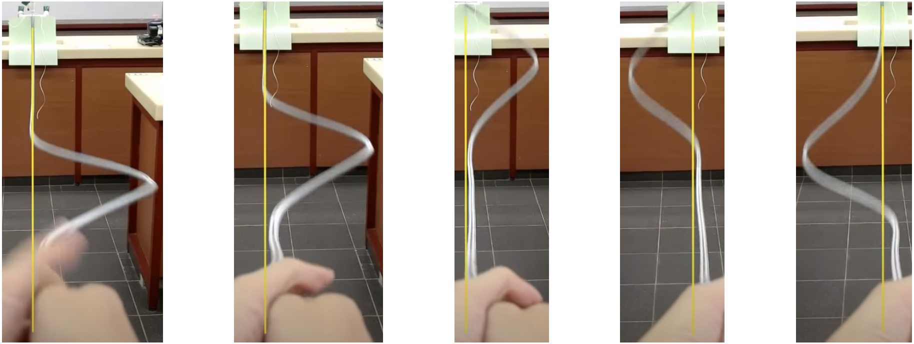
Waves travelling on strings



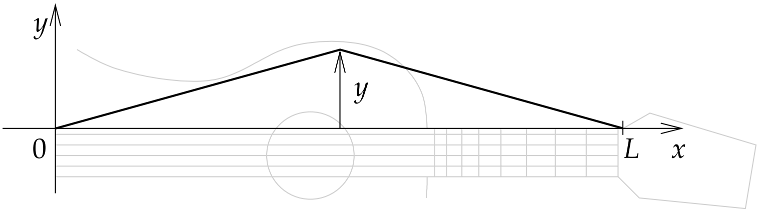
source: https://www.youtube.com/watch?v=PVX4V5Adbzk

guitar strings: https://vine.co/v/0qx7BJgazKZ

Wave Equation: $y_{tt} = c^2 y_{xx}$

Plucked string under tension: $c^2 = \frac{T}{\rho}$

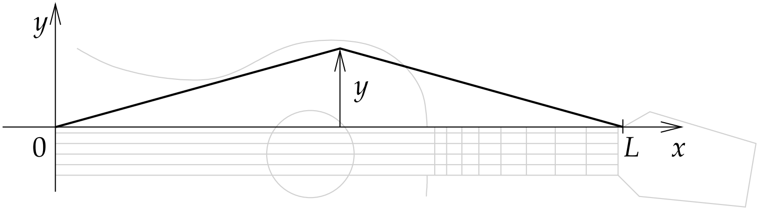
- T =tension
- ρ =mass density of the string (mass/length)



Tapped elastic rod: $c^2 = \frac{E}{\rho}$

- E =Young's Modulus
- ρ =mass density of the rod (mass/length)

Boundary Conditions and Initial Conditions: $y_{tt} = c^2 y_{xx}$



Boundary Conditions: Clamped end-points

$$y(0) = y(L) = 0$$

Initial Conditions:

Now we need two, because of the two time derivatives

$$y(x,0) = f(x)$$
$$y_t(x,0) = g(x)$$

The Wave Equation is Homogeneous: Superposition

Suppose we find two solutions to the wave equation $w(x, t)$ and $z(x, t)$, then a linear combination

$$y(x, t) = c_1 w(x, t) + c_2 z(x, t)$$

is also a solution.

Proof: We are given

$$w_{tt} = c^2 w_{xx} \qquad \underline{\text{and}} \qquad z_{tt} = c^2 z_{xx}$$

with

$$w(0) = w(L) = 0 \qquad \underline{\text{and}} \qquad z(0) = z(L) = 0$$

then

$$\begin{aligned} y_{tt} &= c_1 w_{tt} + c_2 z_{tt} = c_1 c^2 w_{xx} + c_2 c^2 z_{xx} \\ &= c^2 y_{xx} \end{aligned}$$

with

$$y(0) = c_1 w(0) + c_2 z(0) = 0 \qquad \underline{\text{and}} \qquad y(L) = c_1 w(L) + c_2 z(L) = 0$$

