Optimal Control of Ordinary Differential Equations
SOD 311

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Lecture 3:
HJB equation and viscosity solutions

- *Goal:* finding global solutions to optimal control problems (in feedback form), by solving a non-linear PDE.
- *Issues:* characterization of the value function with the Hamilton-Jacobi-Bellman equation.

Bibliography

The following references are related to Chapter 3:

- M. Bardi and I. Capuzzo-Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, 1997.
- F. Bonnans et P. Rouchon, Commande et optimisation de systèmes dynamiques, Editions de l'Ecole Polytechnique, 2005. (Partie 3, Chapitres 3 et 4).

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Introduction

- Our results so far were based on optimality conditions (Pontryagin's principle).
- Now: a different approach, based on **dynamic programming**. In some sense, more specific to optimal control.
- The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to "split" some problems into a family of simpler problems.

- The central tool: the **value function** V.
  - Defined as the value of the optimization problem, expressed as a function of the initial state.
  - Characterized as the unique **viscosity** solution of a non-linear partial differential equation (PDE) called **HJB equation**.

Introduction

- *Interest:* a **globally optimal** solution to the problem can be derived from V.
- *Limitation:* **curse of dimensionality**.
- *Warning:* focus on a specific class of problems.
  All concepts can be extended, in particular to a stochastic framework (finance), and to other nonlinear PDEs.

Problem formulation

Data of the problem and assumptions:

- A parameter $\lambda > 0$.
- A non-empty and compact subset U of $\mathbb{R}^m$.
- A bounded and L_f-Lipschitz continuous mapping $f\colon (u,y) \in U \times \mathbb{R}^n \to \mathbb{R}^n$, i.e.

$$\|f\|_\infty := \sup_{(u,y)\in U\times\mathbb{R}^n} \|f(u,y)\| < \infty,$$

$$\|f(u_2,y_2) - f(u_1,y_1)\| \leq L_f \|(u_2,y_2) - (u_1,y_1)\|,$$

  for all (u_1,y_1) and $(u_2,y_2) \in U \times \mathbb{R}^n$.
- A bounded and L_ℓ-Lipschitz continuous mapping $\ell\colon (u,y) \in U \times \mathbb{R}^n \to \mathbb{R}$.

Problem formulation

- *Notation:* for any $\tau \in [0, \infty]$, $\mathcal{U}_\tau$ is the set of measurable functions from $(0, \tau)$ to U.
- State equation: for $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$, there is a unique solution $y[u, x]$ to the ODE

$$\dot{y}(t) = f(u(t), y(t)), \quad y(0) = x,$$

  by the Picard-Lindelöf theorem (Cauchy-Lipschitz).
- Cost function W, for $u \in \mathcal{U}_\infty$ and $x \in \mathbb{R}^n$:

$$W(u, x) = \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) \, \mathrm{d}t.$$

- Optimal control problem and value function V:

$$V(x) = \inf_{u \in \mathcal{U}_\infty} W(u, x). \tag{$P(x)$}$$

Grönwall's lemma

Lemma 1 (Grönwall's lemma)

Let $\alpha > 0$ and let $\beta > 0$. Let $\theta \colon [0,\infty) \to \mathbb{R}$ be a continuous function such that

$$\theta(t) \leq \alpha + \beta \int_0^t \theta(s)\, ds, \quad \forall t \in [0,\infty).$$

Then, $\theta(t) \leq \alpha e^{\beta t}$, for all $t \in [0,\infty)$.

Corollary 2

Let $u \in \mathcal{U}_\infty$. For all x and $\tilde{x}$, for all $t \geq 0$, it holds:

$$\|y[u,x](t) - y[u,\tilde{x}](t)\| \leq e^{L_f t}\|x - \tilde{x}\|.$$

Proof. Grönwall with $\theta = \|y[u,x] - y[\tilde{u},x]\|$, $\alpha = \|x - \tilde{x}\|$, $\beta = L_f$.

