

Doc: continuous models

Laurent Pagnier and Julian Fritzsche

$$m(\mathbf{x}) \frac{\partial^2}{\partial t^2} \theta(\mathbf{x}, t) + d(\mathbf{x}) \frac{\partial}{\partial t} \theta(\mathbf{x}, t) = p(\mathbf{x}, t) + \nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)], \quad (1)$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned} \theta_{i,j}(t + \Delta t) = & \left[2\chi_{ij} - \frac{\Delta t^2 m_{ij}^{-1} \chi_{ij}}{\Delta x^2} \left(b_{i,j|i+1,j} + b_{i-1,j|i,j} + b_{i,j|i,j+1} + b_{i,j-1|i,j} \right) \right] \theta_{i,j}(t) \\ & - \left[1 - \frac{\gamma_{i,j} \Delta t}{2} \right] \chi_{ij} \theta_{i,j}(t - \Delta t) + \frac{\Delta t^2 \chi_{i,j} m_{ij}^{-1}}{\Delta x^2} \left[b_{i-1,j|i,j} \theta_{i-1,j}(t) + b_{i,j|i+1,j} \theta_{i+1,j}(t) \right. \\ & \left. + b_{i,j-1|i,j} \theta_{i,j-1}(t) + b_{i,j|i,j+1} \theta_{i,j+1}(t) \right] + \Delta t^2 \chi_{ij} m_{ij}^{-1} p_{i,j}, \end{aligned} \quad (2)$$

where $\chi_{i,j} = \left[1 + \frac{\gamma_{i,j} \Delta t}{2} \right]^{-1}$ and $\gamma_{i,j} = d_{i,j}/m_{i,j}$.

The previous equation follows from

$$\nabla \cdot [\mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t)] = \partial_x b_x \partial_x \theta + b_x \partial_x^2 \theta + \partial_y b_y \partial_y \theta + b_y \partial_y^2 \theta, \quad (3)$$

where

$$b_x(\mathbf{x}) \approx \frac{b_{i,j-1}^x + b_{i,j}^x}{2}, \quad (4)$$

$$\partial_x b_x \approx \frac{b_{i,j}^x - b_{i,j-1}^x}{\Delta x}, \quad (5)$$

$$\partial_x \theta \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\Delta x}, \quad (6)$$

$$\partial_x^2 \theta \approx \frac{\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}}{\Delta x^2}, \quad (7)$$

So, finally

$$\partial_x b_x \partial_x \theta + b_x \partial_x^2 \theta + \partial_y b_y \partial_y \theta + b_y \partial_y^2 \theta \approx \quad (8)$$

$$\frac{b_{i,j-1}^x \theta_{i,j-1} + b_{i,j}^x \theta_{i,j+1} + b_{i-1,j}^y \theta_{i-1,j} + b_{i,j}^y \theta_{i+1,j} - \overbrace{(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^x + b_{i,j}^y)}^{=\beta} \theta_{i,j}}{\Delta x^2}, \quad (9)$$

I. STEADY STATE

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)} = \frac{\Delta x^2 p_{i,j} + b_{i,j}^y \theta_{i-1,j}^{(n)} + b_{i,j}^y \theta_{i+1,j}^{(n)} + b_{i,j-1}^x \theta_{i,j-1}^{(n)} + b_{i,j}^x \theta_{i,j+1}^{(n)}}{b_{i,j}^y + b_{i-1,j}^y + b_{i,j}^x + b_{i,j-1}^x} \quad (10)$$

II. BOUNDARY CONDITIONS

$$\int_{\partial\Omega} \mathbf{b}(\mathbf{x}) \circ \nabla \theta(\mathbf{x}, t) \cdot \mathbf{n} \, d\mathbf{x} = 0, \quad (11)$$

