Modeling a competitive demand-responsive transport market

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Abstract

We present a model for a demand-responsive transport (DRT) service in which customers request trips from specified origins to specified destinations and a fleet of competing vehicles serves the customers. The trips are requested without pre-order times and the vehicle routes are generated in real time. Customers seek rides to minimize travel time and competing vehicles aim to maximize profit. We describe the demand and movement of vehicles by logit models and define the network equilibrium as a state in which the demand meets the supply of DRT trips. We show that an equilibrium always exists and provide an equilibration algorithm. Finally, we study the long-run behavior of competitive DRT by varying the number of vehicles and price. Examples suggest that (i) with no price regulation, free entry results in a taxi-type service and (ii) the optimal equilibrium from the customers' perspective is achieved by price regulation and free entry.

Keywords: Demand-responsive transport, Competitive market, Network equilibrium, Transportation network

1. Introduction

Demand responsive transport (DRT) is an advanced, user-oriented form of public transport involving flexible routing of small or medium sized vehicles between the pick-up and drop-off locations of customers. DRT can be seen as a passenger transport service between bus and taxi ranging from less formal community transportation to area-wide service networks (Mageean and Nelson, 2003). The major difference to taxi services is that customers with different pick-up and drop-off locations may share the same vehicle. The major difference to bus services is that

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the vehicle routes and schedules are generated in real time according to the trips requested by customers. Earlier studies suggest that DRT has potential to become a socially and economically remarkable form of public transportation (Mulley and Nelson, 2009; Sihvola et al., 2010; Jokinen et al., 2011). In addition, DRT is seen to be more effective than a fixed route service in minimizing emissions of pollutants (Diana et al., 2007).

Most existing DRT services provide door-to-door transportation for elderly or handicapped people and require customers to book trips at least one hour in advance (Cordeau et al., 2007a; Helsinki Region Traffic, 2010). Conventionally, the trips are organized centrally via *travel dispatch centers*, which have the capability of assigning customers to vehicles and optimizing the routes (Mageean and Nelson, 2003). In contrast to such centralized DRT services, we consider a competitive form of DRT in which each driver providing service attempts to maximize his/her profit. That is, the movement of vehicles is governed by the decisions of individual drivers, instead of a travel dispatch center controlled by a single transport operator. This market structure is in fact similar to conventional taxi-markets, which have been extensively studied, see for example (Hackner and Nyberg, 1995; Arnott, 1996; Cairns and Liston-Heyes, 1996; Flores-Guri, 2003; Lagos, 2003; Wong et al., 2005; Matsushima and Kobayashi, 2006; Fernandez et al., 2006; Moore and Balaker, 2006; Yang et al., 2002, 2005, 2010),

We introduce a network model to characterize the behavior of customers and vehicles in a competitive DRT market. The drivers providing service seek routes that serve as many customers per unit time as possible, and customers seek rides to minimize subjective trip price, defined as a combination of ticket price and travel time. Given these vehicle-customer behavior models, we calculate the *network equilibrium*, defined as a state in which the demand matches the supply of trips in each part of the network.

This approach is somewhat similar to the taxi model proposed by Yang et al. (2010). However, DRT is to some extent a more complex transportation service to model than a taxi service. The main difference is that in a taxi service, customers are delivered to their destinations directly, whereas in a DRT service, a vehicle can serve several customers simultaneously and therefore a customer's trip from an origin to a destination is not necessarily a direct one. From modeling perspective, this means that optimal decisions of customers and vehicles cannot be based only on customer waiting times and vehicle searching times as in the taxi equilibrium model of Yang et al. (2010). Basically, a DRT vehicle can make more profits by serving more customers per unit time, but the number of served customers also increases average travel times, which reduces customers' satisfaction and demand for the DRT service. Therefore, in our DRT model, the level of service of a trip is determined by the total travel time of the trip, which depends on the vehicles' routing decisions.

This work is partially motivated by a demand-responsive transport (DRT) service currently being planned to operate in Helsinki. Helsinki Region Transport board has approved a plan under which the trial period of the service takes place from 2012 to 2014. Similarly as the current flexible service routes (Helsinki Region Traffic, 2010), the new DRT service is designed to operate on a demand-responsive basis, that is, vehicle routes can be modified in real time in order to meet the demand efficiently. The main difference to existing services is that no pre-order times for trips are required and the trips can be booked "on the fly" by means of an interactive user interface.

The rest of this paper is organized as follows. In Section 2, we describe the behavior of customers and the movement of vehicles. In Section 3, we define the network equilibrium for our model and present an algorithm for determining the equilibrium. In Section 4, we study the long-run market behavior as a function of the number of vehicles and price. The conclusions of this work are given in Section 5.

2. Model

Our model of competitive DRT is governed by the following preliminary assumptions.

1. There are N competing vehicles that produce trips between origins and destinations in a specific operating zone. Different trips may have different prices and travel times. A single vehicle can simultaneously serve several customers.
2. The subjective price of a trip is defined as a combination of ticket price and travel time. Customers choose between trips provided by DRT and a virtual mode by comparing the subjective prices of different trips.
3. Customers are assigned to vehicles by means of an electronic booking service. For analysis, we assume that the DRT customers that choose trip r are divided equally[2] among vehicles that produce trip r.

In the following, we will study the behavior of customers (Section 2.1) and drivers providing service (Section 2.2) in more detail.

2.1. Demand

Let us consider a set of *nodes* I representing the origins and destinations of customers in a specific operating zone. Each pair of nodes $(i, j) \in I \times I$ is associated with a specific *direct ride time* t_{ij}. Generally, the direct ride times are not symmetric, that is, t_{ij} and t_{ji} are not necessarily equal for $i, j \in I$.

A *trip* from an origin $i_0 \in I$ to a destination $i_d \in I$ is defined as an acyclic sequence of nodes $(i_0, i_1, \ldots, i_{d-1}, i_d)$ in I, that is, each node in the sequence appears in the sequence exactly once. When a customer takes a trip $(i_0, i_1, \ldots, i_{d-1}, i_d)$, the customer enters a vehicle at i_0, which visits the nodes $i_1, \ldots, i_{d-1}$ before the drop-off of the customer at i_d. That is, the trip denotes the path of the vehicle that transports the customer from i_0 to i_d. For example, a *direct* trip from $i_0 \in I$ to $i_d \in I$, denoted by (i_0, i_d), describes a trip in which a customer enters a vehicle at i_0 and the vehicle drives directly to i_d without stopping between i_0 and i_d. The set of all trips in I is denoted by $\mathcal{R}$ and the set of trips from $i \in I$ to $j \in I$ is denoted by $\mathcal{R}_{ij}$.

2.1.1. Subjective price

Each trip $r \in \mathcal{R}$ has a specific ticket price p_r. The ticket price may be defined equal for all trips with the same origin and destination, that is, for any $i, j \in I$ we have $p_r = p_{r'}$ for all $r, r' \in \mathcal{R}_{ij}$. This pricing idea is used in the numerical examples in Section 3.2. At this point, however, we consider the general case in which the prices of trips with the same origin and destination may be

[2]cf. mean value analysis (Reiser and Lavenberg, 1980).

different. Similarly as in (Yang et al., 2010), each customer seeks a trip to minimize the *subjective price*, defined as a combination of p_r and *travel time*.

The *ride time* of a trip $(i_0, \ldots, i_d) \in \mathcal{R}_{i_0 i_d}$ refers to the time spent inside a vehicle and is defined by

$$q^{\mathrm{DRT}}_{(i_0,\ldots,i_d)} = \sum_{k=1}^{d} t_{i_{k-1} i_k}. \tag{1}$$

Note that since the direct ride times t_{ij} are transitive, the ride time from i_0 to i_d is minimized in the direct trip (i_0, i_d).

For any pair of nodes $i, j \in I$, the demand responsive transport service produces different trips from i to j at different intervals. That is, there may be different *waiting times* for different trips. For example, if a direct trip (i, j) from i to j is produced two times per hour and a trip (i, k, j) is produced three times per hour, we expect that the average waiting times for the trips (i, j) and (i, k, j) satisfy $w^{\mathrm{DRT}}_{(i,j)} > w^{\mathrm{DRT}}_{(i,k,j)}$ (see Section 2.2.5 for a more detailed discussion on waiting times).

