University of St. Gallen

International Macroeconomics

Assignment 2a: 2-Period Model with Currency Risk

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1.) Set up Household optimization problem

The household optimization problem is given by

$$\max_{c_t, c_{s_{t+1}}, b_{t+1}, b^*_{t+1}} \quad u(c_t) + \beta \mathbb{E}_{s_{t+1}} \{ u(c_{s_{t+1}}) \} \tag{1}$$

$$s.t. \quad e_t b^*_{t+1} + b_{t+1} + pc_t = py_t \tag{2}$$

$$pc_{s_{t+1}} = py_{s_{t+1}} + e_{s_{t+1}}(1 + i^*_t) b^*_{t+1} + (1 + i_t) b_{t+1} \tag{3}$$

where b_{t+1} is the domestic bond level, b_{t+1} is the foreign bond level, c_t is the consumption level, e_t is the exchange rate to convert the domestic currency into the foreign currency, and i_t and i^*_t are the domestic and foreign interest rates, respectively. The subscript s_{t+1} denotes the state-dependency of variables in period $t+1$. Hence, $c_{s_{t+1}}$, $e_{s_{t+1}}$, and $y_{s_{t+1}}$ are the uncertain levels of consumption, exchange rate, and income, respectively. $\mathbb{E}_{s_{t+1}}$ declares the expectation formation over these variables.
Constraints (2) and (3) can be rearranged for c_t and $c_{s_{t+1}}$. Moreover, income and the exchange rate are unknown in period $t = 1$. It is assumed that $y_{s_{t+1}}$ and $e_{s_{t+1}}$ are distributed by $f(y_{s_{t+1}}, e_{s_{t+1}})$. Plugging the rearranged constraints into (1), the maximization problem can be rewritten to

$$\begin{aligned} \max_{b_{t+1}, b^*_{t+1}} \quad & u(y_t - \frac{e_t}{p} b^*_{t+1} - \frac{1}{p} b_{t+1}) \\ & + \beta \int_{y_{s_{t+1}}} \int_{e_{s_{t+1}}} u(y_{s_{t+1}} + \frac{e_{s_{t+1}}}{p}(1 + i^*_t) b^*_{t+1} + \frac{1}{p}(1 + i_t) b_{t+1}) \\ & \times f(y_{s_{t+1}}, e_{s_{t+1}}) dy_{s_{t+1}} de_{s_{t+1}} \end{aligned} \tag{4}$$

2.) Derivation of the Euler equations

Maximizing (4) with respect to b_{t+1} yields the *Euler equation for domestic bonds*

$$1 = (1 + i_t) \int_{y_{s_{t+1}}} \int_{e_{s_{t+1}}} \underbrace{\beta \frac{u'(c_{s_{t+1}})}{u'(c_t)}}_{M_{s_{t+1}}} f(y_{s_{t+1}}, e_{s_{t+1}}) dy_{s_{t+1}} de_{s_{t+1}} \tag{5}$$

$$= (1 + i_t) \mathbb{E}_{s_{t+1}} \{ M_{s_{t+1}} \} \tag{6}$$

where $M_{s_{t+1}} := \beta \frac{u'(c_{s_{t+1}})}{u'(c_t)}$ is called the pricing kernel. Multiplying $M_{s_{t+1}}$ with any nominal payment in a given state of the world in period $t + 1$ returns the expected

value of such a payment in period t.
Maximizing with respect to b^*_{t+1} yields the *Euler equation for foreign bonds*

$$1 = (1+i^*_t)\int_{y_{s_{t+1}}}\int_{e_{s_{t+1}}} \beta \frac{u'(c_{s_{t+1}})}{u'(c_t)}\frac{e_{s_{t+1}}}{e_t} f(y_{s_{t+1}}, e_{s_{t+1}}) dy_{s_{t+1}} de_{s_{t+1}} \tag{7}$$

$$= (1+i^*_t)\mathbb{E}_{s_{t+1}}\{M_{s_{t+1}}\frac{e_{s_{t+1}}}{e_t}\}. \tag{8}$$

Interpretation The Euler Equation is a requirement for households to maximize their utility. If it does not hold, then there would exist some alternative intertemporal arrangement of consumption that would make the household better off.
In the context of domestic and foreign bonds, households can either save domestically or in foreign assets, which pay the interest rates i_t and i^*_t, respectively. Households first need to convert domestic into foreign currency at the exchange rate e_t to buy a foreign asset. Returns generated by this asset also have to be converted back at the state-dependent exchange rate $e_{s_{t+1}}$.
The Euler equation for domestic bonds reflects the intertemporal decision between consuming and saving domestically. In equilibrium, households should be indifferent between consuming another unit of income or saving it by buying a domestic bond.
The Euler equation for foreign bonds reflects the intertemporal decision between consuming and saving abroad. In equilibrium, households should be indifferent between consuming another unit of income or saving it by buying a foreign bond, which comes with exchange rate risk, because the future exchange rate $e_{s_{t+1}}$ is unknown in period t.

