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University of St. Gallen

International Macroeconomics

Assignment 1a: Sovereign Default

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1.) State the optimization problem of the government

Definition The maximization problem is given by

$$\max_{B_2} \quad u(c_1) + \beta \mathbb{E}_{y_2}\big\{u(c_2)\big\} \tag{1}$$

$$s.t. \qquad c_1 = y_1 - B_1 + q_1(B_2)B_2 \tag{2}$$

$$u(c_2) = max\big\{u(c_2^d, u_2^r)\big\} \tag{3}$$

where $c_2^d := y_2^d$ and $c_2^r := y_2 - B_2$. Moreover, if we substitute the definitions for c_2^d and c_2^r into the maximization problem, we can rewrite (1) to the following expression

$$\max_{B_2} \; u(y_1 - B_1 + q_1(B_2)B2) + \beta \mathbb{E}_{y_2}\big\{\max\{u(y^d), u(y_2 - B_2)\}\big\} \tag{4}$$

where the expectation can also be defined as

$$\mathbb{E}_{y_2}\big\{\max\{u(y^d), u(y_2 - B_2)\}\big\} = F(y^d + B_2)u(y^d) + \mathbb{E}_{y_2}\big\{u(y_2 - B_2)|y_2 \geq y^d + B_2\big\} \tag{5}$$

$$:= F(y^d + B_2)u(y^d) + \int_{y^d+B2}^{\infty} u(y_2 - B_2)f(y_2)dy_2. \tag{6}$$

where $F(y^d + B_2)u(y^d)$ is the certain utility gain the government can expect. $\int_{y^d+B2}^{\infty} u(y_2 - B_2)f(y_2)dy_2$ is the additional utility gain of not defaulting.

Derivation We have to consider the Leibniz integral rule to take the derivative of the expectation with respect to (w.r.t) B2

$$\frac{\partial \mathbb{E}_{y_2}\big\{\max\{u(y^d), u(y_2 - B_2)\}\big\}]}{\partial B_2} = f(y^d + B_2)u(y^d) + 0 - f(y^d + B_2)u(y^d) \tag{7}$$

[redacted]

$$- \int_{y^d+B_2}^{\infty} u'(y_2 - B_2)f(y_2)dy_2$$

$$= - \int_{y^d+B_2}^{\infty} u'(y_2 - B_2)f(y_2)dy_2. \tag{8}$$

Please note that the derivative of infinity w.r.t B_2 automatically becomes zero, and the derivative of the utility gain from defaulting equals the derivative of the lower bound of the integral. Consequently, we obtain the definition (8).

The derivation of the remaining part can be written as

$$\begin{aligned}\frac{\partial u(c_1)}{\partial B_2} &= u'(c_1)q_1(B_2) + u'(c_1)\frac{\partial q_1(B_2)}{\partial B_2}B_2 && (9)\\ &= u'(c_1)q_1(B_2) + u'(c_1)\underbrace{\frac{\partial q_1(B_2)}{\partial B_2}\frac{B_2}{q_1(B_2)}}_{-\varepsilon_{q_1,B_2}}q_1(B_2) && (10)\\ &= u'(c_1)q_1(B_2)\big(1-\varepsilon_{q_1,B_2}\big). && (11)\end{aligned}$$

Combining (8) and (11), finally yields the Euler Equation[1]

$$u'(y_1 - B_1 + q_1B_2)q_1\big(1-\varepsilon_{q_1,B_2}\big) = \beta\int_{y_2\geq y^d+B_2} u'(y_2-B_2)f(y_2)dy_2 \qquad (12)$$

where $q_1 = q_1(B_2)$ and $\int_{y_2\geq y^d+B_2}\dots = \int_{y^d+B_2}^{\infty}\dots$ are notional simplifications. $\square$

