)^2 X = 0$$

$$X(x) = Ae^{n\pi x} + Be^{-n\pi x}$$

$$\text{BC @ } x=0: \quad 0 = A + B$$

$$B = -A$$

$$X(x) = A(e^{n\pi x} - e^{-n\pi x})$$

$$= a_n \sinh(n\pi x)$$

$$u_n(x, y) = a_n \sin(n\pi y) \sinh(n\pi x)$$



# The East Problem

$$u_n(x, y) = a_n \sin(n\pi y) \sinh(n\pi x)$$

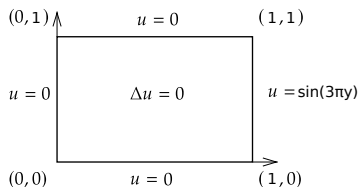
$$u_E = \sum_{n=1}^{\infty} u_n(x, y)$$

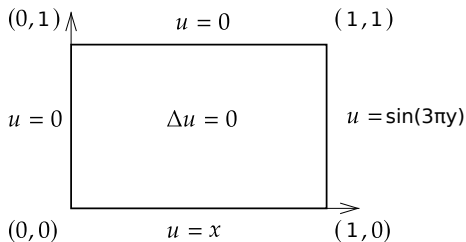
BC @  $x=1$

$$u(1, y) = \sin(3\pi y) = \sum_n a_n \sin(n\pi y) \sinh(n\pi)$$

$$a_n = \begin{cases} \frac{1}{\sinh(3\pi)} & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u_E = u_3(x, y) = \frac{\sinh(3\pi x)}{\sinh(3\pi)} \sin(3\pi y)$$





$$u(x, t) = u_S + u_E + \cancel{u_N}^0 + \cancel{u_W}^0$$

$$\begin{aligned}
 &= -2 \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n\pi \sinh(n\pi)} \sinh(n\pi(1-y)) \\
 &\quad + \frac{\sinh(3\pi x)}{\sinh(3\pi)} \sin(3\pi y)
 \end{aligned}$$

# Separation of Variables on Rectangular Domains

- General Approach:  $u(x, y) = u_N + u_S + u_E + u_W$
- Each  $u_I$  solves a Dirichlet problem with one non-zero BC
  - If a problem has all zero BCs, then  $u_I$  is zero

$$u_N = \sum_n a_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(\frac{n\pi}{w}y\right)$$

$$u_S = \sum_n b_n \sin\left(\frac{n\pi}{w}x\right) \sinh\left(\frac{n\pi}{w}(h-y)\right)$$

$$u_E = \sum_n c_n \sin\left(\frac{n\pi}{h}y\right) \sinh\left(\frac{n\pi}{h}x\right)$$

$$u_W = \sum_n d_n \sin\left(\frac{n\pi}{h}y\right) \sinh\left(\frac{n\pi}{h}(w-x)\right)$$

- Find the arbitrary coefficients by comparing each series solution to a Fourier series of the non-zero BC.