3.) **UIP and the risk premium**

Using equation (8), one can show that

$$\begin{aligned}
1 &= (1+i_t^*)\mathbb{E}_{s_{t+1}}\left\{M_{s_{t+1}}\frac{e_{s_{t+1}}}{e_t}\right\} \\
&= (1+i_t^*)\mathbb{E}_{s_{t+1}}\{M_{s_{t+1}}\}\mathbb{E}_{s_{t+1}}\left\{\frac{e_{s_{t+1}}}{e_t}\right\} \\
&+ (1+i_t^*)Cov_{s_{t+1}}\left(M_{s_{t+1}}, \frac{e_{s_{t+1}}}{e_t}\right) \qquad (9) \\
\overset{(6)}{\Longleftrightarrow} \quad (1+i_t) &= (1+i_t^*)\mathbb{E}_{s_{t+1}}\left\{\frac{e_{s_{t+1}}}{e_t}\right\} \\
&+ (1+i_t^*)\frac{1}{\mathbb{E}_{s_{t+1}}\{M_{s_{t+1}}\}}Cov_{s_{t+1}}\left(M_{s_{t+1}}, \frac{e_{s_{t+1}}}{e_t}\right) \qquad (10)
\end{aligned}$$

where $(1+i_t^*)\frac{1}{\mathbb{E}_{s_{t+1}}\{M_{s_{t+1}}\}}Cov_{s_{t+1}}\left(M_{s_{t+1}}, \frac{e_{s_{t+1}}}{e_t}\right)$ refers to the *currency risk premium*, which reflects the additional risk premium demanded by risk-averse investors due to uncertainty in future exchange rates $e_{s_{t+1}}$.
The traditional *uncovered interest rate parity* (UIP) is given by

$$1 + i_t = (1+i_t^*)\mathbb{E}_{s_{t+1}}\left\{\frac{e_{s_{t+1}}}{e_t}\right\} \qquad (11)$$

and, hence, assumes that investors do not care about risk, $\sigma_{y,e} \equiv Cov_{s_{t+1}}\left(M_{s_{t+1}}, \frac{e_{s_{t+1}}}{e_t}\right) = 0$. This would only be true if and only if investors are risk neutral, $\sigma = 0$. As this is not given by assumption, the UIP fails for any risk-averse investor, $\sigma > 0$.

4.) **Solving the model in MATLAB**

Households are assumed to be identical, which results in a zero domestic bond equilibrium, $b_{t+1} = 0$.

a) - b) *See MATLAB Code*

c) The anonymous function for the bivariate lognormal distribution is given by

$$\begin{aligned} f(y_{s_{t+1}}, e_{s_{t+1}}) = {} & \frac{1}{2\pi\sigma_y\sigma_e\sqrt{1-\rho_{y,e}^2}} \\ & \times \exp\left[-\frac{1}{2(1-\rho_{y,e}^2)} \left(\frac{(ln(y_{s_{t+1}}) - \mu_y)^2}{\sigma_y^2} + \frac{(ln(e_{s_{t+1}}) - \mu_e)^2}{\sigma_e^2} \right.\right. \\ & \left.\left. - 2\rho_{y,e} \frac{(ln(y_{s_{t+1}}) - \mu_y)(ln(e_{s_{t+1}}) - \mu_e)}{\sigma_y\sigma_e} \right) \right] \end{aligned} \tag{12}$$

d) The equilibrium level of foreign bonds b_{t+1}^* is 0.1078 and the equilibrium exchange rate e_t is 2.7347.

e) The equilibrium risk premium is

$$\begin{aligned} RP_t & \equiv 1 + i_t - (1 + i_t^*)\mathbb{E}_{s_{t+1}}\left\{ \frac{e_{s_{t+1}}}{e_t} \right\} \\ & = (1 + i_t^*) \frac{1}{\mathbb{E}_{s_{t+1}}\{M_{s_{t+1}}\}} Cov_{s_{t+1}} \left(M_{s_{t+1}}, \frac{e_{s_{t+1}}}{e_t} \right) = -0.0041. \end{aligned}$$

5.) Numerical comparison

Table 1-4 present all numerical values in the Appendix.

i) variation in the risk aversion coefficient σ

Households face uncertainty in the exchange rate for period $t+1$. The decline in the risk premium in figure 1 captures the negative relationship between the pricing kernel and the relative change of future and current exchange rates. The pricing kernel itself has an inverse relationship to consumption in the next period, e.g., $u'(c_{s_{t+1}}) = 1/c^{\sigma}_{s_{t+1}}$, which implies that exchange rates $e_{s_{t+1}}$ and consumption $c_{s_{t+1}}$ move cyclical. Hence, the foreign bond is a bad hedge against exchange rate risk and households demand higher compensation for this risk if the degree of risk aversion increases linearly.

