Simulation, Semester II 2023-2024

ST3247: Tutorial 1[1]

!! For your convenience, the first two problems are theoretical problems and the last two are coding problems.

1. In the sequential inversion algorithm, to generate Y with $P(Y = k) = p_k$ for $k = 0, 1, 2, \cdots$, we follow the pseudo-code:

   Step 1. Generate $X \sim Unif(0, 1)$

   Step 2. Generate Y by comparing X with the CDF of Y, in the way that

   $$Y = \min\{y : x \leq \sum_{k=0}^{y} p_k\}.$$

   Step 3. Return Y.

   Now the question is, will the generated Y follow the distribution of interest? To answer it, please solve the following questions:

   (a) We start with a concrete example. Consider the distribution that

   $$P(Y = 0) = 0.2,\ P(Y = 1) = 0.3,\ P(Y = 2) = 0.4,\ P(Y = 4) = 0.1.$$

   We apply the sequential inversion algorithm.

   (i) Given X, how do we decide Y? Please specify the numbers we are comparing in each iteration.

   (ii) What is the probability that $Y = 2$?

   (b) Now we consider the general case

   (i) According to the formula $Y = \min\{y : x \leq \sum_{k=0}^{y} p_k\}$, figure out the interval of X on which we will get $Y = k$.

   (ii) By (i), find the probability of $Y = k$.

2. Consider the distribution that

   $$P(Y = k) = \frac{1}{k(k+1)}, k = 1, 2, \cdots$$

   We want to simulate Y by the inversion by truncation method.

   (a) For any $i = 1, 2, 3, \cdots$, find $F(i) = P(Y \leq i)$.

(b) Find out G on $(0, \infty)$, so that $G(i+1) = F(i)$, and G is monotone increasing.

(c) Find out $G^{-1}(x)$.

(d) According to the previous steps, write out the pseudo-code.

3. For the following integrals, please write out the pseudo code to calculate them with the Monte Carlo integration

   (a) $\int_{-2}^{2} e^{x+x^2} dx$;

   (b) $\int_{-\infty}^{\infty} e^{-x^2} dx$;

   (c) $\int_0^1 \int_0^1 e^{(x+y)^2} dy dx$;

   (d) $\int_0^{\infty} \int_0^{x} e^{-(x+y)} dy dx$.

4. Use simulation to approximate $Cov(U, e^U)$, where $U \sim Unif(0, 1)$. Please give the mean and standard deviation of your estimate.