Linear-quadratic optimal control problems

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[Objectives]
- *Goal:* investigating linear-quadratic optimal control problems and their associated linear optimality system.
- *Issues:* existence of a solution, optimality conditions, connexion with the Riccati equation (a non-linear ODE which yields a feedback law).

[Bibliography] The following references are related to Lecture 2:
- E. Trélat, Contrôle optimal : théorie et applications. Version électronique, 2013. (Chapitre 3).
- E. Sontag, Mathematical Control Theory, Springer, 1998. (Chapter 8).
- U. Boscain and Y. Chitour. Introduction à l'automatique (Chapitre 5) / Introduction to automatic control (Chapter 8). Available on U. Boscain's webpage.

1. EXISTENCE OF A SOLUTION

[Linear quadratic optimal control] Consider the following LQ optimal control problem:

$$\inf \ \frac{1}{2}\int_0^T \Big(\langle y(t), Wy(t)\rangle + \|u(t)\|^2 \Big)\,\mathrm{d}t + \frac{1}{2}\langle y(T), Ky(T)\rangle$$
$$\text{st: } \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \\ y(0) = y_0. \end{cases} \qquad (P(y_0))$$

In the above minimization problem, $y \in H^1(0,T;\mathbb{R}^n)$ and $u \in L^2(0,T;\mathbb{R}^m)$.

Data and assumptions:

- Time horizon: $T > 0$.
- Dynamics coefficients: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$
- Cost coefficients: $W \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$, both assumed symmetric positive semi-definite.

The initial condition $y_0 \in \mathbb{R}^n$ is seen as a *parameter* of the problem.

[The generic constant M]
Convention.
All constants M appearing in forthcoming lemmas will depend on A, B, W, K, and T only. They **will not depend** on y_0.
We use **the same name** for (a priori) independent constants. This is acceptable in so far as all statements remain true if the value of M is **increased**.

[The Sobolev space $H^1(0,T;\mathbb{R}^m)$] The space $H^1(0,T;\mathbb{R}^n)$ is defined as follows:

$$H^1(0,T;\mathbb{R}^n) = \Big\{ y \in L^2(0,T;\mathbb{R}^n) \,|\, \dot{y} \in L^2(0,T;\mathbb{R}^n) \Big\}$$

where $\dot{y}$ denotes the weak derivative of y. It is a **Hilbert space**, equipped with the scalar product:

$$\langle y_1, y_2 \rangle = \int_0^T \langle y_1(t), y_2(t)\rangle\,\mathrm{d}t + \int_0^T \langle \dot{y}_1(t), \dot{y}_2(t)\rangle\,\mathrm{d}t$$

and the norm

$$\|y\|_{H^1(0,T;\mathbb{R}^n)} = \Big(\|y\|^2_{L^2(0,T;\mathbb{R}^n)} + \|\dot{y}\|^2_{L^2(0,T;\mathbb{R}^n)} \Big)^{1/2}.$$

Lemma 1. The space $H^1(0,T;\mathbb{R}^m)$ is contained in the set of continuous functions from $[0,T]$ to $\mathbb{R}^n$. Moreover, all usual calculus rules are valid (in particular, **integration by parts**).

[State equation] Given $u \in L^2(0,T;\mathbb{R}^m)$ and $y_0 \in \mathbb{R}^n$, let $y[u,y_0] \in H^1(0,T;\mathbb{R}^n)$ denote the solution to the state equation

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0.$$

Lemma 2. The map $(u,y_0) \in L^2(0,T;\mathbb{R}^m) \times \mathbb{R}^n \mapsto y[u,y_0] \in H^1(0,T;\mathbb{R}^n)$ is **linear**. There exists $M > 0$ such that for all $u \in L^2(0,T;\mathbb{R}^m)$ and for all $y_0 \in \mathbb{R}^n$,

$$\|y[u,y_0]\|_{L^\infty(0,T;\mathbb{R}^n)} \leq M\big(\|y_0\| + \|u\|_{L^2(0,T;\mathbb{R}^n)}\big),$$
$$\|y[u,y_0]\|_{H^1(0,T;\mathbb{R}^n)} \leq M\big(\|y_0\| + \|u\|_{L^2(0,T;\mathbb{R}^n)}\big).$$

Proof. A direct application of Duhamel's formula and Cauchy-Schwarz inequality.