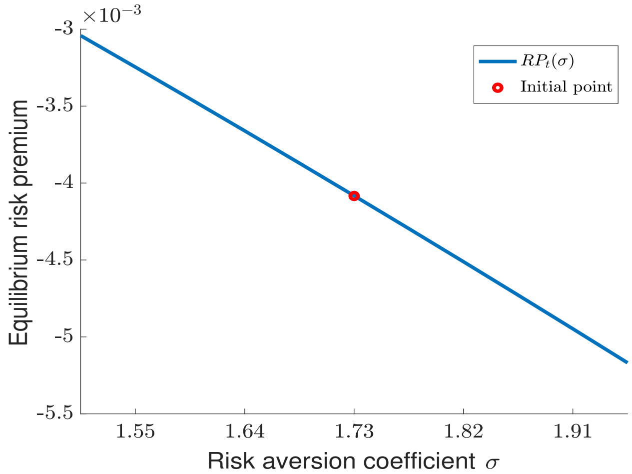

Figure 1: Change in the risk premium due to a change in the risk aversion coefficient σ

The equilibrium foreign bond level b^*_{t+1} increases with the risk aversion coefficient σ. Intuitively, foreign bonds b^*_{t+1} are used to hedge against income risk. Even though households perceive additional exchange rate risk with an increasing degree of risk aversion, the desire for a hedge against income risk overweights.[1] Foreign bonds are the only asset through which households can hedge against income risk. Therefore, higher risk perception also leads to a higher demand for foreign bonds to avoid/counteract income risk in the future. Figure 2 illustrates that this relationship is linear. How-

[1] See figure 4, a variation in the variance of the income has larger effects on the foreign bond level than variations of the exchange rate variance

ever, the increments for b^*_{t+1} are slightly decreasing with increasing values for the risk aversion coefficient.

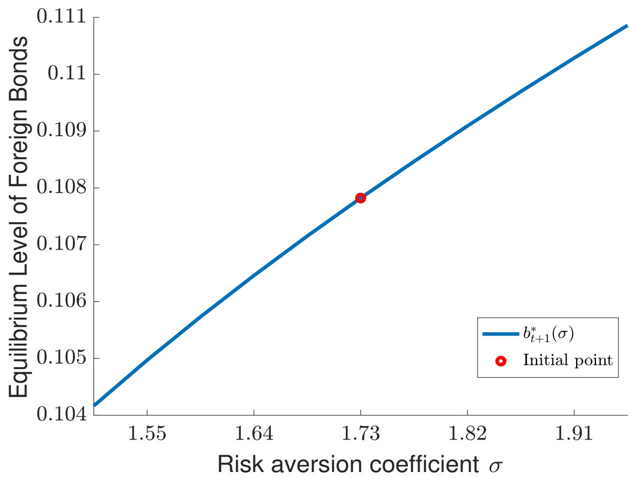

Figure 2: Change in the equilibrium level of foreign bonds due to a change in the risk aversion coefficient σ

As the risk perception tightens ($\sigma \uparrow$), the exchange rate e_t decreases. The UIP (10) decomposes the risk premium into two components - interest rate changes $(1 + i^*_t)$ and $(1 + i_t)$ and expected exchange rate adjustments $\mathbb{E}_{s_{t+1}}\{e_{s_{t+1}}/e_t\}$. Exchange rates are the only channel through which changes in the risk premium and the foreign bond level are prized because interest rates are assumed to stay constant in the model. The effect of exchange rate risk overweights the UIP effects because the overall effect shows less demand for foreign currency. The UIP fails to hold due to the risk premium. The domestic currency appreciates relative to the foreign currency and the exchange rate e_t falls. Figure 3 indicates a linear relationship.

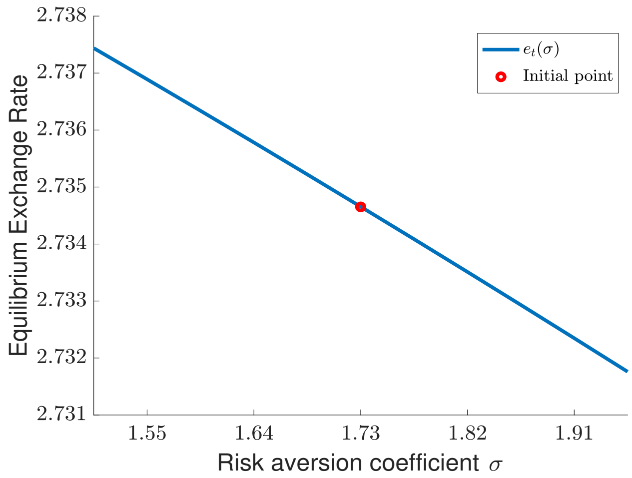

Figure 3: Change in the equilibrium exchange rate due to a change in the risk aversion coefficient σ

ii) variation in the variance of the exchange rate σ_e^2 and iii) variation in the variance of the income σ_y^2

An increment in the variance of the exchange rate σ_e^2 leads to a decline in the foreign bond level b_{t+1}^*. The more exchange rates in the future vary, the more households perceive the foreign asset as a bad hedge against income risk. Subsequently, the level of foreign bonds decreases.

