# SOD 311 — Lecture 2: Linear-quadratic optimal control problems

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**[Objectives]**
- *Goal:* investigating linear-quadratic optimal control problems and their associated linear optimality system.
- *Issues:* existence of a solution, optimality conditions, connexion with the Riccati equation (a non-linear ODE which yields a feedback law).

**[Bibliography]** The following references are related to Lecture 2:
- E. Trélat, Contrôle optimal : théorie et applications. Version électronique, 2013. (Chapitre 3).
- E. Sontag, Mathematical Control Theory, Springer, 1998. (Chapter 8).
- U. Boscain and Y. Chitour. Introduction à l'automatique (Chapitre 5) / Introduction to automatic control (Chapter 8). Available on U. Boscain's webpage.

## 1. LQ OPTIMIZATION IN FINITE DIMENSIONAL VECTOR SPACES

**[LQ problem]** Consider the **linear-quadratic (LQ) optimization problem**:

$$\inf_{x \in \mathbb{R}^n} f(x) := \frac{1}{2}\langle x, Qx \rangle, \quad \text{subject to: } Ax = b. \quad (P[b])$$

Data:
- $Q \in \mathbb{R}^{n \times n}$, assumed symmetric positive definite
- $A \in \mathbb{R}^{m \times n}$, assumed surjective (i.e. $\text{rank}(A) = m$)
- $b \in \mathbb{R}^m$.

*First goal:* characterizing the solution to $(P[b])$ with a linear system (the optimality system), analyzing this system.
*Second goal:* extending the techniques for solving $(P[b])$ to a linear-quadratic optimal control problem.

**[Elementary remarks]**
*Lemma 1.* The map $f$ is **strictly convex** and continuously **differentiable**, with

$$\nabla f(x) = Qx.$$

*Proof.* Let $x \in \mathbb{R}^n$, let $\delta x \in \mathbb{R}^n$. We have:

$$\begin{aligned} f(x+\delta x) &= \frac{1}{2}\Big( \langle x, Qx \rangle + 2\langle \delta x, Qx \rangle + \langle \delta x, Q\delta x \rangle \Big) \\ &= f(x) + \langle Qx, \delta x \rangle + \frac{1}{2} \underbrace{\langle \delta x, Q \delta x \rangle}_{=\mathcal{O}(\|\delta x\|^2)}. \end{aligned}$$

Thus, $\nabla f(x) = Qx$ and $f(x+\delta x) > f(x) + \langle \nabla f(x), \delta x \rangle$ if $\delta x \neq 0$. Therefore $f$ is strictly convex.

**[Elementary remarks]**
*Lemma 2.* Let $\alpha > 0$ denote the smallest eigenvalue of $Q$. Then

$$\langle x, Qx \rangle \geq \alpha \|x\|^2, \quad \forall x \in \mathbb{R}^n.$$

*Proof.* By the spectral theorem, there exists an orthonormal basis $(e_i)_{i=1,\dots,n}$ of eigenvectors of $Q$. Let $(\lambda_i)_{i=1,\dots,n}$ denote the associated eigenvalues.
Let $x = \sum_{i=1}^n x_i e_i$. We have

$$\langle x, Qx \rangle = \Big\langle \sum_{i=1}^n x_i e_i, \sum_{i=1}^n \lambda_i x_i e_i \Big\rangle = \sum_{i=1}^n \lambda_i x_i^2 \geq \alpha \|x\|^2.$$

**[Elementary remarks]**
*Lemma 3.* There exists a matrix $\tilde{A} \in \mathbb{R}^{n \times m}$ such that

$$A\tilde{A} = I.$$

*Proof.* Let $(e_i)_{i=1,\dots,m}$ denote a basis of $\mathbb{R}^m$. For all $i = 1,\dots,m$, let $u_i$ be such that $Au_i = e_i$. Given $x = \sum_{i=1}^m x_i e_i$, define

$$\tilde{A}x = \sum_{i=1}^m x_i u_i.$$

Obviously $\tilde{A}$ is linear and

$$A\tilde{A}x = \sum_{i=1}^m x_i A u_i = \sum_{i=1}^m x_i e_i = x.$$

**[Elementary remarks]**
*Corollary 4.* There exists a constant $M_1$ such that for all $b \in \mathbb{R}^m$, there exists $x \in \mathbb{R}^n$ satisfying

$$Ax = b \quad \text{and} \quad \|x\| \leq M_1 \|b\|.$$

*Proof.* Take $x = \tilde{A}b$ and $M_1 = \|\tilde{A}\|$.

