

## Doc: continuous models

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$$m(\mathbf{x}) \frac{\partial^2}{\partial t^2} \theta(\mathbf{x}, t) + d(\mathbf{x}) \frac{\partial}{\partial t} \theta(\mathbf{x}, t) = p(\mathbf{x}, t) + \nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)], \quad (1)$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned} \theta_{i,j}(t + \Delta t) = & \left[ 2\chi_{ij} - \frac{\Delta t^2 m_{ij}^{-1} \chi_{ij}}{\Delta x^2} \left( b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j} \right) \right] \theta_{i,j}(t) \\ & - \left[ 1 - \frac{\gamma_{i,j} \Delta t}{2} \right] \chi_{ij} \theta_{i,j}(t - \Delta t) + \frac{\Delta t^2 \chi_{i,j} m_{ij}^{-1}}{\Delta x^2} \left[ b_{i-1,j|i,j} \theta_{i-1,j}(t) + b_{i,j|i+1,j} \theta_{i+1,j}(t) \right. \\ & \left. + b_{i,j-1|i,j} \theta_{i,j-1}(t) + b_{i,j|i,j+1} \theta_{i,j+1}(t) \right] + \Delta t^2 \chi_{ij} m_{ij}^{-1} p_{i,j}, \end{aligned} \quad (2)$$

where  $\chi_{i,j} = \left[ 1 + \frac{\gamma_{i,j} \Delta t}{2} \right]^{-1}$  and  $\gamma_{i,j} = d_{i,j}/m_{i,j}$ .

The previous equation follows from

$$\nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)] = \partial_x b_x \partial_x \theta + b_x \partial_x^2 \theta + \partial_y b_y \partial_y \theta + b_y \partial_y^2 \theta, \quad (3)$$

where

$$b_x(\mathbf{x}) \approx \frac{b_{i,j-1}^x + b_{i,j}^x}{2}, \quad (4)$$

$$\partial_x b_x \approx \frac{b_{i,j}^x - b_{i,j-1}^x}{\Delta x}, \quad (5)$$

$$\partial_x \theta \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta x}, \quad (6)$$

$$\partial_x^2 \theta \approx \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2}, \quad (7)$$

So, finally

$$\partial_x b_x \partial_x \theta + b_x \partial_x^2 \theta + \partial_y b_y \partial_y \theta + b_y \partial_y^2 \theta \approx \quad (8)$$

$$\frac{b_{i,j-1}^x \theta_{i,j-1} + b_{i,j}^x \theta_{i,j+1} + b_{i-1,j}^y \theta_{i-1,j} + b_{i,j}^y \theta_{i+1,j} - \overbrace{(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^x + b_{i,j}^y)}^{=\beta} \theta_{i,j}}{\Delta x^2}, \quad (9)$$

### I. STEADY STATE

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)} = \frac{\Delta x^2 p_{i,j} + b_{i,j}^y \theta_{i-1,j}^{(n)} + b_{i,j}^y \theta_{i+1,j}^{(n)} + b_{i,j-1}^x \theta_{i,j-1}^{(n)} + b_{i,j}^x \theta_{i,j+1}^{(n)}}{b_{i,j}^y + b_{i-1,j}^y + b_{i,j}^x + b_{i,j-1}^x} \quad (10)$$

### II. BOUNDARY CONDITIONS

$$\int_{\partial\Omega} \mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t) \cdot \mathbf{n} \, d\mathbf{x} = 0, \quad (11)$$

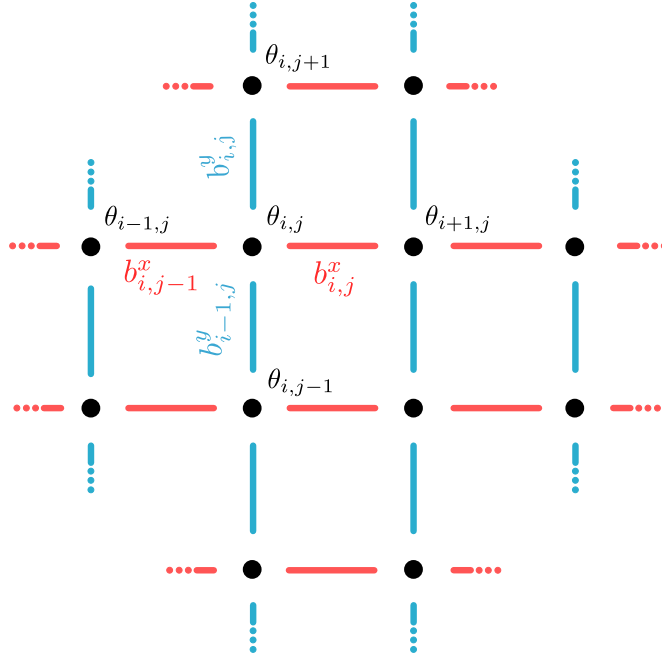


FIG. 1. How it is implemented.

