Recall: Superposition for Linear Homogeneous ODEs

Suppose the linearly independent functions $y_1(t)$ and $y_2(t)$ both solve the 2^{nd} order linear homogeneous ODE given by

$$\mathrm{L}[y] = 0, \qquad \text{then} \qquad y_h = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution to the homogeneous ODE.

Proof:

$$\begin{aligned}
\mathrm{L}[c_1 y_1 + c_2 y_2] &\overset{\text{Linearity 1}}{=} \mathrm{L}[c_1 y_1] + \mathrm{L}[c_2 y_2] \\
&\overset{\text{Linearity 2}}{=} c_1 \mathrm{L}[y_1] + c_2 \mathrm{L}[y_2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 = 0
\end{aligned}$$

Superposition for Linear Inhomogeneous ODEs

Suppose the linearly independent functions $y_1(t)$ and $y_2(t)$ both solve the 2nd order linear homogeneous ODE given by

$$\mathrm{L}[y] = 0, \qquad \text{then} \qquad y_h = c_1 y_1(t) + c_2 y_2(t)$$

is the general solution to the homogeneous ODE. If some particular solution y_p solves

$$\mathrm{L}[y_p] = g(t) \neq 0, \qquad \text{then} \qquad y = \overbrace{c_1 y_1(t) + c_2 y_2(t)}^{\text{complementary solution}} + y_p$$

is the general solution to the inhomogeneous ODE.

Proof:

$$\begin{aligned} \mathrm{L}[c_1 y_1 + c_2 y_2 + y_p] &\overset{\text{Linearity 1}}{=} \mathrm{L}[c_1 y_1] + \mathrm{L}[c_2 y_2] + \mathrm{L}[y_p] \\ &\overset{\text{Linearity 2}}{=} \underbrace{c_1 \mathrm{L}[y_1] + c_2 \mathrm{L}[y_2]}_{0} + \underbrace{\mathrm{L}[y_p]}_{g(t)} = g(t) \end{aligned}$$

For proof of uniqueness of y_p, see DiffQs §2.5.1

Solving Linear Inhomogeneous IVPs

$$\mathrm{L}\left[y(t)\right] = g(t) \neq 0 \quad \Rightarrow \quad y(t) = y_h(t) + y_p(t)$$

ex: Second order $\mathrm{L}\left[y\right] = g(t) \neq 0$ with $y(0) = y_0$, $y'(0) = v_0$

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

$$\begin{aligned} y(0) = y_0 &= c_1 y_1(0) + c_2 y_2(0) + y_p(0) \\ y'(0) = v_0 &= c_1 y_1'(0) + c_2 y_2'(0) + y_p'(0) \end{aligned}$$

Convert to matrix notation:

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 - y_p(0) \\ v_0 - y_p'(0) \end{bmatrix}$$

Same solvability condition as homogeneous IVPs ($W \neq 0$).

Do not forget to include y_p when solving for c_1 and c_2!

Rough idea for finding y_p for $\mathrm{L}[y] = g(t)$

Method of Undeteremined Coefficients:

1. Find the family of functional forms obtained by differentiating $g(t)$.

2. Guess a linear combination of the family members for y_p.

3. Plug guess into ODE and solve the algebraic system.

ex: Find the particular solution to $y'' + 2y' + 2y = 2t^2 - 2$.

Guess: $y_p = At^2 + Bt + C$ (3 Unknown Coefficients)

$$y_p' = 2At + B, \quad y_p'' = 2A$$

$$2A + 2(2At + B) + 2(At^2 + Bt + C) = 2t^2 - 2$$

$$2At^2 + (4A + 2B)t + 2(A + B + C) = 2t^2 - 2$$

group by powers of t: (3 Algebraic Equations)

$\underline{t^2:}$ $\quad 2At^2 = 2t^2 \quad \Rightarrow A = 1$

$\underline{t^1:}$ $\quad 4\cancelto{1}{A}t + 2Bt = 0$

$4 + 2B = 0 \quad \Rightarrow B = -2$

$\underline{t^0:}$ $\quad 2\cancelto{1}{A} + 2\cancelto{-2}{B} + 2C = -2$

$2 - 4 + 2C = -2 \quad \Rightarrow C = 0$

$$\boxed{y_p = t^2 - 2t}$$

ex: Solve $y'' + 2y' + 2y = 2t^2 - 2$ with $y(0) = 1, y'(0) = -3$.