**[Existence of a solution]**
*Lemma 5.* For all $b \in \mathbb{R}^m$, the problem $(P[b])$ has a **unique solution** $\bar{x}[b]$. Moreover, there exists a constant $M_2 > 0$, depending only on $Q$ and $A$, such that

$$\|\bar{x}[b]\| \leq M_2 \|b\|.$$

*Proof.* Let $z \in \mathbb{R}$ denote the value of problem $(P[b])$. Let $\tilde{x} = \tilde{A}b$. Let $(x_k)_{k \in \mathbb{N}}$ be a **minimizing sequence**, i.e. a sequence such that

$$Ax_k = b, \quad \forall k \in \mathbb{N} \quad \text{and} \quad f(x_k) \to z.$$

Without loss of generality, we assume that $f(x_k) \leq f(\tilde{x})$. We have $\frac{1}{2}\alpha\|x_k\|^2 \leq$

$$f(x_k) \leq f(\tilde{x}) \leq \frac{1}{2}\|Q\| \cdot \|\tilde{x}\|^2 \leq \frac{1}{2}\|Q\| \big(\|\tilde{A}\| \cdot \|b\|\big)^2.$$

**[Existence of a solution]** It follows that

$$\|x_k\| \leq \underbrace{\left(\frac{\|Q\|}{\alpha}\right)^{1/2} \|\tilde{A}\|}_{=:M_2} \cdot \|b\|.$$

By the **Bolzano-Weierstrass theorem**, there exists an accumulation point $\bar{x}[b]$ such that

$$A\bar{x}[b] = b, \quad f(\bar{x}[b]) = z, \quad \|\bar{x}[b]\| \leq M_2\|b\|.$$

Thus $\bar{x}[b]$ is **optimal**.
*Uniqueness:* follows of the strict convexity of $f$ and the linearity of the constraints.

**[Optimality conditions]**
*Lemma 6.* For all $b \in \mathbb{R}^m$, there exists a unique $\lambda[b] \in \mathbb{R}^m$ such that

$$Q\bar{x}[b] + A^\top \lambda[b] = 0.$$

Moreover, $(\bar{x}[b], \lambda[b])$ is the unique solution to the following linear system:

$$\begin{pmatrix} Q & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

The variable $\lambda[b]$ is referred to as **Lagrange multiplier**.
The above system is referred to as the **optimality system**.

**[Optimality conditions]** *Proof. Step 1:* existence of the Lagrange multiplier.
Define the **Lagrangian**

$$L(x,\lambda) = f(x) + \langle \lambda, Ax - b\rangle = f(x) + \langle A^\top\lambda, x\rangle - \langle \lambda, b\rangle.$$

By the **Karush-Kuhn-Tucker conditions**, $\exists\, \lambda[b]$ such that

$$0 = \nabla_x L(\bar{x}[b], \lambda[b]) = Q\bar{x}[b] + A^\top \lambda[b].$$

*Step 2:* uniqueness of the Lagrange multiplier.
A direct consequence of the injectivity of $A^\top$.

**[Optimality conditions]** *Step 3:* uniqueness of the solution to the optimality system.
Take a pair $(x,\lambda)$ such that $\begin{pmatrix} Q & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$.
Let $x'$ be such that $Ax' = b$. By the convexity of the Lagrangian with respect to its first variable, we have

$$\begin{aligned} f(x') = L(x',\lambda) &\geq \underbrace{L(x,\lambda)}_{=f(x)} + \langle \underbrace{\nabla_x L(x,\lambda)}_{=Qx + A^\top\lambda = 0}, x' - x\rangle \\ &= f(x). \end{aligned}$$

Therefore $x$ is optimal for $(P[b])$ and thus $x = \bar{x}[b]$ and $\lambda = \lambda[b]$.

**[Analytic solution]**
*Lemma 7.* For all $b \in \mathbb{R}^m$, we have

$$\begin{pmatrix} \bar{x}[b] \\ \lambda[b] \end{pmatrix} = \begin{pmatrix} Q & A^\top \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} Q^{-1}A^\top(AQ^{-1}A^\top)^{-1}b \\ -(AQ^{-1}A^\top)^{-1}b. \end{pmatrix}$$

*Proof.* For all $(x,\lambda)$, we have

$$\begin{cases} Qx + A^\top\lambda = 0 \\ Ax = b \end{cases} \iff \begin{cases} x = -Q^{-1}A^\top\lambda \\ -AQ^{-1}A^\top\lambda = b \end{cases}$$

$$\iff \begin{cases} x = Q^{-1}A^\top(AQ^{-1}A^\top)^{-1}b \\ \lambda = -(AQ^{-1}A^\top)^{-1}b. \end{cases}$$

Note that $\tilde{Q} = AQ^{-1}A^\top$ is sym. positive definite (thus regular).