Decomposition into two simpler problems: Superposition

$$y_{tt} = c^2 y_{xx}, \quad y(0) = y(L) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = g(x)$$

$$y(x,t) = w(x,t) + z(x,t)$$

Problem 1: initial velocity, but no displacement of string

$$\begin{array}{lll} w_{tt} = c^2 w_{xx} & w(x,0) = 0 & \\ w(0,t) = w(L,t) = 0 & w_t(x,0) = g(x) & \text{for } 0 < x < L \end{array}$$

Problem 2: initial displacement, but no velocity of string

$$\begin{array}{lll} z_{tt} = c^2 z_{xx} & z(x,0) = f(x) & \text{for } 0 < x < L \\ z(0,t) = z(L,t) = 0 & z_t(x,0) = 0 & \end{array}$$

Solve each problem by separation of variables

Separation of Variables: Problem 1

$$w_{tt} = c^2 w_{xx} \qquad w(x,0) = 0$$
$$w(0,t) = w(L,t) = 0 \qquad w_t(x,0) = g(x) \qquad \text{for } 0 < x < L$$

Try $w(x,t) = X(x)T(t)$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$
$$\text{BCs} \Rightarrow X_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right)$$
$$T_n(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

$$w(x,0) = 0 \quad \Rightarrow \quad A_n = 0$$
$$\text{arbitrarily choose } B_n = \frac{L}{n\pi c} \qquad \Rightarrow T_n'(0) = 1$$

Separation of Variables + Superposition: Problem 1

$$w_{tt} = c^2 w_{xx} \qquad w(x,0) = 0$$
$$w(0,t) = w(L,t) = 0 \qquad w_t(x,0) = g(x) \qquad \text{for } 0 < x < L$$

with $w_n(x,t) = \frac{L}{n\pi c} \sin\left(\frac{n\pi c}{L} t\right) b_n \sin\left(\frac{n\pi}{L} x\right)$ $\qquad w(x,t) = \sum_{n=1}^{\infty} w_n(x,t)$

Initial condition:

$$w_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) b_n \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

use an odd periodic extension of $g(x)$: $\qquad b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right)$

$$w(x,t) = \sum_{n=1}^{\infty} \frac{L}{n\pi c} \sin\left(\frac{n\pi c}{L} t\right) b_n \sin\left(\frac{n\pi}{L} x\right)$$

Separation of Variables: Problem 2

$$z_{tt} = c^2 z_{xx} \qquad z(x,0) = f(x) \qquad \text{for } 0 < x < L$$
$$z(0,t) = z(L,t) = 0 \qquad z_t(x,0) = 0$$

Try $z(x,t) = X(x)T(t)$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\text{BCs} \Rightarrow X_n(x) = c_n \sin\left(\frac{n\pi}{L}x\right)$$

$$T_n(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

$$z_t(x,0) = 0 \quad \Rightarrow \quad B_n = 0$$

arbitrarily choose $A_n = 1 \qquad \Rightarrow T_n(0) = 1$

Separation of Variables + Superposition: Problem 2

$$z_{tt} = c^2 z_{xx} \qquad z(x,0) = f(x) \qquad \text{for } 0 < x < L$$
$$z(0,t) = z(L,t) = 0 \qquad z_t(x,0) = 0$$

with $z_n(x,t) = \cos\left(\frac{n\pi c}{L} t\right) c_n \sin\left(\frac{n\pi}{L} x\right)$ $\qquad$ $z(x,t) = \sum_{n=1}^{\infty} z_n(x,t)$

Initial condition:

$$z(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right)$$

use an odd periodic extension of $f(x)$: $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$

$$z(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi c}{L} t\right) c_n \sin\left(\frac{n\pi}{L} x\right)$$

Fourier Solution to the Wave Equation

$$y_{tt} = c^2 y_{xx}, \quad y(0) = y(L) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = g(x)$$

$$\begin{aligned} y(x,t) &= w(x,t) + z(x,t) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[b_n \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right) + c_n \cos\left(\frac{n\pi c}{L}t\right)\right] \end{aligned}$$

c_n is obtained from an odd periodic extension of $f(x)$

$$c_n = \tfrac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi}{L}x\right)$$

b_n is obtained from an odd periodic extension of $g(x)$

$$b_n = \tfrac{2}{L}\int_0^L g(x)\sin\left(\tfrac{n\pi}{L}x\right)$$