The average *travel time* for a trip $r \in \mathcal{R}_{ij}$ is defined as the sum of waiting time and ride time, that is,

$$t^{\mathrm{DRT}}_r = w^{\mathrm{DRT}}_r + q^{\mathrm{DRT}}_r. \tag{2}$$

The *subjective price* of a DRT trip $r \in \mathcal{R}_{ij}$ is given by

$$g^{\mathrm{DRT}}_r = p_r + \beta t^{\mathrm{DRT}}_r, \tag{3}$$

where p_r is the ticket price for the trip r and β is the customers' monetary value of unit travel time.

2.1.2. Demand for DRT

The choices of customers are determined by a logit model, similar to the one in (Yang et al., 2010), as follows. The subjective trip price of a trip from i to j provided by the *virtual mode*[3] is denoted by $\bar{g}_{ij}$. The probability that a customer traveling from i to j chooses a DRT trip $r \in \mathcal{R}_{ij}$ is defined by

$$P^{\mathrm{DRT}}_r = \frac{\exp(-\theta g^{\mathrm{DRT}}_r)}{\exp(-\theta \bar{g}_{ij}) + \sum_{r' \in \mathcal{R}_{ij}} \exp(-\theta g^{\mathrm{DRT}}_{r'})}, \tag{4}$$

where θ is a nonnegative parameter describing the uncertainty in transport services and demand from the perspective of customers. Note that when $\theta = 0$, the choice probability is equal for all alternatives. When $\theta \to \infty$, each customer chooses the option with the lowest subjective price with probability 1. Clearly, the above logit model has the property of independence from irrelevant alternatives, that is, the ratio $P_r/P_{r'}$ depends on the subjective prices of trips r and r' but not on the subjective prices of other trips (Small and Verhoef, 2007).

[3]The virtual mode represents alternatives for the DRT service.

The *total demand* from i to j is denoted by Q_{ij} and describes the number of customers willing to travel from i to j per unit time. The corresponding demand for a trip $r \in \mathcal{R}_{ij}$ is given by

$$Q_r^{\mathrm{DRT}} = Q_{ij} P_r^{\mathrm{DRT}}. \tag{5}$$

The *demand for DRT* describing the expected number of customers that choose to travel from i to j by DRT per unit time is given by

$$Q_{ij}^{\mathrm{DRT}} = \sum_{r \in \mathcal{R}_{ij}} Q_r^{\mathrm{DRT}}. \tag{6}$$

2.2. *Vehicle movements*

We assume that there are N vehicles available for transporting customers. At any point in time, each vehicle follows a specific *route* $(i_0, i_1, \ldots, i_m)$ determined by a sequence of nodes in the transportation network. The vehicle starts at the first node i_0 and proceeds by visiting the other nodes i_k for $k = 1, \ldots, m$ in the order determined by the route. Each node corresponds to a stop during which customers may enter or exit the vehicle. We assume that each route is acyclic and consists of a minimum of two nodes, that is, none of the vehicles are idle at any time. Since the routes are acyclic, each route consists of a maximum of $|I|$ nodes.

The total time needed to complete a route is given by

$$d_{(i_0,i_1,\ldots,i_m)}^{\mathrm{route}} = \sum_{h=1}^{m} t_{i_{h-1} i_h} \tag{7}$$

and the total cost of a route equals

$$c_{(i_0,i_1,\ldots,i_m)}^{\mathrm{route}} = \sum_{h=1}^{m} c_{i_{h-1} i_h}, \tag{8}$$

where $c_{i_{h-1} i_h}$ is the cost associated with leg (i_{h-1}, i_h).

2.2.1. *Trip production*

When a vehicle following a route $(i_0, i_1, \ldots, i_m)$ arrives at stop i_k, customers waiting at i_k to be transported to any of the stops $i_{k+1}, \ldots, i_m$ may enter the vehicle. In other words, we say that the vehicle *produces* trips from i_k to the nodes $i_{k+1}, \ldots, i_m$, see Figure 1. Formally, the set of produced trips from the current stop i_k to the remaining stops $i_{k+1}, \ldots, i_m$ is given by

$$\mathcal{R}_{(i_k,\ldots,i_m)} = \bigcup_{h=k+1}^{m} \{(i_k, i_{k+1}, \ldots, i_h)\}. \tag{9}$$

During the execution of the route $(i_0, i_1, \ldots, i_m)$, the vehicle consecutively arrives at stops i_k, where $k = 0, \ldots, m$. The set of produced trips during the execution of the route equals

$$\mathcal{R}_{(i_0,\ldots,i_m)}^{\mathrm{route}} = \bigcup_{k=0}^{m-1} \mathcal{R}_{(i_k,\ldots,i_m)}, \tag{10}$$

that is, the set of subsequences of $(i_0, \ldots, i_m)$. This idea of producing trips is in fact similar as in traditional public transport: a bus with a given route produces trips that are subsequences of the route (see Figure 1).

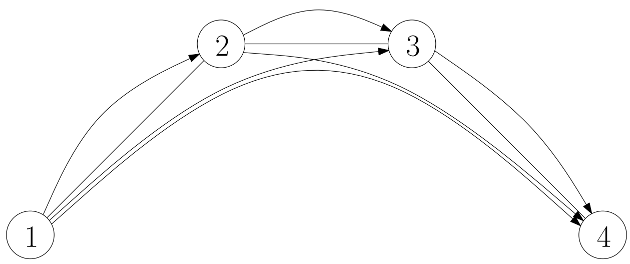


Figure 1: Trip production. A vehicle follows the route $(1, 2, 3, 4)$. Upon arrival at each stop, the vehicle produces trips to the other stops on the route. The set of produced trips during the execution of the route is equal to the set of subsequences of $(1, 2, 3, 4)$, that is, $\mathcal{R}^{\text{route}}_{(1,2,3,4)} = \{(1, 2), (1, 2, 3), (1, 2, 3, 4), (2, 3), (2, 3, 4), (3, 4)\}$.

2.2.2. Route modifications

The most distinguishing feature of demand-responsive transport, which differentiates it from traditional public transport, is that the vehicle routes may be modified in real time in order to meet the demand for transportation efficiently. In this work, we consider the following type of route modifications.

Suppose a vehicle follows a route $(i_0, \ldots, i_m)$ and arrives at node i_k. Naturally, the vehicle produces trips from i_k to all remaining nodes $i_{k+1}, \ldots, i_m$. In order to serve additional customers, the driver may choose to offer transportation from i_k to *destinations that are not included in the current route*. That is, the driver may *extend*[4] the current route by adding a sequence of nodes $i_{m+1}, \ldots, i_q$ to the end of the route (see Figure 2). In this case, the vehicle produces the set of trips $\mathcal{R}_{(i_k,\ldots,i_q)}$ from i_k to the stops $(i_{k+1}, \ldots, i_q)$, see Figure 2c.

When a vehicle following a route $(i_0, \ldots, i_m)$ arrives at i_k, where $0 \le k < m$, there are two options:

- The route is *extended*. The vehicle transfers to a new route beginning at i_k, namely, $(i_k, \ldots, i_q)$.
- The route is *not extended*. The vehicle proceeds by executing the current route $(i_0, \ldots, i_m)$.

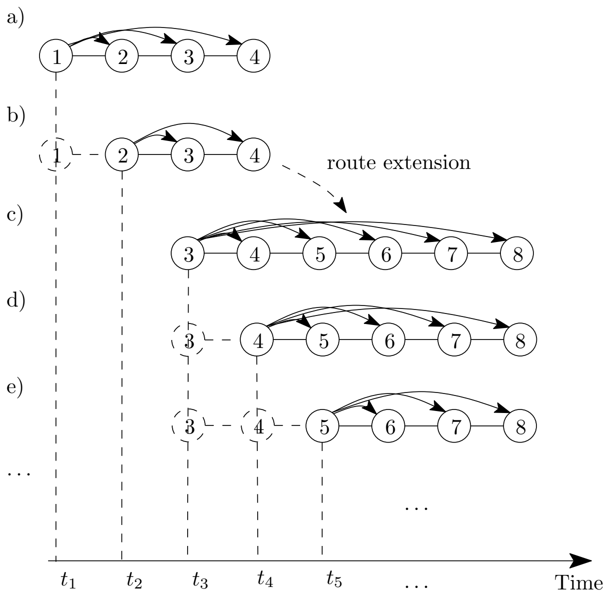
By definition, the minimum number of nodes in a route is two. Thus, when the vehicle arrives at i_m, the route is automatically extended.

