

# Optimal Control of Ordinary Differential Equations

## SOD 311

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## Lecture 3:

# HJB equation and viscosity solutions

- *Goal*: finding global solutions to optimal control problems (in feedback form), by solving a non-linear PDE.
- *Issues*: characterization of the value function with the Hamilton-Jacobi-Bellman equation.

# Bibliography

The following references are related to Chapter 3:



M. Bardi and I. Capuzzo-Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, 1997.



F. Bonnans et P. Rouchon, Commande et optimisation de systèmes dynamiques, Editions de l'Ecole Polytechnique, 2005. (Partie 3, Chapitres 3 et 4).

## 1 Introduction

## 2 Dynamic programming principle

## 3 A first characterization of the value function

## 4 HJB equation: the classical sense

## 5 HJB equation: viscosity solutions

# Introduction

- Our results so far were based on optimality conditions (Pontryagin's principle).
- Now: a different approach, based on **dynamic programming**. In some sense, more specific to optimal control.
- The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to “split” some problems into a family of simpler problems.
- The central tool: the **value function**  $V$ .
  - Defined as the value of the optimization problem, expressed as a function of the initial state.
  - Characterized as the unique **viscosity** solution of a non-linear partial differential equation (PDE) called **HJB equation**.

# Introduction

- *Interest:* a **globally optimal** solution to the problem can be derived from  $V$ .
- *Limitation:* **curse of dimensionality**.
- *Warning:* focus on a specific class of problems.  
All concepts can be extended, in particular to a stochastic framework (finance), and to other nonlinear PDEs.

# Problem formulation

*Data of the problem and assumptions:*

- A parameter  $\lambda > 0$ .
- A non-empty and compact subset  $U$  of  $\mathbb{R}^m$ .
- A mapping  $f: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , such that

$$\begin{aligned}\|f(u, y)\| &\leq \|f\|_\infty, \\ \|f(u', y') - f(u, y)\| &\leq L_f \|(u', y') - (u, y)\|,\end{aligned}$$

for all  $(u, y)$  and  $(u', y') \in U \times \mathbb{R}^n$ .

- A mapping  $\ell: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}$ , such that

$$\begin{aligned}\|\ell(u, y)\| &\leq \|\ell\|_\infty, \\ \|\ell(u', y') - \ell(u, y)\| &\leq L_\ell \|(u', y') - (u, y)\|,\end{aligned}$$

for all  $(u, y)$  and  $(u', y') \in U \times \mathbb{R}^n$ .

# Problem formulation

- *Notation:* for any  $\tau \in [0, \infty]$ ,  $\mathcal{U}_\tau$  is the set of measurable functions from  $(0, \tau)$  to  $U$ .
- State equation: for  $x \in \mathbb{R}^n$  and  $u \in \mathcal{U}_\infty$ , there is a unique solution  $y[u, x]$  to the ODE

$$\dot{y}(t) = f(u(t), y(t)), \quad y(0) = x,$$

by the theorem of Picard-Lindelöf (Cauchy-Lipschitz).

- Cost function  $W$ , for  $u \in \mathcal{U}_\infty$  and  $x \in \mathbb{R}^n$ :

$$W(u, x) = \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) dt.$$

- Optimal control problem and value function  $V$ :

$$V(x) = \inf_{u \in \mathcal{U}_\infty} W(u, x). \quad (P(x))$$



# Grönwall's lemma

## Lemma 1 (Grönwall's lemma)

Let  $\alpha > 0$  and let  $\beta > 0$ . Let  $\theta: [0, \infty) \rightarrow \mathbb{R}$  be a continuous function such that

$$\theta(t) \leq \alpha + \beta \int_0^t \theta(s) ds, \quad \forall t \in [0, \infty).$$

Then,  $\theta(t) \leq \alpha e^{\beta t}$ , for all  $t \in [0, \infty)$ .

