

## Doc: continuous models

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$$m(\mathbf{x}) \frac{\partial^2}{\partial t^2} \theta(\mathbf{x}, t) + d(\mathbf{x}) \frac{\partial}{\partial t} \theta(\mathbf{x}, t) = P(\mathbf{x}, t) + \nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)], \quad (1)$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned} \theta_{i,j}(t + \Delta t) = & \left[ 2\chi_{ij} - \frac{\Delta t^2 m_{ij}^{-1} \chi_{ij}}{\Delta x^2} \left( b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j} \right) \right] \theta_{i,j}(t) \\ & - \xi_{ij} \theta_{i,j}(t - \Delta t) + \frac{\Delta t^2 \chi_{i,j} m_{ij}^{-1}}{\Delta x^2} \left[ b_{i-1,j|i,j} \theta_{i-1,j}(t) + b_{i,j|i+1,j} \theta_{i+1,j}(t) \right. \\ & \left. + b_{i,j-1|i,j} \theta_{i,j-1}(t) + b_{i,j|i,j+1} \theta_{i,j+1}(t) \right] + \Delta t^2 \chi_{ij} m_{ij}^{-1} P_{i,j}, \end{aligned} \quad (2)$$

where  $\chi_{i,j} = \left[ 1 + \frac{\gamma_{i,j} \Delta t}{2} \right]^{-1}$ ,  $\xi_{i,j} = \left[ 1 - \frac{\gamma_{i,j} \Delta t}{2} \right] \chi_{ij}$ , with  $\gamma_{i,j} = d_{i,j}/m_{i,j}$ .

The previous equation follows from

$$\nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)] = \partial_x B_x \partial_x \theta + B_x \partial_x^2 \theta + \partial_y B_y \partial_y \theta + B_y \partial_y^2 \theta, \quad (3)$$

where

$$B_x \approx \frac{b_{i,j|i+1,j} + b_{i-1,j|i,j}}{2}, \quad (4)$$

$$\partial_x B_x \approx \frac{b_{i,j|i+1,j} - b_{i-1,j|i,j}}{\Delta x}, \quad (5)$$

$$\partial_x \theta \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta x}, \quad (6)$$

$$\partial_x^2 \theta \approx \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2}, \quad (7)$$

So, finally

$$\partial_x B_x \partial_x \theta + B_x \partial_x^2 \theta \approx \quad (8)$$

$$\frac{b_{i-1,j|i,j} \theta_{i-1,j} - (b_{i,j|i+1,j} + b_{i-1,j|i,j}) \theta_{i,j} + b_{i,j|i+1,j} \theta_{i+1,j}}{\Delta x^2}, \quad (9)$$

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)} = \frac{\Delta x^2 P_{i,j} + b_{i-1,j|i,j} \theta_{i-1,j}^{(n)} + b_{i,j|i+1,j} \theta_{i+1,j}^{(n)} + b_{i,j-1|i,j} \theta_{i,j-1}^{(n)} + b_{i,j|i,j+1} \theta_{i,j+1}^{(n)}}{b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j}} \quad (10)$$

Boundary conditions:

$$\int_{\partial\Omega} \mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t) \cdot \mathbf{n} \, d\mathbf{x} = 0 \quad (11)$$

which is trivially true, if

$$n_x b_x \partial_x \theta(\mathbf{x}) + n_y b_y \partial_y \theta(\mathbf{x}) = 0, \quad \forall t \text{ and } \mathbf{x} \in \partial\Omega, \quad (12)$$

which leads to

$$\theta_{i,j} = \frac{n_x [(n_x + 1) b_{i,j-1|i,j} \theta_{i,j-1} + (n_x - 1) b_{i,j|i,j+1} \theta_{i,j+1}] + n_y [(n_y + 1) b_{i-1,j|i,j} \theta_{i-1,j} + (n_y - 1) b_{i,j|i+1,j} \theta_{i+1,j}]}{n_x [(n_x + 1) b_{i,j-1|i,j} + (n_x - 1) b_{i,j|i,j+1}] + n_y [(n_y + 1) b_{i-1,j|i,j} + (n_y - 1) b_{i,j|i+1,j}]}, \quad (13)$$

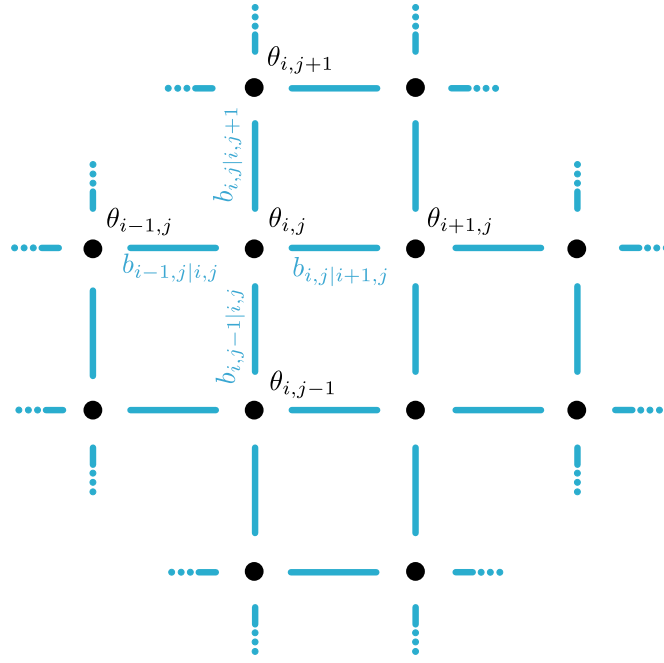


FIG. 1. Lattice with the “easy to read for a human” labels

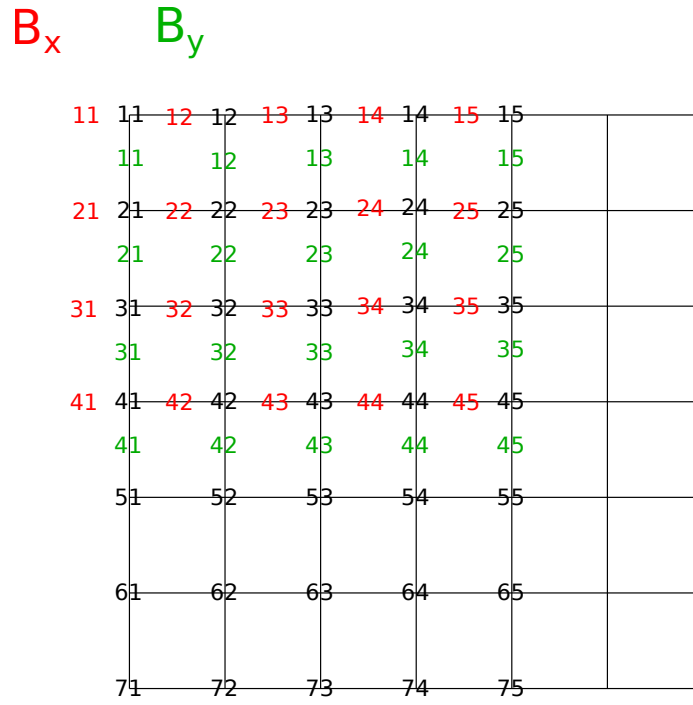


FIG. 2. How it is implemented.