6

Additional Topics in Scalar and Vector Calculus

6.1 Objectives

This chapter presents additional topics in basic calculus that go beyond that of the introductory material in Chapter 5. These topics include more advanced uses of derivatives like partial derivatives and higher order partial derivatives; root finding (locating an important point along some function of interest); analyzing function minima, maxima, and inflection points (points where derivatives change); integrals on functions of multiple variables; and, finally, the idea of an abstract series. In general, the material extends that of the last chapter and demonstrates some applications in the social sciences. A key distinction made in this chapter is the manner in which functions of multiple variables are handled with different operations.

6.2 Partial Derivatives

A **partial derivative** is a regular derivative, just as we have already studied, except that the operation is performed on a function of multiple variables where the derivative is taken only with respect to one of them and the others are treated as constants. We start with the generic function

$$u = f(x, y, z)$$

specified for the three variables, x, y, and z. We can now define the three partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(x, y, z) = g_1(x, y, z)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} f(x, y, z) = g_2(x, y, z)$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z} f(x, y, z) = g_3(x, y, z),$$

where the resulting functions are labeled as $g_i(),\ i = 1, 2, 3$, to indicate that not only are they different from the original function, they are also distinct from each other because different variables are manipulated with the derivative processes. As an example, consider the function

$$f(x, y, z) = x \exp(y) \log(z),$$

which has the three partial derivatives

$$\frac{\partial}{\partial x} f(x, y, z) = \exp(y) \log(z) \frac{\partial}{\partial x} x = \exp(y) \log(z)$$

$$\frac{\partial}{\partial y} f(x, y, z) = x \log(z) \frac{\partial}{\partial y} \exp(y) = x \exp(y) \log(z)$$

$$\frac{\partial}{\partial z} f(x, y, z) = x \exp(y) \frac{\partial}{\partial z} \log(z) = x \exp(y) \frac{1}{z}.$$

Notice that the variables that are not part of the derivation process are simply treated as constants in these operations.

We can also evaluate a more complex function with additional variables:

$$f(u_1, u_2, u_3, u_4, u_5) = u_1^{u_2 u_3} \sin\left(\frac{u_1}{2}\right) \log\left(\frac{u_4}{u_5}\right) u_3^2,$$

which produces

$$\frac{\partial}{\partial u_1} f(u_1, u_2, u_3, u_4, u_5) = \left[u_1^{u_2 u_3} \log\left(\frac{u_4}{u_5}\right) u_3^2 \right] \times \left[\frac{u_2 u_3}{u_1} \sin\left(\frac{u_1}{2}\right) + \frac{1}{2} \cos\left(\frac{u_1}{2}\right) \right]$$

$$\frac{\partial}{\partial u_2} f(u_1, u_2, u_3, u_4, u_5) = \log(u_1) u_1^{u_2 u_3} \sin\left(\frac{u_1}{2}\right) \log\left(\frac{u_4}{u_5}\right) u_3^3$$

$$\frac{\partial}{\partial u_3} f(u_1, u_2, u_3, u_4, u_5) = u_1^{u_2 u_3} \sin\left(\frac{u_1}{2}\right) \log\left(\frac{u_4}{u_5}\right) u_3 \left[u_2 \log(u_1) u_3 + 2 \right]$$

$$\frac{\partial}{\partial u_4} f(u_1, u_2, u_3, u_4, u_5) = u_1^{u_2 u_3} \sin\left(\frac{u_1}{2}\right) \frac{u_3^2}{u_4}$$

$$\frac{\partial}{\partial u_5} f(u_1, u_2, u_3, u_4, u_5) = -u_1^{u_2 u_3} \sin\left(\frac{u_1}{2}\right) \frac{u_3^2}{u_5}.$$

★ **Example 6.1: The Relationship Between Age and Earnings**. Rees and Schultz (1970) analyzed the relationship between workers' ages and their earnings with the idea that earnings increase early in a worker's career but tend to decrease toward retirement. Thus the relationship is parabolic in nature according to their theory. Looking at maintenance electricians, they posited an additive relationship that affects income (in units of \$1,000) according to

$$\begin{aligned} Income = \beta_0 &+ \beta_1(Seniority) + \beta_2(School.Years) \\ &+ \beta_3(Experience) + \beta_4(Training) \\ &+ \beta_5(Commute.Distance) + \beta_6(Age) + \beta_7(Age^2), \end{aligned}$$

where the β values are scalar values that indicate how much each factor individually affects $Income$ (produced by linear regression, which is not critical to our discussion here). Since Age and Age^2 are both included in the analysis, the effect of the workers' age is parabolic, which is exactly as the authors intended: rising early, cresting, and then falling back. If we take the first partial derivative with respect to age, we get

$$\frac{\partial}{\partial Age} Income = \beta_6 + 2\beta_7 Age,$$

where $\beta_6 = 0.031$ and $\beta_7 = -0.00032$ (notice that all the other causal factors disappear due to the additive specification). Thus at age 25 the incremental effect of one additional year is $0.031 + 2(-0.00032)(25) = 0.015$, at age 50 it is $0.031 + 2(-0.00032)(50) = -0.001$, and at 75 it is $0.031 + 2(-0.00032)(75) = -0.017$.

★ **Example 6.2: Indexing Socio-Economic Status (SES)**. Any early measure of socio-economic status by Gordon (1969) reevaluated conventional compilation of various social indicators as a means of measuring "the position of the individual in some status ordering as determined by the individual's characteristics–his education, income, position, in the community, the market place, etc." The basic idea is to combine separate collected variables into a single measure because using any one single measure does not fully provide an overall assessment of status. The means by which these indicators are combined can vary considerably: additive or multiplicative, scaled or unscaled, weighted according to some criteria, and so on. Gordon proposed an alternative multiplicative causal expression of the form for an individual's socio-economic status:

$$SES = AE^b I^c M^d,$$

where

A = all terms not explicitly included in the model

E = years of education

I = amount of income

M = percent of time employed annually

and b, c, d are the associated "elasticities" providing a measure of strength for each causal term. The marginal (i.e., incremental) impacts on SES for each of the three terms of interest are given by partial derivatives with respect to the term of interest:

$$\frac{\partial SES}{\partial E} = AbE^{b-1}I^cM^d$$

$$\frac{\partial SES}{\partial I} = AcE^bI^{c-1}M^d$$

$$\frac{\partial SES}{\partial M} = AdE^bI^cM^{d-1}.$$

The point is that individual effects can be pulled from the general multiplicative specification with the partial derivatives. Note that due to the multiplicative nature of the proposed model, the marginal impacts are still dependent on levels of the other variables in a way that a strictly additive model would not produce: $SES = A + E^b + I^c + M^d$.

6.3 Derivatives and Partial Derivatives of Higher Order

Derivatives of higher order than one (what we have been doing so far) are simply iterated applications of the derivative process. The **second derivative** of a function with respect to x is

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x)\right),$$

also denoted $f''(x)$ for functions of only one variable (since $f''(x, y)$ would be ambiguous). It follows then how to denote, calculate, and notate higher order derivatives:

$$\frac{\partial^3 f(x)}{\partial x^3}, \quad \frac{\partial^4 f(x)}{\partial x^4}, \quad \frac{\partial^5 f(x)}{\partial x^5},$$

(i.e., $f'''(x)$, $f''''(x)$, and $f'''''(x)$, although this notation gets cumbersome). Note the convention with respect to the order designation here, and that it differs by placement in the numerator (∂^5) and denominator (∂x^5). Of course at some point a function will cease to support new forms of the higher order derivatives when the degree of the polynomial is exhausted. For instance, given the function

$f(x) = 3x^3$,

$$\frac{\partial f(x)}{\partial x} = 9x^2 \qquad \frac{\partial^4 f(x)}{\partial x^4} = 0$$

$$\frac{\partial^2 f(x)}{\partial x^2} = 18x \qquad \frac{\partial^5 f(x)}{\partial x^5} = 0$$

$$\frac{\partial^3 f(x)}{\partial x^3} = 18 \qquad \vdots$$

Thus we "run out" of derivatives eventually, and all derivatives of order four or higher remain zero.

So how do we interpret higher order derivatives? Because the first derivative is the rate of change of the original function, the second derivative is the rate of change for the rate of that change, and so on. Consider the simple example of the velocity of a car. The first derivative describes the rate of change of this velocity: very high when first starting out from a traffic light, and very low when cruising on the highway. The second derivative describes how fast this rate of change is changing. Again, the second derivative is very high early in the car's path as it increases speed rapidly but is low under normal cruising conditions. Third-order and higher derivatives work in exactly this same way on respective lower orders, but the interpretation is often less straightforward.

Higher order derivation can also be applied to partial derivatives. Given the function $f(x, y) = 3x^3y^2$, we can calculate

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial}{\partial y}(9x^2y^2) = 18x^2y$$

and

$$\begin{aligned} \frac{\partial^3}{\partial x^2 \partial y} f(x, y) &= \frac{\partial^2}{\partial x \partial y}(9x^2y^2) \\ &= \frac{\partial}{\partial y}(18xy^2) \\ &= 36xy. \end{aligned}$$

Thus the order hierarchy in the denominator gives the "recipe" for how many derivatives of each variable to perform, but the sequence of operations does not change the answer and therefore should be done in an order that makes the problem as easy as possible. Obviously there are other higher order partial derivatives that could be calculated for this function. In fact, if a function has k variables each of degree n, the number of derivatives of order n is given by $\binom{n+k-1}{n}$.

