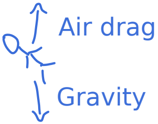
Intro to Linear Systems: Skydiving

Newton's 2nd Law:



$$ma = F(t) = -mg - \mu v$$

noting that $a = x''$ and $v = x'$, we can rewrite this as

$$\left. x'' + \frac{\mu}{m} x' = -g \quad \right\} 2^{nd} \text{ order ODE, one unknown function}$$

or equivalently,

$$\left. \begin{aligned} x' &= v \\ v' &= -\tfrac{\mu}{m} v - g \end{aligned} \right\} 1^{st} \text{ order ODEs, two unknown functions}$$

Q: How do we find two unknown functions simultaneously?

Linear Systems of DEs: Matrix Notation

$$x \to x_1,\ v \to x_2 \quad \Rightarrow \quad \begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{\mu}{m} x_2 - g \end{aligned}$$

Using matrix notation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\mu}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$= \mathbf{A}\vec{x} + \text{constant vector}$$

Linear Systems of DEs: Matrix Notation

General Linear System IVP:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t) \quad \text{with} \quad \vec{x}(t_0) = \vec{x}_0$$

where $\mathbf{A}$ is an $n \times n$ matrix, and both $\vec{x}$ and $\vec{f}$ are $n \times 1$ column vectors.

Operator Form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} - \mathbf{A}(t)\vec{x} = \vec{f}(t) \quad \text{with} \quad \vec{x}(t_0) = \vec{x}_0$$

$$\mathrm{L}\left[\vec{x}\right] = \vec{f}(t)$$

$$\vec{f}(t) = \vec{0} \qquad \Rightarrow \qquad \text{homogeneous system}$$

Equivalence of problems

For every n^{th} order linear ODE, there is a corresponding system of n 1^{st} order linear ODEs.

$$a_n(t)x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) = h(t)$$

can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x} + \vec{f}(t)$$

with

$$\vec{x}(t) = \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}$$

ex: Rewrite $x''' + x' + x = t^2$ in the form $\dfrac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \vec{f}(t)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= x'' \end{aligned} \qquad\qquad \frac{d}{dt}\vec{x} = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} = \mathbf{A}\vec{x} + \vec{f}(t)$$

$$\begin{aligned} x' &= x_2 \\ x'' &= x_3 \\ x''' &= -x - x' + t^2 \\ &= -x_1 - x_2 + t^2 \end{aligned}$$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ t^2 \end{bmatrix}}_{\vec{f}}$$

Skydiving: Homogeneous Solutions

Suppose we want to solve $v'' + \frac{\mu}{m} v' = 0$ Guess $v(t) = e^{rt}$

$$r^2 + \frac{\mu}{m} r = 0 \qquad r\left(r + \frac{\mu}{m}\right) = 0$$

$$r = 0, -\frac{\mu}{m} \qquad v(t) = c_1 + c_2 e^{-\frac{\mu}{m}t}$$

$$= c_1 y_1(t) + c_2 y_2(t)$$

What about for the vector expression?

two LI homogeneous solutions $\rightarrow$ two LI vectors

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-\frac{\mu}{m}t} \begin{bmatrix} 1 \\ -\frac{\mu}{m} \end{bmatrix}$$

$$= c_1 \vec{x}_1(t) \quad + \quad c_2 \vec{x}_2(t)$$

Homogeneous Problem and Superposition

Suppose $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\ldots$, $\vec{x}_k(t)$ all solve the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

Then

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_k\vec{x}_k(t)$$

also solves the same homogeneous problem.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \sum_{i=1}^{n} c_i \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_i &= \sum_{i=1}^{n} c_i \mathbf{A}(t)\vec{x}_i \\ &= \mathbf{A}(t)\sum_{i=1}^{n} c_i\vec{x}_i \\ &= \mathbf{A}(t)\vec{x}(t)\end{aligned}$$

Homogeneous Problem and Superposition

Suppose $\vec{x}_1(t)$, $\vec{x}_2(t)$, $\ldots$, $\vec{x}_k(t)$ all solve the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

Then the set of solutions

$$\{\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_k(t)\}$$

form a vector space V.

Since we are using "standard" addition/multiplication we only need to check 2 properties

1. Additive Closure

$$\vec{x}_i + \vec{x}_j \in V \quad \checkmark$$

2. Scalar Closure:
   For any constant c

$$c\vec{x}_j \in V \quad \checkmark$$

Homogeneous Problem and Superposition

Suppose $\vec{x}_1(t)$, $\vec{x}_2(t)$, …, $\vec{x}_k(t)$ all solve the homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)\vec{x}$$

Then the set of solutions

$$\{\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_k(t)\}$$

form a vector space V.

If $\mathbf{A}$ is $n \times n$, and we can find n linearly independent solution vectors (i.e., $k = n$), then we have a solution basis.