2.) Derivation of the bond price q_1

Probability $F(y_2)$ is defined as the cdf of $f(y_2)$. Thus, we can write it in probability notation $F(\tilde{x}) = \mathbb{P}\{y_2 \leq \tilde{x}\}$ where $\tilde{x}$ defines a fixed point. To calculate the default probability, we have to calculate $\mathbb{P}\{y_2 \leq y^d + B_2\}$ where $\tilde{y} \coloneqq y^d + B_2$ defines the threshold under which the government decides to default. Figure 1 graphically illustrates the definition of this probability. Furthermore, we can plug in the definition of the penalty $y^d = y_2 - a_0\frac{y_2^{1+a_1}}{1+a_1}$ such that we obtain

$\mathbb{P}$ [redacted] $\dfrac{y_2^{1+a_1}}{1+a_1} + B_2\Big\}$ (13)

$$\Leftrightarrow\ \mathbb{P}\left\{a_0\frac{y_2^{1+a_1}}{1+a_1} \leq B_2\right\}. \qquad (14)$$

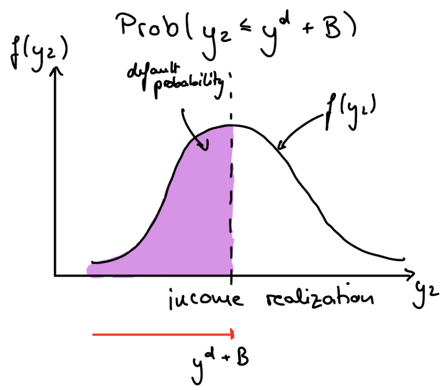

Figure 1: pdf of income in the second period

[1] Be aware that β was not considered in the derivation of (8) and the definition of c_2, given by (2), is substituted

[redacted]

Rearranging for y_2 yields then the final term $y_2 \leq \left[\frac{B_2(1+a_1)}{a_0}\right]^{\frac{1}{1+a_1}}$ and, hence, the definition

$$
\begin{aligned}
&\mathbb{P}\left\{y_2 \leq \left[\frac{B_2(1+a_1)}{a_0}\right]^{\frac{1}{1+a_1}}\right\} && (15)\\
&= F\left(\left[\frac{B_2(1+a_1)}{a_0}\right]^{\frac{1}{1+a_1}}\right) && (16)\\
&= F(\tilde{y}_2(B_2)). && (17)
\end{aligned}
$$

Pricing equation The definition for the risk-free bond in a two-period time framework is universally defined as $\mathbb{E}\{m\} = \frac{1}{1+r}$ in the absence of default (or any other additional) risk in standard asset pricing literature (e.g., Cochrane (2009, p. 11)).[2] In a framework **without** default risk, the maximization problem is defined as

$$
\begin{aligned}
&\max_{B_2} \quad u(c_1) + \beta \mathbb{E}_{y_2}\{u(c_2)\} && (18)\\
s.t. \quad & c_1 = y_1 - B_1 + q_1(B_2)B_2 && (19)\\
& c_2 = y_2 - B_2. && (20)
\end{aligned}
$$

Under straightforward algebra, the Euler Equation can be derived as

$$
\begin{aligned}
u'(c_1)q_1(1-\varepsilon_{q_1,B_2}) &= \beta \mathbb{E}_{y_2}\{u'(c_2)\} && (21)\\
\Leftrightarrow \quad q_1 &= \mathbb{E}_{y_2}\{\underbrace{\beta\frac{u'(c_2)}{u'(c_1)}(1-\varepsilon_{q_1,B_2})^{-1}}_{\text{SDF/pricing kernel}}\} = \frac{1}{1+r} && (22)
\end{aligned}
$$

where (22) is the pricing equation for the risk-free asset.

[redacted]

[2] m is the stochastic discount factor (SDF)

[redacted]