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Dynamic programming principle

Theorem 3 (Dynamic programming (DP) principle)

Let $\tau > 0$. Then for all $x \in \mathbb{R}^n$, abbreviating $y = y[u,x]$,

$$V(x) = \inf_{u \in \mathcal{U}_\tau} \left(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t))\, dt + e^{-\lambda \tau} V(y(\tau)) \right). \quad (DPP)$$

Interpretation:

- $V(x)$ is the value function of an optimal control problem on the interval $(0, \tau)$.
- The original integral has been truncated:

$$\int_\tau^\infty e^{-\lambda t} \ell(u(t), y(t))\, \mathrm{d}t \quad \rightsquigarrow \quad e^{-\lambda \tau} V(y(\tau)).$$

The term $e^{-\lambda \tau} V(y(\tau))$ is the “optimal cost from τ to ∞”.

Flow property

Lemma 4 (Flow property)

Let $x \in \mathbb{R}^n$ and let $u \in \mathcal{U}_\infty$. Define:

- $u_1 = u_{|(0,\tau)} \in \mathcal{U}_\tau$
- $u_2 = u_{|(\tau,\infty)} \in L^\infty(\tau,\infty;U)$
- $\tilde{u}_2 \in \mathcal{U}_\infty$, $\tilde{u}_2(t) = u_2(t+\tau)$.

It holds:

$$y[u,x](t) = y\big[\tilde{u}_2, y[u_1,x](\tau)\big](t-\tau),$$

for any $t \geq \tau$.

Remark. After time τ, one can forget u_1 and only remember $y[x,u_1](\tau)$.

Proof

Proof of the DP-principle. Let us denote

$$\tilde{V}(x) = \inf_{u \in \mathcal{U}_\tau} \Big(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t))\, dt + e^{-\lambda \tau} V(y(\tau)) \Big).$$

Step 1: $V \geq \tilde{V}$. Let u, u_1, u_2, and $\tilde{u}_2$ be as in Lemma 4.

$$\begin{aligned} W(u,x) &= \int_0^\infty e^{-\lambda t} \ell(u(t), y[u,x](t))\, dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u,x](t))\, dt \\ &\qquad + e^{-\lambda \tau} \int_\tau^\infty e^{-\lambda(t-\tau)} \ell(u(t), y[u,x](t))\, dt \\ &= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u,x](t))\, dt \\ &\qquad + e^{-\lambda \tau} \int_0^\infty e^{-\lambda s} \ell(u(s+\tau), y[u,x](s+\tau))\, ds. \end{aligned}$$

Proof

We further have, for the last integral:

$$\begin{aligned}\int_0^\infty e^{-\lambda s}\ell\big(u(s+\tau), y[u,x](s+\tau)\big)\,\mathrm{d}s \\ &= \int_0^\infty e^{-\lambda s}\ell\big(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1,x](\tau)](s)\big)\,\mathrm{d}s \\ &= W(\tilde{u}_2, y[u_1,x](\tau)) \geq V(y[u_1,x](\tau)).\end{aligned}$$

Injecting in the above equality:

$$\begin{aligned}W(u,x) &\geq \int_0^\tau e^{-\lambda t}\ell\big(u_1(t), y[u_1,x](t)\big)\,\mathrm{d}t + e^{-\lambda\tau}V(y[u_1,x](\tau)) \\ &\geq \tilde{V}(x).\end{aligned}$$

Minimizing with respect to u yields $V \geq \tilde{V}$.

Proof

Step 2: $\tilde{V} \leq V$. Let $\varepsilon > 0$. Let $u_1 \in \mathcal{U}_\tau$ be such that

$$\int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1](t))\, \mathrm{d}t + e^{-\lambda \tau} V(y[u_1, x](\tau)) \leq \tilde{V}(x) + \varepsilon/2.$$

Let $\tilde{u}_2 \in \mathcal{U}_\infty$ be such that

$$W(\tilde{u}_2, y[u_1, x](\tau)) \leq V(y[u_1, x](\tau)) + \varepsilon/2.$$

Let u be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau), \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