That is, <u>any</u> solution $\vec{x}(t)$ can be written as

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

See DiffyQs Appendixes A.4 & A.5 for review of vector spaces and bases.

Finding the solution vectors $\vec{x}_i$

Given

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}\vec{x},$$

we want to find one of its solution vectors.

For a scalar equation, if we have $y' = ay$ then we know the solution is $y = ce^{at}$.

Lets guess $\vec{x}(t) = e^{\lambda t}\vec{v}$, and plug it into the ODE

$$\lambda e^{\lambda t}\vec{v} = \mathbf{A}e^{\lambda t}\vec{v}$$
$$\lambda\vec{v} = \mathbf{A}\vec{v}$$

$\vec{v}$ is an eigenvector of $\mathbf{A}$, and λ is its associated eigenvalue.

Eigenvectors/Eigenvalues

- A $n \times n$ matrix has n eigenvalues and eigenvectors
  - except in some special cases

Ignoring those special cases for now, we can write any homogeneous solution as

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \cdots + c_n e^{\lambda_n t} \vec{v}_n$$

where $\vec{v}_i$ is the eigenvector associated with the eigenvalue λ_i.

See DiffyQs §3.7 for details on those special cases.

Eigenvectors/Eigenvalues

- A $n \times n$ matrix has n eigenvalues and eigenvectors
  - except in some special cases

We want to find values λ such that for some non-zero $\vec{v}$

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

Rearrange as

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$$

If $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ exists, then

$$\vec{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\vec{0} = \vec{0}$$

So we need

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

See DiffyQs §A.6 for a review of determinants

Eigenvectors/Eigenvalues

- A $n \times n$ matrix has n eigenvalues and eigenvectors
  - except in some special cases
- The eigenvalues are obtained from the roots of the characteristic polynomial obtained from

  $$\det(\mathbf{A} - \lambda \mathbf{I}) = 0,$$

  where "det" is short for "determinant" and $\mathbf{I}$ is the $n \times n$ identity matrix
- The eigenvectors are computed by solving the linear system

  $$(\mathbf{A} - \lambda \mathbf{I})v = 0$$

  once each of the eigenvalues λ is found.
- the eigenvectors are not unique, defined up to arbitrary mulitplicative constant

Find the eigenvalues/vectors associated with $\frac{dx}{dt} = -3x - 2y$, $\frac{dy}{dt} = -2x - 6y$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -3-\lambda & -2 \\ -2 & -6-\lambda \end{bmatrix}\right) = 0$$

$$(3+\lambda)(6+\lambda) - 4 = 0$$

$$\lambda^2 + 9\lambda + 18 - 4 = 0$$

$$\lambda^2 + 9\lambda + 14 = 0$$

$$\lambda = \frac{-9 \pm \sqrt{81 - 4\cdot 14}}{2} = \frac{-9 \pm \sqrt{81-56}}{2}$$

$$= \frac{-9 \pm \sqrt{25}}{2} = \frac{-9 \pm 5}{2}$$

$$\lambda_{1,2} = -2, -7$$

Find the eigenvalues/vectors associated with $\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - 2y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 6y$

$\underline{\lambda_1 = -2:}$ $\mathbf{A}\vec{v}_1 = -2\vec{v}_1$ $(\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \vec{0}$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \vec{v}_1 = \vec{0}$$

Augmented matrix: $\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 - 2R_1 \to R_2$

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-1x - 2y = 0$$

$$x = -2y$$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \vec{x}_1(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t}$$

Find the eigenvalues/vectors associated with $\begin{aligned}\frac{dx}{dt} &= -3x - 2y \\ \frac{dy}{dt} &= -2x - 6y\end{aligned}$

$\underline{\lambda_2 = -7:}$ $\quad \mathbf{A}\vec{v}_2 = -7\vec{v}_2$ $\qquad (\mathbf{A} + 7\mathbf{I})\vec{v}_2 = \vec{0}$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \vec{v}_2 = \vec{0}$$

Augmented matrix: $\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 1 & -4 & 0 \end{array}\right]$

row 2 and row 1 are linearly dependent: $R_2 + 2R_1 \to R_2$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 4x - 2y = 0$$

$$x = y/2$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{x}_2(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-7t}$$

Summary

- Linear n^{th} order DEs can be converted to a system of n 1^{st} order DEs
- Homogeneous system: $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathbf{A}(t)x$
  - Need to find n linearly independent solutions $\{\vec{x}_1(t),\ \vec{x}_2(t),\ \ldots,\ \vec{x}_n(t)\}$
  - These solution vectors are based on the eigenvectors/eigenvalues of $\mathbf{A}$
  - $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$
- Finding eigenvalues/eignevectors, we need to solve

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})v = 0$$