In a framework **with** default risk, we can start from (13) and show that

$$
\begin{aligned}
u'(y_1 - B_1 + q_1 B_2)q_1\big(1 - \varepsilon_{q_1,B_2}\big) &= \beta \int_{y_2 \geq y^d + B_2} u'(y_2 - B_2) f(y_2) dy_2 && (23)\\
&= \beta \int_{y^d+B_2}^{\infty} u'(y_2 - B_2) f(y_2) dy_2 && (24)\\
&\quad + \beta \int_{-\infty}^{y^d+B_2} u'(y_2 - B_2) f(y_2) dy_2 \\
&\quad - \beta \int_{-\infty}^{y^d+B_2} u'(y_2 - B_2) f(y_2) dy_2 \\
&= \beta \int_{-\infty}^{\infty} u'(y_2 - B_2) d\mathbb{P}(y_2) && (25)\\
&\quad - \beta \int_{-\infty}^{\infty} u'(y_2 - B_2) d\mathbb{P}(y_2 \leq y^d + B_2) \\
&= \beta \mathbb{E}_{y_2}\big\{u'(y_2 - B_2)\big\} && (26)\\
&\quad - \beta \mathbb{E}_{y_2}\big\{u'(y_2 - B_2)\mathbb{1}_{y_2 \leq y^d + B_2}\big\} \\
&= \beta \mathbb{E}_{y_2}\big\{u'(y_2 - B_2)(1 - \mathbb{1}_{y_2 \leq y^d + B_2})\big\} && (27)\\
&= (1 - \underbrace{\mathbb{P}_{y_2 \leq y^d + B_2}(y_2)}_{F(\tilde{y})})\mathbb{E}_{y_2}\big\{\beta u'(y_2 - B_2)\big\} && (28)\\
\Leftrightarrow q_1 &= (1 - F(\tilde{y}))\mathbb{E}_{y_2}\big\{\beta \frac{u'(c_2)}{u'(c_1)}(1 - \varepsilon_{q_1,B_2})^{-1}\big\} && (29)\\
&\overset{(22)}{=} \frac{1 - F(\tilde{y}_2)}{1 + r}. && (30)
\end{aligned}
$$

Plugging (16) into (30) finally gives

$$
q_1 = \frac{1 - F\left(\left[\frac{B_2(1+a_1)}{a_0}\right]^{\frac{1}{1+a_1}}\right)}{1 + r} \tag{31}
$$

where [redacted] to the risk-free rate, $\tilde{y}_2 \coloneqq \left[\frac{B_2(1+a_1)}{a_0}\right]^{\frac{1}{1+a_1}}$ is the threshold value for the governance's default choice, and $q_1 = q_1(B_2)$ is again a notional simplification. □

[redacted]

3.) Formal expression of the elasticity ε_{q_1,B_2}

Derivation Under consideration of (31), the derivative of $q_1(B_2)$ w.r.t B_2 yields

$$\frac{\partial q_1(B_2)}{\partial B_2} = -\frac{F'(\tilde{y}_2)}{(1+r)}\frac{1}{a_0}\left(\frac{B_2(1+a_1)}{a_0}\right)^{-\frac{a_1}{1+a_1}} \tag{32}$$

$$= -\frac{f(\tilde{y})}{1+r}\left(\frac{B_2(1+a_1)}{a_0}\right)^{-\frac{a_1}{1+a_1}}\frac{1}{a_0}. \tag{33}$$

The elasiticity is then defined as

$$\varepsilon_{q_1,B_2} \coloneqq -\frac{\partial q_1(B_2)}{B_2}\frac{B_2}{q_1(B_2)} \tag{34}$$

$$= \frac{f(\tilde{y})}{1+r}\left(\frac{B_2(1+a_1)}{a_0}\right)^{-\frac{a_1}{1+a_1}}\frac{1}{a_0}\frac{B_2}{q_1}. \tag{35}$$

□

[redacted]

[redacted]

4.) **Coding**

a) - g) *See MATLAB Code*

h) The intersection point of the Euler Equation and, thus, the equilibrium is graphically given in figure 2. Distortions in the elasticity of the bond price w.r.t debt, ε_{q_1,B_2}, can explain the deviation around debt values of 0.65 because q_1 and $f(\tilde{y})$ tend towards zero with increasing levels of debt.[3] Likewise, the marginal utility of c_1 tends towards infinity which kinks the left-hand side of the Euler equation for $B_2 > 0.67$. Consequently, we observe stark fluctuations in the blue curve with increasing debt. Values above a certain threshold are not computed because MATLAB calculates only 16 decimal places. Hence, the left-hand side of the Euler equation is only plotted partially.

