Doc: continuous models

Laurent Pagnier and Julian Fritzsch

$$m(\boldsymbol{x})\frac{\partial^2}{\partial t^2}\theta(\boldsymbol{x},t) + d(\boldsymbol{x})\frac{\partial}{\partial t}\theta(\boldsymbol{x},t) = p(\boldsymbol{x},t) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\right], \tag{1}$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned}
\theta_{i,j}(t+\Delta t) &= \left[2\chi_{ij} - \frac{\Delta t^2 m_{ij}^{-1}\chi_{ij}}{\Delta x^2}\left(b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j}\right)\right]\theta_{i,j}(t)\\
&- \left[1 - \frac{\gamma_{i,j}\Delta t}{2}\right]\chi_{ij}\theta_{i,j}(t-\Delta t) + \frac{\Delta t^2\chi_{i,j}m_{ij}^{-1}}{\Delta x^2}\left[b_{i-1,j|i,j}\theta_{i-1,j}(t) + b_{i,j|i+1,j}\theta_{i+1,j}(t)\right.\\
&\left. + b_{i,j-1|i,j}\theta_{i,j-1}(t) + b_{i,j|i,j+1}\theta_{i,j+1}(t)\right] + \Delta t^2\chi_{ij}m_{ij}^{-1}p_{i,j}\,,
\end{aligned} \tag{2}$$

where $\chi_{i,j} = \left[1 + \frac{\gamma_{i,j}\Delta t}{2}\right]^{-1}$ and $\gamma_{i,j} = d_{i,j}/m_{i,j}$.

The previous equation follows from

$$\boldsymbol{\nabla}\cdot\left[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\right] = \partial_x b_x\partial_x\theta + b_x\partial_x^2\theta + \partial_y b_y\partial_y\theta + b_y\partial_y^2\theta\,, \tag{3}$$

where

$$b_x(\boldsymbol{x}) \approx \frac{b_{i,j-1}^x + b_{i,j}^x}{2}\,, \tag{4}$$

$$\partial_x b_x \approx \frac{b_{i,j}^x - b_{i,j-1}^x}{\Delta x}\,, \tag{5}$$

$$\partial_x\theta \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta x}\,, \tag{6}$$

$$\partial_x^2\theta \approx \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2}\,, \tag{7}$$

So, finally

$$\partial_x b_x\partial_x\theta + b_x\partial_x^2\theta + \partial_y b_y\partial_y\theta + b_y\partial_y^2\theta \approx \tag{8}$$

$$\frac{b_{i,j-1}^x\theta_{i,j-1} + b_{i,j}^x\theta_{i,j+1} + b_{i-1,j}^y\theta_{i-1,j} + b_{i,j}^y\theta_{i+1,j} - \left(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^x + b_{i,j}^y\right)\theta_{i,j}}{\Delta x^2}\,, \tag{9}$$

I. STEADY STATE

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)} = \frac{\Delta x^2\,p_{i,j} + b_{i,j}^y\theta_{i-1,j}^{(n)} + b_{i,j}^y\theta_{i+1,j}^{(n)} + b_{i,j-1}^x\theta_{i,j-1}^{(n)} + b_{i,j}^x\theta_{i,j+1}^{(n)}}{b_{i,j}^y + b_{i-1,j}^y + b_{i,j}^x + b_{i,j-1}^x} \tag{10}$$

II. BOUNDARY CONDITIONS

$$\int_{\partial\Omega} b(\boldsymbol{x})\circ\nabla\theta(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 0, \tag{11}$$
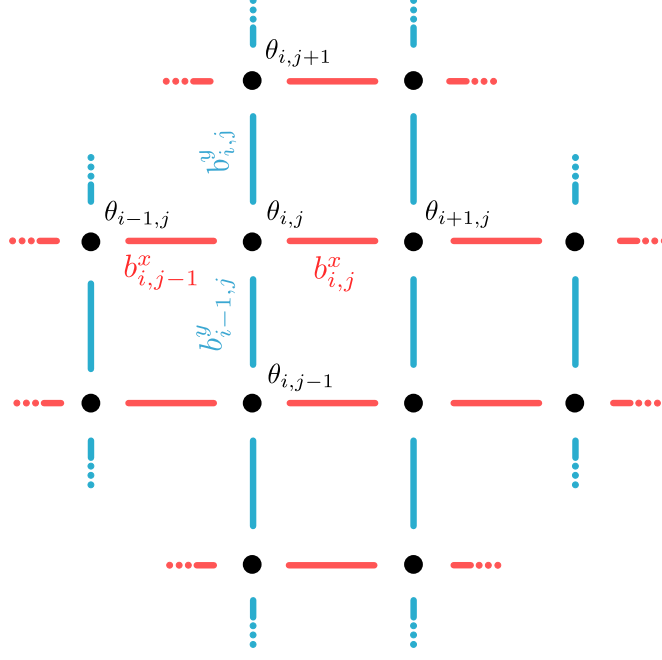
FIG. 1. How it is implemented.

which is trivially true, if

$$n_x b_x \partial_x \theta(\boldsymbol{x}) + n_y b_y \partial_y \theta(\boldsymbol{x}) = 0 \,, \ \forall t \text{ and } \boldsymbol{x} \in \partial\Omega \,, \tag{12}$$

