Problem: Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

Solution: This is a one-dimensional problem with homogeneous Robin boundary conditions on two endpoints ($x = 0$ and $x = 1$). We know we have existence and uniqueness in the solution, but stability is in question. We would like to know if the change in sign in the second boundary condition causes large changes to u.

The general solution to this PDE is

$$u(x,t) = C_1 f(t)\, x + C_2 g(t)$$

.

For the first boundary condition, we have

$$\begin{aligned} &u'(0) + ku(0) = 0 \\ &C_1 f(t) + kC_2 g(t) = 0 \\ &C_1 f(t) = -kC_2 g(t) \end{aligned}$$

For the second boundary condition,

$$\begin{aligned} u'(1) + ku(1) &= 0 \\ C_1 f(t) + k\left(C_1 f(t) + C_2 g(t)\right) &= 0 \\ C_1 f(t)(1+k) + kC_2 g(t) &= 0 \\ -kC_2 g(t)(1+k) + kC_2 g(t) &= 0 \\ k^2 C_2 g(t) &= 0 \\ C_2 &= 0. \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} u'(1) - ku(1) &= 0 \\ C_1 f(t) - k\left(C_1 f(t) + C_2 g(t)\right) &= 0 \\ C_1 f(t)(1-k) - kC_2 g(t) &= 0 \\ -kC_2 g(t)(1-k) - kC_2 g(t) &= 0 \\ k^2 C_2 g(t) - 2kC_2 g(t) &= 0 \\ k(k-2) &= 0. \end{aligned}$$

For the negative sign case, the extra term creates a situation where C_2 can have multiple values. Thus the solution is not stable at $k = 2$. ✓