PGE 305: Test

January 12, 2021

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Exercise 1 We consider the following optimization problem:

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} \frac{1}{2}x_1^2 - 3x_1 + 2x_2^2 - 8x_2, \quad \text{subject to:} \begin{cases} -(x_1+x_2^2)+2 \geq y, \\ x_2 \leq 1+x_1, \\ x_2 \geq 0. \end{cases}$$

where y is a real parameter. Let us take $y=0$.

1. Prove the existence of a solution.
2. Write the KKT conditions. Verify that the point $(1,1)$ is feasible and satisfies the KKT conditions.
3. Does the LICQ hold at $(1,1)$?
4. Is the point $(1,1)$ a global solution to the problem?

We consider now arbitrary values of y. Let us denote by $V(y)$ the value of the optization problem.

5. Assume that the result of Theorem 21 holds true. Calculate a first-order Taylor expansion of V at $y=0$.

Exercise 2 Let $n \geq 2$. Let x and y in $\mathbb{R}^n$. We consider the following problem:

$$\inf_{(a,b)\in\mathbb{R}^2} F(a,b) := \sum_{i=1}^{n}(ax_i+b-y_i)^2.$$

1. Give an interpretation of the problem.
2. Calculate $\nabla F(a,b)$ and $\nabla^2 F(a,b)$.
3. Let us assume that the vectors x and y are linearly independent. It is then easy to check (you do not have to do it) that $\nabla^2 F(a,b)$ is positive definite. Prove that the optimization problem has a unique solution, that can be found by solving a coupled system of linear equations.

Exercise 3 We consider a management problem of $N=3$ connected hydroelectric dams, over $T=4$ time intervals. We denote by $q_{t,i}$ the amount of water contained in the dam i at the end of the time interval t. We denote by $x_{t,i}$ the quantity of water that flows out from the dam i over the time interval t.

As indicated on the graph, the amount of water that flows out from the first dam is conveyed to the two other dams; the proportion of water that is conveyed to each of the two dams is not fixed and must be optimized at each time interval. The water that flows out from the dams 2 and 3 is not exploited anymore.

We denote by $P_{t,i}$ the incoming quantity of water due to rain. The capacity of each dam is given by Q_i. The initial level of water in dam i is K_i. At the end of the time interval $t=T$, the total amount of water contained in the dams should be greater or equal to U for security reasons ($U=10$).

Finally, the revenue of the exploitation of water is given by

$$\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{A\,x_{t,i}}{B+x_{t,i}},$$

where $A=B=1$.

1. Write an optimization problem that models the above situation.
2. Solve the problem with the help of AMPL. The numerical values of P, Q, and K are given below.
3. How does a small variation of U impact the optimal revenue (at the first order)?

$P[t,i]$	$i=1$	$i=2$	$i=3$
$t=1$	0	0	0
$t=2$	1	3	2
$t=3$	2	6	4
$t=4$	0	0	0

	$i=1$	$i=2$	$i=3$
$Q[i]$	12	15	10
$K[i]$	4	4	4

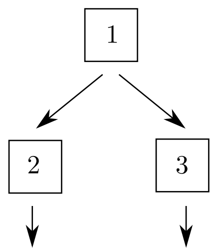

Figure 1: Three connected hydroelectric dams

Exercise 4 We try to model the efficiency of a device in function of its age. Some values of the efficiency, denoted $(y_i)_{i=1,\ldots,N}$, corresponding to various ages, denoted $(x_i)_{i=1,\ldots,N}$, are known and reported below.

For our purposes we consider the problem

$$\inf_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{N} \left(\frac{a}{1+bx_i} - y_i \right)^2 .$$

1. Give an interpretation of a and b.
2. Propose a method for the initialization a and b.
3. Solve the problem with the help of AMPL (initialize a and b). Report the optimal solution on your sheet.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
x_i	0	1	2	3	4	5
y_i	0.80	0.71	0.65	0.60	0.55	0.50