In the following, we discuss the route extensions and production of trips in more detail by representing routes as sequences of *states*.

2.2.3. States

The *state* of a vehicle describes which part of the route the vehicle is currently executing. Each time a vehicle following a route $(i_0, \ldots, i_m)$ arrives at stop i_k, we say that the vehicle *transfers* to a new state. Similarly as routes, the states are defined as sequences of nodes. The difference between

[4]In some services, it might be reasonable to think that the route of a vehicle could be modified *during* the trip of a customer by adding nodes to the sequence before the drop-off point of the customer. In this work, however, we assume that when a customer enters a vehicle, the route to the customer's destination is fixed and the possible route extensions affect the part of the route after the customer's destination.



Route	State	Trips produced
$(1, 2, 3, 4)$	$(1, 2, 3, 4)$	$(1, 2), (1, 2, 3), (1, 2, 3, 4)$
$(1, 2, 3, 4)$	$(2, 3, 4)$	$(2, 3), (2, 3, 4)$
$(3, 4, 5, 6, 7, 8)$	$(3, 4, 5, 6, 7, 8)$	$(3, 4), \ldots, (3, 4, 5, 6, 7, 8)$
$(3, 4, 5, 6, 7, 8)$	$(4, 5, 6, 7, 8)$	$(4, 5), \ldots, (4, 5, 6, 7, 8)$
$(3, 4, 5, 6, 7, 8)$	$(5, 6, 7, 8)$	$(5, 6), (5, 6, 7), (5, 6, 7, 8)$
...	...	...

Figure 2: Routes, states and trips. a) At t_1, a vehicle follows the route $(1, 2, 3, 4)$ consisting of four nodes. Customers are transported from 1 to 2, 3 and 4. b) At time t_2, the vehicle arrives at node 2. At this instant, customers waiting at 2 to be transported to 3 or 4 may enter the vehicle, that is, the vehicle produces trips $(2, 3)$ and $(2, 3, 4)$. The route of the vehicle remains unchanged. c) At t_3, the vehicle arrives at node 3. In addition to node 4, the driver chooses to offer transportation from node 3 to nodes $5, \ldots, 8$. In this case, the vehicle produces trips $(3, 4), (3, 4, 5), (3, 4, 5, 6), (3, 4, 5, 6, 7), (3, 4, 5, 6, 7, 8)$. The route of the vehicle is extended, and the new route is $(3, 4, 5, 6, 7, 8)$. The execution of the new route is represented as a sequence of states, namely $(3, 4, 5, 6, 7, 8), (4, 5, 6, 7, 8), \ldots, (7, 8)$, see Figures c,d,e.

states and routes is that the state of a vehicle corresponds to the *remaining part* of its current route. That is, the vehicle transfers to a new state each time it arrives at a stop, even if the route remains unchanged. During the execution of a route $(i_0, \ldots, i_m)$, the vehicle successively transfers to states $(i_0, \ldots, i_m), (i_1, \ldots, i_m), (i_2, \ldots, i_m), \ldots, (i_{m-1}, i_m)$. For example, in Figure 2, the vehicle follows the same route in both 2a and 2b, but the state in 2a is $(1, 2, 3, 4)$ and the state in 2b is $(2, 3, 4)$. In 2c, the vehicle transfers to state $(3, 4, 5, 6, 7, 8)$.

Generally, the set of states is defined by

$$\mathcal{S} = \{(i_0, i_1, \ldots, i_m) \mid m \geq 1,\ (h \neq g \Rightarrow i_h \neq i_g),\ i_h, i_g \in I \text{ for all } h, g \in \{0, \ldots, m\}\}.$$

The total number of states equals $|\mathcal{S}| = \sum_{l=2}^{|I|} \binom{|I|}{l} l!$. Note that the set of states is equal to the set of trips, that is, $\mathcal{S} = \mathcal{R}$. Clearly, the number of states grows rapidly with the size of the network. In practice, it might be reasonable to consider only a subset of all possible states in a network due to computational issues. However, since the modification of our model to this approximate case is straightforward, we will focus on the general case in which the state space is complete.

When a vehicle arrives at a stop, the route of the vehicle may be extended by adding a sequence of nodes to the end of the route, as dicussed in Section 2.2.2. In other words, the set of states *to* which a transfer from state $s = (i_0, \ldots, i_m)$ is possible is defined by

$$\mathcal{S}_s = \{(s_1, \ldots, i_m, i_{m+1}, \ldots, i_q) \in \mathcal{S} \mid i_{m+1}, \ldots, i_q \in I\}. \tag{11}$$

This set will be referred to as the *successor set* of state s. Note that if the route is not extended, we have $q = m$ in the above equation.

The first two nodes i_0, i_1 of a state $(i_0, \ldots, i_m)$ determine the leg the vehicle is currently traversing. Since the state of the vehicle is modified at the time the vehicle arrives at i_1, the duration of a state $(i_0, \ldots, i_m)$ is given by

$$d_{(i_0,\ldots,i_m)} = t_{i_0 i_1}. \tag{12}$$

Note that the duration of a state $(i_0, \ldots, i_m)$ is generally different from the total time (7) needed to complete the route.

2.2.4. Arrival rate and production rate

The *arrival rate* of vehicles at state $s \in \mathcal{S}$ describes the number of vehicles arriving at state s per unit time and is denoted by T_s. During a finite time period $[0, t]$, we require that the total service time of vehicles at different states equals Nt, that is,

$$\sum_{s\in\mathcal{S}} tT_s d_s = Nt \quad \Leftrightarrow \quad \sum_{s\in\mathcal{S}} T_s d_s = N, \tag{13}$$

where d_s is the duration of state s defined by Equation (12).

Let us consider the *production rate* T_r^p for a trip $r \in \mathcal{R}$, that is, how many times the trip r is produced per unit time. Using the notation of (9), we note that a state s *produces* the set of trips $\mathcal{R}_s$, that is, the set of all subsequences of s beginning from the first node of s. Correspondingly, for any trip $r \in \mathcal{R}$, we define the set of states that produce trip r by

$$\mathcal{S}_r^p = \{s \in \mathcal{S} \mid r \in \mathcal{R}_s\},$$

where $\mathcal{S}$ is the set of all states. That is, a state s is included in S_r^p if r is a subsequence of s beginning from the first node in s.

Thus, the production rate T_r^p for a trip r is given by the total arrival rate of vehicles at states $s \in \mathcal{S}_r^p$ that produce trip r, that is,

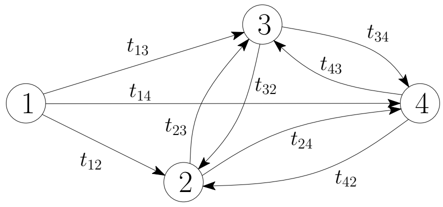
$$T_r^p = \sum_{s\in\mathcal{S}_r^p} T_s, \tag{14}$$

where T_s is the arrival rate of vehicles at state s (see Figure 3).

2.2.5. Average travel time

Let us then consider the *average travel time* t_r^{DRT} for a trip $r \in \mathcal{R}$, defined as the sum of waiting time and ride time, see Equation (2).

The ride time q_r^{DRT} for a trip r is given by Equation (1). The average waiting time w_r^{DRT} for a trip r depends on the production rate T_r^p of the trip, that is, the arrival rate of vehicles at states S_r^p



Trip $r \in \mathcal{R}_{14}$	Ride time q_r	States S_r^p that produce r
$(1,4)$	t_{14}	$(1,4),(1,4,2),(1,4,3),$ $(1,4,2,3),(1,4,3,2)$
$(1,2,4)$	$t_{12}+t_{24}$	$(1,2,4),(1,2,4,3)$
$(1,3,4)$	$t_{13}+t_{34}$	$(1,3,4),(1,3,4,2)$
$(1,2,3,4)$	$t_{12}+t_{23}+t_{34}$	$(1,2,3,4)$
$(1,3,2,4)$	$t_{13}+t_{32}+t_{24}$	$(1,3,2,4)$

Figure 3: The production rate for different trips between two nodes. Let us consider the rate of vehicles producing trips from node 1 to node 4 in the example network presented in the figure. The trips $r \in \mathcal{R}_{14}$ from 1 to 4 are presented in the first column of the table next to the figure. The second column of the table shows the ride time q_r for the trip r and the third column shows the set of states $\mathcal{S}_r^p$ that produce the trip r. The production rate T_r^p for a trip r is defined as the sum of the arrival rates T_s of vehicles at states $s \in \mathcal{S}_r^p$ that produce trip r. The average travel time for a trip r is given by $t_r^{\mathrm{DRT}} = \frac{1}{T_r^p} + q_r$.

that produce trip r. In addition to the production rate T_r^p, the average waiting time w_r^{DRT} depends on the *distribution* of the arrival times of vehicles at states S_r^p.