Proof

The same calculation as above yields:

$$\begin{aligned} W(u,x) = & \int_0^\tau e^{-\lambda t}\ell\big(u_1(t), y[u_1,x](t)\big)\,\mathrm{d}t \\ & \quad + e^{-\lambda\tau}\underbrace{\int_0^\infty e^{-\lambda t}\ell\big(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1,x](\tau)](t)\big)\,\mathrm{d}t}_{=W(\tilde{u}_2, y[u_1,x](\tau)))} \\ \leq & \int_0^\tau e^{-\lambda t}\ell\big(u_1(t), y[u_1,x](t)\big)\,\mathrm{d}t \\ & \quad + e^{-\lambda\tau}\big(V(y[u_1,x](\tau)) + \varepsilon/2\big) \\ \leq & \tilde{V}(x) + \varepsilon. \end{aligned}$$

It follows that

$$V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0.$$

Decoupling

Corollary 5

- *Let $u \in \mathcal{U}_\infty$ be a solution to $P(x)$. Let $\tau > 0$. Let u_1 and $\tilde{u}_2$ be defined as in Lemma 4. Then,*
  - u_1 *is* optimal in the DP principle
  - $\tilde{u}_2$ *is* optimal *for $P(y[u_1, x](\tau))$.*
- *Conversely: let u_1 be a minimizer of (DPP). Let $\tilde{u}_2$ be a solution to $P(y[u_1, x])(\tau)$. Let $u \in \mathcal{U}_\infty$ be defined by*

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau) \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

  Then u is a solution to $P(x)$.

What can we do with the value function? If V is known, then the DP-principle allows to **decouple** the problem in time.

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Regularity of V

Lemma 6

The value function V is bounded. It is also uniformly continuous, that is, for all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all x and $\tilde{x} \in \mathbb{R}^n$,

$$\|\tilde{x} - x\| \leq \alpha \Longrightarrow |V(\tilde{x}) - V(x)| \leq \varepsilon.$$

Proof. Step 1: proof of boundedness. Let $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$. We have

$$|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\|_\infty \, \mathrm{d}t \leq \frac{1}{\lambda} \|\ell\|_\infty.$$

Thus $|V(x)| \leq \frac{1}{\lambda} \|\ell\|_\infty$.

Regularity of V

Step 2: proof of uniform continuity. Let $\varepsilon > 0$. Let $\alpha > 0$. Let x and $\tilde{x}$ be such that $\|\tilde{x} - x\| \leq \alpha$, we will specify α later. We have:

$$
\begin{aligned}
|V(\tilde{x}) - V(x)| &= \Big| \inf_{u \in \mathcal{U}_\infty} W(\tilde{x}, u) - \inf_{u \in \mathcal{U}_\infty} W(x, u) \Big| \\
&\leq \sup_{u \in \mathcal{U}_\infty} \big| W(\tilde{x}, u) - W(x, u) \big| \leq \Delta_1 + \Delta_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1 &= \sup_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} \big| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \big| \, \mathrm{d}t \\
\Delta_2 &= \sup_{u \in \mathcal{U}_\infty} \int_\tau^\infty e^{-\lambda t} \big| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \big| \, \mathrm{d}t,
\end{aligned}
$$

where $\tilde{y} = y[\tilde{x}, u]$ and $y = y[x, u]$ and where $\tau > 0$ is **arbitrary**.

Regularity of V

- Bound of Δ_1. By Corollary 2,

$$\|\tilde{y}(t) - y(t)\| \leq e^{L_f t}\|\tilde{x} - x\| \leq e^{L_f \tau}\alpha, \quad \forall t \in [0, \tau].$$

  Therefore, $\Delta_1 \leq \tau L_\ell e^{L_f \tau}\alpha$.

- Bound of Δ_2. Since ℓ is bounded,

$$\Delta_2 \leq 2\|\ell\|_\infty \int_\tau^\infty e^{-\lambda t}\, \mathrm{d}t = \frac{2\|\ell\|_\infty}{\lambda} e^{-\lambda \tau}.$$

Conclusion: take $\tau > 0$ sufficiently large, so that $\Delta_2 \leq \frac{\varepsilon}{2}$.
Take then α sufficiently small, so that $\Delta_1 \leq \frac{\varepsilon}{2}$.
The construction of α is independent of x and $\tilde{x}$.
We have $|V(x) - V(\tilde{x})| \leq \varepsilon$.