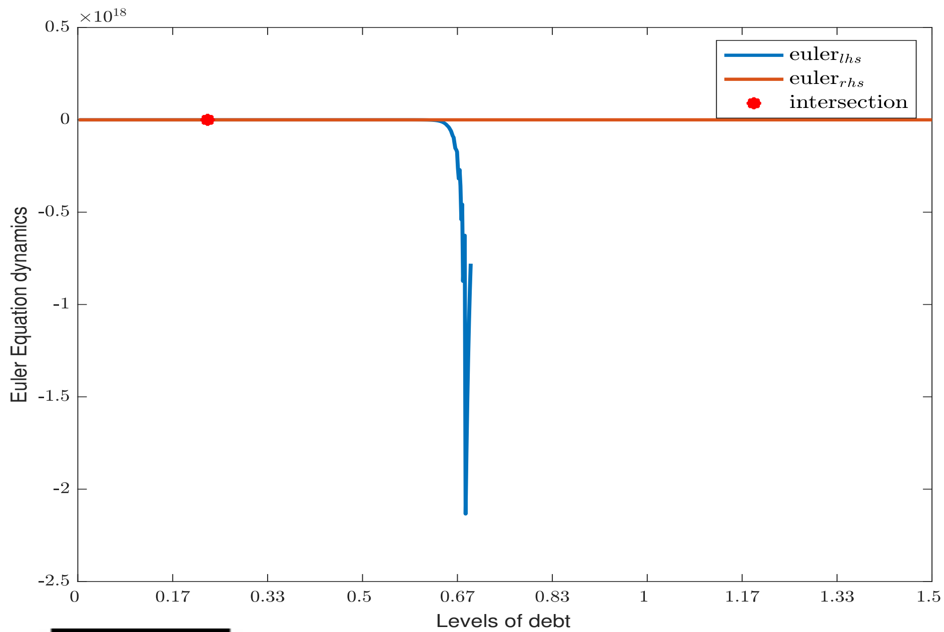

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Figure 2: Dynamics of both sides of the Euler Equation (13)

In equilibrium, the government chooses $B_2 = 0.2267$ values of debt. The corresponding bond price is $q_1(B_2) = 0.9414$, and the default probability is $\delta = 0.0492$.

[3] See figure 7 in the Appendix for a plot of ε_{q_1,B_2}

[redacted]

5.) **Sensitivity of optimal choice of debt B_2, bond price q_1, and default probability δ in equilibrium**

Variation of B_1 and y_1 Table 1 shows a decreasing pattern for B_2 with decreasing initial debt B_1. Hence, equations 17 and 30 indicate that the default probability δ also decreases and the bond price q_1 increases. Intuitively, the government has enough budget in the first period if it has less initial debt and, therefore, it has to shift less income from the second to the first period by creating new debt B_2. Consequently, low debt levels in the second period make the government less likely to default, which increases bond prices because these assets bear less risk.

	B1 =0.1	B1 =0.2	B1 =0.3	B1 =0.5	B1 =0.7	B1=1
B2	0.064565	0.11411	0.16366	0.21171	0.22072	0.22673
q1	0.9901	0.9901	0.9901	0.97844	0.96116	0.94136
prob.(default)	3.1612e-38	3.4649e-15	2.5078e-06	0.011772	0.029232	0.04923

Table 1: Changes of B_2, q_1, and δ with respect to B_1.

One can see the exact same relationship for increasing income endowment. If y_1 increases in the same amount as B_1 decreases, namely $\Delta y_1 = -\Delta B_1$, the government decision is equally affected because the changes in c_1 and the Euler Equation are identical. Table 2 provides the changes of B_2, q_1, and δ for the same absolute changes in y_1 in the opposite direction of B_1.

	y1 =1.9	y1 =1.8	y1 =1.7	y1 =1.6	y1 =1.3	y1=1
B2	0.064565	0.11411	0.16366	0.1997	0.22072	0.22673
q1	0.9901	0.9901	0.9901	0.98746	0.96116	0.94136
prob.(default)	3.1612e-38	3.4649e-15	2.5078e-06	0.0026655	0.029232	0.04923

Table 2: Changes of B_2, q_1, and δ with respect to y_1.

