Doc: continuous models

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$$m(\boldsymbol{x})\frac{\partial^2}{\partial t^2}\theta(\boldsymbol{x},t) + d(\boldsymbol{x})\frac{\partial}{\partial t}\theta(\boldsymbol{x},t) = P(\boldsymbol{x},t) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\right], \tag{1}$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned}\theta_{i,j}(t+\Delta t) = &\left[2\chi_{ij} - \frac{\Delta t^2 m_{ij}^{-1}\chi_{ij}}{\Delta x^2}\Big(b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j}\Big)\right]\theta_{i,j}(t)\\ &- \xi_{ij}\theta_{i,j}(t-\Delta t) + \frac{\Delta t^2\chi_{i,j}m_{ij}^{-1}}{\Delta x^2}\Big[b_{i-1,j|i,j}\theta_{i-1,j}(t) + b_{i,j|i+1,j}\theta_{i+1,j}(t)\\ &+ b_{i,j-1|i,j}\theta_{i,j-1}(t) + b_{i,j|i,j+1}\theta_{i,j+1}(t)\Big] + \Delta t^2\chi_{ij}m_{ij}^{-1}P_{i,j}\,,\end{aligned} \tag{2}$$

where $\chi_{i,j} = \left[1 + \frac{\gamma_{i,j}\Delta t}{2}\right]^{-1}$, $\xi_{i,j} = \left[1 - \frac{\gamma_{i,j}\Delta t}{2}\right]\chi_{ij}$, with $\gamma_{i,j} = d_{i,j}/m_{i,j}$.

The previous equation follows from

$$\boldsymbol{\nabla}\cdot\left[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\right] = \partial_x B_x\partial_x\theta + B_x\partial_x^2\theta + \partial_y B_y\partial_y\theta + B_y\partial_y^2\theta\,, \tag{3}$$

where

$$B_x \approx \frac{b_{i,j|i+1,j} + b_{i-1,j|i,j}}{2}\,, \tag{4}$$

$$\partial_x B_x \approx \frac{b_{i,j|i+1,j} - b_{i-1,j|i,j}}{\Delta x}\,, \tag{5}$$

$$\partial_x\theta \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta x}\,, \tag{6}$$

$$\partial_x^2\theta \approx \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2}\,, \tag{7}$$

So, finally

$$\partial_x B_x\partial_x\theta + B_x\partial_x^2\theta \approx \tag{8}$$

$$\frac{b_{i-1,j|i,j}\theta_{i-1,j} - (b_{i,j|i+1,j} + b_{i-1,j|i,j})\theta_{i,j} + b_{i,j|i+1,j}\theta_{i+1,j}}{\Delta x^2}\,, \tag{9}$$

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)} = \frac{\Delta x^2 P_{i,j} + b_{i-1,j|i,j}\theta_{i-1,j}^{(n)} + b_{i,j|i+1,j}\theta_{i+1,j}^{(n)} + b_{i,j-1|i,j}\theta_{i,j-1}^{(n)} + b_{i,j|i,j+1}\theta_{i,j+1}^{(n)}}{b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j}} \tag{10}$$

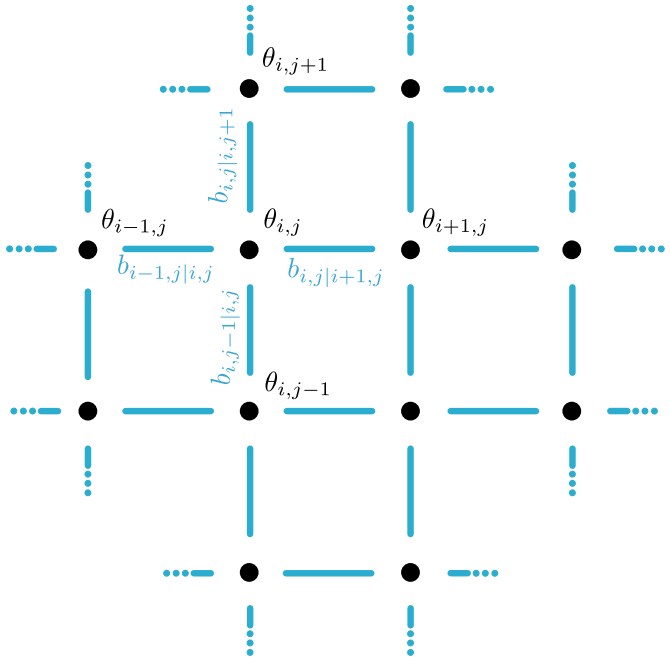

FIG. 1. Lattice with the "easy to read for a human" labels

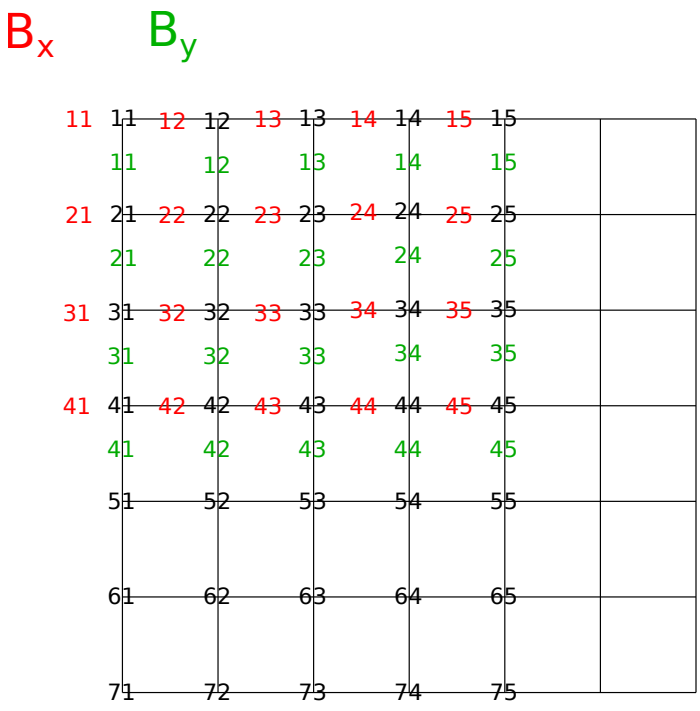

FIG. 2. How it is implemented.