[Reduced problem] Let $J\colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ be defined by

$$J(u) = J_1(u) + J_2(u) + J_3(u),$$

where

$$J_1(u) = \frac{1}{2}\int_0^T \langle y[u,y_0](t), Wy[u,y_0](t)\rangle\,\mathrm{d}t$$
$$J_2(u) = \frac{1}{2}\int_0^T \|u(t)\|^2\,\mathrm{d}t$$
$$J_3(u) = \frac{1}{2}\langle y[u,y_0](T), Ky[u,y_0](T)\rangle.$$

Consider the **reduced problem**, equivalent to $(P(y_0))$,

$$\inf_{u \in L^2(0,T;\mathbb{R}^m)} J(u). \qquad (P'(y_0))$$

[Weak lower semi-continuity]
Definition 3. A map $F\colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ is said to be **weakly lower semi-continuous** (resp. **weakly continuous**) if for any weakly convergent sequence $(u_k)_{k\in\mathbb{N}}$ with weak limit $\bar{u}$, it holds

$$F(\bar{u}) \leq \liminf_{k\in\mathbb{N}} F(u_k) \quad \Big(\text{resp. } F(\bar{u}) = \lim_{k\in\mathbb{N}} F(u_k)\ \Big).$$

Lemma 4. The map J is **strictly convex** and **weakly lower semi-continuous**.

Proof.
- J_1, J_2, and J_3 are convex, J_2 is strictly convex
- J_1 and J_3 are weakly continuous, J_2 is weakly lower semi-continuous

We skip the details of the proof. They are available on the lecture slides.

[Existence result]
Lemma 5. For all $y_0 \in \mathbb{R}^n$, the problem $(P'(y_0))$ has a unique solution $\bar{u}[y_0]$. Moreover, there exists a constant M, independent of y_0, such that

$$\|\bar{u}[y_0]\|_{L^2(0,T;\mathbb{R}^m)} \leq M\|y_0\|.$$

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a **minimizing sequence**. W.l.o.g.,

$$\frac{1}{2}\|u_k\|^2_{L^2(0,T)} = J_2(u_k) \leq J(u_k) \leq J(0) \leq \frac{1}{2}\big(M\|y_0\|\big)^2.$$

Extracting a subsequence, we can assume that $u_k \rightharpoonup \bar{u}$, for some $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. We have $\|\bar{u}\|_{L^2(0,T;\mathbb{R}^m)} \leq M\|y_0\|$, moreover

$$J(\bar{u}) \leq \liminf J(u_k) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} J(u).$$

Thus, $\bar{u}$ is optimal. Strict convexity of $J \Longrightarrow$ uniqueness.

2. PONTRYAGIN'S PRINCIPLE

[Fréchet differentiability]
Definition 6. The map J is said to be **Fréchet differentiable** if for any $u \in L^2(0,T;\mathbb{R}^m)$, there exists a continuous linear form $DJ(u)\colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ such that

$$\frac{|J(u+v) - J(u) - DJ(u)v|}{\|v\|_{L^2(0,T;\mathbb{R}^m)}} \underset{\|v\|_{L^2}\downarrow 0}{\longrightarrow} 0.$$

Remark. A sufficient condition for Fréchet differentiability is to have

$$|J(u+v) - J(u) - DJ(u)v| \leq M\|v\|^2_{L^2(0,T;\mathbb{R}^m)},$$

for all v and for some M independent of v.

[Fréchet differentiability]
Lemma 7. The map J is Fréchet differentiable. Let $\bar{u}$ and $v \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$ and let $z \in y[v,0]$. Omitting the time variable,

$$DJ(\bar{u})v = \int_0^T \langle W\bar{y}, z\rangle + \langle \bar{u}, v\rangle \,\mathrm{d}t + \langle K\bar{y}(T), z(T)\rangle.$$

Proof. First, $y[u+v,y_0] - y[u,y_0] = y[v,0] = z$.
We have

$$J_1(\bar{u}+v) - J_1(\bar{u}) = \underbrace{\int_0^T \langle W\bar{y}, z\rangle \,\mathrm{d}t}_{=DJ_1(\bar{u})v} + \underbrace{\frac{1}{2}\int_0^T \langle z, Wz\rangle \,\mathrm{d}t}_{\substack{=\mathcal{O}\left(\|z\|^2_{L^\infty(0,T;\mathbb{R}^n)}\right) \\ =\mathcal{O}\left(\|v\|^2_{L^2(0,T;\mathbb{R}^m)}\right)}}.$$

