

Optimal Control of Ordinary Differential Equations

SOD 311

Laurent Pfeiffer

Inria and CentraleSupélec

Ensta-Paris
Paris-Saclay University



Lecture on: HJB equation and viscosity solutions

- *Goal*: finding global solutions to optimal control problems (in feedback form), by solving a non-linear PDE.
- *Issues*: characterization of the value function with the Hamilton-Jacobi-Bellman equation.

Bibliography

The following references are related:



M. Bardi and I. Capuzzo-Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, 1997.



F. Bonnans et P. Rouchon, Commande et optimisation de systèmes dynamiques, Editions de l'Ecole Polytechnique, 2005. (Partie 3, Chapitres 3 et 4).

1 Introduction

2 Dynamic programming principle

3 A first characterization of the value function

4 HJB equation: the classical sense

5 HJB equation: viscosity solutions

Introduction

- Our results so far were based on optimality conditions (Pontryagin's principle).
- Now: a different approach, based on **dynamic programming**. In some sense, more specific to optimal control.
- The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to “split” some problems into a family of simpler problems.
- The central tool: the **value function** V .
 - Defined as the value of the optimization problem, expressed as a function of the initial state.
 - Characterized as the unique **viscosity** solution of a non-linear partial differential equation (PDE) called **HJB equation**.

Introduction

- *Interest:* a **globally optimal** solution to the problem can be derived from V .
- *Limitation:* **curse of dimensionality**.
- *Warning:* focus on a specific class of problems.
All concepts can be extended, in particular to a stochastic framework (finance), and to other nonlinear PDEs.

Problem formulation

Data of the problem and assumptions:

- A parameter $\lambda > 0$.
- A non-empty and compact subset U of \mathbb{R}^m .
- A bounded and L_f -Lipschitz continuous mapping $f: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e.

$$\|f\|_\infty := \sup_{(u,y) \in U \times \mathbb{R}^n} \|f(u, y)\| < \infty,$$

$$\|f(u_2, y_2) - f(u_1, y_1)\| \leq L_f \|(u_2, y_2) - (u_1, y_1)\|,$$

for all (u_1, y_1) and $(u_2, y_2) \in U \times \mathbb{R}^n$.

- A bounded and L_ℓ -Lipschitz continuous mapping $\ell: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}$.

Problem formulation

- *Notation:* for any $\tau \in [0, \infty]$, \mathcal{U}_τ is the set of measurable functions from $(0, \tau)$ to U .
- State equation: for $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$, there is a unique solution $y[u, x]$ to the ODE

$$\dot{y}(t) = f(u(t), y(t)), \quad y(0) = x,$$

by the Picard-Lindelöf theorem (Cauchy-Lipschitz).

- Cost function W , for $u \in \mathcal{U}_\infty$ and $x \in \mathbb{R}^n$:

$$W(u, x) = \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) dt.$$

- Optimal control problem and value function V :

$$V(x) = \inf_{u \in \mathcal{U}_\infty} W(u, x). \quad (P(x))$$

Grönwall's lemma

Lemma 1 (Grönwall's lemma)

Let $\alpha > 0$ and let $\beta > 0$. Let $\theta: [0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that

$$\theta(t) \leq \alpha + \beta \int_0^t \theta(s) ds, \quad \forall t \in [0, \infty).$$

Then, $\theta(t) \leq \alpha e^{\beta t}$, for all $t \in [0, \infty)$.

Corollary 2

Let $u \in \mathcal{U}_\infty$. For all x and \tilde{x} , for all $t \geq 0$, it holds:

$$\|y[u, x](t) - y[u, \tilde{x}](t)\| \leq e^{L_f t} \|x - \tilde{x}\|.$$

Proof. Grönwall with $\theta = \|y[u, x] - y[u, \tilde{x}]\|$, $\alpha = \|x - \tilde{x}\|$, $\beta = L_f$.