6.4 Maxima, Minima, and Root Finding

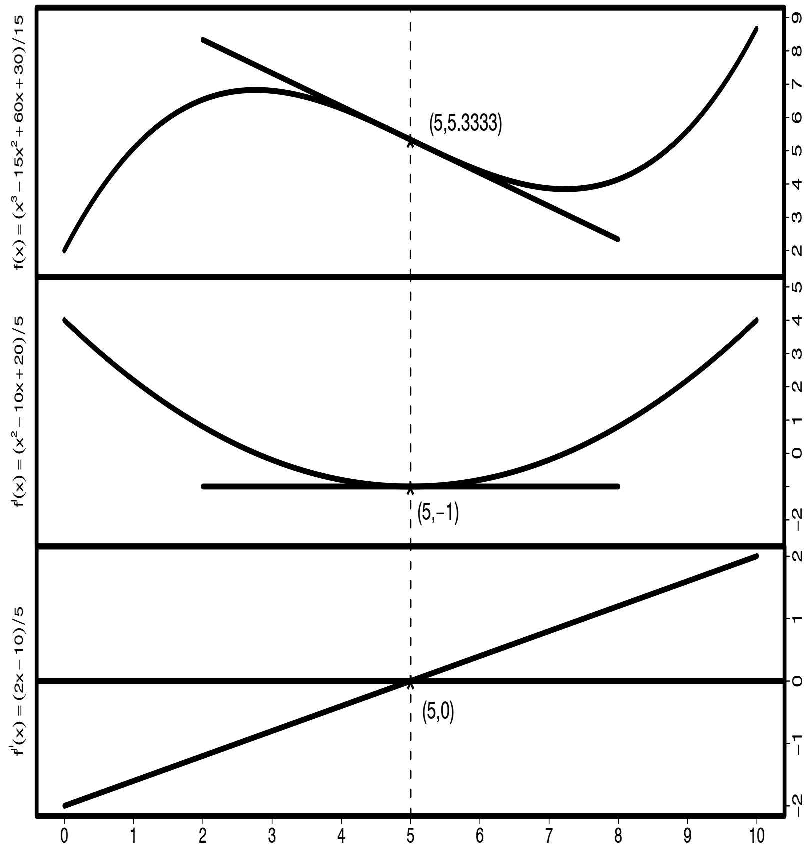
Derivatives can be used to find points of interest along a given function. One point of interest is the point where the "curvature" of the function changes, given by the following definition:

- **[Inflection Point.]** For a given function, $y = f(x)$, a point (x^*, y^*) is called an **inflection point** if the second derivative immediately on one side of the point is signed oppositely to the second derivative immediately on the other side.

So if x^* is indeed an inflection point, then for some small interval around x^*, $[x^* - \delta, x^* + \delta]$, $f''(x)$ is positive on one side and negative on the other. Interestingly, if $f''(x)$ is continuous at the inflection point $f(x^*)$, then $f''(x^*) = 0$. This makes intuitive sense since on one side $f'(x)$ is increasing so that $f''(x)$ must be positive, and on the other side $f'(x)$ is decreasing so that $f''(x)$ must be negative. Making δ arbitrarily small and decreasing it toward zero, x^* is the point where δ vanishes and the second derivative is neither positive nor negative and therefore must be zero.

Graphically, the tangent line at the inflection point crosses the function such that it is on one side before the point and on the other side afterward. This is a consequence of the change of sign of the second derivative of the function and is illustrated in Figure 6.1 with the function $f(x) = (x^3 - 15x^2 + 60x + 30)/15$. The function (characteristically) curves away from the tangent line on one side

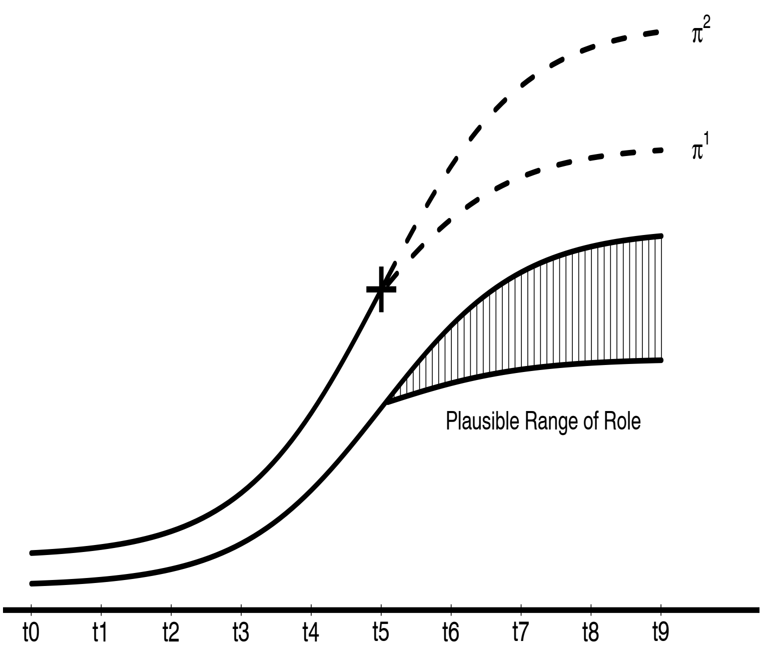
Fig. 6.1. Illustrating the Inflection Point



and curves away in the opposite direction on the other side. In the first panel of this figure the function itself is plotted with the tangent line shown. In the second panel the first derivative function, $f'(x) = (x^2 - 10x + 20)/5$, is plotted with the tangent line shown at the corresponding minima. Finally the third panel shows the second derivative function, $f''(x) = (2x - 10)/5$, with a horizontal line where the function crosses zero on the y-axis. Note that having a zero second derivative is a necessary but not sufficient condition for being an inflection point since the function $f'(5)$ has a zero second derivative, but $f'''(5)$ does not change sign just around 5.

★ **Example 6.3: Power Cycle Theory**. Sometimes inflection points can be substantively helpful in analyzing politics. Doran (1989) looked at critical changes in relative power between nations, evaluating ***power cycle theory***, which asserts that war is caused by changes in the gap between state power and state interest. The key is that when dramatic differences emerge between relative power (capability and prestige) and systematic foreign policy role (current interests developed or allowed by the international system), existing balances of power are disturbed. Doran particularly highlighted the case where a nation ascendent in both power and interest, with power in excess of interest, undergoes a "sudden violation of the prior trend" in the form of an inflection point in their power curve.

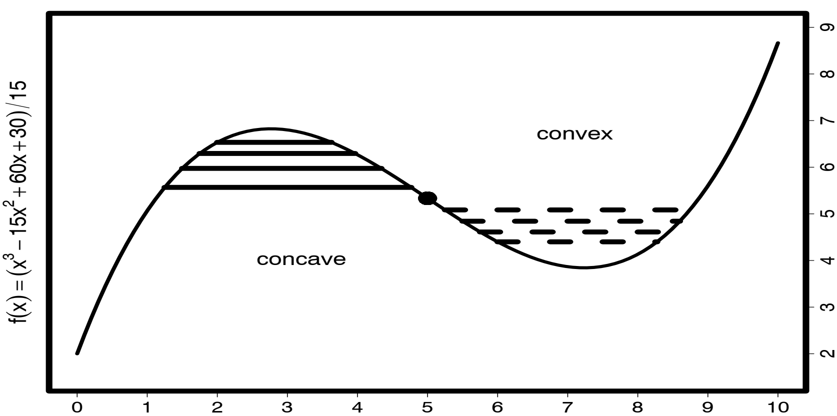
Fig. 6.2. Power/Role Gap Changes Induced by the Inflection Point



This is illustrated in Figure 6.2, where the inflection point at time $t5$ in the power curve introduces uncertainty in the anticipated decline in increasing power (i.e., the difference in π^1 and π^2 in the figure) and the subsequent potential differences between power and the range of plausible roles. Thus the problem introduced by the inflection point is that it creates changes in the slope difference and thus changes in the gap between role and power that the state and international system must account for.

The function $f(x) = (x^3 - 15x^2 + 60x + 30)/15$ in Figure 6.1 has a single inflection point in the illustrated range because there is on point where the concave portion of the function meets the convex portion. If a function (or portion of a function) is **convex** (also called **concave upward**), then every possible chord is above the function, except for their endpoints. A **chord** is just a line segment connecting two points along a curve. If a function (or portion of a function) is **concave** (or **concave downward**), then every possible chord is below the function, except for their endpoints. Figure 6.3 shows chords over the concave portion of the example function, followed by chords over the convex portion.

Fig. 6.3. Concave and Convex Portions of a Function



There is actually a more formal determination of concave and convex forms. If a function $f(x)$ is twice differentiable and concave over some open interval, then $f''(x) \leq 0$, and if a function $f(x)$ is twice differentiable and convex over some open interval then $f''(x) \geq 0$. The reverse statement is also true: A twice differentiable function that has a nonpositive second derivative over some interval is concave over that interval, and a twice differentiable function that has a nonnegative derivative over some interval is convex over that interval. So the sign of the second derivative gives a handy test, which will apply in the next section.

6.4.1 Evaluating Zero-Derivative Points

Repeated application of the derivative also gives us another general test: the **second derivative test**. We saw in Chapter 5 (see, for instance, Figure 5.4 on page 186) that points where the first derivatives are equal to zero are either a maxima or a minima of the function (modes or anti-modes), but without graphing how would we know which? Visually it is clear that the rate of change of the first derivative declines, moving away from a relative maximum point in both directions, and increases, moving away from a relative minimum point. The term "relative" here reinforces that these functions may have multiple maxima and minima, and we mean the behavior in a small neighborhood around the point of interest. This rate of change observation of the first derivative means that we can test with the second derivative: If the second derivative is negative at a point where the first derivative is zero, then the point is a relative maximum, and if the second derivative is positive at such a point, then the point is a relative minimum.

Higher order polynomials often have more than one mode. So, for example, we can evaluate the function

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x + \frac{11}{4}.$$

The first derivative is

$$f'(x) = x^3 - 6x^2 + 11x - 6,$$

which can be factored as

$$\begin{aligned} &= (x^2 - 3x + 2)(x - 3) \\ &= (x - 1)(x - 2)(x - 3). \end{aligned}$$

Since the critical values are obtained by setting this first derivative function equal to zero and solving, it is apparent that they are simply 1, 2, and 3. Substituting these three x values into the original function shows that at the three points $(1, 0.5)$, $(2, 0.75)$, $(3, 0.5)$ there is a function maximum or minimum. To determine whether each of these is a maximum or a minimum, we must first obtain the second derivative form,

$$f''(x) = 3x^2 - 12x + 11,$$

and then plug in the three critical values:

$$f''(1) = 3(1)^2 - 12(1) + 11 = 2$$

$$f''(2) = 3(2)^2 - 12(2) + 11 = -1$$

$$f''(3) = 3(3)^2 - 12(3) + 11 = 2.$$

This means that $(1, 0.5)$ and $(3, 0.5)$ are minima and $(2, 0.75)$ is a maximum. We can think of this in terms of Rolle's Theorem from the last chapter. If we modified the function slightly by subtracting $\frac{1}{2}$ (i.e., add $\frac{9}{4}$ instead of $\frac{11}{4}$), then two minima would occur on the x-axis. This change does not alter the fundamental form of the function or its maxima and minima; it just shifts the function up or down on the y-axis. By Rolle's Theorem there is guaranteed to be another point in between where the first derivative is equal to zero; the point at $x = 2$ here is the only one from the factorization. Since the points at $x = 1$

and $x = 3$ are minima, then the function increases away from them, which means that $x = 2$ has to be a maximum.

6.4.2 Root Finding with Newton-Raphson

A **root** of a function is the point where the function crosses the x-axis: $f(x) = 0$. This value is a "root" of the function $f()$ in that it provides a solution to the polynomial expressed by the function. It is also the point where the function crosses the x-axis in a graph of x versus $f(x)$. A discussion of polynomial function roots with examples was given on page 33 in Chapter 1.

