Recall: $ay'' + by' + cy = f(t)$ with $y(0) = y_0, y'(0) = v_0$

$$\Rightarrow Y(s) = \underbrace{\frac{F(s)}{as^2+bs+c}}_{\text{particular part}} + \underbrace{\frac{ay_0 s + (by_0 + av_0)}{as^2+bs+c}}_{\text{homogeneous part}}$$

We now focus on cases where the forcing function $f(t)$ is not continuous.

Two types of such forcing:

1. Jump Discontinuities
   - Sudden change in forcing

ex: Voltage jumps in a circuit.

Heaviside Function

2. Impulsive Forcing
   - Very brief and strong forcing

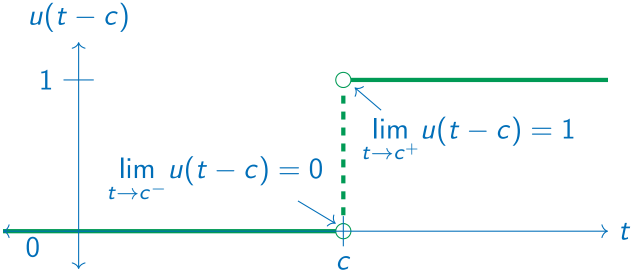
ex: Hitting a nail with a hammer.

Delta Dirac Function

The Heaviside Step Function: $u(t-c)$ or $H(t-c)$ or $u_c(t)$

Used to model effects that "turn-on" at some time c.

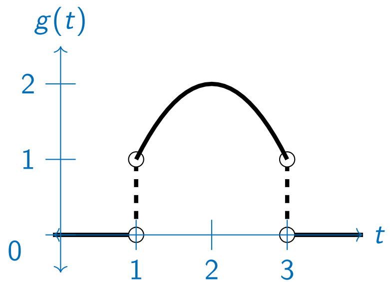
$$u(t-c) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t > c \end{cases}$$



Write the piecewise function

$$g(t) = \begin{cases} 0 & t < 1 \\ -(t-2)^2 + 2 & 1 < t < 3 \\ 0 & t > 3 \end{cases}$$

in terms of the heaviside function.

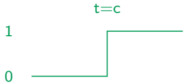


$$g(t) = \left((t-2)^2 + 2\right) \underbrace{u(t-1)}_{\substack{\text{turn on} \\ \text{at } t=1}} \times \underbrace{(1 - u(t-3))}_{\substack{\text{turn off} \\ \text{at } t=3}}$$

or

$$g(t) = \left((t-2)^2 + 2\right) \Big(\underbrace{u(t-1)}_{\substack{\text{turn on} \\ \text{at } t=1}} - \underbrace{u(t-3)}_{\substack{\text{turn off} \\ \text{at } t=3}} \Big)$$

Laplace Tranform of the Heaviside Function



$$\mathcal{L}\{u(t-c)\} = \int_0^{\infty} e^{-st} u(t-c)dt = \lim_{A \to c^+} \int_A^{\infty} e^{-st} dt$$

$$\overset{s \geq 0}{=} \lim_{A \to c^+} \frac{1}{s} e^{-sA}$$

$$= \boxed{e^{-sc}\frac{1}{s}} = e^{-sc}\mathcal{L}\{1\}$$

How about the more general pattern $e^{-sc}F(s)$?

Second Shift Theorem

$$\boxed{\mathcal{L}\{f(t-c)u(t-c)\} = e^{-sc}F(s)}$$

Proof of Second Shift Theorem

$$
\begin{aligned}
\mathcal{L}\{f(t-c)u(t-c)\} &= \int_0^\infty e^{-st}\underbrace{f(t-c)u(t-c)}_{0 \text{ for } t<c}dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&= \int_0^\infty e^{-s(v+c)}f(v)dv \\
&= e^{-sc}\int_0^\infty e^{-sv}f(v)dv = e^{-sc}\mathcal{L}\{f(t)\}
\end{aligned}
$$

$$
\begin{aligned}
v &= t-c \\
dv &= dt
\end{aligned}
$$

$$\boxed{\mathcal{L}\{f(t-c)u(t-c)\} = e^{-sc}F(s)}$$

ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+s^2}$, find $y(t)$.

$$\begin{aligned} y(t) &= \left[u(t)\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\ &= u(t-4)\left[\sin(3t)\right]_{t=t-4} \\ &= u(t-4)\sin(3(t-4)) \end{aligned}$$

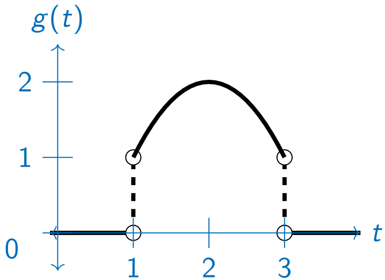
ex: Suppose $Y(s) = e^{-4s}\frac{3}{9+(s+11)^2}$, find $y(t)$.