The more income varies in the period $t+1$, the more households demand a hedge against income risk and, hence, the level of foreign bonds increases.

The blue line in figure 4 corresponds to the change in the foreign bond level due to a change in the variance of the exchange rate. The orange line shows the changes in the foreign bond level due to a change in the variance of the income. Moreover, figure 4 also shows that variations in the variance of the income have larger effects on the foreign bond level, which is in line with the explanation in 5i).[2] Again, the plots indicate a linear relationship.

[2]Desire for a hedge against income risk overweights exchange rate risk, which leads to a higher foreign bond level if σ increases

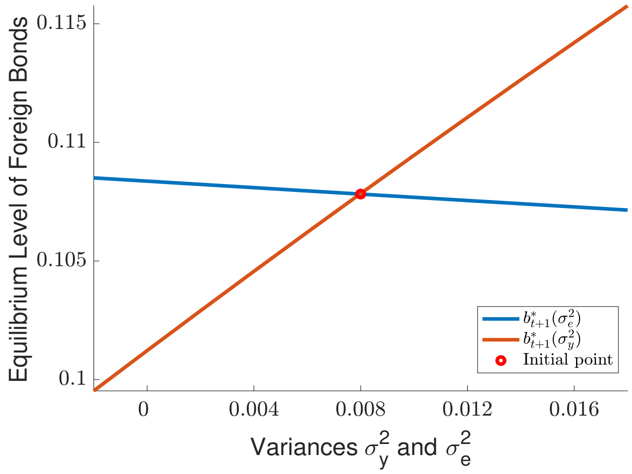

Figure 4: Change in the equilibrium level of foreign bonds due to a change in the variance of the exchange rate σ_e^2 and a change in the variance of income σ_y^2

An increment of both variances decreases the risk premium but to a different extent. As stated in 5i), $c_{s_{t+1}}$ comoves with the exchange rate $e_{s_{t+1}}$, which is captured by the falling curves in figure 5. Therefore, an increment in the exchange rate variance σ_e^2 comes with a higher demand for additional risk compensation because foreign bonds become a less attractive hedge against income risk.

An increment of the income variance σ_y^2 also leads to a higher desired risk compensation because consumption becomes less uncertain in future periods. However, the income variance increment affects the desire for risk compensation less than the incline in exchange rate variance. Effects are again linear.

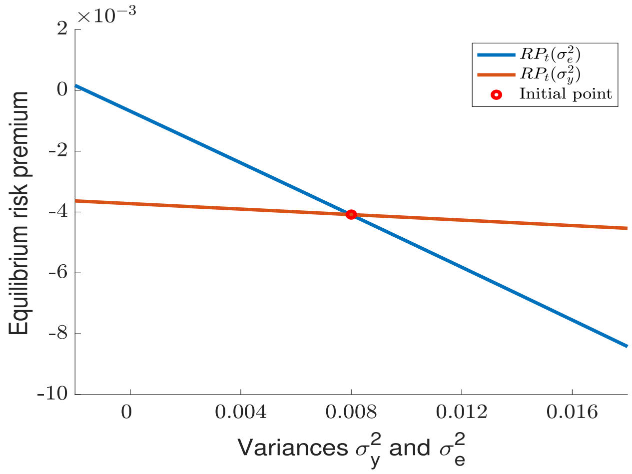

Figure 5: Change in the risk premium due to a change in the variance of the exchange rate σ_e^2 and a change in the variance of income σ_y^2

The exchange rate e_t increases with the exchange rate variance σ_e^2. Foreign bonds become a worse hedge if the uncertainty of the next period's exchange rate increases and the risk premium increases due to a stronger desire for risk compensation. Again, the pricing effects channel through the exchange rate as interest rates stay constant. The effect of the risk premium overweights the impact of the UIP, the exchange rate increases and the domestic currency depreciates relative to foreign currency. The UIP fails to hold.

The exchange rate e_t decreases with the exchange rate variance σ_y^2. Foreign bonds become a better hedge if the income risk increases and the risk premium decreases slightly. Pricing effects channel through the exchange rate as interest rates stay constant. The exchange rate e_t falls because the impact of the UIP on the exchange rate is overweighted by the effect of the risk premium. The domestic currency appreciates relative to the foreign currency because the UIP fails to hold. Both effects are again linear.