which is trivially true, if

$$n_x b_x \partial_x \theta(\mathbf{x}) + n_y b_y \partial_y \theta(\mathbf{x}) = 0, \quad \forall t \text{ and } \mathbf{x} \in \partial\Omega, \quad (12)$$

For the moment, let's assume in 1D and treating the case  $n_x = -1$ .

$$b_x(x) \partial_x \theta(x) = 0 \quad (13)$$

$$b_x(x + \Delta x) \theta(x + \Delta x) = b_x(x) \theta(x) + \left( \partial_x b_x(x) \theta(x) + \cancel{b_x(x) \partial_x \theta(x)} \right) \Delta x + (\dots) \Delta x^2 / 2 + O(\Delta x^3) \quad (14)$$

0, b.c.

$$b_x(x + 2\Delta x) \theta(x + 2\Delta x) = b_x(x) \theta(x) + \left( \partial_x b_x(x) \theta(x) + \cancel{b_x(x) \partial_x \theta(x)} \right) 2\Delta x + 2(\dots) \Delta x^2 + O(\Delta x^3) \quad (15)$$

0, b.c.

(From the 1st equation, we get

$$\theta_{i,j} = \frac{(b_{i,j}^x + b_{i,j+1}^x) \theta_{i,j+1}}{3b_{i,j}^x - \cancel{b_{i,j-1}^x}} + O(\Delta x^2) \quad (16)$$

which is not exactly what was implemented. It might be enough to change that...) Summing four times the 1st equation with minus the second, one gets

$$\theta(x) = \frac{4b_x(x + \Delta x) \theta(x + \Delta x) - b_x(x + 2\Delta x) \theta(x + 2\Delta x)}{3b_x(x) + 2\Delta x \partial_x b_x} + O(\Delta x^3) \quad (17)$$

Converting this expression with Eqs. 4-7, we get

$$\theta_{i,j} = \frac{4(b_{i,j}^x + b_{i,j+1}^x) \theta_{i,j+1} - (b_{i,j+1}^x + b_{i,j+2}^x) \theta_{i,j+2}}{3(b_{i,j-1}^x + b_{i,j}^x) + 4(b_{i,j}^x - b_{i,j-1}^x)} \quad (18)$$

What to do? Set  $b_{i,j-1}^x$  to 0 or to  $b_{i,j}^x$ !? (My opinion is that we should set it to 0. I agree with that.)

Analogously we obtain for  $n_x = 1$  we obtain by changing  $\Delta x \rightarrow -\Delta x$

$$\theta_{i,j} = \frac{4(b_{i,j-2}^x + b_{i,j-1}^x) \theta_{i,j-1} - (b_{i,j-3}^x + b_{i,j-2}^x) \theta_{i,j-2}}{3(b_{i,j-1}^x + b_{i,j}^x) - 4(b_{i,j}^x - b_{i,j-1}^x)} \quad (19)$$

In this case we should set  $b_{i,j}^x = 0$ .

For the corners we can use ( $n_x = -1$ ,  $n_y = -1$ )

$$b_x(x + \Delta, y)\theta(x + \Delta, y) + b_y(x, y + \Delta)\theta(x, y + \Delta) = (b_x(x, y) + b_y(x, y))\theta(x, y) + 0, b.c. \quad (20)$$

$$(b_x(x, y)\partial_x\theta(x, y) + b_y(x, y)\partial_y\theta(x, y) + b_x(x, y)\partial_y\theta(x, y) + b_y(x, y)\partial_x\theta(x, y))\Delta + (\dots)\Delta^2/2 + \mathcal{O}(\Delta^3),$$

$$b_x(x + 2\Delta, y)\theta(x + 2\Delta, y) + b_y(x, y + 2\Delta)\theta(x, y + 2\Delta) = (b_x(x, y) + b_y(x, y))\theta(x, y) + 0, b.c. \quad (21)$$

$$2(b_x(x, y)\partial_x\theta(x, y) + b_y(x, y)\partial_y\theta(x, y) + b_x(x, y)\partial_y\theta(x, y) + b_y(x, y)\partial_x\theta(x, y))\Delta + 2(\dots)\Delta^2 + \mathcal{O}(\Delta^3)$$

This means for the corners the  $\theta_{i,j}$  is given by the sum of (18) and its  $y$  equivalent.