Roots are typically substantively important points along the function, and it is therefore useful to be able to find them without much trouble. Previously, we were able to easily factor such functions as $f(x) = x^2 - 1$ to find the roots. However, this is not always realistically the case, so it is important to have a more general procedure. One such procedure is **Newton's method** (also called **Newton-Raphson**). The general form of Newton's method also (to be derived) is a series of steps according to

$$x_1 \cong x_0 - \frac{f(x_0)}{f'(x_0)},$$

where we move from a starting point x_0 to x_1, which is closer to the root, using characteristics of the function itself.

Newton's method exploits the **Taylor series expansion**, which gives the relationship between the value of a mathematical function at the point x_0 and the function value at another point, x_1, given (with continuous derivatives over the relevant support) as

$$\begin{aligned} f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) &+ \frac{1}{2!}(x_1 - x_0)^2 f''(x_0) \\ &+ \frac{1}{3!}(x_1 - x_0)^3 f'''(x_0) + \cdots, \end{aligned}$$

where f' is the first derivative with respect to x, f'' is the second derivative with respect to x, and so on. Infinite precision between the values $f(x_1)$ and $f(x_0)$ is achieved with the infinite extending of the series into higher order derivatives and higher order polynomials:

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{1}{2!}(x_1 - x_0)^2 f''(x_0) + \cdots$$
$$+ \cdots \frac{1}{\infty!}(x_1 - x_0)^{\infty} f^{(\infty)}(x_0)$$

(of course the factorial component in the denominator means that these are rapidly decreasing increments).

We are actually interested in finding a root of the function, which we will suppose exists at the undetermined point x_1. What we know so far is that for any point x_0 that we would pick, it is possible to relate $f(x_1)$ and $f(x_0)$ with the Taylor series expansion. This is simplified in two ways. First, note that if x_1 is a root, then $f(x_1) = 0$, meaning that the left-hand side of the two above equations is really zero. Second, while we cannot perfectly relate the $f(x_0)$ to the function evaluated at the root because the expansion can never by ***fully*** evaluated, it should be obvious that using some of the first terms at least gets us closer to the desired point. Since the factorial function is increasing rapidly, let us use these last two facts and assert that

$$0 \cong f(x_0) + (x_1 - x_0)f'(x_0),$$

where the quality of the approximation can presumably be improved with better guesses of x_0. Rearrange the equation so that the quantity of interest is on the left-hand side:

$$x_1 \cong x_0 - \frac{f(x_0)}{f'(x_0)},$$

to make this useful for candidate values of x_0. If this becomes "algorithmic" because x_0 is chosen arbitrarily, then repeating the steps with successive approximations gives a process defined for the $(j+1)$th step:

$$x_{j+1} \cong x_j - \frac{f(x_j)}{f'(x_j)},$$

so that progressively improved estimates are produced until $f(x_{j+1})$ is sufficiently close to zero. The process exploits the principle that the first terms of the Taylor series expansion get qualitatively better as the approximation gets closer and the remaining (ignored) steps get progressively less important.

Newton's method converges quadratically in time (number of steps) to a solution provided that the selected starting point is reasonably close to the solution, although the results can be very bad if this condition is not met. The key problem with distant starting points is that if $f''(x)$ changes sign between this starting point and the objective, then the algorithm may even move ***away*** from the root (diverge).

As a simple example of Newton's method, suppose that we wanted a numerical routine for finding the square root of a number, μ. This is equivalent to finding the root of the simple equation $f(x) = x^2 - \mu = 0$. The first derivative is just $\frac{\partial}{\partial x} f(x) = 2x$. If we insert these functions into the $(j+1)$th step:

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)},$$

we get

$$x_{j+1} = x_j - \frac{(x_j)^2 - \mu}{2x_j} = \frac{1}{2}(x_j + \mu x_j^{-1}).$$

Recall that μ is a constant here defined by the problem and x_j is an arbitrary value at the jth step. A very basic algorithm for implementing this in software or on a hand calculator is

```
delta = 0.0000001
x = starting.value
DO:
        x.new = 0.5*(x + mu/x)
        x = x.new
UNTIL:
        abs(x^2 - mu) < delta
```

where `abs()` is the absolute value function and `delta` is the accuracy threshold that we are willing to accept. If we are interested in getting the square root of 99 and we run this algorithm starting at the obviously wrong point of $x = 2$, we get:

Iteration	x	x^2
0	2.0	4.0
1	25.75	663.0625
2	14.7973300970874	218.960978002168
3	10.7438631130069	115.430594591030
4	9.97921289277379	99.5846899593026
5	9.94991749815775	99.0008582201457
6	9.94987437115967	99.00000000186

where further precision could be obtained with more iterations.

6.5 Multidimensional Integrals

As we have previously seen, the integration process measures the area under some function for a given variable. Because functions can obviously have multiple variables, it makes sense to define an integral as measuring the area (volume actually) under a function in more than one dimension. For two variables, the **iterated integral** (also called the **repeated integral**), is given in definite form here by

$$V = \int_a^b \int_c^d f(x,y)dydx,$$

where x is integrated between constants a and b, and y is integrated between constants c and d. The best way to think of this is by its inherent "nesting" of operations:

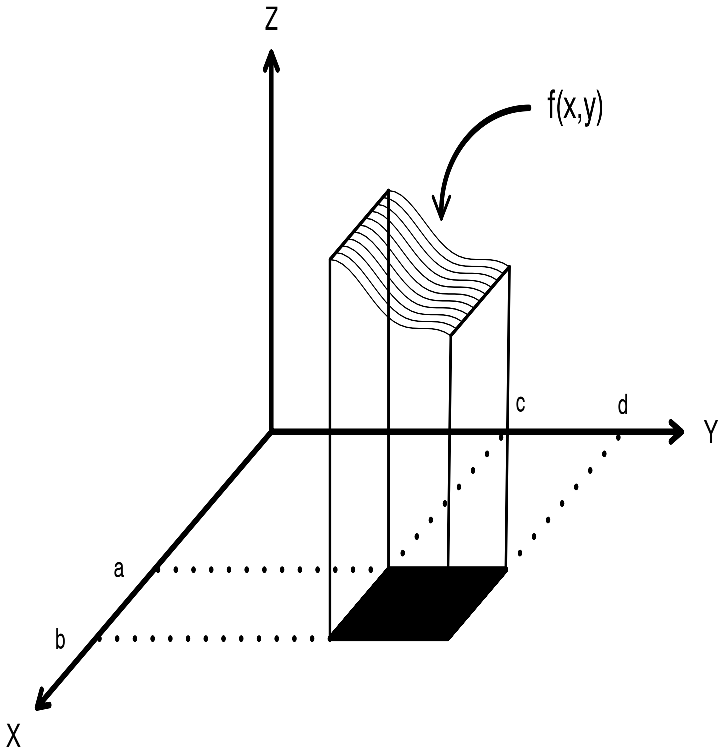
$$V = \int_a^b \left[\int_c^d f(x,y)dy \right] dx,$$

so that after the integration with respect to y there is a single integral left with respect to x of some ***new*** function that results from the first integration:

$$V = \int_a^b g(x)dx,$$

where $g(x) = \int_c^d f(x,y)dy$. That is, in the first (inner) step x is treated as a constant, and once the integration with respect to y is done, y is treated as a constant in the second (outer) step. In this way each variable is integrated

Fig. 6.4. Illustration of Iterated Integration of Volumes



separately with respect to its limits. The idea of integrating under a surface with an iterated intregral is illustrated in Figure 6.4, where the black rectangle shows the region and the stripped region shows $f(x, y)$. The integration provides the volume between this plan and this surface.

★ **Example 6.4: Double Integral with Constant Limits**. In the following simple integral we first perform the operation for the variable y:

$$\begin{aligned}\int_2^3 \int_0^1 x^2 y^3 dy dx &= \int_2^3 \left[\frac{1}{4} x^2 y^4 \bigg|_{y=0}^{y=1} \right] dx \\ &= \int_2^3 \left[\frac{1}{4} x^2 (1)^4 - \frac{1}{4} x^2 (0)^4 \right] dx \\ &= \int_2^3 \frac{1}{4} x^2 dx = \frac{1}{12} x^3 \bigg|_{x=2}^{x=3} = \frac{19}{12}.\end{aligned}$$

It turns out that it is not important which variable we integrate first. Therefore, in our case,

$$
\begin{aligned}
V &= \int_a^b \int_c^d f(x,y)dydx \\
&= \int_a^b \left[\int_c^d f(x,y)dy \right] dx \\
&= \int_c^d \left[\int_a^b f(x,y)dx \right] dy.
\end{aligned}
$$

Sometimes though it pays to be strategic as to which variable integration to perform first, because the problem can be made more or less complicated by this decision.

★ **Example 6.5: Revisiting Double Integrals with Constant Limits**. Now integrate x first with the same function:

$$
\begin{aligned}
\int_0^1 \int_2^3 x^2 y^3 dxdy &= \int_0^1 \left[\frac{1}{3} x^3 y^3 \Big|_{x=2}^{x=3} \right] dy \\
&= \int_0^1 \left[\frac{1}{3}(3)^3 y^3 - \frac{1}{3}(2)^3 y^3 \right] dy = \int_0^1 \frac{19}{3} y^3 dy \\
&= \frac{19}{12} y^4 \Big|_{y=0}^{y=1} = \frac{19}{12}.
\end{aligned}
$$

So far all the limits of the integration have been constant. This is not particularly realistic because it assumes that we are integrating functions over rectangles only to measure the volume under some surface described by $f(x,y)$. More generally, the region of integration is a function of x and y. For instance, consider the unit circle centered at the origin defined by the equation $1 = x^2 + y^2$. The limits of the integral in the x and the y dimensions now depend on the other variable, and an iterated integral over this region measures the cylindrical volume under a surface defined by $f(x,y)$. Thus, we need a means of performing the integration in two steps as before but accounting for this dependency. To generalize the iterated integration process above, we first express the limits of the integral in terms of a single variable. For the circle example we could use either of

$$y = g_y(x) = \sqrt{1 - x^2} \qquad\qquad x = g_x(y) = \sqrt{1 - y^2},$$

depending on our order preference. If we pick the first form, then the integral limits in the inner operation are the expression for y in terms of x, so we label this as the function $g_y(x)$.