1 Introduction

2 Dynamic programming principle

3 A first characterization of the value function

4 HJB equation: the classical sense

5 HJB equation: viscosity solutions

Dynamic programming principle

Theorem 3 (Dynamic programming (DP) principle)

Let $\tau > 0$. Then for all $x \in \mathbb{R}^n$, abbreviating $y = y[u, x]$,

$$V(x) = \inf_{u \in \mathcal{U}_\tau} \left(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) \right). \quad (DPP)$$

Interpretation:

- $V(x)$ is the value function of an optimal control problem on the interval $(0, \tau)$.
- The original integral has been truncated:

$$\int_\tau^\infty e^{-\lambda t} \ell(u(t), y(t)) dt \quad \rightsquigarrow \quad e^{-\lambda \tau} V(y(\tau)).$$

The term $e^{-\lambda \tau} V(y(\tau))$ is the “optimal cost from τ to ∞ ”.

Flow property

Lemma 4 (Flow property)

Let $x \in \mathbb{R}^n$ and let $u \in \mathcal{U}_\infty$. Define:

- $u_1 = u|_{(0,\tau)} \in \mathcal{U}_\tau$
- $u_2 = u|_{(\tau,\infty)} \in L^\infty(\tau, \infty; U)$
- $\tilde{u}_2 \in \mathcal{U}_\infty, \tilde{u}_2(t) = u_2(t + \tau).$

It holds:

$$y[u, x](t) = y[\tilde{u}_2, y[u_1, x](\tau)](t - \tau),$$

for any $t \geq \tau$.

Remark. After time τ , one can forget u_1 and only remember $y[x, u_1](\tau)$.

Proof

Proof of the DP-principle. Let us denote

$$\tilde{V}(x) = \inf_{u \in \mathcal{U}_\tau} \left(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) \right).$$

Step 1: $V \geq \tilde{V}$. Let u , u_1 , u_2 , and \tilde{u}_2 be as in Lemma 4.

$$\begin{aligned} W(u, x) &= \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &\quad + e^{-\lambda \tau} \int_\tau^\infty e^{-\lambda(t-\tau)} \ell(u(t), y[u, x](t)) dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) dt \\ &\quad + e^{-\lambda \tau} \int_0^\infty e^{-\lambda s} \ell(u(s+\tau), y[u, x](s+\tau)) ds. \end{aligned}$$

Proof

We further have, for the last integral:

$$\begin{aligned}
 & \int_0^\infty e^{-\lambda s} \ell(u(s+\tau), y[u, x](s+\tau)) \, ds \\
 &= \int_0^\infty e^{-\lambda s} \ell(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1, x](\tau)](s)) \, ds \\
 &= W(\tilde{u}_2, y[u_1, x](\tau)) \geq V(y[u_1, x](\tau)).
 \end{aligned}$$

Injecting in the above equality:

$$\begin{aligned}
 W(u, x) &\geq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \\
 &\geq \tilde{V}(x).
 \end{aligned}$$

Minimizing with respect to u yields $V \geq \tilde{V}$.

Proof

Step 2: $\tilde{V} \leq V$. Let $\varepsilon > 0$. Let $u_1 \in \mathcal{U}_\tau$ be such that

$$\int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1](t)) dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \leq \tilde{V}(x) + \varepsilon/2.$$

Let $\tilde{u}_2 \in \mathcal{U}_\infty$ be such that

$$W(\tilde{u}_2, y[u_1, x](\tau)) \leq V(y[u_1, x](\tau)) + \varepsilon/2.$$

Let u be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau), \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

Proof

The same calculation as above yields:

$$\begin{aligned} W(u, x) &= \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) dt \\ &\quad + e^{-\lambda \tau} \underbrace{\int_0^\infty e^{-\lambda t} \ell(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1, x](\tau)](t)) dt}_{=W(\tilde{u}_2, y[u_1, x](\tau))} \\ &\leq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) dt \\ &\quad + e^{-\lambda \tau} (V(y[u_1, x](\tau)) + \varepsilon/2) \\ &\leq \tilde{V}(x) + \varepsilon. \end{aligned}$$

It follows that

$$V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0.$$

Decoupling

Corollary 5

- Let $u \in \mathcal{U}_\infty$ be a solution to $P(x)$. Let $\tau > 0$. Let u_1 and \tilde{u}_2 be defined as in Lemma 4. Then,
 - u_1 is optimal in the DP principle
 - \tilde{u}_2 is optimal for $P(y[u_1, x](\tau))$.
- Conversely: let u_1 be a minimizer of (DPP). Let \tilde{u}_2 be a solution to $P(y[u_1, x])(\tau)$. Let $u \in \mathcal{U}_\infty$ be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau) \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

Then u is a solution to $P(x)$.