[Fréchet differentiability] Similarly, we have

$$J_2(\bar{u}+v) - J_2(\bar{u}) = \underbrace{\int_0^T \langle \bar{u}, v\rangle \,\mathrm{d}t}_{=DJ_2(\bar{u})v} + \frac{1}{2}\|v\|^2_{L^2(0,T;\mathbb{R}^m)}.$$

and

$$J_3(\bar{u}+v) - J_3(\bar{u}) = \underbrace{\langle K\bar{y}(T), z(T)\rangle}_{=DJ_3(\bar{u})v} + \langle z(T), Kz(T)\rangle.$$

[Riesz representative] **Pre-hamiltonian:** given $(u,y,p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$,

$$H(u,y,p) = \frac{1}{2}\big(\langle y, Wy\rangle + \|u\|^2\big) + \langle p, Ay + Bu\rangle \in \mathbb{R}.$$

We have

$$\nabla_y H(u,y,p) = Wy + A^\top p$$
$$\nabla_u H(u,y,p) = u + B^\top p.$$

Lemma 8. Let $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$. Let $p \in H^1(0,T;\mathbb{R}^n)$ be the solution to

$$-\dot{p}(t) = \nabla_y H(\bar{u}(t), \bar{y}(t), p(t)), \quad p(T) = K\bar{y}(T).$$

Then,

$$DJ(\bar{u})v = \Big\langle \nabla_u H(\bar{u}(\cdot), \bar{y}(\cdot), p(\cdot)), v\Big\rangle_{L^2(0,T;\mathbb{R}^m)}.$$

[Riesz representative] *Proof.* We have

$$\begin{aligned}
\langle K\bar{y}(T), z(T)\rangle &= \langle p(T), z(T)\rangle - \langle p(0), z(0)\rangle \\
&= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle p(t), z(t)\rangle \,\mathrm{d}t \\
&= \int_0^T \langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle \,\mathrm{d}t \\
&= \int_0^T \langle -A^\top p - W\bar{y}, z\rangle + \langle p, Az + Bv\rangle \,\mathrm{d}t \\
&= \int_0^T -\langle W\bar{y}, z\rangle + \langle B^\top p, v\rangle \,\mathrm{d}t.
\end{aligned}$$

Combined with Lemma 7 and the expression of $\nabla_u H(u,y,p)$, we obtain the result.

[Pontryagin's principle]
Theorem 9. Let $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$. Let $\bar{p}$ be defined by the adjoint equation

$$-\dot{\bar{p}}(t) = \nabla_y H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = A^\top \bar{p}(t) + W\bar{y}(t),$$
$$\bar{p}(T) = K\bar{y}(T).$$

Then, $\bar{u}$ is a **solution** to $(P'(y_0))$ **if and only if**

$$\bar{u}(t) + B^\top \bar{p}(t) = \nabla_u H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = 0,$$

for a.e. $t \in (0,T)$.

Proof. Since J is convex, $\bar{u}$ is optimal if and only if $DJ(\bar{u}) = 0$.

Remark. By convexity of $H(\cdot, \bar{y}(t), \bar{p}(t))$,

$$\nabla_u H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = 0 \iff \bar{u}(t) \in \operatorname*{argmin}_{v \in \mathbb{R}^m} H(v, \bar{y}(t), \bar{p}(t)).$$

3. RICCATI EQUATION

[Linear optimality system]
The numerical resolution of $(P'(y_0))$ boils down to the numerical resolution of the following **linear optimality system**:

$$\begin{cases} \dot{y}(t) - Ay(t) - Bu(t) = 0 & \text{State equation} \\ \dot{p}(t) + A^\top p(t) + Wy(t) = 0 & \text{Adjoint equation} \\ u(t) + B^\top p(t) = 0 & \text{Minimality condition} \\ p(T) - Ky(T) = 0 & \text{Initial condition} \\ y(0) = y_0. & \text{Terminal condition} \end{cases}$$

More precisely: Theorem 17 ensures that the optimality system has a unique solution, which is $(\bar{y}, \bar{u}, \bar{p})$.