[redacted]

Tables 5 and 6 also show that the inverse relationship between B_1 and y_1 holds for changes in the opposite direction.[4] However, the model results in economically implausible values, as c_1 becomes negative for $B_1 > 1.2$ or $y_1 < 0.8$.[5] Restricting consumption to be non-negative might fix this problem. However, the previously explained relationship holds for increments in B_1 until 1.2 or declines in y_1 until 0.8. This can be

[4]Tables are attached in the Appendix

[5]$c_1(B_2 = 1.2) = 0.0134$ and $c_1(B_2 = 1.25) = -0.0366$

[redacted]

briefly explained by the following chain of causation:
More B_1 (less y_1) increases (decreases) debt B_2 and, hence, increases (decreases) the probability of defaulting δ and reduces (increases) the bond price q_1.

Figure 3 also provides visual evidence for the finding explained previously. The left-hand side of the Euler Equation flattens for decreasing levels of B_1. The dashed curves are the Euler equations for the previous initial debt level, namely $B_1 = 1$. Values closer to one have larger fluctuations, which can be explained by the increasing computational impact of the elasticity ε_{q_1,B_2} and the exponential increase in marginal utility in the first period, $c(c_1)$.

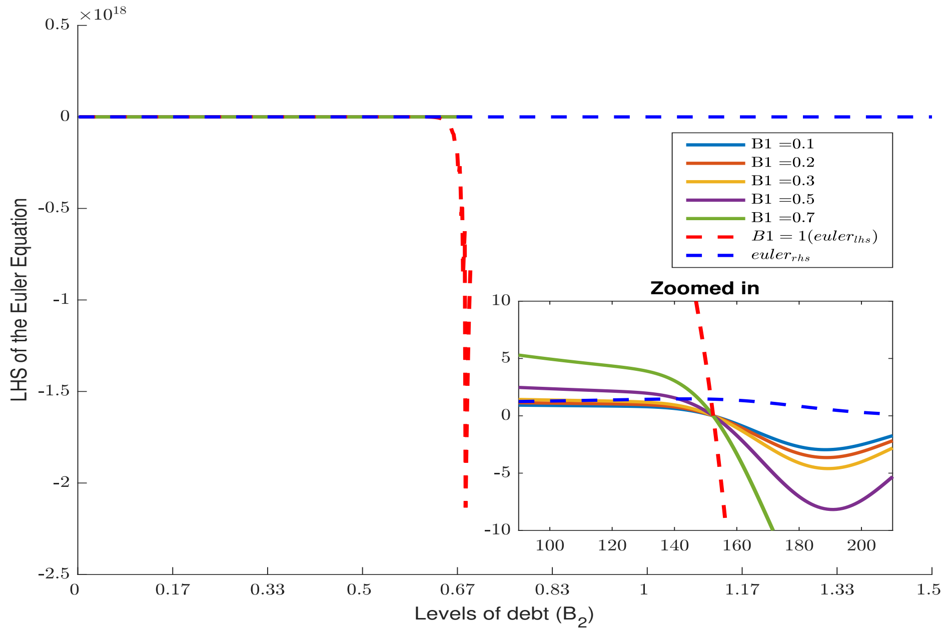

Figure 3: Euler Equations for different values of B_1

[redacted]

Variation of r Table 3 shows that changes in the risk-free rate have (almost) no effect on the optimal amount of debt B_2 and the default probability. Only the bond price is negatively affected by increments in r. Intuitively, it becomes more attractive for investors to invest in government bonds if the risk-free rate decreases. Hence, demand for bonds increases, which increases prices. The opposite holds analogously for declines in r.

[redacted]

	r =0.001	r =0.005	r =0.02	r =0.1	r =0.2	r=0.01
B2	0.22673	0.22673	0.22673	0.22673	0.22673	0.22673
q1	0.94982	0.94604	0.93213	0.86434	0.79231	0.94136
prob.(default)	0.04923	0.04923	0.04923	0.04923	0.04923	0.04923

Table 3: Changes of B_2, q_1, and δ with respect to r.