## Corollary 2

Let  $u \in \mathcal{U}_\infty$ . For all  $x$  and  $\tilde{x}$ , for all  $t \geq 0$ , it holds:

$$\|y[u, x](t) - y[u, \tilde{x}](t)\| \leq e^{L_f t} \|x - \tilde{x}\|.$$

*Proof.* Grönwall with  $\theta = \|y - \tilde{y}\|$ ,  $\alpha = \|x - \tilde{x}\|$ ,  $\beta = L_f$ .

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# Dynamic programming principle

## Theorem 3 (Dynamic programming (DP) principle)

Let  $\tau > 0$ . Then for all  $x \in \mathbb{R}^n$ , abbreviating  $y = y[u, x]$ ,

$$V(x) = \inf_{u \in \mathcal{U}_\tau} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) \right). \quad (DPP)$$

*Interpretation:*

- $V(x)$  is the value function of an optimal control problem on the interval  $(0, \tau)$ .
- The original integral has been truncated:

$$\int_\tau^\infty e^{-\lambda t} \ell(u(t), y(t)) dt \quad \rightsquigarrow \quad e^{-\lambda \tau} V(y(\tau)).$$

The term  $e^{-\lambda \tau} V(y(\tau))$  is the “optimal cost from  $\tau$  to  $\infty$ ”.

# Flow property

## Lemma 4 (Flow property)

Let  $x \in \mathbb{R}^n$  and let  $u \in \mathcal{U}_\infty$ . Define:

- $u_1 = u|_{(0,\tau)} \in \mathcal{U}_\tau$
- $u_2 = u|_{(\tau,\infty)} \in L^\infty(\tau, \infty; U)$
- $\tilde{u}_2 \in \mathcal{U}_\infty$ ,  $\tilde{u}_2(t) = u_2(t + \tau)$ .

It holds:

$$y[u, x](t) = y[\tilde{u}_2, y[u_1, x](\tau)](t - \tau),$$

for any  $t \geq \tau$ .

*Remark.* After time  $\tau$ , one can forget  $u_1$  and only remember  $y[x, u_1](\tau)$ .

# Proof

*Proof of the DP-principle.* Let us denote

$$\tilde{V}(x) = \inf_{u \in \mathcal{U}_\tau} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) \right).$$

Step 1:  $V \geq \tilde{V}$ . Let  $u$ ,  $u_1$ ,  $u_2$ , and  $\tilde{u}_2$  be as in Lemma 4.

$$\begin{aligned} W(u, x) &= \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &\quad + e^{-\lambda \tau} \int_\tau^\infty e^{-\lambda(t-\tau)} \ell(u(t), y[u, x](t)) dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &\quad + e^{-\lambda \tau} \int_0^\infty e^{-\lambda s} \ell(u(s+\tau), y[u, x](s+\tau)) ds. \end{aligned}$$

# Proof

We further have, for the last integral:

$$\begin{aligned}
 & \int_0^\infty e^{-\lambda s} \ell(u(s+\tau), y[u, x](s+\tau)) \, ds \\
 &= \int_0^\infty e^{-\lambda s} \ell(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1, x](\tau)](s)) \, ds \\
 &= W(\tilde{u}_2, y[u_1, x](\tau)) \geq V(y[u_1, x](\tau)).
 \end{aligned}$$

Injecting in the above equality:

$$\begin{aligned}
 W(u, x) &\geq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \\
 &\geq \tilde{V}(x).
 \end{aligned}$$

Minimizing with respect to  $u$  yields  $V \geq \tilde{V}$ .

# Proof

Step 2:  $\tilde{V} \leq V$ . Let  $\varepsilon > 0$ . Let  $u_1 \in \mathcal{U}_\tau$  be such that

$$\int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1](t)) dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \leq \tilde{V}(x) + \varepsilon/2.$$

Let  $\tilde{u}_2 \in \mathcal{U}_\infty$  be such that

$$W(\tilde{u}_2, y[u_1, x](\tau)) \leq V(y[u_1, x](\tau)) + \varepsilon/2.$$

Let  $u$  be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau), \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