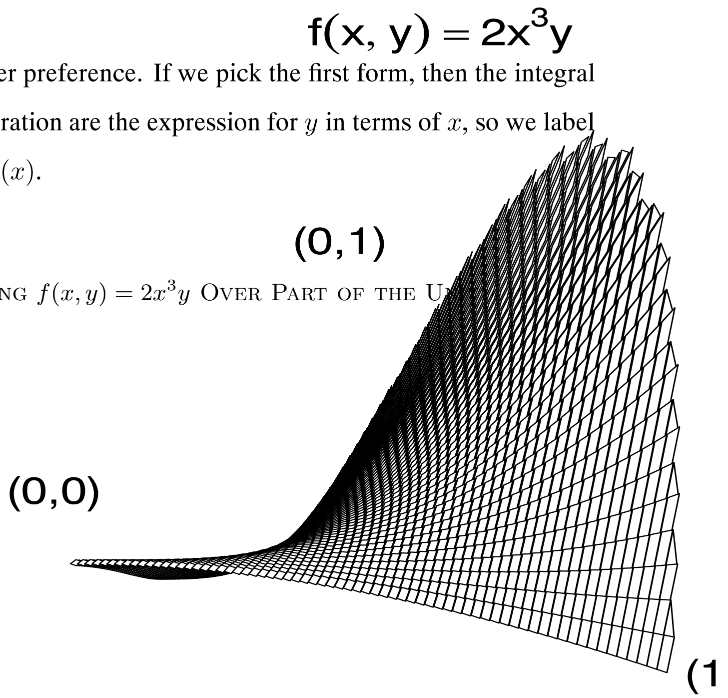


Fig. 6.5. Integrating $f(x, y) = 2x^3y$ Over Part of the U

★ **Example 6.6: Double Integral with Dependent Limits**

integrate the function $f(x, y) = 2x^3y$ over the quarter of th

the first quadrant (x and y both positive), as illustrated in Figur

to integrate y in the inner operation, we have the limits on t and $\sqrt{1-x^2}$. Thus the process proceeds as

$$\begin{aligned}
V &= \int_0^1 \int_0^{\sqrt{1-x^2}} 2x^3 y dy dx \\
&= \int_0^1 x^3 y^2 \bigg|_{y=0}^{y=\sqrt{1-x^2}} dx \\
&= \int_0^1 x^3(1-x^2)dx = \int_0^1 x^3 - x^5 dx \\
&= \frac{1}{4}x^4 - \frac{1}{6}x^6 \bigg|_{x=0}^{x=1} = \frac{1}{12}.
\end{aligned}$$

In the above example we would get identical (and correct) results integrating x first with the limits $(0, \sqrt{1-x^2})$. This leads to the following general theorem.

Iterated Integral Theorem:

- A two-dimensional area of interest, denoted $\mathfrak{A}$, is characterized by either

$$a \leq x \leq b, \qquad g_{y1}(x) \leq y g_{y2}(x)$$

or

$$c \leq y \leq d, \qquad g_{x1}(y) \leq x g_{x2}(y).$$

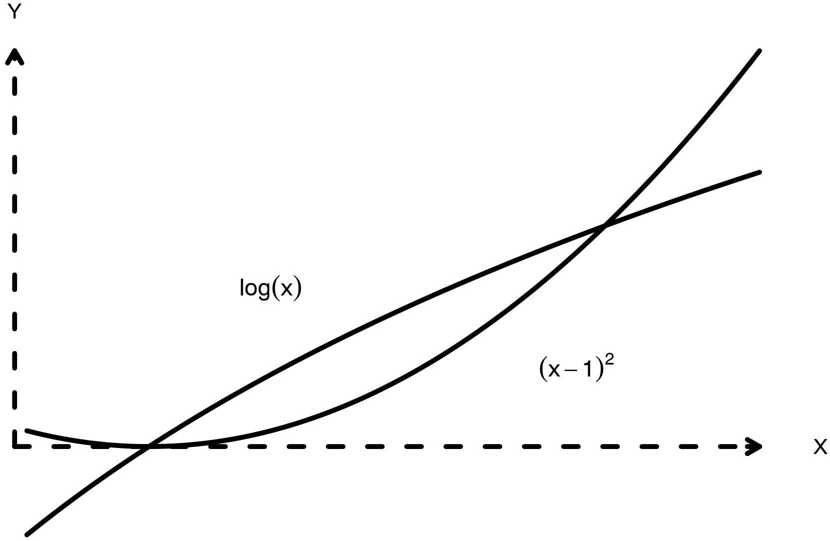
- The function to be integrated, $f(x, y)$, is continuous for all of $\mathfrak{A}$.
- Then the double integral over $\mathfrak{A}$ is equivalent to either of the iterated integrals:

$$\iint\limits_{\mathfrak{A}} f(x,y) d\mathfrak{A} = \int_a^b \int_{g_{y1}(x)}^{g_{y2}(x)} f(x,y) dy dx = \int_c^d \int_{g_{x1}(y)}^{g_{x2}(y)} f(x,y) dx dy.$$

This theorem states that, like the case with constant limits, we can switch the order of integration if we like, and that in both cases the result is equivalent to the motivating double integral.

★ **Example 6.7: More Complicated Double Integral**. Consider the problem of integrating the function $f(x, y) = 3 + x^2$ over the region $\mathfrak{A}$, determined by the intersection of the function $f_1(x) = (x-1)^2$ and the function

Fig. 6.6. Irregular Region Double Integral



$f_2(x) = \log(x)$, as depicted in Figure 6.6. This integral is

$$V = \int_a^b \int_{(x-1)^2}^{log(x)} 3 + x^2 dydx,$$

where a is obviously from the point $(1, 0)$, but it is not clear how to determine b. One way to obtain this point is to define the difference equation and its first derivative:

$$f_d(x) = x^2 - 2x + 1 - \log(x), \qquad f'_d(x) = 2x - 2 - \frac{1}{x},$$

which comes from expanding the square and subtracting $log(x)$. Now we can apply the Newton-Raphson algorithm quite easily, and the result is the point

$(1.7468817, 0.5578323)$ (try it!). Thus the integration procedure becomes

$$\int_{1}^{1.74\ldots} \int_{(x-1)^2}^{\log(x)} 3 + x^2 dy dx$$

$$= \int_{1}^{1.74\ldots} 3y + x^2 y \Big|_{y=(x-1)^2}^{y=\log(x)} dx$$

$$= \int_{1}^{1.74\ldots} \left(3\log(x) - 3(x-1)^2 + x^2\log(x) - x^2(x-1)^2\right) dx$$

$$= \int_{1}^{1.74\ldots} \left(x^2\log(x) + 3\log(x) - x^4 + 2x^3 - 4x^2 + 6x - 3\right) dx$$

$$= \frac{1}{9}x^3(3\log(x) - 1) + 3(x\log(x) - x) - \frac{1}{5}x^5 + \frac{1}{2}x^4$$

$$- \frac{4}{3}x^3 + 3x^2 - 3x \Big|_{1}^{1.7468817}$$

$$= 0.02463552 - (-0.1444444) = 0.1690799,$$

where the antiderivative of $x^2\log(x)$ comes from the rule

$$\int u^n \log(u) du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\log(u) - 1] + k$$

(k is again an arbitrary constant in the case of indefinite integrals).

6.6 Finite and Infinite Series

The idea of finite and infinite series is very important because it underlies many theoretical principles in mathematics, and because some physical phenomena can be modeled or explained through a series. The first distinction that we care about is whether a series converges or diverges. We will also be centrally concerned here with the idea of a limit as discussed in Section 5.2 of Chapter 5.

★ **Example 6.8: The Nazification of German Sociological Genetics**. The Hitlerian regime drove out or killed most of the prominent German sociologists of the time (a group that had enormous influence on the discipline).

The few remaining German sociologists apparently supported Nazi principles of heredity even though these were wrong and the correct chromosonal theory of inheritance had been published and supported in Germany since 1913 (Hager 1949). The motivation was Hitler's fascination with ***Reinrassen*** (pure races) as opposed to ***Mischrassen*** (mixed races), although such distinctions have no scientific basis in genetics whatsoever. These sociologists prescribed to Galton's (1898) theory that "the two parents, between them, contribute ***on average*** one half each inherited faculty,. . . " and thus a person's contributed genetics is related to a previous ancestor by the series

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

(which interestingly sums to 1). This incorrect idea of heredity supported Hitler's goal of passing laws forbidding mixed marriages (i.e., "Aryan" and "non-Aryan" in this case) because then eventually German society could become 100% Aryan since the fraction above goes to zero in the limit (recall that it was claimed to be a "thousand year Reich").

★ **Example 6.9: Measuring Small Group Standing**. Some sociologists care about the relative standing of individuals in small, bounded groups. This can be thought of as popularity, standing, or esteem, broadly defined. Suppose there are N members of the group and $\mathbf{A}$ is the $N \times N$ matrix where a 1 for a_{ij} indicates that individual i (the row) chooses to associate with individual j (the column), and 0 indicates the opposite choice. For convenience, the diagonal values are left to be zero (one cannot choose or not choose in this sense). The early literature (Moreno 1934) posited a ranking of status that simply added up choice reception by individual, which can be done by multiplying the $\mathbf{A}$ matrix by an appropriate unit vector to sum by columns, for instance,

$$\text{status}_1 = \mathbf{A}'u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}' \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

Later it was proposed (Katz 1953) that indirect associations also led to increased standing, so a means of incorporating indirect paths throughout this matrix needed to be included. Of course we would not want to require that indirect paths be equal in weight to direct paths, so we include a weighting factor, $0 < \alpha < 1$, that discounts distance. If we include every possible path, then the matrix that keeps track of these is given by the ***finite matrix series***:

$$B = \alpha\mathbf{A} + \alpha^2\mathbf{A}'\mathbf{A} + \alpha^3\mathbf{A}\mathbf{A}'\mathbf{A} + \alpha^4\mathbf{A}'\mathbf{A}\mathbf{A}'\mathbf{A} + \ldots + \alpha^N\mathbf{A}'\mathbf{A}\cdots\mathbf{A}'\mathbf{A}$$

(assuming N is even) so that the Katz measure of standing for the example above is now $status_2 = B'u =$

$$= \left(0.5 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} + 0.5^2 \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. +0.5^3 \begin{bmatrix} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 2 & 0 \\ 6 & 5 & 4 & 0 \end{bmatrix} + 0.5^4 \begin{bmatrix} 14 & 11 & 7 & 0 \\ 11 & 9 & 6 & 0 \\ 7 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)' \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7500 & 1.3125 & 1.4375 & 0.0000 \\ 2.0625 & 1.3125 & 0.7500 & 0.0000 \\ 1.8125 & 1.6250 & 1.1250 & 0.0000 \\ 1.2500 & 1.1250 & 1.0000 & 0.0000 \end{bmatrix}' \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6.8750 \\ 5.3750 \\ 4.3125 \\ 0.0000 \end{bmatrix}.$$

These calculations obviously used $\alpha = 0.5$. The final column vector shows relative standing that includes these other paths. This new measure apparently improves the standing of the second person.