Figure 4 also supports the finding that changes in r have (almost) no impact on either B_2 or q_1 as the shape of the left-hand side Euler equation is (almost) not affected. Consequently, the intersection point barely shifts.

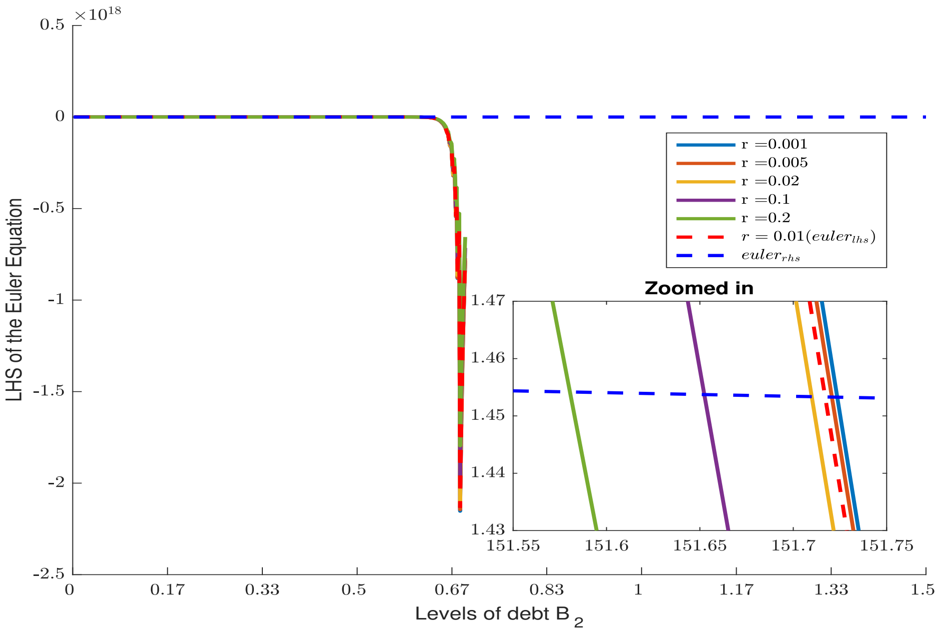

Figure 4: Euler Equations for different values of $r1$

[redacted]

[redacted]

6.) **Sensitivity of optimal choice of debt B_2, bond price q_1, and default probability δ in equilibrium**

Changing the penalty function from $L(y_2) = a_0 \frac{y_2^{1+a1}}{1+a1}$ to $L(y_2) = \alpha y_2$, leads to an increment of debt in the second period, as indicated in figure 5.

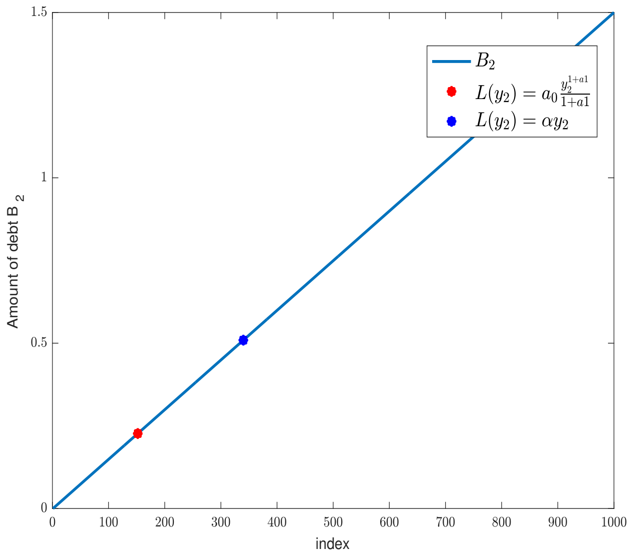

Figure 5: Different level of debt B_2 for $L(y_2) = a_0 \frac{y_2^{1+a1}}{1+a1}$ and $L(y_2) = \alpha y_2$