# Proof

The same calculation as above yields:

$$\begin{aligned} W(u, x) &= \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) dt \\ &\quad + e^{-\lambda \tau} \underbrace{\int_0^\infty e^{-\lambda t} \ell(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1, x](\tau)](t)) dt}_{=W(\tilde{u}_2, y[u_1, x](\tau))} \\ &\leq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) dt \\ &\quad + e^{-\lambda \tau} (V(y[u_1, x](\tau)) + \varepsilon/2) \\ &\leq \tilde{V}(x) + \varepsilon. \end{aligned}$$

It follows that

$$V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0.$$



# Decoupling

## Corollary 5

- *Let  $u \in \mathcal{U}_\infty$  be a solution to  $P(x)$ . Let  $\tau > 0$ . Let  $u_1$  and  $\tilde{u}_2$  be defined as in Lemma 4. Then,*
  - *$u_1$  is optimal in the DP principle*
  - *$\tilde{u}_2$  is optimal for  $P(y[u_1, x](\tau))$ .*
- *Conversely: let  $u_1$  be a minimizer of (DPP). Let  $\tilde{u}_2$  be a solution to  $P(y[u_1, x])(\tau)$ . Let  $u \in \mathcal{U}_\infty$  be defined by*

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau) \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

*Then  $u$  is a solution to  $P(x)$ .*

What can we do with the value function? If  $\mathcal{V}$  is known, then the DP-principle allows to **decouple** the problem in time.

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# Regularity of $V$

## Lemma 6

*The value function  $V$  is bounded. It is also uniformly continuous, that is, for all  $\varepsilon > 0$ , there exists  $\alpha > 0$  such that for all  $x$  and  $\tilde{x} \in \mathbb{R}^n$ ,*

$$\|\tilde{x} - x\| \leq \alpha \implies |V(\tilde{x}) - V(x)| \leq \varepsilon.$$

*Proof. Step 1: proof of boundedness.* Let  $x \in \mathbb{R}^n$  and  $u \in \mathcal{U}_\infty$ . We have

$$|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\|_\infty dt \leq \frac{1}{\lambda} \|\ell\|_\infty.$$

Thus  $|V(x)| \leq \frac{1}{\lambda} \|\ell\|_\infty$ .

# Regularity of $V$

*Step 2:* proof of uniform continuity. Let  $\varepsilon > 0$ . Let  $\alpha > 0$ . Let  $x$  and  $\tilde{x}$  be such that  $\|\tilde{x} - x\| \leq \alpha$ , we will specify  $\alpha$  later. We have:

$$\begin{aligned} |V(\tilde{x}) - V(x)| &= \left| \inf_{u \in \mathcal{U}_\infty} W(\tilde{x}, u) - \inf_{u \in \mathcal{U}_\infty} W(x, u) \right| \\ &\leq \sup_{u \in \mathcal{U}_\infty} |W(\tilde{x}, u) - W(x, u)| \leq \Delta_1 + \Delta_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \sup_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt \\ \Delta_2 &= \sup_{u \in \mathcal{U}_\infty} \int_\tau^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt, \end{aligned}$$

where  $\tilde{y} = y[\tilde{x}, u]$  and  $y = y[x, u]$  and where  $\tau > 0$  is **arbitrary**.

# Regularity of $V$

- Bound of  $\Delta_1$ . By Corollary 2,

$$\|\tilde{y}(t) - y(t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha, \quad \forall t \in [0, \tau].$$

Therefore,  $\Delta_1 \leq \tau L_\ell e^{L_f \tau} \alpha$ .

- Bound of  $\Delta_2$ . Since  $\ell$  is bounded,

$$\Delta_2 \leq 2\|\ell\|_\infty \int_\tau^\infty e^{-\lambda t} dt = 2\|\ell\|_\infty e^{-\lambda \tau}.$$

*Conclusion:* take  $\tau > 0$  sufficiently large, so that  $\Delta_2 \leq \frac{\varepsilon}{2}$ .

Take then  $\alpha$  sufficiently small, so that  $\Delta_1 \leq \frac{\varepsilon}{2}$ .

The construction of  $\alpha$  is independent of  $x$  and  $\tilde{x}$ .