6.6.1 Convergence

The key point of a series is that the consecutively produced values are generated by some sort of a rule or relationship. This can be anything from simply adding some amount to the previous value or a complex mathematical form operating on consecutive terms. Notationally, start with an **infinite series**:

$$S_\infty = \sum_{i=1}^{\infty} x_i = x_1 + x_2 + x_3 + \cdots,$$

which is just a set of enumerated values stretching out to infinity. This is not to say that any one of these values is itself equal to infinity or even that their sum is necessarily infinite, but rather that the ***quantity*** of them is infinite. Concurrently, we can also define a **finite series** of length n:

$$S_n = \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n,$$

which is a series that terminates with the nth value: x_n. This may also simply be the first n values of an infinite series and in this context is called an **nth partial sum** of the larger infinite sequence. The difference in subscript of S on the left-hand side emphasizes that the length of these summations differs.

A series is **convergent** if the limit as n goes to infinity is bounded (noninfinite itself):

$$\lim_{n\to\infty} S_n = \mathfrak{A}, \text{ where } \mathfrak{A} \text{ is bounded.}$$

A series is **divergent** if it is not convergent, that is, if $\mathfrak{A}$ above is positive or negative infinity. Another test is stated in the following theorem.

Integral Test for Series Convergence:

- If S_n converges, then $\lim_{n\to\infty} x_n = 0$, and if $\lim_{n\to\infty} x_n \neq 0$, then S_n diverges.
- If S_n is a series with all positive terms and $f(x)$ is the corresponding function that must be decreasing and everywhere continuous for values of $x \geq 1$, then the series S_n and the integral $\int_1^\infty f(x)dx$ both converge or diverge.

It is important to think about these statements carefully. It is ***not*** true that a zero limit of x_n shows convergence (the logic only goes in one direction). For instance, a harmonic series (see the Exercises) has the property that x_n goes to zero in the limit, but it is a well-known divergent series. Convergence is a handy result because it means that the infinite series can be approximated by a reasonable length finite series (i.e., additional values become unimportant at some point). So how does this test work? Let us now evaluate the limiting term of the series:

$$\sum_{i=1}^{n} \frac{i-1}{i+1} = 0 + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \cdots .$$

We could note that the values get larger, which is clearly an indication of a diverging series. The integral test also shows this, because

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = 1,$$

which is not zero, indicating divergence of the series. The integral part of the statement above relates the characteristic of a series with an integral, so that if we can obtain convergence of one, we can establish convergence of the other.

Consider the simple series and associated integral

$$S_\infty = \sum_{i=1}^{\infty} \frac{1}{i^3}, \qquad I_\infty = \int_1^\infty \frac{1}{x^3} dx.$$

The integral quantity is $\frac{1}{2}$, so we know that the series converges.

Here are some famous examples along with their convergence properties.

★ **Example 6.10: Consecutive Integers**. Given $x_k = x_{k-1}+1$, then $S_\infty = \sum_{i=1}^{\infty} i = 1 + 2 + 3 + \cdots$. This is a divergent series.

★ **Example 6.11: Telescoping Sum**. Assume $S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$. On quick inspection, there seems to be no obvious reason why this sum should converge in the limit, but note that it can be reexpressed as

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{i} + \frac{1}{i}\right) + \frac{1}{i+1} \\
&= 1 + \frac{1}{i+1}.
\end{aligned}$$

So this series gets its name from the way that it expands or contracts with the cancellations like an old-style telescope. Now if we take n to infinity, the result is obvious:

$$\begin{aligned}
S_\infty &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right) \\
&= 1
\end{aligned}$$

since no matter how big n becomes, adjacent terms cancel except for the first and last. Notice also that the limit of the nth term is zero, as per the integral test.

★ **Example 6.12: Geometric Series**. Suppose for some positive integer k and real number r, we have the series

$$S_n = \sum_{i=0}^{n} kr^i = k + kr + kr^2 + kr^3 + \cdots + kr^n.$$

Specific cases include

$$\begin{aligned} k = 1, r = -1: &\qquad 1 - 1 + 1 - 1 \cdots \\ k = 1, r = 2: &\qquad 1 + 2 + 4 + 8 + 16 + \cdots \\ k = 2, r = \frac{1}{2}: &\qquad 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots. \end{aligned}$$

It turns out that the geometric series converges to $S_n = \frac{k}{1-r}$ if $|r| < 1$ but diverges if $|r| > 1$. The series also diverges for $r = 1$ since it is then simply the sum $k + k + k + k + \cdots$.

★ **Example 6.13: Repeating Values as a Geometric Series**. Consider the repeating number

$$0.123123123\ldots = \frac{123}{1000^1} + \frac{123}{1000^2} + \frac{123}{1000^3} + \cdots$$

which is expressed in the second form as a geometric series with $k = 123$ and $r = 0.001$. Clearly this sequence converges because r is (much) less than one.

Because it can sometimes be less than obvious whether a series is convergent, a number of additional tests have been developed. The most well known are listed below for the infinite series $\sum_{i=1}^{\infty} a_i$.

- **Ratio Test**. If every $a_i > 0$ and $\lim\limits_{i\to\infty} \frac{a_{i+1}}{a_i} = A$, then the series converges for $A < 1$, diverges for $A > 1$, and may converge or diverge for $A = 1$.
- **Root Test**. If every $a_i > 0$ and $\lim\limits_{i\to\infty} (a_i)^{\frac{1}{i}} = A$, then the series converges for $A < 1$, diverges for $A > 1$, and may converge or diverge for $A = 1$.
- **Comparison Test**. If there is a convergent series $\sum_{i=1}^{\infty} b_i$ and a positive (finite) integer value J such that $a_i \leq b_i \ \forall i \geq J$, then $\sum_{i=1}^{\infty} a_i$ converges.

Some Properties of Convergent Series

$\rightarrowtail$	Limiting Values	$\lim\limits_{n\to\infty} a_n = 0$ (if $\lim\limits_{n\to\infty} a_n! = 0$, then $\sum_{i=1}^{\infty} a_i$ diverges)
$\rightarrowtail$	Summation	$\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty}(a_i + b_i)$
$\rightarrowtail$	Scalar Multiplication	$\sum_{i=1}^{\infty} ka_i = k\sum_{i=1}^{\infty} a_i$

★ **Example 6.14: An Equilibrium Point in Simple Games**. Consider the basic prisoner's dilemma game, which has many variants, but here two parties obtain **$**10 each for both cooperating, **$**15 dollars for acting opportunistically when the other acts cooperatively, and only **$**5 each for both acting opportunistically. What is the value of this game to a player who intends to act opportunistically at all iterations and expects the other player to do so as well? Furthermore, assume that each player discounts the future value of payoffs by 0.9 per period. Then this player expects a minimum payout of

$$\$5(0.9^0 + 0.9^1 + 0.9^2 + 0.9^3 + \ldots + 0.9^{\infty}).$$

The component in parentheses is a geometric series where $r = 0.9 < 1$, so it converges giving $\$5\frac{1}{1-0.9} = \$50$. Of course the game might be worth slightly more to our player if the opponent was unaware of this strategy on the first or second iteration (presumably it would be quite clear after that).

6.6.1.1 Other Types of Infinite Series

Occasionally there are special characteristics of a given series that allow us to assert convergence. A series where adjacent terms are alternating in sign for the whole series is called an **alternating series**. An alternating series converges if the same series with absolute value terms also convergences. So if $\sum_{i=1}^{\infty} a_i$ is an alternating series, then it converges if $\sum_{i=1}^{\infty} |a_i|$ converges. For instance, the alternating series given by

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^2}$$

converges if $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges since the latter is always greater for some given i value. This series converges if the integral is finite

$$\int_1^{\infty} \frac{1}{x^2} dx = -x^{-1}\Big|_1^{\infty} = -\frac{1}{\infty} - (-\frac{1}{1}) = 1,$$

so the second series converges and thus the original series converges.

Another interesting case is the **power series**, which is a series defined for x of the form

$$\sum_{i=1}^{\infty} a_i x^i,$$

for scalar values $a_1, a_2, \dots, a_\infty$. A special case is defined for the difference operator $(x - x_0)$:

$$\sum_{i=1}^{\infty} a_i (x - x_0)^i.$$

This type of power series has the characteristic that if it converges for the given value of $x_0 \neq 0$, then it converges for $|x| < |x_0|$. Conversely, if the power series diverges at x_0, then it also diverges for $|x| > |x_0|$. There are three power series that converge in important ways:

$$\sum_{i=1}^{\infty} \frac{x^i}{i!} = e^x$$

$$\sum_{i=1}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} = \sin(x)$$

$$\sum_{i=1}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!} = \cos(x).$$

The idea here is bigger than just these special cases (as interesting as they are). It turns out that if a function can be expressed in the form $f(x) = \sum_{i=1}^{\infty} a_i (x - x_0)^i$, then it has derivatives of all orders and a_i can be expressed as the ith derivative divided by the ith factorial. Note that the converse is not necessarily true in terms of guaranteeing the existence of a series expansion. Thus the function can be expressed as

$$f(x) = \frac{1}{0!}(x - x_0)^0 f(x_0) + \frac{1}{1!}(x - x_0)^1 f'(x_0) + \frac{1}{2!}(x_1 - x_0)^2 f''(x_0)$$
$$+ \frac{1}{3!}(x - x_0)^3 f'''(x_0) + \cdots ,$$

which is just the **Taylor series** discussed in Section 6.4.2.

The trick of course is expressing some function of interest in terms of such a series including the sequence of increasing derivatives. Also, the ability to express a function in this form does not guarantee convergence for particular values of x; that must be proven if warranted.

A special case of the Taylor series is the **Maclaurin series**, which is given when $x_0 = 0$. Many well-known functions can be rewritten as a Maclaurin series. For instance, now express $f(x) = \log(x)$ as a Maclaurin series and compare at $x = 2$ to $x = 1$ where $f(x) = 0$. We first note that

$$f'(x) = \frac{1}{x}$$
$$f''(x) = \frac{-1}{x^2}$$
$$f'''(x) = \frac{2}{x^3}$$
$$f''''(x) = \frac{-6}{x^4}$$
$$\vdots$$

which leads to the general order form for the derivative

$$f^{(i)}(x) = \frac{(-1)^{i+1}(i-1)!}{x^i}.$$

So the function of interest can be expressed as follows by plugging in the derivative term and simplifying:

$$\begin{aligned}\log(x) &= \sum_{i=0}^{\infty} \frac{1}{i!}(x-x_0)^i f^{(i)}(x_0) \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}(x-x_0)^i \left(\frac{(-1)^{i+1}(i-1)!}{x_0^i}\right) \\ &= \sum_{i=0}^{\infty}(-1)^{i+1}\frac{1}{i}\frac{(x-x_0)^i}{x_0^i}.\end{aligned}$$

Now set $x_0 = 1$ and $x = 2$:

$$\log(2) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots,$$

which converges to the (correct) value of 0.6931472.

