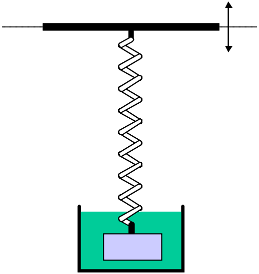
Recall: Spring-dashpot system without forcing

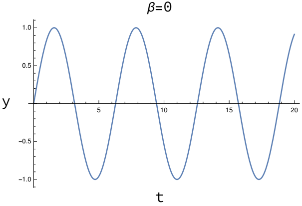


$x(t) =$ displacement from rest position

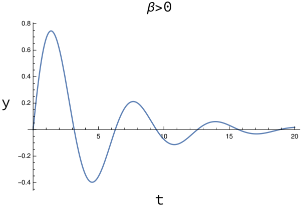
Newton's 2^{nd} Law:

$$F = ma$$
$$x'' + \beta x' + kx = 0$$

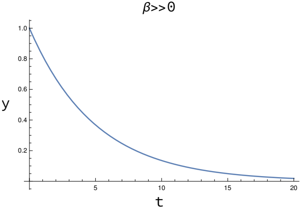
Simple Harmonic Motion



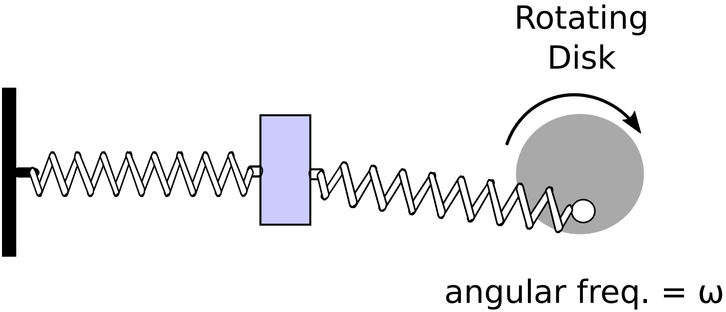
Underdamped Motion



Overdamped Motion



Spring oscillators with periodic forcing: $f(t) = F_0 \cos(\omega t)$



Harmonic motion with forcing

$$\underbrace{x'' + \omega_0^2 x}_{\frac{1}{m}(mx''+kx) \text{ with } \omega_0=\sqrt{k/m}} = \frac{f(t)}{m} = \frac{F_0}{m}\cos(\omega t) \quad \text{with} \quad \begin{array}{l} \omega \neq \omega_0 \\ x(0) = x'(0) = 0 \end{array}$$

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \qquad x_p = A\cos(\omega t) + B\sin(\omega t)$$

$$x_p'' = -\omega^2 A\cos(\omega t)$$

$$-\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

$$\underline{\sin(t):} \quad -\omega^2 B + \omega_0^2 B = 0 \qquad \Rightarrow B = 0$$

$$\underline{\cos(t):} \quad -\omega^2 A + \omega_0^2 A = \frac{F_0}{m} \qquad \Rightarrow A = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}$$

$$x_p = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t)$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \qquad \text{with} \qquad \begin{array}{l} \omega \neq \omega_0 \\ x(0) = x'(0) = 0 \end{array}$$

$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \qquad\qquad x_p = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t)$$

Initial Conditions:

$$x(0) = 0 = c_1 + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}$$

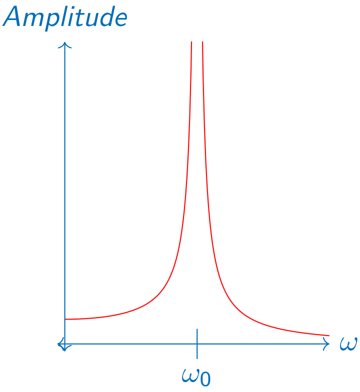
$$c_1 = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}$$

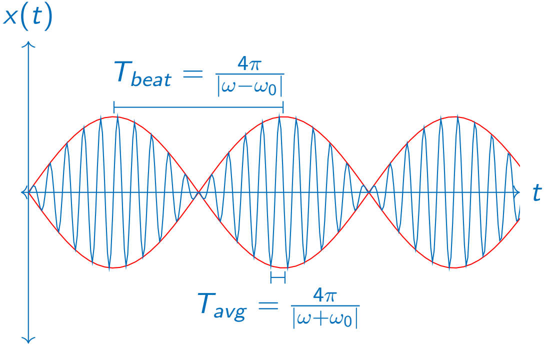
$$x'(0) = 0 = c_2$$

$$x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \underbrace{\left(\cos(\omega t) - \cos(\omega_0 t)\right)}_{\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)}$$

Beat phenomena

$$x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$





What happens to $x(t)$ as $\omega \to \omega_0$?