We have  $|V(x) - V(\tilde{x})| \leq \varepsilon$ .

# (More) regularity of $V$

## Lemma 7

We have

- if  $\lambda < L_f$ , then  $V$  is  $(\lambda/L_f)$ -Hölder continuous
- if  $\lambda = L_f$ , then  $V$  is  $\alpha$ -Hölder continuous for all  $\alpha \in (0, 1)$
- if  $\lambda > L_f$ , then  $V$  is Lipschitz continuous.

Exercise: prove the last statement of the lemma.

# (More) regularity of $V$

*Proof of the last case.* We have

$$\begin{aligned}
 |V(\tilde{x}) - V(x)| &\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt \\
 &\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} L_\ell \|\tilde{y}(t) - y(t)\| dt \\
 &\leq \int_0^\infty e^{-\lambda t} L_\ell e^{L_f t} \|\tilde{x} - x\| dt \\
 &\leq \frac{L_\ell}{\lambda - L_f} \|\tilde{x} - x\|.
 \end{aligned}$$

# DP-mapping

*Notation:*  $BUC(\mathbb{R}^n)$  is the set of **bounded and uniformly continuous** functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

## Lemma 8

*The space  $BUC(\mathbb{R}^n)$ , equipped with the uniform norm (denoted  $\|\cdot\|_\infty$ ) is a Banach space.*

Fix  $\tau > 0$ . Consider the “**DP-mapping**”:

$$\mathcal{T}: v \in BUC(\mathbb{R}^n) \mapsto \mathcal{T}v \in BUC(\mathbb{R}^n),$$

defined by

$$\mathcal{T}v(x) = \inf_{u \in \mathcal{U}_\tau} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} v(y(\tau)) \right),$$

where  $y = y[u, x]$ . *Exercise:* verify that  $\mathcal{T}v \in BUC(\mathbb{R}^n)$ .



# DP-mapping

*Proof.* Let  $v \in BUC(\mathbb{R}^n)$ . Let  $\varepsilon > 0$ . Let  $\alpha_0 > 0$  be such that

$$\|\tilde{x} - x\| \leq \alpha_0 \implies |v(\tilde{x}) - v(x)| \leq \varepsilon/2.$$

Let  $\alpha > 0$ . Let  $x$  and  $\tilde{x} \in \mathbb{R}^n$  be such that  $\|\tilde{x} - x\| \leq \alpha$ . The value of  $\alpha$  will be fixed later.

For all  $u \in U_\tau$ , for all  $t \in [0, \tau]$ , we have

$$\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha.$$

We have  $\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x) \leq \Delta_1 + \Delta_2$ , with...

# DP-mapping

$$\Delta_1 = \sup_{u \in \mathcal{U}_\tau} \left| \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt - \int_0^\tau e^{-\lambda t} \ell(u(t), \tilde{y}(t)) dt \right|,$$
$$\Delta_2 = \sup_{u \in \mathcal{U}_\tau} \left| e^{-\lambda \tau} v(\tilde{y}(\tau)) - e^{-\lambda \tau} v(y(\tau)) \right|.$$

We fix now

$$\alpha = e^{-L_f \tau} \left( \alpha_0, \frac{\varepsilon}{2\tau} \right).$$

We have

$$\Delta_1 \leq L_\ell e^{\tau L_f} \alpha \leq \varepsilon/2 \quad \text{and} \quad \Delta_2 \leq \varepsilon_2,$$

since  $\|\tilde{y}(\tau) - y(\tau)\| \leq e^{L_f \tau} \alpha \leq \alpha_0$ . Therefore,

$$|\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x)| \leq \varepsilon.$$

# DP-mapping

## Lemma 9

*The operator  $\mathcal{T}$  is Lipschitz continuous with modulus  $e^{-\lambda\tau}$ .*