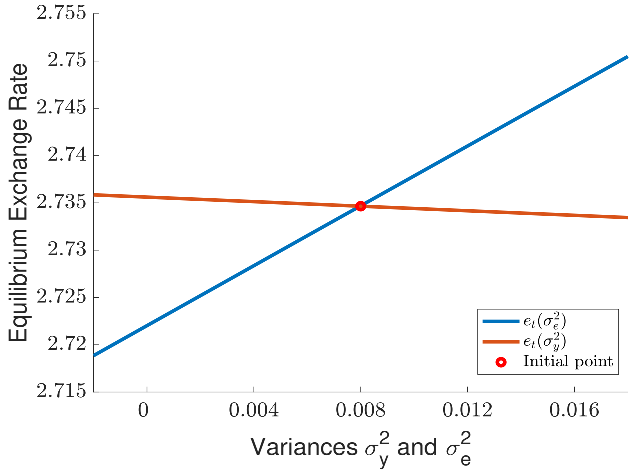

Figure 6: Change in the equilibrium exchange rate due to a change in the variance of the exchange rate σ_e^2 and a change in the variance of income σ_y^2

iv) variation in the covariance of the exchange rate and the income $\sigma_{y,e}$

An increment of the covariance of the exchange rate and the income $\sigma_{y,e}$ increases the the level of the foreign bonds b^*_{t+1}. More comovements of income/consumption and the exchange rate in period $t+1$ lead to higher demand for exchange rate risk compensation. The falling risk premia in figure 7 capture these dynamics as the pricing kernel and consumption have an inverse relationship. The risk premium curve 8 indicates that the effects are linear.

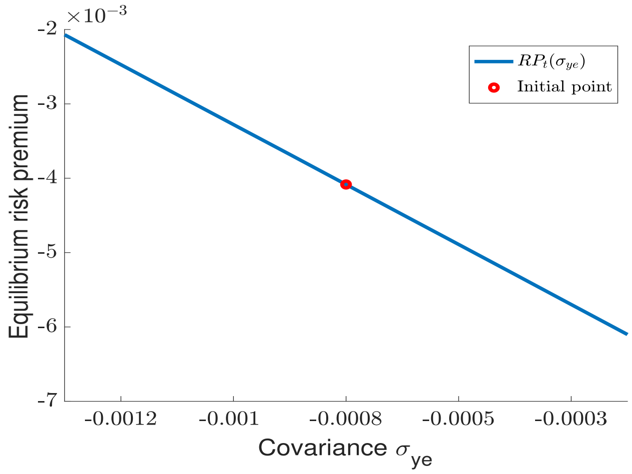

Figure 7: Change in the risk premium due to a change in the covariance of the exchange rate and the income $\sigma_{y,e}$

The increment in the covariance of the exchange rate and the income $\sigma_{y,e}$ leads to an incline of the level of foreign bonds b^*_{t+1}. As foreign bonds are the only asset that hedges against income risk, demand increases because its hedging property worsens with comovements in the exchange rate and income risk.

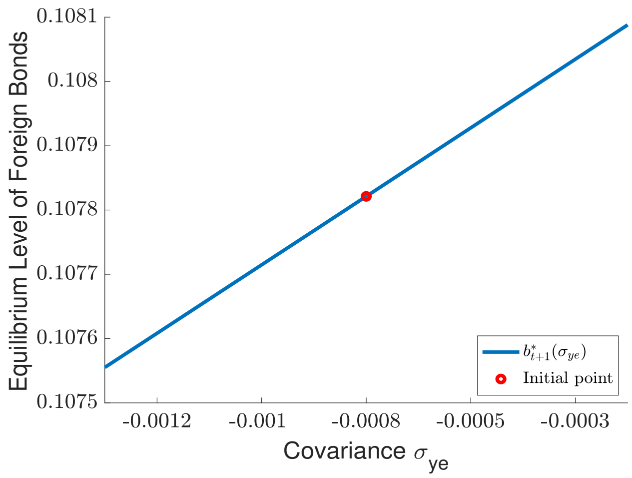

Figure 8: Change in the equilibrium level of foreign bonds due to a change in the covariance of the exchange rate and the income $\sigma_{y,e}$

The exchange rate e_t decreases with the covariance of the exchange rate and the income $\sigma_{y,e}$. Again, the UIP fails because the effects of the risk premium overweight its impact. The domestic currency appreciates relative to the foreign currency.

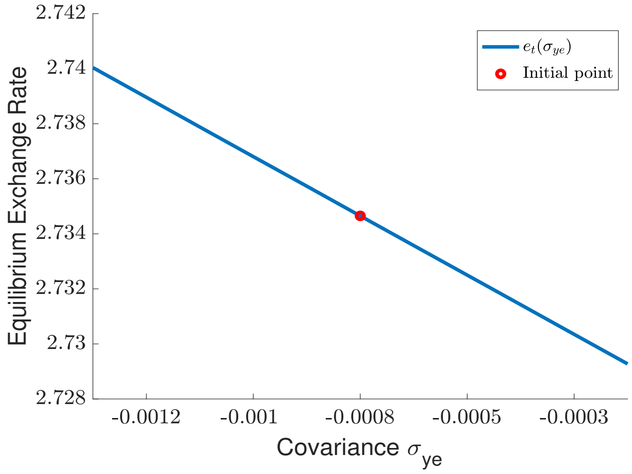

Figure 9: Change in the equilibrium exchange rate due to a change in the covariance of the exchange rate and the income $\sigma_{y,e}$

6.) Comparison to a model with a perfectly anticipated exchange rate

i) variation in the risk aversion coefficient σ

In a model with a perfectly anticipated exchange rate e_{t+1}, the change in the degree of risk aversion σ has a similar effect on the equilibrium level of foreign bonds b^*_{t+1} compared to the uncertainty model assessed in this assignment. Figure 10 shows a parallel trend. The difference between the models can be explained by the definition of $e_{t+1} = exp(1)$ in the perfectly anticipated model. Modifying this term could lead to almost identical results.