Given the production rate T_r^p, the average waiting time for trip r is minimized when the vehicles arrive at *equal time intervals*. In this case, the average waiting time equals $\frac{1}{2T_r^p}$. This idea is widely used in traditional public transport: If vehicles arrive every n minutes, the average waiting time is $n/2$ minutes. In addition to minimizing average waiting time, equal time intervals divide the customers evenly among vehicles providing service.

However, in our competitive DRT model, a driver does not know the exact locations of other vehicles and thus, it may not be reasonable to think that the inter-arrival times of vehicles were equal. In the absence of accurate information, we assume that the vehicles arrive according to a Poisson process, in which arrivals occur continuously and independently of one another (Ross, 1995). In this case, the expected waiting time for a customer departing at a random instant is equal to the expected time interval between arrivals, that is, $w_r^{\mathrm{DRT}} = \frac{1}{T_r^p}$. The average travel time is thus given by

$$t_r^{\mathrm{DRT}} = \begin{cases} \frac{1}{T_r^p} + q_r^{\mathrm{DRT}} & \text{if } T_r^p > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{15}$$

For instance, if during one hour there are three vehicles that produce a trip r for which the *ride time* is 10 minutes, the average *travel time* for that trip is given by adding the average *waiting time*, that is, $t_r^{\mathrm{DRT}} = \frac{1}{\frac{3}{60}} + 10 = 30$ minutes.

Given the production rate T_r^p, the average waiting time in the Poisson model is twofold compared to the equal time intervals model. In other words, the arrival time distribution has a significant effect on the waiting time. This is seen to be a major difference between centralized and competitive transport services[5].

[5] Assuming that the competing divers had some information on the locations of other vehicles, the inter-arrival times could even out since it would result in a better level of service. In practice, we expect that the average waiting time is between $\frac{1}{2T_r^p}$ and $\frac{1}{T_r^p}$.

Note that if the production rate T_r^p for a trip $r \in \mathcal{R}$ is zero, the average travel time for r is infinite. In this case, the demand for the trip is zero, see Equation (4).

2.2.6. Expected profit rate

The drivers attempt to maximize profit by offering transportation to as many customers per unit time as possible. More precisely, the drivers attempt to transfer to states at which the *expected profit rate* is maximized. By using the expected profit rate as a utility measure we can compare the profitability of routes of different lengths.

For example, a long route including a large number of nodes produces more trips than a short one and thus, we expect that the expected profit earned during a long route is greater than in a short one. However, the time needed to execute a long route is also greater. That is, in some cases it may be profitable to execute many short routes successively instead of a single long route, as in the example presented in Figure 4.

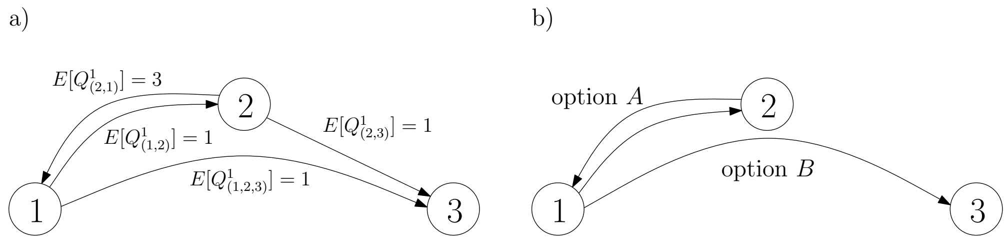


Figure 4: Expected profit rate example. Figure a) shows a simplified example network with three nodes and the demand for different trips. The cost and time of each leg is assumed to be one, that is, $c_{ij} = t_{ij} = 1$ for all $i, j \in \{1, 2, 3\}$. Figure b) shows two alternative route options for a vehicle located at node 1. Option A consists of two short routes $(1, 2)$ and $(2, 1)$, whereas option B consists of a single route $(1, 2, 3)$. By choosing option B, the vehicle produces the trips $(1, 2), (1, 2, 3), (2, 3)$, thus serving three customers in two time units. By choosing option A, the vehicle produces the trips $(1, 2), (2, 1)$ and serves four customers in the same amount of time.

Given that a single vehicle arrives at state $s \in \mathcal{S}_r^p$ that produces a trip $r \in \mathcal{R}_{ij}$ from i to j, let us study the expected number $E[Q_r^1]$ of DRT customers traveling from i to j that enter the vehicle. We determine $E[Q_r^1]$ by dividing the demand $E[Q_r^{\mathrm{DRT}}]$ for the trip r by the total production rate T_r^p of the trip, that is,

$$E[Q_r^1] = \frac{E[Q_r^{\mathrm{DRT}}]}{T_r^p}. \tag{16}$$

This equation is justified by the assumption that the customers that choose the trip r are divided equally among vehicles that produce the trip r. For instance, if during one hour there are three vehicles that produce a trip r and the demand for the trip r is six customers/hour, the expected number of DRT customers for a single vehicle equals $E[Q_r^1] = \frac{6 \text{ customers/h}}{3 \text{ vehicles/h}} = 2$ customers/vehicle.

Let us then consider the expected profit rate for a vehicle arriving at state $s = (i_1, \ldots, i_m)$. For analysis, we calculate the expected profit rate by assuming that the route corresponding to s is not extended during its execution. That is, the vehicle executes the route by successively transferring to states $(i_1, \ldots, i_m), (i_2, \ldots, i_m), \ldots, (i_{m-1}, i_m)$. In this case, the set of trips produced during the execution equals $\mathcal{R}_s^{\text{route}}$ defined by Equation (10), that is, the set of all subsequences of s (see Figure 1).

The expected number of customers that enter the vehicle during the execution of the route is given by $\sum_{r\in\mathcal{R}_s^{\text{route}}} E[Q_r^1]$ and the corresponding expected revenue is given by

$$\sum_{r\in\mathcal{R}_s^{\text{route}}} E[Q_r^1]p_r,$$

where p_r is the ticket price for the trip r.

The corresponding *expected revenue rate* $E[R(s)]$ is given by dividing the expected revenue by the total time (7) needed to execute the route, that is,

$$E[R(s)] = \frac{\sum_{r\in\mathcal{R}_s^{\text{route}}} E[Q_r^1]p_r,}{t_s^{\text{route}}}. \tag{17}$$

The corresponding *cost rate* $C(s)$ of a route s is given by

$$C(s) = \frac{c_s^{\text{route}}}{t_s^{\text{route}}}, \tag{18}$$

where c_s^{route} denotes the total cost of the route s defined by (8).

By combining equations (17) and (18), the expected profit rate at state $s \in \mathcal{S}$ is given by

$$U(s) = E[R(s)] - C(s). \tag{19}$$

By assuming a logit model similar to the one in Equation (4), the probability that a vehicle at state s transfers to state $s' \in \mathcal{S}$ is given by

$$P_{s,s'} = \begin{cases} \frac{\exp(\theta^d U(s'))}{\sum_{s''\in\mathcal{S}_s}\exp(\theta^d U_{s''})}, & \text{if } s' \in \mathcal{S}_s \quad (\mathcal{S}_s = \text{ successor set of } s), \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

where θ^d is a nonnegative parameter reflecting the uncertainty on demand and DRT services from the perspective of drivers.