*Proof.* Let  $x \in \mathbb{R}^n$ . We have

$$\begin{aligned} |\mathcal{T}\tilde{v}(x) - \mathcal{T}v(x)| &\leq \sup_{u \in \mathcal{U}_\tau} |e^{-\lambda\tau} \tilde{v}(y[x, u](\tau)) - e^{-\lambda\tau} v(y[x, u](\tau))| \\ &\leq e^{-\lambda\tau} \|\tilde{v} - v\|_\infty. \end{aligned}$$

We conclude that

$$\|\mathcal{T}\tilde{v} - \mathcal{T}v\|_\infty \leq e^{-\lambda\tau} \|\tilde{v} - v\|_\infty.$$

# A characterization of $V$

## Lemma 10

*The value function  $V$  is the unique solution of the fixed-point equation:*

$$\mathcal{T}v = v, \quad v \in BUC(\mathbb{R}^n).$$

*Proof.*

- Existence: direct consequence of the DP principle ( $V = \mathcal{T}V$ ).
- Uniqueness: for any  $v$  such that  $v = \mathcal{T}v$ , we have

$$\|v - V\|_{\infty} = \|\mathcal{T}v - \mathcal{T}V\|_{\infty} \leq e^{-\lambda\tau} \|v - V\|_{\infty}.$$

Thus  $v = V$ .

*Remark:* the dynamic programming principle entirely characterises the value function!

# Min-plus linearity

*Notation.* Given  $v_1$  and  $v_2 \in BUC(\mathbb{R}^n)$ , we write  $v_1 \leq v_2$  if  $v_1(x) \leq v_2(x)$  for all  $x \in \mathbb{R}^n$ . We define  $\min(v_1, v_2) \in BUC(\mathbb{R}^n)$  by

$$\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.$$

Given  $\alpha \in \mathbb{R}$ , we define  $v_1 + \alpha$  by  $(v_1 + \alpha)(x) = v_1(x) + \alpha$ .

## Lemma 11

Let  $v_1$  and  $v_2 \in BUC(\mathbb{R}^n)$ . Let  $\alpha \in \mathbb{R}$ . The map  $\mathcal{T}$  is monotone:

$$v_1 \leq v_2 \implies \mathcal{T}v_1 \leq \mathcal{T}v_2$$

and min-plus linear:

$$\min(\mathcal{T}v_1, \mathcal{T}v_2) = \mathcal{T} \min(v_1, v_2), \quad \mathcal{T}(v + \alpha) = (\mathcal{T}v) + e^{-\lambda\tau}\alpha.$$

*Proof:* exercise.

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# Hamiltonian

We define the **pre-Hamiltonian**  $H$  and the **Hamiltonian**  $\mathcal{H}$  by

$$\begin{aligned} H(u, x, p) &= \ell(u, x) + \langle p, f(u, x) \rangle, \\ \mathcal{H}(x, p) &= \min_{u \in U} H(u, x, p). \end{aligned}$$

## Lemma 12

*The mapping  $\mathcal{H}$  is continuous, concave with respect to  $p$ , and Lipschitz continuous with respect to  $p$  with modulus  $\|f\|_\infty$ .*

*Proof.* The pre-Hamiltonian  $H$  is affine in  $p$ , thus concave in  $p$ . As an infimum of concave functions,  $\mathcal{H}$  is concave. We have:

$$\begin{aligned} |\mathcal{H}(x, \tilde{p}) - \mathcal{H}(x, p)| &\leq \sup_{u \in U} |H(u, x, \tilde{p}) - H(u, x, p)| \\ &\leq \sup_{u \in U} |\langle \tilde{p} - p, f(u, x) \rangle| \leq \|\tilde{p} - p\| \cdot \|f\|_\infty. \end{aligned}$$

# Informal derivation

*Notation:*  $C^1(\mathbb{R}^n)$ , the set of continuously differentiable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

## Lemma 13

*Let  $\Phi \in C^1(\mathbb{R}^n)$ . Let  $x \in \mathbb{R}^n$ , let  $u \in \mathcal{U}_\infty$ , let  $y = y[u, x]$ . Consider the mapping:*

$$\varphi: \tau \in [0, \infty) \mapsto \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).$$