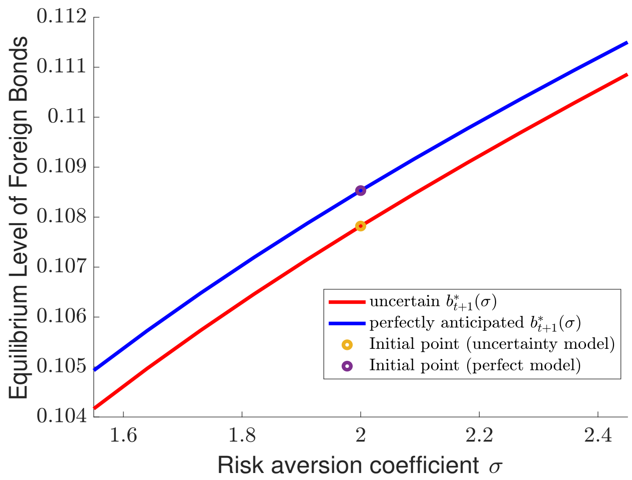

Figure 10: Change in the level of foreign bonds due to a change in the risk aversion coefficient σ

As the UIP holds in a perfectly anticipated model (see equation (11)), the risk premium is zero and the exchange rate stays constant for different values of σ. In the uncertainty model, the exchange rate decreases with the degree of risk aversion, as previously explained. Figure 11 graphically illustrates the impact of the risk aversion coefficient σ on the exchange rate e_t in both models. It also shows that the exchange rate is overall higher in the uncertainty model. However, this is again caused by the definition of e_{t+1}.

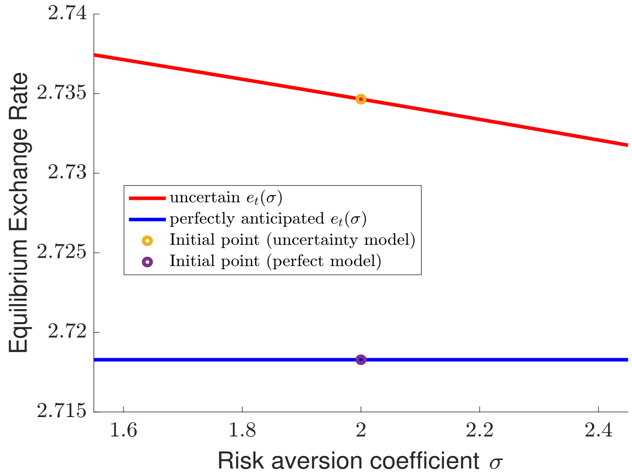

Figure 11: Change in the exchange level due to a change in the risk aversion coefficient σ

iii) variation in the variance of the income σ_y^2

Again, changes in the income variance σ_y^2 have a similar impact on the level of foreign bonds in both models. Intuitively, investors are risk-averse and want to hedge risk by shifting wealth from the initial to the future period in both models. The existence of one uncertainty source has almost the same impact on the foreign bond level in the perfectly anticipated model because the relationship between income/consumption and the exchange rate is weak in the uncertainty model, $\sigma_{y,e} = -0.0001$. Hence, investors demand almost the same amount of savings in foreign bonds to hedge against income risk in both models.

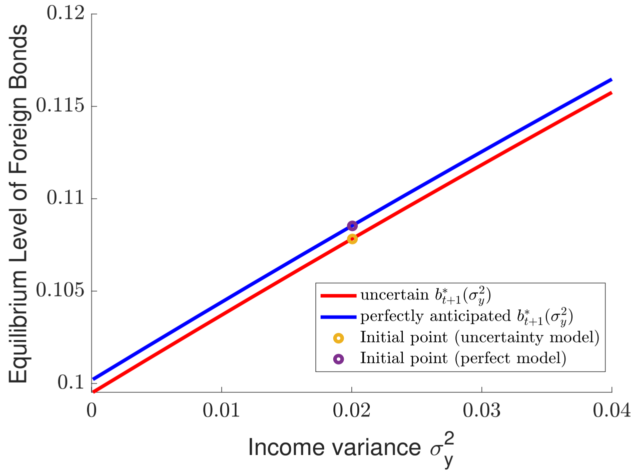

Figure 12: Change in the level of foreign bonds due to a change in the variance of the income σ_y^2

Again, UIP holds and the exchange rate is not affected due to change in the income variance in the perfectly anticipated model. Likewise, the risk premium is zero.