Finally, we state that the arrival rate of vehicles at state $s \in \mathcal{S}$ satisfies

$$T_s = \sum_{s'\in\mathcal{S}} T_{s'} P_{s',s}. \tag{21}$$

By using matrix notation, we can write equation (21) in the form

$$T = T \cdot P, \tag{22}$$

where T is a $1 \times |\mathcal{S}|$ row vector containing the arrival rates T_s of vehicles at different states and P is a $|\mathcal{S}| \times |\mathcal{S}|$ matrix consisting of the transition probabilities (20) between states. Note that the arrival rate vector T satisfying Equation (22) is an eigenvector of the matrix P corresponding to eigenvalue 1. However, the calculation of T is not straightforward since the matrix P depends on the entries of T, see Equation (20). The calculation of T is described in Section 3.1.

3. Network equilibrium

The drivers attempt to maximize profit rate by transporting as many customers per unit time as possible. Thus, we expect that the drivers prefer detours instead of direct routes in order to serve more customers, see Equation (17). In some cases, however, producing only direct trips may be more profitable, as in Figure 4.

We also expect that the customers prefer direct trips instead of detours, see Equation (3). However, if many vehicles produce non-direct trips, the travel times in non-direct trips may be smaller than in direct trips due to small waiting times.

The movement of vehicles is described by means of the *arrival rates* T_s of vehicles at different states $s \in \mathcal{S}$ and the movement of customers is described by means of the demands Q_r^{DRT} for different DRT trips $r \in \mathcal{R}$.

A *network equilibrium* denotes the following combination of arrival rates T_s^* at states $s \in \mathcal{S}$ and demands $Q_r^{\text{DRT}*}$ for trips $r \in \mathcal{R}$:

- The set of arrival rates $\{T_{s_1}^*, \ldots, T_{s_{|\mathcal{S}|}}^*\}$ of vehicles at states $s_i \in \mathcal{S}$ generates the set of demands $\{Q_{r_1}^{\text{DRT}*}, \ldots, Q_{r_{|\mathcal{R}|}}^{\text{DRT}*}\}$ for trips $r_i \in \mathcal{R}$.
- The set of demands $\{Q_{r_1}^{\text{DRT}*}, \ldots, Q_{r_{|\mathcal{R}|}}^{\text{DRT}*}\}$ for trips $r_i \in \mathcal{R}$ generates the set of arrival rates $\{T_{s_1}^*, \ldots, T_{s_{|\mathcal{S}|}}^*\}$ of vehicles at states $s_i \in \mathcal{S}$.

Formally, we define a network equilibrium T^* as a $1 \times |\mathcal{S}|$ vector, containing the arrival rates T_s of vehicles at different states $s \in \mathcal{S}$, that satisfies Equations (22) and (13). This definition is justified by the fact that any arrival rate vector T generates a unique demand vector Q containing the demands for trips $r \in \mathcal{R}$ (see Equations (15) and (5)) and the demand vector Q produces unique transition probabilities (20). Thus, if an equilibrium exists, it is uniquely defined by the arrival rate vector T^*. The following theorem is related to the existence of such an equilibrium.

Theorem 1. *For any finite transportation network I, there exists a network equilibrium T^*.*

Proof. Brouwer's fixed point theorem states that every continuous function f from a convex compact subset of a Euclidean space to itself has a fixed point, that is, a point x_0 for which $f(x_0) = x_0$.

Let $\mathcal{T}$ denote the set of arrival rate vectors that satisfy Equation (13). Clearly, since $\mathcal{T}$ is a plane in $R^{|\mathcal{S}|}$ restricted by $T_s \geq 0$ for $s \in \mathcal{S}$, we see that $\mathcal{T}$ is convex. In addition, since $\mathcal{T}$ is closed and bounded, we know by the Heine-Borel theorem that $\mathcal{T}$ is compact.

Equations (1 - 5) and (15) define a continuous function $g : \mathcal{T} \to Q^{\text{DRT}}$, where Q^{DRT} is the set of demand vectors Q^{DRT} that contain the demands for trips $r \in \mathcal{R}$. Equations (19 - 22) define a continuous function $h : Q^{\text{DRT}} \to \mathcal{T}$. Thus, since $f = g \circ h$ is a continuous function from $\mathcal{T}$ to $\mathcal{T}$, there exists an arrival rate vector T^* for which $f(T^*) = T^*$. The corresponding equilibrium demand is given by $Q^{\text{DRT}*} = g(T^*)$ $\square$.

3.1. Algorithm

If the transition probability matrix P were independent of the arrival rate vector T, the solution to Equation (22) would be achieved by means of a straightforward power method by successively computing $T^{k+1} = T^k \cdot P$ for $k = 1, \ldots$ and normalizing T^{k+1} after each multiplication, as described in (Meyer, 2000). However, since the transition probabilities (20) depend on the arrival rates, the matrix P needs to be updated after each step of the power method. In other words, the idea for finding a steady-state equilibrium is to solve both the customer choice subproblem and the vehicle movement subproblem iteratively until a convergence criterion is met, similarly as in (Yang et al., 2010).

The outline of the procedure is presented in Algorithm 1.

Algorithm 1: Equilibration of DRT.

Data: Set I of nodes, direct ride time t_{ij}, cost c_{ij}, demand Q_{ij}, ticket price p_r, cost of virtual mode $\bar{g}_{ij}$ for all pairs $i, j \in I$ of nodes, number of vehicles N, customers' monetary value for unit time β and the uncertainty parameters θ, θ^d.

Result: Equilibrium arrival rates T_s of vehicles at states $s \in \mathcal{S}$.

Initialize the arrival rates T_s of vehicles by dividing the vehicles evenly among all states $s \in \mathcal{S}$, $T_s = \frac{N}{d_s|\mathcal{S}|}$;

repeat

  Customer demand updating: Calculate $E[Q_r^{\mathrm{DRT}}]$ for all $r \in \mathcal{R}$ given by equation (6) ;

  Vehicle movements: Solve T_s for all $s \in \mathcal{S}$ by equation (21) ;

until *convergence* ;

3.2. A three node example

Let us demonstrate the calculation of the network equilibrium in a simple case in which the network consists of three nodes denoted by 1, 2, 3. The number of vehicles is $N = 30$. The direct ride times (in minutes) are defined by $t_{12} = t_{21} = 3$, $t_{13} = t_{31} = 4$ and $t_{23} = t_{32} = 5$. The distances between the nodes (in kilometers) are equal to the direct ride times, as shown in Figure 5.

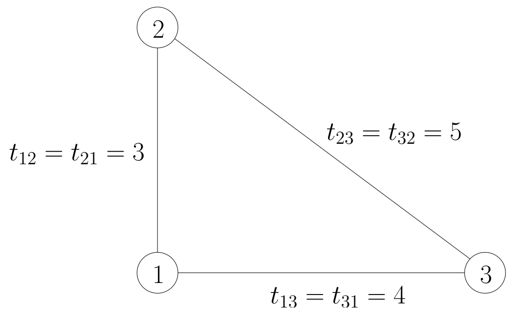


Figure 5: A three node example network. The distances between the nodes (in kilometers) are equal to the direct ride times t_{ij}.

The cost of leg (i, j) is defined by $c_{ij} = \frac{1}{2}t_{ij}$. The customers' monetary value of unit time is set to $\beta = 1$ and the values of the uncertainty parameters are set to $\theta = \theta^d = 1$. For simplicity, the ticket price of all trips from i to j is $p_{ij} = 0.3t_{ij}$, that is, the price per kilometer (in EUR) is 0.30. The subjective cost of the virtual mode is $\bar{g}_{ij} = 2t_{ij}$. Finally, the total demand is $Q_{ij} = 3$ for all $i, j \in I$. The parameters values are presented in Table 1.

Table 1: Parameter values of the three node example.

