Doc: continuous models

Laurent Pagnier and Julian Fritzsch

$$m(\boldsymbol{x})\frac{\partial^2}{\partial t^2}\theta(\boldsymbol{x},t)+d(\boldsymbol{x})\frac{\partial}{\partial t}\theta(\boldsymbol{x},t)=P(\boldsymbol{x},t)+\boldsymbol{\nabla}\cdot\Big[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\Big]\,, \tag{1}$$

Discretizing in time and space, updated state is computed as

$$\begin{aligned}\theta_{i,j}(t+\Delta t)=&\Bigg[2\chi_{ij}-\frac{\Delta t^2 m_{ij}^{-1}\chi_{ij}}{\Delta x^2}\Big(b_{i,j|i+1,j}+b_{i-1,j|i,j}+b_{i,j|i,j+1}+b_{i,j-1|i,j}\Big)\Bigg]\theta_{i,j}(t)\\&-\xi_{ij}\theta_{i,j}(t-\Delta t)+\frac{\Delta t^2\chi_{i,j}m_{ij}^{-1}}{\Delta x^2}\Bigg[b_{i-1,j|i,j}\theta_{i-1,j}(t)+b_{i,j|i+1,j}\theta_{i+1,j}(t)\\&+b_{i,j-1|i,j}\theta_{i,j-1}(t)+b_{i,j|i,j+1}\theta_{i,j+1}(t)\Bigg]+\Delta t^2\chi_{ij}m_{ij}^{-1}P_{i,j}\,,\end{aligned}\tag{2}$$

where $\chi_{i,j}=\Big[1+\frac{\gamma_{i,j}\Delta t}{2}\Big]^{-1}$, $\xi_{i,j}=\Big[1-\frac{\gamma_{i,j}\Delta t}{2}\Big]\chi_{ij}$, with $\gamma_{i,j}=d_{i,j}/m_{i,j}$.

The previous equation follows from

$$\boldsymbol{\nabla}\cdot\Big[\boldsymbol{b}(\boldsymbol{x})\circ\boldsymbol{\nabla}\theta(\boldsymbol{x},t)\Big]=\partial_x B_x\partial_x\theta+B_x\partial_x^2\theta+\partial_y B_y\partial_y\theta+B_y\partial_y^2\theta\,, \tag{3}$$

where

$$b_x(\boldsymbol{x})\approx\frac{b^x_{i,j-1}+b^x_{i,j}}{2}\,, \tag{4}$$

$$\partial_x v_x\approx\frac{b^x_{i,j-1}+b^x_{i,j}}{\Delta x}\,, \tag{5}$$

$$\partial_x\theta\approx\frac{\theta_{i+1,j}-\theta_{i-1,j}}{2\Delta x}\,, \tag{6}$$

$$\partial_x^2\theta\approx\frac{\theta_{i-1,j}-2\theta_{i,j}+\theta_{i+1,j}}{\Delta x^2}\,, \tag{7}$$

So, finally

$$\partial_x b_x\partial_x\theta+b_x\partial_x^2\theta\approx \tag{8}$$

$$\frac{b^x_{i,j-1}\theta_{i,j-1}-(b^x_{i,j-1}+b^x_{i,j})\theta_{i,j}+b^x_{i,j}\theta_{i+1,j}}{\Delta x^2}\,, \tag{9}$$

Steady state is obtained iteratively as

$$\theta_{i,j}^{(n+1)}=\frac{\Delta x^2\,p_{i,j}+b^y_{i,j}\theta^{(n)}_{i-1,j}+b^y_{i,j}\theta^{(n)}_{i+1,j}+b^x_{i,j-1}\theta^{(n)}_{i,j-1}+b^x_{i,j}\theta^{(n)}_{i,j+1}}{b^y_{i,j}+b^y_{i-1,j}+b^x_{i,j}+b^x_{i,j-1}} \tag{10}$$

Boundary conditions:

$$\int\limits_{\partial\Omega} b(\boldsymbol{x})\circ\nabla\theta(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}=0 \tag{11}$$

which is trivially true, if

$$n_x b_x\partial_x\theta(\boldsymbol{x})+n_y b_y\partial_y\theta(\boldsymbol{x})=0\,,\ \forall t \text{ and } \boldsymbol{x}\in\partial\Omega\,, \tag{12}$$

which leads to

$$\theta_{i,j}=\frac{n_x\big[(n_x+1)b_{i,j-1|i,j}\theta_{i,j-1}+(n_x-1)b_{i,j|i,j+1}\theta_{i,j+1}\big]+n_y\big[(n_y+1)b_{i-1,j|i,j}\theta_{i-1,j}+(n_y-1)b_{i,j|i+1,j}\theta_{i+1,j}\big]}{n_x\big[(n_x+1)b_{i,j-1|i,j}+(n_x-1)b_{i,j|i,j+1}\big]+n_y\big[(n_y+1)b_{i-1,j|i,j}+(n_y-1)b_{i,j|i+1,j}\big]}\,, \tag{13}$$