6.7 The Calculus of Vector and Matrix Forms

This last section is more advanced than the rest of the chapter and may be skipped as it is not integral to future chapters. A number of calculus techniques operate or are notated differently enough on matrices and vectors that a separate section is warranted (if only a short one). Sometimes the notation is confusing when one misses the point that derivatives and integrals are operating on these larger, nonscalar structures.

6.7.1 Vector Function Notation

Using standard (Hamiltonian) notation, we start with two orthogonal unit vectors $\mathbf{i}$ and $\mathbf{j}$ starting at the origin and following along the x-axis and y-axis correspondingly. Any vector in two-space ($\Re^2$) can be expressed as a scalar-weighted sum of these two **basis vectors** giving the horizontal and vertical progress:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j}.$$

So, for example, to characterize the vector from the point $(3, 1)$ to the point $(5, 5)$ we use $\mathbf{v} = (5-3)\mathbf{i} + (5-1)\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}$. Now instead of the scalars a

and b, substitute the real-valued functions $f_1(t)$ and $f_2(t)$ for $t \in \Re$. Now we can define the **vector function**:

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j},$$

which gives the x and y vector values $\mathbf{f}(t) = (x, y)$. The parametric representation of the line passing through $(3, 1)$ and $(5, 5)$ is found according to

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \qquad & y &= y_0 + t(y_1 - y_0) \\ x &= 3 + 2t & y &= 1 + 4t, \end{aligned}$$

meaning that for some value of t we have a point on the line. To get the expression for this line in standard slope-intercept form, we first find the slope by getting the ratio of differences $(5-1)/(5-3) = 2$ in the standard fashion and subtracting from one of the two points to get the y value where x is zero: $(0, -5)$. Setting $x = t$, we get $y = -5 + 2x$.

So far this setup has been reasonably simple. Now suppose that we have some curvilinear form in $\Re$ given the functions $f_1(t)$ and $f_2(t)$, and we would like to get the slope of the tangent line at the point $t_0 = (x_0, y_0)$. This, it turns out, is found by evaluating the ratio of first derivatives of the functions

$$\mathbf{R}'(t_0) = \frac{f_2'(t_0)}{f_1'(t_0)},$$

where we have to worry about the restriction that $f_1'(t_0) \neq 0$ for obvious reasons. Why does this work? Consider what we are doing here; the derivatives are producing incremental changes in x and y separately by the construction with $\mathbf{i}$ and $\mathbf{j}$ above. Because of the limits, this ratio is the instantaneous change in y for a change in x, that is, the slope. Specifically, consider this logic in the notation:

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\lim\limits_{\Delta t \to 0} \Delta t}}{\frac{\Delta x}{\lim\limits_{\Delta t \to 0} \Delta t}} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{\partial y}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial y}{\partial x}.$$

For example, we can find the slope of the tangent line to the curve $x =$

$3t^3 + 5t^2 + 7$, $y = t^2 - 2$, at $t = 1$:

$$f_1'(1) = 9t^2 + 10t\Big|_{t=1} = 19$$

$$f_2'(1) = 2t\Big|_{t=1} = 2$$

$$\mathbf{R}'(1) = \frac{2}{19}.$$

We can also find all of the horizontal and vertical tangent lines to this curve by a similar calculation. There are horizontal tangent lines when $f_1'(t) = 9t^2 + 10t = 0$. Factoring this shows that there are horizontal tangents when $t = 0$, $t = -\frac{10}{9}$. Plugging these values back into $x = 3t^3 + 5t^2 + 7$ gives horizontal tangents at $x = 7$ and $x = 9.058$. There are vertical horizontal lines when $f_2'(t) = 2t = 0$, which occurs only at $t = 0$, meaning $y = -2$.

6.7.2 Differentiation and Integration of a Vector Function

The vector function $\mathbf{f}(t)$ is differentiable with domain t if the limit

$$\lim_{\Delta t \to 0} \frac{\mathbf{f}(t + \Delta t) - \mathbf{f}(t)}{\Delta t}$$

exists and is bounded (finite) for all specified t. This is the same idea we saw for scalar differentiation, except that by consequence

$$\mathbf{f}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j},$$

which means that the function can be differentiated by these orthogonal pieces. It follows also that if $\mathbf{f}'(t)$ meets the criteria above, then $\mathbf{f}''(t)$ exists, and so on. As a demonstration, let $\mathbf{f}(t) = e^{5t}\mathbf{i} + \sin(t)\mathbf{j}$, so that

$$\mathbf{f}'(t) = 5e^5t\mathbf{i} + \cos(t)\mathbf{j}.$$

Not surprisingly, integration proceeds piecewise for the vector function just as differentiation was done. For $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}$, the integral is

$$\begin{cases} \int \mathbf{f}(t)dt = \left[\int f_1'(t)dt\right]\mathbf{i} + \left[\int f_2'(t)dt\right]\mathbf{j} + \mathbf{K} & \text{for the indefinite form,} \\ \int_a^b \mathbf{f}(t)dt = \left[\int_a^b f_1'(t)dt\right]\mathbf{i} + \left[\int_a^b f_2'(t)dt\right]\mathbf{j} & \text{for the definite form.} \end{cases}$$

Incidently, we previously saw an arbitrary constant k for indefinite integrals of scalar functions, but that is replaced here with the more appropriate vector-valued form $\mathbf{K}$. This "splitting" of the integration process between the two dimensions can be tremendously helpful in simplifying difficult dimensional problems.

Consider the trigonometric function $\mathbf{f}(t) = \tan(t)\mathbf{i} + \sec^2(t)\mathbf{j}$. The integral over $[0{:}\pi/4]$ is produced by

$$\begin{aligned}
\int_0^{\pi/4} \mathbf{f}(t)dt &= \left[\int_0^{\pi/4} \tan(t)dt\right]\mathbf{i} + \left[\int_0^{\pi/4} \sec^2(t)dt\right]\mathbf{j} \\
&= \left[-\log(|\cos(t)|)\Big|_0^{\pi/4}\right]\mathbf{i} + \left[\tan(t)\Big|_0^{\pi/4}\right]\mathbf{j} \\
&= 0.3465736\mathbf{i} + 0.854509\mathbf{j}.
\end{aligned}$$

Indefinite integrals sometimes come with additional information that makes the problem more complete. If $\mathbf{f}'(t) = t^2\mathbf{i} - t^4\mathbf{j}$, and we know that $\mathbf{f}(0) = 4\mathbf{i} - 2\mathbf{j}$, then a full integration starts with

$$\begin{aligned}
\mathbf{f}(t) &= \int \mathbf{f}'(t)dt \\
&= \int t^2 dt\mathbf{i} - \int t^4 dt\mathbf{j} \\
&= \frac{1}{3}t^3\mathbf{i} + \frac{1}{5}t^5\mathbf{j} + \mathbf{K}.
\end{aligned}$$

Since $\mathbf{f}(0)$ is the function value when the components above are zero, except for $\mathbf{K}$ we can substitute this for $\mathbf{K}$ to complete

$$\mathbf{f}(t) = \left(\frac{1}{3}t^3 + 4\right)\mathbf{i} + \left(\frac{1}{5}t^5 - 2\right)\mathbf{j}.$$

In statistical work in the social sciences, a scalar-valued vector function is important for maximization and description. We will not go into the theoretical derivation of this process (maximum likelihood estimation) but instead will describe the key vector components. Start with a function: $y = f(\mathbf{x}) = f(x_1, x_2, x_3 \ldots, x_k)$ operating on the k-length vector $\mathbf{x}$. The vector of partial derivatives with respect to each x_i is called the **gradient**:

$$\mathbf{g} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \partial y/\partial x_1 \\ \partial y/\partial x_2 \\ \partial y/\partial x_3 \\ \vdots \\ \partial y/\partial x_k \end{bmatrix},$$

which is given by convention as a column vector. The second derivative for this setup is taken in a different manner than one might suspect; it is done by differentiating the complete gradient vector by each x_i such that the result is a $k \times k$ matrix:

$$\mathbf{H} = \left[\frac{\partial}{\partial x_1}\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right), \frac{\partial}{\partial x_2}\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right), \right.$$
$$\left. \frac{\partial}{\partial x_3}\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right), \dots \frac{\partial}{\partial x_k}\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)\right]$$

$$= \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_1 \partial \mathbf{x}_1} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_1 \partial \mathbf{x}_2} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_1 \partial \mathbf{x}_3} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_1 \partial \mathbf{x}_k} \\ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_2 \partial \mathbf{x}_1} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_2 \partial \mathbf{x}_2} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_2 \partial \mathbf{x}_3} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_2 \partial \mathbf{x}_k} \\ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_3 \partial \mathbf{x}_1} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_3 \partial \mathbf{x}_2} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_3 \partial \mathbf{x}_3} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_3 \partial \mathbf{x}_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_k \partial \mathbf{x}_1} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_k \partial \mathbf{x}_2} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_k \partial \mathbf{x}_3} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_k \partial \mathbf{x}_k} \end{bmatrix}$$

$$= \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}}.$$

Note that the partial derivatives in the last (most succinct) form are done on vector quantities. This matrix, called the **Hessian** after its inventor/discover the German mathematician Ludwig Hesse, is square and symmetrical. In the course of normal statistical work it is also positive definite, although serious problems arise if for some reason it is not positive definite because it is necessary to invert the Hessian in many estimation problems.

6.8 Constrained Optimization

This section is considerably more advanced than the previous and need not be covered on the first read-through of the text. It is included because constrained optimization is a standard tool in some social science literatures, notably economics.

We have already seen a similar example in the example on page 187, where a cost function was minimized subject to two terms depending on committee size. The key feature of these methods is using the first derivative to find a point where the slope of the tangent line is zero. Usually this is substantively interesting in that it tells us where some x value leads to the greatest possible $f(x)$ value to maximize some quantity of interest: money, utility, productivity, cooperation, and so on. These problems are usually more useful in higher dimensions, for instance, what values of x_1, x_2, and x_3 simultaneously provide the greatest value of $f(x_1, x_2, x_3)$?

