

Partial Differential Equations (PDEs)

A PDE is a DE for a function $u(x_1, x_2, \dots)$ that depends on 2 or more independent variables and contains partial derivatives taken with respect to at least 2 variables.

ex: 1D Heat Equation

$$\frac{\partial}{\partial t} u(x, t) = \alpha \frac{\partial^2}{\partial x^2} u(x, t)$$

or

$$u_t = \alpha u_{xx}$$

Partial derivatives are used so we get the same results in non-euclidean coordinates (e.g., speherical, polar, etc)

Trick: convert PDE into an (infinite) system of ODEs and solve those ODEs instead.

1D Heat Equation - An Initial Boundary Value Problem

Let $u(x, t)$ be the temperature at the position $x \in \underbrace{[0, L]}_{\text{domain}}$ and the time t

The heat equation tells us the evolution of the temperature distribution throughout the domain (e.g., a 1D rod) from some initial condition $u_0(x)$.

(Homogeneous)

Boundary Conditions

Initial Condition

rod connected to ice baths

$$\overbrace{u(0, t) = u(L, t) = 0}$$

or

$$\underbrace{u_x(0, t) = u_x(L, t) = 0}$$

insulated rod

$$u_t = \alpha u_{xx} \quad \text{with} \\ \text{for } x \in (0, L)$$

$$\text{and } u(x, 0) = u_0(x)$$

$\alpha =$ thermal diffusivity

Separation of Variables

$$\frac{\partial}{\partial t} u(x, t) = \alpha \frac{\partial^2}{\partial x^2} u(x, t)$$

Guess: $u(x, t) = X(x)T(t)$

$$X(x)T'(t) = \alpha X''(x)T(t)$$

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}$$

This can only be true if each fraction is a constant

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad \Leftrightarrow \quad \begin{array}{l} X''(x) + \lambda X(x) = 0 \\ \text{and} \\ T'(t) = -\alpha \lambda T(t) \end{array}$$

$$X'' + \lambda X = 0 \quad \text{with BCs: } X(0) = X(L) = 0 \quad \text{or} \quad X'(0) = X'(L) = 0$$

Non-zero solutions exist only for $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $n \in \mathbb{N}^+$ or $n = 0$.

$$\underline{\lambda = 0}$$

$$X_0'' = 0$$

$$X_0(x) = A_0 + B_0 x \quad \text{does not match BCs}$$

$$\underline{\lambda > 0}$$

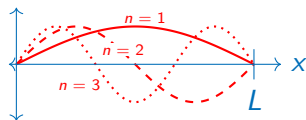
$$X_n'' + \lambda_n X_n = 0 \quad \text{try} \quad X_n(x) = c_n e^{rx}$$

$$r^2 + \lambda_n = 0 \quad \Rightarrow r = \pm i\sqrt{\lambda_n}$$

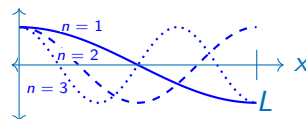
$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)$$

Choose sin or cos based on the BCs of the problem.

$$\sin(n\pi x/L)$$



$$\cos(n\pi x/L)$$



Time-Dependence + Superposition

$$u_n(x, t) = T_n(t)X_n(x) \quad \text{with} \quad T'_n = -\alpha\lambda_n T \quad \Leftrightarrow \quad T_n(t) = c_n e^{-\alpha\lambda_n t}$$

$$u_n(x, t) = c_n e^{-\alpha\omega_n^2 t} (A_n \cos(\omega_n x) + B_n \sin(\omega_n x))$$

We have infinitely many solutions to the linear PDE, any linear combinations of those solutions is also a solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

Initial condition: $u(x, 0) = u_0(x)$ Pick $c_n = 1$ (without loss of generality)

$$u_0(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(\omega_n x) + B_n \sin(\omega_n x)) \quad \omega_n = \frac{n\pi}{L}$$

Find A_n and B_n by taking a Fourier Series expansion of $u_0(x)$

ICs + BCs \Rightarrow Periodic extension

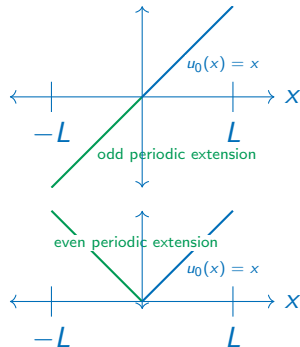
$$u_0(x) = A_0 + \sum_{i=n}^{\infty} (A_n \cos(\omega_i nx) + B_n \sin(\omega_n x)) \quad \omega_n = \frac{n\pi}{L}$$

$u_0(x)$ is defined for $x \in [0, L]$ only.

Our spatial solution is the Fourier Series of a $2L$ periodic function!

BCs: $X(0) = X(L) = 0$ or $X'(0) = X'(L) = 0$

- sin terms satisfy $X(0) = X(L) = 0$
 1. Extend $u_0(x)$ as an odd function
 2. Take its FS: Fourier Sine Series
- cos terms satisfy $X'(0) = X'(L) = 0$
 1. Extend $u_0(x)$ as an even function
 2. Take its FS: Fourier Cosine Series



Solution to $u_t = \alpha u_{xx}$ with $0 < x < L$
and $u(x, 0) = u_0(x)$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} \left(a_n \cos \left(\frac{n\pi}{L} x \right) + b_n \sin \left(\frac{n\pi}{L} x \right) \right)$$

If $u(0, t) = u(L, t) = 0$: Fourier sine series

$$a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L u_0(x) \sin \left(\frac{n\pi}{L} x \right) dx$$

If $u_x(0, t) = u_x(L, t) = 0$: Fourier cosine series

$$a_n = \frac{2}{L} \int_0^L u_0(x) \cos \left(\frac{n\pi}{L} x \right) dx$$

$$b_n = 0$$

ex: $u_t = \alpha u_{xx}$ for $0 < x < L$ with $u(x, 0) = x$

Two possible homogeneous BCs:

$$u(0, t) = u(L, t) = 0$$

$$u_x(0, t) = u_x(L, t) = 0$$

$$u(x, t) =$$

$$\sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} a_n \cos\left(\frac{n\pi}{L} x\right)$$

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= -L \frac{2(-1)^n}{\pi n} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L x \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2L((-1)^n - 1)}{\pi^2 n^2} \end{aligned}$$

$$a_0 = \frac{2}{L} \int_0^L x dx = L$$

ex: $u_t = \alpha u_{xx}$ for $0 < x < L$ with $u(x, 0) = \sin(5\pi x/L)$
 $u(0, t) = u(L, t) = 0$

From the BCs,

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\alpha \frac{n^2 \pi^2}{L^2} t} b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \sin(5\pi x/L) \sin\left(\frac{n\pi}{L} x\right) dx \\ &= \begin{cases} 1 & \text{if } n = 5 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$u(x, t) = e^{-\alpha \frac{25\pi^2}{L^2} t} \sin\left(\frac{5\pi}{L} x\right)$$