(More) regularity of V

Lemma 7

We have

- *if* $\lambda < L_f$, *then* V *is* (λ/L_f)*-Hölder continuous*
- *if* $\lambda = L_f$, *then* V *is* α*-Hölder continuous for all* $\alpha \in (0,1)$
- *if* $\lambda > L_f$, *then* V *is Lipschitz continuous.*

(More) regularity of V

Proof of the last case. We have

$$
\begin{aligned}
|V(\tilde{x}) - V(x)| &\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| \, \mathrm{d}t \\
&\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} L_\ell \|\tilde{y}(t) - y(t)\| \, \mathrm{d}t \\
&\leq \int_0^\infty e^{-\lambda t} L_\ell e^{L_f t} \|\tilde{x} - x\| \, \mathrm{d}t \\
&\leq \frac{L_\ell}{\lambda - L_f} \|\tilde{x} - x\|.
\end{aligned}
$$

DP-mapping

Notation: $BUC(\mathbb{R}^n)$ is the set of **bounded and uniformly continuous** functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 8

The space $BUC(\mathbb{R}^n)$, equipped with the uniform norm (denoted $\|\cdot\|_\infty$) is a Banach *space.*

Fix $\tau > 0$. Consider the "**DP-mapping**" (also called Bellman operator):

$$\mathcal{T}\colon v \in BUC(\mathbb{R}^n) \mapsto \mathcal{T}v \in BUC(\mathbb{R}^n),$$

defined by

$$\mathcal{T}v(x) = \inf_{u \in \mathcal{U}_\tau} \left(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t + e^{-\lambda \tau} v(y(\tau)) \right),$$

where $y = y[u, x]$.

DP-mapping

Let $v \in BUC(\mathbb{R}^n)$. Let us verify that $\mathcal{T}v \in BUC(\mathbb{R}^n)$. Clearly $\mathcal{T}v$ is bounded.

Let $\varepsilon > 0$. Let $\alpha_0 > 0$ be such that

$$\|\tilde{x} - x\| \leq \alpha_0 \Longrightarrow |v(\tilde{x}) - v(x)| \leq \varepsilon/2.$$

Let $\alpha > 0$. Let x and $\tilde{x} \in \mathbb{R}^n$ be such that $\|\tilde{x} - x\| \leq \alpha$. The value of α will be fixed later.

For all $u \in U_\tau$, for all $t \in [0, \tau]$, we have

$$\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{L_f t}\|\tilde{x} - x\| \leq e^{L_f \tau}\alpha.$$

We have $|\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x)| \leq \Delta_1 + \Delta_2$, with...

DP-mapping

$$\Delta_1 = \sup_{u \in \mathcal{U}_\tau} \left| \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t - \int_0^\tau e^{-\lambda t} \ell(u(t), \tilde{y}(t)) \, \mathrm{d}t \right|,$$

$$\Delta_2 = \sup_{u \in \mathcal{U}_\tau} \left| e^{-\lambda\tau} v(\tilde{y}(\tau)) - e^{-\lambda\tau} v(y(\tau)) \right|.$$

We fix now

$$\alpha = e^{-L_f \tau} \min \left(\alpha_0, \frac{\varepsilon}{2\tau} \right).$$

We have

$$\Delta_1 \leq \tau L_\ell e^{\tau L_f} \alpha \leq \varepsilon/2 \quad \text{and} \quad \Delta_2 \leq \varepsilon/2,$$

since $\|\tilde{y}(\tau) - y(\tau)\| \leq e^{L_f \tau} \alpha \leq \alpha_0$. Therefore,

$$|\mathcal{T} v(\tilde{x}) - \mathcal{T} v(x)| \leq \varepsilon.$$

DP-mapping

Lemma 9

The operator $\mathcal{T}$ *is* Lipschitz continuous *with modulus* $e^{-\lambda\tau}$.

