Crank-Nicolson for Continuous Models

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For the Crank-Nicolson method we discretize the spatial derivatives and then do one backwards and one forwards Euler step and average over them.

HOMOGENEOUS CASE

Differential equation:

$$\partial_t \omega(x,y,t) = -\gamma\omega(x,y,y) + m^{-1}P(x,y,t) + c^2(\partial_x^2 + \partial_y^2)\theta(x,y,t), \tag{1}$$

where $\gamma = m^{-1}d$ and $c^2 = m^{-1}b$. (Note the + sign in front of the spatial derivatives is due to the graph Laplacian being the discrete version of $-\nabla^2$.)

Discretization of Spatial Derivatives

Discretize spatial derivatives as

$$\partial_x^2\theta(x,y,t) = \frac{\theta(x+a,y,t) + \theta(x-a,y,t) - 2\theta(x,y,t)}{a^2}, \tag{2}$$

$$\partial_y^2\theta(x,y,t) = \frac{\theta(x,y+a,t) + \theta(x,y-a,t) - 2\theta(x,y,t)}{a^2}, \tag{3}$$

where a is the grid constant.

Forwards Euler

The forwards Euler step is given by

$$\theta(x,y,t+h) = \theta(x,y,t) + h\omega(x,y,t), \tag{4}$$

$$\begin{aligned}\omega(x,y,t+h) = &\omega(x,y,t) + h\Bigg(-\gamma\omega(x,y,t) + m^{-1}P(x,y,t) + \frac{c^2}{a^2}\left(\theta(x+a,y,t)\right.\\ &\left.+\theta(x-a,y,t) + \theta(x,y+a,t) + \theta(x,y-a,t) - 4\theta(x,y,t)\right)\Bigg),\end{aligned} \tag{5}$$

where h is the time step.

Backwards Euler

For the backwards Euler replace $h \to -h$ and after $t \to t + h$.

$$\theta(x, y, t) = \theta(x, y, t + h) - h\omega(x, y, t + h), \tag{6}$$

$$\begin{aligned}\omega(x, y, t) =& \omega(x, y, t + h) - h \left(-\gamma\omega(x, y, t + h) + m^{-1}P(x, y, t + h) + \frac{c^2}{a^2} \left(\theta(x + a, y, t + h)\right.\right. \\ & \left.\left. + \theta(x - a, y, t + h) + \theta(x, y + a, t + h) + \theta(x, y - a, t + h) - 4\theta(x, y, t + h)\right) \right),\end{aligned} \tag{7}$$

Crank-Nicolson

From the averaging we obtain

$$\theta(x, y, t + h) - \frac{1}{2}h\omega(x, y, t + h) = \theta(x, y, t) + \frac{1}{2}h\omega(x, y, t), \tag{8}$$

$$\begin{aligned}&\omega(x, y, t + h) - \frac{1}{2}h \left(-\gamma\omega(x, y, t + h) + m^{-1}P(x, y, t + h) + \frac{c^2}{a^2} \left(\theta(x + a, y, t + h)\right.\right. \\ &\left.\left. + \theta(x - a, y, t + h) + \theta(x, y + a, t + h) + \theta(x, y - a, t + h) - 4\theta(x, y, t + h)\right) \right) \\ =& \omega(x, y, t) + \frac{1}{2}h \left(-\gamma\omega(x, y, t) + m^{-1}P(x, y, t) + \frac{c^2}{a^2} \left(\theta(x + a, y, t)\right.\right. \\ &\left.\left. + \theta(x - a, y, t) + \theta(x, y + a, t) + \theta(x, y - a, t) - 4\theta(x, y, t)\right) \right),\end{aligned} \tag{9}$$

To obtain the values at the next time step we first have to calculate the right side and then solve a linear equation on the left side. To do so, we combine the θ and ω into a vector $\Theta = (\theta_1, \ldots, \theta_N, \omega_1, \ldots, \omega_N)^\top$. We can then write (8) and (9) as

$$\Xi\Theta(x, y, t + h) - \Pi(x, y, t + h) = \Lambda\Theta(x, y, t) + \Pi(x, y, t), \tag{10}$$

where

$$\Xi = \begin{bmatrix} \mathbf{1}_N & -\frac{h}{2}\mathbf{1}_N \\ \frac{hc^2}{2a^2}\left(4\mathbf{1}_N - A\right) & \mathbf{1}_N\left(1 + \frac{h\gamma}{2}\right) \end{bmatrix}, \tag{11}$$

$$\Lambda = \begin{bmatrix} \mathbf{1}_N & \frac{h}{2}\mathbf{1}_N \\ \frac{hc^2}{2a^2}\left(A - 4\mathbf{1}_N\right) & \mathbf{1}_N\left(1 - \frac{h\gamma}{2}\right) \end{bmatrix}, \tag{12}$$

$$\Pi(x, y, t) = (\mathbf{0}_N^\top, \frac{hm^{-1}}{2}P(x, y, t)^\top)^\top, \tag{13}$$

where A is the adjacency matrix of the grid. To enforce the boundary conditions that θ and ω vanish at the boundary, we have to remove the rows and columns corresponding to the boundary nodes from the adjacency matrix.