$$y = c_1 y_1 + c_2 y_2 + \underbrace{y_p}_{t^2-2t}$$

$$y_{1,2} = e^{rt}$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y(t) = e^{-t}\left(c_1 \cos(t) + c_2 \sin(t)\right) + t^2 - 2t$$

initial conditions:

$$y(0) = 1 = c_1 \quad \Rightarrow c_1 = 1$$

$$y'(t) = -e^{t}\left(c_1 \cos(t) + c_2 \sin(t)\right) + e^{t}\left(c_2 \cos(t) - c_1 \sin(t)\right) + 2t - 2$$

$$y'(0) = -3 = -c_1 + c_2 - 2 \Rightarrow c_2 = 0$$

$$\boxed{y(t) = e^{-t}\cos(t) + t^2 - 2t}$$

Method of Undetermined Coefficients: $\mathrm{L}[y] = g(t)$

1. Find the homogeneous solutions:

$$2^{nd}\text{order} \;\Rightarrow\; y_h = c_1 y_1 + c_2 y_2$$

2. Find the family of functional forms obtained by differentiating $g(t)$.
   ex: $g(t) = t^2 e^{-t}$

$$g' = 2\underline{te^{-t}} - t^2 e^{-t} \qquad g'' = \underline{2e^{-t}} - 2te^{-t} - 2te^{-t} + t^2 e^{-t}$$

$$\text{family } = \{t^2 e^{-t}, te^{-t}, e^{-t}\}$$

3. Guess a linear combination of the family members for y_p.

$$y_p = At^2 e^{-t} + Bte^{-t} + Ce^{-t}$$

4. Plug guess into ODE and solve the algebraic system.
   group coeffs. by funcs of t, ex: 3 eqs., 3 unknowns $(A,\ B,\ C)$

Complication: Mathematical Resonance $\quad (\mathrm{L}[y] = g(t))$

1. Find the homogeneous solutions: $y_h = c_1 y_1 + c_2 y_2$
2. Find the family of functional forms obtained by differentiating $g(t)$.
3. Guess a linear combination of the family members for y_p.

$$g(t) = t^2 e^{-t} \Rightarrow \text{naively guess: } y_p = At^2 e^{-t} + Bte^{-t} + Ce^{-t}$$

4. Plug guess into ODE and solve the algebraic system.
   ex: suppose $y_1 = e^{-t}$ and $y_2 = e^{3t}$

$$\mathrm{L}[y_p] = A\mathrm{L}\left[t^2 e^{-t}\right] + B\mathrm{L}\left[te^{-t}\right] + \cancelto{0}{C\mathrm{L}\left[e^{-t}\right]} = t^2 e^{-t}$$

Problem: 3 eqs, 2 unkowns $(A,\ B) \Rightarrow$ impossible to solve
(overdetermined system)

Solution: Mulitply the problematic family member by t until we get a new L.I. family member that is not y_1 or y_2.

Complication: Mathematical Resonance $\quad (\mathrm{L}[y] = g(t))$

1. Find the homogeneous solutions:

$$2^{nd}\text{order} \;\Rightarrow\; y_h = c_1 y_1 + c_2 y_2$$

2. Find the family of functional forms obtained by differentiating $g(t)$.

3. Guess a linear combination of the family members for y_p.

   - Multiply any family member that is L.D. with $y_1(t)$ or $y_2(t)$ by t repeatedly until you obtain a linear combination of L.I. functions that are also L.I. with the fundamental set $\{y_1, y_2\}$

ex: $\begin{array}{l} y_1 = e^{-t} \\ y_2 = e^{3t} \end{array}$ with $g(t) = t^2 e^{-t}$ family $= \{t^2 e^{-t}, te^{-t}, e^{-t}\}$

$$y_p = At^2 e^{-t} + Bte^{-t} + Ct^3 e^{-t}$$

4. Plug guess into ODE and solve the algebraic system.

Practice spotting mathematical resonance

(1) $y' + 6y = \cos t + t^2$

$$y_h = c_1 e^{-6t}$$

$$\text{family} = \left\{\cos t, \sin t, t^2, t, 1\right\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + E$$