Proof. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned}|\mathcal{T}\tilde{v}(x) - \mathcal{T}v(x)| &\leq \sup_{u \in \mathcal{U}_\tau} \left| e^{-\lambda\tau}\tilde{v}(y[x,u](\tau)) - e^{-\lambda\tau}v(y[x,u](\tau))\right| \\ &\leq e^{-\lambda\tau}\|\tilde{v} - v\|_\infty.\end{aligned}$$

We conclude that

$$\|\mathcal{T}\tilde{v} - \mathcal{T}v\|_\infty \leq e^{-\lambda\tau}\|\tilde{v} - v\|_\infty.$$

A characterization of V

Lemma 10

The value function V is the unique solution *of the fixed-point equation:*

$$\mathcal{T}v = v, \quad v \in BUC(\mathbb{R}^n).$$

Proof.

- Existence: direct consequence of the DP principle ($V = \mathcal{T}V$).
- Uniqueness: for any v such that $v = \mathcal{T}v$, we have

$$\|v - V\|_\infty = \|\mathcal{T}v - \mathcal{T}V\|_\infty \leq e^{-\lambda\tau}\|v - V\|_\infty.$$

  Thus $v = V$.

Remark: the dynamic programming principle entirely characterises the value function!

Min-plus linearity

Notation. Given v_1 and $v_2 \in BUC(\mathbb{R}^n)$, we write $v_1 \leq v_2$ if $v_1(x) \leq v_2(x)$ for all $x \in \mathbb{R}^n$. We define $\min(v_1, v_2) \in BUC(\mathbb{R}^n)$ by

$$\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.$$

Given $\alpha \in \mathbb{R}$, we define $v_1 + \alpha$ by $(v_1 + \alpha)(x) = v_1(x) + \alpha$.

Lemma 11

Let v_1 and $v_2 \in BUC(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$. The map $\mathcal{T}$ is monotone:

$$v_1 \leq v_2 \Longrightarrow \mathcal{T} v_1 \leq \mathcal{T} v_2$$

and min-plus linear:

$$\min(\mathcal{T} v_1, \mathcal{T} v_2) = \mathcal{T} \min(v_1, v_2), \quad \mathcal{T}(v + \alpha) = (\mathcal{T} v) + e^{-\lambda \tau} \alpha.$$

Proof: exercise.

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Hamiltonian

We define the **pre-Hamiltonian** H and the **Hamiltonian** $\mathcal{H}$ by

$$\begin{aligned} H(u,x,p) &= \ell(u,x) + \langle p, f(u,x)\rangle, \\ \mathcal{H}(x,p) &= \min_{u \in U} \; H(u,x,p). \end{aligned}$$

Lemma 12

The mapping $\mathcal{H}$ is continuous, concave *with respect to p, and* Lipschitz continuous *with respect to p with modulus $\|f\|_\infty$.*

Proof. The pre-Hamiltonian H is affine in p, thus concave in p. As an infimum of concave functions, $\mathcal{H}$ is concave. We have:

$$\begin{aligned} |\mathcal{H}(x,\tilde{p}) - \mathcal{H}(x,p)| &\leq \sup_{u \in U} \; |H(u,x,\tilde{p}) - H(u,x,p)| \\ &\leq \sup_{u \in U} \; |\langle \tilde{p} - p, f(u,x)\rangle| \leq \; \|\tilde{p} - p\| \cdot \|f\|_\infty. \end{aligned}$$

Informal derivation

Notation: $C^1(\mathbb{R}^n)$, the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 13

Let $\Phi \in C^1(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$, let $u \in \mathcal{U}_\infty$, let $y = y[u,x]$. Consider the mapping:

$$\varphi \colon \tau \in [0,\infty) \mapsto \int_0^\tau e^{-\lambda t} \ell(u(t), y(t))\, dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).$$

Then $\varphi(0) = 0$ and $\varphi \in W^{1,\infty}(0,\infty)$ with

$$\dot{\varphi}(\tau) = e^{-\lambda \tau} \big(H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \big). \qquad (*)$$

In particular: $\dot{\varphi}(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$ (if u is continuous at 0).