Leg ij	t_{ij}	c_{ij}	p_{ij}	$\bar{g}_{ij}$	Q_{ij}
12 and 21	3	1.5	0.90	6	3
13 and 31	4	2	1.20	8	3
23 and 32	5	2.5	1.50	10	3

The number of states equals $|\mathcal{S}| = \sum_{l=2}^{3} \binom{3}{l} l! = 3 \cdot 2 + 1 \cdot 6 = 12$ and the set of states $\mathcal{S}$ is given by

$$\mathcal{S} = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

The set of trips is equal to the set of states, that is, $\mathcal{R} = \mathcal{S}$. The successor sets $\mathcal{S}_s$ of the states are given by

$$\begin{aligned}
\mathcal{S}_{(1,2)} = \mathcal{S}_{(3,2)} &= \{(2,1),(2,3),(2,1,3),(2,3,1)\} \\
\mathcal{S}_{(1,3)} = \mathcal{S}_{(2,3)} &= \{(3,1),(3,2),(3,1,2),(3,2,1)\} \\
\mathcal{S}_{(2,1)} = \mathcal{S}_{(3,1)} &= \{(1,2),(1,3),(1,2,3),(1,3,2)\} \\
\mathcal{S}_{(1,2,3)} = \{(2,3),(2,3,1)\} &\quad \mathcal{S}_{(1,3,2)} = \{(3,2),(3,2,1)\} \\
\mathcal{S}_{(2,1,3)} = \{(1,3),(1,3,2)\} &\quad \mathcal{S}_{(2,3,1)} = \{(3,1),(3,1,2)\} \\
\mathcal{S}_{(3,1,2)} = \{(1,2),(1,2,3)\} &\quad \mathcal{S}_{(3,2,1)} = \{(2,1),(2,1,3)\}.
\end{aligned}$$

For example, a transition from state $(1,3,2)$ to state $(2,3)$ is not possible since $(2,3)$ is not included in the successor set of $(1,3,2)$.

The set of states $\mathcal{S}_r^p$ that produce trip r are given by

$$\begin{array}{lll}
\mathcal{S}^p_{(1,2)} = \{(1,2),(1,2,3)\} & \mathcal{S}^p_{(1,3)} = \{(1,3),(1,3,2)\} & \mathcal{S}^p_{(2,1)} = \{(2,1),(2,1,3)\} \\
\mathcal{S}^p_{(2,3)} = \{(2,3),(2,3,1)\} & \mathcal{S}^p_{(3,1)} = \{(3,1),(3,1,2)\} & \mathcal{S}^p_{(3,2)} = \{(3,2),(3,2,1)\} \\
\mathcal{S}^p_{(1,2,3)} = \{(1,2,3)\} & \mathcal{S}^p_{(1,3,2)} = \{(1,3,2)\} & \mathcal{S}^p_{(2,1,3)} = \{(2,1,3)\} \\
\mathcal{S}^p_{(2,3,1)} = \{(2,3,1)\} & \mathcal{S}^p_{(3,1,2)} = \{(3,1,2)\} & \mathcal{S}^p_{(3,2,1)} = \{(3,2,1)\}.
\end{array}$$

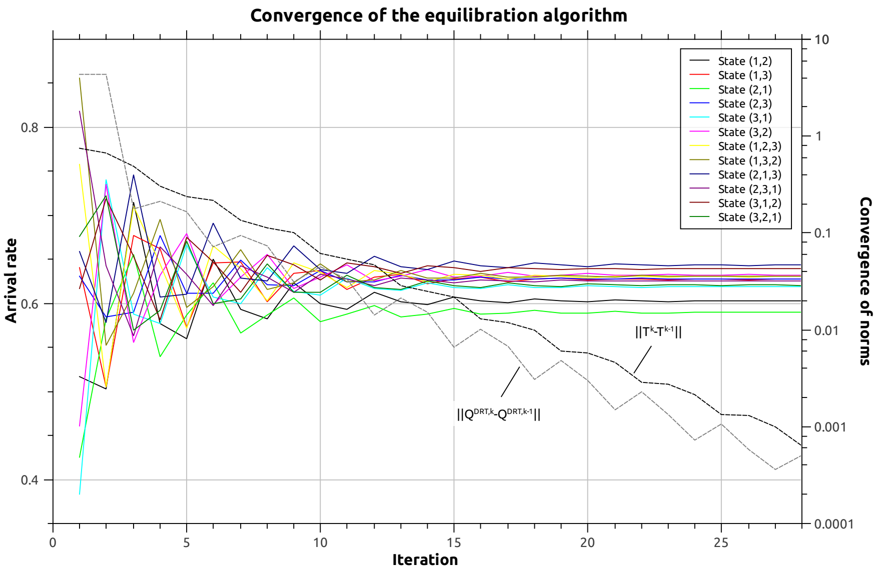
For example, trip $(2,3)$ is produced by states $(2,3),(2,3,1)$ whereas each trip consisting of three nodes is produced by a single state equal to the trip.

Assuming that a vehicle arriving at state $s \in \mathcal{S}$ executes the route corresponding to s without route extensions, the set of trips $\mathcal{R}_s^{\text{route}}$ produced during the execution of routes corresponding to states s are given by

$$\begin{array}{lll}
\mathcal{R}^{\text{route}}_{(1,2)} = \{(1,2)\} & \mathcal{R}^{\text{route}}_{(1,3)} = \{(1,3)\} & \mathcal{R}^{\text{route}}_{(2,1)} = \{(2,1)\} \\
\mathcal{R}^{\text{route}}_{(2,3)} = \{(2,3)\} & \mathcal{R}^{\text{route}}_{(3,1)} = \{(3,1)\} & \mathcal{R}^{\text{route}}_{(3,2)} = \{(3,2)\} \\
\mathcal{R}^{\text{route}}_{(1,2,3)} = \{(1,2),(1,2,3),(2,3)\} & \mathcal{R}^{\text{route}}_{(1,3,2)} = \{(1,3),(1,3,2),(3,2)\} & \mathcal{R}^{\text{route}}_{(2,1,3)} = \{(2,1),(2,1,3),(1,3)\} \\
\mathcal{R}^{\text{route}}_{(2,3,1)} = \{(2,3),(2,3,1),(3,1)\} & \mathcal{R}^{\text{route}}_{(3,1,2)} = \{(3,1),(3,1,2),(1,2)\} & \mathcal{R}^{\text{route}}_{(3,2,1)} = \{(3,2),(3,2,1),(2,1)\}.
\end{array}$$

Note that the routes consisting of three nodes produce more trips than the short routes consisting of two nodes. However, in some cases it may be profitable to execute short routes instead of long ones, as in the example presented in Figure 4.

With the above sets $\mathcal{S}_s, S_r^p$ and $\mathcal{R}_s^{\text{route}}$, we calculated the network equilibrium by using Algorithm 1. The solid lines in Figure 6 show the arrival rates T_s of vehicles at different states $s \in \mathcal{S}$ after each step of the algorithm. The black dashed line shows on a logarithmic scale the convergence of the norm $\|T^k - T^{k-1}\|$ of the difference between two successive arrival rate vectors T^k and T^{k-1}. The grey dashed line shows the corresponding convergence of the demand vector Q^{DRT}. After 28 iterations, the norm $\|T^k - T^{k-1}\|$ was less than 0.001. The equilibration was continued until the norm was less than 0.00001.

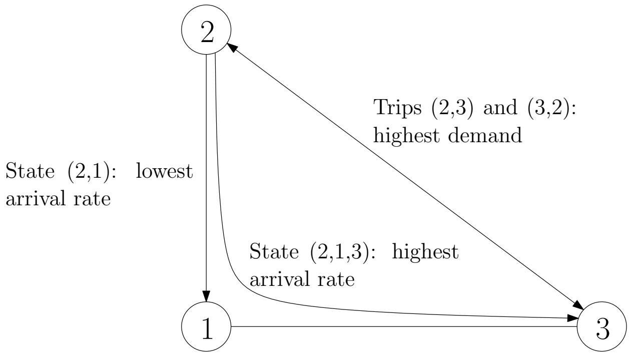


Arrival rates at states in the network equilibrium T^*												
State s	(1,2)	(1,3)	(2,1)	(2,3)	(3,1)	(3,2)	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)
Arrival rate T_s	0.603	0.627	0.590	0.628	0.619	0.632	0.631	0.631	0.644	0.626	0.639	0.621

Figure 6: Convergence of Algorithm 1 in the three node example. The solid lines in show the arrival rates T_s of vehicles at different states $s \in \mathcal{S}$ after each step of the algorithm. The black and grey dashed lines show on a logarithmic scale the convergence of the arrival rate vector T and the demand vector Q^{DRT}, respectively. After 28 iterations, the norm $\|T^k - T^{k-1}\|$ was less than 0.001. The equilibration was continued until the norm was less than 0.00001. The network equilibrium is defined by the arrival rates at different states shown in the table below the figure.