Now treating the full case (setting the  $b$  outside the grid to 0), lets first define

$$\eta_{\pm}(x) = 1 \pm x/2 - x^2/2, \quad (22)$$

Note that  $\eta_+(-1) = 0$ ,  $\eta_+(0) = 1$ ,  $\eta_+(1) = 1$ ,  $\eta_-(-1) = 1$ ,  $\eta_-(0) = 1$  and  $\eta_-(1) = 0$ .

$$\beta = \eta_+(n_x)b_{i,j-1}^x + \eta_-(n_x)b_{i,j}^x + \eta_+(n_y)b_{i,j-1}^y + \eta_-(n_y)b_{i,j}^y, \quad (23)$$

### III. VECTORIZATION

The different quantities are flattened, as follows

$$\theta_{i,j} \rightarrow \tilde{\theta}_k, \quad (24)$$

with  $k = N_y(i - 1) + j$ . With this change in the indexing, the indices of neighbouring nodes become

$$i - 1, j \rightarrow k - 1, \quad (25)$$

$$i + 1, j \rightarrow k + 1, \quad (26)$$

$$i, j - 1 \rightarrow k - N_y, \quad (27)$$

$$i, j + 1 \rightarrow k + N_y, \quad (28)$$

$$\xi_{i,j}(\theta) = -(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^y + b_{i,j}^y)\theta_{i,j} + b_{i,j-1}^x\theta_{i,j-1} + b_{i,j}^x\theta_{i,j+1} + b_{i-1,j}^y\theta_{i-1,j} + b_{i,j}^y\theta_{i+1,j}, \quad (29)$$

The function  $\xi(\theta)$  can be represented as a matrix acting on the flattened  $\tilde{\theta}$ ,

$$\xi(\theta) = \Xi \tilde{\theta}. \quad (30)$$

The elements of  $\Xi$  read

$$\Xi_{kl} = -(\tilde{b}_k^x + \tilde{b}_{k-N_y}^x + \tilde{b}_k^y + \tilde{b}_{k-1}^y)\delta_{k,l} + \tilde{b}_{k-N_y}^x\delta_{k-N_y,l} + \tilde{b}_k^x\delta_{k+N_y,l} + \tilde{b}_{k-1}^y\delta_{k-1,l} + \tilde{b}_k^y\delta_{k+1,l}, \quad (31)$$

with  $\delta_{\cdot,\cdot}$  is the Kronecker product.

### IV. CRANK-NICOLSON

$$\theta_{i,j}(t + \Delta t) - \frac{\Delta t}{2} \omega_{i,j}(t + \Delta t) = \theta_{i,j}(t) + \frac{\Delta t}{2} \omega_{i,j}(t), \quad (32)$$

$$\left(1 + \frac{\gamma_{i,j}\Delta t}{2}\right)\omega_{i,j}(t + \Delta t) - \frac{\Delta t}{2}\xi(\theta_{i,j}(t + \Delta t)) = \left(1 - \frac{\gamma_{i,j}\Delta t}{2}\right)\omega_{i,j}(t) + \frac{\Delta t}{2}\xi(\theta_{i,j}(t)) + (2m_{i,j})^{-1}[p_{i,j}(t + \Delta t) + p_{i,j}(t)], \quad (33)$$

$$\underbrace{\begin{bmatrix} \mathbb{1} & -\frac{\Delta t}{2}\mathbb{1} \\ -\frac{\Delta t}{2\Delta x^2}M^{-1}\Xi & \mathbb{1} + \frac{\Delta t}{2}\Gamma \end{bmatrix}}_{=A} \mathbf{x}(t + \Delta t) = \underbrace{\begin{bmatrix} \mathbb{1} & \frac{\Delta t}{2}\mathbb{1} \\ \frac{\Delta t}{2\Delta x^2}M^{-1}\Xi & \mathbb{1} - \frac{\Delta t}{2}\Gamma \end{bmatrix}}_{=B} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \pi \end{bmatrix}}_{=C}, \quad (34)$$

with  $\mathbf{x}^\top = [\tilde{\theta}^\top \tilde{\omega}^\top]$ .

## V. HIGHER ORDERS

$$\partial_x^2 \theta = \frac{-\theta_{i,j-2} + 16\theta_{i,j-1} - 30\theta_{i,j} + 16\theta_{i,j+1} - \theta_{i,j+2}}{24\Delta x^2} + O(\Delta x^5), \quad (35)$$

$$\partial_x \theta = \frac{\theta_{i,j-2} - 8\theta_{i,j-1} + 8\theta_{i,j+1} - \theta_{i,j+2}}{12\Delta x} + O(\Delta x^5), \quad (36)$$

$$\partial_x b_x = \frac{b_{i,j-2}^x - 27b_{i,j-1}^x + 27b_{i,j}^x - b_{i,j+1}^x}{48\Delta x} + O(\Delta x^5), \quad (37)$$

$$\partial_x b_x \partial_x \theta_x + b_x \partial_x^2 \theta = \left[ (b_{i,j-2}^x - 29b_{i,j-1}^x + 25b_{i,j}^x - b_{i,j+1}^x) \theta_{i,j-2} \right. \quad (38)$$

$$\left. + (b_{i,j-2}^x - 152b_{i,j-1}^x + 280b_{i,j}^x - b_{i,j+1}^x) \theta_{i,j-1} \right. \quad (39)$$

$$\left. - (48b_{i,j-1}^x + 48b_{i,j}^x) \theta_{i,j} \right] / (576\Delta x^2) (FALSETODO) \quad (40)$$