What can we do with the value function? If V is known, then the DP-principle allows to **decouple** the problem in time.

- 1 Introduction
- 2 Dynamic programming principle
- 3 A first characterization of the value function**
- 4 HJB equation: the classical sense
- 5 HJB equation: viscosity solutions

Regularity of V

Lemma 6

The value function V is bounded. It is also uniformly continuous, that is, for all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all x and $\tilde{x} \in \mathbb{R}^n$,

$$\|\tilde{x} - x\| \leq \alpha \implies |V(\tilde{x}) - V(x)| \leq \varepsilon.$$

Proof. Step 1: proof of boundedness. Let $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$. We have

$$|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\|_\infty dt \leq \frac{1}{\lambda} \|\ell\|_\infty.$$

Thus $|V(x)| \leq \frac{1}{\lambda} \|\ell\|_\infty$.

Regularity of V

Step 2: proof of uniform continuity. Let $\varepsilon > 0$. Let $\alpha > 0$. Let x and \tilde{x} be such that $\|\tilde{x} - x\| \leq \alpha$, we will specify α later. We have:

$$\begin{aligned} |V(\tilde{x}) - V(x)| &= \left| \inf_{u \in \mathcal{U}_\infty} W(\tilde{x}, u) - \inf_{u \in \mathcal{U}_\infty} W(x, u) \right| \\ &\leq \sup_{u \in \mathcal{U}_\infty} |W(\tilde{x}, u) - W(x, u)| \leq \Delta_1 + \Delta_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \sup_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt \\ \Delta_2 &= \sup_{u \in \mathcal{U}_\infty} \int_\tau^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt, \end{aligned}$$

where $\tilde{y} = y[\tilde{x}, u]$ and $y = y[x, u]$ and where $\tau > 0$ is **arbitrary**.

Regularity of V

- Bound of Δ_1 . By Corollary 2,

$$\|\tilde{y}(t) - y(t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha, \quad \forall t \in [0, \tau].$$

Therefore, $\Delta_1 \leq \tau L_\ell e^{L_f \tau} \alpha$.

- Bound of Δ_2 . Since ℓ is bounded,

$$\Delta_2 \leq 2\|\ell\|_\infty \int_\tau^\infty e^{-\lambda t} dt = \frac{2\|\ell\|_\infty}{\lambda} e^{-\lambda \tau}.$$

Conclusion: take $\tau > 0$ sufficiently large, so that $\Delta_2 \leq \frac{\varepsilon}{2}$.

Take then α sufficiently small, so that $\Delta_1 \leq \frac{\varepsilon}{2}$.

The construction of α is independent of x and \tilde{x} .

We have $|V(x) - V(\tilde{x})| \leq \varepsilon$.

(More) regularity of V

Lemma 7

We have

- if $\lambda < L_f$, then V is (λ/L_f) -Hölder continuous
- if $\lambda = L_f$, then V is α -Hölder continuous for all $\alpha \in (0, 1)$
- if $\lambda > L_f$, then V is Lipschitz continuous.

(More) regularity of V

Proof of the last case. We have

$$\begin{aligned} |V(\tilde{x}) - V(x)| &\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| dt \\ &\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} L_\ell \|\tilde{y}(t) - y(t)\| dt \\ &\leq \int_0^\infty e^{-\lambda t} L_\ell e^{L_f t} \|\tilde{x} - x\| dt \\ &\leq \frac{L_\ell}{\lambda - L_f} \|\tilde{x} - x\|. \end{aligned}$$

DP-mapping

Notation: $BUC(\mathbb{R}^n)$ is the set of **bounded and uniformly continuous** functions from \mathbb{R}^n to \mathbb{R} .

Lemma 8

The space $BUC(\mathbb{R}^n)$, equipped with the uniform norm (denoted $\|\cdot\|_\infty$) is a Banach space.

Fix $\tau > 0$. Consider the “**DP-mapping**” (also called Bellman operator):

$$\mathcal{T}: v \in BUC(\mathbb{R}^n) \mapsto \mathcal{T}v \in BUC(\mathbb{R}^n),$$

defined by

$$\mathcal{T}v(x) = \inf_{u \in \mathcal{U}_\tau} \left(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} v(y(\tau)) \right),$$

where $y = y[u, x]$.