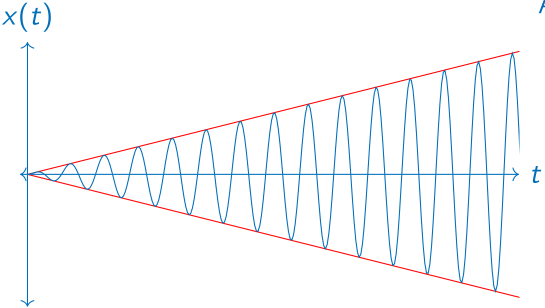
Resonance $(\omega \to \omega_0)$

$$x'' + \omega_0^2 x = f(t) = \frac{F_0}{m}\cos(\omega_0 t)$$

$x_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$ $\qquad$ $x_p = ?$

$y_p \neq A\cos(\omega_0 t) + B\sin(\omega_0 t)$ $\qquad$ (mathematical resonance)

$y_p = \overset{0}{\cancel{A}}t\cos(\omega_0 t) + Bt\sin(\omega_0 t)$ $\qquad$ $B = \dfrac{F_0}{2m\omega_0}$



As $t \to \infty$

Amplitude $\to \infty$

Finite input power

Very large response

(physical resonance)

Damped oscillator with forcing

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

$x(t) = x_h(t) + x_p(t)$

Three cases:

$$x_h(t) = \begin{cases} e^{-\frac{\beta}{2m}t}\left(c_1 \cos\left(\omega_1 t\right) + c_2 \sin\left(\omega_1 t\right)\right) & \beta^2 < 4km \\ c_1 e^{r_1 t} + c_2 e^{r_2 t} & \beta^2 > 4km \\ c_1 e^{rt} + c_2 t e^{rt} & \beta^2 = 4km \end{cases}$$

Since $\beta > 0$, all exponents are negative

$$\Rightarrow \quad x_h \to 0 \text{ as } t \to \infty$$

Damped oscillator with forcing

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

$x(t) = x_h(t) + x_p(t)$

Three cases:

$$x_h(t) = \begin{cases} e^{-\frac{\beta}{2m}t}\left(c_1 \cos\left(\omega_1 t\right) + c_2 \sin\left(\omega_1 t\right)\right) & \beta^2 < 4km \\ c_1 e^{r_1 t} + c_2 e^{r_2 t} & \beta^2 > 4km \\ c_1 e^{rt} + c_2 t e^{rt} & \beta^2 = 4km \end{cases}$$

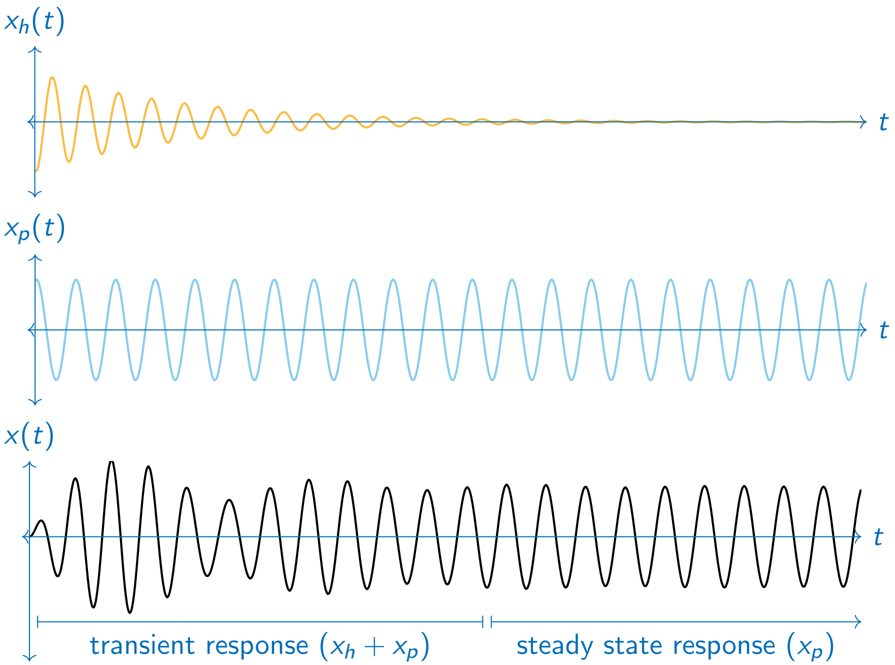
From the Method of Undetermined Coefficients:

$$x_p = A\cos(\omega t) + B\sin(\omega t) \qquad \forall\, \omega \in \mathbb{R}$$

Impossible to have mathematical resonance with periodic forcing

As $t \to \infty$

$$x(t) \to x_p(t)$$


$x_h(t)$
t
$x_p(t)$
t
$x(t)$
t
transient response $(x_h + x_p)$
steady state response (x_p)

$$mx'' + \beta x' + kx = \cos(\omega t)$$

How does the amplitude of the steady state response vary with ω?