Now let us revisit the optimization problem but requiring the additional ***constraint*** that the values of x_1, x_2, and x_3 have to conform to some predetermined relationship. Usually these constraints are expressed as inequalities, say $x_1 > x_2 > x_3$, or with specific equations like $x_1 + x_2 + x_3 = 10$. The procedure we will use is now called **constrained optimization** because we will optimize the given function but with the constraints specified in advance. There is one important underlying principle here. The constrained solution will never be a better solution than the unconstrained solution because we are requiring certain relationships among the terms. At best these will end up being trivial constraints and the two solutions will be identical. Usually, however, the constraints lead to a suboptimal point along the function of interest, and this is done by substantive necessity.

Our task will be to maximize a k-dimensional function $f(\mathbf{x})$ subject to the arbitrary constraints expressed as m functions:

$$c_1(\mathbf{x}) = r_1,\ c_2(\mathbf{x}) = r_2, \ldots,\ c_m(\mathbf{x}) = r_m,$$

where the $r_1, r_2, \ldots, r_m$ values are stipulated constants. The trick is to deliberately include these constraint functions in the maximization process. This method is called the **Lagrange multiplier** , and it means substituting for the standard function, $f(\mathbf{x})$, a modified version of the form

$$L(\mathbf{x},\boldsymbol{\lambda}) = f(\mathbf{x}) + \lambda_1(c_1(\mathbf{x}) - r_1) + \lambda_2(c_2(\mathbf{x}) - r_2) + \ldots + \lambda_m(c_m(\mathbf{x}) - r_m)$$

$$= f(\mathbf{x}) + \underset{1\times m}{\boldsymbol{\lambda}'}\underset{m\times k}{\mathbf{c}(\mathbf{x})},$$

where the second form is just a restatement in vector form in which the $-r$ terms are embedded ($\boldsymbol{\lambda}'$ denotes a transpose not a derivative). Commonly these $r_1, r_2, \ldots, r_m$ values are zero (as done in the example below), which makes the expression of $L(\mathbf{x},\boldsymbol{\lambda})$ cleaner. The λ terms in this expression are called Lagrange multipliers, and this is where the name of the method comes from.

Now we take two (multidimensional) partial derivatives and set them equal to zero just as before, except that we need to keep track of $\boldsymbol{\lambda}$ as well:

$$\frac{d}{d\mathbf{x}}L(\mathbf{x},\boldsymbol{\lambda}) = \frac{d}{d\mathbf{x}}f(\mathbf{x}) + \boldsymbol{\lambda}'\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x}) \equiv \mathbf{0} \quad \Rightarrow \quad \frac{d}{d\mathbf{x}}f(\mathbf{x}) = -\boldsymbol{\lambda}'\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x})$$

$$\frac{d}{d\boldsymbol{\lambda}}L(\mathbf{x},\boldsymbol{\lambda}) = \mathbf{c}(\mathbf{x}) \equiv \mathbf{0}.$$

The derivative with respect to $\boldsymbol{\lambda}$ is simple because there are no λ values in the first term. The term $\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x})$ is just the matrix of partial derivatives of the constraints, and it is commonly abbreviated $\mathbf{C}$. Expressing the constraints in this way also means that the Lagrange multiplier component $-\boldsymbol{\lambda}'\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x})$ is now just $-\boldsymbol{\lambda}'\mathbf{C}$, which is easy to work with. It is interesting to contrast the final part of the first line $\frac{d}{d\mathbf{x}}f(\mathbf{x}) = -\boldsymbol{\lambda}\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x})$ with unconstrained optimization at this step, $\frac{d}{d\mathbf{x}}f(\mathbf{x}) = \mathbf{0}$, because it clearly shows the imposition of the constraints on the function maximization. Finally, after taking these derivatives, we solve for the values of $\mathbf{x}$ and λ that result. This is our constrained answer.

Just about every econometrics book has a numerical example of this process, but it is helpful to have a simple one here. Suppose we have "data" according to the following matrix and vector:

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad \boldsymbol{\Omega} = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 5 & 5 \\ 1 & 5 & 5 \end{bmatrix},$$

and we want to maximize the quadratic function:

$$f(\mathbf{x}) = \mathbf{x}'\boldsymbol{\Omega}\mathbf{x} - 2\boldsymbol{\omega}'\mathbf{x} + 5.$$

This is great justification for using matrix algebra because the same function $f(\mathbf{x})$ written out in scalar notation is

$$f(\mathbf{x}) = 5x_1^2 + 5x_2^2 + 5x_3^2 + 5x_1x_2 + 6x_1x_3 + 10x_2x_3 + 2x_1 + 4x_2 + 6x_3 + 5.$$

We now impose the two constraints:

$$2x_1 + x_2 = 0 \qquad x_1 - x_2 - x_3 = 0,$$

which gives

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} c_1(\mathbf{x}) \\ c_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 2x_1 & x_2 & 0 \\ x_1 & -x_2 & 0 \end{bmatrix}$$

Taking the partial derivatives $\frac{d}{d\mathbf{x}}\mathbf{c}(\mathbf{x})$ (i.e., with respect to x_1, x_2, and x_3) produces the $\mathbf{C}$ matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix},$$

which is just the matrix-collected multipliers of the $\mathbf{x}$ terms in the constraints since there were no higher order terms to worry about here. This step can be somewhat more involved with more complex (i.e., nonlinear constraints).

With the specified constraints we can now specify the Lagrange multiplier version of the function:

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\lambda}) &= f(\mathbf{x}) + \boldsymbol{\lambda}'\mathbf{c}(\mathbf{x}) \\ &= \underset{(1\times3)}{\mathbf{x}'}\underset{(3\times3)}{\boldsymbol{\Omega}}\underset{(3\times1)}{\mathbf{x}} - 2\underset{(1\times3)}{\boldsymbol{\omega}'}\underset{(3\times1)}{\mathbf{x}} + 5 + \underset{(1\times2)}{\boldsymbol{\lambda}'}\underset{(2\times3)}{\mathbf{C}}\underset{(3\times1)}{\mathbf{x}}. \end{aligned}$$

Note that $m = 2$ and $k = 3$ in this example. The next task is to take the two derivatives and set them equal to zero:

$$\frac{d}{d\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) = 2\mathbf{x}'\boldsymbol{\Omega} - 2\boldsymbol{\omega}' + \boldsymbol{\lambda}'\mathbf{C} \equiv \mathbf{0} \quad \Rightarrow \quad \mathbf{x}'\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\lambda}'\mathbf{C} = \boldsymbol{\omega}'$$

$$\frac{d}{d\boldsymbol{\lambda}} L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{c}(\mathbf{x}) = \mathbf{C}\mathbf{x} \equiv \mathbf{0},$$

where $\mathbf{c}(\mathbf{x}) = \mathbf{C}\mathbf{x}$ comes from the simple form of the constraints here. This switch can be much more intricate in more complex specifications of restraints. These final expressions allow us to stack the equations (they are multidimensional anyway) into the following single matrix statement:

$$\begin{bmatrix} \boldsymbol{\Omega}' & \frac{1}{2}\mathbf{C}' \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{0} \end{bmatrix}$$

where we used the transpose property such that $(\boldsymbol{\lambda}'\mathbf{C})' = \mathbf{C}'\boldsymbol{\lambda}$ and $(\mathbf{x}'\boldsymbol{\Omega})' = \boldsymbol{\Omega}'\mathbf{x}$ (given on page 116) since we want the column vector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}' = \mathbf{x}'\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\lambda}'\mathbf{C}$$

$$\boldsymbol{\omega} = (\mathbf{x}'\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\lambda}'\mathbf{C})' = \boldsymbol{\Omega}'\mathbf{x} + \frac{1}{2}\mathbf{C}'\boldsymbol{\lambda}$$

$$= \begin{bmatrix} \boldsymbol{\Omega}' & \frac{1}{2}\mathbf{C}' \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}$$

(the second row is done exactly the same way). This order of multiplication on the left-hand side is essential so that the known quantities are in the matrix and the unknown quantities are in the first vector. If we move the matrix to the right-hand side by multiplying both sides by its inverse (presuming it is nonsingular of course), then all the unknown quantities are expressed by the known quantities. So a solution for $[\mathbf{x}\ \boldsymbol{\lambda}]'$ can now be obtained by matrix inversion and then pre-multiplication:

$$\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}' & \frac{1}{2}\mathbf{C}' \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 1 & 1.0 & 0.5 \\ 1 & 5 & 5 & 0.5 & -0.5 \\ 5 & 5 & 5 & 0.0 & -0.5 \\ 2 & 1 & 0 & 0.0 & 0.0 \\ 1 & -1 & -1 & 0.0 & 0.0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{18} & -\frac{1}{9} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{9} & \frac{2}{9} & -\frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{4}{9} & \frac{10}{9} & -\frac{2}{3} & -\frac{4}{3} & \frac{8}{3} \\ \frac{10}{9} & -\frac{20}{9} & \frac{4}{3} & \frac{10}{3} & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\frac{1}{3}, -\frac{2}{3}, 1, \frac{2}{3}, \frac{2}{3}\right]',$$

meaning that $x_1 = \frac{1}{3}$, $x_2 = -\frac{2}{3}$, $x_3 = 1$, and $\boldsymbol{\lambda}' = [\frac{2}{3}, \frac{2}{3}]$. Since the λ values are nonzero, we know that the constraints changed the optimization solution. It is not essential, but we can check that the solution conforms to the constraints:

$$2\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right) = 0 \qquad \left(\frac{1}{3}\right) - \left(-\frac{2}{3}\right) - (1) = 0.$$

Suppose we now impose just one constraint: $3x_1 + x_2 - x_3 = 0$. Thus the $\mathbf{C}$ matrix is a vector according to $[3, 1, -1]$ and the $\boldsymbol{\Omega}$ matrix is smaller. The calculation of the new values is

$$\begin{bmatrix} 5 & 1 & 5 & -0.5 \\ 1 & 5 & 5 & 0.5 \\ 5 & 5 & 5 & -0.5 \\ 3 & 1 & -1 & 0.0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & -\frac{7}{20} & \frac{1}{4} & -\frac{3}{10} \\ -\frac{9}{25} & \frac{17}{25} & -\frac{2}{5} & \frac{26}{25} \\ \frac{6}{25} & -\frac{37}{100} & \frac{7}{20} & -\frac{43}{50} \\ \frac{4}{5} & -\frac{2}{5} & 0 & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{5} \\ \frac{11}{20} \\ 0 \end{bmatrix}.$$

Notice that the $\lambda = 0$ here. What this means is that our restriction actually made no impact: The solution above is the unconstrained solution. So we imposed a constraint that would have been satisfied anyway.