DP-mapping

Let $v \in BUC(\mathbb{R}^n)$. Let us verify that $\mathcal{T}v \in BUC(\mathbb{R}^n)$. Clearly $\mathcal{T}v$ is bounded.

Let $\varepsilon > 0$. Let $\alpha_0 > 0$ be such that

$$\|\tilde{x} - x\| \leq \alpha_0 \implies |v(\tilde{x}) - v(x)| \leq \varepsilon/2.$$

Let $\alpha > 0$. Let x and $\tilde{x} \in \mathbb{R}^n$ be such that $\|\tilde{x} - x\| \leq \alpha$. The value of α will be fixed later.

For all $u \in U_\tau$, for all $t \in [0, \tau]$, we have

$$\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha.$$

We have $|\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x)| \leq \Delta_1 + \Delta_2$, with...

DP-mapping

$$\Delta_1 = \sup_{u \in \mathcal{U}_\tau} \left| \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt - \int_0^\tau e^{-\lambda t} \ell(u(t), \tilde{y}(t)) dt \right|,$$
$$\Delta_2 = \sup_{u \in \mathcal{U}_\tau} \left| e^{-\lambda \tau} v(\tilde{y}(\tau)) - e^{-\lambda \tau} v(y(\tau)) \right|.$$

We fix now

$$\alpha = e^{-L_f \tau} \min \left(\alpha_0, \frac{\varepsilon}{2\tau} \right).$$

We have

$$\Delta_1 \leq \tau L_\ell e^{\tau L_f} \alpha \leq \varepsilon/2 \quad \text{and} \quad \Delta_2 \leq \varepsilon/2,$$

since $\|\tilde{y}(\tau) - y(\tau)\| \leq e^{L_f \tau} \alpha \leq \alpha_0$. Therefore,

$$|\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x)| \leq \varepsilon.$$

DP-mapping

Lemma 9

The operator \mathcal{T} is Lipschitz continuous with modulus $e^{-\lambda\tau}$.

Proof. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned} |\mathcal{T}\tilde{v}(x) - \mathcal{T}v(x)| &\leq \sup_{u \in \mathcal{U}_\tau} |e^{-\lambda\tau} \tilde{v}(y[x, u](\tau)) - e^{-\lambda\tau} v(y[x, u](\tau))| \\ &\leq e^{-\lambda\tau} \|\tilde{v} - v\|_\infty. \end{aligned}$$

We conclude that

$$\|\mathcal{T}\tilde{v} - \mathcal{T}v\|_\infty \leq e^{-\lambda\tau} \|\tilde{v} - v\|_\infty.$$

A characterization of V

Lemma 10

The value function V is the unique solution of the fixed-point equation:

$$\mathcal{T}v = v, \quad v \in BUC(\mathbb{R}^n).$$

Proof.

- Existence: direct consequence of the DP principle ($V = \mathcal{T}V$).
- Uniqueness: for any v such that $v = \mathcal{T}v$, we have

$$\|v - V\|_{\infty} = \|\mathcal{T}v - \mathcal{T}V\|_{\infty} \leq e^{-\lambda\tau} \|v - V\|_{\infty}.$$

Thus $v = V$.

Remark: the dynamic programming principle entirely characterises the value function!

Min-plus linearity

Notation. Given v_1 and $v_2 \in BUC(\mathbb{R}^n)$, we write $v_1 \leq v_2$ if $v_1(x) \leq v_2(x)$ for all $x \in \mathbb{R}^n$. We define $\min(v_1, v_2) \in BUC(\mathbb{R}^n)$ by

$$\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.$$

Given $\alpha \in \mathbb{R}$, we define $v_1 + \alpha$ by $(v_1 + \alpha)(x) = v_1(x) + \alpha$.

Lemma 11

Let v_1 and $v_2 \in BUC(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$. The map \mathcal{T} is monotone:

$$v_1 \leq v_2 \implies \mathcal{T}v_1 \leq \mathcal{T}v_2$$

and min-plus linear:

$$\min(\mathcal{T}v_1, \mathcal{T}v_2) = \mathcal{T} \min(v_1, v_2), \quad \mathcal{T}(v + \alpha) = (\mathcal{T}v) + e^{-\lambda\tau}\alpha.$$

Proof: exercise.