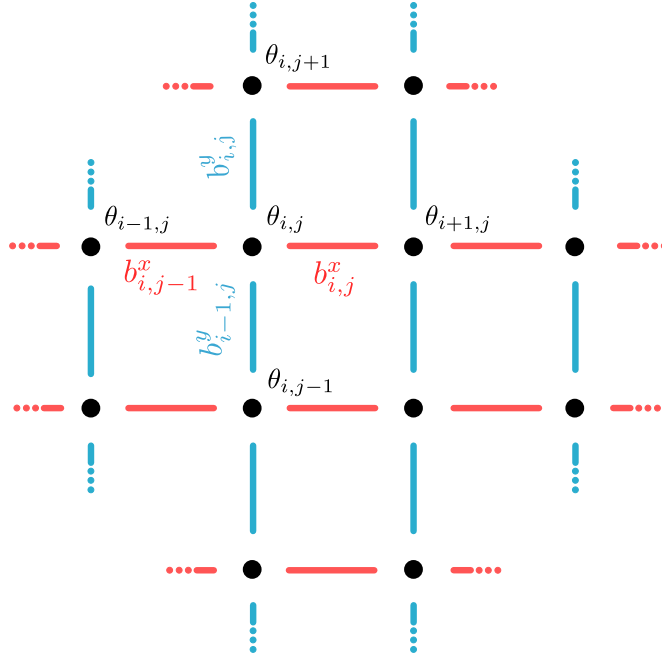


FIG. 1. How it is implemented.

which is trivially true, if

$$n_x b_x \partial_x \theta(\mathbf{x}) + n_y b_y \partial_y \theta(\mathbf{x}) = 0, \quad \forall t \text{ and } \mathbf{x} \in \partial\Omega, \quad (12)$$

For the moment, let's assume in 1D and treating the case $n_x = -1$.

$$b_x(x) \partial_x \theta(x) = 0 \quad (13)$$

$$b_x(x + \Delta x) \theta(x + \Delta x) = b_x(x) \theta(x) + \left(\partial_x b_x(x) \theta(x) + b_x(x) \partial_x \theta(x) \right) \Delta x + (\dots) \Delta x^2 / 2 + O(\Delta x^3) \quad (14)$$

$$b_x(x + 2\Delta x) \theta(x + 2\Delta x) = b_x(x) \theta(x) + \left(\partial_x b_x(x) \theta(x) + b_x(x) \partial_x \theta(x) \right) 2\Delta x + 2(\dots) \Delta x^2 + O(\Delta x^3) \quad (15)$$

(From the 1st equation, we get

$$\theta_{i,j} = \frac{(b_{i,j}^x + b_{i,j+1}^x) \theta_{i,j+1}}{3b_{i,j}^x} + O(\Delta x^2) \quad (16)$$

which is not exactly what was implemented. It might be enough to change that...) Summing four times the 1st equation with minus the second, one gets

$$\theta(x) = \frac{4b_x(x + \Delta x) \theta(x + \Delta x) - b_x(x + 2\Delta x) \theta(x + 2\Delta x)}{3b_x(x) + 2\Delta x \partial_x b_x} + O(\Delta x^3) \quad (17)$$

Converting this expression with Eqs. 4-7, we get

$$\theta_{i,j} = \frac{4(b_{i,j}^x + b_{i,j+1}^x) \theta_{i,j+1} - (b_{i,j+1}^x + b_{i,j+2}^x) \theta_{i,j+2}}{3(b_{i,j-1}^x + b_{i,j}^x) + 4(b_{i,j}^x - b_{i,j-1}^x)} \quad (18)$$

What to do? Set $b_{i,j-1}^x$ to 0 or to $b_{i,j}^x$! (My opinion is that we should set it to 0.) Now treating the full case, let's first define

$$\eta_{\pm}(x) = 1 \pm x/2 - x^2/2, \quad (19)$$

Note that $\eta_+(-1) = 0$, $\eta_+(0) = 1$, $\eta_+(1) = 1$, $\eta_-(-1) = 1$, $\eta_-(0) = 1$ and $\eta_-(1) = 0$.