**[Value function]**
*Lemma 8.* For all $b \in \mathbb{R}^m$,

$$\begin{aligned} V(b) &:= f(\bar{x}[b]) = \frac{1}{2}\langle b, (AQ^{-1}A^\top)^{-1}b\rangle, \\ \nabla V(b) &= -\lambda[b]. \end{aligned}$$

*Proof.* Direct calculation following Lemma 7. We have

$$\begin{aligned} V(b) &= \frac{1}{2}\langle Q^{-1}A^\top\tilde{Q}^{-1}b, QQ^{-1}A^\top\tilde{Q}^{-1}b\rangle \\ &= \frac{1}{2}\langle b, \tilde{Q}^{-1}(AQ^{-1}A^\top)\tilde{Q}^{-1}b\rangle = \frac{1}{2}\langle b, \tilde{Q}^{-1}b\rangle. \end{aligned}$$

Thus, $\nabla V(b) = \tilde{Q}^{-1}b = -\lambda[b]$.

**[Summary]** *Remember:*

- Problem $(P[b])$ has a **unique solution** with a unique associated Lagrange multiplier.
- The pair $(\bar{x}[b], \lambda[b])$ is characterized by a **well-posed linear system**.
- In the analytical expression, we get a relation between $\lambda[b]$ and $b$, involving a **symmetric matrix**.

We adapt next the previous analysis to LQ optimal control problems.

## 2. EXISTENCE OF A SOLUTION

**[Linear quadratic optimal control]** Consider the following LQ optimal control problem:

$$\begin{aligned} &\inf\ \frac{1}{2}\int_0^T \Big(\langle y(t), Wy(t)\rangle + \|u(t)\|^2\Big)\,\mathrm{d}t + \frac{1}{2}\langle y(T), Ky(T)\rangle \\ &\text{st: } \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \\ y(0) = y_0. \end{cases} \end{aligned} \tag{$P(y_0)$}$$

In the above minimization problem, $y \in H^1(0,T;\mathbb{R}^n)$ and $u \in L^2(0,T;\mathbb{R}^m)$.
*Data and assumptions:*

- Time horizon: $T > 0$.
- Dynamics coefficients: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$
- Cost coefficients: $W \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$, both assumed symmetric positive semi-definite.

The initial condition $y_0 \in \mathbb{R}^n$ is seen as a *parameter* of the problem.

**[The generic constant $M$]**
*Convention.*
All constants $M$ appearing in forthcoming lemmas will depend on $A$, $B$, $W$, $K$, and $T$ only. They **will not depend** on $y_0$.
We use **the same name** for (a priori) independent constants. This is acceptable in so far as all statements remain true if the value of $M$ is **increased**.

**[The Sobolev space $H^1(0,T;\mathbb{R}^m)$]** The space $H^1(0,T;\mathbb{R}^n)$ is defined as follows:

$$H^1(0,T;\mathbb{R}^n) = \Big\{ y \in L^2(0,T;\mathbb{R}^n) \,|\, \dot{y} \in L^2(0,T;\mathbb{R}^n) \Big\}$$

where $\dot{y}$ denotes the weak derivative of $y$. It is a **Hilbert space**, equipped with the scalar product:

$$\langle y_1, y_2 \rangle = \int_0^T \langle y_1(t), y_2(t) \rangle \,\mathrm{d}t + \int_0^T \langle \dot{y}_1(t), \dot{y}_2(t) \rangle \,\mathrm{d}t$$

and the norm

$$\|y\|_{H^1(0,T;\mathbb{R}^n)} = \Big( \|y\|^2_{L^2(0,T;\mathbb{R}^n)} + \|\dot{y}\|^2_{L^2(0,T;\mathbb{R}^n)} \Big)^{1/2}.$$

*Lemma 9.* The space $H^1(0,T;\mathbb{R}^m)$ is contained in the set of continuous functions from $[0,T]$ to $\mathbb{R}^n$. Moreover, all usual calculus rules are valid (in particular, **integration by parts**).