*Then  $\varphi(0) = 0$  and  $\varphi \in W^{1,\infty}(0, \infty)$  with*

$$\dot{\varphi}(\tau) = e^{-\lambda \tau} (H(u(\tau), y(\tau), \nabla V(y(\tau))) - \lambda \Phi(y(\tau))). \quad (*)$$

*In particular:  $\dot{\varphi}(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$  (if  $u$  is continuous at 0).*



# Informal derivation

*Proof.* To simplify, we only consider the case where  $u$  is continuous, so that  $y$  is  $C^1$  and  $\varphi$  is  $C^1(\mathbb{R}^n)$ . We have then:

$$\begin{aligned}\dot{\varphi}(\tau) &= e^{-\lambda\tau} \ell(u(\tau), y(\tau)) + e^{-\lambda\tau} \langle \nabla \Phi(y(\tau)), \dot{y}(\tau) \rangle \\ &\quad - \lambda e^{-\lambda\tau} \Phi(y(\tau)) \\ &= e^{-\lambda\tau} [\ell(u(\tau), y(\tau)) + \langle \nabla \Phi(y(\tau)), f(u(\tau), y(\tau)) \rangle] \\ &\quad - \lambda e^{-\lambda\tau} \Phi(y(\tau)) \\ &= e^{-\lambda\tau} [H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau))].\end{aligned}$$

# HJB in the classical sense

## Theorem 14

Let  $x \in \mathbb{R}^n$ . Assume that

- $V$  is continuously differentiable in a neighborhood of  $x$
- $P(x)$  has a solution  $\bar{u}$  which is continuous at time 0.

Then,

$$\lambda V(x) - \mathcal{H}(x, \nabla V(x)) = 0,$$

$$\bar{u}(0) \in \operatorname{argmin}_{u_0 \in U} H(u_0, x, \nabla V(x)).$$

# HJB in the classical sense

*Proof. Step 1.* Let  $u_0 \in U$ , let  $u$  be the constant control equal to  $u_0$ , let  $y = y[u, x]$ . By the **dynamic programming** principle, we have:

$$0 \leq \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) - V(x),$$

for all  $\tau$ . Since  $\varphi(0) = 0$ , we deduce from (\*) that:

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).$$

Therefore,

$$0 \leq H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.$$

# HJB in the classical sense

*Step 2.* Let us apply the **dynamic programming principle** again. Redefining  $\varphi$  and setting  $\bar{y} = y[\bar{u}, x]$ , we obtain:

$$0 = \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x),$$

for all  $\tau \geq 0$ . It follows that

$$0 = H(\bar{u}(0), x, \nabla V(x)) - \lambda V(x).$$

*Step 3.* It follows that

$$H(\bar{u}(0), x, \nabla V(x)) = \lambda V(x) \leq H(u_0, x, \nabla V(x)), \quad \forall u_0 \in U.$$

Therefore,  $H(\bar{u}(0), x, \nabla V(x)) = \mathcal{H}(x, \nabla V(x))$ .

# HJB in the classical sense

## Corollary 15

Let  $t \geq 0$ , assume that  $\bar{u}$  is continuous in a neighborhood of  $t$  and that  $V$  is  $C^1$  in a neighborhood of  $\bar{y}(t)$ , where  $\bar{y} := y[\bar{u}, x](t)$ . Then,

$$\bar{u}(t) \in \operatorname{argmin}_{u_0 \in U} H(u_0, \bar{y}, \nabla V(\bar{y})).$$

# HJB in the classical sense

## Remarks.

Let us define the Q-function by  $Q(u, y) := H(u, y, \nabla V(y))$ , assuming that  $V \in C^1(\mathbb{R}^n)$ .

- If the minimizer is unique in the following relation, we have a **feedback law**:

$$\bar{u}(t) = \underset{u}{\operatorname{argmin}} Q(\cdot, \bar{y}(t)).$$

- In some cases, one can show that  $\nabla V(\bar{y}(t)) = p(t)$ , where  $p$  is defined by some adjoint equation  $\rightarrow$  **Pontryagin's principle**.
- In **Reinforcement Learning**, the approximation of  $Q$  is a central objective.

