## Notes on Markov Chains

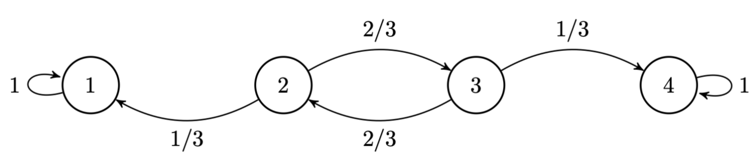
A Markov chain is a model of a stochastic process where the next step is determined purely by the current state. In other words, it only matters where you are, not how you got there. This assumption makes it easy to compute the conditional probabilities of reaching a certain state in the next step, since it only depends on the current state and not the full history of the process.

### Transition Matrices

Such a process is essentially a sequence of random variables $X_0, X_1, \cdots, X_n$. Each $X_i$ is drawn based on a probability distribution over a state space $\{0, 1, \cdots, M\}$. Each $X_i$ has its own probability distribution, and this distribution depends on the value of $X_{i-1}$ took on.

These probability distributions can be expressed through transition matrices $Q$. They show the probability of transitioning from one state into another. The row of a transition matrix is the current state $i$ and the column is the target state $j$. The entry itself is then the probability that we reach state $j$ from state $i$, namely $P(X_2 = j | X_1 = i)$.

A five-state Markov chain could have the following transition matrix representation.




$$Q = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

Two properties to keep in mind:

1. Each row shows us all the possible next states of a Markov process. Logically, the entries must sum to $1$.
2. For a finite state space of size $M$, the dimensions of $Q$ are always $M \times M$.

These matrices can be generalised to $n$ steps, in which case $Q^n$ would show the probability of moving from $i \to j$ over the course of $n$ moves.

### Initial States

To simulate a Markov process, you need to specify an initial state $X_0$. The simple way to do this would be to assign $X_0$ to a certain state. However, it's often desirable to choose a random initial state based on some distribution given as a vector:

$$\mathbf{s} = (s_1, s_2, \cdots, s_M)$$

where each entry is the probability of the initial state $X_0$ being a state $i$, more formally: $s_i = P(X_0 = i)$.

### Transient and Recurrent States