For the moment, lets assume in 1D and treating the case $n_x = -1$.

$$b_x(x)\partial_x \theta(x) = 0 \tag{13}$$

$$b_x(x+\Delta x)\theta(x+\Delta x) = b_x(x)\theta(x) + \Big(\partial_x b_x(x)\theta(x) + b_x(x)\partial_x\theta(x)\Big)\Delta x + (\cdots)\Delta x^2/2 + O(\Delta x^3) \tag{14}$$

$$b_x(x+2\Delta x)\theta(x+2\Delta x) = b_x(x)\theta(x) + \Big(\partial_x b_x(x)\theta(x) + b_x(x)\partial_x\theta(x)\Big)2\Delta x + 2(\cdots)\Delta x^2 + O(\Delta x^3) \tag{15}$$

Summing four times the 1st equation with minus the second, one gets

$$\theta(x) = \frac{4b_x(x+\Delta x)\theta(x+\Delta x) - b_x(x+2\Delta x)\theta(x+2\Delta x)}{3b_x(x) + 2\Delta x \partial_x b_x} + O(\Delta x^3) \tag{16}$$

Converting this expression with Eqs. 4-7, we get

$$\theta_{i,j} = \frac{4(b_{i,j}^x + b_{i,j+1}^x)\theta_{i,j+1} - (b_{i,j+1}^x + b_{i,j+2}^x)\theta_{i,j+2}}{3(b_{i,j-1}^x + b_{i,j}^x) + 4(b_{i,j}^x - b_{i,j-1}^x)} \tag{17}$$

What to do? Set $b_{i,j-1}^x$ to 0 or to $b_{i,j}^x$!? (My opinion is that we should set it to 0.)

III.   VECTORIZATION

The different quantities are flattened, as follows

$$\theta_{i,j} \to \tilde{\theta}_k \,, \tag{18}$$

with $k = N_y(i-1) + j$. With this change in the indexing, the indices of neighbouring nodes become

$$i-1, j \rightarrow k-1\,, \tag{19}$$
$$i+1, j \rightarrow k+1\,, \tag{20}$$
$$i, j-1 \rightarrow k-N_y\,, \tag{21}$$
$$i, j+1 \rightarrow k+N_y\,, \tag{22}$$

$$\xi_{i,j}(\boldsymbol{\theta}) = -\big(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^y + b_{i,j}^y\big)\theta_{i,j} + b_{i,j-1}^x\theta_{i,j-1} + b_{i,j}^x\theta_{i,j+1} + b_{i-1,j}^y\theta_{i-1,j} + b_{i,j}^y\theta_{i+1,j}\,, \tag{23}$$

$$\boldsymbol{\xi}(\boldsymbol{\theta}) = \boldsymbol{\Xi}\,\tilde{\boldsymbol{\theta}} \tag{24}$$

$$\Xi_{kl} = \big(\tilde{b}_k^x + \tilde{b}_{k-N_y}^x + \tilde{b}_k^y + \tilde{b}_{k-1}^y\big)\delta_{k,l} + \tilde{b}_{k-N_y}^x\delta_{k-N_y,l} + \tilde{b}_k^x\delta_{k+N_y,l} + \tilde{b}_{k-1}^y\delta_{k-1,l} + \tilde{b}_k^y\delta_{k+1,l}\,, \tag{25}$$

with $\delta_{.,.}$ is the Kronecker product.

IV.   CRANK–NICOLSON

$$\theta_{i,j}(t+\Delta t) - \frac{\Delta t}{2}\,\omega_{i,j}(t+\Delta t) = \theta_{i,j}(t) + \frac{\Delta t}{2}\,\omega_{i,j}(t)\,, \tag{26}$$

$$\Big(1+\frac{\gamma_{i,j}\Delta t}{2}\Big)\omega_{i,j}(t+\Delta t) - \frac{\Delta t}{2}\xi\big(\theta_{i,j}(t+\Delta t)\big) = \Big(1-\frac{\gamma_{i,j}\Delta t}{2}\Big)\omega_{i,j}(t) + \frac{\Delta t}{2}\xi\big(\theta_{i,j}(t)\big) + (2m_{i,j})^{-1}\Big[p_{i,j}(t+\Delta t) + p_{i,j}(t)\Big]\,, \tag{27}$$

$$\underbrace{\begin{bmatrix} \mathbb{1} & -\frac{\Delta t}{2}\mathbb{1} \\ -\frac{\Delta t}{2\Delta x^2}\boldsymbol{M}^{-1}\boldsymbol{\Xi} & \mathbb{1}+\frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}}_{=\boldsymbol{A}}\boldsymbol{x}(t+\Delta t) = \underbrace{\begin{bmatrix} \mathbb{1} & \frac{\Delta t}{2}\mathbb{1} \\ \frac{\Delta t}{2\Delta x^2}\boldsymbol{M}^{-1}\boldsymbol{\Xi} & \mathbb{1}-\frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}}_{=\boldsymbol{B}}\boldsymbol{x}(t) + \underbrace{\begin{bmatrix} \mathbb{0} \\ \boldsymbol{\pi} \end{bmatrix}}_{=\boldsymbol{C}}\,, \tag{28}$$

with $\boldsymbol{x}^\top = [\tilde{\boldsymbol{\theta}}^\top\ \tilde{\boldsymbol{\omega}}^\top]$.