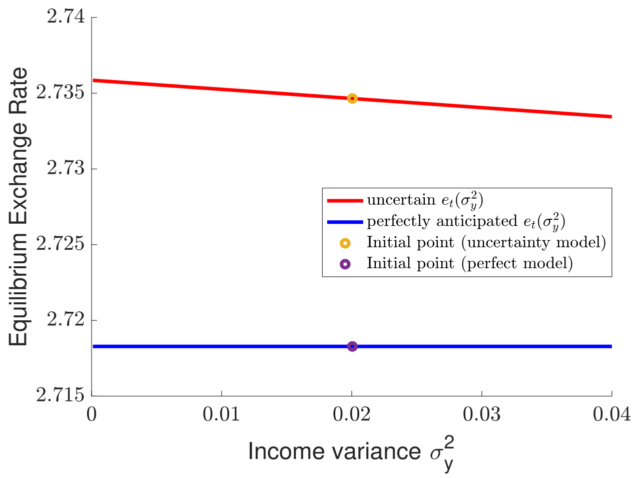

Figure 13: Change in the level of foreign bonds due to a change in the variance of the income σ_y^2

ii) variation in the variance of the exchange rate σ_e^2 and iv) variation in the covariance of the exchange rate and the income $\sigma_{y,e}$

Both parameters do not show up in the perfectly anticipated model and, hence, variations do not have an impact on the variables of interest within this model.

Conclusion Changes in the parameters, which appear in both models, namely σ and σ_y^2, have almost identical impacts on the level of foreign bonds.[3] Other parameters, e.g., σ_e^2 and $\sigma_{y,e}$, only affect the level of foreign bonds in the uncertainty model. The exchange rate is never affected and the risk premium is zero in the perfectly anticipated model because the UIP holds.

[3]Tables a presented in the Appendix.

7.) **The risk premium**

Figure 14 graphically illustrates that the risk premium approaches zero if the variance of the income σ_y^2 approaches zero. Table 5 in the Appendix gives the exact numbers for this task.

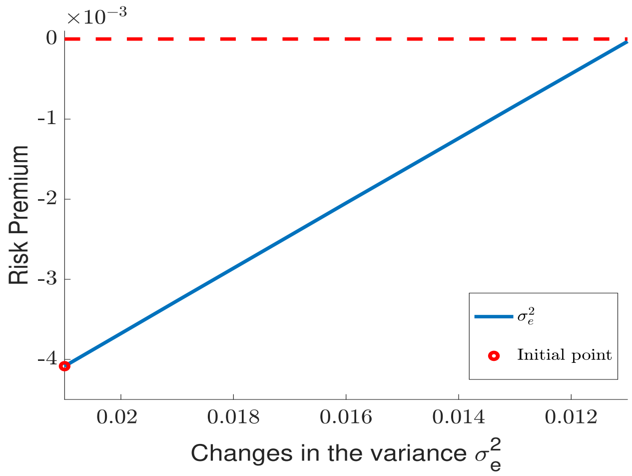

Figure 14: Change in the risk premium due to σ_y^2 approaching zero.

A Appendix: Figures and Tables

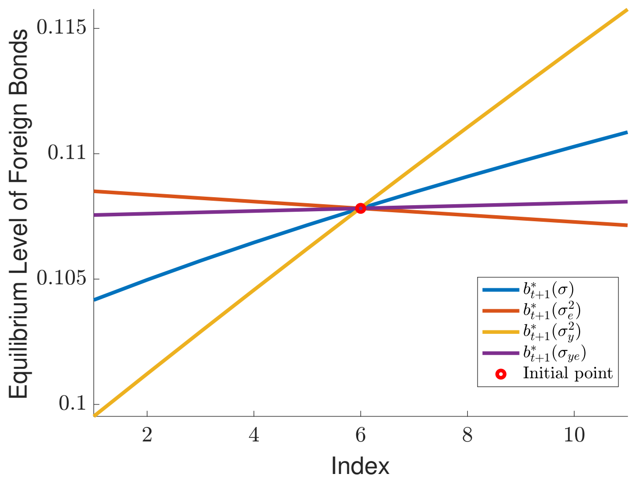

Figure 15: Changes in the level of foreign bonds (i-iv)

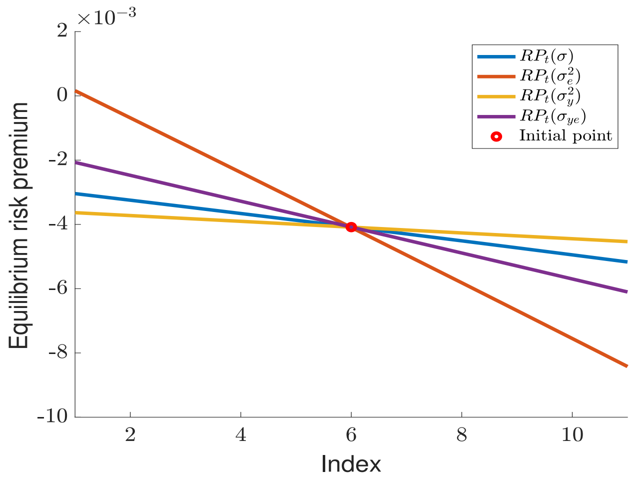

Figure 16: Changes in the risk premium (i-iv)