(2) $y'' = t^2$

$$y_h = c_1 + c_2 t$$

$$\text{family} = \left\{t^2, \underline{t}, \underline{1}\right\}$$

$$y_p = At^2 + Bt^3 + Ct^4$$

(3) $y'' + 3y' + 2y = 5e^{-t}$

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

$$\text{family} = \left\{\underline{e^{-t}}\right\}$$

$$y_p = Ate^{-t}$$

(4) $y'' + 2y' + y = 12e^{-t}$

$$y_h = c_1 e^{-t} + c_2 te^{-t}$$

$$\text{family} = \left\{\underline{e^{-t}}\right\}$$

$$y_p = At^2 e^{-t}$$

(5) $y'' + 6y' = \cos t + t^2$

$$y_h = c_1 e^{-6t} + c_2$$

$$\text{family} = \left\{\cos t, \sin t, t^2, t, \underline{1}\right\}$$

$$y_p = A\cos t + B\sin t + Ct^2 + Dt + Et^3$$

Find the general solution of $y'' + 5y' + 4y = e^{-4t}$

$$r_{1,2} = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

$$y_h = c_1 e^{-t} + \underbrace{c_2 e^{-4t}}_{\propto g(t)}$$

Try:

$$y_p = Ate^{-4t}$$

$$y_p' = A\left(e^{-4t} - 4te^{-4t}\right)$$

$$y_p'' = -Ae^{-4t} - 4A\left(e^{-4t} - 4te^{-4t}\right)$$

$$= -8Ae^{-4t} + 16Ate^{-4t}$$

plug into DE:

$$-8Ae^{-4t} + 16Ate^{-4t} + 5Ae^{-4t} - 20Ate^{-4t} + 4Ate^{-4t} = e^{-4t}$$
$$(-8+5)Ae^{-4t} + (20-20)te^{-4t} = e^{-4t}$$
$$-3Ae^{-4t} = e^{-4t}$$

$$A = -\frac{1}{3}$$

$$y = c_1e^{-4t} + c_2e^{-t} - \frac{1}{3}te^{-4t}$$

Find the general solution of $y'' + 4y' + 4y = e^{-2t}$

$$r_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = -2$$

$$y_h = \underbrace{c_1 e^{-2t}}_{\propto g(t)} + c_2 t e^{-2t}$$

Try: $$y_p = At^2 e^{-2t}$$

$$y_p' = A\left(2te^{-2t} - 2t^2 e^{-2t}\right)$$

$$y_p'' = 2A\left(e^{-2t} - 2te^{-2t}\right) - 2A\left(2te^{-2t} - 2t^2 e^{-2t}\right)$$

$$= 4At^2 e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t}$$

plug into DE:

$$4At^2e^{-2t} - 8Ate^{-2t} + 2Ae^{-2t} + 8Ate^{-2t} - 8At^2e^{-2t} + 4At^2e^{-2t} = e^{-2t}$$

$$(-8+8)At^2e^{-2t} + (-8+8)te^{-2t} + 2Ae^{-2t} = e^{-2t}$$

$$2Ae^{-2t} = e^{-2t}$$

$$2A = 1 \qquad \Rightarrow A = \frac{1}{2}$$

$$y = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{2}t^2e^{2t}$$

Method of Undetermined Coefficients $\quad L[y_p] = g(t)$

1. Find the homogeneous solutions:

$$2^{nd}\text{order} \;\Rightarrow\; y_h = c_1 y_1 + c_2 y_2$$

2. Find the family of functional forms obtained by differentiating $g(t)$.

3. Guess a linear combination of the family members for y_p.

   - Multiply any family member that is L.D. with $y_1(t)$ or $y_2(t)$ by t repeatedly until you obtain a linear combination of L.I. functions that are also L.I. with the fundamental set $\{y_1, y_2\}$

4. Plug guess into ODE and solve the algebraic system.