Informal derivation

Proof. To simplify, we only consider the case where u is continuous, so that y is C^1 and φ is $C^1(\mathbb{R}^n)$. We have then:

$$
\begin{aligned}
\dot{\varphi}(\tau) &= e^{-\lambda\tau}\ell(u(\tau), y(\tau)) + e^{-\lambda\tau}\langle \nabla\Phi(y(\tau)), \dot{y}(\tau)\rangle \\
&\qquad - \lambda e^{-\lambda\tau}\Phi(y(\tau)) \\
&= e^{-\lambda\tau}\Big[\ell(u(\tau), y(\tau)) + \langle \nabla\Phi(y(\tau)), f(u(\tau), y(\tau))\rangle\Big] \\
&\qquad - \lambda e^{-\lambda\tau}\Phi(y(\tau)) \\
&= e^{-\lambda\tau}\Big[H(u(\tau), y(\tau), \nabla\Phi(y(\tau))) - \lambda\Phi(y(\tau))\Big].
\end{aligned}
$$

HJB in the classical sense

Theorem 14

Let $x \in \mathbb{R}^n$. Assume that

- *V is continuously differentiable in a neighborhood of x*
- *$P(x)$ has a solution $\bar{u}$ which is continuous at time 0.*

Then,

$$\lambda V(x) - \mathcal{H}(x, \nabla V(x)) = 0,$$

$$\bar{u}(0) \in \operatorname*{argmin}_{u_0 \in U} H(u_0, x, \nabla V(x)).$$

HJB in the classical sense

Proof. Step 1. Let $u_0 \in U$, let u be the constant control equal to u_0, let $y = y[u,x]$. By the **dynamic programming** principle, we have:

$$0 \leq \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t + e^{-\lambda \tau} V(y(\tau)) - V(x),$$

for all τ. Since $\varphi(0) = 0$, we deduce from $(*)$ that:

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).$$

Therefore,

$$0 \leq H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.$$

HJB in the classical sense

Step 2. Let us apply the **dynamic programming principle** again. Redefining φ and setting $\bar{y} = y[\bar{u}, x]$, we obtain:

$$0 = \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) \, \mathrm{d}t + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x),$$

for all $\tau \geq 0$. It follows that

$$0 = H(\bar{u}(0), x, \nabla V(x)) - \lambda V(x).$$

Step 3. It follows that

$$H(\bar{u}(0), x, \nabla V(x)) = \lambda V(x) \leq H(u_0, x, \nabla V(x)), \quad \forall u_0 \in U.$$

Therefore, $H(\bar{u}(0), x, \nabla V(x)) = \mathcal{H}(x, \nabla V(x))$.

HJB in the classical sense

Corollary 15

Let $t \geq 0$, assume that $\bar{u}$ is continuous in a neighborhood of t and that V is C^1 in a neighborhood of $\bar{y}(t)$, where $\bar{y} := y[\bar{u}, x]$. Then,

$$\bar{u}(t) \in \operatorname*{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))).$$

HJB in the classical sense

Remarks.
Let us define the Q-function by $Q(u, y) := H(u, y, \nabla V(y))$, assuming that $V \in C^1(\mathbb{R}^n)$.

- If the minimizer is unique in the following relation, we have a **feedback law**:

$$\bar{u}(t) = \underset{U}{\operatorname{argmin}}\ Q(\cdot, \bar{y}(t)).$$

- In some cases, one can show that $\nabla V(\bar{y}(t)) = p(t)$, where p is defined by some adjoint equation $\to$ **Pontryagin's principle**.
- In **Reinforcement Learning**, the approximation of Q is a central objective.

We will call the equation

$$\lambda v(x) - \mathcal{H}(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n \qquad \text{(HJB)}$$

Hamilton-Jacobi-Bellman equation, with unknown $v \colon \mathbb{R}^n \to \mathbb{R}$.

Remarks.

- In general V is not differentiable $\to$ in **which sense** is the HJB equation to be understood?
- In Theorem 14, we have shown that

$$\bar{u}(t) \in \operatorname{argmin} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),$$

  (under restrictive assumptions). We will see next that this necessary condition is also **sufficient**.