- 1 Introduction
- 2 Dynamic programming principle
- 3 A first characterization of the value function
- 4 HJB equation: the classical sense**
- 5 HJB equation: viscosity solutions

Hamiltonian

We define the **pre-Hamiltonian** H and the **Hamiltonian** \mathcal{H} by

$$\begin{aligned} H(u, x, p) &= \ell(u, x) + \langle p, f(u, x) \rangle, \\ \mathcal{H}(x, p) &= \min_{u \in U} H(u, x, p). \end{aligned}$$

Lemma 12

The mapping \mathcal{H} is continuous, concave with respect to p , and Lipschitz continuous with respect to p with modulus $\|f\|_\infty$.

Proof. The pre-Hamiltonian H is affine in p , thus concave in p . As an infimum of concave functions, \mathcal{H} is concave. We have:

$$\begin{aligned} |\mathcal{H}(x, \tilde{p}) - \mathcal{H}(x, p)| &\leq \sup_{u \in U} |H(u, x, \tilde{p}) - H(u, x, p)| \\ &\leq \sup_{u \in U} |\langle \tilde{p} - p, f(u, x) \rangle| \leq \|\tilde{p} - p\| \cdot \|f\|_\infty. \end{aligned}$$

Informal derivation

Notation: $C^1(\mathbb{R}^n)$, the set of continuously differentiable functions from \mathbb{R}^n to \mathbb{R} .

Lemma 13

Let $\Phi \in C^1(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$, let $u \in \mathcal{U}_\infty$, let $y = y[u, x]$. Consider the mapping:

$$\varphi: \tau \in [0, \infty) \mapsto \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).$$

Then $\varphi(0) = 0$ and $\varphi \in W^{1,\infty}(0, \infty)$ with

$$\dot{\varphi}(\tau) = e^{-\lambda \tau} (H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau))). \quad (*)$$

In particular: $\dot{\varphi}(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$ (if u is continuous at 0).

Informal derivation

Proof. To simplify, we only consider the case where u is continuous, so that y is C^1 and φ is $C^1(\mathbb{R}^n)$. We have then:

$$\begin{aligned}\dot{\varphi}(\tau) &= e^{-\lambda\tau} \ell(u(\tau), y(\tau)) + e^{-\lambda\tau} \langle \nabla \Phi(y(\tau)), \dot{y}(\tau) \rangle \\ &\quad - \lambda e^{-\lambda\tau} \Phi(y(\tau)) \\ &= e^{-\lambda\tau} [\ell(u(\tau), y(\tau)) + \langle \nabla \Phi(y(\tau)), f(u(\tau), y(\tau)) \rangle] \\ &\quad - \lambda e^{-\lambda\tau} \Phi(y(\tau)) \\ &= e^{-\lambda\tau} [H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau))].\end{aligned}$$

HJB in the classical sense

Theorem 14

Let $x \in \mathbb{R}^n$. Assume that

- V is continuously differentiable in a neighborhood of x
- $P(x)$ has a solution \bar{u} which is continuous at time 0.

Then,

$$\lambda V(x) - \mathcal{H}(x, \nabla V(x)) = 0,$$

$$\bar{u}(0) \in \operatorname{argmin}_{u_0 \in U} H(u_0, x, \nabla V(x)).$$

HJB in the classical sense

Proof. Step 1. Let $u_0 \in U$, let u be the constant control equal to u_0 , let $y = y[u, x]$. By the **dynamic programming** principle, we have:

$$0 \leq \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) dt + e^{-\lambda \tau} V(y(\tau)) - V(x),$$

for all τ . Since $\varphi(0) = 0$, we deduce from (*) that:

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).$$

Therefore,

$$0 \leq H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.$$

HJB in the classical sense

Step 2. Let us apply the **dynamic programming principle** again. Redefining φ and setting $\bar{y} = y[\bar{u}, x]$, we obtain:

$$0 = \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x),$$

for all $\tau \geq 0$. It follows that

$$0 = H(\bar{u}(0), x, \nabla V(x)) - \lambda V(x).$$

Step 3. It follows that

$$H(\bar{u}(0), x, \nabla V(x)) = \lambda V(x) \leq H(u_0, x, \nabla V(x)), \quad \forall u_0 \in U.$$

Therefore, $H(\bar{u}(0), x, \nabla V(x)) = \mathcal{H}(x, \nabla V(x))$.