**[State equation]** Given $u \in L^2(0,T;\mathbb{R}^m)$ and $y_0 \in \mathbb{R}^n$, let $y[u,y_0] \in H^1(0,T;\mathbb{R}^n)$ denote the solution to the state equation

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0.$$

*Lemma 10.* The map $(u,y_0) \in L^2(0,T;\mathbb{R}^m) \times \mathbb{R}^n \mapsto y[u,y_0] \in H^1(0,T;\mathbb{R}^n)$ is **linear**. There exists $M > 0$ such that for all $u \in L^2(0,T;\mathbb{R}^m)$ and for all $y_0 \in \mathbb{R}^n$,

$$\|y[u,y_0]\|_{L^\infty(0,T;\mathbb{R}^n)} \leq M\big(\|y_0\| + \|u\|_{L^2(0,T;\mathbb{R}^n)}\big),$$
$$\|y[u,y_0]\|_{H^1(0,T;\mathbb{R}^n)} \leq M\big(\|y_0\| + \|u\|_{L^2(0,T;\mathbb{R}^n)}\big).$$

*Proof.* A direct application of Duhamel's formula and Cauchy-Schwarz inequality.

**[Reduced problem]** Let $J \colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ be defined by

$$J(u) = J_1(u) + J_2(u) + J_3(u),$$

where

$$J_1(u) = \frac{1}{2} \int_0^T \langle y[u,y_0](t), Wy[u,y_0](t) \rangle \,\mathrm{d}t$$
$$J_2(u) = \frac{1}{2} \int_0^T \|u(t)\|^2 \,\mathrm{d}t$$
$$J_3(u) = \frac{1}{2} \langle y[u,y_0](T), Ky[u,y_0](T) \rangle.$$

Consider the **reduced problem**, equivalent to $(P(y_0))$,

$$\inf_{u \in L^2(0,T;\mathbb{R}^m)} J(u). \qquad (P'(y_0))$$

**[Weak lower semi-continuity]**
*Definition 11.* A map $F \colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ is said to be **weakly lower semi-continuous** (resp. **weakly continuous**) if for any weakly convergent sequence $(u_k)_{k\in\mathbb{N}}$ with weak limit $\bar{u}$, it holds

$$F(\bar{u}) \leq \liminf_{k\in\mathbb{N}} F(u_k) \quad \Big( \text{resp. } F(\bar{u}) = \lim_{k\in\mathbb{N}} F(u_k) \Big).$$

*Lemma 12.* The map $J$ is **strictly convex** and **weakly lower semi-continuous**.
*Proof.*

- $J_1$, $J_2$, and $J_3$ are convex, $J_2$ is strictly convex
- $J_1$ and $J_3$ are weakly continuous, $J_2$ is weakly lower semi-continuous

**[Regularity of $J$]**
Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(0,T;\mathbb{R}^m)$, let $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. Assume that $u_k \rightharpoonup \bar{u}$. Let $y_k = y[u_k,y_0]$ and $\bar{y} = y[\bar{u},y_0]$. Then,

- $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;\mathbb{R}^m)$
- by Lemma 10, $y_k$ is bounded in $L^\infty(0,T;\mathbb{R}^n)$.

With the help of Duhamel's formula, we obtain that

$$y[u_k,y_0](t) \to y[\bar{u},y_0](t), \quad \text{for all } t \in [0,T].$$

*Step 1:* This directly implies that

$$J_3(u_k) = \frac{1}{2}\langle y_k(T), Ky_k(T) \rangle \to \frac{1}{2}\langle \bar{y}(T), K\bar{y}(T) \rangle = J_3(\bar{u}).$$

Thus $J_3$ is weakly continuous.

**[Regularity of $J$]** *Step 2:* By the dominated convergence theorem,

$$J_1(u_k) = \frac{1}{2} \int_0^T \langle y_k(t), Wy_k(t) \rangle \,\mathrm{d}t \to = J_1(\bar{u}).$$

*Step 3:* Finally, we have:

$$J_2(u_k) - J_2(\bar{u}) = \frac{1}{2} \int_0^T \|u_k(t)\|^2 - \|\bar{u}(t)\|^2 \,\mathrm{d}t$$
$$= \underbrace{\int_0^T \langle \bar{u}(t), u_k(t) - \bar{u}(t) \rangle \,\mathrm{d}t}_{\to 0} + \underbrace{\frac{1}{2} \int_0^T \|u_k(t) - \bar{u}(t)\|^2 \,\mathrm{d}t}_{\geq 0}.$$

Therefore, $\liminf J_2(u_k) - J_2(\bar{u}) \geq 0$ and $J_2$ is weakly lower semi-continuous.