Theorem 16 (Verification)

Let us assume the assumptions of Theorem 14 hold for all $x \in \mathbb{R}^n$, so that the HJB equation is satisfied in the classical sense. *Let $x \in \mathbb{R}^n$. Assume that there exists a control $\bar{u}$ such that*

$$\bar{u}(t) \in \operatorname*{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))),$$

where $\bar{y} = y[\bar{u}, x]$. Then $\bar{u}$ is globally optimal.

Proof. Consider the function:

$$\varphi(\tau) = \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t))\, \mathrm{d}t + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x).$$

We have $\varphi(0) = 0$. Using $(*)$ and Theorem 14, we obtain:

$$\begin{aligned} \dot{\varphi}(\tau) &= e^{-\lambda\tau} \big[H(\bar{u}(\tau), \bar{y}(\tau), \nabla V(\bar{y}(\tau)) - V(\bar{y}(\tau)) \big] \\ &= e^{-\lambda\tau} \big[\mathcal{H}(\bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau)) \big] \\ &= 0. \end{aligned}$$

Thus φ is constant, equal to 0. Its limit is given by:

$$0 = \int_0^\infty e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t))\, \mathrm{d}t - V(x) = W(x, \bar{u}) - V(x),$$

proving the optimality of $\bar{u}$.

1 Introduction

2 Dynamic programming principle

3 A first characterization of the value function

4 HJB equation: the classical sense

5 HJB equation: viscosity solutions

Abstract PDE

We consider an abstract PDE of the form:

$$\mathcal{F}(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n,$$

where $\mathcal{F} \colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

It contains the HJB equation with

$$\mathcal{F}(x, v, p) = \lambda v - \mathcal{H}(x, p).$$

Goal of the section: showing that V is a viscosity solution to the HJB equation.

Sub- and super-differentials

Definition 17

Let $v\colon \mathbb{R}^n \to \mathbb{R}$. The following sets are called **sub- and superdifferential**, respectively:

$$D^- v(x) = \left\{ p \in \mathbb{R}^n \,\middle|\, \liminf_{y \to x} \frac{v(y) - v(x) - \langle p, y - x\rangle}{\|y - x\|} \geq 0 \right\}$$

$$D^+ v(x) = \left\{ p \in \mathbb{R}^n \,\middle|\, \limsup_{y \to x} \frac{v(y) - v(x) - \langle p, y - x\rangle}{\|y - x\|} \leq 0 \right\}.$$

Exercise. Let $v(x) = |x|$. Show that $D^- v(0) = [-1, 1]$.

Sub- and super-differentials

We have the following characterization.

Lemma 18

Let $v\colon \mathbb{R}^n \to \mathbb{R}$ be continuous. Let $p \in \mathbb{R}^n$.

- $p \in D^- v(x) \iff$ *there exists* $\Phi \in C^1(\mathbb{R}^n)$ *such that* $\nabla\Phi(x) = p$ *and* $v - \Phi$ *has a local minimum in* x.
- $p \in D^+ v(x) \iff$ *there exists* $\Phi \in C^1(\mathbb{R}^n)$ *such that* $\nabla\Phi(x) = p$ *and* $v - \Phi$ *has a local maximum in* x.

Proof. The implication $\Longrightarrow$ is admitted. The implication $\Longleftarrow$ is left as an exercise.

Sub- and super-differentials

Remark. In the above lemma, one can chose $\Phi(x) = v(x)$ without loss of generality. Thus, we have:

- $(v - \Phi)$ has a local minimum in $x \iff v - \Phi$ is nonnegative in a neighborhood of $x \iff v$ is locally bounded from below by Φ
- $(v - \Phi)$ has a local maximum in $x \iff v - \Phi$ is nonpositive in a neighborhood of $x \iff v$ is locally bounded from above by Φ

Remark. If v is Fréchet differentiable at x, then the sub- and superdifferential are equal to $\{\nabla v(x)\}$.