$$\beta = \eta_+(n_x) b_{i,j-1}^x + \eta_-(n_x) b_{i,j}^x + \eta_+(n_y) b_{i,j-1}^y + \eta_-(n_y) b_{i,j}^y, \quad (20)$$

III. VECTORIZATION

The different quantities are flattened, as follows

$$\theta_{i,j} \rightarrow \tilde{\theta}_k, \quad (21)$$

with $k = N_y(i-1) + j$. With this change in the indexing, the indices of neighbouring nodes become

$$i-1, j \rightarrow k-1, \quad (22)$$

$$i+1, j \rightarrow k+1, \quad (23)$$

$$i, j-1 \rightarrow k-N_y, \quad (24)$$

$$i, j+1 \rightarrow k+N_y, \quad (25)$$

$$\xi_{i,j}(\boldsymbol{\theta}) = -(b_{i,j-1}^x + b_{i,j}^x + b_{i-1,j}^y + b_{i,j}^y)\theta_{i,j} + b_{i,j-1}^x\theta_{i,j-1} + b_{i,j}^x\theta_{i,j+1} + b_{i-1,j}^y\theta_{i-1,j} + b_{i,j}^y\theta_{i+1,j}, \quad (26)$$

The function $\boldsymbol{\xi}(\boldsymbol{\theta})$ can be represented as a matrix acting on the flattened $\tilde{\boldsymbol{\theta}}$,

$$\boldsymbol{\xi}(\boldsymbol{\theta}) = \boldsymbol{\Xi} \tilde{\boldsymbol{\theta}}. \quad (27)$$

The elements of $\boldsymbol{\Xi}$ read

$$\Xi_{kl} = -(\tilde{b}_k^x + \tilde{b}_{k-N_y}^x + \tilde{b}_k^y + \tilde{b}_{k-1}^y)\delta_{k,l} + \tilde{b}_{k-N_y}^x\delta_{k-N_y,l} + \tilde{b}_k^x\delta_{k+N_y,l} + \tilde{b}_{k-1}^y\delta_{k-1,l} + \tilde{b}_k^y\delta_{k+1,l}, \quad (28)$$

with $\delta_{\cdot,\cdot}$ is the Kronecker product.

IV. CRANK-NICOLSON

$$\theta_{i,j}(t+\Delta t) - \frac{\Delta t}{2} \omega_{i,j}(t+\Delta t) = \theta_{i,j}(t) + \frac{\Delta t}{2} \omega_{i,j}(t), \quad (29)$$

$$\left(1 + \frac{\gamma_{i,j}\Delta t}{2}\right)\omega_{i,j}(t+\Delta t) - \frac{\Delta t}{2}\xi(\theta_{i,j}(t+\Delta t)) = \left(1 - \frac{\gamma_{i,j}\Delta t}{2}\right)\omega_{i,j}(t) + \frac{\Delta t}{2}\xi(\theta_{i,j}(t)) + (2m_{i,j})^{-1}\left[p_{i,j}(t+\Delta t) + p_{i,j}(t)\right], \quad (30)$$

$$\underbrace{\begin{bmatrix} \mathbb{1} & -\frac{\Delta t}{2}\mathbb{1} \\ -\frac{\Delta t}{2\Delta x^2}\mathbf{M}^{-1}\boldsymbol{\Xi} & \mathbb{1} + \frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}}_{=\mathbf{A}} \mathbf{x}(t+\Delta t) = \underbrace{\begin{bmatrix} \mathbb{1} & \frac{\Delta t}{2}\mathbb{1} \\ \frac{\Delta t}{2\Delta x^2}\mathbf{M}^{-1}\boldsymbol{\Xi} & \mathbb{1} - \frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}}_{=\mathbf{B}} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \boldsymbol{\pi} \end{bmatrix}}_{=\mathbf{C}}, \quad (31)$$

with $\mathbf{x}^\top = [\tilde{\boldsymbol{\theta}}^\top \tilde{\boldsymbol{\omega}}^\top]$.