By looking at the solid lines, we see the oscillatory nature of the arrival rates: When the arrival rate of vehicles in a specific state s increases during the equilibration, the number of customers available for a single vehicle in that state decreases. This causes the drivers to choose other states instead of s. When the arrival rate at state s decreases, it becomes more profitable for individual vehicles and results in drivers choosing state s more often. In the network equilibrium, the arrival rate is highest at state $(2, 1, 3)$ and lowest at state $(2, 1)$.

Referring to the dashed lines, which are roughly straight lines on a logarithmic scale, we see that the norms of the arrival rate and demand vectors converge exponentially with respect to the



Trip	(1,2)	(1,3)	(2,1)	(2,3)	(3,1)	(3,2)	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)
Demand	2.350	2.625	2.350	2.653	2.625	2.655	0.022	0.003	0.169	0.003	0.167	0.021
Travel time ratio	1.270	1.199	1.270	1.160	1.199	1.160	2.396	3.528	1.711	3.533	1.713	2.403
Waiting time	0.811	0.795	0.811	0.798	0.795	0.798	1.586	1.585	1.554	1.598	1.564	1.611
Subjective price	4.711	5.995	4.711	7.298	5.995	7.298	10.786	11.485	10.054	11.498	10.064	10.811
- virtual mode	6	8	6	10	8	10	8	6	10	6	10	8

Figure 7: Equilibrium statistics. The table shows the demand, travel time ratio, waiting time, subjective price and the subjective price of the virtual mode in the network equilibrium for all trips in the three-node example.

number of iterations. Decreasing the difference $||T^k - T^{k-1}||$ by a factor of $F \in \mathbb{R}$ requires $G \in \mathbb{N}$ iterations. That is, $||T^k - T^{k-1}|| = \frac{1}{F^K}||T^{k-GK} - T^{k-1-GK}||$ for $K \in \mathbb{N}$. By looking at the slope of the dark dashed line, we see that the difference $||T^k - T^{k-1}||$ decreases by a factor of 10 in approximately 10 iterations, that is, $||T^k - T^{k-1}|| = \frac{1}{10^K}||T^{k-10K} - T^{k-1-10K}||$ for $K \in \mathbb{N}$.

The trip statistics of the example are presented in the table in Figure 7. Generally, the demand for detours is low due to the fact that the ride times are long compared to the direct ride times. For example, the ride time in trips $(1,3,2), (2,3,1)$ is $4+5 = 9$ and the direct ride time is $t_{12} = t_{21} = 3$. Thus, the subjective price is at least $9 + 0.30 \cdot 3 = 9.90$. Since the subjective price of the virtual mode is $\bar{g}_{12} = \bar{g}_{21} = 6$, the demand for these DRT trips is low. The most popular detours are $(2,1,3)$ and $(3,1,2)$, since the ratio of ride time ($= 7$) to direct ride time $t_{13} = t_{31} = 5$ is relatively small. This can be seen also by looking at the arrival rates at states $(2,1,3), (3,1,2)$ in Figure 6, which are the highest among all states.

4. Long-run market equilibria

In a competitive demand responsive market with no entry limits, we expect that the number of vehicles increases as long as the profit rate of vehicles is positive. Regulating the number of vehicles and ticket price could improve the service from the customers' point of view as well as from the perspective of drivers. In the following, we study different characteristics of the competitive DRT service as a function of the number of vehicles N and average price per kilometer p. We identify four *long-run market equilibria* (24), (26), (28), (27), defined by combinations of N and p.

4.1. Free entry equilibrium

In the absence of entry limits, new vehicles arrive as long as the total profit rate $\Pi(N, p)$ of vehicles is positive[6]. Thus, the feasible set for the free entry equilibrium is given by

$$\mathcal{F} = \{(N, p) \mid \Pi(N, p) \geq 0\}. \tag{23}$$

In the free entry market, we assume that the drivers may agree to modify the ticket price of the service. Thus, when the number of vehicles increases and the total profit rate approaches zero with a given price, the price is modified in order to keep the service profitable for the drivers. These type of modifications result in ticket prices that attract additional vehicles to the market. Thus, we define the free entry equilibrium by means of the condition

$$(N^{\mathrm{f}}, p^{\mathrm{f}}) \in \{(N, p) \in \mathcal{F} \mid N \geq N' \ \text{ for all } (N', p) \in \mathcal{F}\}. \tag{24}$$

That is, both the number of vehicles and price per kilometer are chosen in a way that the number of vehicles is maximized.

4.2. Customer welfare equilibrium

Public authorities might consider regulating the DRT service in a way that it would improve the service provided to customers as much as possible, compared to existing transport services. We define the *customer welfare effect* as the difference between the subjective price of the virtual mode and the subjective price of DRT. Formally, the customer welfare effect is given by

$$W(N, p) = \sum_{(i,j)\in I\times I} \left(Q_{ij} - Q_{ij}^{DRT}\right)\bar{g}_{ij} + \sum_{(i,j)\in I\times I}\sum_{r\in\mathcal{R}_{ij}} Q_r^{DRT} g_r = \sum_{(i,j)\in I\times I}\sum_{r\in\mathcal{R}_{ij}} Q_r^{DRT}(\bar{g}_{ij} - g_r), \tag{25}$$

where Q_{ij}^{DRT} is the total demand for DRT trips from i to j defined by Equation (6), $Q_{ij} - Q_{ij}^{DRT}$ is the demand for the virtual mode from i to j, $\mathcal{R}_{ij}$ is the set of DRT trips from i to j, g_r is the subjective price of trip r and $\bar{g}_{ij}$ is the subjective price of the virtual mode from i to j.

Similarly as in the free entry equilibrium, we require that the total profit rate of vehicles is positive. Thus, the customer welfare equilibrium is defined by

$$(N^{\mathrm{W}}, p^{\mathrm{W}}) = \arg\max_{(N,p)\in\mathcal{F}} W(N, p). \tag{26}$$

In this case, the number of vehicles and price are regulated in a way that the customer welfare effect is maximized.

4.3. Maximum profit rate

Maximizing the *total profit rate* of vehicles results in the equilibrium

$$(N^{\mathrm{tot}}, p^{\mathrm{tot}}) = \arg\max_{(N,p)\in\mathcal{F}} \Pi(N, p). \tag{27}$$

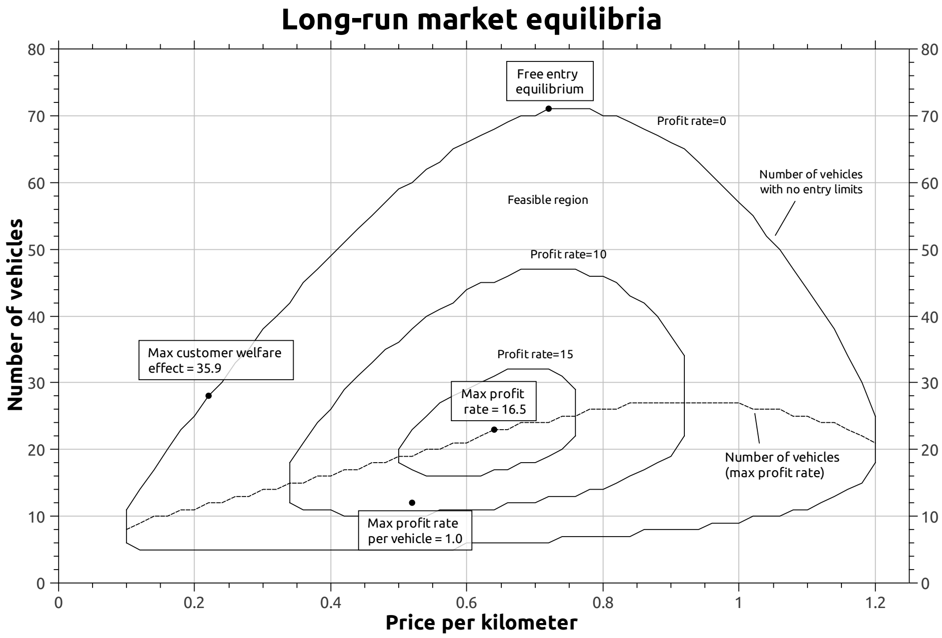
However, from the perspective of individual drivers, the optimal combination of the number of vehicles and price is such that the *profit rate per vehicle* is maximized, that is

$$(N^1, p^1) = \arg\max_{(N,p)\in\mathcal{F}} \frac{\Pi(N, p)}{N}. \tag{28}$$

[6] We assume that the profits are divided equally among vehicles on average.