HJB in the classical sense

Corollary 15

Let $t \geq 0$, assume that \bar{u} is continuous in a neighborhood of t and that V is C^1 in a neighborhood of $\bar{y}(t)$, where $\bar{y} := y[\bar{u}, x]$. Then,

$$\bar{u}(t) \in \operatorname{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))).$$

HJB in the classical sense

Remarks.

Let us define the Q-function by $Q(u, y) := H(u, y, \nabla V(y))$, assuming that $V \in C^1(\mathbb{R}^n)$.

- If the minimizer is unique in the following relation, we have a **feedback law**:

$$\bar{u}(t) = \underset{u}{\operatorname{argmin}} Q(\cdot, \bar{y}(t)).$$

- In some cases, one can show that $\nabla V(\bar{y}(t)) = p(t)$, where p is defined by some adjoint equation \rightarrow **Pontryagin's principle**.
- In **Reinforcement Learning**, the approximation of Q is a central objective.

We will call the equation

$$\lambda v(x) - \mathcal{H}(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n \quad (\text{HJB})$$

Hamilton-Jacobi-Bellman equation, with unknown $v: \mathbb{R}^n \rightarrow \mathbb{R}$.

Remarks.

- In general V is not differentiable \rightarrow in **which sense** is the HJB equation to be understood?
- In Theorem 14, we have shown that

$$\bar{u}(t) \in \operatorname{argmin} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),$$

(under restrictive assumptions). We will see next that this necessary condition is also **sufficient**.

Theorem 16 (Verification)

Let us assume the assumptions of Theorem 14 hold for all $x \in \mathbb{R}^n$, so that the HJB equation is satisfied in the classical sense. Let $x \in \mathbb{R}^n$. Assume that there exists a control \bar{u} such that

$$\bar{u}(t) \in \operatorname{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))),$$

where $\bar{y} = y[\bar{u}, x]$. Then \bar{u} is globally optimal.

Proof. Consider the function:

$$\varphi(\tau) = \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x).$$

We have $\varphi(0) = 0$. Using $(*)$ and Theorem 14, we obtain:

$$\begin{aligned}\dot{\varphi}(\tau) &= e^{-\lambda \tau} [H(\bar{u}(\tau), \bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau))] \\ &= e^{-\lambda \tau} [\mathcal{H}(\bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau))] \\ &= 0.\end{aligned}$$

Thus φ is constant, equal to 0. Its limit is given by:

$$0 = \int_0^\infty e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) dt - V(x) = W(x, \bar{u}) - V(x),$$

proving the optimality of \bar{u} .

- 1 Introduction
- 2 Dynamic programming principle
- 3 A first characterization of the value function
- 4 HJB equation: the classical sense
- 5 HJB equation: viscosity solutions**

Abstract PDE

We consider an abstract PDE of the form:

$$\mathcal{F}(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n,$$

where $\mathcal{F}: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous.

It contains the HJB equation with

$$\mathcal{F}(x, v, p) = \lambda v - \mathcal{H}(x, p).$$

Goal of the section: showing that V is a viscosity solution to the HJB equation.

Sub- and super-differentials

Definition 17

Let $v: \mathbb{R}^n \rightarrow \mathbb{R}$. The following sets are called **sub- and superdifferential**, respectively:

$$D^- v(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

$$D^+ v(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \leq 0 \right\}.$$

Exercise. Let $v(x) = |x|$. Show that $D^- v(0) = [-1, 1]$.

Sub- and super-differentials

We have the following characterization.

Lemma 18

Let $v: \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. Let $p \in \mathbb{R}^n$.

- $p \in D^-v(x) \iff$ there exists $\Phi \in C^1(\mathbb{R}^n)$ such that $\nabla\Phi(x) = p$ and $v - \Phi$ has a local minimum in x .
- $p \in D^+v(x) \iff$ there exists $\Phi \in C^1(\mathbb{R}^n)$ such that $\nabla\Phi(x) = p$ and $v - \Phi$ has a local maximum in x .