We will call the equation

$$\lambda v(x) - \mathcal{H}(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n \quad (\text{HJB})$$

**Hamilton-Jacobi-Bellman** equation, with unknown  $v: \mathbb{R}^n \rightarrow \mathbb{R}$ .

*Remarks.*

- In general  $V$  is not differentiable  $\rightarrow$  in **which sense** is the HJB equation to be understood?
- In Theorem 14, we have shown that

$$\bar{u}(t) \in \operatorname{argmin} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),$$

(under restrictive assumptions). We will see next that this necessary condition is also **sufficient**.

## Theorem 16 (Verification)

*Let us assume the assumptions of Theorem 14 hold for all  $x \in \mathbb{R}^n$ , so that the HJB equation is satisfied in the classical sense. Let  $x \in \mathbb{R}^n$ . Assume that there exists a control  $\bar{u}$  such that*

$$\bar{u}(t) \in \operatorname{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))),$$

*where  $\bar{y} = y[\bar{u}, x]$ . Then  $\bar{u}$  is globally optimal.*



*Proof.* Consider the function:

$$\varphi(\tau) = \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x).$$

We have  $\varphi(0) = 0$ . Using  $(*)$  and Theorem 14, we obtain:

$$\begin{aligned}\dot{\varphi}(\tau) &= e^{-\lambda \tau} [H(\bar{u}(\tau), \bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau))] \\ &= e^{-\lambda \tau} [\mathcal{H}(\bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau))] \\ &= 0.\end{aligned}$$

Thus  $\varphi$  is constant, equal to 0. Its limit is given by:

$$0 = \int_0^\infty e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt - V(x) = W(x, \bar{u}) - V(x),$$

proving the optimality of  $\bar{u}$ .

- 1 Introduction
- 2 Dynamic programming principle
- 3 A first characterization of the value function
- 4 HJB equation: the classical sense
- 5 HJB equation: viscosity solutions**

# Abstract PDE

We consider an abstract PDE of the form:

$$\mathcal{F}(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n,$$

where  $\mathcal{F}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.

It contains the HJB equation with

$$\mathcal{F}(x, v, p) = \lambda v - \mathcal{H}(x, p).$$

*Goal of the section:* showing that  $V$  is a viscosity solution to the HJB equation.

# Sub- and super-differentials

## Definition 17

Let  $v: \mathbb{R}^n \rightarrow \mathbb{R}$ . The following sets are called **sub- and superdifferential**, respectively:

$$D^- v(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

$$D^+ v(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \leq 0 \right\}.$$

*Exercise.* Let  $v(x) = |x|$ . Show that  $D^- v(0) = [-1, 1]$ .

# Sub- and super-differentials

We have the following characterization.

## Lemma 18

Let  $v: \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous. Let  $p \in \mathbb{R}^n$ .

- $p \in D^-v(x) \iff$  there exists  $\Phi \in C^1(\mathbb{R}^n)$  such that  $\nabla\Phi(x) = p$  and  $v - \Phi$  has a local minimum in  $x$ .
- $p \in D^+v(x) \iff$  there exists  $\Phi \in C^1(\mathbb{R}^n)$  such that  $\nabla\Phi(x) = p$  and  $v - \Phi$  has a local maximum in  $x$ .

*Proof.* The implication  $\implies$  is admitted. The implication  $\impliedby$  is left as an exercise.

# Sub- and super-differentials

*Remark.* In the above lemma, one can chose  $\Phi(x) = v(x)$  without loss of generality. Thus, we have:

- $(v - \Phi)$  has a local minimum in  $x \iff v - \Phi$  is nonnegative in a neighborhood of  $x \iff v$  is locally bounded from below by  $\Phi$
- $(v - \Phi)$  has a local maximum in  $x \iff v - \Phi$  is nonpositive in a neighborhood of  $x \iff v$  is locally bounded from above by  $\Phi$

*Remark.* If  $v$  is Fréchet differentiable at  $x$ , then the sub- and superdifferential are equal to  $\{\nabla v(x)\}$ .