$$\begin{aligned} y(t) &= \left[u(t)\mathcal{L}^{-1}\left\{\frac{3}{9+(s+11)^2}\right\}\right]_{t=t-4} \\ &= u(t-4)\left[e^{-11t}\mathcal{L}^{-1}\left\{\frac{3}{9+s^2}\right\}\right]_{t=t-4} \\ &= u(t-4)e^{-11(t-4)}\sin(3(t-4)) \end{aligned}$$

Write the piecewise function

$$g(t) = \begin{cases} 0 & t < 1 \\ -(t-2)^2 + 2 & 1 < t < 3 \\ 0 & t > 3 \end{cases}$$

in terms of the heaviside function.



$$\begin{aligned} g(t) &= \left((t-2)^2 + 2\right) u(t-1) - \left((t-2)^2 + 2\right) u(t-3) \\ &= f_1(t-1)u(t-1) + f_2(t-3)u(t-3) \end{aligned}$$

$$G(s) = e^{-s}F_1(s) - e^{-3s}F_2(s)$$

Need to find f_1 and f_2. To find f_1, let $z_1 = t - 1 \;\Rightarrow\; t = z_1 + 1$

$$f_1(t) = -(t-2)^2 + 2 = -(z_1 - 1)^2 + 2$$

$$f_1(z_1) = -z_1^2 + 2z_1 + 1 \quad \Rightarrow F_1(s) = -\frac{2!}{s^3} + 2\frac{1!}{s^2} + \frac{1}{s}$$

Let $z_2 = t - 3 \;\;\Rightarrow\;\; t = z_2 + 3$

$$\begin{aligned} f_2(t-3) &= -(t-2)^2 + 2 = (z_2+1)^2 + 2 \\ &= -z_2^2 - 2z_2 + 1 \Rightarrow F_2(s) = -\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} \end{aligned}$$

$$G(s) = \left(\frac{s^2 + 2s - 2}{s^3}\right) e^{-s} - \left(\frac{s^2 - 2s - 2}{s^3}\right) e^{-3s}$$

ex: $y'' + y = \begin{cases} 0 & t < 1 \\ -(t-2)^2 + 2 & t > 1 \end{cases}$ with $y(0) = 1,\ y'(0) = 0$

$$s^2 Y(s) + 1Y(s) - s = \left(\frac{s^2 + 2s - 2}{s^3}\right) e^{-s}$$

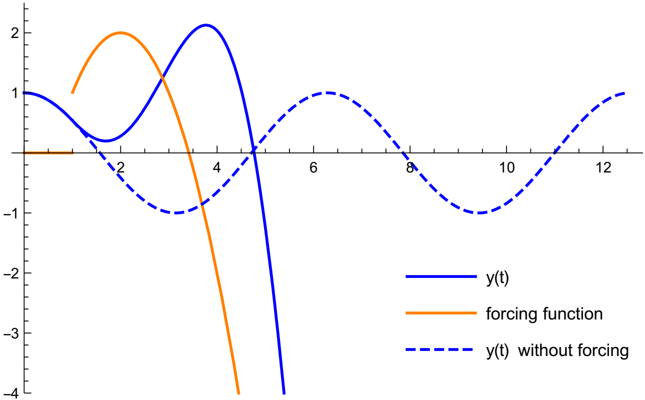
$$Y(s) = \left(\frac{s^2 + 2s - 2}{(s^2+1)s^3}\right) e^{-s} + \underbrace{\frac{s}{s^2+1}}_{\mathcal{L}\{\cos(t)\}}$$

Apply partial fraction decomp....

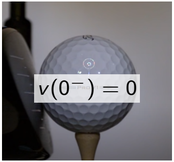
$$Y(s) = \Bigg(\underbrace{\frac{-3s-2}{s^2+1}}_{\mathcal{L}\{-2\sin(t)-3\cos(t)\}} + \underbrace{\frac{2}{s^2}}_{\mathcal{L}\{2t\}} - \underbrace{\frac{2}{s^3}}_{\mathcal{L}\{t^2\}} + \underbrace{\frac{3}{s}}_{\mathcal{L}\{3\}} \Bigg) e^{-s} + \mathcal{L}\{\cos(t)\}$$

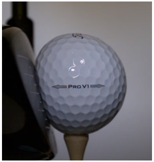
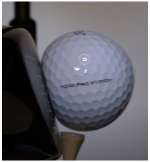
$$y(t) = u(t-1)\Big[-2\sin(t-1) - 3\cos(t-1) + 2(t-1) - (t-1)^2 + 3\Big] + \cos(t)$$

$$y(t) = u(t-1)\big[-2\sin(t-1) - 3\cos(t-1) + 2(t-1) - (t-1)^2 + 3\big] + \cos(t)$$