**[Existence result]**
*Lemma 13.* For all $y_0 \in \mathbb{R}^n$, the problem $(P'(y_0))$ has a unique solution $\bar{u}[y_0]$. Moreover, there exists a constant $M$, independent of $y_0$, such that

$$\|\bar{u}[y_0]\|_{L^2(0,T;\mathbb{R}^m)} \leq M\|y_0\|.$$

*Proof.* Let $(u_k)_{k\in\mathbb{N}}$ be a **minimizing sequence**. W.l.o.g.,

$$\frac{1}{2}\|u_k\|^2_{L^2(0,T)} = J_2(u_k) \leq J(u_k) \leq J(0) \leq \frac{1}{2}\big(M\|y_0\|\big)^2.$$

Extracting a subsequence, we can assume that $u_k \rightharpoonup \bar{u}$, for some $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. We have $\|\bar{u}\|_{L^2(0,T;\mathbb{R}^m)} \leq M\|y_0\|$, moreover

$$J(\bar{u}) \leq \liminf J(u_k) = \inf_{u \in L^2(0,T;\mathbb{R}^m)} J(u).$$

Thus, $\bar{u}$ is optimal. Strict convexity of $J \Longrightarrow$ uniqueness.

**[Fréchet differentiability]**
*Definition 14.* The map $J$ is said to be **Fréchet differentiable** if for any $u \in L^2(0,T;\mathbb{R}^m)$, there exists a continuous linear form $DJ(u)\colon L^2(0,T;\mathbb{R}^m) \to \mathbb{R}$ such that

$$\frac{|J(u+v) - J(u) - DJ(u)v|}{\|v\|_{L^2(0,T;\mathbb{R}^m)}} \underset{\|v\|_{L^2}\downarrow 0}{\longrightarrow} 0.$$

*Remark.* A sufficient condition for Fréchet differentiability is to have

$$|J(u+v) - J(u) - DJ(u)v| \leq M\|v\|^2_{L^2(0,T;\mathbb{R}^m)},$$

for all $v$ and for some $M$ independent of $v$.

**[Fréchet differentiability]**
*Lemma 15.* The map $J$ is Fréchet differentiable. Let $\bar{u}$ and $v \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$ and let $z \in y[v, 0]$. Omitting the time variable,

$$DJ(\bar{u})v = \int_0^T \langle W\bar{y}, z\rangle + \langle \bar{u}, v\rangle \,\mathrm{d}t + \langle K\bar{y}(T), z(T)\rangle.$$

*Proof.* First, $y[u+v, y_0] - y[u, y_0] = y[v,0] = z$.
We have

$$J_1(\bar{u}+v) - J_1(\bar{u}) = \underbrace{\int_0^T \langle W\bar{y}, z\rangle \,\mathrm{d}t}_{=DJ_1(\bar{u})v} + \underbrace{\frac{1}{2}\int_0^T \langle z, Wz\rangle \,\mathrm{d}t}_{\substack{=\mathcal{O}\left(\|z\|^2_{L^\infty(0,T;\mathbb{R}^n)}\right) \\ =\mathcal{O}\left(\|v\|^2_{L^2(0,T;\mathbb{R}^m)}\right)}}.$$

**[Fréchet differentiability]** Similarly, we have

$$J_2(\bar{u}+v) - J_2(\bar{u}) = \underbrace{\int_0^T \langle \bar{u}, v\rangle \,\mathrm{d}t}_{=DJ_2(\bar{u})v} + \frac{1}{2}\|v\|^2_{L^2(0,T;\mathbb{R}^m)}.$$

and

$$J_3(\bar{u}+v) - J_3(\bar{u}) = \underbrace{\langle K\bar{y}(T), z(T)\rangle}_{=DJ_3(\bar{u})v} + \langle z(T), Kz(T)\rangle.$$

**[Riesz representative]** **Pre-hamiltonian:** given $(u,y,p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$,

$$H(u,y,p) = \frac{1}{2}\big(\langle y, Wy\rangle + \|u\|^2\big) + \langle p, Ay + Bu\rangle \in \mathbb{R}.$$

We have

$$\nabla_y H(u,y,p) = Wy + A^\top p$$
$$\nabla_u H(u,y,p) = u + B^\top p.$$

*Lemma 16.* Let $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$. Let $p \in H^1(0,T;\mathbb{R}^n)$ be the solution to

$$-\dot{p}(t) = \nabla_y H(\bar{u}(t), \bar{y}(t), p(t)), \quad p(T) = K\bar{y}(T).$$

Then,

$$DJ(\bar{u})v = \Big\langle \nabla_u H(\bar{u}(\cdot), \bar{y}(\cdot), p(\cdot)), v\Big\rangle_{L^2(0,T;\mathbb{R}^m)}.$$

**[Riesz representative]** *Proof.* We have

$$\begin{aligned}\langle K\bar{y}(T), z(T)\rangle &= \langle p(T), z(T)\rangle - \langle p(0), z(0)\rangle \\ &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle p(t), z(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle \dot{p}(t), z(t)\rangle + \langle p(t), \dot{z}(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle -A^\top p - W\bar{y}, z\rangle + \langle p, Az + Bv\rangle \,\mathrm{d}t \\ &= \int_0^T -\langle W\bar{y}, z\rangle + \langle B^\top p, v\rangle \,\mathrm{d}t.\end{aligned}$$

Combined with Lemma 15 and the expression of $\nabla_u H(u,y,p)$, we obtain the result.