The new penalization rule changes the probability from (16) to the following equation

$$\mathbb{P}\left\{ y_2 \leq \frac{1}{\alpha} \right\}. \tag{36}$$

Under the given parameters, the probability of default shifts to the right, as indicated by the left figure in 6, decreasing the default probability for a higher debt B_2. Intuitively, [redacted] the penalization rule is equal to a decline in the parameter of the loss fu[redacted] the given parameters, which would shift the probability curve to the right and the bond price curve to the left. Moreover, the curves are shaped differently under a linear penalization rule and become less steep. Intuitively, the shift in $L(y_2)$ and its new shape is interpreted as an increasing incentive for governments to acquire more debt B_2. Less penalization also leads to a smaller probability of default simultaneously as the government's default probability shrinks, the bond price

[redacted]

increases. Figure 6 presents all changes in δ and q_1.[6]

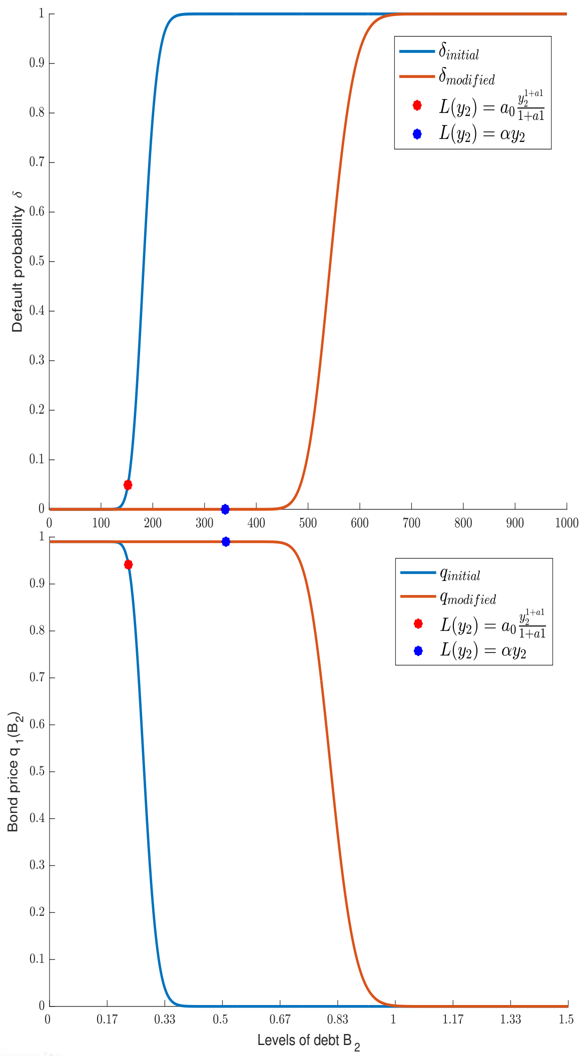

Figure [redacted]ault probabilities δ (left) and bond prices q_1 (right) for $L(y_2) = a_0 \frac{y_2^{1+a1}}{1+a1}$ and $L(y_2) = \alpha y_2$

Moreover, foreign creditors should be (almost) indifferent to investments in government bonds or risk-free bonds when the default probability approaches zero.

[6] Tables are left out as graphs are sufficient to explain the mechanics and intuition of a change in the penalization rule.

[redacted]

7.) Multiple equilibria and their conditions in a model of sovereign default

Ayres et al. (2018) make the following additional assumptions to a standard model of sovereign default: i) *changes in the timing assumption*, ii) *decision over debt at maturity or current debt*, and iii) *changes in the income distribution.*
i) differs between models where creditors move first and offer funds at a specific interest rate and those models where borrowers move first and pick the amount of debt. ii) determines the decision variable of borrowers. Either they choose current debt or debt at maturity. iii) changes the distribution of the stochastic income process in period two from a standard distribution (e.g., normal distribution) to a bimodal distribution. Table 4 summarizes the findings of unique and multiple equilibria.[7]

	debt at maturity	**current debt**
borrower first	unique	depending on schedule (multiple)
creditor first	multiple	multiple

Table 4: Four cases in Ayres et al. (2018)