[Linear optimality system] After elimination of $u = -B^\top p$, we obtain the **coupled** system:

$$\begin{cases} \dot{y}(t) - Ay(t) + BB^\top p(t) = 0 \\ \dot{p}(t) + A^\top p(t) + Wy(t) = 0 \\ p(T) - Ky(T) = 0 \\ y(0) = y_0. \end{cases} \qquad (OS(y_0))$$

[Key idea] A key idea is to **decouple** the linear system, by constructing a map

$$E\colon [0,T] \to \mathbb{R}^{n \times n},$$

independent of y_0, such that for any solution (y,p) to $(OS(y_0))$, we have

$$p(t) = -E(t)y(t).$$

Roadmap. Once E has been constructed, we have:

$$\dot{y} = Ay + Bu = Ay - BB^\top p = (A + BB^\top E)y$$

together we the initial condition $y(0) = y_0$. Thus, y can be computed by solving a linear differential system. Then, p and u are obtained via

$$p = -Ey \quad \text{and} \quad u = -B^\top p.$$

[Derivation of the Riccati equation] Wanted: $p = -Ey$. The terminal condition $p(T) = Ky(T)$ yields

$$E(T) = -K.$$

Next, by differentiation, we have:

$$\dot{p} = -\dot{E}y - E\dot{y},$$

therefore,

$$\begin{aligned} -\dot{E}y &= \dot{p} + E\dot{y} \\ &= \big[-A^\top p - Wy\big] + \big[E(Ay - BB^\top p)\big] \\ &= \big[A^\top Ey - Wy\big] + \big[E(Ay + BB^\top Ey)\big] \\ &= \big(A^\top E + EA - W + EBB^\top E\big)y. \end{aligned}$$

[Riccati equation]
Theorem 10. There exists a unique smooth solution to the following matrix differential equation, called **Riccati equation:**

$$\begin{cases} -\dot{E}(t) = A^\top E(t) + E(t)A - W + E(t)BB^\top E(t) \\ E(T) = -K. \end{cases} \qquad (RE)$$

Moreover, for all $y_0 \in \mathbb{R}^n$, the optimal trajectory $\bar{y}$ for $(P'(y_0))$ is the solution to the closed-loop system

$$\dot{y}(t) = (A + BB^\top E(t))y(t), \quad y(0) = y_0.$$

It also holds:

$$\bar{p}(t) = -E(t)\bar{y}(t) \quad \text{and} \quad \bar{u}[y_0](t) = \underbrace{B^\top E(t)\bar{y}(t)}_{\text{Feedback law!}}. \qquad (1)$$

4. SHOOTING METHOD

[Optimality system] Recall the **optimality system** to be solved:

$$\begin{cases} \dot{y} = Ay - BB^\top p, & y(0) = y_0, \\ \dot{p} = -A^\top p - Wy, & p(T) = Ky(T). \end{cases}$$

Equivalently:

$$\begin{pmatrix} \dot{y} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BB^\top \\ -W & -A^\top \end{pmatrix}}_{=:R} \begin{pmatrix} y \\ p \end{pmatrix}, \ y(0) = y_0, \ p(T) = Ky(T).$$

The optimality system is a **two-point boundary value problem**.
If $p(0)$ was known, then the differential system could be solved numerically.
Shooting method: find $p(0)$ such that $p(T) = Ky(T)$.

[Shooting] Setting $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = e^{TR}$, we have the equivalent formulation:

$$\begin{pmatrix} y(T) \\ p(T) \end{pmatrix} = e^{TR} \begin{pmatrix} y(0) \\ p(0) \end{pmatrix}, \ y(0) = y_0, \ p(T) = Ky(T).$$

The optimality system reduces to the **shooting equation**:

$$\begin{aligned} & X_3 y_0 + X_4 p(0) = K(X_1 y_0 + X_2 p(0)) \\ \iff & p(0) = (X_4 - KX_2)^{-1}\big(KX_1 - X_3\big)y_0. \end{aligned} \qquad (SE)$$

[Shooting algorithm]
In the LQ case, the shooting algorithm consists then in the following steps:

- Compute e^{TR}, by solving the **matrix differential equation**
$$\dot{X}(t) = RX(t), \quad X(0) = I,$$
in $\mathbb{R}^{2n\times 2n}$.
- Solve the **shooting equation** (SE) and find p_0.
- Solve the **differential equation**
$$\begin{pmatrix} \dot{y} \\ \dot{p} \end{pmatrix} = R \begin{pmatrix} y \\ p \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}.$$
- The optimal control is given by $u = -B^\top p$.