**[Pontryagin's principle]**
*Theorem 17.* Let $\bar{u} \in L^2(0,T;\mathbb{R}^m)$. Let $\bar{y} = y[\bar{u}, y_0]$. Let $\bar{p}$ be defined by the adjoint equation

$$-\dot{\bar{p}}(t) = \nabla_y H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = A^\top \bar{p}(t) + W\bar{y}(t),$$
$$\bar{p}(T) = K\bar{y}(T).$$

Then, $\bar{u}$ is a **solution** to $(P'(y_0))$ **if and only if**

$$\bar{u}(t) + B^\top \bar{p}(t) = \nabla_u H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = 0,$$

for a.e. $t \in (0,T)$.
*Proof.* Since $J$ is convex, $\bar{u}$ is optimal if and only if $DJ(\bar{u}) = 0$.
*Remark.* By convexity of $H(\cdot, \bar{y}(t), \bar{p}(t))$,

$$\nabla_u H(\bar{u}(t), \bar{y}(t), \bar{p}(t)) = 0$$
$$\Longleftrightarrow \bar{u}(t) \in \underset{v\in\mathbb{R}^m}{\operatorname{argmin}}\ H(v, \bar{y}(t), \bar{p}(t)).$$

**[Estimate of $p$]**
*Lemma 18.* Let $\bar{u}$ denote the solution to $(P'(y_0))$, let $\bar{y} = y[\bar{u}, y_0]$, and let $\bar{p}$ be the associated costate. Then, there exists a constant $M$, independent of $y_0$, such that

$\|\bar{p}\|_{L^\infty(0,T;\mathbb{R}^m)} \leq M\|y_0\|$ and $\|\bar{p}\|_{H^1(0,T;\mathbb{R}^m)} \leq M\|y_0\|$.
*Proof.* We know that

$\|\bar{u}\|_{L^2(0,T;\mathbb{R}^m)} \leq M\|y_0\|$ and $\|\bar{y}\|_{L^\infty(0,T;\mathbb{R}^m)} \leq M\|y_0\|$.

Denote $\tilde{p}(t) = \bar{p}(T-t)$. Then $\tilde{p}$ is solution to

$$\dot{\tilde{p}}(t) = A^\top \tilde{p}(t) + W\bar{y}(T-t), \quad \tilde{p}(0) = K\bar{y}(T).$$

Duhamel $\Longrightarrow$ bounds of $\tilde{p}$ in $L^\infty(0,T;\mathbb{R}^n)$ and $H^1(0,T;\mathbb{R}^n)$.

**[A last formula]**
*Lemma 19.* Let $\bar{u} = \bar{u}[y_0]$, let $\bar{y} = y[\bar{u}, y_0]$, and let $\bar{p}$ be the associated costate. Then,

$$V(y_0) := \Big(\inf_{u\in L^2(0,T;\mathbb{R}^m)} J(u)\Big) = J(\bar{u}) = \frac{1}{2}\langle \bar{p}(0), y_0\rangle.$$

*Proof.* We have

$$\begin{aligned}2J_3(\bar{u}) &= \langle \bar{y}(T), K\bar{y}(T)\rangle = \langle \bar{p}(T), \bar{y}(T)\rangle \\ &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle \bar{p}(t), \bar{y}(t)\rangle \,\mathrm{d}t + \langle \bar{p}(0), y_0\rangle \,\mathrm{d}t.\end{aligned}$$

**[A last formula]** We further have

$$\begin{aligned}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle \bar{p}(t), \bar{y}(t)\rangle \,\mathrm{d}t &= \int_0^T \langle \dot{p}, \bar{y}\rangle + \langle \bar{p}, \dot{\bar{y}}\rangle \,\mathrm{d}t \\ &= \int_0^T \langle -A^\top \bar{p} - W\bar{y}, \bar{y}\rangle + \langle \bar{p}, A\bar{y} + B\bar{u}\rangle \,\mathrm{d}t \\ &= \int_0^T -\langle W\bar{y}, \bar{y}\rangle + \langle B^\top \bar{p}, \bar{u}\rangle \,\mathrm{d}t \\ &= \int_0^T -\langle W\bar{y}, \bar{y}\rangle - \|\bar{u}\|^2 \,\mathrm{d}t \\ &= -2J_1(\bar{u}) - 2J_2(\bar{u}).\end{aligned}$$