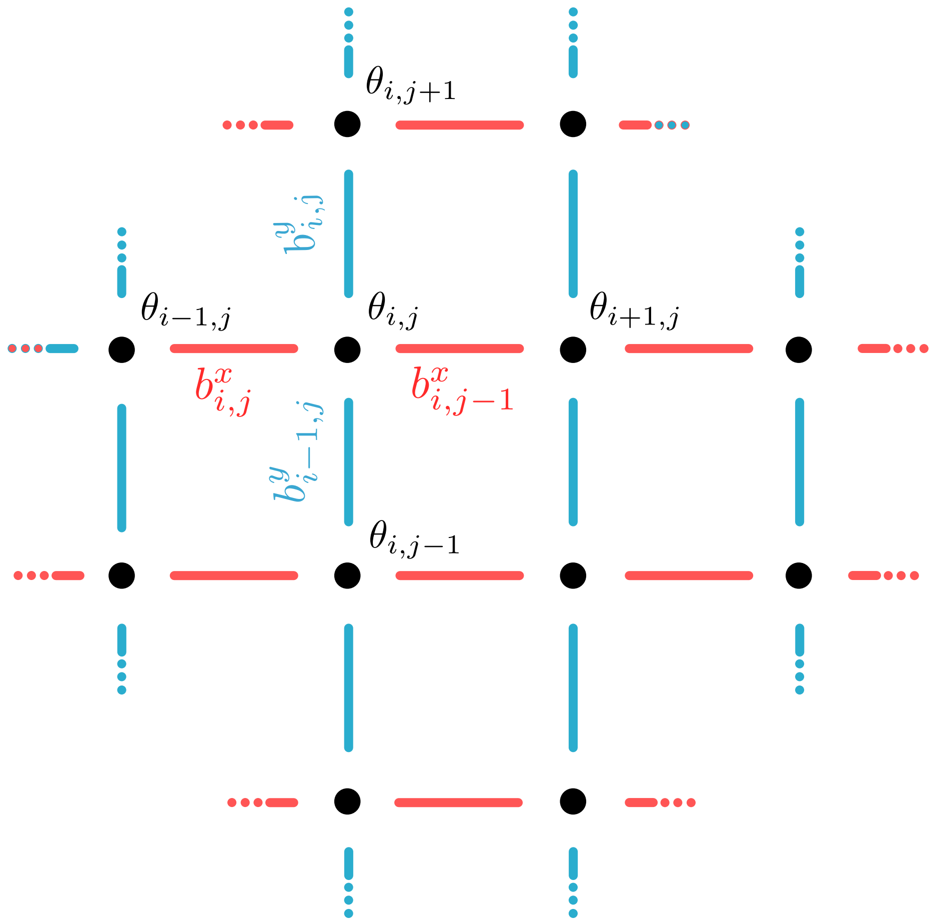

FIG. 1. How it is implemented.

I. CRANK-NICOLSON

$$\frac{\Delta t}{2}\,\omega_{ij}(t+\Delta t)+\theta_{ij}(t+\Delta t)=\frac{\Delta t}{2}\,\omega_{ij}(t)+\theta_{ij}(t) \tag{14}$$

$$\left(1+\frac{\gamma_{ij}\Delta t}{2}\right)\omega_{ij}(t+\Delta t)-\frac{\Delta t}{2}\xi\big(\theta_{ij}(t+\Delta t)\big)=\left(1-\frac{\gamma_{ij}\Delta t}{2}\right)\omega_{ij}(t)+\frac{\Delta t}{2}\xi\big(\theta_{ij}(t)\big)+(2m_{ij})^{-1}\Big[p_{ij}(t+\Delta t)+p_{ij}(t)\Big] \tag{15}$$

$$\begin{bmatrix} \mathbb{1} & -\frac{\Delta t}{2}\mathbb{1} \\ -\frac{\Delta t}{2}\boldsymbol{A} & \mathbb{1}+\frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}\boldsymbol{x}(t+\Delta t)=\begin{bmatrix} \mathbb{1} & \frac{\Delta t}{2}\mathbb{1} \\ \frac{\Delta t}{2}\boldsymbol{A} & \mathbb{1}-\frac{\Delta t}{2}\boldsymbol{\Gamma} \end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix} \mathbb{0} \\ \boldsymbol{\Pi} \end{bmatrix} \tag{16}$$