Viscosity solutions

Definition 19

Let $v\colon \mathbb{R}^n \to \mathbb{R}$. We call v a **viscosity subsolution** if

$$\mathcal{F}(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n, \ \forall p \in D^+ v(x)$$

or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local maximum in x,

$$\mathcal{F}(x, v(x), \nabla\Phi(x)) \leq 0.$$

Viscosity solutions

Definition 20

Let $v\colon \mathbb{R}^n \to \mathbb{R}$. We call v a **viscosity supersolution** if

$$\mathcal{F}(x, v(x), p) \geq 0, \quad \forall p \in D^- v(x)$$

or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local minimum in x,

$$\mathcal{F}(x, v(x), \nabla\Phi(x)) \geq 0.$$

We call v a **viscosity solution** if it is a sub- and a supersolution.

Viscosity solutions

Theorem 21

The value function V is a viscosity solution *of the HJB equation.*

Step 1: V is a subsolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local maximizer in x and $V(x) = \Phi(x)$.
We have to prove that

$$\lambda v(x) - \mathcal{H}(x, \nabla\Phi(x)) \leq 0.$$

Let $u_0 \in U$, let u be the constant control equal to u_0 and let $y = y[u, x]$. By the DPP, we have:

$$V(x) \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t))\, \mathrm{d}t + e^{-\lambda\tau} V(y(\tau)).$$

If τ is sufficiently small, we have $V(y(\tau)) \leq \Phi(y(\tau))$.

Viscosity solutions

This implies that for τ sufficiently small,

$$0 \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) \, \mathrm{d}t + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x) =: \varphi(\tau).$$

Since $\varphi(0) = 0$, we deduce with $(*)$ that

$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla \Phi(x)) - \lambda V(x).$$

Minimizing with respect to $u_0 \in U$, we obtain:

$$0 \leq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x),$$

as was to be proved.

Viscosity solutions

Step 2: V is supersolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local minimizer in x and such that $V(x) = \Phi(x)$. We have to prove that

$$\lambda V(x) - \mathcal{H}(x, \nabla\Phi(x)) \leq 0.$$

It follows from the dynamic programming principle that for $\tau > 0$ small enough

$$\Phi(x) \geq \inf_{u \in \mathcal{U}_\tau} \underbrace{\int_0^\tau e^{-\lambda t} \ell(u(t), y[x,u](t))\, \mathrm{d}t + e^{-\lambda t} \Phi(y[x,u](\tau))}_{=:\varphi[u](\tau)}.$$

Viscosity solutions

Thus by Lemma 13,

$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \dot{\varphi}[u](t)\, \mathrm{d}t \\ &= \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} \big(H(u(t), y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t)) \big)\, \mathrm{d}t \\ &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \underbrace{e^{-\lambda t} \big(\mathcal{H}(y[u](t), \nabla \Phi(y[u](t)) - \lambda \Phi(y[u](t)) \big)}_{=:\psi[u](t)}\, \mathrm{d}t. \end{aligned}$$

We have $\psi[u](0) = \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x)$, in particular, $\psi[u](0)$ does not depend on u.

Viscosity solutions

Let $\varepsilon > 0$. There exists (exercise!) $\tau > 0$ such that

$$|\psi[u](t) - \psi[u](0)| \leq \varepsilon, \quad \forall t \in [0,\tau], \ \forall u \in \mathcal{U}_\infty.$$

The previous inequality yields

$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \big(\psi[u](0) - \varepsilon\big)\, \mathrm{d}t \\ &\geq \tau(\mathcal{H}(x, \nabla\Phi(x)) - \lambda V(x) - \varepsilon). \end{aligned}$$

Dividing by τ and sending ε to 0, we obtain that

$$\lambda V(x) - \mathcal{H}(x, \nabla\Phi(x)) \leq 0.$$

Viscosity solutions

Theorem 22 (Comparison principle)

Let v_1 be a subsolution to the HJB equation. Let v_2 be a supersolution to the HJB equation. Then

$$v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.$$

Proof: admitted.

Corollary 23

The value function V is the unique *viscosity solution.*

Proof. By the comparison principle, any viscosity solution v is such that $v \leq V$ and $v \geq V$.