$$\begin{aligned} x_p &= A\cos(\omega t) + B\sin(\omega t) \\ x_p' &= -\omega A\sin(\omega t) + \omega B\cos(\omega t) \\ x_p'' &= -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) \end{aligned}$$

$$\begin{aligned} m\left(-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)\right) + \beta\left(-\omega A\sin(\omega t) + \omega B\cos(\omega t)\right) \\ +k\left(A\cos(\omega t) + B\sin(\omega t)\right) = \cos(\omega t) \end{aligned}$$

$$\begin{aligned} (Ak + B\beta\omega - mA\omega^2)\cos(\omega t) + (Bk - A\beta\omega - mB\omega^2)\sin(\omega t) \\ = \cos(\omega t) \end{aligned}$$

Match coefficients

$$\underline{\cos:} \qquad (Ak + B\beta\omega - mA\omega^2) = 1$$

$$\underline{\sin:} \qquad (Bk - A\beta\omega - mB\omega^2) = 0$$

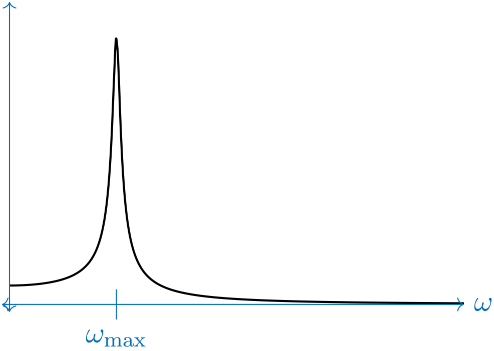
$$B = A\frac{\beta\omega}{k - m\omega^2}$$

$$\left(Ak + A\frac{(\beta\omega)^2}{k - m\omega^2} - mA\omega^2\right) = 1$$

$$A = \frac{k - m\omega^2}{(\beta\omega)^2 - (k - m\omega^2)^2} \qquad B = \frac{\beta\omega}{(\beta\omega)^2 - (k - m\omega^2)^2}$$

Amplitude:

$$\begin{aligned} R &= \sqrt{A^2 + B^2} \\ &= \frac{\sqrt{(k - m\omega^2)^2 + (\beta\omega^2)}}{(\beta\omega)^2 - (k - m\omega^2)^2} \\ &= \frac{1}{\sqrt{k^2 - 2km\omega^2 + m^2\omega^4 + \omega^2\beta^2}} \end{aligned}$$



Amplitude vs Forcing Frequency

Steady-state response: $x_p = R\cos(\omega t - \phi)$ with

$$R(\omega) = \frac{1}{\sqrt{k^2 - 2km\omega^2 + m^2\omega^4 + \omega^2\beta^2}}$$
$$= \frac{1}{\sqrt{\beta^2\omega^2 + (k - m\omega^2)^2}}$$

What value of ω creates the largest amplitude response?

$$\frac{\mathrm{d}}{\mathrm{d}\omega}R = -\frac{\omega\left(\beta^2 - 2km + 2m^2\omega^2\right)}{\left(\beta^2\omega^2 + (k - m\omega^2)^2\right)^{3/2}} = 0$$

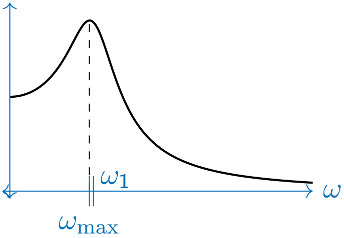
Critical points: $\omega^* = 0, \quad \pm\sqrt{\frac{k}{m} - \frac{1}{2}\left(\frac{\beta}{m}\right)^2}$

Amplitude vs Forcing Frequency

Underdamped:

$$x_h = e^{-\frac{\beta}{2m}t}(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$$

$$\omega_1 = \sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}$$

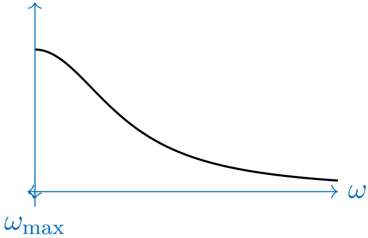


$$\omega_{\max} = \sqrt{\frac{k}{m} - \frac{1}{2}\left(\frac{\beta}{m}\right)^2} \approx \omega_1$$

Overdamped:

$$x_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$



$$\omega_{\max} = 0$$

Resonance with damped oscillators?

$$mx'' + \beta x' + kx = F_0 \cos(\omega t)$$

What happens as $\omega \to \omega_{\max}$?

- Transient response ($x_h + x_p$ for $t \approx 0$)
  - amplitude and duration both grow larger
- Steady state response (x_p)
  - amplitude grows larger

This phenomenon is called quasi-resonance or practical resonance.

$\omega_{\max}$ is called the quasi-resonance frequency