(borrower first; debt at maturity): The borrower will face an expected return on the debt schedule[8] $h(R; a)$, which is monotonically increasing in debt at maturity a. In such an equilibrium, the borrower will always choose a low-interest and low-debt **unique** equilibrium to minimize debt repayments (= debt at maturity). For the borrower, choosing debt at maturity amounts to choosing the default probability and the interest rate. [redacted]
(borrower first; current debt): In Ayres et al. (2018), the schedule is a Laffer-type curve with a low-interest and a high-interest equilibrium. The low-interest equilibrium is equal to the one described in the previous scenario. The second equilibrium lies on the "wrong" side of the curve, where the interest rate decreases with debt. This result is

[7] As in Ayres et al. (2018), iii) is an additional assumption made in the case of *(borrower first; current debt)*

[8] In the following, the term expected return on the debt schedule will be abbreviated with schedule

[redacted]

economically counter-intuitive and faces striking properties as borrowers and lenders benefit jointly from lowering debt or interest. As a consequence, the default probability would decrease, and profits increase. Ayres et al. (2018) point out that many different schedules exist, which will determine if the model of sovereign default has unique or multiple equilibria. The model will have multiple equilibria and a Laffer-type curve for standard distributions like the normal. Finally, the authors also point out that the multiplicity of equilibria becomes stable for bimodal output because it incorporates two local interior maxima into the maximization problem. Moreover, the bimodal output will always lead to at least two solutions and multiplicity if debt levels are not too high.

(creditor first; debt at maturity) and *(creditor first; current debt)*: If the creditors move first, the equilibrium will be the same for models where borrowers choose either debt at maturity or current debt because they will act like price takers. In the models explained by Ayres et al. (2018), debt at maturity will be equal to the gross service of the debt, namely $a = Rb$.

[redacted]

[redacted]

References

Ayres, J., Navarro, G., Nicolini, J. P., and Teles, P. (2018). Sovereign default: The role of expectations. *Journal of Economic Theory*, 175:803–812.

Cochrane, J. (2009). *Asset pricing: Revised edition*. Princeton university press.

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A Appendix: Tables and Figures

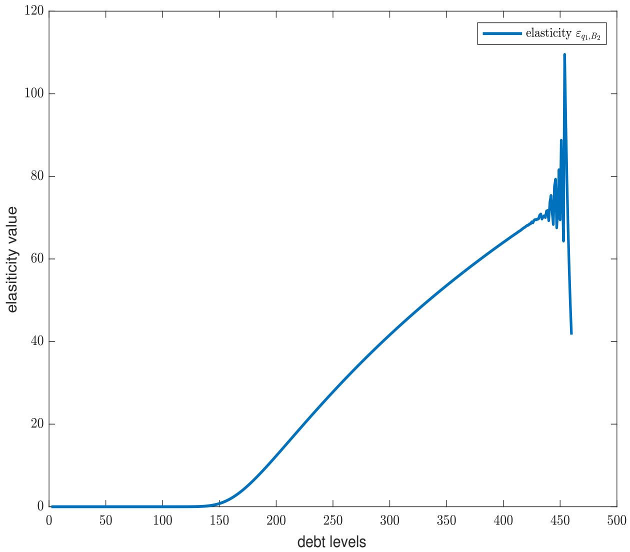

Figure 7: Elasticity ε_{q_1,B_2} for different level of debts

	B1 =1.9	B1 =1.6	B1 =1.3	B1 =1.2	B1 =1.1	B1=1
B2	0.21171	0.22372	0.22823	0.22823	0.22823	0.22673
q1	0.97844	0.95224	0.93511	0.93511	0.93511	0.94136
prob.(default)	0.011772	0.03824	0.055536	0.055536	0.055536	0.04923

Table 5: Changes of B_2, q_1, and δ with respect to B_1.

	y1 =0.1	y1 =0.4	y1 =0.7	y1 =0.8	y1 =0.9	y1=1
B2	0.21171	0.22372	0.22823	0.22823	0.22823	0.22673
q1	0.97844	0.95224	0.93511	0.93511	0.93511	0.94136
prob[redacted]	11772	0.03824	0.055536	0.055536	0.055536	0.04923

Table 6: Changes of B_2, q_1, and δ with respect to y_1.

[redacted]

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(Lauritz Storch)

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