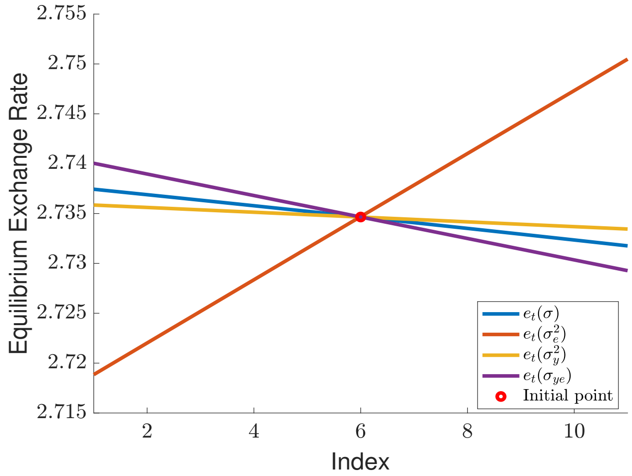

Figure 17: Changes in the exchange rate (i-iv)

Table 1: Changes of b^*_{t+1}, e_t, and RP_t with respect to the risk aversion coefficient σ

σ	1.55	1.64	1.73	1.82	1.91	2	2.09	2.18	2.27	2.36	2.45
b^*_{t+1}	0.1042	0.1050	0.1057	0.1065	0.1072	0.1078	0.1085	0.1091	0.1097	0.1103	0.1109
e_t	2.7374	2.7369	2.7363	2.7358	2.7352	2.7347	2.7341	2.7335	2.7329	2.7323	2.7318
RP_t	-0.0030	-0.0032	-0.0035	-0.0037	-0.0039	-0.0041	-0.0043	-0.0045	-0.0047	-0.0049	-0.0052

Table 2: Changes of b^*_{t+1}, e_t, and RP_t with respect to the variance of the exchange rate in period σ^2_e

σ^2_e	0.0001	0.0041	0.0081	0.0121	0.0161	0.02	0.024	0.028	0.032	0.036	0.04
b^*_{t+1}	0.1085	0.1084	0.1082	0.1081	0.1080	0.1078	0.1077	0.1075	0.1074	0.1073	0.1071
e_t	2.7188	2.7220	2.7252	2.7284	2.7315	2.7347	2.7379	2.7410	2.7442	2.7473	2.7505
RP_t	0.0002	-0.0007	-0.0015	-0.0024	-0.0032	-0.0041	-0.0050	-0.0058	-0.0067	-0.0076	-0.0084

Table 3: Changes of b^*_{t+1}, e_t, and RP_t with respect to the variance of the income in period σ^2_y

σ^2_y	0.0001	0.0041	0.0081	0.0121	0.0161	0.02	0.024	0.028	0.032	0.036	0.04
b^*_{t+1}	0.0995	0.1012	0.1029	0.1046	0.1062	0.1078	0.1095	0.1111	0.1126	0.1142	0.1158
e_t	2.7359	2.7356	2.7354	2.7351	2.7349	2.7346	2.7344	2.7342	2.7339	2.7337	2.7334
RP_t	-0.0036	-0.0037	-0.0038	-0.0039	-0.0040	-0.0041	-0.0042	-0.0043	-0.0044	-0.0044	-0.0045

Table 4: Changes of b^*_{t+1}, e_t, and RP_t with respect to the covariance of the exchange rate and the income $\sigma_{y,e}$

$\sigma_{y,e}$	-0.0012	-0.0010	-0.0008	-0.0005	-0.0003	-0.0001	0.0001	0.0003	0.0006	0.0008	0.0010
b^*_{t+1}	0.1076	0.1076	0.1077	0.1077	0.1078	0.1078	0.1079	0.1079	0.1080	0.1080	0.1081
e_t	2.7400	2.7390	2.7379	2.7368	2.7357	2.7347	2.7336	2.7325	2.7314	2.7303	2.7293
RP_t	-0.0021	-0.0025	-0.0029	-0.0033	-0.0037	-0.0041	-0.0045	-0.0049	-0.0053	-0.0057	-0.0061

Table 5: Change in the risk premium due to σ^2_y approaching zero. Rounded to 4 decimal digits.

σ^2_e	0.0200	0.0181	0.0162	0.0143	0.0124	0.0105	0.0086	0.0067	0.0048	0.0029	0.0010
RP_t	-0.0041	-0.0037	-0.0033	-0.0029	-0.0025	-0.0020	-0.0016	-0.0012	-0.0008	-0.0004	0

Declaration of Authorship

I hereby declare

- that I have written this essay without any help from others and without the use of documents and aids other than those stated in the references;
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(Lauritz Storch)