ENT 305 A: Programming exercises

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Exercice 1 (Minimizing an unbounded function)**.** Consider the problem

$$\inf_{(x,y)\in\mathbb{R}^2} f(x), \tag{P}$$

where

$$f\colon (x,y) \in \mathbb{R}^2 \mapsto \frac{x^3}{3} + \frac{x^2}{2} + 2xy + \frac{y^2}{2} - y + 9.$$

Does problem (P) has a global solution? Calculate all stationary points of f. With the help of AMPL, try to minimize f, taking initial points more or less close to the stationary points.

Expected results.

Initialization of (x,y)	Result
$(0,0)$	unbounded (or badly scaled)
$(1,-1)$	$(1,-1)$
$(1.001,-1.001)$	$(2,-3)$
$(2,-3)$	$(2,-3)$
$(2.001,-3.001)$	$(2,-3)$

Exercice 2 (Projection on the simplex)**.** Let $(x_0,y_0)\in\mathbb{R}^2$. Consider the problem:

$$\inf_{(x,y)\in\mathbb{R}^2} \frac{1}{2}\big((x-x_0)^2 + (y-y_0)^2\big), \quad \text{sous la contrainte :} \begin{cases} x+y \leq 1 \\ x \geq 0 \\ y \geq 0. \end{cases}$$

Solve the problem graphically. In particular, calculate the solution for the following values of (x_0,y_0):

$$(x_0,y_0) = (1,1), \quad (x_0,y_0) = (0,2), \quad (x_0,y_0) = (-1,-1).$$

For each case, check that the KKT conditions are satisfied. Solve the problem with AMPL for these different values of (x_0,y_0).

Expected results.

(x_0,y_0)	Result	Lagrange multiplier
$(1,1)$	$(0.5,0.5)$	$(0.5,0,0)$
$(0,2)$	$(0,1)$	$(1,1,0)$
$(-1,-1)$	$(0,0)$	$(0,1,1)$

Exercice 3 (Polynomial interpolation)**.** We consider a set of N measurements $(x_i,y_i)_{i=1,\ldots,N}$, where $x_i\in\mathbb{R}$ and $y_i\in\mathbb{R}$, for all $i=1,\ldots,N$. We aim at finding a heuristic relation between x_i and y_i, in the form of a second-order polynomial function:

$$y_i \approx f(x_i;a,b,c), \quad \text{where: } f(x;a,b,c) = ax^2+bx+c.$$

For this purpose, we consider the following least-square problem:

$$\inf_{(a,b,c)\in\mathbb{R}^3} \sum_{i=1}^{N} (f(x_i;a,b,c)-y_i)^2.$$

Write an AMPL program for solving the problem with arbitrary values of N, x, and y. Solve the problem for the following values:

$$N=21, \quad x_i=(i-1)/20, \quad y_i=\exp(x_i).$$

Optional: write a program computing a polynomial approximation of any order.

Expected results: $a=0,84$, $b=0,85$, $c=1,01$.

Exercice 4 (Hanging chain)**.** We consider a necklace, made of N pearls of identical mass, connected by a chain of negligible mass. The distance between two consecutive pearls is taken equal to 1. The chain is hanging, suspended by its extremities. The resulting configuration is such that the total gravity energy is minimized.

The problem can be mathematically formulated as follows:

$$\inf_{\substack{x\in\mathbb{R}^N\\ y\in\mathbb{R}^N}} \sum_{i=1}^{N} y_i, \quad \text{subject to :} \begin{cases} \|(x_{i+1}, y_{i+1}) - (x_i, y_i)\|^2 \leq 1, \quad \forall i = 1, ..., N-1 \\ (x_1, y_1) = (x_I, y_I) \\ (x_N, y_N) = (x_F, y_F), \end{cases}$$

where (x_I, y_I) and (x_F, y_F) are given parameters.

1. Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ be a feasible point satisfying the KKT conditions. Is it a global solution to the problem ?

2. Write a program with AMPL that allows to solve the problem for an arbitrary number of pearls N and arbitrary points (x_I, y_I) and (x_F, y_F), to be specified in a data file.

Expected results, with $(x_I, y_I) = (0, 0)$, $(x_F, y_F) = (6, 0)$, $N = 20$.

i	1	2	3	4	5	6	7	8	9	10
$x(i)$	0	0, 11	0, 24	0, 38	0, 55	0, 75	1, 00	1, 32	1, 78	2, 5
$y(i)$	0	−0, 99	−1, 98	−2, 97	−3, 96	−4, 94	−5, 90	−6, 85	−7, 74	−8, 44

i	11	12	13	14	15	16	17	18	19	20
$x(i)$	3, 5	4, 21	4, 67	4, 99	5, 24	5, 44	5, 61	5, 75	5, 88	6
$y(i)$	−8, 44	−7, 74	−6, 85	−5, 90	−4, 94	−3, 96	−2, 97	−1, 98	−0, 99	0

Exercice 5 (Economic dispatch)**.** A company must satisfy the energetic demand along the day, divided in $T = 24$ time slots. The demand at time t is denoted L_t (with $t \in \{1, ..., T\}$). The company has n production units. The production of the unit i during the time slot t is denoted $P_{i,t}$ (with $i \in \{1, ..., n\}$).

The economic problem is modelled a follows:

- The production cost of unit i, at any time slot t, is given by
$$C_i(P_{i,t}) = a_i P_{i,t}^2 + b_i P_{i,t} + c_i.$$
- The production of the unit i, on the time slot t, is bounded from below and from above as follows:
$$P_i^{\min} \leq P_{i,t} \leq P_i^{\max}.$$
- The variation of production of unit i, from the times slot $t-1$ to the time slot t, is also bounded from below and from above:
$$R_i^{\min} \leq P_{i,t} - P_{i,t-1} \leq R_i^{\max}.$$
- The demand must be satisfied at all time slots:
$$\sum_{i=1}^{n} P_{i,t} \geq L_t.$$

The values of the parameters a_i, b_i, c_i, $P_i^{\min}$, $P_i^{\max}$, $R_i^{\min}$, $R_i^{\max}$, and L_t are given below.

Unit i	a_i	b_i	c_i	$P_i^{\min}$	$P_i^{\max}$	$R_i^{\min}$	$R_i^{\max}$
1	0.12	14.8	89	28	200	−40	40
2	0.17	16.57	83	20	290	−30	30
3	0.15	15.55	100	30	190	−30	30
4	0.19	16.21	70	20	260	−50	50

Time slot t	1	2	3	4	5	6	7	8	9	10	11	12
Demand L_t	510	530	516	510	515	544	646	686	741	734	748	760

Time slot t	13	14	15	16	17	18	19	20	21	22	23	24
Demand D_t	754	700	686	720	714	761	727	714	618	584	578	544

1. List the parameters and optimization variables, indicate their dimension.
2. Solve the problem with AMPL. Is the result a global solution to the problem?
3. For each time slot, compute (with the help of AMPL) the augmentation of cost generated by a (small) augmentation of demand at time t.

Expected resuts (production, first five time steps).

Time \ Unit	1	2	3	4
1	166, 191	112, 105	130, 452	101, 252
2	172.565	116.605	135.552	105.278
3	168.103	113.455	131.982	102.46
4	166.191	112.105	130.452	101.252
5	167.784	113.23	131.727	102.258
⋮	⋮	⋮	⋮	⋮