4.4. A three node example

We determined the long-run market equilibria described above for the three node example case introduced in Section 3.2. The network equilibrium was calculated for different numbers of vehicles N and prices per kilometer p by means of Algorithm 1. For each combination of N and p, we calculated the total profit rate, profit rate per vehicle and customer welfare effect. The results are shown in Figure 8.



Equilibrium	Number of vehicles	price per kilometer	Demand for DRT (customers/min)	Tot. profit rate / per vehicle (EUR/min)	Welfare effect (EUR/min)	Travel time ratio	Average occupancy
Max. customer welfare effect	28	0.22	16.1	0.13 / 0.005	35.9	1.24	2.4
Maximum total profit rate	23	0.64	10.8	16.5 / 0.7	3.9	1.28	1.9
Maximum profit rate per vehicle	12	0.52	8.4	12.6 / 1.0	0.2	1.50	3.0
Free entry equilibrium	71	0.72	12.4	0.1 / 0.001	8.7	1.11	0.7

Figure 8: Long-run market equilibria for the three node example. The first column shows the four studied equilibrium points. The second and third columns show the number of vehicles and price per kilometer with which the equilibrium is achieved. The remainder of the columns show the total demand for DRT, total profit rate and profit rate per vehicle, customer welfare effect, the average ratio of travel time to direct travel time and the average number of customers in a single vehicle.

The solid curves represent contour lines in which the total profit rate $\Pi(N, p)$ of vehicles is equal $(= 0, 10, 20)$. In particular, the outermost contour line corresponding to $\Pi(N, p) = 0$ encloses the *feasible region*, that is, the area in which the service is profitable for drivers. The dashed curve shows the number of vehicles for different prices for which the total profit rate is maximized. The four points represent the long-run market equilibria defined in the previous section.

The table in Figure 8 shows the demand for DRT, total profit rate, profit rate per vehicle, customer welfare effect, the average ratio of travel time to direct travel time and the average number of customers in a single vehicle in the four equilibrium points.

By looking at the figure, we note that there is a significant difference between the four equilibria. In the free entry equilibrium, the number of vehicles is significantly higher than in the other points. The average travel time ratio and the average occupancy are extremely low. This indicates that with no regulation, the DRT service would approach a taxi-type service, in which all customers are transported privately and all trips are direct trips.

The difference in price between the free entry equilibrium and the point in which the total profit rate is maximized is small. In addition, the number of served customers (demand for DRT) is only slightly smaller in the maximum profit rate case compared to the free entry case. However, maximizing the profit rate of vehicles would decrease the customer welfare effect and decrease the average level of service, as can be seen by looking at the average travel time ratio and average occupancy.

The profit rate per vehicle is maximized with an extremely small number of vehicles. In this case, only a small number of customers could be served and the level of service would be poor.

The customer welfare effect is maximized by using a significantly lower ticket price than would be optimal from the perspective of drivers. This is mainly due to the fact that the low price results in a high demand for DRT. Moreover, we note that the customer welfare equilibrium is achieved by using price regulation exclusively. That is, the results suggest that the optimal solution from the customers' point of view would be to regulate price and allow free entry.

5. Conclusions

In this work, we present a model for a competitive demand-responsive transport (DRT) market. In our model, customers seek trips to minimize travel times and drivers attempt to maximize profit rate by dynamically choosing routes that serve a large number of customers. We define the network equilibrium as a state in which the demand for trips matches the supply of trips: the choices of drivers do not change if the demand remains constant and the choices of customers do not change if the distribution of the vehicles at different routes remains constant. We show that such an equilibrium always exists in finite transportation networks and provide an algorithm for determining the equilibrium. In addition, we study long-run market equilibria for competitive DRT by varying the number of vehicles and price per kilometer.

Numerical examples conducted on a simple network suggest that with no entry limits or price regulation, the DRT service approaches a taxi-type service providing private transport and direct trips. The optimal equilibrium from the customers' perspective is achieved by regulating price and allowing free entry. This type of regulation could substantially increase the service rate.

The proposed model can be used to simulate the operations of a DRT service in a wide range of scenarios, from paratransit services for the elderly and disabled to large-scale urban DRT services designed to compete with private car traffic. Such calculations can provide valuable information to public authorities and planners of transportation services, regarding, for example, regulation and the magnitude of investments.

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References

Arnott, R., 1996. Taxi travel should be subsidized. Journal of Urban Economics 40 (3), 316–333.

Cairns, R., Liston-Heyes, C., 1996. Competition and regulation in the taxi industry. Journal of Public Economics 59 (1), 1–15.

Cordeau, J.-F., Laporte, G., Potvin, J.-Y., Savelsbergh, M., 2007a. Transportation on demand. In: Transportation. Amsterdam: North-Holland, pp. 429–466.

Diana, M., Quadrifoglio, L., Pronelloa, C., 2007. Emissions of demand responsive services as an alternative to conventional transit systems. Transportation Research Part D: Transport and Environment 12 (3), 183–188.

Fernandez, J., De Cea, J., Briones, J., 2006. A diagrammatic analysis of the market for cruising taxis. Transportation Research Part E 42 (6), 498–526.

Flores-Guri, D., 2003. An economic analysis of regulated taxicab markets. Review of Industrial Organization 23 (3-4), 255–266.

Hackner, J., Nyberg, S., 1995. Deregulating taxi services: a word of caution. Journal of Transport Economics and Policy 29 (2), 195–207.

Helsinki Region Traffic, Oct. 2010. Jouko neighbourhood routes and service routes.
URL `http://www.hsl.fi/EN/passengersguide/ServiceroutesandJoukoroutes/Pages/default.aspx`

Jokinen, J.-P., Sihvola, T., Hyytiä, E., Sulonen, R., June-July 2011. Why urban mass demand responsive transport? In: IEEE Forum on Integrated and Sustainable Transportation Systems (FISTS). Vienna, Austria, to appear.

Lagos, R., 2003. An analysis of the market for taxicab rides in new york city. International Economic Review 44 (2), 423–434.

Mageean, J., Nelson, J. D., 2003. The evaluation of demand responsive transport services in europe. Journal of Transport Geography 11 (4), 255–270.

Matsushima, K., Kobayashi, K., 2006. Endogenous market formation with matching externality: an implication for taxi spot markets. In: Structural Change in Transportation and Communications in the Knowledge Economy. Edward Elgar, pp. 313–336.

Moore, A., Balaker, T., 2006. Do economists reach a conclusion on taxi deregulation? Econ Journal Watch 3 (1), 109–132.

Mulley, C., Nelson, J. D., 2009. Flexible transport services: A new market opportunity for public transport. Research in Transportation Economics 25 (1), 39–45.

Reiser, M., Lavenberg, S. S., 1980. Mean value analysis of closed multi-chain queueing networks. JACM 27.

Ross, S. M., 1995. Stochastic Processes. Wiley. ISBN 978-0-471-12062-9.

Sihvola, T., Häme, L., Sulonen, R., 2010. Passenger Pooling and Trip Combining Potential of High-Density Demand Responsive Transport. Presented at the Annual Meeting of the Transportation Research Board, Washington D.C.

Small, K. A., Verhoef, E. T., 2007. The Economics of Urban Transportation. Routledge, 2 Park Square, Milton Park, Abingdon, OX14 4RN.

Wong, K., Wong, S., Bell, M., Yang, H., 2005. Modeling the bilateral micro-searching behavior for urban taxi services using the absorbing markov chain approach. Journal of Advanced Transportation 39 (1), 81–104.

Yang, H., Leung, C. W. Y., Wong, S. C., Bell, M. G. H., 2010. Equilibria of bilateral taxi-customer searching and meeting on networks. Transportation Research Part B 44, 1067–1083.

Yang, H., Wong, S., Wong, K., 2002. Demand-supply equilibrium of taxi services in a network under competition and regulation. Transportation Research Part B 36, 799–819.

Yang, H., Ye, M., Tang, W. H.-C., Wong, S. C., 2005. A multiperiod dynamic model of taxi services with endogenous service intensity. Operations Research 53 (3), 501–515.