Proof. The implication \implies is admitted. The implication \impliedby is left as an exercise.

Sub- and super-differentials

Remark. In the above lemma, one can chose $\Phi(x) = v(x)$ without loss of generality. Thus, we have:

- $(v - \Phi)$ has a local minimum in $x \iff v - \Phi$ is nonnegative in a neighborhood of $x \iff v$ is locally bounded from below by Φ
- $(v - \Phi)$ has a local maximum in $x \iff v - \Phi$ is nonpositive in a neighborhood of $x \iff v$ is locally bounded from above by Φ

Remark. If v is Fréchet differentiable at x , then the sub- and superdifferential are equal to $\{\nabla v(x)\}$.

Viscosity solutions

Definition 19

Let $v: \mathbb{R}^n \rightarrow \mathbb{R}$. We call v a **viscosity subsolution** if

$$\mathcal{F}(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall p \in D^+ v(x)$$

or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local maximum in x ,

$$\mathcal{F}(x, v(x), \nabla \Phi(x)) \leq 0.$$

Viscosity solutions

Definition 20

Let $v: \mathbb{R}^n \rightarrow \mathbb{R}$. We call v a **viscosity supersolution** if

$$\mathcal{F}(x, v(x), p) \geq 0, \quad \forall p \in D^- v(x)$$

or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local minimum in x ,

$$\mathcal{F}(x, v(x), \nabla \Phi(x)) \geq 0.$$

We call v a **viscosity solution** if it is a sub- and a supersolution.

Viscosity solutions

Theorem 21

The value function V is a viscosity solution of the HJB equation.

Step 1: V is a subsolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local maximizer in x and $V(x) = \Phi(x)$.

We have to prove that

$$\lambda v(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$

Let $u_0 \in U$, let u be the constant control equal to u_0 and let $y = y[u, x]$. By the DPP, we have:

$$V(x) \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) dt + e^{-\lambda \tau} V(y(\tau)).$$

If τ is sufficiently small, we have $V(y(\tau)) \leq \Phi(y(\tau))$.

Viscosity solutions

This implies that for τ sufficiently small,

$$0 \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x) =: \varphi(\tau).$$

Since $\varphi(0) = 0$, we deduce with (*) that

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla \Phi(x)) - \lambda V(x).$$

Minimizing with respect to $u_0 \in U$, we obtain:

$$0 \leq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x),$$

as was to be proved.

Viscosity solutions

Step 2: V is supersolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local minimizer in x and such that $V(x) = \Phi(x)$. We have to prove that

$$\lambda V(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$

It follows from the dynamic programming principle that for $\tau > 0$ small enough

$$\Phi(x) \geq \inf_{u \in \mathcal{U}_\tau} \underbrace{\int_0^\tau e^{-\lambda t} \ell(u(t), y[x, u](t)) dt + e^{-\lambda \tau} \Phi(y[x, u](\tau))}_{=: \varphi[u](\tau)}.$$

Viscosity solutions

Thus by Lemma 13,

$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \dot{\varphi}[u](t) dt \\ &= \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} (H(u(t), y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t))) dt \\ &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \underbrace{e^{-\lambda t} (\mathcal{H}(y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t)))}_{=: \psi[u](t)} dt. \end{aligned}$$

We have $\psi[u](0) = \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x)$, in particular, $\psi[u](0)$ does not depend on u .

Viscosity solutions

Let $\varepsilon > 0$. There exists (exercise!) $\tau > 0$ such that

$$|\psi[u](t) - \psi[u](0)| \leq \varepsilon, \quad \forall t \in [0, \tau], \quad \forall u \in \mathcal{U}_\infty.$$

The previous inequality yields

$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau (\psi[u](0) - \varepsilon) \, dt \\ &\geq \tau(\mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x) - \varepsilon). \end{aligned}$$

Dividing by τ and sending ε to 0, we obtain that

$$\lambda V(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$

Viscosity solutions

Theorem 22 (Comparison principle)

Let v_1 be a subsolution to the HJB equation. Let v_2 be a supersolution to the HJB equation. Then

$$v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.$$

Proof: admitted.

Corollary 23

The value function V is the unique viscosity solution.

Proof. By the comparison principle, any viscosity solution v is such that $v \leq V$ and $v \geq V$.