# Viscosity solutions

## Definition 19

Let  $v: \mathbb{R}^n \rightarrow \mathbb{R}$ . We call  $v$  a **viscosity subsolution** if

$$\mathcal{F}(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall p \in D^+ v(x)$$

or, equivalently, if for all  $\Phi \in C^1(\mathbb{R}^n)$  such that  $v - \Phi$  has a local maximum in  $x$ ,

$$\mathcal{F}(x, v(x), \nabla \Phi(x)) \leq 0.$$

# Viscosity solutions

## Definition 20

Let  $v: \mathbb{R}^n \rightarrow \mathbb{R}$ . We call  $v$  a **viscosity supersolution** if

$$\mathcal{F}(x, v(x), p) \geq 0, \quad \forall p \in D^- v(x)$$

or, equivalently, if for all  $\Phi \in C^1(\mathbb{R}^n)$  such that  $v - \Phi$  has a local minimum in  $x$ ,

$$\mathcal{F}(x, v(x), \nabla \Phi(x)) \geq 0.$$

We call  $v$  a **viscosity solution** if it is a sub- and a supersolution.



# Viscosity solutions

## Theorem 21

*The value function  $V$  is a viscosity solution of the HJB equation.*

*Step 1:*  $V$  is a subsolution. Let  $x \in \mathbb{R}^n$ , let  $\Phi \in C^1(\mathbb{R}^n)$  be such that  $V - \Phi$  has a local maximizer in  $x$  and  $V(x) = \Phi(x)$ .

We have to prove that

$$\lambda v(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$

Let  $u_0 \in U$ , let  $u$  be the constant control equal to  $u_0$  and let  $y = y[u, x]$ . By the DPP, we have:

$$V(x) \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) dt + e^{-\lambda \tau} V(y(\tau)).$$

If  $\tau$  is sufficiently small, we have  $V(y(\tau)) \leq \Phi(y(\tau))$ .

# Viscosity solutions

This implies that for  $\tau$  sufficiently small,

$$0 \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x) =: \varphi(\tau).$$

Since  $\varphi(0) = 0$ , we deduce with (\*) that

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla \Phi(x)) - \lambda V(x).$$

Minimizing with respect to  $u_0 \in U$ , we obtain:

$$0 \leq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x),$$

as was to be proved.

# Viscosity solutions

*Step 2:*  $V$  is supersolution. Let  $x \in \mathbb{R}^n$ , let  $\Phi \in C^1(\mathbb{R}^n)$  be such that  $V - \Phi$  has a local minimizer in  $x$  and such that  $V(x) = \Phi(x)$ . We have to prove that

$$\lambda v(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$

It follows from the dynamic programming principle that for  $\tau > 0$  small enough

$$\Phi(x) \geq \inf_{u \in \mathcal{U}_\tau} \underbrace{\int_0^\tau e^{-\lambda t} \ell(u(t), y[x, u](t)) dt + e^{-\lambda \tau} \Phi(y[x, u](\tau))}_{=:\phi[u](\tau)}.$$

# Viscosity solutions

Thus by Lemma 13,

$$\begin{aligned}
 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \dot{\phi}[u](t) \, dt \\
 &= \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} (H(u(t), y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t))) \, dt \\
 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} (\mathcal{H}(y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t))) \, dt.
 \end{aligned}$$

Dividing by  $\tau$  and passing to the limit as  $\tau \downarrow 0$ , we obtain (exercise!) that

$$0 \geq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x).$$

# Viscosity solutions

## Theorem 22 (Comparison principle)

*Let  $v_1$  be a subsolution to the HJB equation. Let  $v_2$  be a supersolution to the HJB equation. Then*

$$v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.$$

*Proof:* admitted.

## Corollary 23

*The value function  $V$  is the unique viscosity solution.*

*Proof.* By the comparison principle, any viscosity solution  $v$  is such that  $v \leq V$  and  $v \geq V$ .