Combining the last two equalities, we obtain

$$J(\bar{u}) = J_1(\bar{u}) + J_2(\bar{u}) + J_3(\bar{u}) = \frac{1}{2}\langle \bar{p}(0), y_0\rangle.$$

## 4. RICCATI EQUATION

**[Linear optimality system]**
The numerical resolution of $(P'(y_0))$ boils down to the numerical resolution of the following **linear optimality system**:

$$\begin{cases} \dot{y}(t) - Ay(t) - Bu(t) = 0 & \text{State equation} \\ \dot{p}(t) + A^\top p(t) + Wy(t) = 0 & \text{Adjoint equation} \\ u(t) + B^\top p(t) = 0 & \text{Minimality condition} \\ p(T) - Ky(T) = 0 & \text{Initial condition} \\ y(0) = y_0. & \text{Terminal condition} \end{cases}$$

More precisely: Theorem 17 ensures that the optimality system has a unique solution, which is $(\bar{y}, \bar{u}, \bar{p})$.

**[Linear optimality system]** After elimination of $u = -B^\top p$, we obtain the **coupled** system:

$$\begin{cases} \dot{y}(t) - Ay(t) + BB^\top p(t) = 0 \\ \dot{p}(t) + A^\top p(t) + Wy(t) = 0 \\ p(T) - Ky(T) = 0 \\ y(0) = y_0. \end{cases} \qquad (OS(y_0))$$

**[Key idea]** A key idea is to **decouple** the linear system, by constructing a map

$$E\colon [0,T] \to \mathbb{R}^{n\times n},$$

independent of $y_0$, such that for any solution $(y,p)$ to $(OS(y_0))$, we have

$$p(t) = -E(t)y(t).$$

*Roadmap.* Once $E$ has been constructed, we have:

$$\dot{y} = Ay + Bu = Ay - BB^\top p = (A + BB^\top E)y$$

together we the initial condition $y(0) = y_0$. Thus, $y$ can be computed by solving a linear differential system. Then, $p$ and $u$ are obtained via

$$p = -Ey \quad \text{and} \quad u = -B^\top p.$$

**[Derivation of the Riccati equation]** Wanted: $p = -Ey$. The terminal condition $p(T) = Ky(T)$ yields

$$E(T) = -K.$$

Next, by differentiation, we have:

$$\dot{p} = -\dot{E}y - E\dot{y},$$

therefore,

$$\begin{aligned}-\dot{E}y &= \dot{p} + E\dot{y} \\ &= \big[-A^\top p - Wy\big] + \big[E(Ay - BB^\top p)\big] \\ &= \big[A^\top Ey - Wy\big] + \big[E(Ay + BB^\top Ey)\big] \\ &= \big(A^\top E + EA - W + EBB^\top E\big)y.\end{aligned}$$

**[Riccati equation]**
*Theorem 20.* There exists a unique smooth solution to the following matrix differential equation, called **Riccati equation:**

$$\begin{cases} -\dot{E}(t) = A^\top E(t) + E(t)A - W + E(t)BB^\top E(t) \\ E(T) = -K. \end{cases} \tag{RE}$$

Moreover, for all $y_0 \in \mathbb{R}^n$, the optimal trajectory $\bar{y}$ for $(P'(y_0))$ is the solution to the closed-loop system

$$\dot{y}(t) = (A + BB^\top E(t))y(t), \quad y(0) = y_0.$$

It also holds:

$$p(t) = -E(t)\bar{y}(t) \quad \text{and} \quad \bar{u}[y_0](t) = \underbrace{B^\top E(t)\bar{y}(t)}_{\text{Feedback law!}}. \tag{1}$$

**[Riccati equation]** *Proof. Step 1.* The only difficulty is to prove that $(RE)$ is **well-posed**. Once we have a solution $E$, the closed-loop system and relation (1) define a triplet $(u,y,p)$ which satisfies the linear optimality system:

- $(y,u)$ satisfies the state equation
- $u$ satisfies the minimality condition
- $p$ satisfies the adjoint equation:

$$-\dot{p} = \dot{E}y + E\dot{y} = ... = A\top p + Wy.$$

Thus $(u,y,p) = (\bar{u}, \bar{y}, \bar{p})$.

