1 Derivation of a continuous HCR

In a discontinuous setting, an example harvest control rule (HCR) is given by

$$F(B) = \begin{cases} 0 & \text{if } B < B_{lim} \\ a + bB & \text{if } B_{lim} < B < B_{trig} \\ F_{MSY} & \text{if } B > B_{trig} \end{cases} \tag{1}$$

where $b = \dfrac{F_{msy} - F_{lim}}{B_{trig} - B_{lim}}$ and $a = F_{msy} - bB_{trig}$. The discontinuous derivative of this goes from zero to b to zero again. A continuous analogue can be represented by the double logistic function

$$\frac{dF}{dB} = \frac{1}{1 + e^{-(B - B_{lim})}} + \frac{1}{1 + e^{-(B_{trig} - B)}}, \tag{2}$$

which is on $[1, 2]$ for $B_{trig} > B_{lim}$. Scaling it to $[0, b]$ is achieved by including

$$\frac{dF}{dB} = b\left(-1 + \frac{1}{1 + e^{-(B - B_{lim})}} + \frac{1}{1 + e^{-(B_{trig} - B)}}\right) \tag{3}$$

Integrating both sides w.r.t. B provides

$$\begin{aligned} F &= b\left(\ln\left(e^{B_{lim}} + e^{B}\right) - \ln\left(e^{B - B_{trig}} + 1\right)\right) + C \\ &= b\ln\left(\frac{e^{B_{lim}} + e^{B}}{e^{B - B_{trig}} + 1}\right) + C \end{aligned} \tag{4}$$

Setting $F(B = 0) = 0$ (i.e., $F_{lim} = 0$) provides the constant of integration

$$C = -b\ln\left(\frac{e^{B_{lim}} + 1}{e^{-B_{trig}} + 1}\right) \tag{5}$$

Thus providing

$$F = b\left(\ln\left(\frac{e^{B_{lim}} + e^{B}}{e^{B - B_{trig}} + 1}\right) - \ln\left(\frac{e^{B_{lim}} + 1}{e^{-B_{trig}} + 1}\right)\right) \tag{6}$$

This form provides a useful continuous HCR (Figure 1) that will be used in the phase plane analysis.

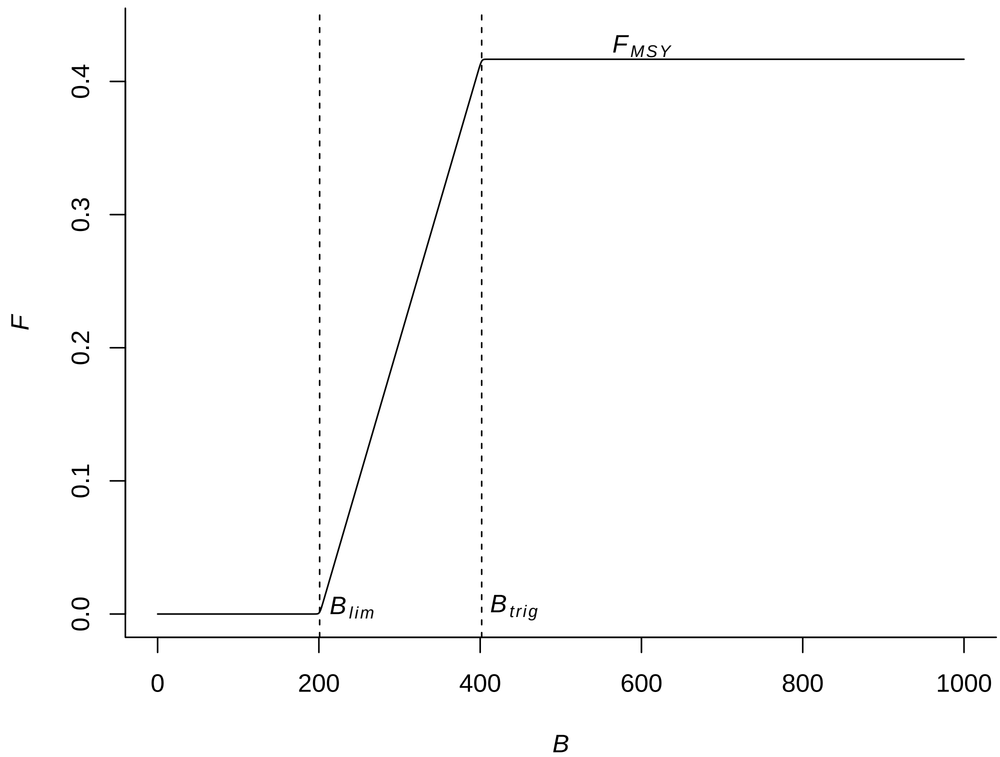

Figure 1: Solution to a continuous HCR with derivative given by Equation (3). Note that the transitions at the breakpoints are continuous but appear almost discontinuous at this scale.