The Lagrange multiplier method is actually more general than is implied here. We used very simple constraints and only a quadratic function. Much more complicated problems can be solved with this approach to constrained optimization.

6.9 New Terminology

alternating series, 263
basis vectors, 266
chord, 244
concave, 244
concave downwards, **??**
concave upwards, **??**
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Taylor series, 265
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vector function, 267

Exercises

6.1 For the following functions, determine the interval over which the function is convex or concave (if at all), and identify any inflection points:

$$f(x) = \frac{1}{x} \qquad\qquad f(x) = x^3$$

$$f(x) = x^2 + 4x + 8 \qquad\qquad f(x) = -x^2 - 9x + 16$$

$$f(x) = \frac{1}{1+x^2} \qquad\qquad f(x) = \frac{\exp(x)}{(1+\exp(x))}$$

$$f(x) = 1 - \exp(-\exp(x)) \qquad\qquad f(x) = \frac{x^{7/2}}{2+x^2}$$

$$f(x) = (x-1)^4(x+1)^3 \qquad\qquad f(x) = \log(x).$$

6.2 Using Newton's method, find one root of each of the following functions:

$$f(x) = x^2 - 5, x > 0 \qquad\qquad f(x) = x^2 + 4x - 8$$

$$f(x) = \log(x^2), x < 0 \qquad\qquad f(x) = x^4 - x^3 - 5$$

$$f(x) = x^2 - 2\sqrt{x}x > 0 \qquad\qquad f(x) = x^5 - 2\tan(x).$$

6.3 To test memory retrieval, Kail and Nippold (1984) asked 8-, 12-, and 21-year-olds to name as many animals and pieces of furniture as possible in separate 7–minute intervals. They found that this number increased across the tested age range but that the rate of retrieval slowed down as the period continued. In fact, the responses often came in "clusters" of related responses ("lion," "tiger," "cheetah," etc.), where the relation of time in seconds to cluster size was fitted to be $cs(t) = at^3 + bt^2 + ct + d$, where time is t, and the others are estimated parameters (which differ by topic, age group, and subject). The researchers were very interested in the inflection point of this function because it suggests a change of cognitive process. Find it for the unknown parameter values analytically by taking the second derivative. Verify that it is an inflection point and not a maxima or minima. Now graph this function for the points supplied in the authors' graph of one particular case for an 8-year-old: $cs(t) = [1.6, 1.65, 2.15, 2.5, 2.67, 2.85, 3.1, 4.92, 5.55]$ at the points $t = [2, 3, 4, 5, 6, 7, 8, 9, 10]$. They do not give parameter values for this case, but plot the function on the same graph for the values $a = 0.04291667$, $b = -0.7725$, $c = 4.75$, and $d = -7.3$. Do these values appear to satisfy your result for the inflection point?

6.4 For the function $f(x, y) = \frac{\sin(xy)}{\cos(x+y)}$, calculate the partial derivatives with respect to x and y.

6.5 Smirnov and Ershov (1992) chronicled dramatic changes in public opinion during the period of "Perestroika" in the Soviet Union (1985 to 1991). They employed a creative approach by basing their model on the principles of thermodynamics with the idea that sometimes an encapsulated liquid is immobile and dormant and sometimes it becomes turbulent and pressured, literally letting off steam. The catalyst for change is hypothesized to be radical economic reform confronted by conservative counter-reformist policies. Define p as some policy on a metric $[-1 : 1]$ representing different positions over this range from conservative ($p < 1$) to liberal ($p > 1$). The resulting public opinion support, S, is a function that can have single or multiple modes over this range, inflection points and monotonic areas, where the number and variety of these reflect divergent opinions in the population. Smirnov and Ershov found that the most convenient mathematical form here was

$$S(p) = \sum_{i=1}^{4} \lambda_i p^i,$$

where the notation on p indicates exponents and the λ_i values are a series of specified scalars. Their claim was that when there are two approximately equal modes (in $S(p)$), this represents the situation where "the government ceases to represent the majority of the electorate." Specify λ_i values to give this shape; graph over the domain of p; and use the first and second derivatives of $S(p)$ to identify maxima, minima, and inflection points.

6.6 Derive the five partial derivatives for $u_1 \ldots u_5$ from the function on page 236. Show all steps.

6.7 For the function $f(u, v) = \sqrt{u + v^2}$, calculate the partial derivatives with respect to u and v and provide the value of these functions at the point $(\frac{1}{2}, \frac{1}{3})$.

6.8 Using the function

$$f(x, y, z) = zy^4 - xy^3 + x^3yz^2,$$

show that

$$\frac{\partial^3}{\partial x \partial y \partial z} f(x, y, z) = \frac{\partial^3}{\partial z \partial y \partial x} f(x, y, z).$$

6.9 Obtain the first, second, and third derivatives of the following functions:

$$f(x) = 5x^4 + 3x^3 - 11x^2 + x - 7 \qquad h(z) = 111z^3 - 121z$$

$$f(y) = \sqrt{y} + \frac{1}{y^{\frac{7}{2}}} \qquad f(x) = (x^9)^{-2}$$

$$h(u) = \log(u) + k \qquad g(z) = \sin(z) - \cos(z).$$

6.10 Graph the function given on page 245, the first derivative function, and the second derivative function over $[0\!:\!4]$. Label the three points of interest.

6.11 Evaluate the following iterated integrals:

$$\int_2^4 \int_3^5 dydx \qquad \int_0^1 \int_0^1 x^{\frac{3}{2}} y^{\frac{2}{3}} dxdy$$

$$\int_1^3 \int_1^x \frac{x}{y} dydx \qquad \int_0^\pi \int_0^x y\cos(x) dydx$$

$$\int_0^1 \int_0^y (x+y^2) dxdy \qquad \int_1^2 \int_1^{\sqrt{2-y}} ydxdy$$

$$\int_0^1 \int_0^1 \int_0^1 \sqrt{1-x-y-z}dxdydz \qquad \int_2^e \int_e^3 \frac{1}{u}\frac{1}{v} dudv$$

$$\int_0^1 \int_0^{2-2x} \int_0^{x^2y^2} dzdydx \qquad \int_{-1}^1 \int_0^{1-y/3} xydxdy.$$

6.12 A well-known series is the **harmonic series** defined by

$$S_\infty = \sum_{i=1}^\infty \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Prove that this sequence diverges. Does this series converge for $\frac{1}{i^k}$, for all $k > 1$?

6.13 Show whether the following series are convergent or divergent. If the series is convergent, find the sum.

$$\sum_{i=0}^\infty \frac{1}{3^i} \qquad \sum_{r=0}^\infty \frac{1}{(r+1)(r+2)} \qquad \sum_{i=1}^\infty \frac{1}{100i}$$

$$\sum_{k=1}^\infty k^{-\frac{1}{2}} \qquad \sum_{i=i}^\infty \left(\frac{i}{i-1}\right)^{\frac{1}{i}} \qquad \sum_{r=1}^\infty \sin\left(\frac{1}{r}\right)$$

$$\sum_{i=1}^\infty \frac{i^3+2i^2-i+3}{2i^5+3i-3} \qquad \sum_{r=1}^\infty \frac{2r+1}{(\log(r)^r} \qquad \sum i = 1^\infty \frac{2^i}{i^2}$$

6.14 Calculate the area between $f(x) = x^2$ and $f(x) = x^3$ over the domain $[0:1]$. Be careful about the placement of functions of x in y's integral. Plot this area.

6.15 Write the following repeating decimals forms in series notation:

$0.3333\ldots$ $\quad 0.43114311\ldots$ $\quad 0.484848\ldots$

$0.1234512345\ldots$ $\quad 555551555551\ldots$ $\quad 0.221222223221222223\ldots.$

6.16 Evaluate the Maclaurin series for

- e^x at 1.
- $\sqrt{x}$ at 4.
- $\log(x)$ at 7.389056.

6.17 Find the Maclaurin series for $\sin(x)$ and $\cos(x)$. What do you observe?

6.18 The Mercator series is defined by

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

which converges for $-1 < x \geq 1$. Write a general expression for this series using summation notation.

6.19 Find the vertical and horizontal tangent lines to the ellipse defined by

$$x = 6 + 3\cos(t) \qquad y = 5 + 2\sin(t).$$

6.20 Express a hyperbola with $a = 9$ and $b = 8$ in $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j}$ notation, and give the slope-intercept forms for the two vertical tangents.

6.21 Given $\mathbf{f}(t) = \frac{1}{t}\mathbf{i} + \frac{1}{t^3}$, find the first three orders of derivatives. Solve for $t = 2$.

6.22 For the function $\mathbf{f}(t) = e^{-2t}\mathbf{i} + \cos(t)\mathbf{j}$, calculate the integral from 1 to 2.

6.23 A number of seemingly counterintuitive voting principles can actually be proven mathematically. For instance, Brams and O'Leary (1970) claimed that "If three kinds of votes are allowed in a voting body, the probability that two randomly selected members disagree on a roll call will be maximized when one-third of the members vote 'yes,' one-third 'no,' and one third 'abstain."' The proof of this statement rests on the premise that their probability of disagreement function is maximized when $y = n = a = t/3$, where y is the number voting yes, n is the number voting no, a is the number abstaining, and these are assumed to divide equally into the total number of voters t. The disagreement function is given by

$$p(DG) = \frac{2(yn + ya + na)}{(y + n + a)(y + n + a - 1)}.$$

Use the Lagrange multiplier method to demonstrate their claim by first taking four partial derivatives of $p(DG) - \lambda(t - y - n - a)$ with respect to y, n, a, λ (the Lagrange multiplier); setting these equations equal to zero; and solving the four equations for the four unknowns.

6.24 Doreian and Hummon (1977) gave applications of differential equation models in empirical research in sociology with a focus transforming the data in useful ways. Starting with their equation (14):

$$\frac{X_2 - X_{20}}{q} = \frac{X_1 - X_{10}}{p} \cos\varphi - \left[1 - \frac{(X_1 - X_{10})^2}{p}\right]^{\frac{1}{2}} \sin\varphi,$$

substitute in

$$p = \frac{\beta_4^2 - \beta_0}{\beta_1 - \beta_2^2} + X_{10}^2 \qquad q = p\beta_1^{\frac{1}{2}} \qquad \cos\varphi = -\frac{\beta_2}{\beta_1^{\frac{1}{2}}}$$

$$X_{20} = \frac{\beta_2\beta_3 - \beta_1\beta_4}{\beta_1 - \beta_2^2} \qquad X_{10} = \frac{\beta_2\beta_4 - \beta_3}{\beta_1 - \beta_2^2}$$

to produce an expression with only β and X terms on the left-hand side and zero on the right-hand side. Show the steps.