**[Riccati equation]** *Step 2.* The Riccati equation has the abstract form:

$$-\dot{E}(t) = \mathcal{F}(E(t)), \quad E(T) = -K.$$

The map $\mathcal{F}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is polynomial, thus **locally Lipschitz** continuous (but not globally Lipschitz continuous!).
By the Picard-Lindelöf theorem, there exists $\tau \in [-\infty, T)$ such that $(RE)$ has a unique solution on $(\tau, T]$. If $\tau \in \mathbb{R}$, then

$$\lim_{t\downarrow\tau} \|E(t)\| = \infty.$$

**[Riccati equation]** *Step 3.* Assume that $\tau \geq 0$. Let $s \in (\tau, T]$. Let $y_s \in \mathbb{R}^n$, consider

$$\begin{aligned} &\inf\ \frac{1}{2}\int_s^T \Big(\langle y(t), Wy(t)\rangle + \|u(t)\|^2\Big)\,\mathrm{d}t + \frac{1}{2}\langle y(T), Ky(T)\rangle \\ &\text{s.t.:} \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \\ y(s) = y_s. \end{cases} \end{aligned}$$

Adapting the theory developed previously, we prove the existence of a unique solution $(\bar{u}, \bar{y})$ with associated costate $p$, such that

$$p(s) = -E(s)y_s \quad \text{and} \quad \|p(s)\| \leq M\|y_s\|. \tag{2}$$

Here the constant $M$ is independent of $y_s$, $(\bar{u}, \bar{y})$ and $p$, it can also be shown to be independent of $s$.

**[Riccati equation]** *Conclusion.* Let $s > 0$ be such that

$$\|E(s)\| \geq M + 2,$$

where $\|\cdot\|$ denotes the operator norm and where $M$ is the constant appearing in (2).
Let $y_s \in \mathbb{R}^n \backslash \{0\}$ be such that

$$\|E(s)y_s\| \geq (M+1)\|y_s\|.$$

Therefore,

$$\|p(s)\| \geq (M+1)\|y_s\| > M\|y_s\|.$$

A contradiction.

**[Additional properties]**
*Lemma 21.* (1) For all $y_0 \in \mathbb{R}^n$,

$$V(y_0) := \Big(\inf_{u \in L^2(0,T;\mathbb{R}^m)} J(u)\Big) = -\frac{1}{2}\langle y_0, E(0)y_0\rangle.$$

(2) For all $t \in [0, T]$, $E(t)$ is **symmetric negative semi-definite**.
(3) For all $y_0 \in \mathbb{R}^n$, $\nabla V(y_0) = \bar{p}(0)$.

*Proof.*
(1) We have $V(y_0) = \frac{1}{2}\langle \bar{p}(0), y_0\rangle = -\frac{1}{2}\langle y_0, E(0)y_0\rangle$.
(2) Verify that $E^\top$ is the solution $(RE)$. Moreover, $V(y_0) \geq 0$.
(3) We have $\nabla V(y_0) = -E(0)y_0 = \bar{p}(0)$.

## 5. SHOOTING METHOD

**[Optimality system]** Recall the **optimality system** to be solved:

$$\begin{cases} \dot{y} = Ay - BB^\top p, & y(0) = y_0, \\ \dot{p} = -A^\top p - Wy, & p(T) = Ky(T). \end{cases}$$

Equivalently:

$$\begin{pmatrix} \dot{y} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BB^\top \\ W & A^\top \end{pmatrix}}_{=:R} \begin{pmatrix} y \\ p \end{pmatrix},\ y(0) = y_0,\ p(T) = Ky(T).$$

The optimality system is a **two-point boundary value problem**.
If $p(0)$ was known, then the differential system could be solved numerically.
**Shooting method:** find $p(0)$ such that $p(T) = Ky(T)$.

**[Shooting]** Setting $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = e^{TR}$, we have the equivalent formulation:

$$\begin{pmatrix} y(T) \\ p(T) \end{pmatrix} = e^{TR} \begin{pmatrix} y(0) \\ p(0) \end{pmatrix},\ y(0) = y_0,\ p(T) = Ky(T).$$

The optimality system reduces to the **shooting equation**:

$$\begin{aligned} &X_3 y_0 + X_4 p(0) = K(X_1 y_0 + X_2 p(0)) \\ &\quad \Longleftrightarrow\ p(0) = (X_4 - KX_2)^{-1}\big(KX_1 - X_3\big)y_0. \end{aligned} \tag{SE}$$

**[Shooting algorithm]**
In the LQ case, the shooting algorithm consists then in the following steps:

- Compute $e^{TR}$, by solving the **matrix differential equation**
$$\dot{X}(t) = RX(t), \quad X(0) = I,$$
in $\mathbb{R}^{2n \times 2n}$.
- Solve the **shooting equation** $(SE)$ and find $p_0$.
- Solve the **differential equation**
$$\begin{pmatrix} \dot{y} \\ \dot{p} \end{pmatrix} = R\begin{pmatrix} y \\ p \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}.$$
- The optimal control is given by $u = -B^\top p$.