Imagine hitting a golf ball from rest at $t = c$

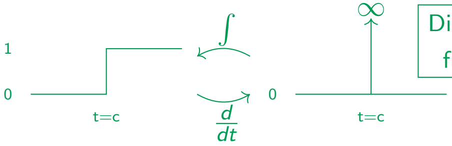


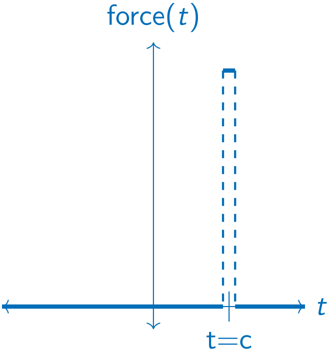

source: `https://www.youtube.com/watch?v=6TA1s1oNpbk&t=80s`

Suddenly a large force is applied to the ball for a very brief period of time.

Idea: Describe the force as $\frac{d}{dt}u(t-c)$.



Dirac delta function



Not a well-defined function, but has a well-defined integral.

Delta Dirac Function: $\delta(t-c) = \frac{d}{dt}u(t-c)$

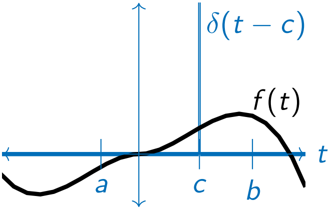
Theorem: For any function $f(t)$ that is integrable in some neighbourhood around c

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = f(c)$$

Integrating a function $f(t)$ against $\delta(t-c)$ essentially "selects" the value of $f(t)$ at $t = c$

More generally

$$\int_{a}^{b} \delta(t-c)f(t)dt = \begin{cases} f(c) & a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$$



Sketch of “proof”:

Approximate the Dirac delta function as normalized pulse:

Let $\delta_\tau = \dfrac{u(t+\tau)-u(t-\tau)}{2\tau}$, and use the approximation $\delta(t) \approx \lim_{\tau\to 0} \delta_\tau(t)$

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(t-c) f(t)dt &= \int_{-\infty}^{\infty} \left(\lim_{\tau\to 0} \delta_\tau(t-c)\right) f(t)dt \\
&= \lim_{\tau\to 0} \int_{-\infty}^{\infty} \delta_\tau(t-c) f(t)dt \\
&= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{c-\tau}^{c+\tau} f(t)dt \\
&= \lim_{\tau\to 0} \frac{F(c+\tau)-F(c-\tau)}{2\tau} = f(c)
\end{aligned}$$

Laplace transform of $\delta(t)$

Integrating a function $f(t)$ against $\delta(t-c)$ essentially "selects" the value of $f(t)$ at $t = c$

Assuming $c > 0$

$$\begin{aligned}\mathcal{L}\{\delta(t-c)\} &= \int_0^\infty e^{-st}\delta(t-c)dt \\ &= \int_{-\infty}^\infty e^{-st}\delta(t-c)dt = e^{-sc}\end{aligned}$$

Special case: $c = 0$

$$\mathcal{L}\{\delta(t)\} = \lim_{c\to 0^+} \mathcal{L}\{\delta(t-c)\} = \lim_{c\to 0^+} e^{-sc} = 1$$

Solve: $y'' + 6y' + 45y = 6\delta(t-5)$ with $\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$

$$(s^2 + 6s + 45)Y(s) - (6 + s) = 6e^{-5s} \quad \Rightarrow \quad Y(s) = \frac{6e^{-5s} + s + 6}{s^2 + 6s + 45}$$

$$Y(s) = e^{-5s} \underbrace{\frac{6}{(s+3)^2 + 36}}_{\mathcal{L}\{e^{-3t}\sin(6t)\}} + \underbrace{\frac{s+3}{(s+3)^2 + 36}}_{\mathcal{L}\{e^{-3t}\cos(6t)\}} + \frac{3}{(s+3)^2 + 36}$$

$$\begin{aligned} y(t) &= \left[e^{-3t}\sin(6t)\right]_{t=t-5} + e^{-3t}\left(\cos(6t) + \frac{1}{2}\sin(6t)\right) \\ &= e^{-3(t-5)}\sin(6(t-5)) + e^{-3t}\left(\cos(6t) + \frac{1}{2}\sin(6t)\right) \end{aligned}$$

$$y(t) = e^{-3(t-5)}\sin(6(t-5)) + e^{-3t}\left(\cos(6t) + \frac{1}{2}\sin(6t)\right)$$